ISyE 6644 — Spring 2019 — Test #2 Solutions

(Revised 7/5/19)

This test is 120 minutes. You're allowed *two* cheat sheets (4 sides total).

This test requires a proctor. All questions are 3 points, except #33, which is 4 points.

1. Consider the following joint p.m.f.

$f(x,y)$	$X=1$	$X=2$
$Y=0$	0.06	0.24
$Y=1$	0.14	0.56

YES or NO? Are X and Y independent?

Solution: Let's re-do the table to include the marginals:

$f(x,y)$	$X=1$	$X=2$	$f_Y(y)$
$Y=0$	0.06	0.24	0.3
$Y=1$	0.14	0.56	0.7
$f_X(x)$	0.2	0.8	1

Note that $f(x,y) = f_X(x)f_Y(y)$ for all four combinations of x and y, so the answer is YES. $\square$

2. TRUE or FALSE? Most discrete-event simulations proceed by moving the simulation clock to the most-imminent event on the future events list; executing that event (including any adds, deletes, or swaps to the FEL); and then repeating this cycle.

Solution: TRUE. $\square$

3. TRUE or FALSE? In Arena, it is *not possible* to `Seize` and `Release` a *specific* member of a resource set, and then `Seize` that same guy a *second* time.

Solution: FALSE. $\square$

4. In Arena, where do we specify the maximum buffer size for a certain queue?

   (a) In the `Queue` spreadsheet found in the Basic Process template.

   (b) In the `Queue` module found in the Advanced Process template.

   (c) In the `Queue` block found in the Blocks template.

   (d) In the `Process` module found in the Basic Process template.

   (e) Both (c) and (d).

   Solution: Just (c). □

5. TRUE or FALSE? The Arena resource that we've named `Barber` can belong to 3 different resource sets.

   Solution: TRUE. □

6. TRUE or FALSE? Arena resources within a particular resource set can each have different schedules.

   Solution: TRUE. □

7. What is `TNOW` in Arena?

   (a) An attribute that keeps track of the current time.

   (b) A variable that keeps track of the current time.

   (c) An attribute that keeps track of the current customer's time in the system.

   (d) A variable that keeps track of the current customer's time in the system.

   Solution: (b). □

8. Which statement best describes Arena arrivals?

   (a) Arrivals can be generated in batches of random size.

   (b) Arrivals can be generated via arbitrary interarrival distributions.

   (c) Arrivals can follow a nonhomogeneous Poisson process.

(d) `EXPO` arrivals constitute a Poisson process.

(e) All of the above.

Solution: (e). $\square$

9. What is the variance of the Arena expression `(0 == 0)*NORM(0,4)`?

   (a) 0

   (b) 1

   (c) 4

   (d) 16

   (e) 64

   Solution: In Arena, a $\texttt{NORM}(a,b)$ random variable has variance b^2. Therefore,

$$\mathsf{Var}\Big((0 == 0) * \texttt{NORM}(10,4)\Big) \;=\; \mathsf{Var}\Big(1 * \texttt{NORM}(10,4)\Big) \;=\; 16.$$

   So the answer is (d). $\square$

10. TRUE or FALSE? In Arena, the expression `NORM(0,1)+NORM(0,1)` has the same distribution as the expression `TRIA(0,1,2)`.

    Solution: FALSE. In fact, the Tria(0,1,2) is the sum of two Unif(0,1)'s, not the sum of two Nor(0,1)'s. $\square$

11. Which statement below describing Arena arrivals is *false*?

    (a) Arrivals can be generated in batches of random size.

    (b) Arrivals can be generated via arbitrary interarrival distributions.

    (c) Arrivals can follow a nonhomogeneous Poisson process.

    (d) `EXPO` arrivals constitute a Poisson process.

    (e) You can use `NORM` random variables to generate *negative* interarrival times.

