Week 10 Homework

Due Mar 29 at 11:59pm **Points** 9 **Questions** 10
Available after Mar 13 at 8am **Time Limit** None

Attempt History

	Attempt	Time	Score
LATEST	Attempt 1	22,313 minutes	5 out of 9

Score for this quiz: **5** out of 9
Submitted Mar 29 at 11:59pm
This attempt took 22,313 minutes.

Question 1

0 / 1 pts

(Lesson 7.11: Composition.) BONUS: It's Raining Cats and Dogs is a pet store with 60% cats and 40% dogs. The weights of cats are Nor(12,4), and the weights of dogs are Nor(30,25). How would we use composition to simulate the weight W of a random pet from the store? (Let $\Phi(\cdot)$ denote the standard normal c.d.f., and let U_i's denote PRN's.)

a. $W = 0.6(12 + 4\Phi^{-1}(U_1)) + 0.4(30 + 25\Phi^{-1}(U_2))$

b. $W = 0.6(12 + 2\Phi^{-1}(U_1)) + 0.4(30 + 5\Phi^{-1}(U_2))$

ou Answered

c. If $U_1 < 0.6$, then $W = 12 + 4\Phi^{-1}(U_2)$; otherwise, $W = 30 + 25\Phi^{-1}(U_2)$

orrect Answer

d. If $U_1 < 0.6$, then $W = 12 + 2\Phi^{-1}(U_2)$; otherwise, $W = 30 + 5\Phi^{-1}(U_2)$

e. If $U_1 < 0.4$, then $W = 12 + 4\Phi(U_2)$; otherwise, $W = 30 + 25\Phi(U_2)$

By inverse transform, the weight of a cat all by itself is $C \sim Nor(12,4) \sim 12 + 2Nor(0,1) \sim 12 + 2\Phi^{-1}(U)$ Similarly, the weight of a dog all by itself is $D = 30 + 5\Phi^{-1}(U)$. These facts eliminate choices (a), (c), and (e). In addition, (a) and (b) are some kind of mutant cat-dog, both of which sort of combine 0.6 of a cat with 0.4 of a dog; so those are wrong. What we really want is to take $W = C$ with probability 0.6, and otherwise $W = D$. This is choice (d)!

Question 2

1 / 1 pts

(Lesson 7.12: The Box-Muller Method.) Suppose U_1 and U_2 are i.i.d. Unif(0,1) with $U_1 = 0.1$ and $U_2 = 0.8$. Use the "cosine" version of Box-Muller to generate a single Nor(-1,4) random variate. Don't forget to use radians instead of degrees!

a. -0.326

b. 0

Correct!

c. 0.326

d. 0.663

e. 1.96

Box-Muller immediately gives the following Nor(0,1) random variate:
$Z = \sqrt{-2\ln(U_1)}\cos(2\pi U_2) = 2.146(0.309) = 0.663.$
To obtain the realization of the Nor($-$1,4), we simply apply the transform
$X = \mu + \sigma Z = -1 + 2Z = -1 + 2(0.663) = 0.326$
which is choice (c).

Question 3

0 / 1 pts

(Lesson 7.13: Generating Order Statistics.) Consider i.i.d. Exp(λ) random variables $X_1, X_2, \ldots, X_n$, and let $Y = \max_i(X_i)$. How can we generate Y using just *one* PRN?

a. $-\frac{1}{\lambda}\ln(1-U)$

b. $-\frac{1}{\lambda}\ln(U)$

orrect Answer

c. $-\frac{1}{\lambda}\ln(1-U^{1/n})$

ou Answered

d. $-\frac{1}{\lambda}\ln(U^{1/n})$

e. $-\left[\frac{1}{\lambda}\ln(1-U)\right]^{1/n}$

Let $F(x) = 1 - e^{-\lambda x}$ denote the Exp(λ) c.d.f. Recall that the inverse is
$F^{-1}(U) = -\frac{1}{\lambda}\ln(1-U) \qquad \clubsuit$
a fact that we'll use in a minute.

Meanwhile, as also explained in class, the c.d.f.\ of Y is

$$\begin{aligned} G(y) &= \Pr(Y \le y) \\ &= \Pr(\max\{X_1, X_2, \ldots, X_n\} \le y) \\ &= \Pr(X_1 \le y, X_2 \le y, \ldots, X_n \le y) \\ &= [\Pr(X_1 \le y)]^n \quad \text{(since the } X_i\text{'s are i.i.d.)} \\ &= [F(y)]^n. \end{aligned}$$

Now, by inverse transform, $G(Y) = U \sim \text{Unif}(0,1)$, and so $[F(Y)]^n = U$. This implies that $F(Y) = U^{1/n}$, and thus $Y = F^{-1}(U^{1/n}) = -\frac{1}{\lambda}\ln(1-U^{1/n})$, where the last equality follows from $\clubsuit$. This is choice (c).

