Image Inpainting Based on Coherence Transport

Tom März

Joint work with Folkmar Bornemann (TUM)

Oxford Centre for Collaborative Applied Mathematics

maerz@maths.ox.ac.uk

Oxford Conference on Challenges in Applied Mathematics

4 July 2013

What is Image Inpainting?

For some people it is this ...

Restoration work on Camposanto in Pisa

Inpainting: Removing undesired picture elements

437×296 px

$\rightarrow$

inpainted, CPU-time: 1 sec

given:

i) Region $\Omega \subset \Omega_0$

ii) Image u_0 on $\Omega_0 \setminus \Omega$

find:

i) Interpolant u on Ω

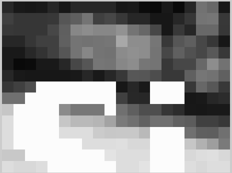
ii) Completed image

$$\mathbb{1}_{\Omega} \cdot u + \mathbb{1}_{\Omega_0 \setminus \Omega} \cdot u_0$$

shall look nice

Outline

- Telea's Basic Algorithm
- Analysis of the Basic Algorithm
- Improvement 1: Coherence Transport
- Improvement 2: Distance Adaption



Gray scale image

$\longleftrightarrow$

40	25	23	25	39	23	39	39	39	39	23	25	11	9	25	9	38	114	104	23
64	54	54	54	56	86	95	116	64	71	56	39	39	23	23	39	55	116	118	56
56	54	54	54	71	95	135	141	135	135	146	116	103	71	23	56	38	104	104	55
39	39	54	64	64	96	116	114	116	118	168	141	141	118	56	86	95	71	55	23
25	25	40	51	64	86	116	135	116	114	139	144	141	116	71	96	81	71	88	56
25	9	11	25	25	39	71	86	118	116	135	141	118	116	56	71	71	104	141	114
39	25	23	23	23	23	25	39	25	39	56	87	86	77	56	56	96	118	114	96
54	64	78	252	252	252	252	252	252	252	23	39	25	252	252	252	56	103	87	56
112	112	252	252	252	252	252	252	252	252	64	40	39	252	252	252	23	25	39	44
198	252	252	252	252	139	119	119	119	252	112	95	78	78	54	44	24	23	25	25
221	252	252	252	252	198	185	185	181	163	122	112	97	112	112	97	92	64	64	56
221	252	252	252	252	221	221	221	203	199	200	200	185	252	252	252	112	95	94	112
221	252	252	252	252	252	252	221	221	221	223	216	216	252	252	252	198	177	158	122
200	200	252	252	252	252	252	252	252	227	221	221	221	252	252	252	216	221	223	216
221	216	216	221	252	252	252	252	252	252	221	221	227	252	252	252	221	216	221	221

Digital image is a matrix

Digital images!

- Digital image is a matrix $u_h : \Omega_h \to \{0, \ldots, 255\}$
- Gray scales $\{0, \ldots, 255\} \leftrightarrow \{\text{black}, \ldots, \text{white}\}$
- Pixel domain $\Omega_h = \{1, \ldots, N\} \times \{1, \ldots, M\}$

Continuous images?

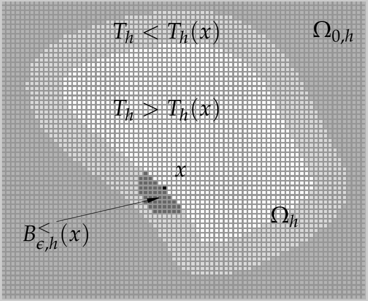
- Continuous image is a function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$
- Digital u_h $=$ discrete version of continuous u

