ISyE 6644 — Spring 2018 — Practice Test #2

Hi Again, Gentle Class!

Our Test 2 will cover material up to and including some material on multivariate normal and nonhomogeneous Poisson random variate generation (but nothing after that) — i.e., towards the end of Module 7 (or, equivalently, the first part of HW 10). You'll be responsible for all of the material from Test 1, but I'll emphasize mostly the new stuff since Test 1.

Practice Test 2 below is actually an amalgamation of Test 2 and part of Test 3 originally given in my "live" 6644 class during the Fall 2017 semester. Your Test 2 won't be as long, and I'll be more generous with time and cheat sheets. ☺

Also, all of our test questions will be multiple choice, in spite of what you see below.

Dave

1. Consider a Poisson process with rate $\lambda = 2$. Find the probability that the time between the 3rd and 5th arrivals is at least 1.

   Solution: The times between Poisson arrivals are i.i.d. Exp(2). Thus, the sum of the 2 interarrival times between the 3rd and 5th arrivals is $X \sim \text{Erlang}_2(2)$. We want

$$\mathsf{P}(X \geq 1) \;=\; \sum_{i=0}^{k-1} \frac{e^{-\lambda t}(\lambda t)^i}{i!} \;=\; \sum_{i=0}^{1} \frac{e^{-2(1)}(2(1))^i}{i!} \;=\; 3e^{-2} \;=\; 0.4060. \quad \square$$

2. Consider the following joint p.m.f.

$f(x,y)$	$X=1$	$X=2$
$Y=0$	0.2	0.5
$Y=1$	0.0	0.3

   Find $\mathsf{E}[XY]$.

Solution:

$$\begin{aligned}\mathsf{E}[XY] &= \sum_x \sum_y xyf(x,y) \\ &= (1)(0)(0.2) + (1)(1)(0.0) + (2)(0)(0.5) + (2)(1)(0.3) \\ &= 0.6. \quad \Box\end{aligned}$$

3. TRUE or FALSE? (Almost) every simulation language utilizes a future events list to process events in order by time.

   Solution: TRUE. $\Box$

4. TRUE or FALSE? In Arena, variables are global.

   Solution: TRUE. $\Box$

5. TRUE or FALSE? In Arena, you can use the `SEIZE - DELAY - RELEASE` module series instead of a single `PROCESS` module.

   Solution: TRUE. $\Box$

6. TRUE or FALSE? In Arena, you can find a `SEIZE` (i) inside a `PROCESS` module, (ii) in the Advanced Process template, and (iii) in the Primitive Blocks template.

   Solution: TRUE. $\Box$

7. In Arena, what kind of module would you use to give a new value to an attribute?

   (a) `RECORD`
   (b) `SET`
   (c) `ASSIGN`
   (d) `TALLY`

Solution: (c) `ASSIGN`. □

8. TRUE or FALSE? In Arena, certain resources can be in more than one resource set.

   Solution: TRUE. □

9. TRUE or FALSE? An Arena `DECIDE` module can route customers probabilistically or conditionally to multiple locations.

   Solution: TRUE. □

10. In Arena, where would you find the `Expression` spreadsheet?

    (a) `Basic Process` panel

    (b) `Advanced Process` panel

    (c) `Basic Transfer` panel

    (d) `Advanced Transfer` panel

    Solution: (b) `Advanced Process`. □

11. TRUE or FALSE? In a particular Arena `PROCESS` module, it is possible to initiate a `SEIZE-DELAY` pair *without* a `RELEASE`.

    Solution: TRUE, though you'd have to do a `RELEASE` in a later module. □

12. What is the name of the Arena module that can be used to split one customer into two or more clones?

    Solution: `Separate` or `Clone`. □

13. TRUE or FALSE? In Arena, you can assign different customers completely different sequences of visitation stations.

Solution: TRUE. $\square$

14. What is the variance of the random variable generated by the Arena expression `2*DISC(0.5,-1,1,1) + (1==1)`?

    Solution: Let $X = \texttt{DISC}(0.5, -1, 1, 1)$. Then $\mathsf{P}(X = -1) = \mathsf{P}(X = 1) = 0.5$, so that $\mathsf{E}[X] = 0$ and $\mathsf{E}[X^2] = 1$. Thus, $\mathsf{Var}(X) = 1$, and we have

    $$\mathsf{Var}(2 * \texttt{DISC}(0.5, -1, 1, 1) + (1 == 1)) \;=\; \mathsf{Var}(2X + 1) \;=\; 4\mathsf{Var}(X) \;=\; 4. \quad \square$$