Question 4

1 / 1 pts

(Lesson 7.14: Multivariate Normal Distribution.) Suppose I have a matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$. Find the lower triangular matrix C such that $A = CC'$ and tell me what the entry c_{21} is.

- a. -1
- b. -0.7071 — Correct!
- c. 0
- d. 0.7071
- e. 1

From our class notes on multivariate normal random variate generation, we know that the Cholesky matrix we need is

$$L = \begin{pmatrix} \sqrt{a_{11}} & 0 \\ \frac{a_{12}}{\sqrt{a_{11}}} & \sqrt{a_{22} - \frac{a_{12}^2}{a_{11}}} \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ \frac{-1}{\sqrt{2}} & \sqrt{4 - \frac{1}{2}} \end{pmatrix} = \begin{pmatrix} 1.4142 & 0 \\ -0.7071 & 1.8708 \end{pmatrix}.$$

Therefore, the correct answer is (b).

Question 5

0 / 1 pts

(Lesson 7.15: Baby Stochastic Processes.) BONUS: Consider a Markov chain in which $X_i = 0$ if it rains on day i; and otherwise, $X_i = 1$. Denote the day-to-day transition probabilities by

$$P_{jk} = \Pr(\text{state } k \text{ on day } i \mid \text{state } j \text{ on day } i-1), \quad j,k = 0,1.$$

Suppose that the probability state transition matrix is

$$\mathbf{P} = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix}.$$

Suppose that it rains on Monday, e.g., $X_0 = 0$. Use simulation to find the probability that it rains on Wednesday, e.g., estimate $\Pr(X_2 = 0 | X_0 = 0)$. [You may have to simulate the process a bunch of times in order to estimate this probability.]

a. 0

b. 0.64

orrect Answer

c. 0.72

ou Answered

d. 0.8

e. 1

I'll actually give the analytical solution.

$$\begin{aligned}
&P(X_2 = 0 | X_0 = 0) \\
=&P(X_2 = 0, X_1 = 0 | X_0 = 0) + P(X_2 = 0, X_1 = 1 | X_0 = 0) \\
=&\frac{P(X_2 = 0, X_1 = 0, X_0 = 0)}{P(X_0 = 0)} + \frac{P(X_2 = 0, X_1 = 1, X_0 = 0)}{P(X_0 = 0)} \\
=&P(X_2 = 0 | X_1 = 0)P(X_1 = 0 | X_0 = 0) + P(X_2 = 0 | X_1 = 1)P(X_1 = 1 | X_0 = 0) \\
=&P_{00}P_{00} + P_{01}P_{10} = 0.8^2 + 0.4 \cdot 0.2 = 0.72
\end{aligned}$$

This is answer (c).

Question 6

1 / 1 pts

(Lesson 7.16: Nonhomogeneous Poisson Processes.) Suppose that the arrival pattern to a parking lot over a certain time period is an NHPP with $\lambda(t) = 2t$. Use simulation to find the probability that there will be exactly 3 arrivals between times $t = 0$ and 2.

- a. 0
- b. 0.195 — Correct!
- c. 0.5
- d. 0.805
- e. 1

I'll give the analytical solution. First of all, the number of arrivals in that time interval is

$$N(2) - N(0) \sim \text{Pois}\left(\int_0^2 2t\,dt\right) \sim \text{Pois}(4)$$

Thus,

$$\Pr\left(N(2) - N(0) = 3\right) = \frac{e^{-4}4^3}{3!} = 0.195.$$

This is answer (b).

Question 7

1 / 1 pts

(Lesson 7.17: Time Series Generation.) BONUS: Suppose that $Y_0 \sim \text{Nor}(0,1)$ and consider the time series $Y_i = 0.7Y_{i-1} + \epsilon_i$, $i = 1, 2, \ldots$, where the ϵ_i's are i.i.d. Nor$\left(0, 1 - (0.7)^2\right)$. (The funny variance of ϵ_i guarantees that $Var(Y_i) = 1$ for all i). Use simulation to find $Cov(Y_2, Y_5)$. Hint: Simulate $Y_0, Y_1, \ldots, Y_5$ many times. For each run of the simulation, save the pair (Y_2, Y_5). Then use those pairs to estimate the covariance.

