

OpenCMISS-iron examples and tests used by OpenCMISS developers at University of Stuttgart, Germany

Christian Bleiler*, Dr.-Ing. Nehzat Emamy[†],
Andreas Hessenthaler*, Thomas Klotz*,
Aaron Krämer[‡], Benjamin Maier[†], Sergio Morales*,
Mylena Mordhorst*, Harry Saini*

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CONTENTS

1	Test summary	7
1.1	Details	7
2	Introduction	11
2.1	Cogui files for cmgui-2.9	11
2.2	Variations to consider	11
2.3	Folder structure	12
3	Progress	13
3.1	Equations to test	13
3.2	Setting up a new test	13
3.3	Long-term goals	14
4	Diffusion equation	15
4.1	Equation in general form	15
4.2	Example-0001 [VALIDATED]	16
4.2.1	Mathematical model - 2D	16
4.2.2	Mathematical model - 3D	16
4.2.3	Computational model	16
4.2.4	Result summary	17
4.3	Example-0001-u [PLAUSIBLE]	20
4.3.1	Mathematical model - 2D	20
4.3.2	Mathematical model - 3D	20
4.3.3	Computational model	20

* Institute of Applied Mechanics (CE), University of Stuttgart, Pfaffenwaldring 7, 70569 Stuttgart, Germany

† Institute for Parallel and Distributed Systems, University of Stuttgart, Universitätsstraße 38, 70569 Stuttgart, Germany

‡ Lehrstuhl Mathematische Methoden für komplexe Simulation der Naturwissenschaft und Technik, University of Stuttgart, Allmandring 5b, 70569 Stuttgart, Germany

4.3.4	Result summary	21
4.4	Example-0002 [VALIDATED]	24
4.4.1	Mathematical model - 2D	24
4.4.2	Mathematical model - 3D	24
4.4.3	Computational model	24
4.4.4	Result summary	25
4.5	Example-0003 [COMPILES]	28
4.5.1	Mathematical model - 2D	28
4.5.2	Mathematical model - 3D	28
4.5.3	Computational model	28
4.5.4	Result summary	29
4.6	Example-0004 [VALIDATED]	32
4.6.1	Mathematical model - 2D	32
4.6.2	Computational model	32
4.6.3	Result summary	32
4.7	Example-0005 [VALIDATED]	34
4.7.1	Mathematical model - 2D	34
4.7.2	Mathematical model - 3D	34
4.7.3	Computational model	34
4.7.4	Result summary	35
4.8	Example-0011 [VALIDATED]	38
4.8.1	Mathematical model - 2D	38
4.8.2	Mathematical model - 3D	38
4.8.3	Computational model	38
4.8.4	Result summary	39
4.9	Example-0012 [VALIDATED]	42
4.9.1	Mathematical model - 2D	42
4.9.2	Mathematical model - 3D	42
4.9.3	Computational model	42
4.9.4	Result summary	43
4.10	Example-0013 [VALIDATED]	46
4.10.1	Mathematical model - 2D	46
4.10.2	Mathematical model - 3D	46
4.10.3	Computational model	46
4.10.4	Result summary	48
	4.10.5 Misc	48
5	Linear elasticity	51
5.1	Equation in general form	51
5.2	Example-0101 [PLAUSIBLE]	52
5.2.1	Mathematical model	52
5.2.2	Computational model	52
5.2.3	Validation	53
5.3	Example-0102 [PLAUSIBLE]	56
5.3.1	Mathematical model	56
5.3.2	Computational model	56
5.3.3	Validation	57
5.4	Example-0111 [PLAUSIBLE]	61
5.4.1	Mathematical model	61
5.4.2	Computational model	61
5.4.3	Validation	62
5.5	Example-0112 [PLAUSIBLE]	65
5.5.1	Mathematical model	65
5.5.2	Computational model	65

5.5.3	Validation	66
6	Finite elasticity	68
7	Navier-Stokes flow	69
7.1	Equation in general form	69
7.2	Example-0302-u [COMPILES]	70
7.2.1	Mathematical model - 2D	70
7.2.2	Mathematical model - 3D	71
7.2.3	Computational model	71
7.2.4	Result summary	72
8	Monodomain	73
8.1	Example-0401 [PLAUSIBLE]	74
8.1.1	Mathematical model	74
8.1.2	Computational model	74
8.1.3	Results	75
8.1.4	Validation	75
8.2	Example-0402 [PLAUSIBLE]	79
8.2.1	Mathematical model	79
8.2.2	Computational model	79
8.2.3	Results	80
8.2.4	Validation	80
8.3	Example-0404-c [PLAUSIBLE]	84
8.3.1	Mathematical model	84
8.3.2	Computational model	84
8.3.3	Results	85
8.3.4	Validation	85
9	CellML model	88

LIST OF FIGURES

Figure 1	2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].	17
Figure 2	2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].	18
Figure 3	3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].	18
Figure 4	3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].	19
Figure 5	2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].	22
Figure 6	2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].	22
Figure 7	3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].	23
Figure 8	3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].	23
Figure 9	2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].	25
Figure 10	2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].	26
Figure 11	3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].	26

Figure 12	3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].	27
Figure 13	2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].	30
Figure 14	2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].	30
Figure 15	3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].	31
Figure 16	3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].	31
Figure 17	2D results, iron reference w/ command line arguments [8 4 0 2 0].	33
Figure 18	2D results, current run w/ command line arguments [8 4 0 2 0].	33
Figure 19	2D geometry and mesh.	35
Figure 20	2D results, iron reference w/ command line arguments [2 2 0].	35
Figure 21	2D results, current run w/ command line arguments [2 2 0].	36
Figure 22	3D geometry (unit cube) and mesh.	36
Figure 23	3D results, iron reference w/ command line arguments [3 2 0].	37
Figure 24	3D results, current run w/ command line arguments [3 2 0].	37
Figure 25	2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0 1 1].	40
Figure 26	2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0 1 1].	40
Figure 27	3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0 1 1 1].	41
Figure 28	3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0 1 1 1].	41
Figure 29	2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0 2 3 0 0 0 0].	44
Figure 30	2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0 2 3 0 0 0 0].	44
Figure 31	3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0 2 3 7 0 0 0].	45
Figure 32	3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0 2 3 7 0 0 0].	45
Figure 33	2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0 2 3 0 0.523598775598299 0 0].	48
Figure 34	2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0 2 3 0 0.523598775598299 0 0].	49
Figure 35	3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0 2 3 7 0.523598775598299 0.698131700797732 0.174532925199433].	49
Figure 36	3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0 2 3 7 0.523598775598299 0.698131700797732 0.174532925199433].	50
Figure 37	Results, Abaqus 2D fine mesh.	53
Figure 38	Results, abaqus 3D fine mesh.	54

Figure 39	Results, iron 2D fine mesh	55
Figure 40	Results, iron 3D fine mesh	55
Figure 41	Results, Abaqus 2D fine mesh.	58
Figure 42	Results, abaqus 3D fine mesh.	59
Figure 43	Results, iron 2D fine mesh.	60
Figure 44	Results, iron 3D fine mesh.	60
Figure 45	Results, Abaqus 2D fine mesh.	63
Figure 46	Results, abaqus 3D fine mesh.	63
Figure 47	Results, iron 2D fine mesh.	63
Figure 48	Results, iron 3D fine mesh.	63
Figure 49	Results, Abaqus 2D fine mesh.	67
Figure 50	Results, abaqus 3D fine mesh.	67
Figure 51	Results, iron 2D fine mesh.	67
Figure 52	Results, iron 3D fine mesh.	67
Figure 53	Result of scenario with 10×10 elements, $t = 20$, direct solver, $p = 1$ process	75
Figure 54	Result of scenario with 24×24 elements, $t = 20$, direct solver, $p = 1$ process	76
Figure 55	Result of scenario with 24×24 elements, $t = 20$, iterative solver, $p = 1$ process	76
Figure 56	Result of scenario with 24×24 elements, $t = 20$, iterative solver, $p = 2$ processes	77
Figure 57	Result of scenario with 24×24 elements, $t = 20$, iterative solver, $p = 8$ processes	77
Figure 58	Result of scenario with 2×2 elements, $t = 20$, direct solver, $p = 1$ process	78
Figure 59	Result of scenario with 2×2 elements, $t = 20$, direct solver, $p = 2$ processes	78
Figure 60	Result of scenario with 10×10 elements, $t = 20$, direct solver, $p = 1$ process	80
Figure 61	Result of scenario with 24×24 elements, $t = 20$, direct solver, $p = 1$ process	81
Figure 62	Result of scenario with 24×24 elements, $t = 20$, iterative solver, $p = 1$ process	81
Figure 63	Result of scenario with 24×24 elements, $t = 20$, iterative solver, $p = 2$ processes	82
Figure 64	Result of scenario with 24×24 elements, $t = 20$, iterative solver, $p = 8$ processes	82
Figure 65	Result of scenario with 2×2 elements, $t = 20$, direct solver, $p = 1$ process	83
Figure 66	Result of scenario with 2×2 elements, $t = 20$, direct solver, $p = 2$ processes	83
Figure 67	V_m for time $t = 1.0$, different time step widths $dt \in \{0.01, 0.005, 0.001, 0.0005, 0.00025\}$	85
Figure 68	Error at $t = 1.0$ for different time steps widths. The slope (=experimental order of convergence) should be around 1.	86
Figure 69	V_m for time $t = 3.0$, different time step widths $dt \in \{0.01, 0.005, 0.001, 0.0005, 0.00025\}$	87
Figure 70	Error at $t = 3.0$ for different time steps widths. The slope (=experimental order of convergence) should be around 1.	87

LIST OF TABLES

Table 1	Quantitative error between Abaqus 2017 and iron simulations for linear elastic uniaxial extensions	54
Table 2	Quantitative error between Abaqus 2017 and iron simulations for linear elastic shear	59
Table 3	Quantitative error between Abaqus 2017 and iron simulations for linear elastic uniaxial extensions	64
Table 4	Quantitative error between Abaqus 2017 and iron simulations for linear elastic shear	66

1 TEST SUMMARY

Passed tests: 212 / 246

1.1 Details

Content of: example-0001/results/failed.tests

No failed tests.

Content of: example-0001-u/results/failed.tests

current_run/l2x1x0_n8x4x0_i8_s0/Example.part0.exnode
current_run/l2x1x0_n8x4x0_i8_s1/Example.part0.exnode

| CHeart - Iron | $_2$ = 0.05804
| CHeart - Iron | $_2$ = 0.05804

Content of: example-0002/results/failed.tests

No failed tests.

Content of: example-0003/results/failed.tests

Failed tests:

current_run/l2x1x0_n2x1x0_i1_s0/Example.part0.exnode
current_run/l2x1x0_n4x2x0_i1_s0/Example.part0.exnode
current_run/l2x1x0_n8x4x0_i1_s0/Example.part0.exnode
current_run/l2x1x0_n2x1x0_i2_s0/Example.part0.exnode
current_run/l2x1x0_n4x2x0_i2_s0/Example.part0.exnode
current_run/l2x1x0_n8x4x0_i2_s0/Example.part0.exnode
current_run/l2x1x0_n2x1x0_i1_s1/Example.part0.exnode
current_run/l2x1x0_n4x2x0_i1_s1/Example.part0.exnode
current_run/l2x1x0_n8x4x0_i1_s1/Example.part0.exnode
current_run/l2x1x0_n2x1x0_i2_s1/Example.part0.exnode
current_run/l2x1x0_n4x2x0_i2_s1/Example.part0.exnode
current_run/l2x1x0_n8x4x0_i2_s1/Example.part0.exnode
current_run/l2x1x0_n2x1x0_i1_s0/Example.part0.exnode
current_run/l2x1x0_n4x2x0_i1_s0/Example.part0.exnode
current_run/l2x1x0_n8x4x0_i1_s0/Example.part0.exnode
current_run/l2x1x1_n2x1x1_i1_s0/Example.part0.exnode
current_run/l2x1x1_n4x2x2_i1_s0/Example.part0.exnode
current_run/l2x1x1_n8x4x4_i1_s0/Example.part0.exnode
current_run/l2x1x1_n2x1x1_i2_s0/Example.part0.exnode
current_run/l2x1x1_n4x2x2_i2_s0/Example.part0.exnode
current_run/l2x1x1_n8x4x4_i2_s0/Example.part0.exnode
current_run/l2x1x1_n2x1x1_i1_s1/Example.part0.exnode
current_run/l2x1x1_n4x2x2_i1_s1/Example.part0.exnode
current_run/l2x1x1_n8x4x4_i1_s1/Example.part0.exnode
current_run/l2x1x1_n2x1x1_i2_s1/Example.part0.exnode
current_run/l2x1x1_n4x2x2_i2_s1/Example.part0.exnode
current_run/l2x1x1_n8x4x4_i2_s1/Example.part0.exnode

| CHeart - Iron | $_2$ = 44.2627
| CHeart - Iron | $_2$ = 37.2770
| CHeart - Iron | $_2$ = 32.2165
| CHeart - Iron | $_2$ = 27.3358
| CHeart - Iron | $_2$ = 22.1869
| CHeart - Iron | $_2$ = 19.7449
| CHeart - Iron | $_2$ = 44.2627
| CHeart - Iron | $_2$ = 37.2770
| CHeart - Iron | $_2$ = 32.2165
| CHeart - Iron | $_2$ = 27.3358
| CHeart - Iron | $_2$ = 22.1869
| CHeart - Iron | $_2$ = 19.7449
| CHeart - Iron | $_2$ = 124.749
| CHeart - Iron | $_2$ = 128.672
| CHeart - Iron | $_2$ = 143.606
| CHeart - Iron | $_2$ = 94.2619
| CHeart - Iron | $_2$ = 98.7606
| CHeart - Iron | $_2$ = 118.047
| CHeart - Iron | $_2$ = 124.749
| CHeart - Iron | $_2$ = 128.672
| CHeart - Iron | $_2$ = 143.606
| CHeart - Iron | $_2$ = 94.2619
| CHeart - Iron | $_2$ = 98.7606
| CHeart - Iron | $_2$ = 118.047

Content of: example-0004/results/failed.tests

No failed tests.

Content of: example-0005/results/failed.tests

No failed tests.

Content of: example-0011/results/failed.tests

No failed tests.

Content of: example-0012/results/failed.tests

No failed tests.

Content of: example-0013/results/failed.tests
No failed tests.

Content of: example-0101/results/failed.tests
No failed tests.

Content of: example-0102/results/failed.tests
Failed tests:

```
1 =====
    LOADING      = SHEAR
    REFINEMENT   = COARSE
    DIMENSION    = 2D
    INTERPOLATION = 1
    CONTROL      = DISPLACEMENT
    norm(aba-iron, 2) = 7e-03 >= 1.000000e-10
2 =====
    LOADING      = SHEAR
    REFINEMENT   = COARSE
    DIMENSION    = 2D
    INTERPOLATION = 2
    CONTROL      = DISPLACEMENT
    norm(aba-iron, 2) = 3e-04 >= 1.000000e-10
3 =====
    LOADING      = SHEAR
    REFINEMENT   = COARSE
    DIMENSION    = 3D
    INTERPOLATION = 1
    CONTROL      = DISPLACEMENT
    norm(aba-iron, 2) = 4e-04 >= 1.000000e-10
4 =====
    LOADING      = SHEAR
    REFINEMENT   = COARSE
    DIMENSION    = 3D
    INTERPOLATION = 2
    CONTROL      = DISPLACEMENT
    norm(aba-iron, 2) = 3e-05 >= 1.000000e-10
5 =====
    LOADING      = SHEAR
    REFINEMENT   = MEDIUM
    DIMENSION    = 2D
    INTERPOLATION = 1
    CONTROL      = DISPLACEMENT
    norm(aba-iron, 2) = 1e-03 >= 1.000000e-10
6 =====
    LOADING      = SHEAR
    REFINEMENT   = MEDIUM
    DIMENSION    = 2D
    INTERPOLATION = 2
    CONTROL      = DISPLACEMENT
    norm(aba-iron, 2) = 6e-05 >= 1.000000e-10
7 =====
```

```

LOADING      = SHEAR
REFINEMENT   = MEDIUM
DIMENSION    = 3D
INTERPOLATION = 1
CONTROL      = DISPLACEMENT
norm(aba-iron, 2) = 4e-05 >= 1.000000e-10
8 =====
LOADING      = SHEAR
REFINEMENT   = MEDIUM
DIMENSION    = 3D
INTERPOLATION = 2
CONTROL      = DISPLACEMENT
norm(aba-iron, 2) = 5e-06 >= 1.000000e-10

```

Content of: example-0111/results/failed.tests
Failed tests:

```

1 =====
LOADING      = UNIAX
REFINEMENT   = COARSE
DIMENSION    = 3D
INTERPOLATION = 1
CONTROL      = FORCE
norm(aba-iron, 2) = 3e-02 >= 1.000000e-10
2 =====
LOADING      = UNIAX
REFINEMENT   = MEDIUM
DIMENSION    = 3D
INTERPOLATION = 1
CONTROL      = FORCE
norm(aba-iron, 2) = 1e-02 >= 1.000000e-10
3 =====
LOADING      = UNIAX
REFINEMENT   = FINE
DIMENSION    = 3D
INTERPOLATION = 1
CONTROL      = FORCE
norm(aba-iron, 2) = 3e-03 >= 1.000000e-10

```

Content of: example-0112/results/failed.tests
Failed tests:

```

1 =====
LOADING      = SHEAR
REFINEMENT   = COARSE
DIMENSION    = 2D
INTERPOLATION = 1
CONTROL      = FORCE
norm(aba-iron, 2) = 3e+00 >= 1.000000e-10

```

Content of: example-0302-u/results/failed.tests
All tests failed.

Content of: example-0401/results/failed.tests
No failed tests.

Content of: example-0402/results/failed.tests
No failed tests.

2 INTRODUCTION

This document contains information about examples used for testing *OpenCMISS-iron*. Read: How-to¹ and [1].

2.1 Cogui files for cmgui-2.9

2.2 Variations to consider

- Geometry and topology
 - 1D, 2D, 3D
 - Length, width, height
 - Number of elements
 - Interpolation order
 - Generated or user meshes
 - quad/hex or tri/tet meshes
- Initial conditions
- Load cases
 - Dirichlet BC
 - Neumann BC
 - Volume force
 - Mix of previous items
- Sources, sinks
- Time dependence
 - Static
 - Quasi-static
 - Dynamic
- Material laws
 - Linear
 - Nonlinear (Mooney-Rivlin, Neo-Hookean, Ogden, etc.)
 - Active (Stress, strain)
- Material parameters, anisotropy
- Solver
 - Direct
 - Iterative
- Test cases
 - Numerical reference data
 - Analytical solution
- A mix of previous items

¹ <https://bitbucket.org/hessenthaler/opencmiss-howto>

2.3 Folder structure

TBD..

3 PROGRESS

People working on setting up tests in alphabetical order (surnames) with initials:

- CB : Christian Bleiler
- NE : Dr.-Ing. Nehzat Emamy
- AH : Andreas Hessenthaler
- TK : Thomas Klotz
- AK : Aaron Krämer
- BM : Benjamin Maier
- SM : Sergio Morales
- MM : Mylena Mordhorst
- HS : Harry Saini

3.1 Equations to test

Test single-physics problems before multi-physics problems!

- Diffusion equation (Laplace, Poisson, Generalized Laplace, ALE Diffusion, etc.)
- Linear elasticity equation (compressible and incompressible)
- Finite elasticity equation (compressible and incompressible Mooney-Rivlin, etc.)
- Navier-Stokes equation (ALE, Stokes, etc.)
- Monodomain equation
- CellML models
- Skeletal muscle models
- Fluid-structure interaction
- etc.

3.2 Setting up a new test

Use the following guideline to set up a new test:

1. Check if it is already there
2. Talk to other developers
3. Create a new subfolder examples/example-xxxx
4. Document the setup (computational domain, etc.) in examples/example-xxxx/doc/example.tex
5. Set up example with all parameters as command line arguments, see Section [2.2](#)

6. Set up reference results (CHeart, Abaqus, analytical solution, etc.)
7. Set up script to run all tests in your example directory
8. Set up script to perform comparison between iron results and reference results
9. Set up visualization scripts
10. Compile, run, test, visualize your example
11. Compile, run, test, visualize all examples

For each example, progress is documented in the respective section titles with the following **TAG**:

- **DOCUMENTED**: finish the documentation of the example (spatial domain, number of time steps, boundary conditions, etc.)
- **COMPILES**: example compiles (for default parameters)
- **RUNS**: example runs (for default parameters)
- **CONVERGES**: no convergence issues (for default parameters, results not plausible)
- **PLAUSIBLE**: results look sensible (for default parameters)
- **VALIDATED**: for all parameter sets it gives the correct results as compared to CHeart/Abaqus/analytical solution (includes visualization scripts, run scripts, comparison scripts, documentation!, ...)

Move all tags **CONVERGE**, **PLAUSIBLE** to **VALIDATED**.

Next steps include:

- Everybody runs everything!
- Meeting with Oliver
- Meeting with Auckland

3.3 Long-term goals

- Different testing targets
 - SMALL : small, fast tests
 - BIG : same as before; further, bigger and more complex geometries, convergence analysis
 - PARALLEL : same as before but in parallel
- Add more examples/those which were on the agenda but not started
- Jenkins continuous testing, integration and deployment
 - test SMALL/BIG/PARALLEL targets
 - integrate with GitHub (pull-requests triggers Jenkins, merge on success)

4 DIFFUSION EQUATION

4.1 Equation in general form

The governing equation is,

$$\partial_t u + \nabla \cdot [\sigma \nabla u] = f, \quad (1)$$

with conductivity tensor σ . The conductivity tensor is,

- defined in material coordinates (fibre direction),
- diagonal,
- defined per element.

