Solutions to Real and Complex Analysis

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1 Chapter 1

1.1 Ex 1

Does there exist an infinite σ-algebra which has only countably many members?

No. Assume Σ is a countable σ-algebra on a space X, and for any $x \in X$ define

$$U_x = \bigcap_{\substack{U \in \Sigma \\ x \in U}} U$$

Since σ-algebras are closed under countable intersection, $U_x \in \Sigma$. Observe that given any $U \in \Sigma$,

$$U = \bigcup_{x \in U} U_x$$

so that Σ is generated by $\{U_x\}$.

We further note that if $y \notin U_x$, then U_y^c is an element of Σ that contains x, in which case $U_x \subseteq U_y^c$, so that U_x and U_y are disjoint. This implies that given two $x, y \in X$, then either U_x and U_y are equal, or they are disjoint.

Since finite collections generate finite σ-algebras, $\{U_x\}$ must be countably infinite. We can therefore assume that $\{U_x\} = \{U_n\}_{n=1}^{\infty}$, and that U_i and U_j are disjoint whenever $i \neq j$.

This implies that there exists an injection $2^{\mathbb{N}} \to \Sigma$, given by

$$(n_1, n_2, \cdots) \mapsto \bigcap_{n_i = 1} U_{n_i} \in \Sigma$$

However, $|2^{\mathbb{N}}| > \aleph_0$, which contradicts the assumption that Σ is countably infinite.

1.2 Ex 2

This proof is entirely analogous to the proof in the book. Let $f = (u_1(x), \cdots, u_n(x))$. It suffices to show that f is measurable. Note that the product topology on $\mathbb{R}^n$

is generated by a countable union of products of the form $R = I_1 \times \cdots \times I_n$. Since $f^{-1}(R) = u_1^{-1}(R) \cap \cdots \cap u_n^{-1}(R)$ is measurable, we are done.