Computer Simulation

Module 3: Hand and Spreadsheet Simulations

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Stepping Through a Differential Equation

Module Overview

Last Module: We reviewed Calculus, Probability, and Statistics – Whew!

This Module: This module will go through a bunch of simulation examples that you can do by hand.

Idea: Give a flavor of what simulation can do from some teensy little examples.

Module Overview

1. Solving a differential equation
2. Monte Carlo integration
3. Making some pi
4. Single-server queue
5. (s,S) inventory system
6. Simulating random variables
7. Spreadsheet simulation

This Lesson: We'll solve a differential equation "by hand".

Stepping Thru a Diff Eq Numerically

Recall: If $f(x)$ is continuous, then it has the *derivative*

$$\frac{d}{dx} f(x) \equiv f'(x) \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists and is well-defined for any given x. Think of the derivative as the slope of the function.

Then for small h,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

and

$$f(x+h) \approx f(x) + hf'(x).$$

Example: Suppose you have a differential equation of a population growth model, $f'(x) = 2f(x)$ with $f(0) = 10$. Let's "solve" this using a fixed-increment time approach with $h = 0.01$. (This is known as *Euler's method*.) By (1), we have

$$f(x+h) \;\approx\; f(x) + hf'(x) \;=\; f(x) + 2hf(x) \;=\; (1+2h)f(x).$$

Similarly,

$$f(x+2h) \;=\; f((x+h)+h) \;\approx\; (1+2h)f(x+h) \;\approx\; (1+2h)^2 f(x).$$

$$f(x+ih) \;\approx\; (1+2h)^i f(x) \quad i = 0, 1, 2, \ldots,$$

though the approximation may deteriorate as i gets large.

Plugging in $f(0) = 10$ and $h = 0.01$, we have

$$f(0.01i) \;\approx\; 10(1.02)^i, \quad i = 0, 1, 2, \ldots. \tag{2}$$

Now, I happen to know that the true solution to the differential equation is $f(x) = 10e^{2x}$. So the approximation (2) makes sense since for small y,

$$e^y \;=\; \sum_{\ell=0}^{\infty} \frac{y^\ell}{\ell!} \;\approx\; 1 + y \;\approx\; (1+y)^i \text{ for small } i.$$

In any case, let's see how well the approximation does....

$x = ih = 0.01i$	0	0.01	0.02	0.03	0.04	$\cdots$	0.10
approx $f(x) \approx 10(1.02)^i$	10	10.20	10.40	10.61	10.82	$\cdots$	12.19
true $f(x) = 10e^{2x}$	10	10.20	10.41	10.62	10.83	$\cdots$	12.21

Not bad at all (at least for small i)! $\square$

Summary

This Time: New module, new goal: Hand simulation! This lesson looked at an easy example with no randomness at all: Solving a differential equation.

Next Time: How to do an integral without using any calculus! You will soon be popular at parties!

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Monte Carlo Integration

Lesson Overview

Last Time: We “solved” a differential equation by hand. (Didn’t use any random numbers.)

This Time: Let’s go the other way and do integration using nothing but random numbers!

This material is used in several disciplines ranging from Physics all the way to Finance.

Integration

Definition The function $F(x)$ having derivative $f(x)$ is called the *antiderivative*. The antiderivative is denoted $F(x) = \int f(x)\,dx$; and this is also called the *indefinite integral* of $f(x)$.

Fundamental Theorem of Calculus: If $f(x)$ is continuous, then the area under the curve for $x \in [a, b]$ is denoted and given by the *definite integral* [3]

$$\int_a^b f(x)\,dx \equiv F(x)\Big|_a^b \equiv F(b) - F(a).$$

[3] "I'm *really* an integral!"

Monte Carlo Integration

Let's integrate

$$I = \int_a^b g(x)\, dx = (b-a)\int_0^1 g(a+(b-a)u)\, du,$$

where we get the last equality by substituting $u = (x-a)/(b-a)$.

