# 0. Course Introduction + Bootcamps

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## Outline of Lessons

1. Syllabus
2. Introduction to Probability and Statistics
3. The Joy of Sets Bootcamp
4. Calculus Bootcamp: Introduction + Derivatives
5. Calculus Bootcamp: Integration and Beyond

## Lesson 0.1 — Syllabus

### ISYE 6739 — Probability and Statistics

**Instructor**: Dave Goldsman
webpage: `www.isye.gatech.edu/~sman`

**Course Objectives**: Provide introduction to probability and statistics, emphasizing applications in science and engineering.

**Prerequisites**: You should be familiar with a spreadsheet package like Excel. You should also know enough calculus to be able to integrate any easy function. But if not, don't panic — we'll have bootcamps for you!

## Syllabus

0. Set Theory and Calculus Bootcamps
1. Getting Started with Probability
2. Random Variables
3. Bivariate Random Variables
4. Distributions
5. Getting Started with Statistics
6. Confidence Intervals
7. Hypothesis Testing
8. Other Goodies (time permitting)

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# Lesson 0.2 — Introduction to Probability and Statistics

## Next Few Lessons:

- This Probability and Statistics Intro
- Set Theory Bootcamp
- Calculus Bootcamps

Mathematical Models for describing observable phenomena:

- Deterministic
- Probabilistic

Deterministic Models

- Ohm's Law ($I = E/R$) (There's no place like Ohm.)
- Drop an object from height $h_0$. After $t$ seconds, height $h(t) = h_0 - 16t^2$.
- Deposit \$1000 in a checking account, continuously compounding at 3%. At time $t$, it's worth \$$1000e^{0.03t}$.

**Probabilistic Models** — Involve uncertainty

- How much snow will fall tomorrow?
- Will IBM make a profit this year?
- Should I buy a call or put option?
- Can I win in blackjack if I use a certain strategy?
- What is the cost-effectiveness of a new drug?
- Which horse will win the Kentucky Derby?

## Some Cool Examples

- Birthday Problem — Assume all 365 days have equal probability of being a person's birthday (ignore freaks born on Feb. 29). Then...

  If there are **23** students in a class, the odds are better than 50–50 that there will be a match.

  If there are **50** students, the probability is about 97%!

- Monopoly — In the long run, the property having the highest probability of being landed on is Illinois Ave.

- Stock Market — Monkeys randomly selecting stocks could have outperformed most market analysts during the past year.

- Poker — Pick 5 cards from a standard deck. Then

$$P(\text{exactly 2 pairs}) \approx 0.0475,$$

$$P(\text{full house}) \approx 0.00144,$$

$$P(\text{flush}) \approx 0.00198.$$

- A couple has two kids and at least one is a boy. What's the probability that BOTH are boys?

  Possibilities: GG, BG, GB, BB. Eliminate GG since we know that there's at least one boy. Then $P(\text{BB}) = 1/3$.

- Ask Marilyn. You are a contestant at a game show. Behind one of three doors is a car; behind the other two are goats. You pick door A. Monty Hall opens door B and reveals a goat. Monty offers you a chance to switch to door C. What should you do? Answer: SWITCH!

- Vietnam Draft Lottery — not as “fair” as you might think!
- Which is the most popular soft drink? Well, in Atlanta, we know the answer to that one!
- Why are some election polls so incredibly wrong?
- How do they do real-time updates of win probabilities as a basketball game progresses?
- How can you simulate randomness on a computer, and what can you use it for?
- How can you tell if the quality of an item that your manufacturing plant is producing has started to get worse?

## Working Definitions

**Probability** — Methodology that describes the random variation in systems. (We'll spend about 50% of our time on this.)

**Statistics** — Uses sample data to draw general conclusions about the population from which the sample was taken. (50% of our time.)

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## Lesson 0.3 — The Joy of Sets Bootcamp

**Definition:** A **set** is a collection of objects. Members of a set are called **elements**.

**Notation:**

$A, B, C, \ldots$ for sets

$a, b, c, \ldots$ for elements

$\in$ for membership, e.g., $x \in A$

$\notin$ for non-membership, e.g., $x \notin A$

$U$ is the **universal set** (i.e., everything)

$\emptyset$ is the **empty set**.

## Examples:

$A = \{1, 2, \ldots, 10\}$. $2 \in A$; $49 \notin A$.