4.2 Example-0001 [VALIDATED]

Example uses generated regular meshes and solves a static problem, i.e., applies the boundary conditions in one step.

4.2.1 Mathematical model - 2D

We solve the following scalar equation,

$$\nabla \cdot \nabla u = 0 \quad \Omega = [0, 2] \times [0, 1], \quad (2)$$

with boundary conditions

$$u = 0 \quad x = y = 0, \quad (3)$$

$$u = 1 \quad x = 2, y = 1. \quad (4)$$

No material parameters to specify.

4.2.2 Mathematical model - 3D

We solve the following scalar equation,

$$\nabla \cdot \nabla u = 0 \quad \Omega = [0, 2] \times [0, 1] \times [0, 1], \quad (5)$$

with boundary conditions

$$u = 0 \quad x = y = z = 0, \quad (6)$$

$$u = 1 \quad x = 2, y = z = 1. \quad (7)$$

No material parameters to specify.

4.2.3 Computational model

- Commandline arguments are:

float: length along x-direction

float: length along y-direction

float: length along z-direction (set to zero for 2D)

integer: number of elements in x-direction

integer: number of elements in y-direction

integer: number of elements in z-direction (set to zero for 2D)

integer: interpolation order (1: linear; 2: quadratic)

integer: solver type (0: direct; 1: iterative)

- Commandline arguments for tests are:

2.0 1.0 0.0 2 1 0 1 0

2.0 1.0 0.0 4 2 0 1 0

2.0 1.0 0.0 8 4 0 1 0

2.0 1.0 0.0 2 1 0 2 0

2.0 1.0 0.0 4 2 0 2 0

2.0 1.0 0.0 8 4 0 2 0

2.0 1.0 0.0 2 1 0 1 1

2.0 1.0 0.0 4 2 0 1 1

```

2.0 1.0 0.0 8 4 0 1 1
2.0 1.0 0.0 2 1 0 2 1
2.0 1.0 0.0 4 2 0 2 1
2.0 1.0 0.0 8 4 0 2 1
2.0 1.0 1.0 2 1 1 1 0
2.0 1.0 1.0 4 2 2 1 0
2.0 1.0 1.0 8 4 4 1 0
2.0 1.0 1.0 2 1 1 2 0
2.0 1.0 1.0 4 2 2 2 0
2.0 1.0 1.0 8 4 4 2 0
2.0 1.0 1.0 2 1 1 1 1
2.0 1.0 1.0 4 2 2 1 1
2.0 1.0 1.0 8 4 4 1 1
2.0 1.0 1.0 2 1 1 2 1
2.0 1.0 1.0 4 2 2 2 1
2.0 1.0 1.0 8 4 4 2 1

```

4.2.4 Result summary

We use CHeart rev. 6292 to produce numerical reference solutions.

Passed tests: 24 / 24

No failed tests.

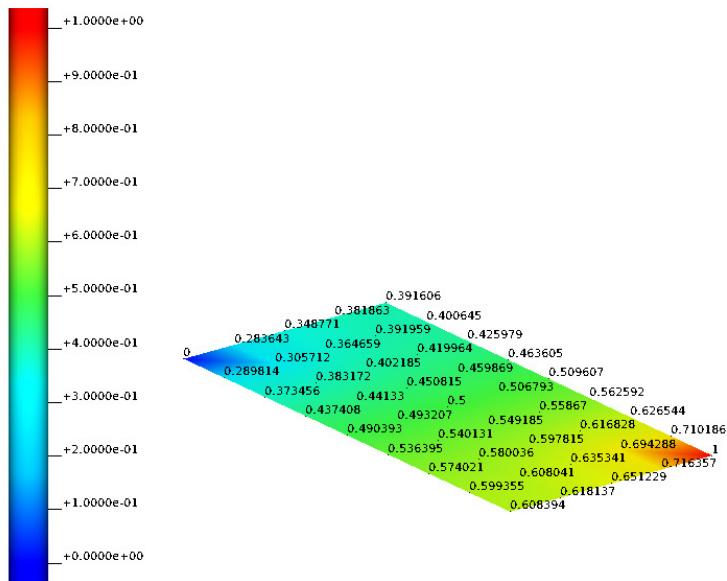


Figure 1: 2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].

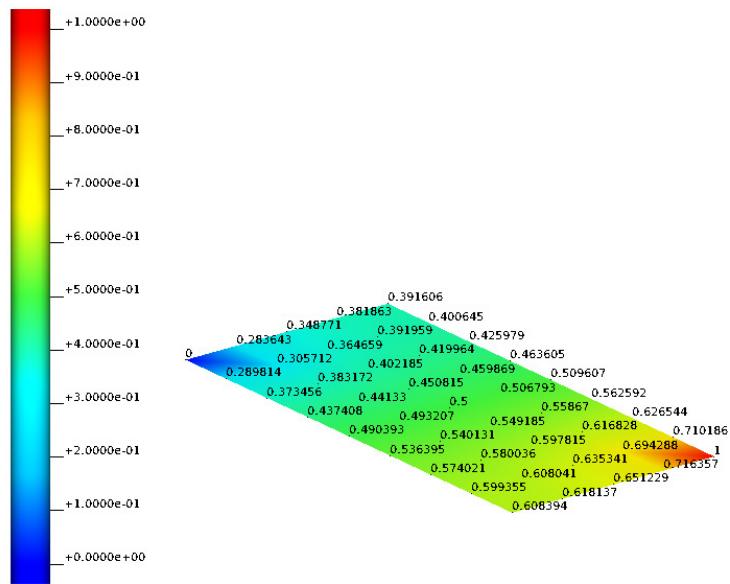


Figure 2: 2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].

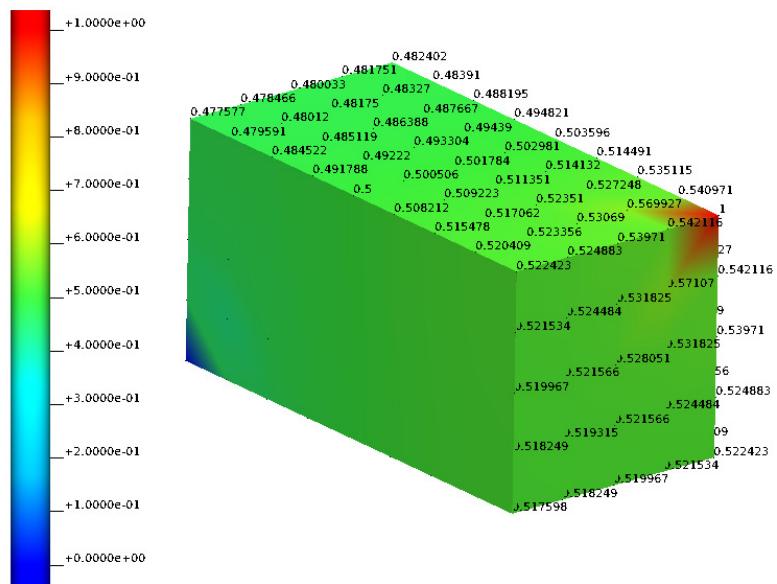


Figure 3: 3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].

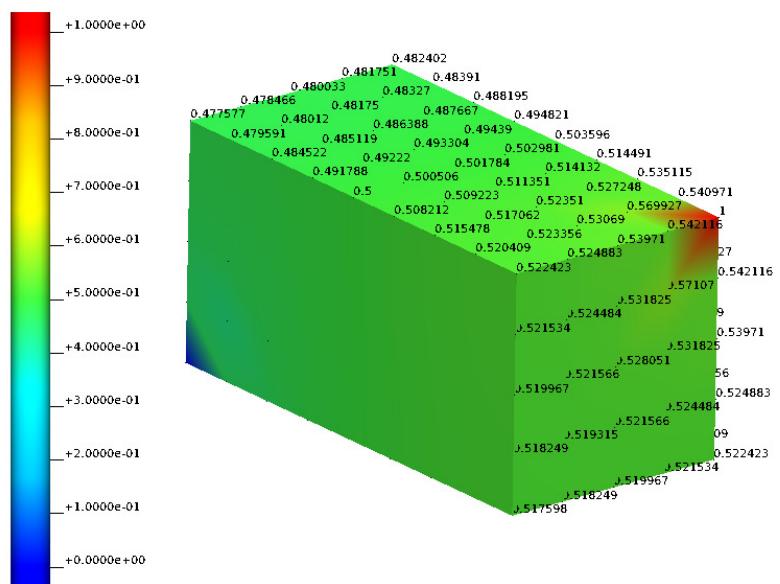


Figure 4: 3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].

4.3 Example-0001-u [PLAUSIBLE]

Example uses user-defined regular meshes in CHearm mesh format and solves a static problem, i.e., applies the boundary conditions in one step.

Issues: Interpolation type 8 (quadratic simplex) gives significantly different results compared to CHearm whereas all other simplex mesh results correspond rather well..

4.3.1 Mathematical model - 2D

We solve the following scalar equation,

$$\nabla \cdot \nabla u = 0 \quad \Omega = [0, 2] \times [0, 1], \quad (8)$$

with boundary conditions

$$u = 0 \quad x = y = 0, \quad (9)$$

$$u = 1 \quad x = 2, y = 1. \quad (10)$$

No material parameters to specify.

4.3.2 Mathematical model - 3D

We solve the following scalar equation,

$$\nabla \cdot \nabla u = 0 \quad \Omega = [0, 2] \times [0, 1] \times [0, 1], \quad (11)$$

with boundary conditions

$$u = 0 \quad x = y = z = 0, \quad (12)$$

$$u = 1 \quad x = 2, y = z = 1. \quad (13)$$

No material parameters to specify.

4.3.3 Computational model

- Commandline arguments are:

float: length along x-direction

float: length along y-direction

float: length along z-direction (set to zero for 2D)

integer: number of elements in x-direction

integer: number of elements in y-direction

integer: number of elements in z-direction (set to zero for 2D)

integer: interpolation order (1: linear; 2: quadratic)

integer: solver type (0: direct; 1: iterative)

- Commandline arguments for tests are:

2.0 1.0 0.0 2 1 0 1 0

2.0 1.0 0.0 4 2 0 1 0

2.0 1.0 0.0 8 4 0 1 0

2.0 1.0 0.0 2 1 0 2 0

```

2.0 1.0 0.0 4 2 0 2 0
2.0 1.0 0.0 8 4 0 2 0
2.0 1.0 0.0 8 4 0 7 0
2.0 1.0 0.0 8 4 0 8 0
2.0 1.0 0.0 2 1 0 1 1
2.0 1.0 0.0 4 2 0 1 1
2.0 1.0 0.0 8 4 0 1 1
2.0 1.0 0.0 2 1 0 2 1
2.0 1.0 0.0 4 2 0 2 1
2.0 1.0 0.0 8 4 0 2 1
2.0 1.0 0.0 8 4 0 7 1
2.0 1.0 0.0 8 4 0 8 1
2.0 1.0 1.0 2 1 1 1 0
2.0 1.0 1.0 4 2 2 1 0
2.0 1.0 1.0 8 4 4 1 0
2.0 1.0 1.0 2 1 1 2 0
2.0 1.0 1.0 4 2 2 2 0
2.0 1.0 1.0 8 4 4 2 0
2.0 1.0 1.0 8 4 4 7 0
2.0 1.0 1.0 8 4 4 8 0
2.0 1.0 1.0 2 1 1 1 1
2.0 1.0 1.0 4 2 2 1 1
2.0 1.0 1.0 8 4 4 1 1
2.0 1.0 1.0 2 1 1 2 1
2.0 1.0 1.0 4 2 2 2 1
2.0 1.0 1.0 8 4 4 2 1
2.0 1.0 1.0 8 4 4 7 1
2.0 1.0 1.0 8 4 4 8 1

```

- Note: Binary uses command line arguments to search for the relevant mesh files.

4.3.4 Result summary

We use CHeart rev. 6292 to produce numerical reference solutions.

Passed tests: 30 / 32

current_run/l2x1x0_n8x4x0_i8_s0/Example.part0.exnode	CHeart	- Iron	_2 = 0.05804
current_run/l2x1x0_n8x4x0_i8_s1/Example.part0.exnode	CHeart	- Iron	_2 = 0.05804

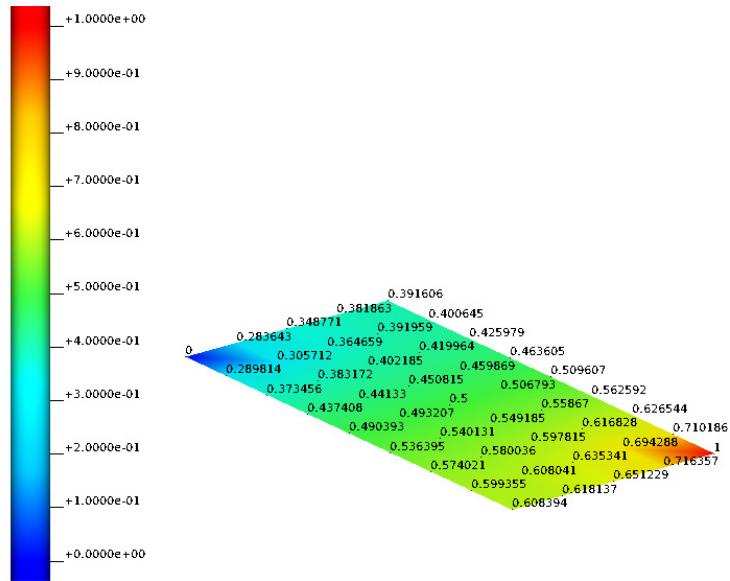


Figure 5: 2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].

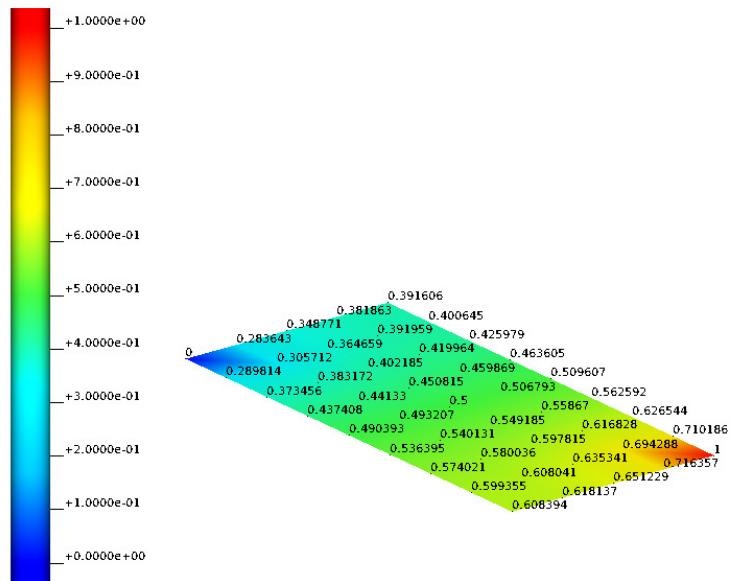


Figure 6: 2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].

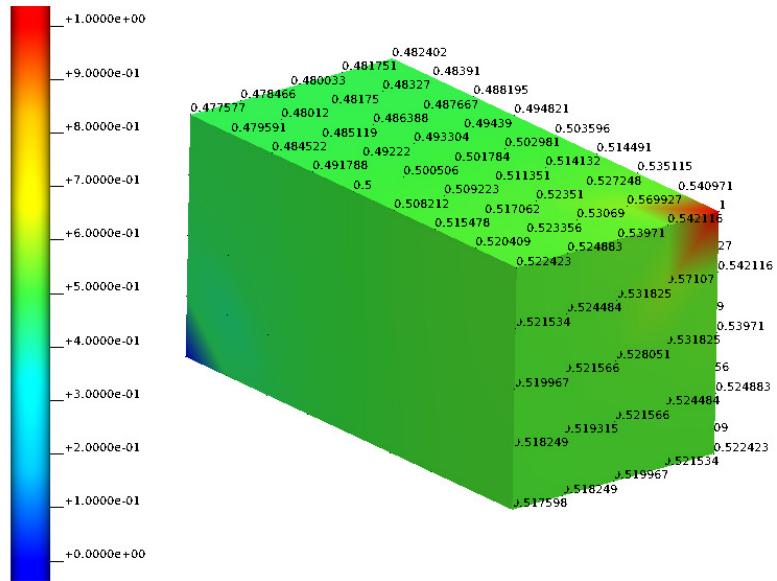


Figure 7: 3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].

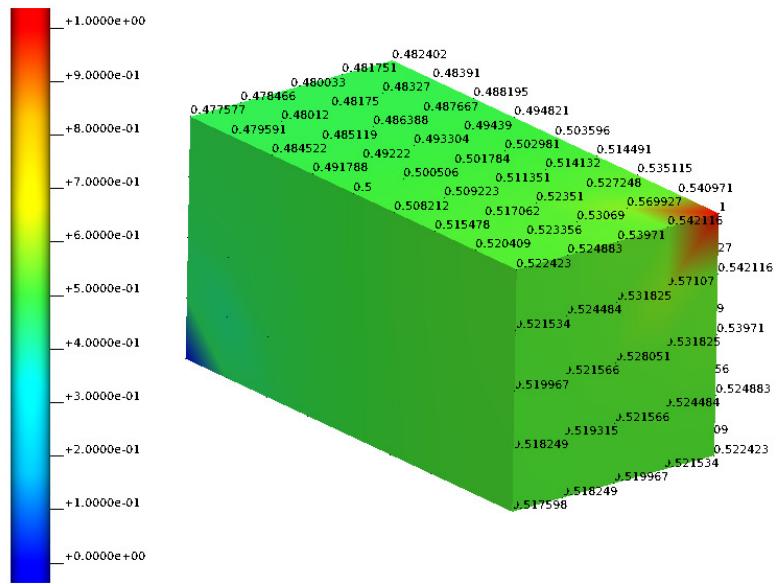


Figure 8: 3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].

4.4 Example-0002 [VALIDATED]

Example uses generated regular meshes and solves a static problem, i.e., applies the boundary conditions in one step.

4.4.1 Mathematical model - 2D

We solve the following scalar equation,

$$\nabla \cdot \nabla u = 0 \quad \Omega = [0, 2] \times [0, 1], \quad (14)$$

with boundary conditions

$$u = 15y \quad x = 0, \quad (15)$$

$$u = 25 - 18y \quad x = 2. \quad (16)$$

No material parameters to specify.

4.4.2 Mathematical model - 3D

We solve the following scalar equation,

$$\nabla \cdot \nabla u = 0 \quad \Omega = [0, 2] \times [0, 1] \times [0, 1], \quad (17)$$

with boundary conditions

$$u = 15y \quad x = 0, \quad (18)$$

$$u = 25 - 18y \quad x = 2. \quad (19)$$

No material parameters to specify.

4.4.3 Computational model

- Commandline arguments are:

float: length along x-direction

float: length along y-direction

float: length along z-direction (set to zero for 2D)

integer: number of elements in x-direction

integer: number of elements in y-direction

integer: number of elements in z-direction (set to zero for 2D)

integer: interpolation order (1: linear; 2: quadratic)

integer: solver type (0: direct; 1: iterative)

- Commandline arguments for tests are:

2.0 1.0 0.0 2 1 0 1 0

2.0 1.0 0.0 4 2 0 1 0

2.0 1.0 0.0 8 4 0 1 0

2.0 1.0 0.0 2 1 0 2 0

2.0 1.0 0.0 4 2 0 2 0

2.0 1.0 0.0 8 4 0 2 0

2.0 1.0 0.0 2 1 0 1 1

2.0 1.0 0.0 4 2 0 1 1

```

2.0 1.0 0.0 8 4 0 1 1
2.0 1.0 0.0 2 1 0 2 1
2.0 1.0 0.0 4 2 0 2 1
2.0 1.0 0.0 8 4 0 2 1
2.0 1.0 1.0 2 1 1 1 0
2.0 1.0 1.0 4 2 2 1 0
2.0 1.0 1.0 8 4 4 1 0
2.0 1.0 1.0 2 1 1 2 0
2.0 1.0 1.0 4 2 2 2 0
2.0 1.0 1.0 8 4 4 2 0
2.0 1.0 1.0 2 1 1 1 1
2.0 1.0 1.0 4 2 2 1 1
2.0 1.0 1.0 8 4 4 1 1
2.0 1.0 1.0 2 1 1 2 1
2.0 1.0 1.0 4 2 2 2 1
2.0 1.0 1.0 8 4 4 2 1

```

4.4.4 Result summary

We use CHeart rev. 6292 to produce numerical reference solutions.

Passed tests: 24 / 24

No failed tests.

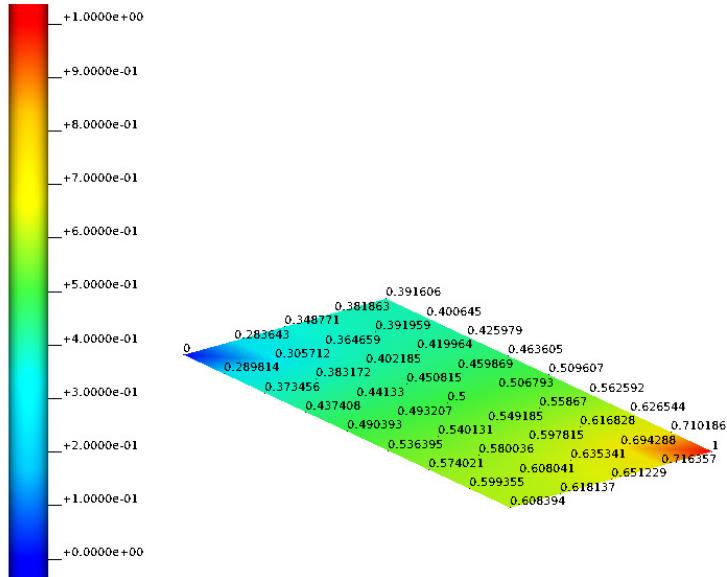


Figure 9: 2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].