15. YES or NO? Is it *ever* OK to use the midsquare random number generator?

    Solution: NO. $\square$

16. What does "LCG" mean?

    Solution: Linear Congruential Generator. $\square$

17. What's the period of the generator $X_i = (3X_{i-1} + 2) \bmod(5)$ if we use the seed $X_0 = 3$?

    Solution: Note that $X_0 = 3$, $X_1 = 1$, $X_2 = 0$, $X_3 = 2$, $X_4 = 3$, so the period is 4. $\square$

    (Interestingly, $X_0 = 4$ would have caused an immediate degenerate cycle.)

18. TRUE or FALSE? The generator $X_i = (32X_{i-1} + 124)\text{mod}(2048)$ has full period.

    Solution: FALSE. (You get all evens.) $\square$

19. Which uniform generator was recommended in class, at least as a "desert island" generator?

    (a) $X_i = 16807X_{i-1} \bmod(2^{31})$

(b) $X_i = 16807 X_{i-1} \bmod (2^{31}-1)$

(c) $X_i = 16807 (X_{i-1}-1) \bmod (2^{31})$

(d) $X_i = 16807 (X_{i-1}-1) \bmod (2^{31}-1)$

Solution: (b). $\square$

20. Suppose that a Tausworthe generator gave you the series of bits 1010101. If you use all 7 bits, what Unif(0,1) random number would that translate to?

Solution: Using the usual base-2 notation, we have

$$\frac{1010101_2}{2^7} = \frac{64+16+4+1}{128} = \frac{85}{128} = 0.6641. \quad \square$$

21. Suppose we observe 1000 numbers to obtain the following data.

interval	$[0.00, 0.25)$	$[0.25, 0.50)$	$[0.50, 0.75)$	$[0.75, 1.0]$
number observed	240	255	243	262

Conduct a χ^2 goodness-of-fit test to see if these numbers are approximately Unif(0,1). Use level of significance $\alpha = 0.05$. Here are some table entries that you may need: $\chi^2_{0.05,3} = 7.81$, $\chi^2_{0.05,4} = 9.49$, and $\chi^2_{0.05,5} = 11.1$. ACCEPT or REJECT?

Solution: The expected number of observations per equal-probability cell is $E_i = n/k = 1000/4 = 250$. This gives us the following augmented table.

interval i	$[0.00, 0.25)$	$[0.25, 0.50)$	$[0.50, 0.75)$	$[0.75, 1.0]$
number observed O_i	240	255	243	262
expected number E_i	250	250	250	250

Thus, the χ^2 goodness-of-fit statistic is

$$\chi^2_0 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} = \frac{100+25+49+144}{250} = 1.272.$$

Meanwhile, the appropriate quantile $\chi^2_{\alpha,k-1} = \chi^2_{0.05,3} = 7.81$. Since $\chi^2_0 < \chi^2_{\alpha,k-1}$, we ACCEPT H_0. In other words, we are willing to assume that the numbers are approximately Unif(0,1). $\square$

22. Consider the following $n = 30$ PRNs. (Read from left to right, and then down.)

0.79	0.68	0.46	0.69	0.90	0.93	0.99	0.86	0.33	0.22
0.60	0.18	0.59	0.38	0.69	0.76	0.91	0.62	0.22	0.19
0.11	0.45	0.72	0.88	0.65	0.55	0.31	0.27	0.46	0.89

Let's conduct a runs *up and down* test to test H_0 : the U_i's are independent with level $\alpha = 0.05$. ACCEPT or REJECT?

Solution: Let's associate $+$ and $-$ for up and down, respectively. Then the 30 PRNs translate to

$$-- \quad ++++ \quad --- \quad + \quad - \quad + \quad - \quad +++ \quad ---- \quad +++ \quad ---- \quad ++$$

so that we have $A = 12$ runs up and down.

Recall that

$$A \sim \text{Nor}\left(\frac{2n-1}{3}, \frac{16n-29}{90}\right) \sim \text{Nor}\,(19.67,\, 5.01)\,.$$

so that the test stat is

$$Z_0 = \frac{A - \mathsf{E}[A]}{\sqrt{\mathsf{Var}(A)}} = \frac{12 - 19.67}{\sqrt{5.01}} = -3.43.$$

Meanwhile, the appropriate quantile is $z_{\alpha/2} = 1.96$.