$B = \{$basketball, baseball$\}$

$C = \{x \,|\, 0 \le x \le 1\}$ ("$|$" means "such that")

$D = \{x \,|\, x^2 = 9\} = \{\pm 3\}$ (either is fine)

$E = \{x \,|\, x \in \mathbb{R}, x^2 = -1\} = \emptyset$ ($\mathbb{R}$ is the real line)

**Definition:** If every element of set $A$ is an element of set $B$ then $A$ is a **subset** of $B$, i.e., $A \subseteq B$.

**Definition:** $A = B$ iff (if and only if) $A \subseteq B$ and $B \subseteq A$.

**Properties:**
$\emptyset \subseteq A$; $A \subseteq U$; $A \subseteq A$
$(A \subseteq B$ and $B \subseteq C) \Rightarrow$ (implies) $A \subseteq C$

**Remark:** The order in which the elements of a set are listed is immaterial, e.g., $\{a, b, c\} = \{b, c, a\}$.

**Definitions:** The **complement** of $A$ with respect to $U$ is $\bar{A} \equiv \{x \,|\, x \in U \text{ and } x \notin A\}$.

**Remark**: Don't confuse complement with compliment! ("You are one fine-lookin' set!")

The **intersection** of $A$ and $B$ is $A \cap B \equiv \{x \,|\, x \in A \text{ and } x \in B\}$.

The **union** of $A$ and $B$ is $A \cup B \equiv \{x \,|\, x \in A \text{ or } x \in B \text{ (or both)}\}$.

(Remember **Venn diagrams**?)

**Example:**
Suppose $U = \{\text{letters of the alphabet}\}$, $A = \{\text{vowels}\}^\star$, and $B = \{a, b, c\}$.

$^\star$We'll ignore the fact that $y$ and $w$ are sometimes vowels (believe it or not)!

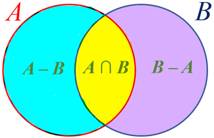
Then

$\bar{A} = \{\text{consonants}\}$
$A \cap B = \{a\}$
$A \cup B = \{a, b, c, e, i, o, u\}$. $\square$

**Definition**: If $A \cap B = \emptyset$, then $A$ and $B$ are **disjoint** (or **mutually exclusive**).

## More Definitions:

**Minus**: $A - B \equiv A \cap \bar{B}$




**Symmetric difference** or **XOR**:

$$A \,\Delta\, B \;\equiv\; (A - B) \cup (B - A) \;=\; (A \cup B) - (A \cap B)$$

The **cardinality** of $A$, denoted by $|A|$, is the number of elements in $A$. $A$ is **finite** if $|A| < \infty$.

**Examples:**

$A = \{3, 4\}$ is finite, since $|A| = 2$.

$B = \{1, 2, 3, \ldots\}$ is **countably infinite**, i.e., $|B| = \aleph_0$ (look up this symbol)!

$C = \{x \,|\, x \in [0, 1]\}$ is **uncountably infinite**, i.e., $|C| = \aleph_1$ (look it up)!

## Laws of Operation:

1. **Complement Law:** $A \cup \bar{A} = U$, $A \cap \bar{A} = \emptyset$, $\bar{\bar{A}} = A$
2. **Commutative:** $A \cup B = B \cup A$, $A \cap B = B \cap A$
3. **Associative:** $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$
4. **Distributive:** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. **DeMorgan's:** $\overline{A \cup B} = \bar{A} \cap \bar{B}$, $\overline{A \cap B} = \bar{A} \cup \bar{B}$

**Proofs:** Easy. Could use Venn diagrams or many other ways.

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## Lesson 0.4 — Calculus Bootcamp: Introduction + Derivatives

**Definition**: The **function** $f(x)$ maps values of $x$ from a certain **domain** $X$ to a certain **range** $Y$, which we can denote by the shorthand $f : X \to Y$.

**Example**: If $f(x) = x^2$, then the function takes $x$-values from the real line $\mathbb{R}$ to the nonnegative portion of the real line $\mathbb{R}^+$.

**Definition**: We say that $f(x)$ is a **continuous** function if, for any $x_0$ and $x \in X$, we have $\lim_{x \to x_0} f(x) = f(x_0)$, where "lim" denotes a **limit** and $f(x)$ is assumed to exist for all $x \in X$.