    Solution: (e). $\square$

12. Consider the generator $X_{i+1} = (3X_i + 5)\text{mod}(8)$. Using $X_0 = 0$, calculate the PRN U_{2001}.

    (a) 0
    (b) 0.125
    (c) 0.5
    (d) 0.625
    (e) 1
    (f) 3

    Solution: If $X_0 = 0$, then we have $X_1 = 5$, $X_2 = 4$, $X_3 = 1$, and $X_4 = 0$, so it cycles after 4 iterations. Thus, $X_{2001} = X_{1997} = \cdots = X_1 = 5$, so that $U_{2001} = 5/8$. So the answer is (d). $\square$

13. Consider our desert island generator $X_{i+1} = 16807\, X_i \,\text{mod}(2^{31} - 1)$. If $X_0 = 3456789$, find the value of X_1.

    (a) 0
    (b) 0.61359
    (c) 25,152,971
    (d) 116,194,254
    (e) 1,993,604,588
    (f) $2^{31} - 582$

    Solution: By hook or by crook (e.g., by the algorithm given in the notes), we find that $X_1 = 116{,}194{,}254$, so that the answer is (d). $\square$

14. TRUE or FALSE? There are perfectly good PRN generators out there having periods greater than 2^{19900}!

    Solution: Crazy, but TRUE. $\square$

15. Consider the following 20 PRN's.

0.16	0.26	0.96	0.09	0.11	0.85	0.59	0.77	0.93	0.60
0.71	0.38	0.49	0.18	0.91	0.62	0.41	0.70	0.11	0.07

How many runs up and down do you get from this sequence?

(a) 8
(b) 14
(c) 15
(d) 20
(e) 33

Solution: Using the usual notation from class, we have the following sequence of $+$'s and $-$'s:

$$\begin{array}{cccccccccc} & + & + & - & + & + & - & + & + & - \\ + & - & + & - & + & - & - & + & - & - \end{array}$$

This corresponds to $A = 14$ runs, i.e., choice (b). $\Box$

16. Suppose that we have a sequence of $n = 100$ PRN's, and we observe 56 runs up and down. Using $\alpha = 0.05$, do we ACCEPT or REJECT the null hypothesis of independence?

Solution: By class notes, under the null hypothesis of independence, $A \approx \text{Nor}(\frac{2n-1}{3}, \frac{16n-29}{90}) \sim \text{Nor}(66.33, 17.46)$. Thus, the test statistic is

$$Z_0 = \frac{A - \mathsf{E}[A]}{\sqrt{\mathsf{Var}(A)}} = \frac{56 - 66.33}{\sqrt{17.46}} = -2.47.$$

Since $|Z_0| > z_{\alpha/2} = 1.96$, we REJECT H_0. Thus, we conclude that they're *dependent*. $\Box$

17. Suppose we sample 1000 PRN's and we wish to conduct a χ^2 goodness-of-fit test at level $\alpha = 0.10$ of the hypothesis that the numbers are Unif(0,1). Here are the results, divided into 5 intervals.

interval	O_i
$[\,0.0, 0.2\,]$	250
$(\,0.2, 0.4\,]$	120
$(\,0.4, 0.6\,]$	110
$(\,0.6, 0.8\,]$	270
$(\,0.8, 1.0\,]$	250

Use the tabled results to find the value of the g-o-f statistic, χ^2_0.

(a) 1.22

(b) 12.2

(c) 122

(d) 1220

(e) 24400

Solution: Since we have $k = 5$ equiprobable intervals, we obtain $E_i = n/k = 200$ for all i. Thus,

$$\chi^2_0 \;=\; \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \;=\; \sum_{i=1}^{5} \frac{(O_i - 200)^2}{200} \;=\; 122,$$

and so the answer is (c). $\Box$

18. Similar to Problem 17, let's do a χ^2 test for uniformity with $n = 1000$ PRN's, $k = 5$ equi-probability intervals, and level $\alpha = 0.10$. But now suppose it turns out that $\chi^2_0 = 56$ (instead of whatever answer you got before). Do we ACCEPT or REJECT the null hypothesis of uniformity?

