SIAM REVIEW
Vol. 31, No. 2, pp. 299–302, June 1989

005

CLASSROOM NOTES

EDITED BY MURRAY S. KLAMKIN

This section contains brief notes which are essentially self-contained applications of mathematics that can be used in the classroom. New applications are preferred, but exemplary applications not well known or readily available are accepted.

Both "modern" and "classical" applications are welcome, especially modern applications to current real world problems.

Notes should be submitted to M. S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

COMPUTING THE PROJECTED AREA OF A CONE*

S. PENNELL† AND J. DEIGNAN‡

Abstract. Techniques from linear algebra and calculus are used to compute the projected area of an arbitrarily oriented right circular cone. This type of computation is important in the design of interceptor missile systems, which use infrared sensors to distinguish between warheads and decoys.

Key words. projection, radiant heat transfer

AMS(MOS) subject classifications. primary 15A04; secondary 26A06, 80A20

1. Introduction. In this note we use techniques from linear algebra and calculus to compute the area of the projection onto a plane Π of an arbitrarily oriented right circular cone C. This problem arose in the design of an interceptor missile system. Interceptor missiles are used to intercept nuclear warheads. These missiles are aided by an infrared sensing device, or infrared spectrophotometer, which provides a thermal image, or "thermal fingerprint," of a nuclear warhead traveling outside the earth's atmosphere. This thermal image is necessary because today's nuclear warheads are accompanied by several "decoys." With conventional radar, it is very difficult, if not impossible, to differentiate between the decoys and the real warheads, and due to the large number of accompanying decoys the successful interception of all the nuclear warheads is unlikely with the aid of radar alone. With infrared spectrophotometry, the interceptor missiles can distinguish the warheads from the decoys.

The infrared spectrophotometer senses the radiation emanating from the warheads and decoys. Due to the difference in mass, or thermal inertia, between warheads and decoys, there is a marked difference in the level of their radiation output over time. This difference is used to differentiate between warheads and decoys. However, since the quantity of radiation received by the sensor at any given instant depends on the projected area of the emitting object, a change in the detected level of an object's radiation output can also be caused by a change in the object's spatial orientation. In order to determine the magnitude of this effect, the maximum and minimum projected areas of the warhead must be calculated for any orientation in three space dimensions. Although we consider only right circular cones in this note, the procedure outlined here can be extended to other types of bodies as well.

* Received by the editors July 10, 1987; accepted for publication (in revised form) May 24, 1988.
† Department of Mathematics, University of Lowell, Lowell, Massachusetts 01854.
‡ 272 School Street, Watertown, Massachusetts 02172.

2. Formulation. Let C be a right circular cone with base in the x-y plane and axis along the z axis. Let r denote the radius of the base, and let h denote the cone's height. (We take the vertex of C to be $(0, 0, h)$.) Let Π be the plane through the origin with normal vector $(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$, where $0 \leqq \varphi < \pi$ and $0 \leqq \theta < 2\pi$. We wish to find the area of C's "shadow," i.e., the area of the region in Π occupied by the projection of C.

3. Formula for the projection operator. In this section we let $P: R^3 \to R^3$ denote the linear transformation that maps a vector to its projection on Π. We use the notation $[P; \alpha, \beta]$ to mean the matrix representing P relative to the ordered bases α and β. (See [1, p. 70].)

In deriving a formula for the projection operator P and in computing the area of the cone's shadow, it is convenient to use an orthonormal basis for R^3 consisting of two vectors in Π and a vector normal to Π. We therefore introduce the basis $\beta = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$, where $\mathbf{u}_1 = (-\sin\theta, \cos\theta, 0)$, $\mathbf{u}_2 = (-\cos\varphi\cos\theta, -\cos\varphi\sin\theta, \sin\varphi)$, and $\mathbf{u}_3 = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$. Since $\mathbf{u}_1$ and $\mathbf{u}_2$ lie in Π, $P(\mathbf{u}_1) = \mathbf{u}_1$ and $P(\mathbf{u}_2) = \mathbf{u}_2$; since $\mathbf{u}_3$ is normal to Π, $P(\mathbf{u}_3) = \mathbf{0}$. Thus,