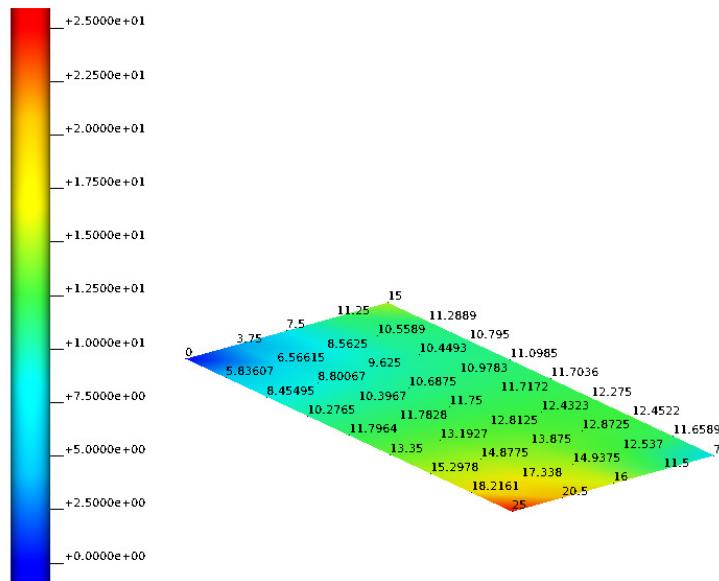


Figure 10: 2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].

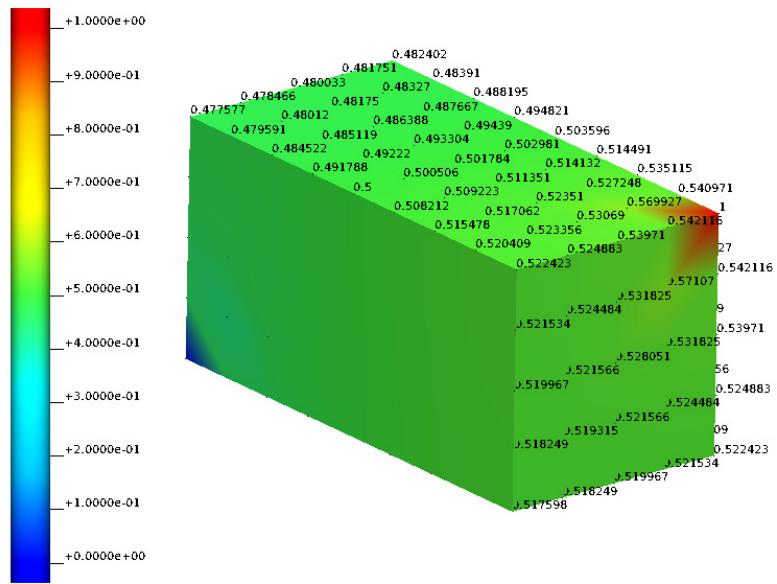


Figure 11: 3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].

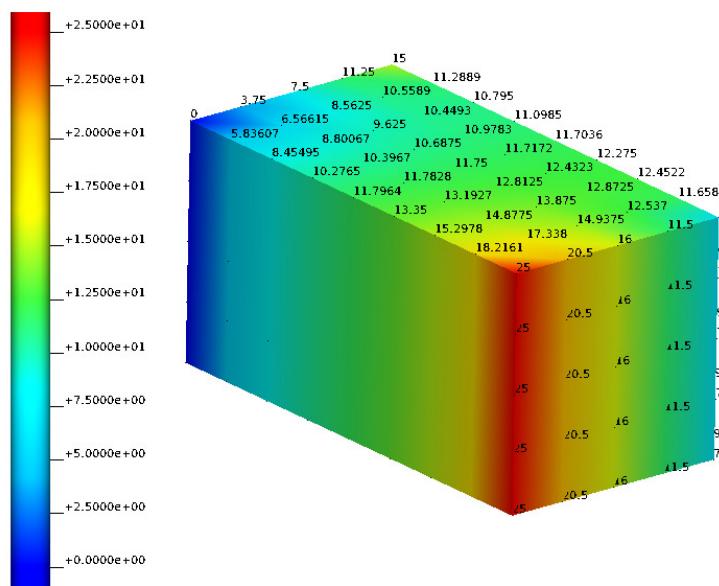


Figure 12: 3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].

4.5 Example-0003 [COMPILES]

Example uses generated regular meshes and solves a static problem, i.e., applies the boundary conditions in one step.

Issues: Not clear how to prescribe Neumann BC. Results seem weird. Reference results are set up.

4.5.1 Mathematical model - 2D

We solve the following scalar equation,

$$\nabla \cdot \nabla u = 0 \quad \Omega = [0, 2] \times [0, 1], \quad (20)$$

with boundary conditions

$$u = 15y \quad x = 0, \quad (21)$$

$$\partial_n u = 25 - 18y \quad x = 2. \quad (22)$$

No material parameters to specify.

4.5.2 Mathematical model - 3D

We solve the following scalar equation,

$$\nabla \cdot \nabla u = 0 \quad \Omega = [0, 2] \times [0, 1] \times [0, 1], \quad (23)$$

with boundary conditions

$$u = 15y \quad x = 0, \quad (24)$$

$$\partial_n u = 25 - 18y \quad x = 2. \quad (25)$$

No material parameters to specify.

4.5.3 Computational model

- Commandline arguments are:

float: length along x-direction

float: length along y-direction

float: length along z-direction (set to zero for 2D)

integer: number of elements in x-direction

integer: number of elements in y-direction

integer: number of elements in z-direction (set to zero for 2D)

integer: interpolation order (1: linear; 2: quadratic)

integer: solver type (0: direct; 1: iterative)

- Commandline arguments for tests are:

2.0 1.0 0.0 2 1 0 1 0

2.0 1.0 0.0 4 2 0 1 0

2.0 1.0 0.0 8 4 0 1 0

2.0 1.0 0.0 2 1 0 2 0

2.0 1.0 0.0 4 2 0 2 0

```

2.0 1.0 0.0 8 4 0 2 0
2.0 1.0 0.0 2 1 0 1 1
2.0 1.0 0.0 4 2 0 1 1
2.0 1.0 0.0 8 4 0 1 1
2.0 1.0 0.0 2 1 0 2 1
2.0 1.0 0.0 4 2 0 2 1
2.0 1.0 0.0 8 4 0 2 1
2.0 1.0 1.0 2 1 1 1 0
2.0 1.0 1.0 4 2 2 1 0
2.0 1.0 1.0 8 4 4 1 0
2.0 1.0 1.0 2 1 1 2 0
2.0 1.0 1.0 4 2 2 2 0
2.0 1.0 1.0 8 4 4 2 0
2.0 1.0 1.0 2 1 1 1 1
2.0 1.0 1.0 4 2 2 1 1
2.0 1.0 1.0 8 4 4 1 1
2.0 1.0 1.0 2 1 1 2 1
2.0 1.0 1.0 4 2 2 2 1
2.0 1.0 1.0 8 4 4 2 1

```

4.5.4 Result summary

We use CHeart rev. 6292 to produce numerical reference solutions.

Passed tests: 0 / 24

Failed tests:

```

current_run/l2x1x0_n2x1x0_i1_s0/Example.part0.exnode
current_run/l2x1x0_n4x2x0_i1_s0/Example.part0.exnode
current_run/l2x1x0_n8x4x0_i1_s0/Example.part0.exnode
current_run/l2x1x0_n2x1x0_i2_s0/Example.part0.exnode
current_run/l2x1x0_n4x2x0_i2_s0/Example.part0.exnode
current_run/l2x1x0_n8x4x0_i2_s0/Example.part0.exnode
current_run/l2x1x0_n2x1x0_i1_s1/Example.part0.exnode
current_run/l2x1x0_n4x2x0_i1_s1/Example.part0.exnode
current_run/l2x1x0_n8x4x0_i1_s1/Example.part0.exnode
current_run/l2x1x0_n2x1x0_i2_s1/Example.part0.exnode
current_run/l2x1x0_n4x2x0_i2_s1/Example.part0.exnode
current_run/l2x1x0_n8x4x0_i2_s1/Example.part0.exnode
current_run/l2x1x1_n2x1x1_i1_s0/Example.part0.exnode
current_run/l2x1x1_n4x2x2_i1_s0/Example.part0.exnode
current_run/l2x1x1_n8x4x4_i1_s0/Example.part0.exnode
current_run/l2x1x1_n2x1x1_i2_s0/Example.part0.exnode
current_run/l2x1x1_n4x2x2_i2_s0/Example.part0.exnode
current_run/l2x1x1_n8x4x4_i2_s0/Example.part0.exnode
current_run/l2x1x1_n2x1x1_i1_s1/Example.part0.exnode
current_run/l2x1x1_n4x2x2_i1_s1/Example.part0.exnode
current_run/l2x1x1_n8x4x4_i1_s1/Example.part0.exnode
current_run/l2x1x1_n2x1x1_i2_s1/Example.part0.exnode

```

CHeart	- Iron	$ _2 = 44.2627$
CHeart	- Iron	$ _2 = 37.2770$
CHeart	- Iron	$ _2 = 32.2165$
CHeart	- Iron	$ _2 = 27.3358$
CHeart	- Iron	$ _2 = 22.1869$
CHeart	- Iron	$ _2 = 19.7449$
CHeart	- Iron	$ _2 = 44.2627$
CHeart	- Iron	$ _2 = 37.2770$
CHeart	- Iron	$ _2 = 32.2165$
CHeart	- Iron	$ _2 = 27.3358$
CHeart	- Iron	$ _2 = 22.1869$
CHeart	- Iron	$ _2 = 19.7449$
CHeart	- Iron	$ _2 = 124.749$
CHeart	- Iron	$ _2 = 128.672$
CHeart	- Iron	$ _2 = 143.606$
CHeart	- Iron	$ _2 = 94.2619$
CHeart	- Iron	$ _2 = 98.7606$
CHeart	- Iron	$ _2 = 118.047$
CHeart	- Iron	$ _2 = 124.749$
CHeart	- Iron	$ _2 = 128.672$
CHeart	- Iron	$ _2 = 143.606$
CHeart	- Iron	$ _2 = 94.2619$

```
current_run/l2x1x1_n4x2x2_i2_s1/Example.part0.exnode | CHeart - Iron |_2 = 98.7606
current_run/l2x1x1_n8x4x4_i2_s1/Example.part0.exnode | CHeart - Iron |_2 = 118.047
```

Figure 13: 2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].

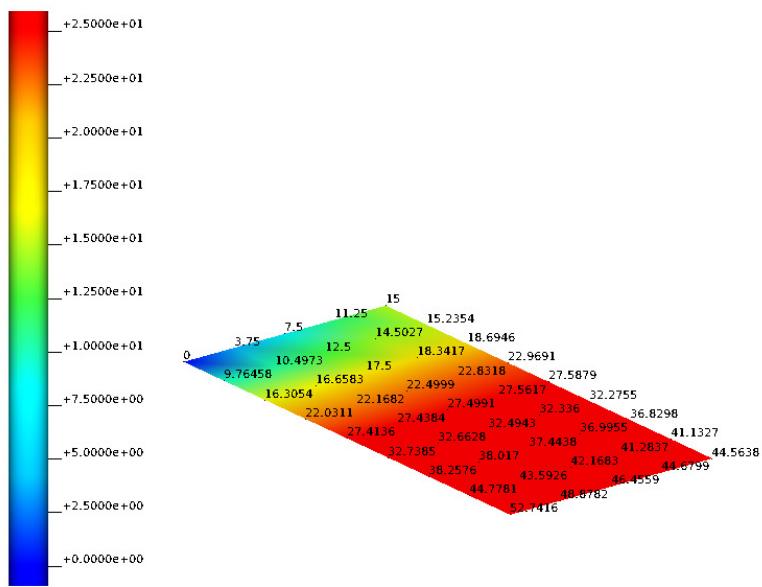


Figure 14: 2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0].

Figure 15: 3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 1 0].

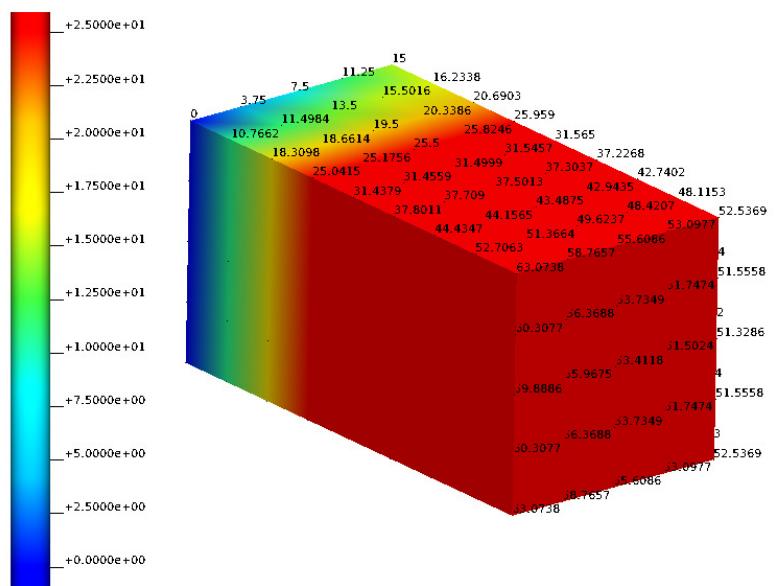


Figure 16: 3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0].

4.6 Example-0004 [VALIDATED]

Example uses generated regular meshes and solves a static problem, i.e., applies the boundary conditions in one step.

4.6.1 Mathematical model - 2D

We solve the following scalar equation,

$$\nabla \cdot \nabla u = 0 \quad \Omega = [0, 2] \times [0, 1], \quad (26)$$

with boundary conditions

$$u = 2.0e^x \cdot \cos(y) \quad \text{on } \partial\Omega. \quad (27)$$

No material parameters to specify.

4.6.2 Computational model

- Commandline arguments are:

integer: number of elements in x-direction
 integer: number of elements in y-direction
 integer: number of elements in z-direction (set to zero for 2D)
 integer: interpolation order (1: linear; 2: quadratic)
 integer: solver type (0: direct; 1: iterative)

- Commandline arguments for tests are:

```
4 2 0 1 0
8 4 0 1 0
2 1 0 2 0
4 2 0 2 0
8 4 0 2 0
4 2 0 1 1
8 4 0 1 1
2 1 0 2 1
4 2 0 2 1
8 4 0 2 1
100 50 0 1 0 (not tested yet..)
100 50 0 2 0 (not tested yet..)
100 50 0 1 1 (not tested yet..)
100 50 0 2 1 (not tested yet..)
```

4.6.3 Result summary

We use CHeart rev. 6292 to produce numerical reference solutions.

Passed tests: 10 / 10

No failed tests.

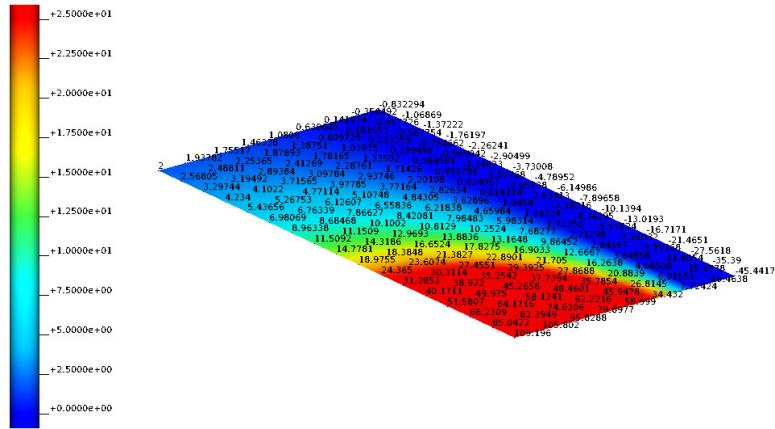


Figure 17: 2D results, iron reference w/ command line arguments [8 4 0 2 0].

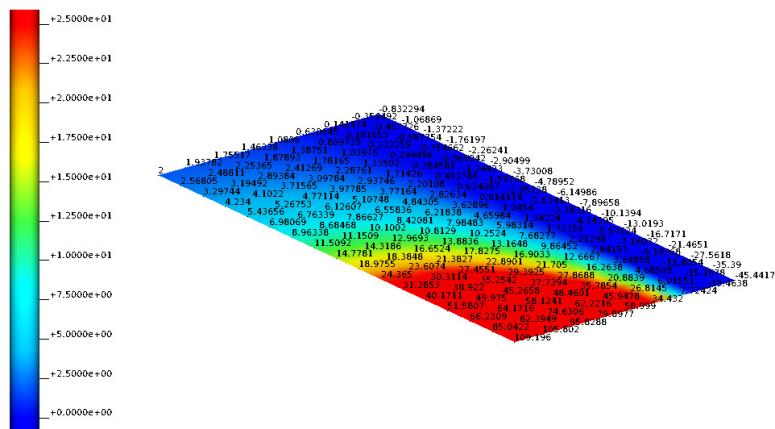


Figure 18: 2D results, current run w/ command line arguments [8 4 0 2 0].

4.7 Example-0005 [VALIDATED]

Example uses two user-defined (2d and 3d) unregular meshes and solves a patch test in form of a static problem, i.e., applies the boundary conditions in one step. Here, a Laplace problem with two Dirichlet boundary conditions is solved. The analytical solution for this setting is known and results in a linear distribution of the scalar Laplace parameter u . The numerical solution has to recover this the linear distribution in order to pass the Patch test. 2d and 3d meshes are according to MacNeil and Harder (1985).

4.7.1 Mathematical model - 2D

We solve the following scalar equation,

$$\nabla \cdot \nabla u = 0 \quad \Omega = [0, 0.24] \times [0, 0.12], \quad (28)$$

with boundary conditions

$$u = 0 \quad x = 0, \quad (29)$$

$$u = 1 \quad x = 0.24. \quad (30)$$

No material parameters to specify.

4.7.2 Mathematical model - 3D

We solve the following scalar equation,

$$\nabla \cdot \nabla u = 0 \quad \Omega = [0, 1] \times [0, 1] \times [0, 1], \quad (31)$$

with boundary conditions

$$u = 0 \quad x = 0, \quad (32)$$

$$u = 1 \quad x = 1. \quad (33)$$

No material parameters to specify.

4.7.3 Computational model

- Commandline arguments are:

integer: dimension (2: 2d; 3: 3d)

integer: interpolation order (1: linear; 2: quadratic)

integer: solver type (0: direct; 1: iterative)

- Commandline arguments for tests are:

2 1 0

2 2 0

2 1 1

2 2 1

3 1 0

3 2 0

3 1 1

3 2 1

4.7.4 Result summary

Since the analytical result is known and the numerical results have to recover the linear distribution of u , the comparisons are done with the analytical solution.

Passed tests: 8 / 8

No failed tests.

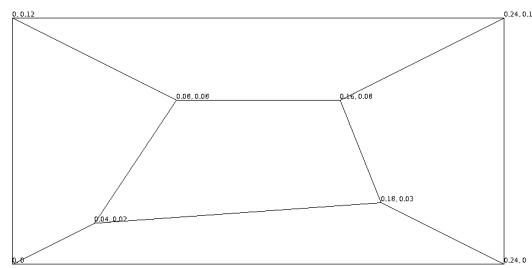


Figure 19: 2D geometry and mesh.

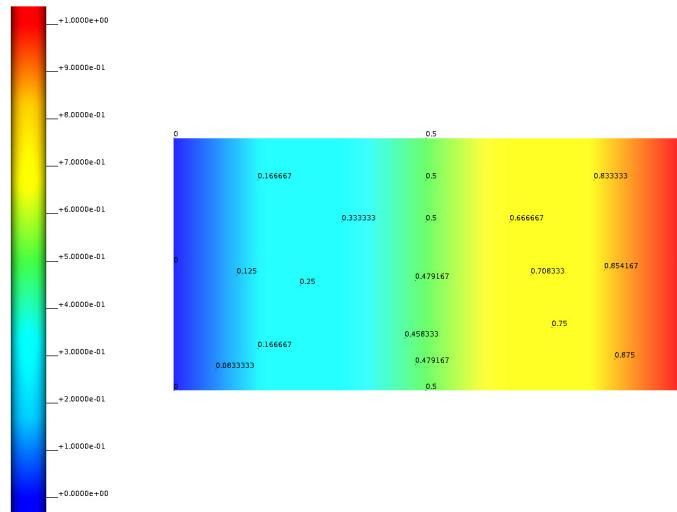


Figure 20: 2D results, iron reference w/ command line arguments [2 2 0].

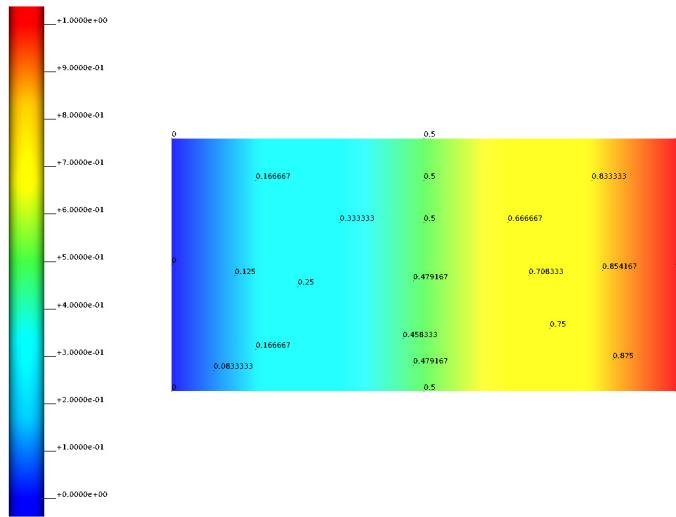


Figure 21: 2D results, current run w/ command line arguments [2 2 0].

