Week 3 Homework

Due Feb 2 at 11:59pm **Points** 14 **Questions** 14
Available after Jan 24 at 8am **Time Limit** None

Instructions

Please answer all the questions below.

Attempt History

	Attempt	Time	Score
LATEST	Attempt 1	10,140 minutes	9 out of 14

Score for this quiz: **9** out of 14
Submitted Feb 2 at 4:52pm
This attempt took 10,140 minutes.

Question 1

0 / 1 pts

(Lesson 2.10: Conditional Expectation.) BONUS: Suppose that

$f(x,y) = 6x$ for $0 \leq x \leq y \leq 1$. Hint (you may have seen this someplace): the marginal p.d.f. of X turns out to be

$f_X(x) = 6x(1-x)$ for $0 \leq x \leq 1$. Find the conditional p.d.f. of Y given that $X = x$.

orrect Answer

a. $f(y|x) = \frac{1}{1-x}, \quad 0 \leq x \leq y \leq 1$

ou Answered

b. $f(y|x) = \frac{1}{1-x}, \quad 0 \leq x \leq 1$

c. $f(y|x) = \frac{1}{1-y}, \quad 0 \leq y \leq 1$

d. $f(x|y) = \frac{1}{1-x}, \quad 0 \leq x \leq y \leq 1$

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{6x}{6x(1-x)} = \frac{1}{1-x}, \quad 0 \le x \le y \le 1$$

So the answer is (a).

Question 2

1 / 1 pts

(Lesson 2.10: Conditional Expectation.) BONUS: Again suppose that $f(x, y) = 6x$ for $0 \le x \le y \le 1$. Hint (you may already have seen this someplace): the the marginal p.d.f. of X turns out to be $f_X(x) = 6x(1-x)$ for $0 \le x \le 1$. Find $\mathsf{E}[Y|X = x]$.

a. $\mathsf{E}[Y|X = x] = 1/2, \quad 0 \le x \le 1$

Correct!

b. $\mathsf{E}[Y|X = x] = \frac{1+x}{2}, \quad 0 \le x \le 1$

By the definition of conditional expectation, we have
$\mathsf{E}[Y|X = x] = \int_{-\infty}^{\infty} y f(y|x)\,dy = \int_x^1 \frac{y}{1-x}\,dy = \frac{1+x}{2}, \quad 0 \le x \le 1.$
So the answer is (b).

c. $\mathsf{E}[Y|X = x] = \frac{1+y}{2}, \quad 0 \le y \le 1$

d. $\mathsf{E}[X|Y = y] = \frac{1+y}{2}, \quad 0 \le y \le 1$

By the definition of conditional expectation, we have
$\mathsf{E}[Y|X = x] = \int_{-\infty}^{\infty} y f(y|x)\,dy = \int_x^1 \frac{y}{1-x}\,dy = \frac{1+x}{2}, \quad 0 \le x \le 1.$
So the answer is (b).

Question 3

1 / 1 pts

(Lesson 2.10: Conditional Expectation.) BONUS: Yet again suppose that $f(x,y) = 6x$ for $0 \le x \le y \le 1$. Hint (you may already have seen this someplace): the the marginal p.d.f. of X turns out to be $f_X(x) = 6x(1-x)$ for $0 \le x \le 1$. Find $\mathsf{E}[\mathsf{E}[Y|X]]$.

a. 1/2

b. 2/3

Correct!

c. 3/4

By Law of the Unconscious Statistician and Question 21,

$$\mathsf{E}[\mathsf{E}[Y|X]] \;=\; \int_0^1 \mathsf{E}[Y|x]\, f_X(x)\, dx \;=\; \int_0^1 \frac{1+x}{2}\, 6x(1-x)\, dx \;=\; 3/4.$$

So the answer is (c).

Let's check this answer. First of all, the marginal p.d.f. of Y is:

$$f_Y(y) \;=\; \int_0^y f(x,y)\, dx \;=\; \int_0^y 6x\, dx \;=\; 3y^2,\; 0 \le y \le 1.$$

Then $\mathsf{E}[Y] \;=\; \int_0^1 y\, f_Y(y)\, dy \;=\; \int_0^1 3y^3\, dy \;=\; 3/4.$

Finally, by double expectation, $\mathsf{E}[\mathsf{E}[Y|X]] \;=\; \mathsf{E}[Y] \;=\; 3/4$, so the check works!

d. 1

By Law of the Unconscious Statistician and Question 21,

$$\mathsf{E}[\mathsf{E}[Y|X]] = \int_0^1 \mathsf{E}[Y|x]\, f_X(x)\, dx = \int_0^1 \frac{1+x}{2}\, 6x(1-x)\, dx = 3/4.$$

So the answer is (c).

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Finally, by double expectation, $\mathsf{E}[\mathsf{E}[Y|X]] = \mathsf{E}[Y] = 3/4$, so the check works!

