ISyE 6644 — Sp/Su 2018 — Test #1 Solutions

(revised 6/17/18 to correct Question 7)

This test is 120 minutes. You're allowed one cheat sheet (both sides). I also give you permission to use your computer (if you want) for Questions 30 and 34.

In any case, this test requires a proctor. All questions are 3 points, except 34, which is 1 point — and I'll probably give you that 1 point as long as you give me any interesting answer. ☺

Good luck! I want you to make this test wish that it had never been born!!!

1. TRUE or FALSE? Simulation can be used to analyze queueing models that are too complicated to solve analytically.

   Solution: TRUE (of course!) $\Box$

2. Use bisection (or any other method) to find x such that $e^{-x} = x$.

   (a) $x = 0$
   (b) $x = e$
   (c) $x = e/2$
   (d) $x \doteq 0.567$
   (e) None of the above.

   Solution: Let's use bisection to find the zero of $g(x) = e^{-x} - x$.

x	$g(x)$	comments
0	1	
1	-0.6321	look in $[0, 1]$
0.5	0.1065	look in $[0.5, 1]$
0.75	-0.2776	look in $[0.5, 0.75]$
0.625	-0.0897	look in $[0.5, 0.625]$
0.5625	0.0073	look in $[0.5625, 0.625]$
0.59375	-0.0415	look in $[0.5625, 0.59375]$
0.578125	-0.0172	look in $[0.5625, 0.578125]$
0.5703125	-0.0050	OK, stop here.

If we keep going, we get closer and closer to $x \doteq 0.5671$; so the answer is (d). $\square$

3. Suppose that X is a continuous random variable with p.d.f. $f(x) = 2x$ for $0 < x < 1$. Find $\Pr(X < 1/2 \,|\, X > 1/4)$.

   (a) 0
   (b) 0.2
   (c) 0.5
   (d) 0.8
   (e) 1/16

   Solution: We have

$$\begin{aligned}
\Pr(X < 1/2 \,|\, X > 1/4) &= \frac{\Pr(X < 1/2 \cap X > 1/4)}{\Pr(X > 1/4)} \\
&= \frac{\Pr(1/4 < X < 1/2)}{\Pr(X > 1/4)} \\
&= \frac{\int_{1/4}^{1/2} 2x\,dx}{\int_{1/4}^{1} 2x\,dx} \\
&= 0.2,
\end{aligned}$$

   after the smoke clears. So the answer is (b). $\square$

4. Suppose I conduct a series of 4 independent experiments, each of which has a 20% chance of success. What's the probability that I'll see at least 3 successes?

   (a) 0.027
   (b) 0.181
   (c) 0.819
   (d) 0.973
   (e) 1

Solution: The number of successes $X \sim \text{Bin}(4, 0.2)$. Thus,

$$\mathsf{Pr}(X \geq 3) \;=\; \sum_{x=3}^{4} \binom{4}{x} (0.2)^x (0.8)^{4-x} \;=\; 0.0272.$$

So the answer is (a). $\square$

5. If $X \sim \text{Bern}(0.5)$, find $\mathsf{E}[\ell\text{n}(X+1)]$.
   (a) 1
   (b) $e/2$
   (c) 0.347
   (d) 1.38
   (e) None of the above.

Solution: By the Unconscious Statistician and the fact that $X \sim \text{Bern}(0.5)$, we have

$$\mathsf{E}[\ell\text{n}(X+1)] \;=\; \sum_{x=0}^{1} \ell\text{n}(x+1)\,\Pr(X=x) \;=\; \frac{1}{2}\ell\text{n}(1) + \frac{1}{2}\ell\text{n}(2) \;=\; 0.347,$$

so the answer is (c). $\square$

6. If X has a mean of -2 and a variance of 3, find $\mathsf{Var}(-2X+1)$.
   (a) -5
   (b) -6
   (c) -12
   (d) 12
   (e) 13

Solution: $\mathsf{Var}(-2X+1) = 4\mathsf{Var}(X) = 12$, so the answer is (d). $\square$

7. Toss a 6-sided die repeatedly. What is the *variance* of the number of tosses until you observe a 3?

   (a) 6.
   (b) 36.
   (c) 1/6.
   (d) 5/6.
   (e) 30.

   Solution: Let $X \sim \text{Geom}(p = 1/6)$ denote the number of tosses. Thus,

   $$\mathsf{Var}(X) \;=\; q/p^2 \;=\; (1-p)/p^2 \;=\; 30,$$

   so that the answer is (e). $\square$

8. Suppose that X and Y are identically distributed with a mean of -2, a variance of 3, and $\mathsf{Cov}(X, Y) = 1$. Find $\mathsf{Corr}(X, Y)$.