Of course, we can often do this by analytical methods that we learned back in calculus class, or by numerical methods like the trapezoid rule or something like Gauss-Laguerre integration. But if these methods aren't possible, you can always use MC integration. . . .

Suppose $U_1, U_2, \ldots$ are iid Unif(0,1), and define

$$I_i \equiv (b-a)g(a+(b-a)U_i) \quad \text{for } i = 1, 2, \ldots, n.$$

We can use the sample average as an estimator for I:

$$\bar{I}_n \equiv \frac{1}{n}\sum_{i=1}^{n} I_i = \frac{b-a}{n}\sum_{i=1}^{n} g(a+(b-a)U_i).$$

Why is this okey dokey? Let's appeal to our old friend, the Law of Large Numbers: If an estimator is asymptotically unbiased and its variance goes to zero, then things are good.

First, by the Law of the Unconscious Statistician, notice that

$$\begin{aligned}
\mathrm{E}[\bar{I}_n] &= (b-a)\mathrm{E}[g(a+(b-a)U_i)] \\
&= (b-a)\int_{\mathbb{R}} g(a+(b-a)u)f(u)\,du \\
&\quad \text{(where } f(u) \text{ is the Unif(0,1) pdf)} \\
&= (b-a)\int_0^1 g(a+(b-a)u)\,du \;=\; I.
\end{aligned}$$

So $\bar{I}_n$ is unbiased for I.

Since it can also be shown that $\mathrm{Var}(\bar{I}_n) = O(1/n)$, the LLN implies $\bar{I}_n \to I$ as $n \to \infty$.

Approximate Confidence Interval for I:

By the CLT, we have

$$\bar{I}_n \;\approx\; \mathrm{Nor}\big(\mathrm{E}[\bar{I}_n], \mathrm{Var}(\bar{I}_n)\big) \;\sim\; \mathrm{Nor}\big(I, \mathrm{Var}(I_i)/n\big).$$

This suggests that a reasonable $100(1-\alpha)\%$ confidence interval for I is

$$I \;\in\; \bar{I}_n \pm z_{\alpha/2}\sqrt{S_I^2/n}, \tag{3}$$

where $z_{\alpha/2}$ is the usual standard normal quantile, and $S_I^2 \equiv \frac{1}{n-1}\sum_{i=1}^{n}(I_i - \bar{I}_n)^2$ is the sample variance of the I_i's.

Example: Suppose $I = \int_0^1 \sin(\pi x)\, dx$ (and pretend we don't know the actual answer, $2/\pi \doteq 0.6366$).

Let's take $n = 4$ Unif(0,1) observations:

$$U_1 \;=\; 0.79 \quad U_2 \;=\; 0.11 \quad U_3 \;=\; 0.68 \quad U_4 \;=\; 0.31$$

Since

$$I_i \;=\; (b-a)g(a+(b-a)U_i) \;=\; g(U_i) \;=\; \sin(\pi U_i),$$

we obtain

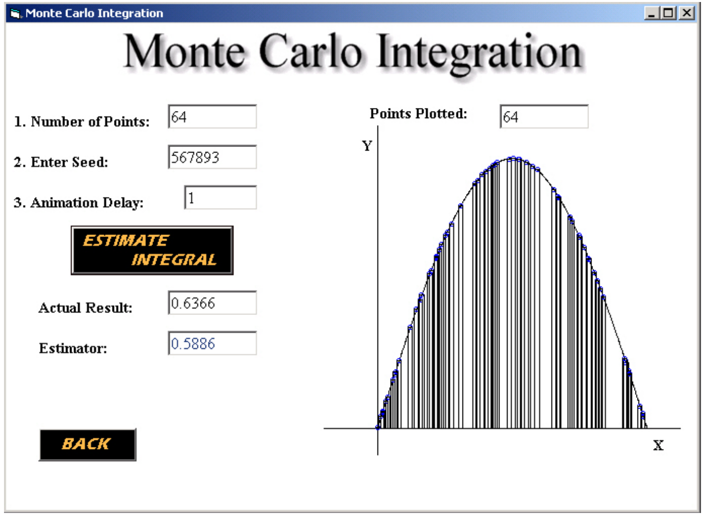
$$\bar{I}_n \;=\; \frac{1}{4}\sum_{i=1}^{4} I_i \;=\; \frac{1}{4}\sum_{i=1}^{4}\sin(\pi U_i) \;=\; 0.656,$$

which is close to $2/\pi$! (Actually, we got lucky.)