Question 4

1 / 1 pts

(Lesson 2.11: Covariance and Correlation.)
Suppose that the correlation between December snowfall and temperature in Siberacuse, NY is -0.5. Further suppose that $\mathsf{Var}(S) =$ 100 in^2 and $\mathsf{Var}(T) = 25$ (degrees F)^2. Find $\mathsf{Cov}(S,T)$ (in units of degree inches, whatever those are).

Correct!

- a. -25

$\mathsf{Cov}(S,T) = \mathsf{Corr}(S,T)\sqrt{\mathsf{Var}(S)\mathsf{Var}(T)} = -0.5(10)(5) = -25.$

- b. -5
- c. 5
- d. 25

$$\text{Cov}(S,T) = \text{Corr}(S,T)\sqrt{\text{Var}(S)\text{Var}(T)} = -0.5(10)(5) = -25.$$

Question 5

0 / 1 pts

(Lesson 2.11: Covariance and Correlation.) If X and Y both have mean -7 and variance 4, and $\text{Cov}(X,Y) = 1$, find $\text{Var}(3X - Y)$.

orrect Answer

a. 34

ou Answered

b. 36

c. 40

d. 41

$$\text{Var}(3X - Y) \;=\; 3^2\,\text{Var}(X) + (-1)^2\,\text{Var}(Y) + 2(3)(-1)\,\text{Cov}(X,Y) \;=\; 36 + 4 - 6 \;=\; 34.$$

Question 6

1 / 1 pts

(Lesson 2.12: Probability Distributions.)

You may recall that the p.m.f. of the Geometric (p) distribution is $f(x) = (1-p)^{x-1}p, x = 1, 2, \ldots$

If the number of orders at a production center this month is a Geom(0.7) random variable, find the probability that we'll have at most 3 orders.

a. 0.027

b. 0.14

c. 0.86

Correct!

d. 0.973

Denote $X \sim \text{Geom}(0.7)$. Then

$$\mathsf{P}(X \le 3) \;=\; \sum_{x=1}^{3} \mathsf{P}(X = x) \;=\; \sum_{x=1}^{3} (0.3)^{x-1}(0.7) \;=\; \mathbf{0.973.}$$

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Question 7

0 / 1 pts

(Lesson 2.12: Probability Distributions.) Suppose the SAT math score of a University of Georgia student can be approximated by a normal distribution with mean 400 and variance 225. Find the probability that the UGA Einstein will score at least a 415.

a. 0.5

orrect Answer

b. 0.1587

c. 0.975

ou Answered

d. 0.8413

This answer is (b). To see why, let X denote his score and let Z denote a standard normal random variable. Then

$$\mathsf{P}(X \geq 415) = \mathsf{P}(\frac{X-400}{\sqrt{225}} \geq \frac{415-400}{15}) = \mathsf{P}(Z \geq 1) = 0.1587,$$

where you could've used the back of the book or the **NORMSDIST** function in Excel to look up that last probability. (Actually, this is a famous one, so you may have memorized it).

Question 8

1 / 1 pts

(Lesson 2.13: Limit Theorems.)
What is the most-important theorem in the universe?

a. Eastern Limit Theorem

Correct!

b. Central Limit Theorem

c. Central Limit Serum

d. Central Simit Theorem (simit is a tasty Turkish bagel)

Question 9

1 / 1 pts

(Lesson 2.13: Limit Theorems.) If $X_1, \ldots, X_{400}$ are i.i.d. from some distribution with mean 1 and variance 400, find the approximate probability that the sample mean $\bar{X}$ is between 0 and 2.

a. 0.1587

b. 0.3174

Correct!

c. 0.6826

First of all, note that $\mathsf{E}[\bar{X}] = \mathsf{E}[X_i] = 1$ and $\mathsf{Var}(\bar{X}) = \mathsf{Var}(X_i)/n = 1.$ Then by the CLT, we have $\bar{X} \approx$ Nor (1,1). Thus,
$\mathsf{P}(0 \le \bar{X} \le 2) \approx \mathsf{P}(-1 \le Z \le 1) = 2\Phi(1) - 1 = 2(0.8413) - 1 = 0.6826.$
So the answer is (c).

d. 0.8413

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$\mathsf{P}(0 \le \bar{X} \le 2) \approx \mathsf{P}(-1 \le Z \le 1) = 2\Phi(1) - 1 = 2(0.8413) - 1 = 0.6826.$
So the answer is (c).

Question 10

1 / 1 pts

(Lesson 2.14: Estimation.)
Suppose we collect the following observations: 7, -2, 1, 6. What is the sample variance?

a. 13

b. $\sqrt{13}$

Correct!

c. 18

First of all, the same mean is $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = 3$. Then the sample variance is

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = 18,$$

So that the answer is (c).

d. 28

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So that the answer is (c).