   (a) 0
   (b) 1/9
   (c) 1/3
   (d) 1
   (e) 3

   Solution: $\mathsf{Corr} = \mathsf{Cov}/\sqrt{\mathsf{Var}(X)\mathsf{Var}(Y)} = 1/3$, so (c) is the answer. $\square$

9. Again suppose that X and Y are identically distributed with a mean of -2, a variance of 3, and $\mathsf{Cov}(X, Y) = 1$. Find $\mathsf{Var}(X - Y)$.

   (a) 3
   (b) 4
   (c) 5
   (d) 6

(e) 8

Solution: $\mathsf{Var}(X) + \mathsf{Var}(Y) - 2\mathsf{Cov} = 4$, so the answer is (b). $\square$

10. Consider a Poisson process with rate $\lambda = 1$. What is the probability that the time between the 4th and 5th arrivals is less than 2?

(a) $1/e$
(b) $1 - (1/e)$
(c) $1/e^2$
(d) $1 - (1/e^2)$
(e) 0.95

Solution: All interarrivals are i.i.d. $\text{Exp}(\lambda)$. In particular, let $X \sim \text{Exp}(\lambda = 1)$ denote the time between the 4th and 5th arrivals. Then $\Pr(X < 2) = 1 - e^{-\lambda x} = 1 - e^{-2} = 0.865$. Thus, the answer is (d). $\square$

11. If X is Nor(5,4), find $\mathsf{Pr}(X > 3)$.

(a) 0.05
(b) 0.159
(c) 0.5
(d) 0.841
(e) 0.95

Solution: $\Pr(X > 3) = \Pr(Z > \frac{3-5}{\sqrt{4}}) = \Pr(Z > -1) = 0.8413$. So the answer is (d). $\square$

12. If X and Y are i.i.d. standard normal random variables, find $\mathsf{Pr}(X - Y > 1)$.

(a) 0.159
(b) 0.24

(c) 0.5

(d) 0.76

(e) 0.841

Solution: Note that $X - Y \sim \text{Nor}(0, 2)$. Then

$$\Pr(X - Y > 1) \;=\; \Pr\left(Z > \frac{1-0}{\sqrt{2}}\right) \;\approx\; 0.24.$$

So (b) is the answer. $\square$

13. Suppose X and Y are i.i.d. $\text{Exp}(\lambda = 2)$. Find $\mathsf{Pr}(X + Y \leq 1)$.

(a) $1/e^2$

(b) $1 - (1/e^2)$

(c) 0.5

(d) 0.406

(e) 0.594

Solution: Note that $S = X + Y \sim \text{Erlang}_{k=2}(\lambda = 2)$. Thus,

$$\Pr(S \leq 1) \;=\; 1 - \sum_{i=0}^{k-1} \frac{e^{-\lambda s}(\lambda s)^i}{i!} \;=\; 1 - \sum_{i=0}^{1} \frac{e^{-2}2^i}{i!} \;=\; 1 - 3e^{-2} \;=\; 0.594.$$

So the answer is (e). $\square$

14. Suppose $X_1, \ldots, X_n$ are i.i.d. from a $\text{Pois}(\lambda = 3)$ distribution. What is the *approximate* distribution of the sample mean $\bar{X}$ for $n = 300$?

(a) Pois(3)

(b) Pois(0.01)

(c) $\text{Nor}(3, 3)$

(d) $\text{Nor}(3, 0.01)$

(e) $\text{Nor}(0.01, 0.01)$

Solution: The Central Limit Theorem implies that

$$\bar{X} \approx \text{Nor}\Big(\mu, \frac{\sigma^2}{n}\Big) \sim \text{Nor}\Big(3, \frac{3}{n}\Big).$$

So the answer is (d). $\square$

15. Suppose U and V are i.i.d. Unif(0,1) random variables. What does $\lceil 6U \rceil + \lceil 6V \rceil$ do? (Recall that $\lceil x \rceil$ is the "ceiling" function.)

    (a) This gives a continuous Unif(0,12) random variate.

    (b) This gives a continuous triangular random variate.

    (c) This gives a normal random variate.

    (d) This is a simulated 6-sided die toss.

    (e) This simulates the sum of two 6-sided dice tosses.

    Solution: (e). $\square$

16. If U and V are i.i.d. Unif(0,1) random variables, what is the distribution of $-\frac{1}{3}\ell\text{n}(U) - \frac{1}{3}\ell\text{n}(1-V)$?