Basic Algorithm

1. Serialize pixels:
   - From boundary inwards (onion peeling !)
   - induced by

     $$T_h(x_j) < T_h(x_k) \quad \Rightarrow \quad j < k$$

   - e.g. $T_h = \mathrm{dist}(., \partial\Omega_h)$



2. Inpaint along pixel sequence:

$$u_h(x_k) = \frac{\sum\limits_{y \in B^<_{\varepsilon,h}(x_k)} w(x_k, y) u_h(y)}{\sum\limits_{y \in B^<_{\varepsilon,h}(x_k)} w(x_k, y)} \quad \forall x_k \in \Omega_h$$

Telea (2004)

- $T_h(x) = \mathrm{dist}(x, \partial\Omega_h)$
- $w(x,y) = \frac{1}{|x-y|} \cdot \left| \left\langle \nabla_h T_h(x),\ \frac{x-y}{|x-y|} \right\rangle \right|$

241 × 159 px

$\rightarrow$

Telea's method $\varepsilon = 5$, CPU-time: 0.1 sec

Telea’s Choice

Telea (2004)

- $T_h(x) = \mathrm{dist}(x, \partial\Omega_h)$
- $w(x,y) = \frac{1}{|x-y|} \cdot \left| \left\langle \nabla_h T_h(x),\ \frac{x-y}{|x-y|} \right\rangle \right|$

300 × 180 px

$\longrightarrow$

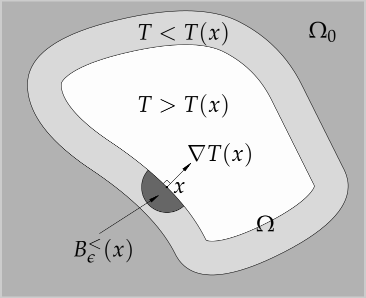
Telea’s method $\varepsilon = 6$, CPU-time: 0.05 sec

Step 1: $h \to 0$ (pixel size $\to 0$)

1. Serialization ?

   hidden in map T

   onion peels = level lines



2. For all $x \in \Omega$:

$$u(x) = \frac{\int_{B_\varepsilon^<(x)} w(x,y)u(y)dy}{\int_{B_\varepsilon^<(x)} w(x,y)dy}$$

$$\frac{1}{\pi\varepsilon^2} \int_{B_\varepsilon^<(x)} (u(x) - u(y))w(x,y)dy = 0 \qquad (IE)$$

Step 2 : $\varepsilon \to 0$ (Localization)

Theorem: if $w(x,y) = |x-y|^{-1} \cdot k(x, \epsilon^{-1}(x-y))$
and $\nabla T(x)$ non-singular

1. IE $\longrightarrow$ PDE: $\langle c(x), \nabla u(x) \rangle = 0,$
2. Causality: $\langle c(x), \nabla T(x) \rangle \geqslant \beta |\nabla T(x)|\,, \quad \beta = \frac{k_{\min}}{\sqrt{2} k_{\max}}$

Induces map $w \mapsto c$

Telea's method:

$T_h = \mathrm{dist}(., \partial\Omega_h)$,

$w(x,y) = \frac{1}{|x-y|} \cdot \left| \left\langle \nabla_h T_h(x),\ \frac{x-y}{|x-y|} \right\rangle \right|$

Apply analysis to Telea's method:

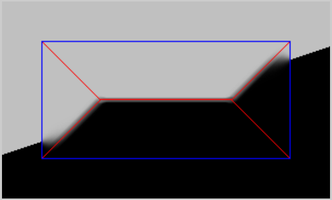
get $\quad w \to c = \frac{1}{4} \cdot \nabla T$

PDE $\quad \langle \nabla T(x), \nabla u(x) \rangle = 0, \quad u|_{\partial\Omega} = u_0$

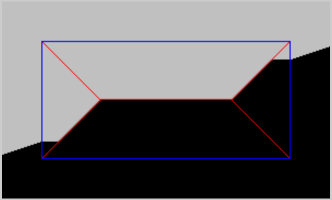
solution $\quad u(x) = u_0(\ cp(x)\ )$

Observe:

propagation of information depends **only on geometry of domain** not on image



$\varepsilon = 6$



$\varepsilon = 1$

The painter/artist ODE: brush strokes as trajectories

$$\begin{aligned} x' &= c(x)\,, & x(0) &= x_0 \in \partial\Omega \\ u' &= 0\,, & u(0) &= u_0(x_0) \end{aligned}$$

gives characteristics of

$$\langle c(x), \nabla u(x) \rangle = 0\,, \quad x \in \Omega \setminus \Sigma\,, \quad u|_{\partial\Omega} = u_0$$

