Computer Simulation

Module 8: Input Data Analysis

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Weibull Example

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Lesson Overview

Last Time: Learned how to carry out a goodness-of-fit test for observations supposedly coming from an Expo distribution.

This Time: Same thing, but now with Weibull.

This takes more work than the Expo, but it's a more-general test…so it's worth it (or so says me)!

Example: Now let's make things more interesting.

$$H_0 : X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} \text{Weibull}(r, \lambda).$$

The Weibull has c.d.f. $F(x) = 1 - \exp\big[-(\lambda x)^r\big]$, for $x \geq 0$. Note that $r = 1$ yields the Exp(λ) as a special case.

Again do a χ^2 g-o-f test with *equal-probability intervals*. We need

$$F(a_i) \;=\; 1 - e^{-(\lambda a_i)^r} \;=\; i/k \quad \Rightarrow$$

$$a_i \;=\; \frac{1}{\lambda}\left[-\ell\text{n}\left(1 - \frac{i}{k}\right)\right]^{1/r}, \quad i = 1, 2, \ldots, k.$$

Now λ and r are unknown, so we'll get $s = 2$ MLEs.

After a little algebra (a couple of chain rules), the p.d.f. is

$$f(x) \;=\; \lambda r(\lambda x)^{r-1} e^{-(\lambda x)^r}, \quad x \geq 0.$$

So the likelihood function for an i.i.d. sample of size n is

$$L(r, \lambda) \;=\; \prod_{i=1}^{n} f(x_i) \;=\; \lambda^{nr} r^n \prod_{i=1}^{n} x_i^{r-1} \exp\left[-\lambda^r \sum_{i=1}^{n} x_i^r\right].$$

$$\ell\mathrm{n}(L) \;=\; n\,\ell\mathrm{n}(r) + (r-1)\,\ell\mathrm{n}\left(\prod_{i=1}^{n} x_i\right) + nr\,\ell\mathrm{n}(\lambda) - \lambda^r \sum_{i=1}^{n} x_i^r.$$

Now maximize with respect to r and λ by setting

$$\frac{\partial}{\partial r}\ell\text{n}(L) \;=\; 0 \quad \text{and} \quad \frac{\partial}{\partial \lambda}\ell\text{n}(L) \;=\; 0.$$

After tons of algebra, get

$$\lambda \;=\; \left(\sum_{i=1}^{n} x_i^r\right)^{-1/r} \qquad \text{and}$$

$$g(r) \;=\; \frac{n}{r} + \sum_{i=1}^{n} \ell\text{n}(x_i) - \frac{n\sum_i x_i^r\,\ell\text{n}(x_i)}{\sum_i x_i^r} \;=\; 0.$$

The equation for λ looks easy enough, if only we could solve for r!

Recall: How to solve for a zero.

- trial-and-error — blech.
- bisection — OK.
- Newton's method — let's try it here!

To use Newton, we need…

$$g'(r) \;=\; -\frac{n}{r^2} - \frac{n\sum_i x_i^r\,[\ell\mathrm{n}(x_i)]^2}{\sum_i x_i^r} + \frac{n\big[\sum_i x_i^r\,\ell\mathrm{n}(x_i)\big]^2}{\big[\sum_i x_i^r\big]^2}.$$

Here's a reasonable implementation of Newton.

1. Initialize $\hat{r}_0 = \bar{X}/S$, where $\bar{X}$ is the sample mean and S^2 is the sample variance.

2. Update

$$\hat{r}_j \leftarrow \hat{r}_{j-1} - \frac{g(\hat{r}_{j-1})}{g'(\hat{r}_{j-1})}.$$

3. If $|g(\hat{r}_{j-1})| < 0.001$, then stop and set the MLE $\hat{r} = \hat{r}_j$. Otherwise, let $j \leftarrow j+1$ and goto Step 2.

Newton usually converges pretty quickly — maybe after 3 or 4 iterations.