Since $Z_0 > z_{\alpha/2}$, we REJECT H_0. In other words, these fellas probably ain't indep. $\square$

23. Suppose that we have two PRNs, $U_1 = 0.7$ and $U_2 = 0.4$. Use these to generate a realization from an $\text{Erlang}_{k=2}(\lambda = 3)$ distribution.

Solution:

$$-\frac{1}{\lambda}\ell\text{n}\Big(\prod_{i=1}^{k} U_i\Big) = -\frac{1}{3}\ell\text{n}(0.28) = 0.424. \quad \square$$

Other answers are possible, e.g.,

$$-\frac{1}{\lambda}\ell\text{n}\Big(\prod_{i=1}^{k}(1-U_i)\Big) = -\frac{1}{3}\ell\text{n}(0.18) = 0.572. \quad \square$$

24. If X is a Nor(0,1) random variate, and $\Phi(x)$ is the Nor(0,1) c.d.f., what is the distribution of $\Phi(X)$?

Solution: By the Inverse Transform Theorem, $\Phi(X) \sim \text{Unif}(0,1)$. $\Box$

25. If U is a Unif(0,1) random variate, and $\Phi(x)$ is the Nor(0,1) c.d.f., what is the distribution of $2\Phi^{-1}(U)+3$?

Solution: By the Inverse Transform Theorem,

$$2\Phi^{-1}(U)+3 \;\sim\; 2\,\text{Nor(0,1)}+3 \;\sim\; \text{Nor(3,4)}. \quad \Box$$

26. If U and V are i.i.d. Unif(0,1), what's the distribution of $-3\ell\text{n}(U^2V^2)$?

Solution: Note that

$$\begin{aligned} -3\ell\text{n}(U^2V^2) &= -6\ell\text{n}(U) - 6\ell\text{n}(V) \\ &\sim \text{Exp}(1/6) + \text{Exp}(1/6) \\ &\sim \text{Erlang}_2(1/6). \quad \Box \end{aligned}$$

27. Suppose the random variable X has the following c.d.f.

$$F(x) = \begin{cases} \frac{1}{2}e^{3x} & \text{if } x \le 0 \\ 1 - \frac{1}{2}e^{-5x} & \text{if } x > 0 \end{cases} .$$

Give an inverse transform method for generating realizations of X.

Solution: Two cases:

For $X \le 0$ (i.e., $0 \le U \le 1/2$), set $F(X) = \frac{1}{2}e^{3X} = U$, so that $X = \frac{1}{3}\ell\text{n}(2U)$.

Similarly, for $X > 0$ (i.e., $1/2 < U \le 1$), set $F(X) = 1 - \frac{1}{2}e^{-5X} = U$, so that $X = -\frac{1}{5}\ell\text{n}(2(1-U))$.

Putting all of this together, we have

$$X = \begin{cases} \frac{1}{3}\ell\text{n}(2U) & \text{if } U \le 1/2 \\ -\frac{1}{5}\ell\text{n}(2(1-U)) & \text{if } U > 1/2 \end{cases} . \quad \Box$$

28. Consider your inverse transform solution to Problem 27. Use the Unif(0,1) random number 0.6 to generate a realization of X.

Solution: Have to use the second case above, i.e.,

$$X = -\frac{1}{5}\ell\text{n}(2(1-U)) = -0.2\ell\text{n}(0.8) = 0.0446. \quad \Box$$

29. If U_1 and U_2 are i.i.d. Unif(0,1) with $U_1 = 0.75$ and $U_2 = 0.75$, use Box-Muller to generate two i.i.d. Nor(0,1) realizations. [Ha! I made U_1 and U_2 the same value so that the problem will be easier for me to grade! But note that you should still get two different Z_1 and Z_2 values.]

Solution: We have

$$\begin{aligned} Z_1 &= \sqrt{-2\ell\text{n}(U_1)}\cos(2\pi U_2) = \sqrt{-2\ell\text{n}(0.75)}\cos(1.5\pi) = 0 \\ Z_2 &= \sqrt{-2\ell\text{n}(U_1)}\sin(2\pi U_2) = \sqrt{-2\ell\text{n}(0.75)}\sin(1.5\pi) = -0.759. \quad \Box \end{aligned}$$

30. If Z_1, Z_2, and Z_3 are i.i.d. Nor(0,1) random variables, find the value of c such that $\mathsf{P}(Z_1^2 + Z_2^2 + Z_3^2 < c) = 0.99$.