**Example**: The function $f(x) = 3x^2$ is continuous for all $x$. The function $f(x) = \lfloor x \rfloor$ (round down to the nearest integer, e.g., $\lfloor 3.4 \rfloor = 3$) has a "jump" discontinuity at any integer $x$. $\square$

**Definition**: The **inverse** of a function $f : X \to Y$ is (informally) the "reverse" mapping $g : Y \to X$ such that $f(x) = y$ if and only if $g(y) = x$ for all appropriate $x$ and $y$. The inverse of $f$ is often written as $f^{-1}$, and is especially useful if $f(x)$ is a strictly increasing or strictly decreasing function. Note that $f^{-1}(f(x)) = x$.

**Examples**: If $f(x) = x^3$, then we have $f^{-1}(y) = y^{1/3}$. If $h(x) = e^x$, then $h^{-1}(y) = \ell\text{n}(y)$. $\square$

**Definition**: If $f(x)$ is continuous, then it is **differentiable** (has a **derivative**) if

$$\frac{d}{dx}f(x) \;\equiv\; f'(x) \;\equiv\; \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is well-defined for any given $x$. Think of the derivative as the slope of the function.

**Examples**: Some well-known derivatives are:

$$[x^k]' = kx^{k-1},$$

$$[c^x]' = c^x \,\ell\text{n}(c), \quad \text{and so} \quad [e^x]' = e^x \quad (\text{since } \ell\text{n}(e) = 1),$$

$$[\sin(x)]' = \cos(x),$$

$$[\cos(x)]' = -\sin(x),$$

$$[\ell\text{n}(x)]' = \frac{1}{x},$$

$$[\arctan(x)]' = \frac{1}{1+x^2}. \quad \Box$$

**Theorem**: Some well-known properties of derivatives are:

$$[af(x)+b]' = af'(x),$$

$$[f(x)+g(x)]' = f'(x)+g'(x),$$

$$[f(x)g(x)]' = f'(x)g(x)+f(x)g'(x) \quad \text{(product rule)},$$

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)} \quad \text{(quotient rule)}^1,$$

$$[f(g(x))]' = f'(g(x))g'(x) \quad \text{(chain rule)}^2.$$

[1] Ho dee Hi minus Hi dee Ho over Ho Ho.

[2] www.youtube.com/watch?v=gGAiW5dOnKo

**Example**: Suppose that $f(x) = x^2$ and $g(x) = \ell\mathrm{n}(x)$. Then

$$\begin{aligned}
[f(x)g(x)]' &= f'(x)g(x) + f(x)g'(x) \\
&= 2x\ell\mathrm{n}(x) + x(1/x) \\
&= 2x\ell\mathrm{n}(x) + x,
\end{aligned}$$

$$\begin{aligned}
\left[\frac{f(x)}{g(x)}\right]' &= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \\
&= \frac{[\ell\mathrm{n}(x)]2x - x^2(1/x)}{\ell\mathrm{n}^2(x)} \\
&= \frac{2x\ell\mathrm{n}(x) - x}{\ell\mathrm{n}^2(x)},
\end{aligned}$$

$$[f(g(x))]' = \left[(g(x))^2\right]' = 2g(x)g'(x) = \frac{2\ell\mathrm{n}(x)}{x}. \qquad \square$$

**Remark**: The second derivative $f''(x) \equiv \frac{d}{dx} f'(x)$ and is the "slope of the slope." If $f(x)$ is "position," then $f'(x)$ can be regarded as "velocity," and $f''(x)$ as "acceleration."

The minimum or maximum of $f(x)$ can only occur when the slope of $f(x)$ is zero, i.e., only when $f'(x) = 0$, say at the **critical point** $x = x_0$. Exception: Check the endpoints of your interval of interest as well.

Then if $f''(x_0) < 0$, you get a max; if $f''(x_0) > 0$, you get a min; and if $f''(x_0) = 0$, you get a **point of inflection**.

**Example**: Find the value of $x$ that minimizes $f(x) = e^{2x} + e^{-x}$. The minimum can only occur when $f'(x) = 2e^{2x} - e^{-x} = 0$. After a little algebra, we find that this occurs at $x_0 = -(1/3)\ell\text{n}(2) \approx -0.231$. It's also easy to show that $f''(x) > 0$ for all $x$; and so $x_0$ yields a minimum. $\square$

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## Lesson 0.5 — Calculus Bootcamp: Integration and Beyond

### Integration

**Definition**: The function $F(x)$ having derivative $f(x)$ is called the **antiderivative** (or **indefinite integral**). It is denoted by $F(x) = \int f(x)\,dx$.