    (a) $\text{Exp}(\lambda = 1/3)$

    (b) $\text{Exp}(\lambda = 2/3)$

    (c) $\text{Exp}(\lambda = 3)$

    (d) $\text{Erlang}_2(\lambda = 1/3)$

    (e) $\text{Erlang}_2(\lambda = 3)$

    Solution: By symmetry of the Unif(0,1), we note that U and $1-V$ are i.i.d. Unif(0,1). Therefore,

$$\begin{aligned} -\frac{1}{3}\ell\text{n}(U) - \frac{1}{3}\ell\text{n}(1-V) &\sim -\frac{1}{3}\ell\text{n}(\text{Unif(0,1)}) - \frac{1}{3}\ell\text{n}(\text{Unif(0,1)}) \\ &\sim \text{Exp}(3) + \text{Exp}(3) \\ &\sim \text{Erlang}_2(3). \end{aligned}$$

So the answer is (e). $\square$

17. Suppose X has the Weibull distribution with c.d.f. $F(x)$, which is too nasty to show here. You'll recall from class that the Inverse Transform Theorem states that $F(X) \sim$ Unif(0,1). Love ♡ those great memories! Now suppose that U is a Unif(0,1) number. What is the distribution of the inverse $F^{-1}(U)$?

    (a) Exponential
    (b) Normal
    (c) Unif(0,1)
    (d) Weibull
    (e) None of the above.

Solution: Since $F(X) \sim U \sim \text{Unif}(0,1)$, we have

$$F^{-1}(U) \;\sim\; F^{-1}(F(X)) \;=\; X \;\sim\; \text{Weibull}.$$

(This is why Inverse Transform is so nice!) Thus, the answer is (d). $\square$

18. If X is a continuous random variable with p.d.f. $f(x)$ and c.d.f. $F(x)$, find $\mathsf{E}[e^{F(X)}]$. (Hint: Don't panic on this problem. One approach might be to use LOTUS.)

    (a) e
    (b) $e - 1$
    (c) 1/2
    (d) 0.876
    (e) 1.876

Solution: The answer turns out to be (b). Since it's Springtime in Atlanta, I'll give you *two* methods to prove this!

Method (i): By LOTUS and the Chain Rule for integration,

$$\mathsf{E}[e^{F(X)}] \;=\; \int_{-\infty}^{\infty} e^{F(x)} f(x)\,dx \;=\; e^{F(x)}\Big|_{-\infty}^{\infty} \;=\; e^{F(\infty)} - e^{F(-\infty)} \;=\; e - 1. \quad \square$$

Method (ii): By Inverse Transform, $F(X) \sim$ Unif(0,1). Thus, if U denotes a Unif(0,1) random variable with p.d.f. $f_U(u)$, then LOTUS gives

$$\mathsf{E}[e^{F(X)}] \;=\; \mathsf{E}[e^U] \;=\; \int_{-\infty}^{\infty} e^u f_U(u)\,du \;=\; \int_0^1 e^u\,du \;=\; e^1 - e^0 \;=\; e-1. \quad \Box$$

19. Consider the ridiculous pseudo-random number generator $X_{i+1} = (5X_i + 1)\text{mod}(8)$. If $X_0 = 3$, calculate X_2.

    (a) 0

    (b) 1

    (c) 3

    (d) 8

    (e) 16

Solution: We have

$$X_1 \;=\; (5X_0 + 1)\text{mod}(8) \;=\; 16\,\text{mod}(8) \;=\; 0,$$

and then

$$X_2 \;=\; (5X_1 + 1)\text{mod}(8) \;=\; 1\,\text{mod}(8) \;=\; 1.$$

So the answer is (b). $\Box$

20. What does the following algorithm do?

    `Initialize` X_0 `(integer) and` $i \leftarrow 1$
    `Repeat`
    `Set` $X_i \leftarrow 16807X_{i-1}\,\texttt{mod}(2^{31}-1)$
    `Set` $U_i \leftarrow X_i/(2^{31}-1)$
    `Set` $i \leftarrow i+1$

    (a) $X_1, X_2, \ldots$ is a sequence of normal random variables.

    (b) $U_1, U_2, \ldots$ is a sequence of PRNs.

    (c) $X_1, X_2, \ldots$ will appear to be i.i.d. Unif(0,1).

    (d) $U_1, U_2, \ldots$ will appear to be i.i.d. Unif(0,1).

(e) Both (b) and (d).