Given that we have the MLE $\hat{r}$, we immediately have the MLE for λ,

$$\widehat{\lambda} = \left(\sum_{i=1}^{n} x_i^{\hat{r}} \right)^{-1/\hat{r}}.$$

Then by invariance, we (finally) have the MLEs for the equal-probability points,

$$\hat{a}_i = \frac{1}{\widehat{\lambda}} \left[-\ell\text{n}\left(1 - \frac{i}{k}\right) \right]^{1/\hat{r}}, \quad i = 1, 2, \ldots, k.$$

Example (from BCNN): Suppose that we take $n = 50$ observations and divide them into $k = 8$ equal-prob intervals.

Moreover, suppose it turns out that $\hat{r} = 0.525$ and $\widehat{\lambda} = 0.161$.

Thus,

$$\hat{a}_i \;=\; 6.23\left[-\ell\text{n}\left(1-\frac{i}{8}\right)\right]^{1.905}, \quad i = 1, 2, \ldots, k.$$

Further suppose that we get the following O_i's.

$(\hat{a}_{i-1}, \hat{a}_i]$	O_i	$E_i = \frac{n}{k}$
(0, 0.134]	6	6.25
(0.134, 0.578]	5	6.25
$\vdots$		
(11.54, 24.97]	5	6.25
(24.97, ∞)	6	6.25
	50	50

$$\chi_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} = 1.20 \quad \text{and}$$

$$\chi^2_{\alpha,k-1-s} = \chi^2_{0.05,5} = 11.1$$

So we "accept" H_0. These observations are sort of Weibull. $\square$

Summary

This Time: We gave details on a chi-square goodness-of-fit test for the Weibull distribution. We even got to use Newton's method along the way!

Next Time: We'll look at several other g-o-f tests that sometimes work in a pinch.

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Still More Goodness-of-Fit Tests

Lesson Overview

Last Time: Learned how to carry out a goodness-of-fit test for observations supposedly coming from Weibull distribution.

This Time: We've done lots of chi-square tests… now it's time to mention other types, especially Kolmogorov-Smirnov, which is good for small-sample situations.

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Kolmogorov-Smirnov Goodness-of-Fit Test

There are plenty of g-o-f tests that you can use instead of a χ^2 test. The K-S test is one that works well in low-data situations.

We'll test

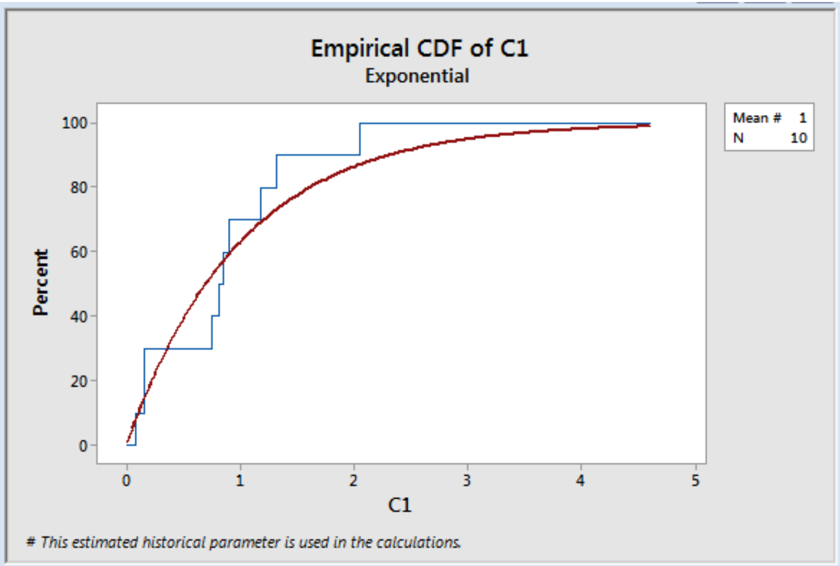
$$H_0 : X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} \text{some distribution with c.d.f. } F(x).$$

Recall the definition of the *empirical c.d.f.* (also called the *sample c.d.f.*) of the data, $X_1, X_2, \ldots, X_n$,

$$\hat{F}_n(x) \equiv \frac{\text{number of } X_i\text{'s} \leq x}{n}.$$

Note that $\hat{F}_n(x)$ is a step function with jumps of height $1/n$ (every time an observation occurs).