Question 11

0 / 1 pts

(Lesson 2.14: Estimation.) BONUS: Consider two estimators, T_1 and T_2, for an unknown parameter θ. Suppose that the $\text{Bias}(T_1) = 0$, $Bias\,(T_2) = \theta$, $\text{Var}(T_1) = 4\theta^2$, and $\text{Var}(T_2) = \theta^2$. Which estimator might you decide to use and why?

a. T_1 - it has lower expected value

b. T_1 - it has lower MSE

ou Answered

c. T_2 - it has lower variance

orrect Answer

d. T_2 - it has lower MSE

Although low bias and variance are just great individually, we are usually interested in the estimator with the lower MSE, which balances bias and variance.
Of course, MSE = Bias² + Variance. Thus, MSE $(T_1) = 4\theta^2$ and MSE $(T_2) = 2\theta^2$. So the answer is (d).
Note that (a) is incorrect because lower expected value doesn't necessarily tell us about bias. (b) is just plain wrong, wrong, wrong. (c) is close, but variance isn't as important as MSE.

Question 12

1 / 1 pts

(Lesson 2.15: Maximum Likelihood Estimation.) BONUS: Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. $\text{Pois}(\lambda)$. Find $\hat{\lambda}$, the MLE of λ. (Don't panic --- it's not that difficult.)

Correct!

a. $\overline{X}$

The likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{n\lambda}\lambda \sum_{i=1}^{n} x_i}{\prod_{i=1}^{n}(x_i!)}$$

To simplify things (as per the suggestion in the lesson), take logs:

$$\ell n(L(\lambda)) = -n\lambda + \sum_{i=1}^{n} x_i \ell n(\lambda) + C,$$

where $C = -\ell n(\prod_{i=1}^{n}(x_i!))$ is a constant with respect to λ.

The recipe says that you now set the derivative $= 0$, and solve for λ:

$$\frac{d}{d\lambda}\ell n(L(\lambda)) = -n + \frac{\sum_{i=1}^{n} x_i}{\lambda} = 0.$$

Solving, we get $\hat{\lambda} = \sum_{i=1}^{n} X_i/n = \bar{X}$. (I won't do a second derivative test because I'm lazy.) Thus, the correct answer is (a).

b. $1/\bar{X}$

c. $n/\sum_{i=1}^{x} X_i$

d. S^2

The likelihood function is

$$L(\lambda) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{n\lambda}\lambda \sum_{i=1}^{n} x_i}{\prod_{i=1}^{n}(x_i!)}$$

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Solving, we get $\hat{\lambda} = \sum_{i=1}^{n} X_i/n = \bar{X}$. (I won't do a second derivative test because I'm lazy.) Thus, the correct answer is (a).

Question 13

0 / 1 pts

(Lesson 2.15: Maximum Likelihood Estimation.) BONUS: Suppose that we are looking at i.i.d. Exp(λ) customer service times. We observe times of 2, 4, and 9 minutes. What's the maximum likelihood estimator of λ^2?

a. 5

b. 1/5

ou Answered

c. 25

orrect Answer

d. 1/25

From the lesson, we know that the MLE of λ is $\hat{\lambda} = 1/\bar{X} = 1/5$.

Therefore, the Invariance Property states that the MLE of λ^2 is $\hat{\lambda}^2 = 1/25$. This is choice (d).

Question 14

1 / 1 pts

(Lesson 2.16: Confidence Intervals.) BONUS: Suppose we collect the following observations: 7, -2, 1, 6 (as in a previous question in this homework). Let's assume that these guys are i.i.d. from a normal distribution with $unknown$ variance σ^2. Give me a two-sided 95% confidence interval for the mean μ.

a. $[-2, 7]$

Correct!

b. $[-3.75,\ 9.75]$

The confidence interval will be of the form
$\mu \in \bar{X} \pm t_{\alpha/2,n-1}\sqrt{S^2/n}$,

where $n = 4$, the sample mean $\bar{X} = 3$, the sample variance $S^2 = 18$, and $\alpha = 0.05$, so that the t-distribution quantile (which you have to look up) is $t_{0.025,3} = 3.182$. All of this stuff yields
$\mu \in 3 \pm 3.182\sqrt{18/4}3 \pm 6.75 = [-3.75, 9.75]$.
This is choice (b).

c. $[-6.75, 6.75]$

d. $[3.75, 9.75]$

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$\mu \in \bar{X} \pm t_{\alpha/2,n-1}\sqrt{S^2/n}$,

where $n = 4$, the sample mean $\bar{X} = 3$, the sample variance $S^2 = 18$, and $\alpha = 0.05$, so that the t-distribution quantile (which you have to look up) is $t_{0.025,3} = 3.182$. All of this stuff yields
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This is choice (b).

Quiz Score: **9** out of 14