Action of painter depends on u:

$c[u]$ should depend on u $\Rightarrow$ $w[u]$ should depend on u

(One) Goal of Inpainting: Continue broken color lines!

- $\rightsquigarrow$ Guidance $g[u](x)$: estimated tangent to color line.
- $\rightsquigarrow$ Estimation: structure tensor analysis (Weickert 1998, Aach et al 2006)

wanted: $\langle g, \nabla u(y) \rangle^2 \approx 0 \quad \text{in} \quad \mathcal{N}(x)$

least squares: $g^T \cdot S[u](x) \cdot g \to \min \,, \quad |g| = 1$

$$S[u](x) = \int K_\rho(x, x-y) \nabla u(y) \nabla u(y)^T \, dy$$

$$\Rightarrow \qquad g[u](x) = \text{ minimal eigenvector of } S[u](x)$$

Wanted:

1. Reuse basic algorithm of Telea,
2. Transport along g, (g guidance),
3. Regularization, if necessary

1 & 2 is : w with $w \mapsto c$ and $c \,||\, g$

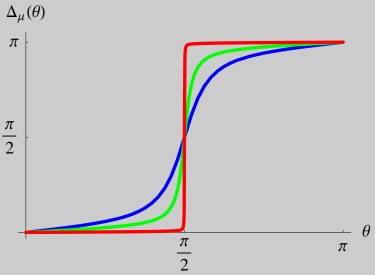
Proposal (Bornemann/M. 2007):

$$w_\mu(x,y) = |x-y|^{-1} \cdot k_\mu(x, \epsilon^{-1}(x-y))$$

$$k_\mu(x,\xi) = \mu \exp\left(-\frac{\mu^2}{2}\left\langle g^\perp(x), \xi\right\rangle^2\right)$$

Transport along g possible?

$$\Delta_\mu(\theta) := \angle(g, c_\mu)\ , \quad \theta := \angle(g, \nabla T)$$
$$\theta = \pi/2 \quad \Leftrightarrow \nabla T \perp g$$

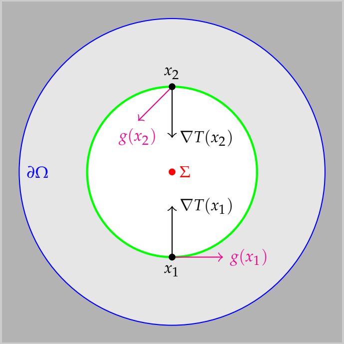


$\mu = 5$, $\mu = 10$, $\mu = 100$

$$\lim_{\mu \to \infty} c_\mu(x) \parallel \begin{cases} \pm g(x), & \theta \lessgtr \pi/2 \\ \nabla T(x), & \theta = \pi/2 \end{cases}$$

Regularization!

Transport along g may be ill-posed $\theta = \pi/2$



Transport along c_μ allways well-posed

$$\angle(c_\mu, \nabla T) \neq \frac{\pi}{2}\ , \qquad \langle c_\mu, \nabla T \rangle \geqslant \beta_\mu > 0$$

Comparison

ε	radius
μ	penalizer
σ, ρ	struc. tens.
T_h	$= \mathrm{dist}(., \partial\Omega_h)$

Telea's Method

Coherence Transport

$\varepsilon = 6$

$\varepsilon = 6, \mu = 50, \sigma = 5, \rho = 100$

Comparison 2

ε	radius
μ	penalizer
σ, ρ	struc. tens.
T_h	$= \mathrm{dist}(., \partial\Omega_h)$