Moreover, the approximate 95% confidence interval for I from (3) is

$$I \in 0.656 \pm 1.96\sqrt{0.0557/4} = [0.596, 0.716].$$

And we'll usually do better as n gets big — though sometimes the convergence is choppy due to good or bad luck. $\square$



Summary

Now we can integrate “anything” without having to do that annoying calculus stuff! ☺

Next Time: We’ll celebrate our success by making some delicious π, using random numbers as our main ingredient.

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Making Some π

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Lesson Overview

Last Time: Learned about Monte Carlo integration and did an example to show how easy it is to use.

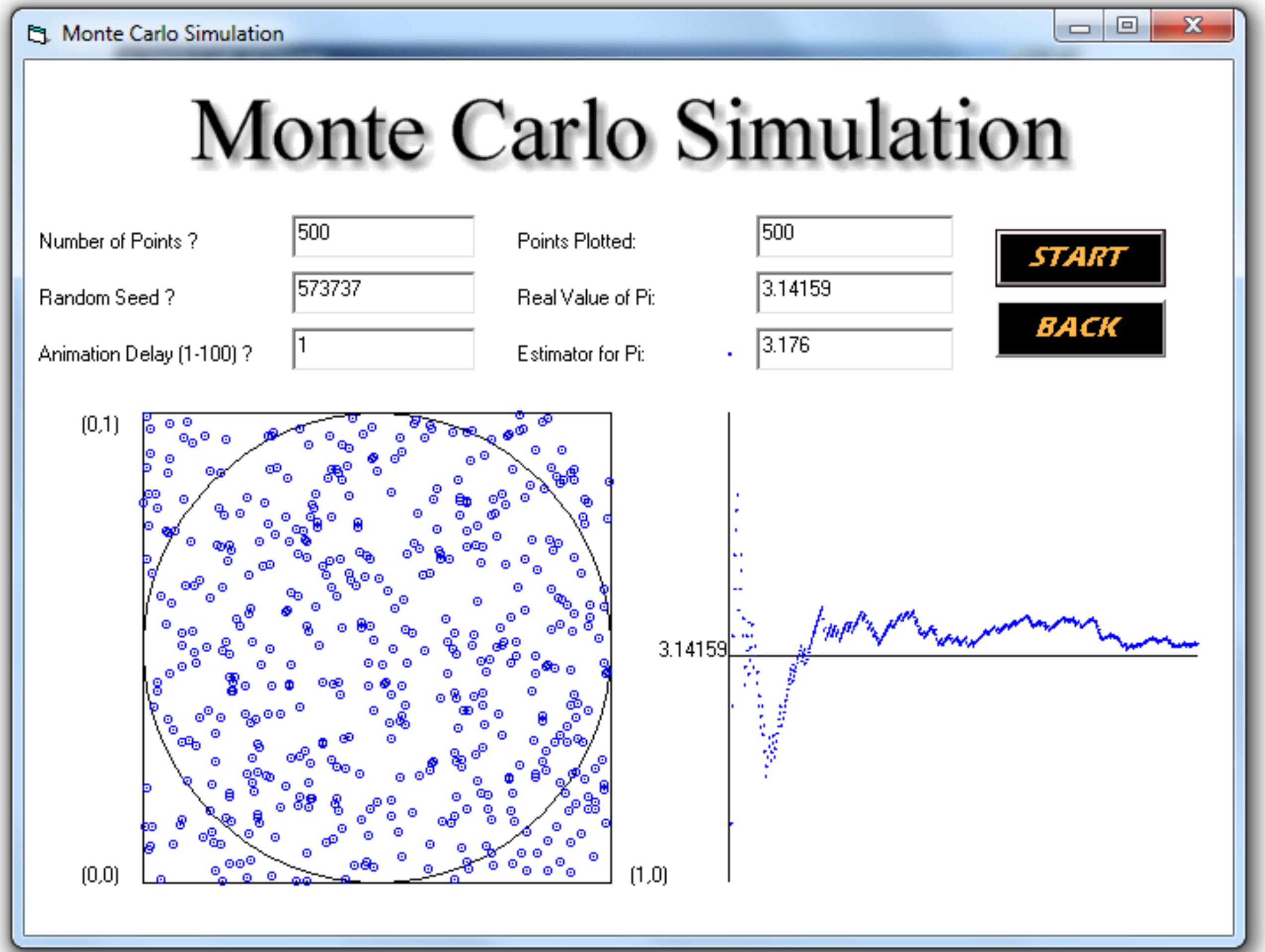
This Time: We'll do a simple MC example to show how to estimate π, which will certainly be food for thought!