$$[P; \beta, \beta] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

The change of coordinate matrix that transforms coordinates relative to the standard basis $\alpha = ((1, 0, 0), (0, 1, 0), (0, 0, 1))$ into coordinates relative to β is simply $[I; \alpha, \beta]$, where $I: R^3 \to R^3$ is the identity transformation. (See [1, p. 98].) Since $I(1, 0, 0) = (1, 0, 0) = (-\sin\theta)\mathbf{u}_1 + (-\cos\varphi\cos\theta)\mathbf{u}_2 + (\sin\varphi\cos\theta)\mathbf{u}_3$, $I(0, 1, 0) = (0, 1, 0) = (\cos\theta)\mathbf{u}_1 + (-\cos\varphi\sin\theta)\mathbf{u}_2 + (\sin\varphi\sin\theta)\mathbf{u}_3$, and $I(0, 0, 1) = (0, 0, 1) = 0\mathbf{u}_1 + (\sin\varphi)\mathbf{u}_2 + (\cos\varphi)\mathbf{u}_3$, this change of coordinate matrix is given by

$$[I; \alpha, \beta] = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\varphi\cos\theta & -\cos\varphi\sin\theta & \sin\varphi \\ \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \end{bmatrix}. \tag{2}$$

Let $A = [P; \alpha, \beta]$. Then $A = [PI; \alpha, \beta] = [P; \beta, \beta][I; \alpha, \beta]$. (See [1, p. 78].) Hence,

$$A = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\varphi\cos\theta & -\cos\varphi\sin\theta & \sin\varphi \\ 0 & 0 & 0 \end{bmatrix}. \tag{3}$$

It follows that if $(x, y, z)^T$ is the coordinate vector of $\mathbf{x}$ relative to the basis α and if $(u, v, w)^T$ is the coordinate vector of $\mathbf{y} = P(\mathbf{x})$ relative to the basis β, then $(u, v, w)^T = A(x, y, z)^T$. Thus,

$$\begin{aligned} u &= (-\sin\theta)x + (\cos\theta)y, \\ v &= (-\cos\varphi\cos\theta)x + (-\cos\varphi\sin\theta)y + (\sin\varphi)z, \\ w &= 0. \end{aligned} \tag{4}$$

For example, the image of the vertex of the cone, $(0, 0, h)$, is the point $(0, h\sin\varphi)$ in the u-v plane. (The u-v plane is the projection plane Π with the basis $\gamma = (\mathbf{u}_1, \mathbf{u}_2)$.)

4. Determination of the shadow region. Now that we have derived (4) describing the projection operator, it is easy to determine the region in the u-v plane Π occupied by the cone's shadow. The base of the cone consists of the family of concentric circles

$$x = \rho\cos\omega, \quad y = \rho\sin\omega, \quad z = 0, \tag{5}$$

$0 \leqq \omega \leqq 2\pi$, $0 \leqq \rho \leqq r$. The images of these circles are the ellipses

$$\begin{aligned} u &= -\rho \sin\theta \cos\omega + \rho \cos\theta \sin\omega = -\rho \sin(\theta-\omega), \\ v &= -\rho \cos\varphi \cos\theta \cos\omega - \rho \cos\varphi \sin\theta \sin\omega \\ &= -\rho \cos\varphi \cos(\theta-\omega), \end{aligned} \tag{6}$$

or

$$(\cos^2\varphi)u^2 + v^2 = \rho^2 \cos^2\varphi. \tag{7}$$

The lateral surface of the cone consists of the family of lines joining the vertex $(0, 0, h)$ to the points on the boundary of the base:

$$x = (r\cos\omega)t, \quad y = (r\sin\omega)t, \quad z = h(1-t), \tag{8}$$

$0 \leqq \omega \leqq 2\pi$, $0 \leqq t \leqq 1$. The images of these lines are lines in the u-v plane joining $(0, h\sin\varphi)$ to the points on the ellipse given by $\rho = r$ in (6):

$$\begin{aligned} u &= [-r\sin(\theta-\omega)]t, \\ v &= [-r\cos\varphi\cos(\theta-\omega)]t + (h\sin\varphi)(1-t), \end{aligned} \tag{9}$$

$0 \leqq \omega \leqq 2\pi$, $0 \leqq t \leqq 1$.

Thus, the shadow of the cone is an elliptical region if $(0, h\sin\varphi)$ lies inside the image of the cone's base (Fig. 1a). The shadow is an elliptical region with a "hat" if $(0, h\sin\varphi)$ lies outside the image of the cone's base (Fig. 1b).