Solution: First of all, the appropriate test quantile is $\chi^2_{\alpha,k-1} = \chi^2_{0.10,4} = 7.78$. Since $\chi^2_0 \gg \chi^2_{\alpha,k-1}$, we easily REJECT uniformity. $\Box$

19. Suppose the random variable X has p.d.f. $f(x) = 3(x-1)^2$ for $1 \le x \le 2$. Find the inverse of its c.d.f., i.e., $F^{-1}(U)$, where U is a PRN.

(a) $3(X-1)^2$

(b) $3(U-1)^2$

(c) $(X-1)^3$

(d) $(U-1)^3$

(e) $U^{1/3}+1$

Solution: We have

$$F(x) \;=\; \int_1^x 3(t-1)^2\,dt \;=\; (x-1)^3, \quad \text{for } 1 \le x \le 2.$$

Set $F(X) = (X-1)^3 = U$. Solving, we have $X = F^{-1}(U) = U^{1/3}+1$. So the answer is (e). $\square$

20. Use the *inverse transform method* with $U = 0.27$ to generate a realization of $Z \sim \text{Nor}(0,1)$. (You'll need the normal tables for this.) And then generate $X \sim \text{Nor}(3,4)$ by making the appropriate linear transformation $X = \mu + \sigma Z$. What's your final value for X?

(a) 0.73

(b) 1.77

(c) 2.73

(d) 3.27

(e) 5.27

Solution: We take

$$X \;=\; \mu+\sigma Z \;=\; \mu+\sigma\Phi^{-1}(U) \;=\; 3+2\Phi^{-1}(0.27) \;=\; 3+2(-0.6128) \;=\; 1.77.$$

Thus, the answer is (b). $\square$

21. Suppose that X has the Gamma(α,λ) distribution with (messy) c.d.f. $F(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}e^{-\lambda t}\,dt$. What is the distribution of the random variable $3F(X)+2$?

(a) Unif(0,1)

(b) Unif(0,3)

(c) Unif(2,5)

(d) Gamma(α,λ)

(e) $3\,\text{Gamma}(\alpha, \lambda) + 2$

Solution: By the Inverse Transform Theorem, $F(X) \sim \text{Unif}(0,1)$. Thus,

$$3F(X) + 2 \;\sim\; \text{Unif}(0,3) + 2 \;\sim\; \text{Unif}(2,5).$$

So the answer is (c). $\square$

22. The number of tries until a UGA student gets one of a series of multiple choice questions correct is $X \sim \text{Geom}(0.3)$. Use the PRN $U = 0.50$ to generate X via inverse transform.

(a) 0
(b) 1
(c) 2
(d) 8
(e) 15
(f) 22

Solution: We have

$$X \;=\; \left\lceil \frac{\ell\text{n}(1-U)}{\ell\text{n}(1-p)} \right\rceil \;=\; \left\lceil \frac{\ell\text{n}(0.50)}{\ell\text{n}(0.7)} \right\rceil \;=\; 2.$$

So the answer is (c). $\square$

23. Suppose that $U_1 = 0.70$ and $U_2 = 0.41$ are realizations of two i.i.d. Unif(0,1)'s. Use the Box–Muller method to generate two i.i.d. standard normals, Z_1 and Z_2. [Note that there are a number of ways to do this problem, but I expect that you will use the standard method; and that answer is reflected in the choices below.]

(a) -0.713 and 0.453
(b) 0 and 1.96
(c) -0.844 and 0.038
(d) 0.844 and 0.038

(e) 1.788 and 0.076

Solution: Being very careful to use radians (not degrees), we have

$$\begin{aligned} Z_1 &= \sqrt{-2\ell\text{n}(U_1)}\cos(2\pi U_2) &= -0.713 \\ Z_2 &= \sqrt{-2\ell\text{n}(U_1)}\sin(2\pi U_2) &= 0.453. \end{aligned}$$

This is answer (a). $\Box$

24. If Z_1 and Z_2 are i.i.d. Nor(0,1), then we say that $(Z_1 + \delta)/Z_2$ has the *noncentral t distribution* with 1 degree of freedom and *noncentrality parameter* δ. If $Z_1 = -0.843$ and $Z_2 = 0.053$, generate a noncentral t random variate with one degree of freedom and noncentrality parameter 2. [Note that there are a number of ways to do this problem, but I expect that you will simply follow the directions given in the problem; it is that answer that is reflected in the choices below.]

