Numerical Analysis Summer Term 2014
Lecture 1: Lagrange Interpolation

This lecture is adapted from the numerical analysis textbook by Süli and Mayers, Chapter 6.

Notation: $\Pi_n = \{\text{real polynomials of degree} \leq n\}$

Setup: given data f_i at distinct x_i, $i = 0, 1, \ldots, n$, with $x_0 < x_1 < \cdots < x_n$, can we find a polynomial p_n such that $p_n(x_i) = f_i$? Such a polynomial is said to **interpolate** the data.

E.g.: $n = 1$,

linear constant

Theorem. $\exists p_n \in \Pi_n$ such that $p_n(x_i) = f_i$ for $i = 0, 1, \ldots, n$.

Proof. Consider, for $k = 0, 1, \ldots, n$, the "cardinal polynomial"

$$L_{n,k}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} \in \Pi_n. \tag{1}$$

Then

$$L_{n,k}(x_i) = 0 \text{ for } i = 0, \ldots, k-1, k+1, \ldots, n \text{ and } L_{n,k}(x_k) = 1.$$

So now define

$$p_n(x) = \sum_{k=0}^{n} f_k L_{n,k}(x) \in \Pi_n \tag{2}$$

$\Longrightarrow$

$$p_n(x_i) = \sum_{k=0}^{n} f_k L_{n,k}(x_i) = f_i \text{ for } i = 0, 1, \ldots, n. \qquad \square$$

The polynomial (2) is the **Lagrange interpolating polynomial**.

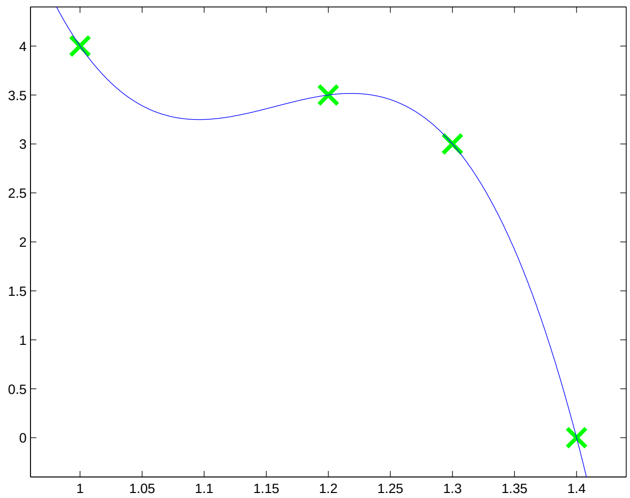
Theorem. The interpolating polynomial of degree $\leq n$ is unique.

Proof. Consider two interpolating polynomials $p_n, q_n \in \Pi_n$. Their difference $d_n = p_n - q_n \in \Pi_n$ satisfies $d_n(x_k) = 0$ for $k = 0, 1, \ldots, n$. i.e., d_n is a polynomial of degree at most n but has at least $n + 1$ distinct roots. Algebra $\Longrightarrow$ $d_n \equiv 0 \Longrightarrow p_n = q_n$. $\square$

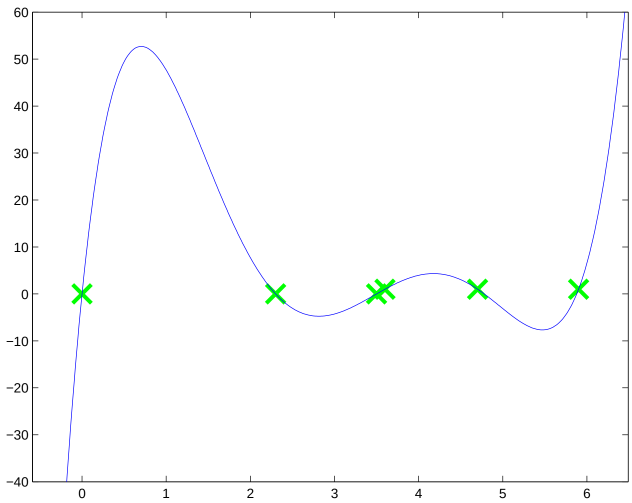
Matlab:

```
>> help lagrange
 LAGRANGE  Plots the Lagrange polynomial interpolant for the
 given DATA at the given KNOTS

>> lagrange([1,1.2,1.3,1.4],[4,3.5,3,0]);
```




```
>> lagrange([0,2.3,3.5,3.6,4.7,5.9],[0,0,0,1,1,1]);
```