Remember Buffon's needle??

Consider a unit square (with area one). Inscribe in the square a circle with radius 1/2 (with area $\pi/4$). Suppose we toss darts randomly at the square. The probability that a particular dart will land in the inscribed circle is obviously $\pi/4$ (the ratio of the two areas). We can use this fact to estimate π.

Toss n such darts at the square and calculate the proportion $\hat{p}_n$ that land in the circle. Then an estimate for π is $\hat{\pi}_n = 4\hat{p}_n$, which converges to π as n becomes large by the LLN.

For instance, suppose that we throw $n = 500$ darts at the square and 397 of them land in the circle. Then $\hat{p}_n = 0.794$, and our estimate for π is $\hat{\pi}_n = 3.176$ — not so bad.


Monte Carlo Simulation
Monte Carlo Simulation
Number of Points ?
500
Random Seed ?
573737
Animation Delay (1-100) ?
1
Points Plotted:
500
Real Value of Pi:
3.14159
Estimator for Pi:
3.176
START
BACK
(0,1)
(0,0)
(1,0)
3.14159

How would we actually conduct such an experiment?

To simulate a dart toss, suppose U_1 and U_2 are iid Unif(0,1). Then (U_1, U_2) represents the random position of the dart on the unit square. The dart lands in the circle if

$$\left(U_1 - \frac{1}{2}\right)^2 + \left(U_2 - \frac{1}{2}\right)^2 \leq \frac{1}{4}.$$

Generate n such pairs of uniforms and count up how many of them fall in the circle. Then plug into $\hat{\pi}_n$. $\square$

Summary

We made some delicious π using random numbers as our main ingredient.

Next Time: We'll simulate how we queue up at the single-server bakery.

Old Quiz: What's the volume of a pizza with radius z and height a?

Answer: $\pi z^2 a$ = pizza!

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A Single-Server Queue

Lesson Overview

Last Time: We enjoyed some π.

This Time: Now will simulate the line that forms in front of the single server at the π bakery.

This is pretty much our first "real" simulation model involving non-static customers arriving and getting served by a resource.

Customers arrive at a single-server queue with iid interarrival times and iid service times. Customers must wait in a FIFO line if the server is busy.

We will estimate the expected customer waiting time, the expected number of people in the system, and the server utilization (proportion of busy time). First, some notation.

Interarrival time between customers $i-1$ and i is I_i

Customer i's arrival time is $A_i = \sum_{j=1}^{i} I_j$

Customer i starts service at time $T_i = \max(A_i, D_{i-1})$

Customer i's waiting time is $W_i^Q = T_i - A_i$

Customer i's time in the system is $W_i = D_i - A_i$

Customer i's service time is S_i

Customer i's departure time is $D_i = T_i + S_i$

Example: Suppose we have the following sequence of events…

i	I_i	A_i	T_i	W_i^Q	S_i	D_i
1	3	3	3	0	7	10
2	1	4	10	6	6	16
3	2	6	16	10	4	20
4	4	10	20	10	6	26
5	5	15	26	11	1	27
6	5	20	27	7	2	29