**Fundamental Theorem of Calculus:** If $f(x)$ is continuous, then the area under the curve for $x \in [a, b]$ is denoted and given by the **definite integral** [3]

$$\int_a^b f(x)\,dx \;\equiv\; F(x)\Big|_a^b \;\equiv\; F(b) - F(a).$$

[3] "I'm *really* an integral!"

**Example**: Some well-known indefinite integrals are:

$$\int x^k\,dx \;=\; \frac{x^{k+1}}{k+1}+C \;\;\text{for } k\neq -1,$$

where $C$ is an arbitrary constant,

$$\int \frac{dx}{x} \;=\; \ell\mathrm{n}|x|+C,$$

$$\int e^x\,dx \;=\; e^x+C,$$

$$\int \cos(x)\,dx \;=\; \sin(x)+C,$$

$$\int \frac{dx}{1+x^2} \;=\; \arctan(x)+C. \quad \square$$

**Example**: It is easy to see that

$$\int \frac{d\,\text{cabin}}{\text{cabin}} = \ell\text{n}|\text{cabin}| + C$$

$$= \text{houseboat}.$$

**Theorem**: Some well-known properties of definite integrals are:

$$\int_a^a f(x)\,dx = 0,$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

**Theorem**: Some other properties of general integrals are:

$$\int [f(x) + g(x)]\,dx \;=\; \int f(x)\,dx + \int g(x)\,dx,$$

$$\int f(x)g'(x)\,dx \;=\; f(x)g(x) - \int g(x)f'(x)\,dx \quad \text{(integration by parts)}^4,$$

$$\int f(g(x))g'(x)\,dx \;=\; \int f(u)\,du \quad \text{(substitution rule with } u = g(x))^5.$$

[4] www.youtube.com/watch?v=OTzLVIc-O5E
[5] www.youtube.com/watch?v=eswQl-hcvU0

**Example**: To demonstrate integration by parts on a definite integral, let $f(x) = x$ and $g'(x) = e^{2x}$, so that $g(x) = e^{2x}/2$. Then

$$
\begin{aligned}
\int_0^1 xe^{2x}\,dx &= \int_0^1 f(x)g'(x)\,dx \\
&= f(x)g(x)\Big|_0^1 - \int_0^1 g(x)f'(x)\,dx \quad \text{(parts)} \\
&= \frac{xe^{2x}}{2}\Big|_0^1 - \int_0^1 \frac{e^{2x}}{2}\,dx \\
&= \frac{e^2}{2} - \frac{e^{2x}}{4}\Big|_0^1 \\
&= \frac{e^2+1}{4}. \quad \Box
\end{aligned}
$$

**Definition**: Derivatives of arbitrary order $k$ can be written as $f^{(k)}(x)$ or $\frac{d^k}{dx^k}f(x)$. By convention, $f^{(0)}(x) = f(x)$.

The **Taylor series expansion** of $f(x)$ about a point $a$ is given by

$$f(x) \;=\; \sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}.$$

The **Maclaurin series** is simply Taylor expanded around $a = 0$.

**Example**: Here are some famous Maclaurin series:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!},$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

**Example**: And while we're at it, here are some miscellaneous sums that you should know:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

$$\sum_{k=0}^{\infty} p^k = \frac{1}{1-p} \text{ (for } -1 < p < 1).$$

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**Theorem**: Occasionally, we run into trouble when taking indeterminate ratios of the form $0/0$ or $\infty/\infty$. In such cases, **L'Hôspital's Rule**[6] is useful: If the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both go to 0 or both go to $\infty$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

**Example**: L'Hôspital shows that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1. \quad \Box$$

[6] This rule makes me sick.

## Double Integration

We'll have occasion to calculate several double integrals. Whereas our usual (single) integrals get us the area under a curve, double integrals represent the *volume* under a three-dimensional function.

**Example**: The volume under $f(x,y) = 8xy$ over the region $0 < x < y < 1$ is given by

$$\int_0^1 \int_0^y f(x,y)\,dx\,dy = \int_0^1 \int_0^y 8xy\,dx\,dy = \int_0^1 4y^3\,dy = 1. \quad \Box$$

We can usually swap the order of integration to get the same answer:

$$\int_0^1 \int_x^1 8xy\,dy\,dx \;=\; \int_0^1 4x(1-x^2)\,dx \;=\; 1. \quad \Box$$

## Saved by Zero! (How to Solve for a Root)

Suppose that we want to solve some equation $g(r) = 0$ for a root $r^\star$, where $g(r)$ is a nicely behaved continuous function.