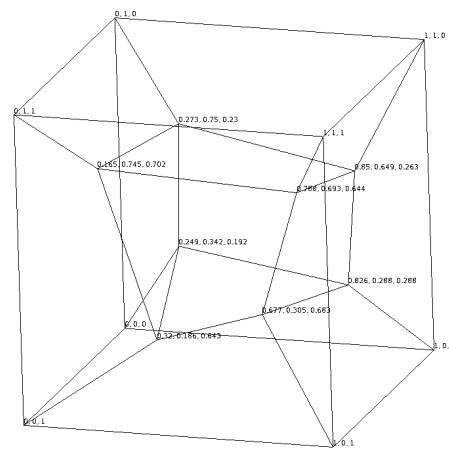


Figure 22: 3D geometry (unit cube) and mesh.

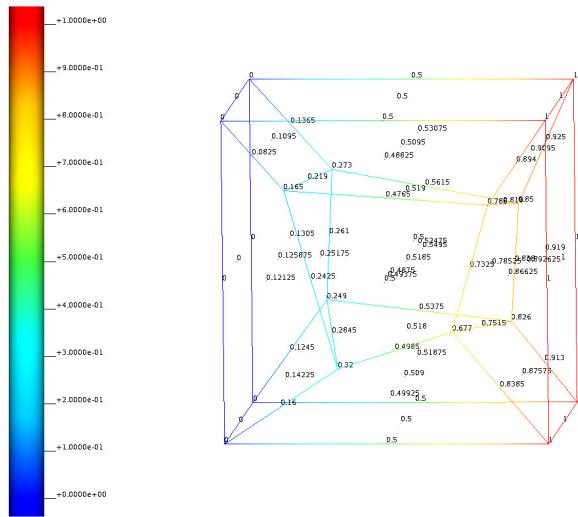


Figure 23: 3D results, iron reference w/ command line arguments [3 2 0].

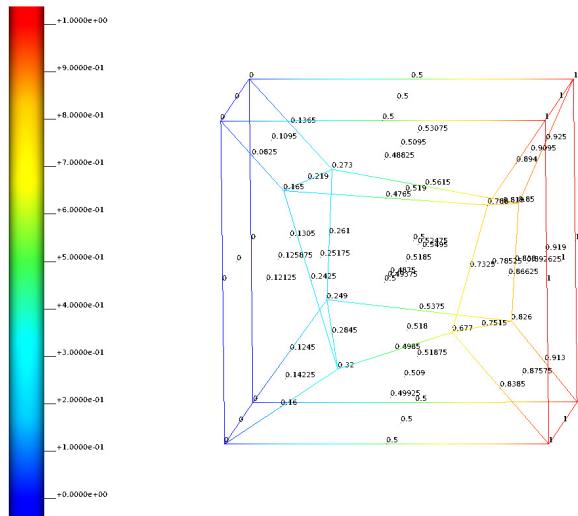


Figure 24: 3D results, current run w/ command line arguments [3 2 0].

4.8 Example-0011 [VALIDATED]

Example uses generated regular meshes and solves a static problem, i.e., applies the boundary conditions in one step.

4.8.1 Mathematical model - 2D

We solve the following scalar equation,

$$\nabla \cdot [\sigma \nabla u] = 0 \quad \Omega = [0, 2] \times [0, 1], \quad (34)$$

with boundary conditions

$$u = 0 \quad x = y = 0, \quad (35)$$

$$u = 1 \quad x = 2, y = 1. \quad (36)$$

The conductivity tensor is defined as,

$$\sigma(x, t) = \sigma = I. \quad (37)$$

4.8.2 Mathematical model - 3D

We solve the following scalar equation,

$$\nabla \cdot [\sigma \nabla u] = 0 \quad \Omega = [0, 2] \times [0, 1] \times [0, 1], \quad (38)$$

with boundary conditions

$$u = 0 \quad x = y = z = 0, \quad (39)$$

$$u = 1 \quad x = 2, y = z = 1. \quad (40)$$

The conductivity tensor is defined as,

$$\sigma(x, t) = \sigma = I. \quad (41)$$

4.8.3 Computational model

- Commandline arguments are:

float: length along x-direction

float: length along y-direction

float: length along z-direction (set to zero for 2D)

integer: number of elements in x-direction

integer: number of elements in y-direction

integer: number of elements in z-direction (set to zero for 2D)

integer: interpolation order (1: linear; 2: quadratic)

integer: solver type (0: direct; 1: iterative)

float: σ_{11}

float: σ_{22}

float: σ_{33} (ignored for 2D)

- Commandline arguments for tests are:

```

2.0 1.0 0.0 2 1 0 1 0 1 1
2.0 1.0 0.0 4 2 0 1 0 1 1
2.0 1.0 0.0 8 4 0 1 0 1 1
2.0 1.0 0.0 2 1 0 2 0 1 1
2.0 1.0 0.0 4 2 0 2 0 1 1
2.0 1.0 0.0 8 4 0 2 0 1 1
2.0 1.0 0.0 2 1 0 1 1 1 1
2.0 1.0 0.0 4 2 0 1 1 1 1
2.0 1.0 0.0 8 4 0 1 1 1 1
2.0 1.0 0.0 2 1 0 2 1 1 1
2.0 1.0 0.0 4 2 0 2 1 1 1
2.0 1.0 0.0 8 4 0 2 1 1 1
2.0 1.0 1.0 2 1 1 1 0 1 1 1
2.0 1.0 1.0 4 2 2 1 0 1 1 1
2.0 1.0 1.0 8 4 4 1 0 1 1 1
2.0 1.0 1.0 2 1 1 2 0 1 1 1
2.0 1.0 1.0 4 2 2 2 0 1 1 1
2.0 1.0 1.0 8 4 4 2 0 1 1 1
2.0 1.0 1.0 2 1 1 1 1 1 1 1
2.0 1.0 1.0 4 2 2 1 1 1 1 1
2.0 1.0 1.0 8 4 4 1 1 1 1 1
2.0 1.0 1.0 2 1 1 2 1 1 1 1
2.0 1.0 1.0 4 2 2 2 1 1 1 1
2.0 1.0 1.0 8 4 4 2 1 1 1 1

```

4.8.4 *Result summary*

We use CHeart rev. 6292 to produce numerical reference solutions.

Passed tests: 24 / 24

No failed tests.

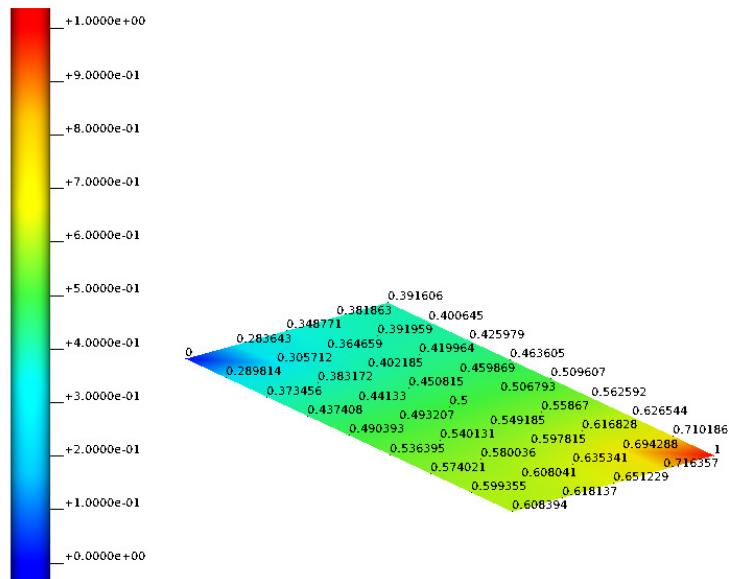


Figure 25: 2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0 1].

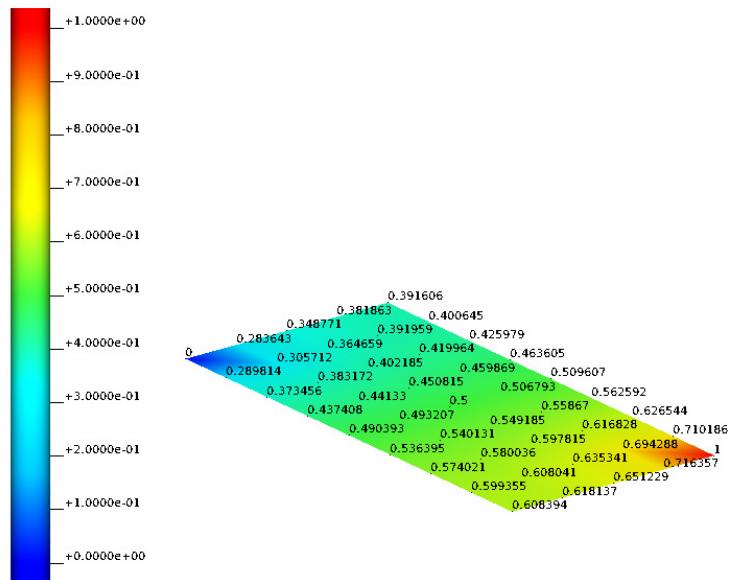


Figure 26: 2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0 1].

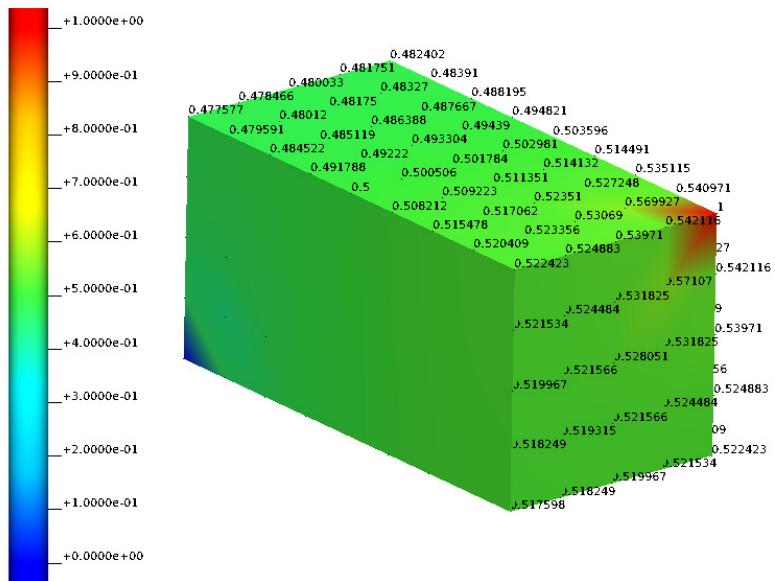


Figure 27: 3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0 1 1 1].

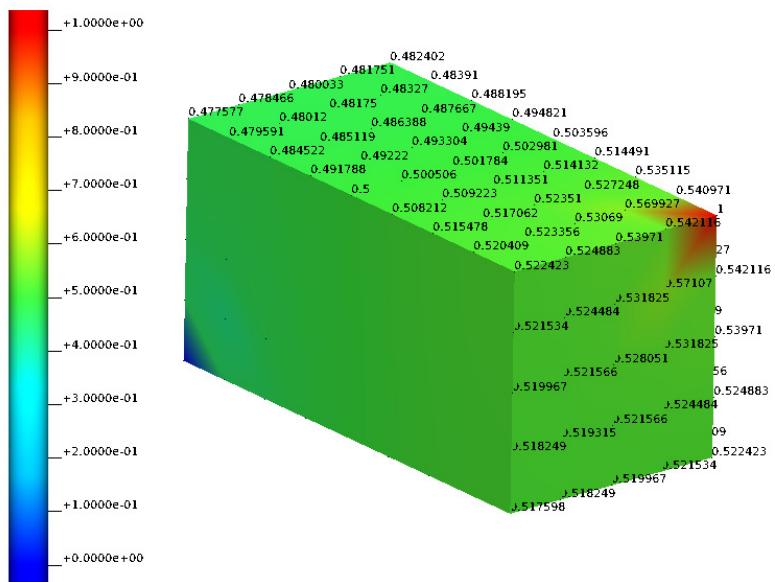


Figure 28: 3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0 1 1 1].

4.9 Example-0012 [VALIDATED]

Example uses generated regular meshes and solves a static problem, i.e., applies the boundary conditions in one step.

4.9.1 Mathematical model - 2D

We solve the following scalar equation,

$$\nabla \cdot [\sigma \nabla u] = 0 \quad \Omega = [0, 2] \times [0, 1], \quad (42)$$

with boundary conditions

$$u = 0 \quad x = y = 0, \quad (43)$$

$$u = 1 \quad x = 2, y = 1. \quad (44)$$

The conductivity tensor is defined as,

$$\sigma(x, t) = \sigma = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}. \quad (45)$$

4.9.2 Mathematical model - 3D

We solve the following scalar equation,

$$\nabla \cdot [\sigma \nabla u] = 0 \quad \Omega = [0, 2] \times [0, 1] \times [0, 1], \quad (46)$$

with boundary conditions

$$u = 0 \quad x = y = z = 0, \quad (47)$$

$$u = 1 \quad x = 2, y = z = 1. \quad (48)$$

The conductivity tensor is defined as,

$$\sigma(x, t) = \sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}. \quad (49)$$

4.9.3 Computational model

- Commandline arguments are:

float: length along x-direction

float: length along y-direction

float: length along z-direction (set to zero for 2D)

integer: number of elements in x-direction

integer: number of elements in y-direction

integer: number of elements in z-direction (set to zero for 2D)

integer: interpolation order (1: linear; 2: quadratic)

integer: solver type (0: direct; 1: iterative)

float: σ_{11}

float: σ_{22}

float: σ_{33} (ignored for 2D)

float: angle 1

float: angle 2

float: angle 3

- Commandline arguments for tests are:

```

2.0 1.0 0.0 2 1 0 1 0 2 3 0 0 0 0
2.0 1.0 0.0 4 2 0 1 0 2 3 0 0 0 0
2.0 1.0 0.0 8 4 0 1 0 2 3 0 0 0 0
2.0 1.0 0.0 2 1 0 2 0 2 3 0 0 0 0
2.0 1.0 0.0 4 2 0 2 0 2 3 0 0 0 0
2.0 1.0 0.0 8 4 0 2 0 2 3 0 0 0 0
2.0 1.0 0.0 2 1 0 1 1 2 3 0 0 0 0
2.0 1.0 0.0 4 2 0 1 1 2 3 0 0 0 0
2.0 1.0 0.0 8 4 0 1 1 2 3 0 0 0 0
2.0 1.0 0.0 2 1 0 2 1 2 3 0 0 0 0
2.0 1.0 0.0 4 2 0 2 1 2 3 0 0 0 0
2.0 1.0 0.0 8 4 0 2 1 2 3 0 0 0 0
2.0 1.0 1.0 2 1 1 1 0 2 3 7 0 0 0
2.0 1.0 1.0 4 2 2 1 0 2 3 7 0 0 0
2.0 1.0 1.0 8 4 4 1 0 2 3 7 0 0 0
2.0 1.0 1.0 2 1 1 2 0 2 3 7 0 0 0
2.0 1.0 1.0 4 2 2 2 0 2 3 7 0 0 0
2.0 1.0 1.0 8 4 4 2 0 2 3 7 0 0 0
2.0 1.0 1.0 2 1 1 1 1 2 3 7 0 0 0
2.0 1.0 1.0 4 2 2 1 1 2 3 7 0 0 0
2.0 1.0 1.0 8 4 4 1 1 2 3 7 0 0 0
2.0 1.0 1.0 2 1 1 2 1 2 3 7 0 0 0
2.0 1.0 1.0 4 2 2 2 1 2 3 7 0 0 0
2.0 1.0 1.0 8 4 4 2 1 2 3 7 0 0 0

```

4.9.4 *Result summary*

We use CHeart rev. 6292 to produce numerical reference solutions.

Passed tests: 24 / 24

No failed tests.

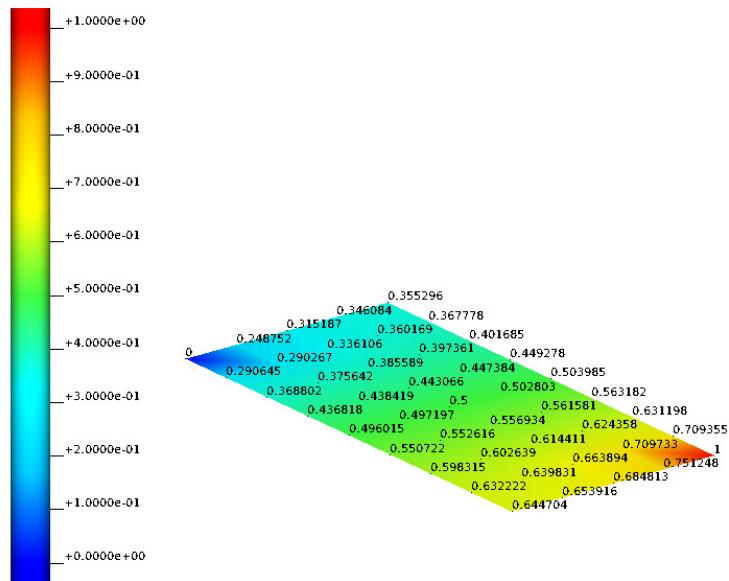


Figure 29: 2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0 2 3 0 0 0 0].

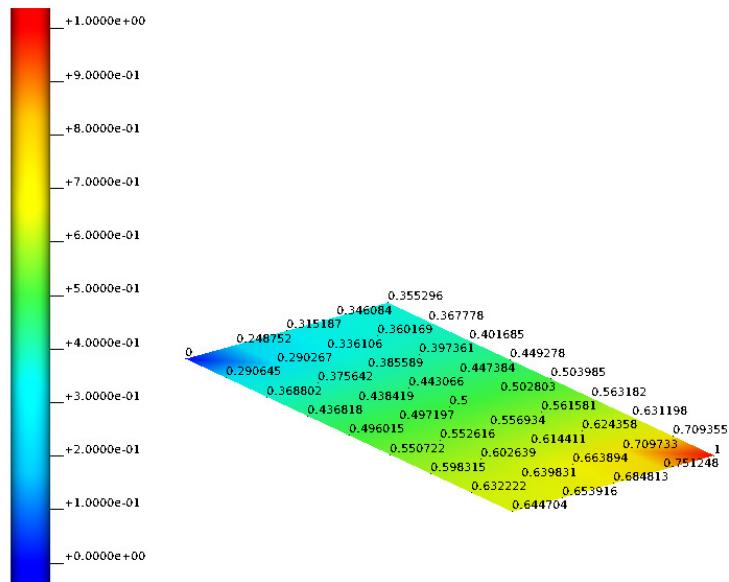


Figure 30: 2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0 2 3 0 0 0 0].

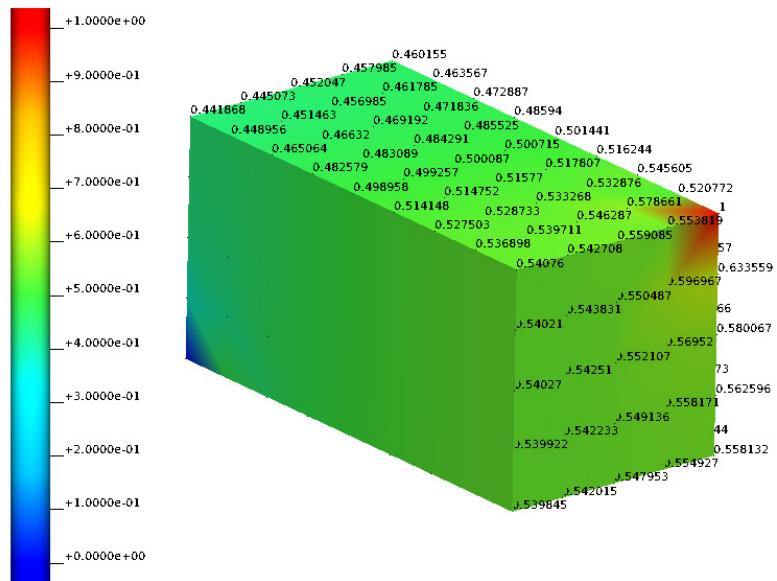


Figure 31: 3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0 2 3 7 0 0 0].

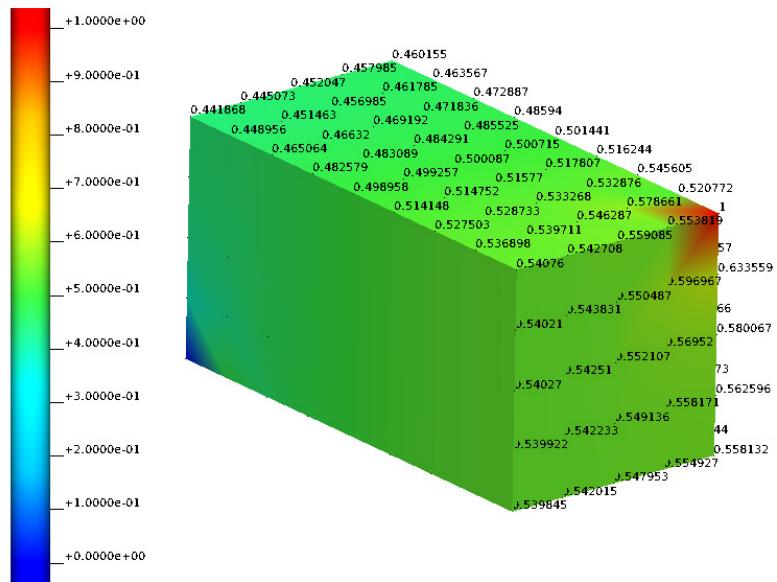


Figure 32: 3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0 2 3 7 0 0 0].