- a. 0
- b. 0.7
- c. 0.49
- Correct! d. 0.7^3
- e. 0.7^4

By class notes, the analytical answer is $(0.7)^{5-2}$, so that (d) is correct.

Question 8

1 / 1 pts

(Lesson 7.19: Brownian Motion.) Let $\mathcal{W}(t)$ denote a Brownian motion process at time t. Calculate $\text{Cov}(\mathcal{W}(3),\ \mathcal{W}(5))$.

a. 0

b. 2

Correct!

c. 3

d. 5

e. 8

We have $\text{Cov}(\mathcal{W}(3),\ \mathcal{W}(5)) = \min\{3, 5\} = 3$, so that the answer is (c).

Question 9

0 / 1 pts

(Lesson 7.19: Brownian Motion.) Let $\mathcal{W}(t)$ denote a Brownian motion process at time t and define a Brownian bridge by$\mathcal{B}(t) = \mathcal{W}(t) - t\mathcal{W}(1)$ for $0 < t < 1$. Find the variance of the area under a bridge, i.e., $Var\left(\int_0^1 \mathcal{B}(t)\, dt \right)$. I'm a nice guy, so I'll get you started...

$$\begin{aligned} Var\Big(\int_0^1 \mathcal{B}(t)\,dt\Big) &= Cov\Big(\int_0^1 \mathcal{B}(s)\,ds,\ \int_0^1 \mathcal{B}(t)\,dt\Big) \\ &= \int_0^1\int_0^1 Cov\big(\mathcal{B}(s),\ \mathcal{B}(t)\big)\,ds\,dt \end{aligned}$$

a. -1/2

b. 0

orrect Answer

c. 1/12

d. 1/2

ou Answered

e. 1

Solution: We have

$$\begin{aligned} \mathsf{Cov}(\mathcal{B}(s),\ \mathcal{B}(t)) &= \mathsf{Cov}\Big(\mathcal{W}(s) - s\mathcal{W}(1),\ \mathcal{W}(t) - t\mathcal{W}(1)\Big) \\ &= \mathsf{Cov}(\mathcal{W}(s),\ \mathcal{W}(t)) - t\,\mathsf{Cov}(\mathcal{W}(s),\ \mathcal{W}(1)) \\ &\quad - s\,\mathsf{Cov}(\mathcal{W}(1),\ \mathcal{W}(t)) + st\,\mathsf{Var}(\mathcal{W}(1)) \\ &= \min(s,t) - t\min(s,1) - s\min(1,t) + st \\ &= \min(s,t) - ts - st + st \ = \ \min(s,t) - st. \end{aligned}$$

Plugging into the Hint, we have

$$\begin{aligned} \mathsf{Var}\Big(\int_0^1 \mathcal{B}(t)\,dt\Big) &= \mathsf{Cov}\Big(\int_0^1 \mathcal{B}(s)\,ds,\ \int_0^1 \mathcal{B}(t)\,dt\Big) \\ &= \int_0^1\int_0^1 \mathsf{Cov}(\mathcal{B}(s),\ \mathcal{B}(t))\,ds\,dt \\ &= \int_0^1\int_0^1 \big[\min(s,t) - st\big]\,ds\,dt \\ &= \int_0^1\int_0^1 \min(s,t)\,ds\,dt - \int_0^1\int_0^1 st\,ds\,dt \\ &= \int_0^1\int_0^t s\,ds\,dt - \frac{1}{4} \quad \text{(by symmetry)} \\ &= 1/12. \end{aligned}$$

Thus, the answer is (c). $\square$

Question 10

0 / 0 pts

(Lesson 7.19: Brownian Motion.) As we discussed in class, you can use Brownian motion to estimate option prices for stocks. I'm not going to have you simulate that, but I'm going to give you a quick look-up assignment. As I write this on Aug 16, 2019, IBM is currently selling for about $134 per share. Suppose I'm interested in guaranteeing that I can buy a share of IBM for at most $145 on Nov. 15, 2019. Look up (maybe using something like FaceTube on the internets) the corresponding stock option price. Did you look up the answer? (Just write "YES".)

As I was writing this solution sheet, the option price was $2.02 --- but this is obviously subject to change depending on how the market does.

Correct!

- a. Yes
- b. No

Quiz Score: **5** out of 9