Telea's Method $\swarrow$

Coherence Transport $\searrow$

$\varepsilon = 5$, CPU-time: 0.1 sec

$\varepsilon = 5$, $\mu = 25$, $\sigma = 1.4$, $\rho = 4$, CPU-time: 0.2 sec

Telea's Method

$\varepsilon = 5$

Coherence Transport

$\varepsilon = 5, \mu = 25, \sigma = 1.4, \rho = 4$

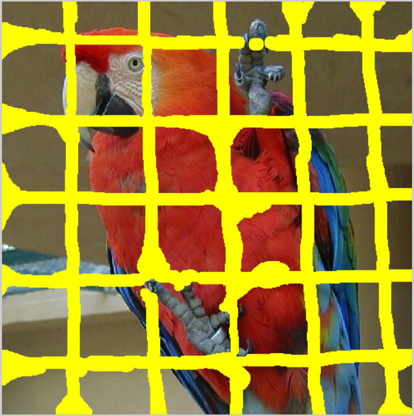
Uncage the parrot

495×498 px

$\longrightarrow$

inpainted, CPU-time: 20 sec

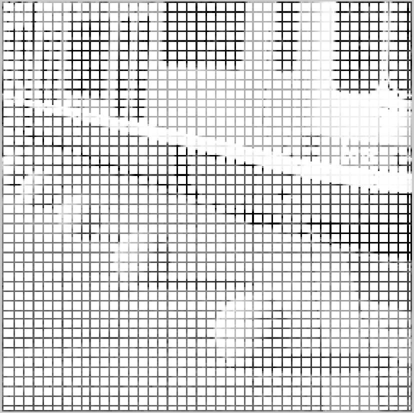
Uncage the parrot


masked, 495 × 498 px

→


inpainted, CPU-time: 20 sec

Image interpolation

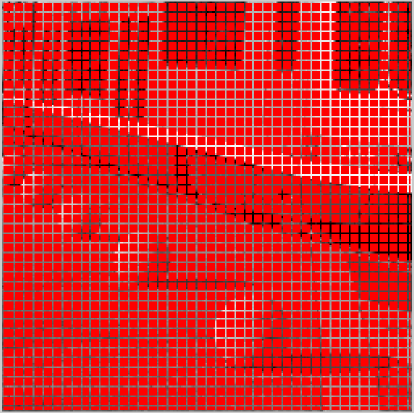


76% dropped, 254×254 px

$\rightarrow$

inpainted, CPU-time: 1 sec

Image interpolation

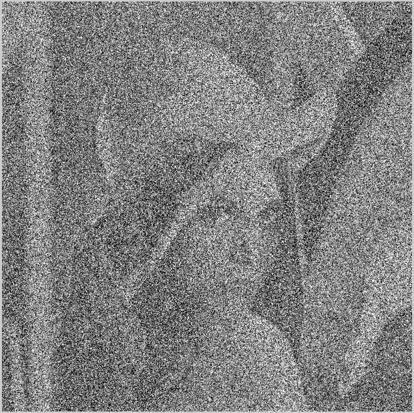


masked, 254×254 px

$\rightarrow$

inpainted, CPU-time: 1 sec

Denoising



80% salt & pepper noise, 1024×1024 px

$\longrightarrow$

inpainted, CPU-time: 16 sec

Denoising

masked, 1024×1024 px

$\longrightarrow$

inpainted, CPU-time: 16 sec

T tells / yields:

1. Start: $\partial\Omega = \{x : T(x) = 0\}$
2. Inside: ∇T points inwards
3. Stop: $\{x : T(x) = \text{maximal}\} \subseteq \Sigma$