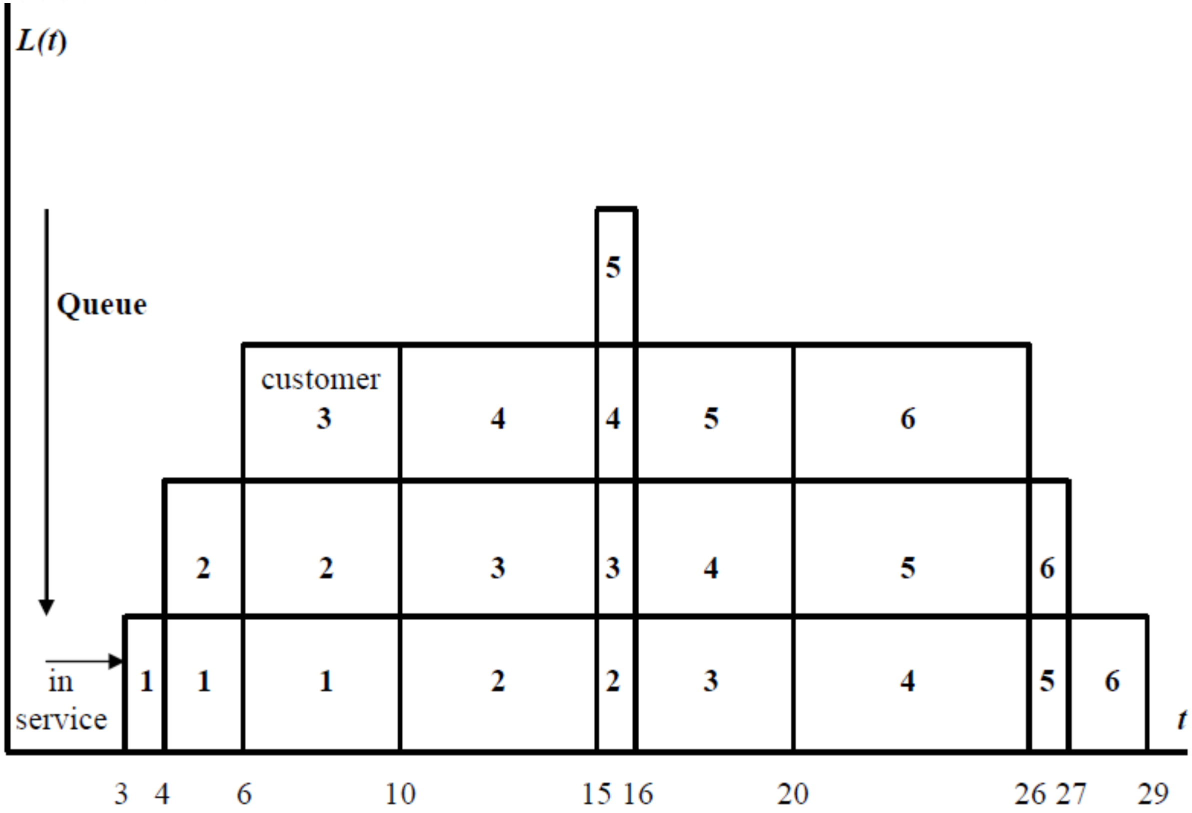
The average waiting time for the six customers is obviously $\sum_{i=1}^{6} W_i^Q/6 = 7.33$.

How do we get the average number of people in the system (in line + in service)?

Note that arrivals and departures are the only possible times for the number of people in the system, $L(t)$, to change.

These times (and the associated things that happen) are called events.

time t	event	$L(t)$
0	simulation begins	0
3	customer 1 arrives	1
4	2 arrives	2
6	3 arrives	3
10	1 departs; 4 arrives	3
15	5 arrives	4
16	2 departs	3
20	3 departs; 6 arrives	3
26	4 departs	2
27	5 departs	1
29	6 departs	0



The average number in the system is $\bar{L} = \frac{1}{29}\int_0^{29} L(t)\,dt = \frac{70}{29}$.