    (a) -40.70
    (b) 0
    (c) 1.96
    (d) 21.83
    (e) 40.70

Solution: $(-0.843 + 2)/0.053 = 21.83$, so that the answer is (d). $\Box$

25. TRUE or FALSE? If Z_1 and Z_2 are i.i.d. Nor(0,1)'s, e.g., resulting from the Box–Muller method, then $Z_1^2 + Z_2^2 \sim \text{Exp}(1/2)$.

Solution: As explained in our class notes,

$$Z_2^2 + Z_2^2 \;=\; -2\ell\text{n}(U_1)[\cos^2(2\pi U_2) + \sin^2(2\pi U_2)] \;=\; -2\ell\text{n}(U_1) \;\sim\; \text{Exp}(1/2),$$

where the distributional result follows by inverse transform (which we've used a million times). Thus, the assertion is TRUE. $\Box$

26. Consider the following 20 PRN's.

0.16	0.26	0.96	0.09	0.11	0.85	0.59	0.77	0.93	0.60
0.71	0.38	0.49	0.18	0.91	0.62	0.41	0.70	0.11	0.45

Use the sum of these PRN's to generate a single approximately Nor(0,1) random variate via our "desert island" technique.

(a) -0.527
(b) 0
(c) 0.217
(d) 0.527
(e) 1.96

Solution: From our class notes, we have

$$Z = \frac{\sum_{i=1}^{n} U_i - \frac{n}{2}}{\sqrt{\frac{n}{12}}} = \frac{10.28 - 10}{\sqrt{20/12}} = 0.217.$$

Thus, the answer is (c). $\Box$

27. Suppose that $U_1 = 0.65$, $U_2 = 0.45$, $U_3 = 0.82$, $U_4 = 0.09$, and $U_5 = 0.26$. Use our acceptance-rejection technique from class to generate a Pois($\lambda = 1.7$) random variate. (You may not need to use all of the uniforms.)

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

Solution: Define $p_n \equiv \prod_{i=1}^{n+1} U_i$. We'll stop as soon as $p_n < e^{-1.7} = 0.1827$. Let's make the following convenient table.

n	U_{n+1}	p_n	Stop?
0	0.65	0.65	nope
1	0.45	0.2925	nope
2	0.82	0.2399	nope
3	0.09	0.0216	yup

So we take $N = 3$, i.e., answer (d). $\square$

28. Suppose U_1 and U_2 are PRN's and $\Phi(x)$ is the standard normal c.d.f. Find $\Pr\left(\Phi^{-1}(U_1) + \Phi^{-1}(U_2) \le \sqrt{2}\right)$.

    (a) 0
    (b) 0.159
    (c) 0.240
    (d) 0.760
    (e) 0.841

Solution: By Inverse Transform, $\Phi^{-1}(U_1)$ and $\Phi^{-1}(U_2)$ are i.i.d. Nor(0,1). Thus,

$$\begin{aligned}\Pr\left(\Phi^{-1}(U_1) + \Phi^{-1}(U_2) \le \sqrt{2}\right) &= \Pr\left(\mathrm{Nor}(0,2) \le \sqrt{2}\right) \\ &= \Pr(\mathrm{Nor}(0,1) \le 1) \\ &= \Phi(1) \;=\; 0.8413.\end{aligned}$$

This is (e). $\square$

29. Suppose Z_1 and Z_2 are i.i.d. standard normal with c.d.f. $\Phi(x)$. Find the value of $\Pr(\Phi(Z_1) + \Phi(Z_2) \le 2)$.