Solution: The answer is (e) (though you get partial credit if you wrote only (b) or only (d)). Note that (c) isn't correct because the X_i's are *integers* — not Unif(0,1)'s. □

21. What does FEL stand for?
    (a) I go to UGA
    (b) Future Events List
    (c) Future Events Lineup
    (d) Free-form Events List
    (e) Free-form Events Lineup

    Solution: (b). □

22. Consider the arrival of a customer in a queueing system simulation. What can (possibly) happen to the FEL at this point?
    (a) That arrival is deleted from the FEL.
    (b) A subsequent arrival is scheduled.
    (c) Some future arrivals are deleted.
    (d) Some future arrivals are re-ordered.
    (e) All of the above.

    Solution: (e). An arrival can cause all sorts of chaos. □

23. TRUE or FALSE? Arena primarily uses the event-scheduling modeling approach.

    Solution: FALSE. Arena takes the process-interaction point of view. □

24. TRUE or FALSE? In an Arena `PROCESS` module, it is possible to do a `SEIZE-DELAY` *without* an accompanying `RELEASE`.

    Solution: TRUE. □

25. TRUE or FALSE? An Arena `DECIDE` module can be used to probabilistically or conditionally to route entities to more than 2 destinations.

    Solution: TRUE. □

26. Calculate the integral $I = \frac{1}{\sqrt{2\pi}} \int_0^2 e^{-x^2/2}\,dx$ (any way you can).

    (a) $\pi/2$
    (b) $2/\pi$
    (c) 0.477
    (d) 0.523
    (e) π

    Solution: This is the integral of the Nor(0,1) p.d.f. Thus, $I = \Phi(2) - \Phi(0) = 0.4773$. So the answer is (c). □

27. Again consider the integral I from Question 26. Now use the following four Unif$(0,1)$ random numbers to compute a Monte Carlo estimate of I:

    $$0.78 \quad 0.15 \quad 0.33 \quad 0.84$$

    In order to make grading easier (since I'm incredibly lazy), I'd like you to use the usual approximation from class,

    $$\hat{I}_n = \frac{b-a}{n} \sum_{i=1}^{n} g(a + (b-a)U_i),$$

    where I leave it to you to recall (or figure out) the relevant notation.

    (a) 1.583

(b) 0.632

(c) 0.459

(d) 0.563

(e) None of the above.

Solution: We have

$$\begin{aligned}
\hat{I}_n &= \frac{b-a}{n}\sum_{i=1}^{n} g(a+(b-a)U_i) \\
&= \frac{2-0}{4}\sum_{i=1}^{4} g(2U_i) \\
&= \frac{1}{2}\sum_{i=1}^{4}\frac{1}{\sqrt{2\pi}}e^{-4U_i^2/2} \\
&= 0.459.
\end{aligned}$$

So the answer is (c). $\square$

28. Now re-do Question 28, except this time use the following "antithetic" Unif(0,1)'s (i.e., $1-U$):

0.22 0.85 0.67 0.16

What's the "antithetic" estimate for I?

(a) 1.583

(b) 0.632

(c) 0.590

(d) 0.459

(e) 0.499

Remark / Hint: It *should* turn out that the difference between your "exact" answer from Question 26 and the *average* of your Monte Carlo answers from Questions 27 and 28 will be small. The reason is that the antithetic run "balances out" the original run, so that the average of the two runs moves closer to the exact answer.

Solution: Using the U_i's from Question 27 and the same manipulations, except with $1 - U_i$, we have

$$\begin{aligned} \hat{I}'_n &= \frac{b-a}{n}\sum_{i=1}^{n} g(a+(b-a)(1-U_i)) \\ &= \frac{1}{2}\sum_{i=1}^{4}\frac{1}{\sqrt{2\pi}}e^{-4(1-U_i)^2/2} \\ &= 0.499. \end{aligned}$$

So the answer is (e). $\Box$

Indeed, as per the Hint, we see that the average of the previous two MC answers minus the exact answer is $0.479 - 0.477 = 0.002$, a remarkably small difference! ☺

29. Suppose I inscribe a circle of radius 1/2 in a unit square. Now I randomly toss 100 darts in the square and 77 happen to land in the circle. Use this sample in conjunction with the example we did in class to give me an estimate of π.

    (a) π
    (b) $\pi/2$
    (c) 3.04
    (d) 3.08
    (e) 3.18

    Solution: Let $\hat{p}_n = 0.77$ be the proportion of darts that hit the circle. Then we know that the estimate $\hat{\pi}_n = 4\hat{p}_n = 3.08$. The answer is therefore (d). $\Box$

30. If X and Y are i.i.d. Unif(0,1), it turns out that the p.d.f. of the nasty joint random variable $W \equiv X/(X - Y)$ is interesting looking.