4.10 Example-0013 [VALIDATED]

Example uses generated regular meshes and solves a static problem, i.e., applies the boundary conditions in one step.

4.10.1 Mathematical model - 2D

We solve the following scalar equation,

$$\nabla \cdot [\sigma \nabla u] = 0 \quad \Omega = [0, 2] \times [0, 1], \quad (50)$$

with boundary conditions

$$u = 0 \quad x = y = 0, \quad (51)$$

$$u = 1 \quad x = 2, y = 1. \quad (52)$$

The conductivity tensor is defined as,

$$\sigma(x, t) = \sigma = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}. \quad (53)$$

Rotation of fibres by 30 degrees (first angle).

4.10.2 Mathematical model - 3D

We solve the following scalar equation,

$$\nabla \cdot [\sigma \nabla u] = 0 \quad \Omega = [0, 2] \times [0, 1] \times [0, 1], \quad (54)$$

with boundary conditions

$$u = 0 \quad x = y = z = 0, \quad (55)$$

$$u = 1 \quad x = 2, y = z = 1. \quad (56)$$

The conductivity tensor is defined as,

$$\sigma(x, t) = \sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}. \quad (57)$$

Rotation of fibres by (30, 40, 10) degrees.

4.10.3 Computational model

- Commandline arguments are:

float: length along x-direction

float: length along y-direction

float: length along z-direction (set to zero for 2D)

integer: number of elements in x-direction

integer: number of elements in y-direction

integer: number of elements in z-direction (set to zero for 2D)

integer: interpolation order (1: linear; 2: quadratic)

integer: solver type (0: direct; 1: iterative)

float: σ_{11}

```

float:  $\sigma_{22}$ 
float:  $\sigma_{33}$  (ignored for 2D)
float: angle 1
float: angle 2
float: angle 3

```

- Commandline arguments for tests are:

```

2.0 1.0 0.0 2 1 0 1 0 2 3 0 0.523598775598299 0 0
2.0 1.0 0.0 4 2 0 1 0 2 3 0 0.523598775598299 0 0
2.0 1.0 0.0 8 4 0 1 0 2 3 0 0.523598775598299 0 0
2.0 1.0 0.0 2 1 0 2 0 2 3 0 0.523598775598299 0 0
2.0 1.0 0.0 4 2 0 2 0 2 3 0 0.523598775598299 0 0
2.0 1.0 0.0 8 4 0 2 0 2 3 0 0.523598775598299 0 0
2.0 1.0 0.0 2 1 0 1 1 2 3 0 0.523598775598299 0 0
2.0 1.0 0.0 4 2 0 1 1 2 3 0 0.523598775598299 0 0
2.0 1.0 0.0 8 4 0 1 1 2 3 0 0.523598775598299 0 0
2.0 1.0 0.0 2 1 0 2 1 2 3 0 0.523598775598299 0 0
2.0 1.0 0.0 4 2 0 2 1 2 3 0 0.523598775598299 0 0
2.0 1.0 0.0 8 4 0 2 1 2 3 0 0.523598775598299 0 0
2.0 1.0 1.0 2 1 1 1 0 2 3 7 0.523598775598299 0.698131700797732
0.174532925199433
2.0 1.0 1.0 4 2 2 1 0 2 3 7 0.523598775598299 0.698131700797732
0.174532925199433
2.0 1.0 1.0 8 4 4 1 0 2 3 7 0.523598775598299 0.698131700797732
0.174532925199433
2.0 1.0 1.0 2 1 1 2 0 2 3 7 0.523598775598299 0.698131700797732
0.174532925199433
2.0 1.0 1.0 4 2 0 2 1 2 3 7 0.523598775598299 0.698131700797732
0.174532925199433
2.0 1.0 1.0 8 4 4 1 1 2 3 7 0.523598775598299 0.698131700797732
0.174532925199433
2.0 1.0 1.0 2 1 1 1 2 3 7 0.523598775598299 0.698131700797732
0.174532925199433
2.0 1.0 1.0 4 2 2 1 1 2 3 7 0.523598775598299 0.698131700797732
0.174532925199433
2.0 1.0 1.0 8 4 4 1 1 2 3 7 0.523598775598299 0.698131700797732
0.174532925199433
2.0 1.0 1.0 2 1 1 2 1 2 3 7 0.523598775598299 0.698131700797732
0.174532925199433
2.0 1.0 1.0 4 2 2 2 1 2 3 7 0.523598775598299 0.698131700797732
0.174532925199433
2.0 1.0 1.0 8 4 4 2 1 2 3 7 0.523598775598299 0.698131700797732
0.174532925199433

```

4.10.4 Result summary

We use CHeart rev. 6292 to produce numerical reference solutions.

Passed tests: 24 / 24

No failed tests.

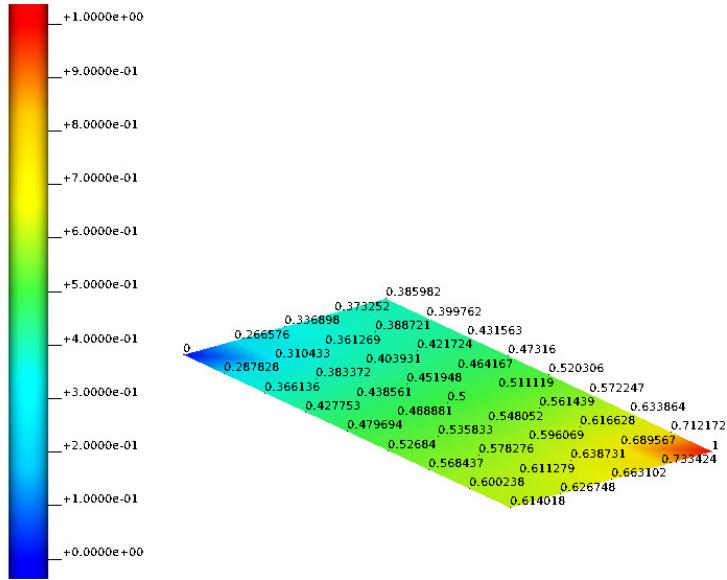


Figure 33: 2D results, iron reference w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0 2 3 0 0.523598775598299 0 0].

4.10.5 Misc

OpenCMISS-iron assumes the conductivity tensor defined in local element Ξ coordinates and uses the fibre angles to rotate the conductivity tensor internally. The script `src/matlab/compute_conductivity_tensor.m` can be used to compute the conductivity tensor in world coordinates, since CHeart assumes the conductivity tensor to be defined in world coordinates. Thus, if one sets up a new test, one has to compute the conductivity tensor in world coordinates as input for CHeart to derive numerical reference data.

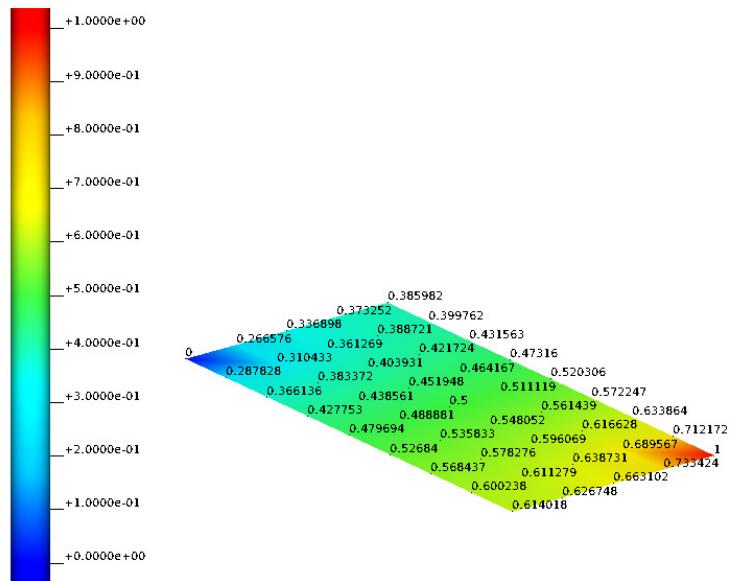


Figure 34: 2D results, current run w/ command line arguments [2.0 1.0 0.0 8 4 0 1 0 2 3 0 0.523598775598299 0 0].

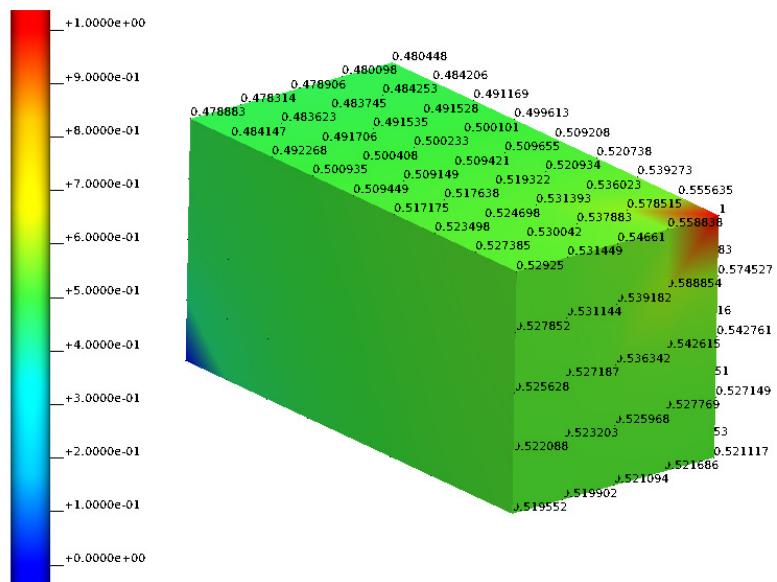


Figure 35: 3D results, iron reference w/ command line arguments [2.0 1.0 1.0 8 4 1 0 2 3 7 0.523598775598299 0.698131700797732 0.174532925199433].

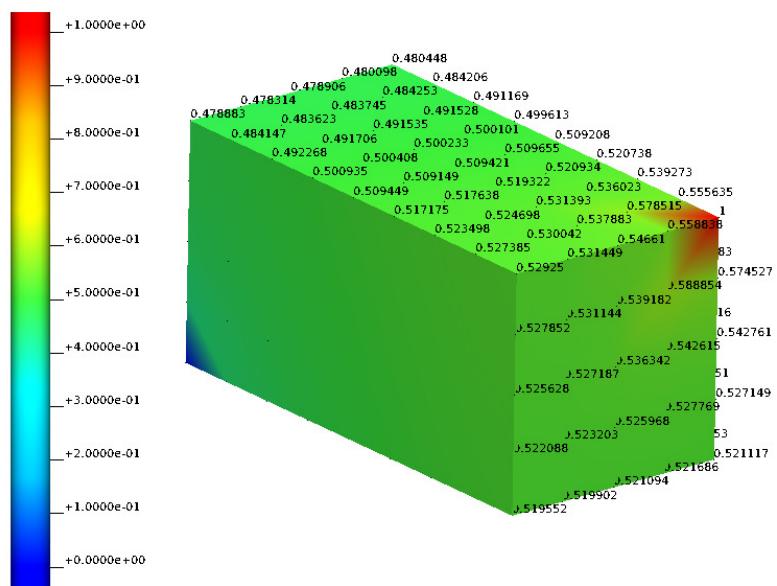


Figure 36: 3D results, current run w/ command line arguments [2.0 1.0 1.0 8 4 4 1 0 2 3 7 0.523598775598299 0.698131700797732 0.174532925199433].

5 LINEAR ELASTICITY

5.1 Equation in general form

$$\partial_{tt} \mathbf{u} + \nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, t) = \mathbf{f}(\mathbf{u}, t) \quad (58)$$

5.2 Example-0101 [PLAUSIBLE]

5.2.1 Mathematical model

We solve the following equation (both 2D and 3D domains are considered),

$$\nabla \cdot \sigma(\mathbf{u}, t) = 0 \quad \Omega = [0, 160] \times [0, 120] \times [0, 120], t \in [0, 5], \quad (59)$$

with time step size $\Delta t = 1$ and $\mathbf{u} = [u_x, u_y]$ in 2D $\mathbf{u} = [u_x, u_y, u_z]$ in 3D. The boundary conditions in 2D are given by

$$u_x = 0 \quad x = 0, \quad (60)$$

$$u_y = 0 \quad y = 0, \quad (61)$$

$$u_x = 8 \quad x = 160, \quad (62)$$

and in 3D by

$$u_x = 0 \quad x = 0, \quad (63)$$

$$u_y = 0 \quad y = 0, \quad (64)$$

$$u_z = 0 \quad z = 0, \quad (65)$$

$$u_x = 8 \quad x = 160. \quad (66)$$

The material parameters are

$$E = 10000 \text{ MPa}, \quad (67)$$

$$\nu = 0.3, \quad (68)$$

$$(69)$$

5.2.2 Computational model

- Commandline arguments are:

float: length along x-direction

float: length along y-direction

float: length along z-direction (set to zero for 2D)

integer: number of elements in x-direction

integer: number of elements in y-direction

integer: number of elements in z-direction (set to zero for 2D)

integer: interpolation order (1: linear; 2: quadratic)

integer: solver type (0: direct; 1: iterative)

float: elastic modulus

float: Poisson ratio

float: displacement percentage load

- Command line arguments for tests are:

160 120 0 8 6 0 1 0 10000 0.3 0.05

160 120 0 16 12 0 1 0 10000 0.3 0.05

160 120 0 32 24 0 1 0 10000 0.3 0.05

160 120 120 8 6 6 1 0 10000 0.3 0.05

160 120 120 16 12 12 1 0 10000 0.3 0.05

160 120 120 32 24 24 1 0 10000 0.3 0.05

```

160 120 0 8 6 0 2 0 10000 0.3 0.05
160 120 0 16 12 0 2 0 10000 0.3 0.05
160 120 0 32 24 0 2 0 10000 0.3 0.05
160 120 120 8 6 6 2 0 10000 0.3 0.05
160 120 120 16 12 12 2 0 10000 0.3 0.05
160 120 120 32 24 24 2 0 10000 0.3 0.05

```

5.2.3 Validation

Passed tests: 12 / 12

No failed tests.

The iron results are compared to those from Abaqus (version 2017). The figures below show selected results from the validation simulations carried out in Abaqus and provide a qualitative validation. A quantitative validation was carried out by comparing the horizontal displacement u_y along the free-edge ($y = 120$ for 2D and $y = z = 120$ for 3D) and computing the L₂-norm according to

$$L_2\text{-norm} = \frac{1}{N} \times \sum_{i=1}^N \sqrt{(u_{y, \text{abaqus}}^i - u_{y, \text{iron}}^i)^2}, \quad (70)$$

where N is the total number of nodes along the free-edge. The results over the mesh refinements are given in Table 1.

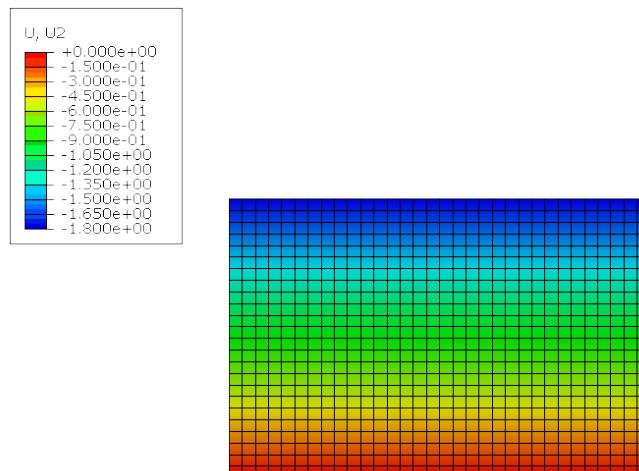


Figure 37: Results, Abaqus 2D fine mesh.

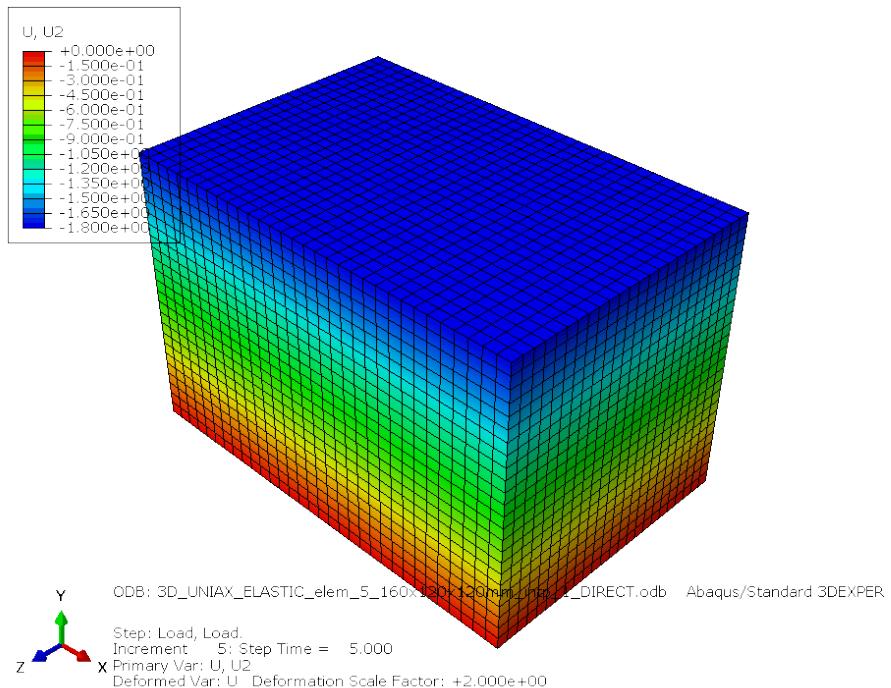


Figure 38: Results, abaqus 3D fine mesh.

Dimension	Mesh	L_2 -norm	Interpolation
2D	Coarse	5.322×10^{-16}	Linear
2D	Medium	1.559×10^{-15}	Linear
2D	Fine	2.900×10^{-15}	Linear
3D	Coarse	3.071×10^{-17}	Linear
3D	Medium	2.125×10^{-17}	Linear
3D	Fine	2.924×10^{-17}	Linear
2D	Coarse	9.728×10^{-16}	Quadratic
2D	Medium	2.039×10^{-15}	Quadratic
2D	Fine	2.159×10^{-15}	Quadratic
3D	Coarse	6.687×10^{-16}	Quadratic
3D	Medium	...	Quadratic
3D	Fine	...	Quadratic

Table 1: Quantitative error between Abaqus 2017 and iron simulations for linear elastic uniaxial extensions

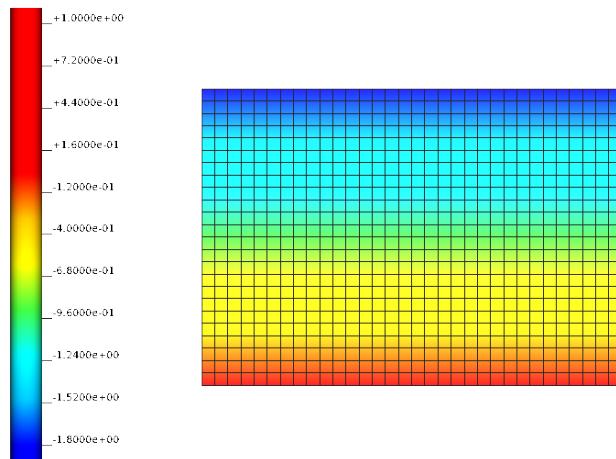


Figure 39: Results, iron 2D fine mesh.

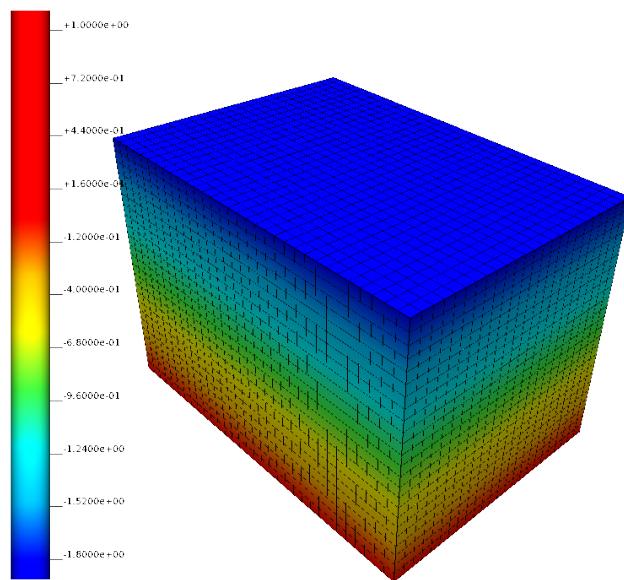


Figure 40: Results, iron 3D fine mesh.