Another way to get the average number in the system is to calculate

$$\begin{aligned}
\bar{L} &= \frac{\text{total person-time in system}}{29} \\
&= \frac{\sum_{i=1}^{6}(D_i - A_i)}{29} \\
&= \frac{7+12+14+16+12+9}{29} = \frac{70}{29}.
\end{aligned}$$

Finally, to obtain the estimated server utilization, we easily see (from the picture) that the proportion of time that the server is busy is $\hat{\rho} = \frac{26}{29}$. $\square$

Example: Same events, but *last*-in-first-out (LIFO) services...

i	I_i	A_i	T_i	W_i^Q	S_i	D_i
1	3	3	3	0	7	10
2	1	4	23	19	6	29
3	2	6	17	11	4	21
4	4	10	10	0	6	16
5	5	15	16	1	1	17
6	5	20	21	1	2	23

The average waiting time for the six customers is 5.33, and the average number of people in the system turns out to be $\frac{58}{29} = 2$, which in this case turn out to better than FIFO.

Summary

We simulated a single-server queueing system and showed how to collect certain important performance statistics. Even did a LIFO version!

Next Time: A more-complicated example involving an inventory problem.

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An (s,S) Inventory System

Lesson Overview

Last Time: Simulated a single-server queue and estimated expected waits, utilizations, etc.

This Time: Let's do something more challenging – an (s,S) inventory system.

You'll start to get the idea why we don't want to do this by hand!

Description of (*s*,*S*)

- A store sells a product at $\$d$ / unit.
- Our inventory policy is to have at least s units in stock at the start of each day.
- If the stock slips to less than s by the end of the day, we place an order with our supplier to push the stock back up to S by the beginning of the next day.
- Various costs.

Let I_i denote the inventory at the *end* of day i, and let Z_i denote the order that we place at the end of day i. Clearly,

$$Z_i = \begin{cases} S - I_i & \text{if } I_i < s \\ 0 & \text{otherwise} \end{cases} .$$

If an order is placed to the supplier at the end of day i, it costs the store $K + cZ_i$. It costs \$$h$/unit for the store to hold unsold inventory overnight, and a penalty cost of \$$p$/unit if demand can't be met. No backlogs are allowed. Demand on day i is D_i.

How much money does the store make on day i?

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$$
\begin{aligned}
&\text{Total} \\
&= \quad \text{Sales} - \text{Ordering Cost} - \text{Holding Cost} - \text{Penalty Cost} \\
&= \quad d \min(D_i, \text{inventory at beginning of day } i) \\
&\qquad - \begin{cases} K + cZ_i & \text{if } I_i < s \\ 0 & \text{otherwise} \end{cases} \\
&\qquad -hI_i - p \max(0, D_i - \text{inventory at beginning of day } i) \\
&= \quad d \min(D_i, I_{i-1} + Z_{i-1}) \\
&\qquad - \begin{cases} K + cZ_i & \text{if } I_i < s \\ 0 & \text{otherwise} \end{cases} \\
&\qquad -hI_i - p \max(0, D_i - (I_{i-1} + Z_{i-1})).
\end{aligned}
$$

Example: Suppose

$$d = 10, \quad s = 3, \quad S = 10, \quad K = 2, \quad c = 4, \quad h = 1, \quad p = 2.$$

Consider the following sequence of demands:

$$D_1 = 5, \quad D_2 = 2, \quad D_3 = 8, \quad D_4 = 6, \quad D_5 = 2, \quad D_6 = 1.$$

Suppose that we start out with an initial stock of $I_0 + Z_0 = 10$.

Day i	begin stock	D_i	I_i	Z_i	sales rev	order cost	hold cost	penalty cost	TOTAL rev
1	10	5	5	0	50	0	−5	0	45
2	5	2	3	0	20	0	−3	0	17
3	3	8	0	10	30	−42	0	−10	−22
4	10	6	4	0	60	0	−4	0	56
5	4	2	2	8	20	−34	−2	0	−16
6	10	1	9	0	10	0	−9	0	1

Summary

We simulated our first “tricky” model – an (*s*,*S*) inventory system.