    YES or NO? Is it possible to ever have $0 < W < 1$?

Hint: Maybe use Monte Carlo to figure this out. Generate a bunch of X's and Y's (and therefore W's) to see what happens.

For this question, I *give you permission to use your computer* — you can use Excel, Matlab, whatever. Have fun!

Solution: The answer is NO. $\square$

It is possible to prove this analytically, but it takes a few pages of algebra. If you were to do so, you would see that the p.d.f. tails off at $\pm\infty$, and has asymptotes at 0 and 1; but the p.d.f. is zero on [0,1].

An easier way is to use Monte Carlo: Generate i.i.d. Unif(0,1)'s X and Y. calculate W. You'll see that it lies outside of $[0, 1]$. Do this many times — W will never be in that interval.

31. Suppose that the probability that the Georgia Tech basketball team will win its first game of the season is 0.5. Also suppose that if the team wins game i ($i = 1, 2, \ldots$), then the team becomes very confident and will win game $i+1$ with probability 0.8. However, if the team loses game i, it becomes discouraged and will win game $i+1$ with probability of only 0.5. Name the probability distribution corresponding to the number of games the team will have to play before they get their first victory.

    (a) Bern(0.5)

    (b) Binomial(n,0.5)

    (c) Geom(0.5)

    (d) Exp(0.5)

    (e) Exp(2)

    Solution: (c). $\square$

32. Consider the basketball set-up in Question 31. We will conduct Monte Carlo sampling to see how many games GT wins. To do so, suppose that I generously give you the following 10 Unif$(0, 1)$ random numbers; call them $U_1, U_2, \ldots, U_{10}$:

0.834 0.168 0.958 0.474 0.374 0.656 0.773 0.203 0.142 0.139

Our simulation will declare that GT wins game i if $U_i < p_i$, where p_i is the conditional probability that GT wins game i (as discussed in Question 31). Using the above random numbers, how many games will GT have to play until our beloved Jackets capture our 3rd victory?

(a) 3

(b) 4

(c) 5

(d) 6

(e) 7

Solution: $p_1 = 0.5$, so $U_1 = 0.834$ corresponds to a loss.
Then $p_2 = 0.5$, so $U_2 = 0.168$ corresponds to a win.
Then $p_3 = 0.8$, so $U_3 = 0.958$ corresponds to a loss.
Then $p_4 = 0.5$, so $U_4 = 0.474$ corresponds to a win.
Then $p_5 = 0.8$, so $U_5 = 0.374$ corresponds to a win.

Thus, it took 5 games, so the answer is (c). $\square$

33. Suppose that two types of customers arrive at a single-server queue: Type-A's from UGA and Type-B's from Georgia Tech. It goes without saying that the wonderful Type-B customers have priority over the loser Type-A customers (though nobody gets pre-empted if they're already being served). Otherwise, service is FIFO within each type class. Assume the system starts out empty and idle.

Customer	Type	Interarrival time	Service time
1	A	3	13
2	A	6	8
3	B	5	4
4	B	3	6
5	B	11	2
6	A	6	8

When does the last customer leave the system?

(a) 33

(b) 36

(c) 41

(d) 44

(e) 51

Solution: Let's construct the following table, where we give priority to Type-B's.

Cust.	Type	Arrival time	Start Serv.	Service time	Depart	Time in Sys.
1	A	3	3	13	16	13
2	A	9	26	8	34	25
3	B	14	16	4	20	6
4	B	17	20	6	26	9
5	B	28	34	2	36	8
6	A	34	36	8	44	10

Thus, the last customer leaves at time 44, choice (d). □

34. Musical Treasure Hunt! Plagiarism is very, very bad, but sometimes it's the best form of flattery. Here's a case in point....

You may already know that my favorite music group is The Zombies, and one of their pretty singles from 1965 is "I Must Move",

www.youtube.com/watch?v=5-AzBkK9Z1A

What other 1960s group has a different song that sounds plagiaristically close to "I Must Move"?

(a) The Beatles

(b) The Pretty Things

(c) Phil and the Frantics

(d) The Poets

(e) Nirvana (the 1960's group, not the more-recent Seattle group)

For this question, I *give you permission to use the internet.* It's very important to me to keep my classes culturally attuned!

Solution: (c) www.youtube.com/watch?v=ZgDtyHdcwdc $\square$