5.3 Example-0102 [PLAUSIBLE]

5.3.1 Mathematical model

We solve the following equation (both 2D and 3D domains are considered),

$$\nabla \cdot \sigma(\mathbf{u}, t) = 0 \quad \Omega = [0, 160] \times [0, 120] \times [0, 120], t \in [0, 5], \quad (71)$$

with time step size $\Delta t = 1$ and $\mathbf{u} = [u_x, u_y]$ in 2D $\mathbf{u} = [u_x, u_y, u_z]$ in 3D. The boundary conditions in 2D are given by

$$u_x = u_y = 0 \quad x = 0, \quad (72)$$

$$u_x = 0 \quad x = 160, \quad (73)$$

$$u_y = 8 \quad x = 160, \quad (74)$$

and in 3D by

$$u_x = u_y = u_z = 0 \quad x = 0, \quad (75)$$

$$u_x = u_z = 0 \quad x = 160, \quad (76)$$

$$u_y = 8 \quad x = 160. \quad (77)$$

The material parameters are

$$E = 10000 \text{ MPa}, \quad (78)$$

$$\nu = 0.3, \quad (79)$$

$$(80)$$

5.3.2 Computational model

- Commandline arguments are:

float: length along x-direction

float: length along y-direction

float: length along z-direction (set to zero for 2D)

integer: number of elements in x-direction

integer: number of elements in y-direction

integer: number of elements in z-direction (set to zero for 2D)

integer: interpolation order (1: linear; 2: quadratic)

integer: solver type (0: direct; 1: iterative)

float: elastic modulus

float: Poisson ratio

float: displacement percentage load

- Command line arguments for tests are:

160 120 0 8 6 0 1 0 10000 0.3 0.05

160 120 0 16 12 0 1 0 10000 0.3 0.05

160 120 0 32 24 0 1 0 10000 0.3 0.05

160 120 120 8 6 6 1 0 10000 0.3 0.05

160 120 120 16 12 12 1 0 10000 0.3 0.05

160 120 120 32 24 24 1 0 10000 0.3 0.05

160 120 0 8 6 0 2 0 10000 0.3 0.05

```

160 120 0 16 12 0 2 0 10000 0.3 0.05
160 120 0 32 24 0 2 0 10000 0.3 0.05
160 120 120 8 6 6 2 0 10000 0.3 0.05
160 120 120 16 12 12 2 0 10000 0.3 0.05
160 120 120 32 24 24 2 0 10000 0.3 0.05

```

5.3.3 Validation

Passed tests: 0 / 8

Failed tests:

```

1 =====
    LOADING      = SHEAR
    REFINEMENT   = COARSE
    DIMENSION    = 2D
    INTERPOLATION = 1
    CONTROL      = DISPLACEMENT
    norm(aba-iron, 2) = 7e-03 >= 1.000000e-10

2 =====
    LOADING      = SHEAR
    REFINEMENT   = COARSE
    DIMENSION    = 2D
    INTERPOLATION = 2
    CONTROL      = DISPLACEMENT
    norm(aba-iron, 2) = 3e-04 >= 1.000000e-10

3 =====
    LOADING      = SHEAR
    REFINEMENT   = COARSE
    DIMENSION    = 3D
    INTERPOLATION = 1
    CONTROL      = DISPLACEMENT
    norm(aba-iron, 2) = 4e-04 >= 1.000000e-10

4 =====
    LOADING      = SHEAR
    REFINEMENT   = COARSE
    DIMENSION    = 3D
    INTERPOLATION = 2
    CONTROL      = DISPLACEMENT
    norm(aba-iron, 2) = 3e-05 >= 1.000000e-10

5 =====
    LOADING      = SHEAR
    REFINEMENT   = MEDIUM
    DIMENSION    = 2D
    INTERPOLATION = 1
    CONTROL      = DISPLACEMENT
    norm(aba-iron, 2) = 1e-03 >= 1.000000e-10

6 =====
    LOADING      = SHEAR
    REFINEMENT   = MEDIUM
    DIMENSION    = 2D
    INTERPOLATION = 2
    CONTROL      = DISPLACEMENT
    norm(aba-iron, 2) = 6e-05 >= 1.000000e-10

```

```

7 =====
  LOADING      = SHEAR
  REFINEMENT   = MEDIUM
  DIMENSION    = 3D
  INTERPOLATION = 1
  CONTROL      = DISPLACEMENT
  norm(aba-iron, 2) = 4e-05 >= 1.000000e-10
8 =====
  LOADING      = SHEAR
  REFINEMENT   = MEDIUM
  DIMENSION    = 3D
  INTERPOLATION = 2
  CONTROL      = DISPLACEMENT
  norm(aba-iron, 2) = 5e-06 >= 1.000000e-10

```

The iron results are compared to those from Abaqus (version 2017). The figures below show selected results from the validation simulations carried out in Abaqus and provide a qualitative validation. A quantitative validation was carried out by comparing the horizontal displacement u_x along the free-edge ($y = 120$ for 2D and $y = z = 120$ for 3D) and computing the L₂-norm according to

$$L_2\text{-norm} = \frac{1}{N} \times \sum_{i=1}^N \sqrt{(u_{y, \text{abaqus}}^i - u_{y, \text{iron}}^i)^2}, \quad (81)$$

where N is the total number of nodes along the free-edge. The results over the mesh refinements are given in Table 2.

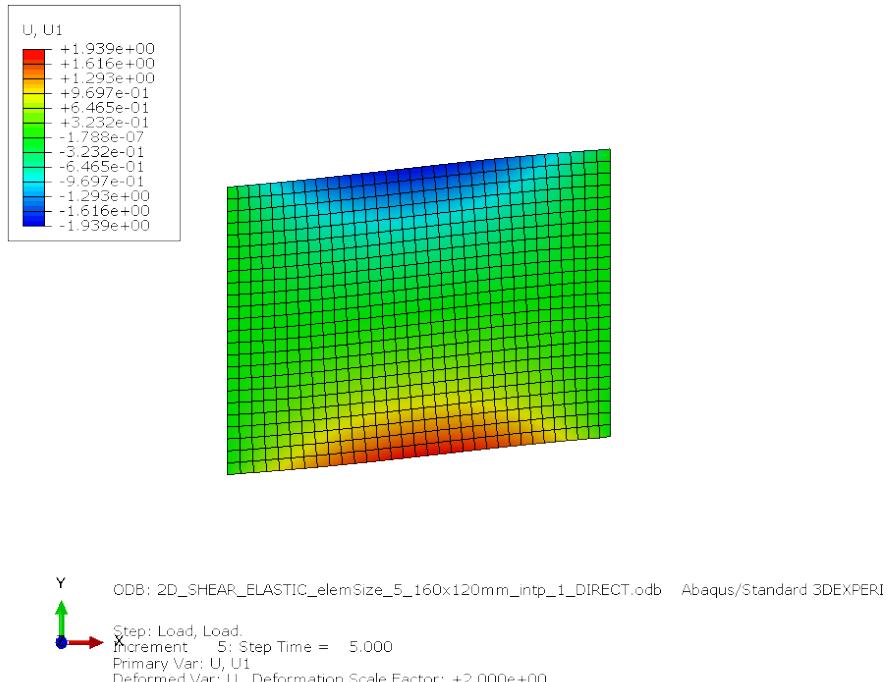


Figure 41: Results, Abaqus 2D fine mesh.

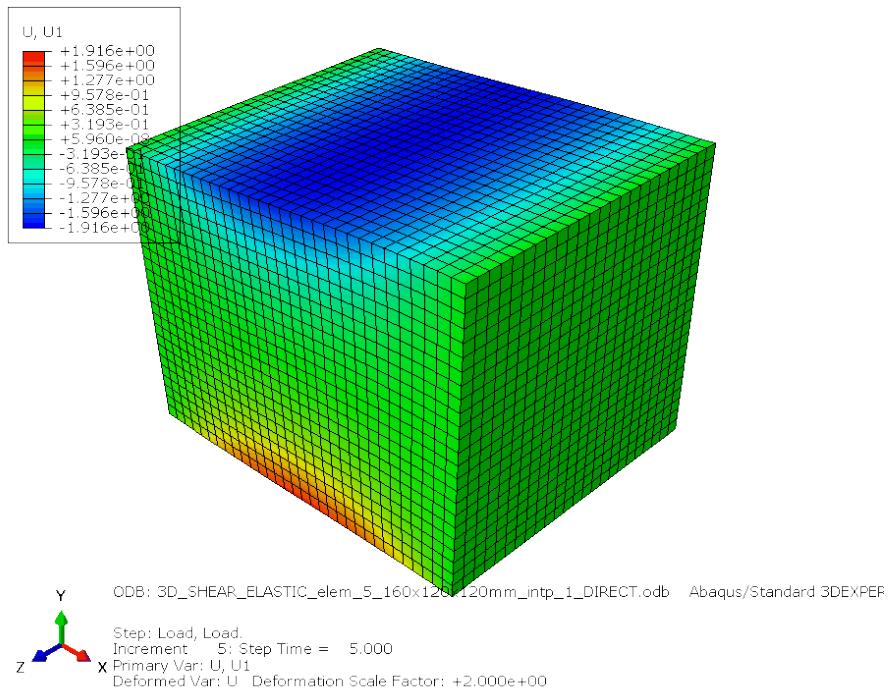


Figure 42: Results, abaqus 3D fine mesh.

Dimension	Mesh	L_2 -norm	Interpolation
2D	Coarse	6.696×10^{-3}	Linear
2D	Medium	1.273×10^{-3}	Linear
2D	Fine	2.489×10^{-4}	Linear
3D	Coarse	4.234×10^{-4}	Linear
3D	Medium	4.184×10^{-5}	Linear
3D	Fine	3.781×10^{-6}	Linear
2D	Coarse	3.036×10^{-4}	Quadratic
2D	Medium	6.099×10^{-5}	Quadratic
2D	Fine	1.089×10^{-5}	Quadratic
3D	Coarse	...	Quadratic
3D	Medium	...	Quadratic
3D	Fine	...	Quadratic

Table 2: Quantitative error between Abaqus 2017 and iron simulations for linear elastic shear

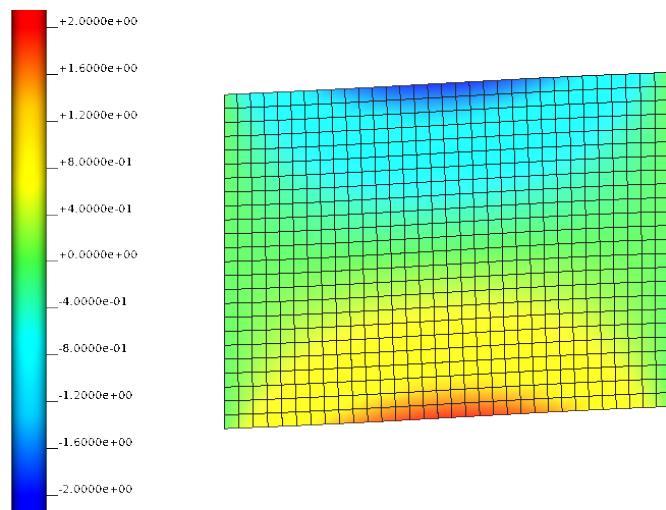


Figure 43: Results, iron 2D fine mesh.

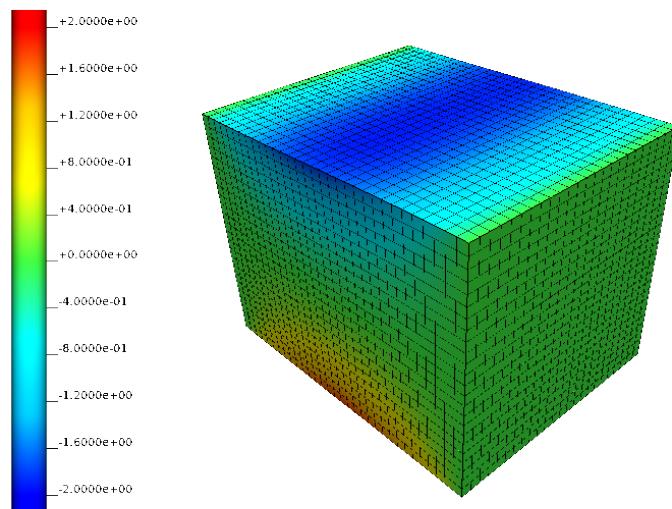


Figure 44: Results, iron 3D fine mesh.

5.4 Example-0111 [PLAUSIBLE]

5.4.1 Mathematical model

We solve the following equation (both 2D and 3D domains are considered),

$$\nabla \cdot \sigma(\mathbf{u}, t) = \mathbf{f}(\mathbf{u}, t) \quad \Omega = [0, 160] \times [0, 120] \times [0, 120], t \in [0, 5], \quad (82)$$

with time step size $\Delta t = 1$ (used for load stepping) and $\mathbf{u} = [u_x, u_y]$ in 2D $\mathbf{u} = [u_x, u_y, u_z]$ in 3D. The boundary conditions in 2D are given by

$$u_x = 0 \quad x = 0, \quad (83)$$

$$u_y = 0 \quad y = 0, \quad (84)$$

$$f(u_x) = 6.0 \times 10^4 \quad x = 160, \quad (85)$$

and in 3D by

$$u_x = 0 \quad x = 0, \quad (86)$$

$$u_y = 0 \quad y = 0, \quad (87)$$

$$u_z = 0 \quad z = 0, \quad (88)$$

$$f(u_x) = 7.2 \times 10^6 \quad x = 160. \quad (89)$$

The material parameters are

$$E = 10000 \text{ MPa}, \quad (90)$$

$$\nu = 0.3, \quad (91)$$

$$(92)$$

5.4.2 Computational model

- Commandline arguments are:

float: length along x-direction

float: length along y-direction

float: length along z-direction (set to zero for 2D)

integer: number of elements in x-direction

integer: number of elements in y-direction

integer: number of elements in z-direction (set to zero for 2D)

integer: interpolation order (1: linear; 2: quadratic)

integer: solver type (0: direct; 1: iterative)

float: elastic modulus

float: Poisson ratio

float: XXX

- Command line arguments for tests are:

160 120 0 8 6 0 1 0 10000 0.3 XXX

160 120 0 16 12 0 1 0 10000 0.3 XXX

160 120 0 32 24 0 1 0 10000 0.3 XXX

160 120 120 8 6 6 1 0 10000 0.3 XXX

160 120 120 16 12 12 1 0 10000 0.3 XXX

```

160 120 120 32 24 24 1 0 10000 0.3 XXX
160 120 0 8 6 0 2 0 10000 0.3 XXX
160 120 0 16 12 0 2 0 10000 0.3 XXX
160 120 0 32 24 0 2 0 10000 0.3 XXX
160 120 120 8 6 6 2 0 10000 0.3 XXX
160 120 120 16 12 12 2 0 10000 0.3 XXX
160 120 120 32 24 24 2 0 10000 0.3 XXX

```

5.4.3 Validation

Passed tests: 3 / 6

Failed tests:

```

1 =====
  LOADING      = UNIAX
  REFINEMENT   = COARSE
  DIMENSION    = 3D
  INTERPOLATION = 1
  CONTROL      = FORCE
  norm(aba-iron, 2) = 3e-02 >= 1.000000e-10

2 =====
  LOADING      = UNIAX
  REFINEMENT   = MEDIUM
  DIMENSION    = 3D
  INTERPOLATION = 1
  CONTROL      = FORCE
  norm(aba-iron, 2) = 1e-02 >= 1.000000e-10

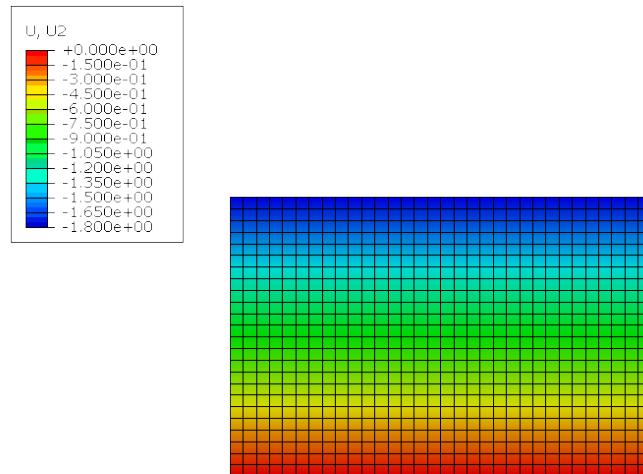
3 =====
  LOADING      = UNIAX
  REFINEMENT   = FINE
  DIMENSION    = 3D
  INTERPOLATION = 1
  CONTROL      = FORCE
  norm(aba-iron, 2) = 3e-03 >= 1.000000e-10

```

The iron results are compared to those from Abaqus (version 2017). The figures below show selected results from the validation simulations carried out in Abaqus and provide a qualitative validation. A quantitative validation was carried out by comparing the horizontal displacement u_y along the free-edge ($y = 120$ for 2D and $y = z = 120$ for 3D) and computing the L₂-norm according to

$$L_2\text{-norm} = \frac{1}{N} \times \sum_{i=1}^N \sqrt{\left(u_{y, \text{abaqus}}^i - u_{y, \text{iron}}^i\right)^2}, \quad (93)$$

where N is the total number of nodes along the free-edge. The results over the mesh refinements are given in Table 3.



Y
ODB: 2D_UNIAX_FORCE_ELASTIC_elem_5_160x120mm_intp_1_DIRECT.odb Abaqus/Standard 3DEXF
Step: Load, Load.
Increment 5: Step Time = 5.000
Primary Var: U, U2
Deformed Var: U Deformation Scale Factor: +2.000e+00

Figure 45: Results, Abaqus 2D fine mesh.

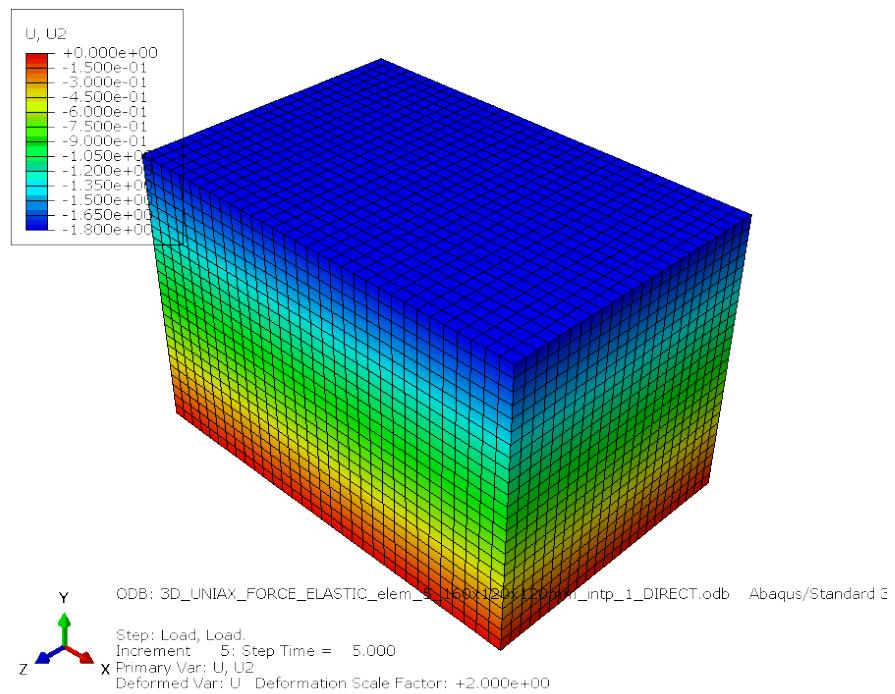


Figure 46: Results, abaqus 3D fine mesh.

Figure 47: Results, iron 2D fine mesh.

Figure 48: Results, iron 3D fine mesh.

Dimension	Mesh	L_2 -norm	Interpolation
2D	Coarse	...	Linear
2D	Medium	...	Linear
2D	Fine	...	Linear
3D	Coarse	...	Linear
3D	Medium	...	Linear
3D	Fine	...	Linear
2D	Coarse	...	Quadratic
2D	Medium	...	Quadratic
2D	Fine	...	Quadratic
3D	Coarse	...	Quadratic
3D	Medium	...	Quadratic
3D	Fine	...	Quadratic

Table 3: Quantitative error between Abaqus 2017 and iron simulations for linear elastic uniaxial extensions

5.5 Example-0112 [PLAUSIBLE]

5.5.1 Mathematical model

We solve the following equation (both 2D and 3D domains are considered),

$$\nabla \cdot \sigma(\mathbf{u}, t) = \mathbf{f}(\mathbf{u}, t) \quad \Omega = [0, 160] \times [0, 120] \times [0, 120], t \in [0, 5], \quad (94)$$

with time step size $\Delta t = 1$ and $\mathbf{u} = [u_x, u_y]$ in 2D $\mathbf{u} = [u_x, u_y, u_z]$ in 3D. The boundary conditions in 2D are given by

$$u_x = u_y = 0 \quad x = 0, \quad (95)$$

$$u_x = 0 \quad x = 160, \quad (96)$$

$$f(u_y) = 6.0 \times 10^4 \quad x = 160, \quad (97)$$

and in 3D by

$$u_x = u_y = 0 \quad x = 0, \quad (98)$$

$$u_x = 0 \quad x = 160, \quad (99)$$

$$u_z = 0 \quad z = 0, \quad (100)$$

$$f(u_y) = 7.2 \times 10^6 \quad x = 160. \quad (101)$$

The material parameters are

$$E = 10000 \text{ MPa}, \quad (102)$$

$$\nu = 0.3, \quad (103)$$

$$(104)$$

5.5.2 Computational model

- Commandline arguments are:

float: length along x-direction

float: length along y-direction

float: length along z-direction (set to zero for 2D)

integer: number of elements in x-direction

integer: number of elements in y-direction

integer: number of elements in z-direction (set to zero for 2D)

integer: interpolation order (1: linear; 2: quadratic)

integer: solver type (0: direct; 1: iterative)

float: elastic modulus

float: Poisson ratio

float: XXX

- Command line arguments for tests are:

160 120 0 8 6 0 1 0 10000 0.3 XXX

160 120 0 16 12 0 1 0 10000 0.3 XXX

160 120 0 32 24 0 1 0 10000 0.3 XXX

160 120 120 8 6 6 1 0 10000 0.3 XXX

160 120 120 16 12 12 1 0 10000 0.3 XXX

Dimension	Mesh	L_2 -norm	Interpolation
2D	Coarse	...	Linear
2D	Medium	...	Linear
2D	Fine	...	Linear
3D	Coarse	...	Linear
3D	Medium	...	Linear
3D	Fine	...	Linear
2D	Coarse	...	Quadratic
2D	Medium	...	Quadratic
2D	Fine	...	Quadratic
3D	Coarse	...	Quadratic
3D	Medium	...	Quadratic
3D	Fine	...	Quadratic

Table 4: Quantitative error between Abaqus 2017 and iron simulations for linear elastic shear

```

160 120 120 32 24 24 1 0 10000 0.3 XXX
160 120 0 8 6 0 2 0 10000 0.3 XXX
160 120 0 16 12 0 2 0 10000 0.3 XXX
160 120 0 32 24 0 2 0 10000 0.3 XXX
160 120 120 8 6 2 0 10000 0.3 XXX
160 120 120 16 12 12 2 0 10000 0.3 XXX
160 120 120 32 24 24 2 0 10000 0.3 XXX

```

5.5.3 Validation

Failed tests:

```

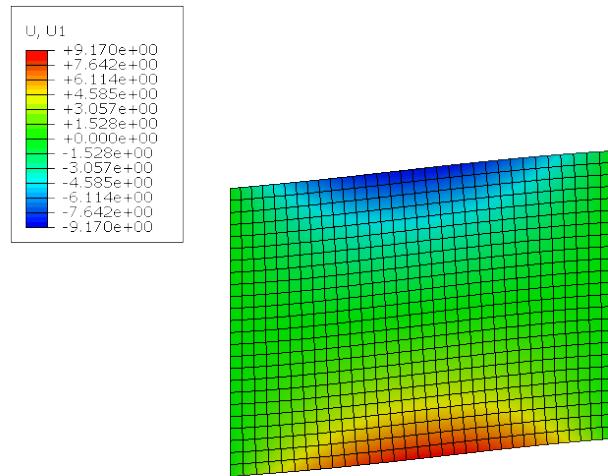
1 =====
LOADING      = SHEAR
REFINEMENT   = COARSE
DIMENSION    = 2D
INTERPOLATION = 1
CONTROL      = FORCE
norm(aba-iron, 2) = 3e+00 >= 1.000000e-10

```

The iron results are compared to those from Abaqus (version 2017). The figures below show selected results from the validation simulations carried out in Abaqus and provide a qualitative validation. A quantitative validation was carried out by comparing the horizontal displacement u_x along the free-edge ($y = 120$ for 2D and $y = z = 120$ for 3D) and computing the L_2 -norm according to

$$L_2\text{-norm} = \frac{1}{N} \times \sum_{i=1}^N \sqrt{(u_{y, \text{abaqus}}^i - u_{y, \text{iron}}^i)^2}, \quad (105)$$

where N is the total number of nodes along the free-edge. The results over the mesh refinements are given in Table 4.