Next Time: The last couple of lessons required us to generate certain random variates. That’s what we’ll look at forthwith!

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Simulating Random Variables

Lesson Overview

Last Time: Simulated an (s,S) inventory model.

This Time: Along the way, we've needed to generate certain RV's. We'll look into that a bit now.

This is just a preview of much more-sophisticated stuff coming up later. Note: Some of this material was covered back in Module 2.

Example (Discrete Uniform): Consider a D.U. on $\{1, 2, \ldots, n\}$, i.e., $X = i$ with probability $1/n$ for $i = 1, 2, \ldots, n$. (Think of this as an n-sided dice toss for you Dungeons and Dragons fans.)

If $U \sim \mathrm{Unif}(0, 1)$, we can obtain a D.U. random variate simply by setting $X = \lceil nU \rceil$, where $\lceil \cdot \rceil$ is the "ceiling" (or "round up") function.

For example, if $n = 10$ and we sample a Unif(0,1) random variable $U = 0.73$, then $X = \lceil 7.3 \rceil = 8$. $\square$

Example (Another Discrete Random Variable):

$$P(X = x) = \begin{cases} 0.25 & \text{if } x = -2 \\ 0.10 & \text{if } x = 3 \\ 0.65 & \text{if } x = 4.2 \\ 0 & \text{otherwise} \end{cases}$$

Can't use a die toss to simulate this random variable. Instead, use what's called the *inverse transform method*.

x	$f(x)$	$P(X \le x)$	Unif(0,1)'s
-2	0.25	0.25	$[0.00, 0.25]$
3	0.10	0.35	$(0.25, 0.35]$
4.2	0.65	1.00	$(0.35, 1.00)$

Sample $U \sim \text{Unif}(0, 1)$. Choose the corresponding x-value, i.e., $X = F^{-1}(U)$. For example, $U = 0.46$ means that $X = 4.2$. $\square$

Now we'll use the *inverse transform method* to generate a continuous random variable. We'll talk about the following result a little later…

Theorem: If X is a continuous random variable with cdf $F(x)$, then the random variable $F(X) \sim \text{Unif}(0,1)$.

This suggests a way to generate realizations of the RV X. Simply set $F(X) = U \sim \text{Unif}(0,1)$ and solve for $X = F^{-1}(U)$.

Example: Suppose $X \sim \text{Exp}(\lambda)$. Then $F(x) = 1 - e^{-\lambda x}$ for $x > 0$. Set $F(X) = 1 - e^{-\lambda X} = U$. Solve for X,

$$X = \frac{-1}{\lambda}\ell n(1-U) \sim \text{Exp}(\lambda). \quad \square$$

Example (Generating Uniforms): The above RV generation examples required us to generate "practically" independent and identically distributed (iid) Unif(0,1) RV's.

If you don't like programming, you can use Excel function `RAND()` or something similar to generate Unif(0,1)'s.

Here's an algorithm to generate *pseudo-random numbers (PRN's)*, i.e., a series $R_1, R_2, \ldots$ of *deterministic* numbers that *appear* to be iid Unif(0,1). Pick a *seed* integer X_0, and calculate

$$X_i = 16807 X_{i-1} \bmod (2^{31} - 1), \quad i = 1, 2, \ldots.$$

Then set $R_i = X_i/(2^{31} - 1)$, $i = 1, 2, \ldots$.

Here's an easy FORTRAN implementation of the above algorithm (from Bratley, Fox, and Schrage).

```
FUNCTION UNIF(IX)
K1 = IX/127773     (this division truncates, e.g., 5/3 = 1.)
IX = 16807*(IX - K1*127773) - K1*2836      (update seed)
IF(IX.LT.0)IX = IX + 2147483647
UNIF = IX * 4.656612875E-10
RETURN
END
```

We input a positive integer IX and the function returns the PRN UNIF, as well as an updated IX that we can use again. □

Summary

Showed how to generate a few very easy RVs (this was also covered in Module 2, and it won’t be the Last Time, as The Stones would say).

