Computer Simulation

Module 8: Input Data Analysis

Dave Goldsman, Ph.D.

Professor

Stewart School of Industrial and Systems Engineering

Invariance Property of MLEs

Lesson Overview

Last Time: Worked on some tricky MLE examples.

This Time: We'll expand our MLE vocabulary via the amazing Invariance Property.

Idea: If you give me the MLE for some parameter, I can give you the MLE for any reasonable function of that parameter!

Theorem (Invariance Property): If $\hat{\theta}$ is the MLE of some parameter θ and $h(\cdot)$ is a one-to-one function, then $h(\hat{\theta})$ is the MLE of $h(\theta)$.

Example: Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Bern}(p)$.

Recall that the MLE of p is $\widehat{p} = \bar{X}$ (which also happens to be unbiased). If we consider the 1:1 function $h(\theta) = \theta^2$ (for $\theta > 0$), then the Invariance Property says that the MLE of p^2 is $\bar{X}^2$. $\square$

Remark: Recall that such a property does *not* hold for unbiasedness. E.g., $\mathrm{E}[S^2] = \sigma^2$ but $\mathrm{E}[\sqrt{S^2}] \neq \sigma$.

Example: Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Nor}(\mu, \sigma^2)$.

We saw that the MLE for σ^2 is $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$. The good news is that we can still get the MLE for σ...

If we consider the 1:1 function $h(\theta) = +\sqrt{\theta}$, then the Invariance Property says that the MLE of σ is

$$\widehat{\sigma} = \sqrt{\widehat{\sigma^2}} = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}}. \quad \Box$$

Example: Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Exp}(\lambda)$. We saw that the MLE for λ is $\hat{\lambda} = 1/\bar{X}$.

Meanwhile, we define the *survival function* as

$$\bar{F}(x) \;=\; P(X > x) \;=\; 1 - F(x) \;=\; e^{-\lambda x}.$$

Then the Invariance Property says that the MLE of $\bar{F}(x)$ is

$$\widehat{\bar{F}(x)} \;=\; e^{-\hat{\lambda} x} \;=\; e^{-x/\bar{X}}. \quad \square$$

This kind of thing is used all of the time in the actuarial sciences.

Summary

This Time: We discussed the Invariance Property of MLEs, along with some of its implications.

Next Time: There's nobody as sweet as your MoM – the Method of Moments estimator!

Let's estimate stuff, hunny!

Computer Simulation

Module 8: Input Data Analysis

Dave Goldsman, Ph.D.
Professor
Stewart School of Industrial and Systems Engineering

The Method of Moments

Lesson Overview

Last Time: We talked about the Invariance property of MLEs.

This Time: We'll finish off our estimator dictionary with the Method of Moments.

Idea: If you always remember your MoM, you'll be really great at simulation input analysis!

The Method of Moments

Recall: The kth *moment* of a random variable X is

$$\mathrm{E}[X^k] = \begin{cases} \sum_x x^k f(x) & \text{if } X \text{ is discrete} \\ \\ \int_{\mathbb{R}} x^k f(x)\, dx & \text{if } X \text{ is cts} \end{cases}$$

Definition: Suppose $X_1, \ldots, X_n$ are i.i.d. from p.m.f. / p.d.f. $f(x)$. Then the *method of moments* (MOM) estimator for $\mathrm{E}[X^k]$ is $m_k \equiv \frac{1}{n}\sum_{i=1}^n X_i^k$.

Examples:

The MOM estimator for $\mu = \mathrm{E}[X_i]$ is $m_1 = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

The MOM estimator for $\mathrm{E}[X_i^2]$ is $m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

The MOM estimator for $\mathrm{Var}(X_i) = \mathrm{E}[X_i^2] - (\mathrm{E}[X_i])^2$ is

$$m_2 - m_1^2 \;=\; \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2 \;=\; \frac{n-1}{n} S^2.$$

(Of course, it's also OK to use S^2.)

Example: Suppose $X_1, \ldots, X_n \stackrel{\text{iid}}{\sim} \text{Pois}(\lambda)$.

Since $\lambda = \text{E}[X_i]$, a MOM estimator for λ is $\bar{X}$.

But also note that $\lambda = \text{Var}(X_i)$, so another MOM estimator for λ is $\frac{n-1}{n} S^2$ (or plain old S^2).

Usually use the easier-looking estimator if you have a choice. $\square$

Example: Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Nor}(\mu, \sigma^2)$.

MOM estimators for μ and σ^2 are $\bar{X}$ and $\frac{n-1}{n} S^2$ (or S^2), respectively.

For this example, these estimators are the same as the MLEs. $\Box$

Let's finish up with a less-trivial example…

Example: Suppose $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Beta}(a, b)$. The p.d.f. is

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1.$$

It turns out (after lots of algebra) that

$$\text{E}[X] = \frac{a}{a+b} \quad \text{and}$$

$$\text{Var}(X) = \frac{ab}{(a+b)^2(a+b+1)} = \frac{\text{E}[X]b}{(a+b)(a+b+1)}.$$

Let's estimate a and b via MOM.

We have

$$\mathrm{E}[X] = \frac{a}{a+b} \Rightarrow a = \frac{b\mathrm{E}[X]}{1-\mathrm{E}[X]} \approx \frac{b\bar{X}}{1-\bar{X}}. \qquad (*)$$

Plug the following into the previous equation for $\mathrm{Var}(X)$: $\bar{X}$ for $\mathrm{E}[X]$, S^2 for $\mathrm{Var}(X)$, and $\frac{b\bar{X}}{1-\bar{X}}$ for a.

Then after lots of algebra, we can solve for b:

$$b \approx \frac{(1-\bar{X})^2\bar{X}}{S^2} - 1 + \bar{X}.$$

To finish up, plug back in $(*)$ to get the MOM estimator for a.

Example (Hayter): Suppose we take a bunch of observations from a Beta distribution and it turns out that $\bar{X} = 0.3007$ and $S^2 = 0.01966$.

Then (try it yourself and see!) the MOM estimators for a and b are 2.92 and 6.78, respectively. $\square$

Summary

This Time: Went over the Method of Moments. These usually intuitive and often easy to compute.

This completes our discussion (for now) on point estimation.

Next Time: Now we're ready to conduct goodness-of-fit tests for potential input distributions!