Y
Step: Load, Load.
Increment 5: Step Time = 5.000
Primary Var: U, U1
Deformed Var: U' Deformation Scale Factor: +4.221e-01

Figure 49: Results, Abaqus 2D fine mesh.

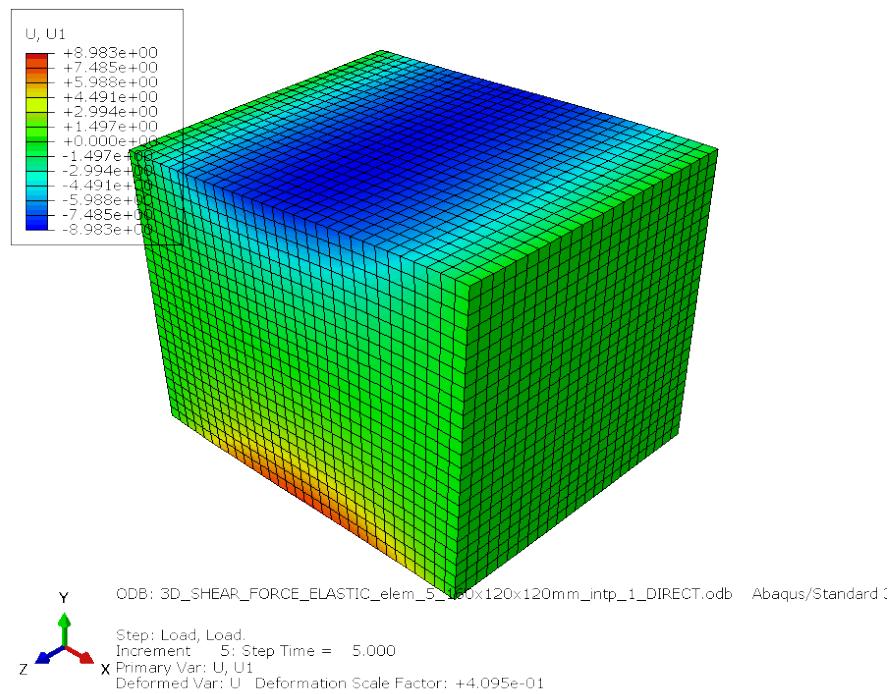


Figure 50: Results, abaqus 3D fine mesh.

Figure 51: Results, iron 2D fine mesh.

Figure 52: Results, iron 3D fine mesh.

6 FINITE ELASTICITY

7 NAVIER-STOKES FLOW

7.1 Equation in general form

$$\partial_t(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + p \mathbf{I}) = \rho \mathbf{f} \quad (106)$$

7.2 Example-0302-u [COMPILES]

Example uses user-defined simplex meshes in CHart mesh format with quadratic/linear interpolation for velocity/pressure and solves a dynamic problem.

Setup is the well-known lid-driven cavity problem on the unit square or unit cube in two and three dimensions.

Current issue: does not converge after 30 some time iterations (2D and 3D).

Visualization issue: In exelem-file, replace

1. constant(2)*constant, no modify, grid based.
#xi1=0, #xi2=0
2. constant(2)*constant, no modify, grid based.

with

1. constant*constant, no modify, grid based.
#xi1=0, #xi2=0
2. constant*constant, no modify, grid based.

and likewise for 3D, replace

1. constant(2;3)*constant*constant, no modify, grid based.
#xi1=0, #xi2=0, #xi3=0
2. constant(2;3)*constant*constant, no modify, grid based.

with

1. constant*constant*constant, no modify, grid based.
#xi1=0, #xi2=0, #xi3=0
2. constant*constant*constant, no modify, grid based.

7.2.1 Mathematical model - 2D

We solve the incompressible Navier-Stokes equation,

$$\partial_t(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) - \nabla \cdot (\mu \nabla \mathbf{v} - p \mathbf{I}) = \rho \mathbf{f} \quad \Omega = [0, 1] \times [0, 1], \quad (107)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (108)$$

with boundary conditions

$$\mathbf{v} = 0 \quad x = 0, \quad (109)$$

$$\mathbf{v} = 0 \quad x = 1, \quad (110)$$

$$\mathbf{v} = 0 \quad y = 0, \quad (111)$$

$$\mathbf{v} = [1, 0]^T \quad y = 1. \quad (112)$$

Viscosity $\mu = 0.0025$, density $\rho = 1$. Thus, Reynolds number $Re = 400$.

7.2.2 Mathematical model - 3D

We solve the incompressible Navier-Stokes equation,

$$\partial_t(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) - \nabla \cdot (\mu \nabla \mathbf{v} - p \mathbf{I}) = \rho \mathbf{f} \quad \Omega = [0, 1] \times [0, 1] \times [0, 1], \quad (113)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (114)$$

with boundary conditions

$$\mathbf{v} = 0 \quad x = 0, \quad (115)$$

$$\mathbf{v} = 0 \quad x = 1, \quad (116)$$

$$\mathbf{v} = 0 \quad y = 0, \quad (117)$$

$$\mathbf{v} = [1, 0]^T \quad y = 1, \quad (118)$$

$$\mathbf{v} = 0 \quad z = 0, \quad (119)$$

$$\mathbf{v} = 0 \quad z = 1. \quad (120)$$

Viscosity $\mu = 0.01$, density $\rho = 1$. Thus, Reynolds number $Re = 100$.

7.2.3 Computational model

- Commandline arguments are:

integer: number of dimensions (2: 2D, 3: 3D)

integer: mesh refinement level (1, 2, 3, ...)

float: start time

float: stop time

float: time step size

float: density

float: viscosity

integer: solver type (0: direct; 1: iterative)

- Commandline arguments for tests are:

2 1 0.0 1.0 0.001 0.0025 1.0 0

2 2 0.0 1.0 0.001 0.0025 1.0 0

2 3 0.0 1.0 0.001 0.0025 1.0 0

2 1 0.0 1.0 0.001 0.0025 1.0 1

2 2 0.0 1.0 0.001 0.0025 1.0 1

2 3 0.0 1.0 0.001 0.0025 1.0 1

3 1 0.0 1.0 0.001 0.01 1.0 0

3 2 0.0 1.0 0.001 0.01 1.0 0

3 3 0.0 1.0 0.001 0.01 1.0 0

3 1 0.0 1.0 0.001 0.01 1.0 1

3 2 0.0 1.0 0.001 0.01 1.0 1

3 3 0.0 1.0 0.001 0.01 1.0 1

- Note: Binary uses command line arguments to search for the relevant mesh files.

7.2.4 *Result summary*

We use CHeart rev. 6292 to produce numerical reference solutions.

Passed tests: 0 / 12

All tests failed.

8 MONODOMAIN

8.1 Example-0401 [PLAUSIBLE]

8.1.1 Mathematical model

We solve the Monodomain Equation

$$\sigma \Delta V_m(t) = A_m \left(C_m \frac{\partial V_m}{\partial t} + I_{\text{ionic}}(V_m) \right) \quad \Omega = [0, 1] \times [0, 1], \quad t \in [0, 3.0] \quad (121)$$

where $V_m(t)$ is given by the Hodgkin-Huxley system of ODEs [2]
with boundary conditions

$$V_m = 0 \quad x = y = 0, \quad (122)$$

$$V_m = 0 \quad x = y = 1. \quad (123)$$

and initial values

$$V_m(t = 0) = -75$$

Additionally a stimulation current I_{stim} is applied for $t_{\text{stim}} = [0, 0.1]$ at the center node of the domain (i.e. at $(x, y) = (\frac{1}{2}, \frac{1}{2},)$).

Material parameters:

$$\sigma = 3.828$$

$$A_m = 500$$

$$C_m = 0.58 \quad \text{for the slow-twitch case}, \quad C_m = 1.0 \quad \text{for the fast-twitch case}$$

$$I_{\text{stim}} = 1200 \quad \text{for the slow-twitch case}, \quad I_{\text{stim}} = 2000.0 \quad \text{for the fast-twitch case}$$

8.1.2 Computational model

- This example uses generated meshes

- Commandline arguments are:

number elements X

number elements Y

interpolation order (1: linear; 2: quadratic)

solver type (0: direct; 1: iterative)

PDE step size

stop time

output frequency

CellML Model URL

slow-twitch

ODE time-step

- Commands for tests are:

```
./folder/src/example 24 24 1 0 0.005 3.0 1 hodgkin_huxley_1952.cellml F 0.0001
./folder/src/example 24 24 1 0 0.005 3.0 1 hodgkin_huxley_1952.cellml F 0.005
./folder/src/example 10 10 1 0 0.005 3.0 1 hodgkin_huxley_1952.cellml F 0.0001
mpirun -n 2 ./folder/src/example 24 24 1 0 0.005 3.0 1 hodgkin_huxley_1952.cellml
```

```
mpirun -n 8 ./folder/src/example 24 24 1 0 0.005 3.0 1 hodgkin_huxley_1952.cellml
./folder/src/example 2 2 1 0 0.005 3.0 1 hodgkin_huxley_1952.cellml F 0.0001
mpirun -n 2 ./folder/src/example 2 2 1 0 0.005 3.0 1 hodgkin_huxley_1952.cellml F
```

- This is a dynamic problem.

8.1.3 Results

Passed tests: 20 / 20

No failed tests.

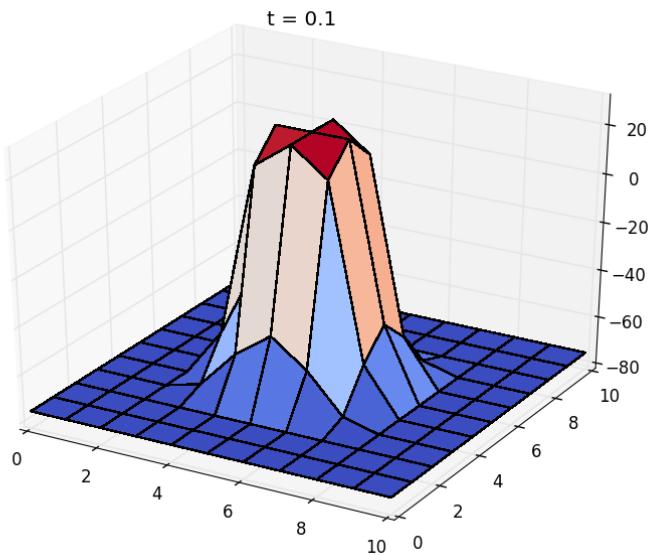


Figure 53: Result of scenario with 10×10 elements, $t = 20$, direct solver, $p = 1$ process

With the 'big' target there will be animations created. You get a better understanding of the solutions by looking at them in `iron-tests/examples/example-0401/doc/figures`.

8.1.4 Validation

We compare with a Matlab implementation as well as with reference iron files.

The matlab scripts use finite difference discretization instead of finite elements. The results are qualitatively the same but exactly. The compare script also tests for matlab reference data which is only included for the 2 examples with 24×24 elements. There is a big L_2 -error. The tolerance is set to a high value to allow for the tests to succeed. With this the comparing mechanism is tested. Maybe in the future someone succeeds to generate suitable matlab data that then can just be exchanged without having to rewrite the compare script.

The iron files to compare with are the output of the simulation as of Aug. 2017. In that way we can check if the simulation brakes with respect to the current state. In order to keep file sizes minimal the comparision is only conducted for time steps $t = 0.01, 0.1, 0.2, 1, 2, 3$ for the 'big' target and $t = 0.1, 0.2, 1$ for the 'fast' target.

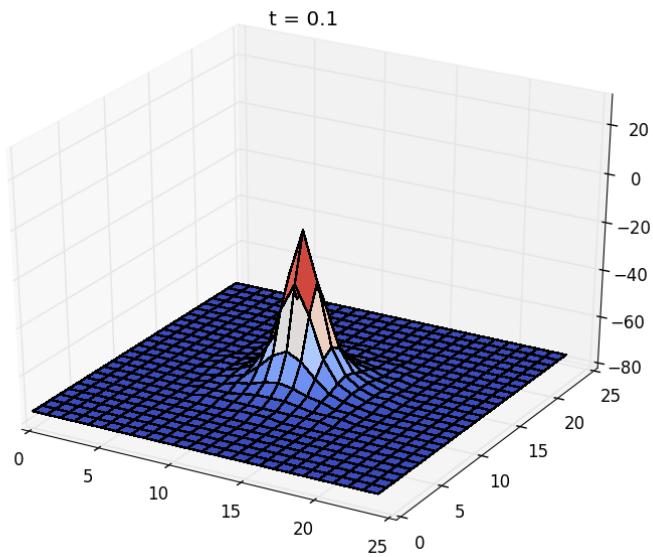


Figure 54: Result of scenario with 24×24 elements, $t = 20$, direct solver, $p = 1$ process

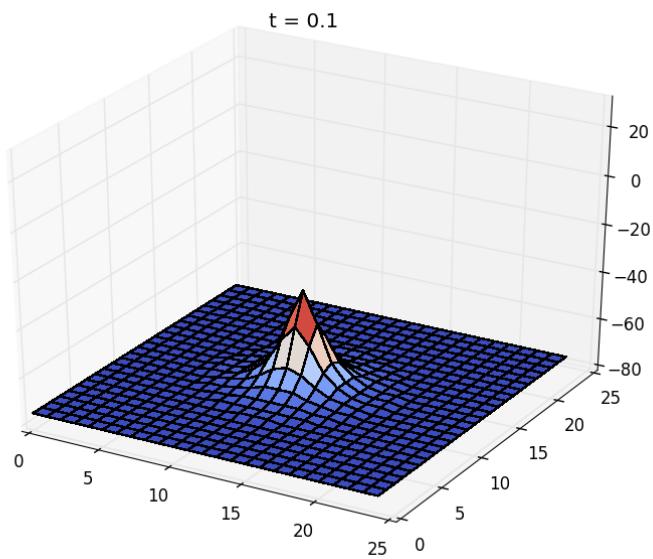


Figure 55: Result of scenario with 24×24 elements, $t = 20$, iterative solver, $p = 1$ process

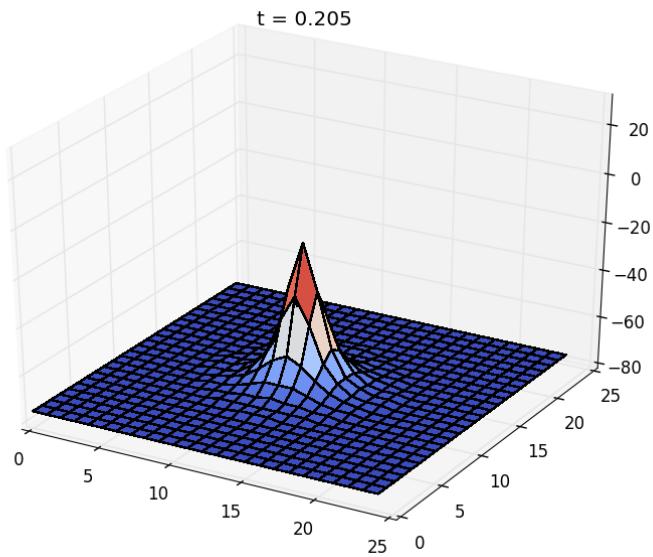


Figure 56: Result of scenario with 24×24 elements, $t = 20$, iterative solver, $p = 2$ processes

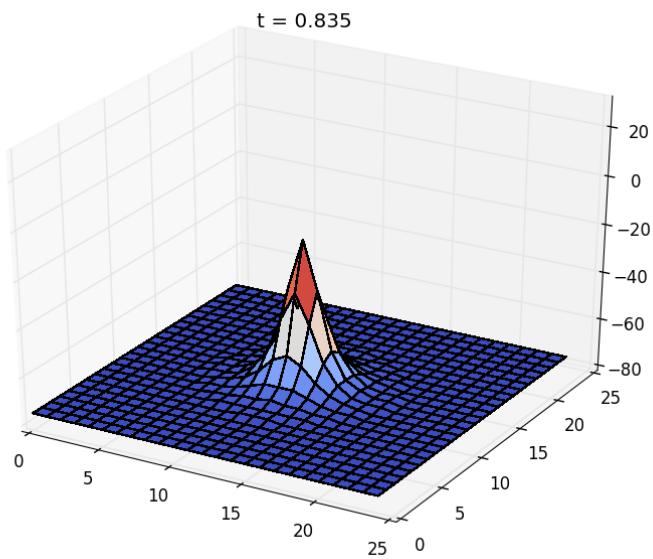


Figure 57: Result of scenario with 24×24 elements, $t = 20$, iterative solver, $p = 8$ processes

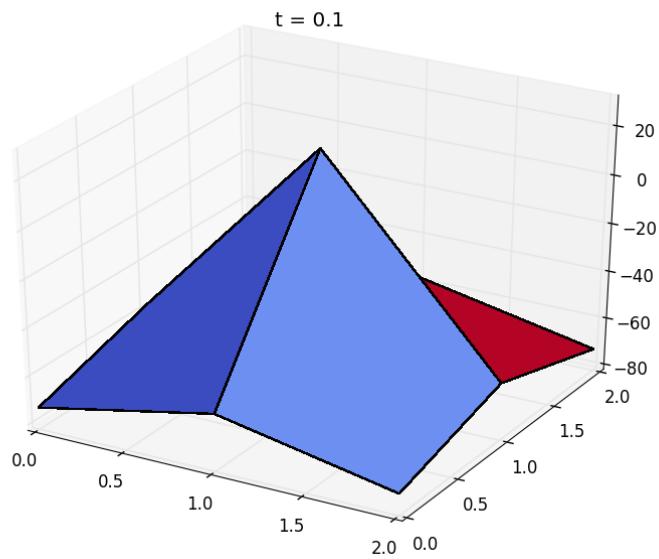


Figure 58: Result of scenario with 2×2 elements, $t = 20$, direct solver, $p = 1$ process

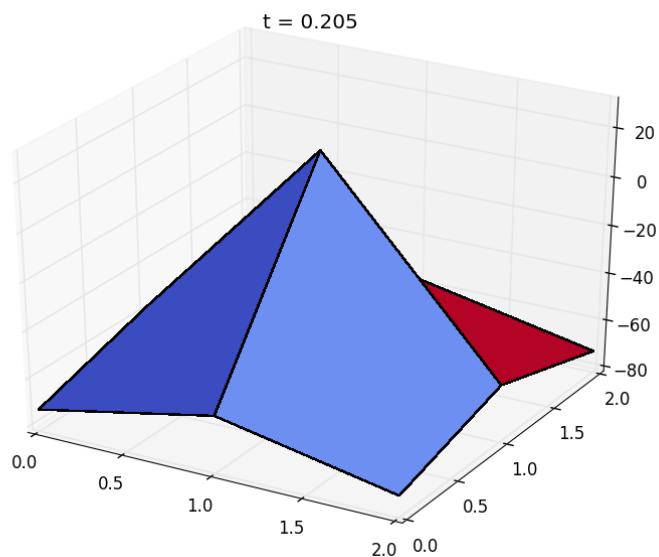


Figure 59: Result of scenario with 2×2 elements, $t = 20$, direct solver, $p = 2$ processes

8.2 Example-0402 [PLAUSIBLE]

8.2.1 Mathematical model

We solve the Monodomain Equation

$$\sigma \Delta V_m(t) = A_m \left(C_m \frac{\partial V_m}{\partial t} + I_{ionic}(V_m) \right) \quad \Omega = [0, 1] \times [0, 1], \quad t \in [0, 3.0] \quad (124)$$

where $V_m(t)$ is given by the CellML description of Noble's 1998 improved guinea-pig ventricular cell model system of ODEs [3]

with boundary conditions

$$V_m = 0 \quad x = y = 0, \quad (125)$$

$$V_m = 0 \quad x = y = 1. \quad (126)$$

and initial values

$$V_m(t = 0) = -75$$

Additionally a stimulation current I_{stim} is applied for $t_{stim} = [0, 0.1]$ at the center node of the domain (i.e. at $(x, y) = (\frac{1}{2}, \frac{1}{2},)$).