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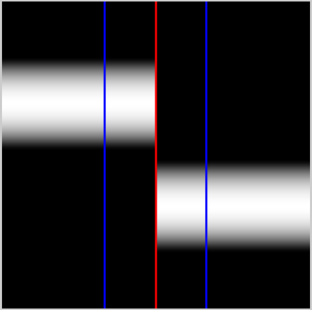
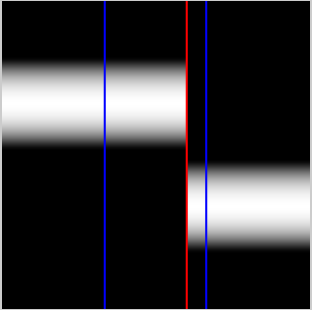
4. Unique solution, $(c = e_1)$



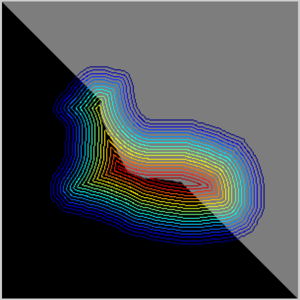


Coherence Transport Inpainting of a broken diagonal

using $T = \mathrm{dist}(., \partial\Omega)$

Problem

Result



Contours of T

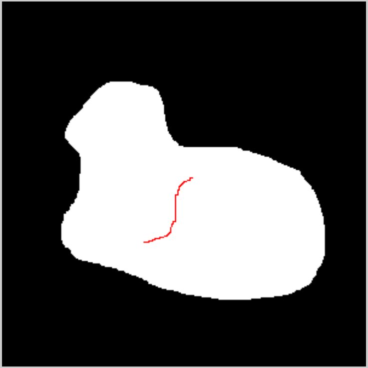
Only condition: T must induce onion peeling

set $T = 0$ on $\partial\Omega$

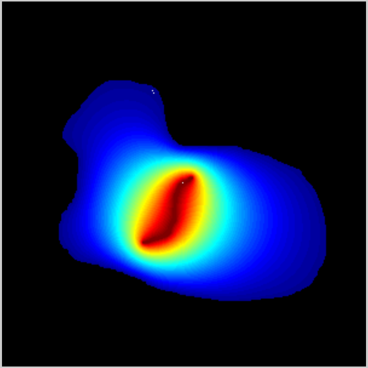
$T = 1$ on Σ

solve

$\Delta T = 0$ in Ω



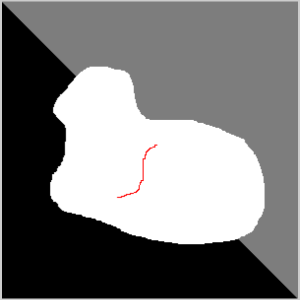
White: Ω, Red: Σ



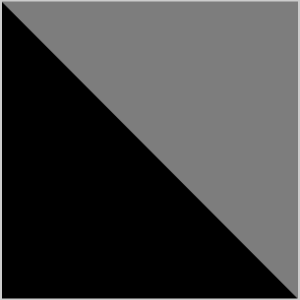
Heat plot of T

Coherence Transport Inpainting of a broken diagonal

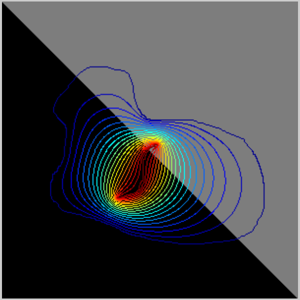
using new T



Problem



Result



Contours of T

And It's Helpful in Practice

And It's Helpful in Practice

$T = \mathrm{dist}(., \partial\Omega)$

T from harmonic interpolation

- F. Bornemann, T. März, *Fast Image Inpainting Based on Coherence Transport*, Journal of Mathematical Imaging and Vision, 28, pp. 259-278, 2007.
- T. März, *Image Inpainting Based on Coherence Transport with Adapted Distance Functions*, SIAM Journal on Imaging Sciences, 4, pp. 981-1000, 2011.
- T. März, *A Well-posedness Framework for Inpainting Based on Coherence Transport*, submitted, 2013.

- Code available at `https://github.com/maerztom/inpaintBCT`