We can use:

- trial-and-error or some sort of linear search — that's for losers! ☹
- **bisection method** — pretty fast! ☺
- **Newton's method** — really fast! ☺

**Intermediate Value Theorem (IVT)**: If $g(\ell)g(u) < 0$, then there is a zero $r^\star \in [\ell, u]$. In other words, if (i) $g(\ell) < 0$ and $g(u) > 0$ or (ii) $g(\ell) > 0$ and $g(u) < 0$, then $g(r)$ crosses 0 somewhere between $\ell$ and $u$.

**Bisection** uses the IVT to hone in on a zero via sequential bisecting:

- Initialization: Find lower and upper bounds $\ell_0 < u_0$ such that $g(\ell_0)g(u_0) < 0$. Then the IVT implies that $r^\star \in [\ell_0, u_0]$.
- For $i = 1, 2, \ldots$,
  - Let the midpoint of the current interval be $r_{i+1} \leftarrow (\ell_i + u_i)/2$.
  - If $g(r_{i+1})$ is sufficiently close to 0, or the interval width $u_i - \ell_i$ is sufficiently small, or your iteration budget is exceeded, then set $r^\star \leftarrow r_{i+1}$ and STOP.
  - If the sign of $g(r_{i+1})$ matches that of $g(\ell_i)$, this means that $r^\star \in [r_{i+1}, u_i]$; so set $\ell_{i+1} \leftarrow r_{i+1}$ and $u_{i+1} \leftarrow u_i$. Otherwise, $r^\star \in [\ell_i, r_{i+1}]$; so set $\ell_{i+1} \leftarrow \ell_i$ and $u_{i+1} \leftarrow r_{i+1}$.

Each iteration of the algorithm chops the search area in two and therefore converges to $r^\star$ pretty quickly.

**Example**: Use bisection to find $\sqrt{2}$ by solving $g(x) = x^2 - 2 = 0$.

In order to initialize the bisection algorithm, we note that $g(1) = -1$ and $g(2) = 2$. So there's a zero in $[1, 2]$ just itching to be found!

| step | $\ell_i$ | $g(\ell_i)$ | $u_i$ | $g(u_i)$ | $r_{i+1}$ | $g(r_{i+1})$ |
|---|---|---|---|---|---|---|
| 0 | 1 | −1 | 2 | 2 | 1.5 | 0.25 |
| 1 | 1 | −1 | 1.5 | 0.25 | 1.25 | −0.4375 |
| 2 | 1.25 | −0.4375 | 1.5 | 0.25 | 1.375 | −0.1094 |
| 3 | 1.375 | −0.1094 | 1.5 | 0.25 | 1.4375 | 0.0664 |
| 4 | 1.375 | −0.1094 | 1.4375 | 0.0664 | 1.40625 | −0.0225 |
| $\vdots$ | | | | | | |

You can see that $r_{i+1}$ seems to be converging to $\sqrt{2} \doteq 1.4142$. $\square$

**Newton's method**. It's usually a lot faster than bisection. Here's a reasonable implementation.

1. Initialize $r_0$ as some first guess of the root. Set $j \leftarrow 0$.
2. Update $r_{j+1} \leftarrow r_j - g(r_j)/g'(r_j)$.
3. If $|g(r_{j+1})|$ or $|r_{j+1} - r_j|$ or your budget is suitably small, then STOP and set $r^\star \leftarrow r_{j+1}$. Otherwise, let $j \leftarrow j+1$ and goto Step 2.

**Example**: Use Newton to find $\sqrt{2}$ by solving $g(x) = x^2 - 2 = 0$.

To do so, note that

$$r_{j+1} \leftarrow r_j - \frac{g(r_j)}{g'(r_j)} \;=\; r_{j+1} \leftarrow r_j - \frac{r_j^2 - 2}{2r_j} \;=\; \frac{r_j^2 + 2}{2r_j}.$$

If $r_0 = 1$, then we find that $r_1 = 3/2$, $r_2 = 17/12 \doteq 1.4167$, $r_3 = 577/408 \doteq 1.4142, \ldots$. Wow, that's fast convergence! ☺