Material parameters:

$$\sigma = 3.828$$

$$A_m = 500$$

$$C_m = 0.58 \quad \text{for the slow-twitch case,} \quad C_m = 1.0 \quad \text{for the fast-twitch case}$$

$$I_{stim} = 1200 \quad \text{for the slow-twitch case,} \quad I_{stim} = 2000.0 \quad \text{for the fast-twitch case}$$

8.2.2 Computational model

- This example uses generated meshes

- Commandline arguments are:

number elements X

number elements Y

interpolation order (1: linear; 2: quadratic)

solver type (0: direct; 1: iterative)

PDE step size

stop time

output frequency

CellML Model URL

slow-twitch

ODE time-step

- Commands for tests are:

`./folder/src/example 24 24 1 0 0.005 3.0 1 n98.xml F 0.0001`

`./folder/src/example 24 24 1 0 0.005 3.0 1 n98.xml F 0.005`

`./folder/src/example 10 10 1 0 0.005 3.0 1 n98.xml F 0.0001`

```

mpirun -n 2 ./folder/src/example 24 24 1 0 0.005 3.0 1 n98.xml F 0.0001
mpirun -n 8 ./folder/src/example 24 24 1 0 0.005 3.0 1 n98.xml F 0.0001
./folder/src/example 2 2 1 0 0.005 3.0 1 n98.xml F 0.0001
mpirun -n 2 ./folder/src/example 2 2 1 0 0.005 3.0 1 n98.xml F 0.0001

```

- This is a dynamic problem.

8.2.3 Results

Passed tests: 12 / 12

No failed tests.

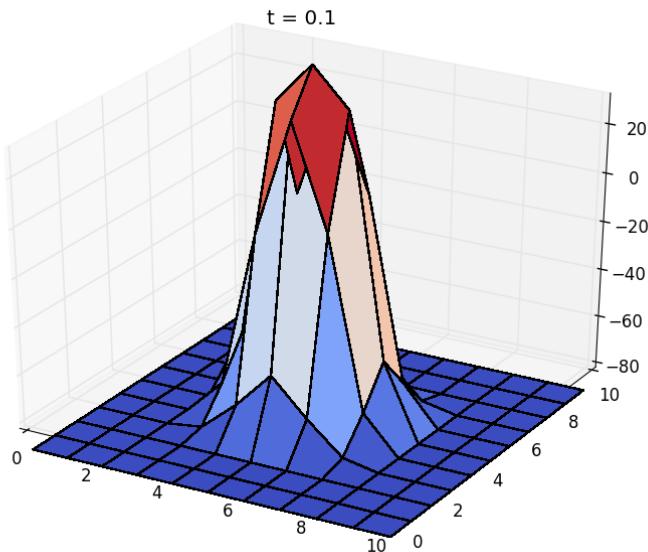


Figure 60: Result of scenario with 10×10 elements, $t = 20$, direct solver, $p = 1$ process

With the 'big' target there will be animations created. You get a better understanding of the solutions by looking at them in `iron-tests/examples/example-0402/doc/figures`.

8.2.4 Validation

We compare with reference iron files of Aug. 2017. See also the notes on `example-0401`.

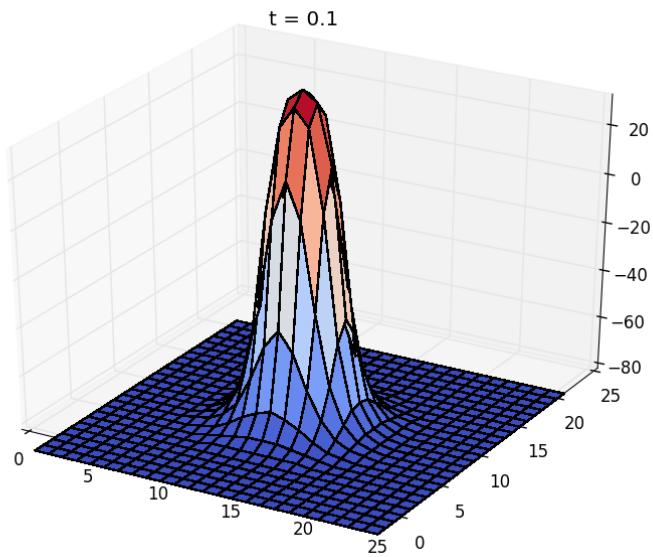


Figure 61: Result of scenario with 24×24 elements, $t = 20$, direct solver, $p = 1$ process

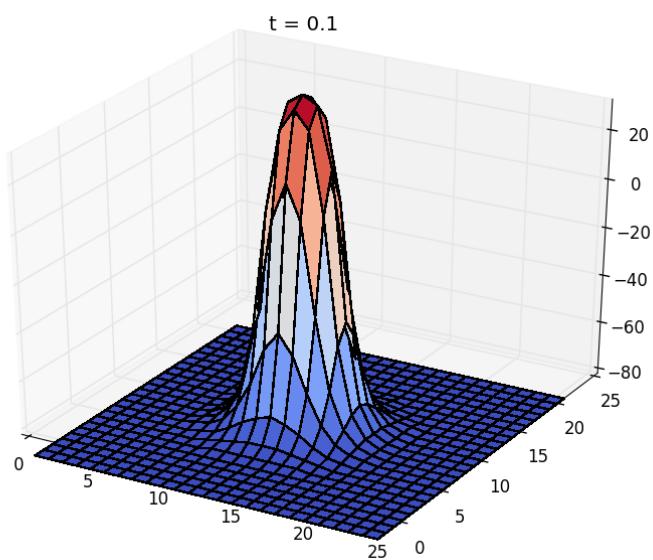


Figure 62: Result of scenario with 24×24 elements, $t = 20$, iterative solver, $p = 1$ process

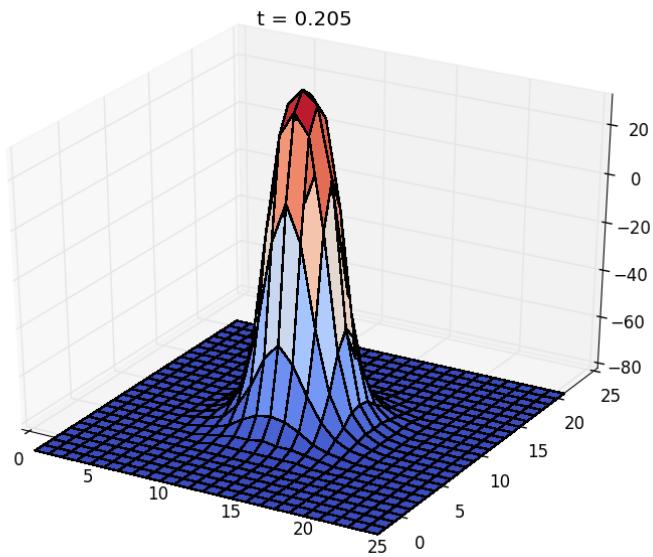


Figure 63: Result of scenario with 24×24 elements, $t = 20$, iterative solver, $p = 2$ processes

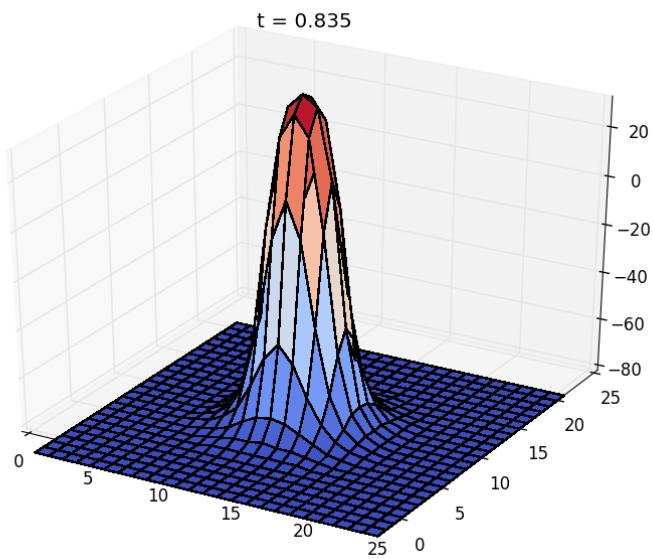


Figure 64: Result of scenario with 24×24 elements, $t = 20$, iterative solver, $p = 8$ processes

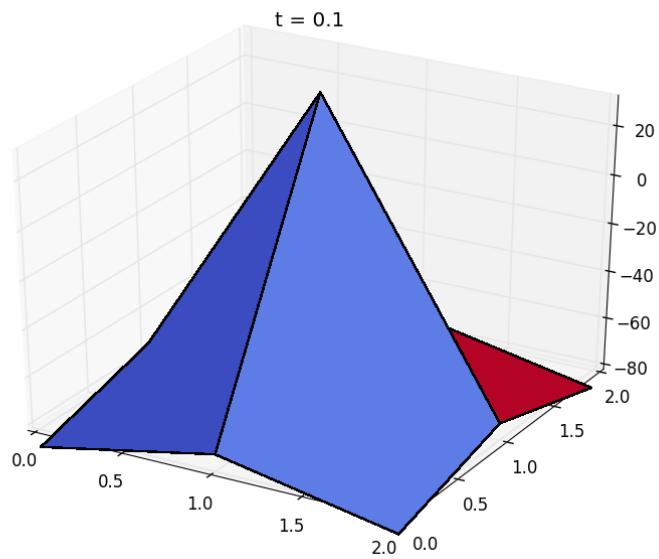


Figure 65: Result of scenario with 2×2 elements, $t = 20$, direct solver, $p = 1$ process

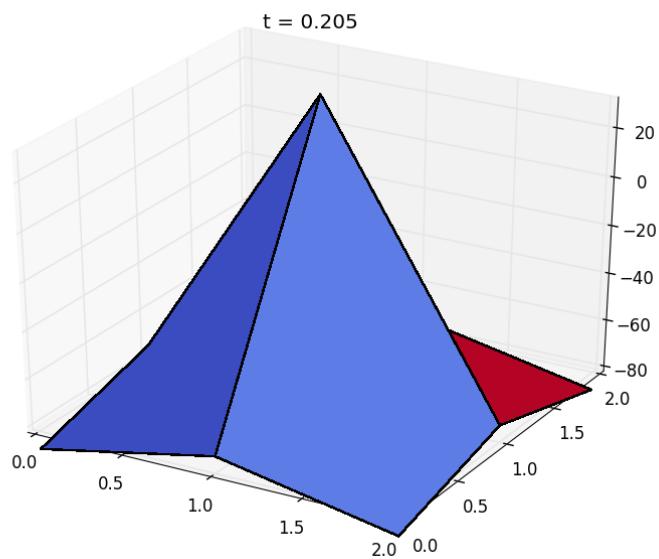


Figure 66: Result of scenario with 2×2 elements, $t = 20$, direct solver, $p = 2$ processes

8.3 Example-0404-c [PLAUSIBLE]

8.3.1 Mathematical model

We solve the Monodomain Equation

$$\sigma \Delta V_m(t) = A_m \left(C_m \frac{\partial V_m}{\partial t} + I_{ionic}(V_m) \right) \quad \Omega = [0, 1], \quad t \in [0, 10.0] \quad (127)$$

where $V_m(t)$ is given by the Hodgkin-Huxley system of ODEs [2]
with Neumann boundary conditions

$$\frac{\partial u}{\partial n} = 0 \quad x = 0, \quad (128)$$

$$\frac{\partial u}{\partial n} = 0 \quad x = 1. \quad (129)$$

and initial values

$$V_m(t = 0) = -75$$

Additionally a stimulation current I_{stim} is applied for $t_{stim} = [0, 0.5]$ at the center node of the domain (i.e. at $(x, y) = (\frac{1}{2}, \frac{1}{2},)$).

Material parameters:

$$\sigma = 3.828$$

$$A_m = 500$$

$$C_m = 0.58 \quad \text{for the slow-twitch case,} \quad C_m = 1.0 \quad \text{for the fast-twitch case}$$

$$I_{stim} = \begin{cases} 75/10 \cdot (2X) & \text{for } X \geq 10 \text{ reference elements} \\ 75 & \text{for } < 10 \text{ reference elements} \end{cases} \quad \text{for the slow-twitch case,}$$

$$I_{stim} = \begin{cases} 75/12 \cdot (2X) & \text{for } X \geq 12 \text{ reference elements} \\ 75 & \text{for } < 12 \text{ reference elements} \end{cases} \quad \text{for the fast-twitch case}$$

8.3.2 Computational model

- This example uses generated meshes

- Commandline arguments are:

number of elements

order of interpolation

solver type (0: direct; 1: iterative)

time step PDE

end time

output file stride

cellml model file

if slow-twitch (T: slow-twitch, F: fast-twitch)

time step ODE

- Commandline arguments for tests are:

64 2 0 0.01 10 5 hodgkin_huxley_1952.cellml F 0.01

64 2 0 0.005 10 10 hodgkin_huxley_1952.cellml F 0.005

```

64 2 0 0.001 10 50 hodgkin_huxley_1952.cellml F 0.001
64 2 0 0.0005 10 100 hodgkin_huxley_1952.cellml F 0.0005
64 2 0 0.00025 10 200 hodgkin_huxley_1952.cellml F 0.00025

```

- This is a dynamic problem.
- More test cases for 2nd order should be constructed.

8.3.3 Results

We run the scenario for different time step widths and examine the experimental order of convergence.

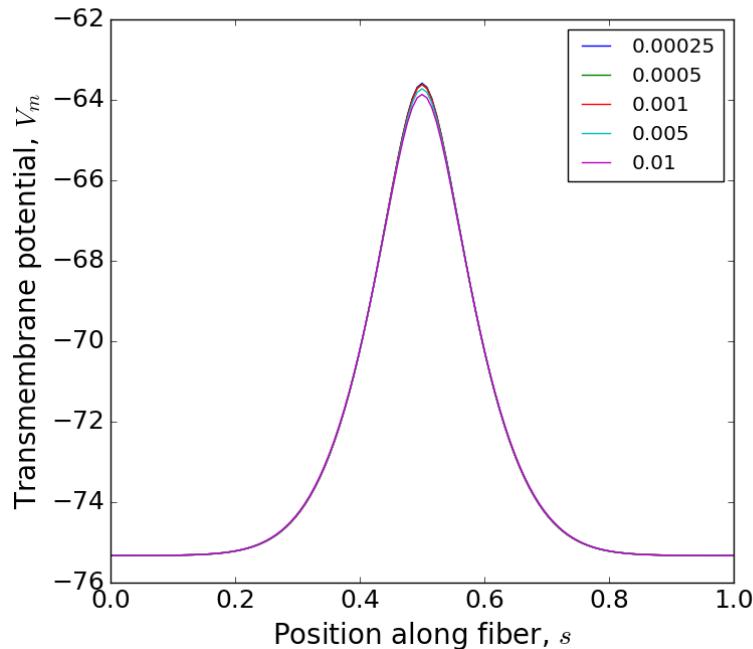


Figure 67: V_m for time $t = 1.0$, different time step widths
 $dt \in \{0.01, 0.005, 0.001, 0.0005, 0.00025\}$

8.3.4 Validation

The purpose of this test case is to see if convergence orders are as expected, no actual validation of the output takes place.

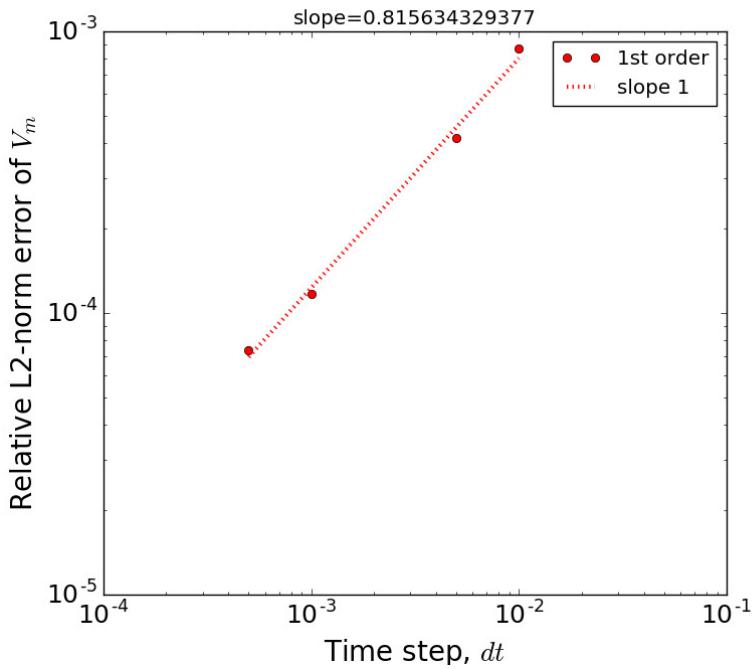


Figure 68: Error at $t = 1.0$ for different time steps widths. The slope (=experimental order of convergence) should be around 1.

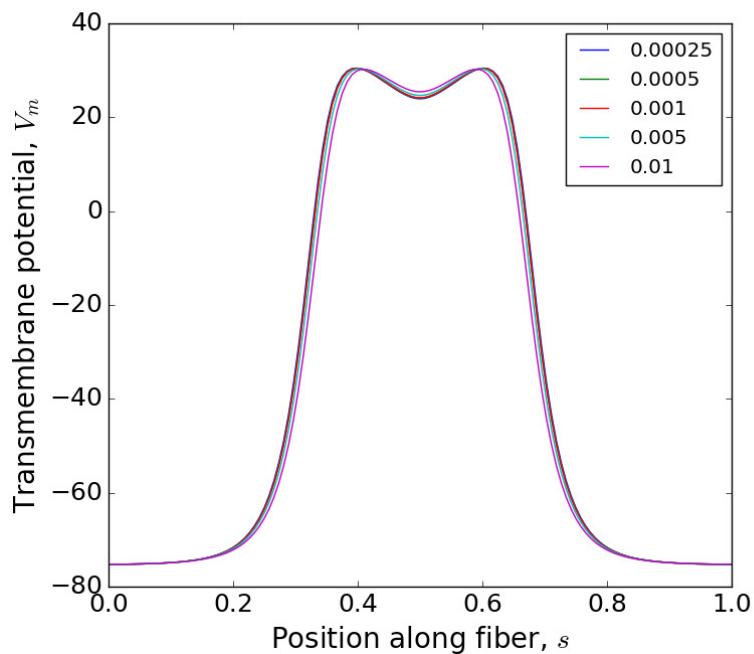


Figure 69: V_m for time $t = 3.0$, different time step widths $dt \in \{0.01, 0.005, 0.001, 0.0005, 0.00025\}$

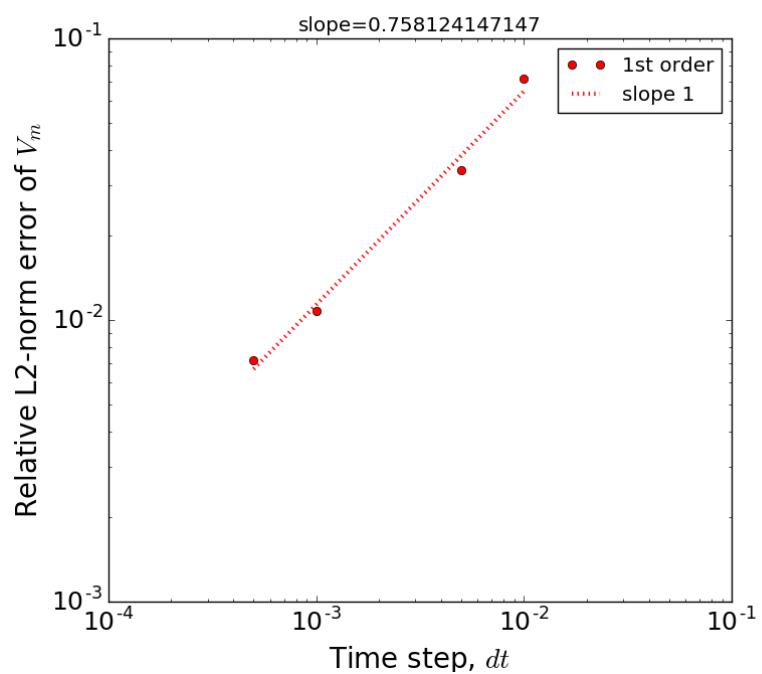


Figure 70: Error at $t = 3.0$ for different time steps widths. The slope (=experimental order of convergence) should be around 1.

9 CELLML MODEL

REFERENCES

- [1] Chris Bradley, Andy Bowery, Randall Britten, Vincent Budelmann, Oscar Camara, Richard Christie, Andrew Cookson, Alejandro F Frangi, Thiranja Babarenda Gamage, Thomas Heidlauf, et al. OpenCMISS: a multi-physics & multi-scale computational infrastructure for the vph-/physiome project. *Progress in biophysics and molecular biology*, 107(1):32–47, 2011.
- [2] Alan L Hodgkin and Andrew F Huxley. Propagation of electrical signals along giant nerve fibres. *Proceedings of the Royal Society of London. Series B, Biological Sciences*, pages 177–183, 1952.
- [3] Denis Noble, Anthony Varghese, Peter Kohl, and Penelope Noble. Improved guinea-pig ventricular cell model incorporating a diadic space, ikr and iks, and length-and tension-dependent processes. *Canadian Journal of Cardiology*, 14(1):123–134, 1998.