Solution:

$$0.99 = \mathsf{P}(Z_1^2 + Z_2^2 + Z_3^2 < c) = \mathsf{P}(\chi^2(1) + \chi^2(1) + \chi^2(1) < c) = \mathsf{P}(\chi^2(3) < c),$$

which implies that $c = \chi^2_{0.01,3} = 11.34$. $\Box$

31. If U_1 and U_2 are PRNs, what is the distribution of $-4(U_1 + U_2) - 2$?

Solution: By class notes,

$$-4(U_1+U_2)-2 \sim -4\text{Tria}(0,1,2)-2 \sim \text{Tria}(-8,-4,0)-2 \sim \text{Tria}(-10,-6,-2). \quad \Box$$

32. If a random variable X has the beta distribution, then its p.d.f. is of the form $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$, $0 < x < 1$, for parameters α and $\beta > 0$, and where $\Gamma(\cdot)$ is the gamma function. How might you generate such a random variate? Pick the best answer.

(a) Ask a UGA student.

(b) Convolution (add up appropriate random variates).

(c) Inversion.

(d) Acceptance-Rejection.

(e) Composition.

Solution: (a) is a sick joke; (b) isn't right because there aren't any sums involved; (c) is wrong because you can't invert the resulting c.d.f. in closed-form; (e) won't work in any natural way (except in a couple of special cases). In fact, from a similar example given in the class notes, we know that the correct answer is...

(d) acceptance-rejection. □

33. Consider a bivariate normal random variable (X, Y), for which $\mathsf{E}[X] = 3$, $\mathsf{Var}(X) = 4$, $\mathsf{E}[Y] = -2$, $\mathsf{Var}(Y) = 9$, and $\mathsf{Cov}(X, Y) = -2$. Find the Cholesky matrix associated with (X, Y), i.e., the lower-triangular matrix C such that $\Sigma = CC'$, where Σ is the variance-covariance matrix.

Solution: By class notes, we saw that for $k = 2$, we obtain

$$C = \begin{pmatrix} \sqrt{\sigma_{11}} & 0 \\ \frac{\sigma_{12}}{\sqrt{\sigma_{11}}} & \sqrt{\sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}} \end{pmatrix} = \begin{pmatrix} \sqrt{4} & 0 \\ \frac{-2}{\sqrt{4}} & \sqrt{9 - \frac{(-2)^2}{4}} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 2\sqrt{2} \end{pmatrix}. \quad \square$$

34. Suppose $Z_1 = 1$ and $Z_2 = 2$ are i.i.d. Nor(0,1) realizations. Using the notation from and your answer to Question 33, generate (X, Y).

Solution: By class notes, we have

$$\begin{aligned} X &= \mathsf{E}[X] + c_{11}Z_1 = 3 + 2(1) = 5 \\ Y &= \mathsf{E}[Y] + c_{21}Z_1 + c_{22}Z_2 = -2 - 1(1) + 2\sqrt{2}(2) = 2.657. \end{aligned}$$

35. Consider a nonhomogeneous Poisson arrival process with rate function $\lambda(t) = t/2$ for $t \geq 0$. Find the probability that there will be exactly 3 arrivals before time $t = 2$.

Solution: The distribution of the number of arrivals by time 2 is

$$N(2) \sim \text{Pois}\left(\int_0^2 \lambda(t)\,dt\right) \sim \text{Pois}\left(\int_0^2 (t/2)\,dt\right) \sim \text{Pois}(1).$$

Thus,

$$\mathsf{P}(N(2) = 3) = \mathsf{P}(\text{Pois}(1) = 3) = \frac{e^{-1}(1)^3}{3!} = 0.0613. \quad \Box$$

36. What is thinning good for? Circle the best choice.

(a) It's used to generate independent exponential interarrivals when the rate changes over time.

(b) It's a method to generate i.i.d. normal observations.

(c) It's a technique for generating multivariate normal observations.

(d) It's used to produce Brownian motion paths.

Solution: (a) Thinning is used to generate the independent exponential interarrivals arising from a nonhomogeneous Poisson process. $\Box$

37. The Zombies or Justin Bieber?

Solution: Gimme a break. www.youtube.com/watch?v=FmuswTEGF-U $\heartsuit$