Next Time: We’ll do a simple “spreadsheet” simulation. Things are starting to get a little too tedious to do by hand.

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Spreadsheet Simulation

Lesson Overview

Last Time: Simulated a few easy random variables.

This Time: We'll do some spreadsheet simulation.

Very useful in business and many other applications. Can even be used in certain discrete-event scenarios.

Let's simulate a fake stock portfolio consisting of 6 stocks from different sectors, as laid out in my Excel file Spreadsheet Stock Portfolio. We start out with \$5000 worth of each stock, and each increases or decreases in value each year according to

$$\text{Previous Value} * \max\Big[\, 0,\, \text{Nor}\big(\mu_i,\, \sigma_i^2\big) * \text{Nor}\big(1, (0.2)^2\big) \,\Big]\,,$$

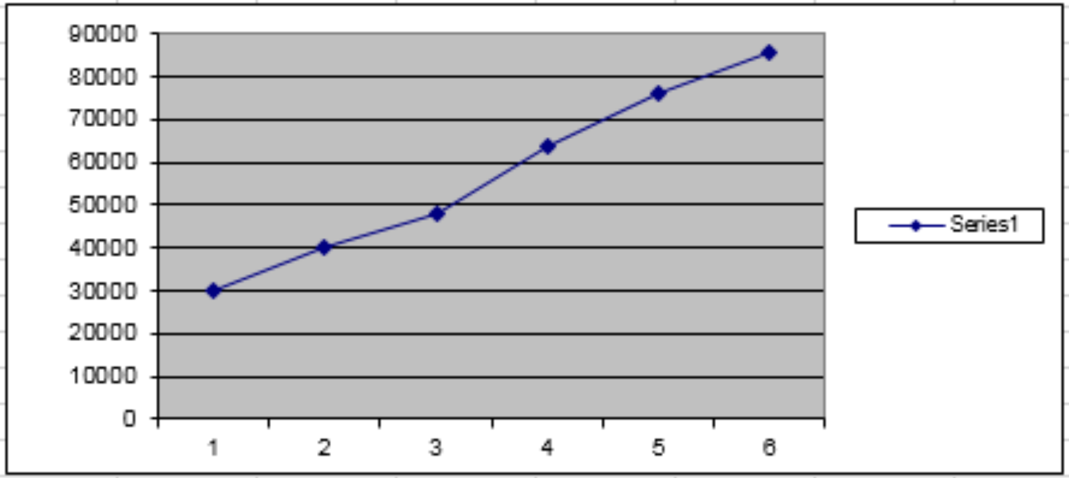
where the first normal term denotes the natural stock fluctuation for stock i, and the second normal denotes natural market conditions (that affect all stocks).

To generate a normal in Excel, you can use

$$\texttt{NORM.INV(RAND(),}\mu\texttt{,}\sigma\texttt{)} \quad ,$$

where `RAND()` is Unif(0,1), so that `NORM.INV` uses the inverse transform method.

	mean	sd	year0	year1	year2	year3	year4	year5
general year performance (so stocks are correlated)				1.18	0.92	0.96	1.11	0.85
energy	0.05	0.30	5000	7295	6636	9095	9509	11482
pharmaceuticals	0.06	0.20	5000	7015	6851	5312	5604	4287
entertainment	0.04	0.10	5000	4702	5997	5822	5630	5534
insurance	0.07	0.05	5000	5941	6004	6505	7368	6666
banking	0.06	0.30	5000	5349	6652	6915	8771	8552
computer technology	0.18	0.50	5000	9814	15901	30057	39026	48933
Totals			30000	40115.8	48040.1	63707.2	75907.7	85454



Summary

Did some spreadsheet simulation. You'll be seeing more of this in upcoming HWs!

This completes Module 3, where we discussed hand simulation (of very easy systems) along with spreadsheet simulation.

Module 4 concerns general principles – what makes a simulation tick?