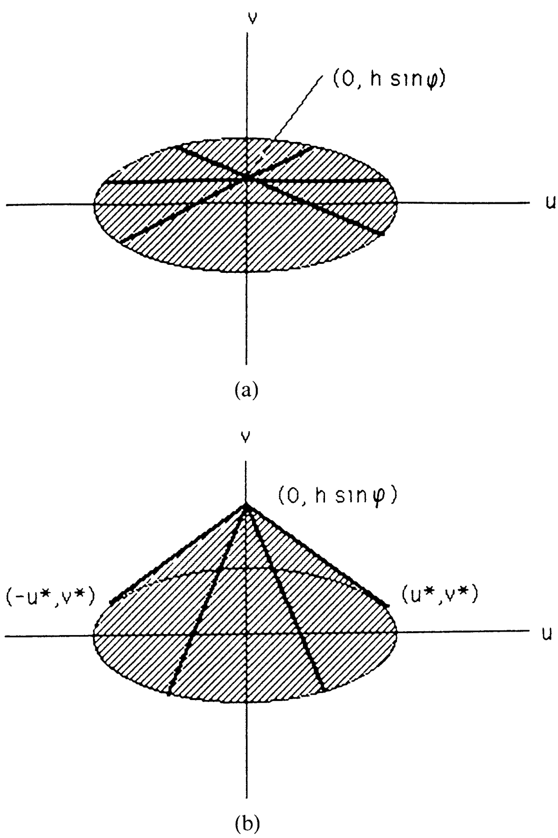

FIG. 1. (a) *The shaded area is the cone's shadow if* $h\sin\varphi < r|\cos\varphi|$. (b) *The shaded area is the cone's shadow if* $h\sin\varphi > r|\cos\varphi|$.

5. Computation of the projected area. The computation of the area of the cone's shadow is now straightforward. The elliptical region obviously has area $\pi r^2\,|\cos\varphi|$. The area of the "hat," which is present if $h\sin\varphi > r\,|\cos\varphi|$, can be found by means of elementary calculus. We first find the points $(\pm u^*, v^*)$ on the ellipse with the property that the line from $(0, h\sin\varphi)$ to $(\pm u^*, v^*)$ is tangent to the ellipse. The upper portion of the ellipse is given by $v = |\cos\varphi|\sqrt{r^2-u^2}$, while the line from $(0, h\sin\varphi)$ to (u^*, v^*) is given by $v = h\sin\varphi + mu$, where $m = (v^* - h\sin\varphi)/u^*$. (Note that $v^* = |\cos\varphi|\sqrt{r^2-(u^*)^2}$.) Equating the slopes of these two curves, we find that $u^* = (r/h)\sqrt{h^2 - r^2\cot^2\varphi}$ and $m = -u^*(h\sin\varphi)/r^2$. The area of the "hat," a_h, is obtained by integrating the difference between the straight segment and the upper part of the ellipse. Thus,

$$\begin{aligned} a_h &= 2\int_0^{u^*} \left[(h\sin\varphi + mu) - |\cos\varphi|\,\sqrt{r^2-u^2}\right] du \\ &= (2h\sin\varphi)u^* + m(u^*)^2 - |\cos\varphi|\left[r^2\sin^{-1}(u^*/r) + u^*\sqrt{r^2-(u^*)^2}\right] \\ &= (2h\sin\varphi)u^* - (u^*)^3(h\sin\varphi)/r^2 - |\cos\varphi|\left[r^2\sin^{-1}(u^*/r) + u^*\sqrt{r^2-(u^*)^2}\right]. \end{aligned}$$

Hence, the area of the cone's shadow is

$$a = \begin{cases} \pi r^2|\cos\varphi| & \text{if } h\sin\varphi \leqq r\,|\cos\varphi|, \quad \text{(10a)} \\ \pi r^2|\cos\varphi| + a_h(\varphi) & \text{if } h\sin\varphi > r\,|\cos\varphi|. \quad \text{(10b)} \end{cases}$$

Note that a is independent of θ, as might have been anticipated from the symmetry of the cone.

Formulas (10) yield the expected results in the special cases $\varphi = 0$ and $\varphi = \pi/2$. When $\varphi = 0$, Π is the x-y plane and C's shadow is simply its base, a circle of radius r. In this case $a = \pi r^2$, as predicted by (10a). When $\varphi = \pi/2$, Π is the y-z plane (if $\theta = 0$) and C's shadow is a triangle of base $2r$ and height h, with area $a = rh$. This value is obtained from (10b), with $u^* = r$.

REFERENCE

[1] S. Friedberg, A. Insel, and L. Spence, *Linear Algebra*, Prentice–Hall, Englewood Cliffs, NJ, 1979.