For example, here's the empirical c.d.f. of 10 Exp(1) observations that I generated (along with the Exp(1) c.d.f.).



The *Glivenko-Cantelli Lemma* says that $\hat{F}_n(x) \to F(x)$ for all x as $n \to \infty$. So if H_0 is true, then $\hat{F}_n(x)$ should be a good approximation to the true c.d.f., $F(x)$, for large n.

The main question: Does the empirical distribution actually support the assumption that H_0 is true?

The K-S test rejects H_0 if

$$D \equiv \max_{x \in \mathbb{R}} |F(x) - \hat{F}_n(x)| > D_{\alpha,n},$$

where α is the level of significance, and $D_{\alpha,n}$ is a K-S quantile that depends on the hypothesized c.d.f. $F(x)$.

Baby K-S Example: Let's test

$$H_0 : X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} \text{Unif}(0,1).$$

Under the Unif(0,1) assumption for $F(x)$, the K-S statistic simplifies:

$$D \equiv \max_{x \in \mathbb{R}} |F(x) - \hat{F}_n(x)| = \max_{0 \le x \le 1} |x - \hat{F}_n(x)|.$$

It's kinda easy to see that the max can only occur when x equals one of the observations, $X_1, X_2, \ldots, X_n$, i.e., at one of the jump points of $\hat{F}_n(x)$. In fact, at the ith jump point, $\hat{F}_n(x)$ increases from $\frac{i-1}{n}$ to $\frac{i}{n}$.

Before giving an easy algorithm to calculate D, let's first define the *ordered* points, $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$. For example, if $X_1 = 4$, $X_2 = 1$, and $X_3 = 6$, then $X_{(1)} = 1$, $X_{(2)} = 4$, and $X_{(3)} = 6$.

Then we compute

$$D^+ \equiv \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - X_{(i)} \right\}, \quad D^- \equiv \max_{1 \leq i \leq n} \left\{ X_{(i)} - \frac{i-1}{n} \right\},$$

and finally it turns out that $D = \max(D^+, D^-)$.

Numerical Example (from BCNN):

X_i	0.039	0.706	0.016	0.198	0.793
$X_{(i)}$	0.016	0.039	0.198	0.706	0.793
$\frac{i}{n}$	0.2	0.4	0.6	0.8	1.0
$\frac{i-1}{n}$	0	0.2	0.4	0.6	0.8
$\frac{i}{n} - X_{(i)}$	0.184	0.361	**0.402**	0.094	0.207
$X_{(i)} - \frac{i-1}{n}$	0.016	–	–	**0.106**	–

Thus, $D^+ = 0.402$, $D^- = 0.106$, and then $D = 0.402$.
If we go to a K-S table for the uniform distribution, we have $D_{\alpha,n} = D_{0.05,5} = 0.565$. So we fail to reject uniformity. $\square$

Remarks: K-S is conservative in the sense that it takes a lot of bad news to reject H_0.

Can easily apply K-S to other distributions.

Many other g-o-f tests: Anderson-Darling, Cramér-von Mises, Shapiro-Wilk, etc.

S-W is especially appropriate for testing normality. Can also use graphical techniques such as Q–Q plots to evaluate normality.

Summary

This Time: Discussed other g-o-f methods, primarily Kolmogorov-Smirnov.

Next Time: There’s always some bad apples in every family. We’ll be talking about problem children when it comes to input analysis.