    (a) 0
    (b) 0.25
    (c) 0.5
    (d) 0.75
    (e) 1

Solution: By Inverse Transform, $\Phi(Z_1)$ and $\Phi(Z_2)$ are i.i.d. Unif(0,1). Thus,

$$\Pr(\Phi(Z_1) + \Phi(Z_2) \le 2) \;=\; \Pr(\mathrm{Tria}(0,1,2) \le 2) \;=\; 1.$$

So the answer is (e). $\square$

30. Consider a RV X having p.d.f. $f(x) = cx^3(1-x)^2e^{-x}$, for $0 < x < 1$, where c is the constant that makes this monster integrate to 1 (and which I'm too lazy to calculate right now). Which method would you most likely use to generate X?

    (a) inversion

    (b) convolution

    (c) Box–Muller

    (d) acceptance-rejection

    (e) composition

    (f) nonhomogeneous Poisson

    Solution: This is an obvious case for A-R, choice (d). $\square$

31. Suppose that X_1, X_2, X_3 are i.i.d. with p.d.f. $f(x) = 2x$ for $0 < x < 1$. Give an expression involving a *single* PRN U that you can use to generate a realization of $\max\{X_1, X_2, X_3\}$.

    (a) $\max\{U_1^{1/2}, U_2^{1/2}, U_3^{1/2}\}$

    (b) $\max\{U_1^2, U_2^2, U_3^2\}$

    (c) $U^{1/2}$

    (d) $U^{1/6}$

    (e) $U^{1/8}$

    (f) U^8

    Solution: I'll first do a general derivation. Consider $X_1, X_2, \ldots, X_n$ i.i.d. with c.d.f. $F(x)$ and let $Y \equiv \max\{X_1, X_2, \ldots, X_n\}$. Then (as explained in class) the c.d.f. of Y is

$$
\begin{aligned}
G(y) &= \Pr(Y \le y) \\
&= \Pr(\max\{X_1, X_2, \ldots, X_n\} \le y) \\
&= \Pr(X_1 \le y, X_2 \le y, \ldots, X_n \le y) \\
&= [\Pr(X_1 \le y)]^n \quad \text{(since the } X_i\text{'s are i.i.d.)} \\
&= [F(y)]^n.
\end{aligned}
$$

Now, by Inverse Transform, $G(Y) = U \sim \text{Unif}(0,1)$, and so $[F(Y)]^n = U$. Since the c.d.f. of X is clearly $F(x) = x^2$ for $0 < x < 1$, and since $n = 3$ here, we have

$$U \;=\; G(Y) \;=\; [F(Y)]^n \;=\; (Y^2)^n \;=\; Y^6,$$

so that $Y = U^{1/6}$. This is choice (d). $\square$

32. Consider a 2×2 covariance matrix $\Sigma = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. Find a lower-triangular matrix C such that $CC' = \Sigma$, and tell me what the entry c_{22} is.

(a) 0

(b) 5/3

(c) $\sqrt{5/3}$

(d) $2/\sqrt{3}$

(e) 4/3

Solution: By class notes, we saw that for $k = 2$,

$$C \;=\; \begin{pmatrix} \sqrt{\sigma_{11}} & 0 \\ \frac{\sigma_{12}}{\sqrt{\sigma_{11}}} & \sqrt{\sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}} \end{pmatrix} \;=\; \begin{pmatrix} \sqrt{3} & 0 \\ \frac{2}{\sqrt{3}} & \sqrt{3 - \frac{2^2}{3}} \end{pmatrix} \;=\; \begin{pmatrix} \sqrt{3} & 0 \\ \frac{2}{\sqrt{3}} & \sqrt{\frac{5}{3}} \end{pmatrix}.$$

(As a check, you'll see that $CC' = \Sigma$, as desired.) Thus, $c_{22} = \sqrt{5/3} = 1.291$, answer (c). $\square$

33. BONUS: How many trombones are there in the big parade?

(a) 0

(b) 1 (since it's the loneliest number)

(c) 2 (which can be as sad as 1)

(d) 13 (the unluckiest number)

(e) 38 (the average UGA IQ)

(f) 76

Solution: 76 (with 110 cornets close at hand), answer (f). $\square$