Data from an underlying smooth function: Suppose that $f(x)$ has at least $n+1$ smooth derivatives in the interval (x_0, x_n). Let $f_k = f(x_k)$ for $k = 0, 1, \ldots, n$, and let p_n be the Lagrange interpolating polynomial for the data (x_k, f_k), $k = 0, 1, \ldots, n$.

Error: how large can the error $f(x) - p_n(x)$ be on the interval $[x_0, x_n]$?

Theorem. For every $x \in [x_0, x_n]$ there exists $\xi = \xi(x) \in (x_0, x_n)$ such that

$$e(x) \stackrel{\text{def}}{=} f(x) - p_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n) \frac{f^{(n+1)}(\xi)}{(n+1)!},$$

where $f^{(n+1)}$ is the $(n+1)$-st derivative of f.

Proof. Trivial for $x = x_k$, $k = 0, 1, \ldots, n$ as $e(x) = 0$ by construction. So suppose $x \neq x_k$. Let

$$\phi(t) \stackrel{\text{def}}{=} e(t) - \frac{e(x)}{\pi(x)} \pi(t),$$

where

$$\begin{aligned}\pi(t) &\stackrel{\text{def}}{=} (t-x_0)(t-x_1)\cdots(t-x_n)\\ &= t^{n+1} - \left(\sum_{i=0}^{n} x_i\right) t^n + \cdots (-1)^{n+1} x_0 x_1 \cdots x_n\\ &\in \Pi_{n+1}.\end{aligned}$$

Now note that ϕ vanishes at $n+2$ points x and x_k, $k = 0, 1, \ldots, n$. $\Longrightarrow$ ϕ' vanishes at $n+1$ points $\xi_0, \ldots, \xi_n$ between these points $\Longrightarrow$ ϕ'' vanishes at n points between these new points, and so on until $\phi^{(n+1)}$ vanishes at an (unknown) point ξ in (x_0, x_n). But

$$\phi^{(n+1)}(t) = e^{(n+1)}(t) - \frac{e(x)}{\pi(x)}\pi^{(n+1)}(t) = f^{(n+1)}(t) - \frac{e(x)}{\pi(x)}(n+1)!$$

since $p_n^{(n+1)}(t) \equiv 0$ and because $\pi(t)$ is a monic polynomial of degree $n+1$. The result then follows immediately from this identity since $\phi^{(n+1)}(\xi) = 0$.

□

Example: $f(x) = \log(1+x)$ on $[0,1]$. Here, $|f^{(n+1)}(\xi)| = n!/(1+\xi)^{n+1} < n!$ on $(0,1)$. So $|e(x)| < |\pi(x)| n!/(n+1)! \leq 1/(n+1)$ since $|x - x_k| \leq 1$ for each x, x_k, $k = 0, 1, \ldots, n$, in $[0,1] \Longrightarrow |\pi(x)| \leq 1$. This is probably pessimistic for many x, e.g. for $x = \frac{1}{2}$, $\pi(\frac{1}{2}) \leq 2^{-(n+1)}$ as $|\frac{1}{2} - x_k| \leq \frac{1}{2}$.

This shows the important fact that the error can be large at the end points, an effect known as the "Runge phenomena" (Carl Runge, 1901). There is a famous example due to Runge, where the error from the interpolating polynomial approximation to $f(x) = (1+x^2)^{-1}$ for $n+1$ equally-spaced points on $[-5, 5]$ diverges near ± 5 as n tends to infinity: try `runge` from the website in Matlab.

Building Lagrange interpolating polynomials from lower degree ones.

Notation: Let $Q_{i,j}$ be the Lagrange interpolating polynomial at x_k, $k = i, \ldots, j$.

Theorem.

$$Q_{i,j}(x) = \frac{(x - x_i)Q_{i+1,j}(x) - (x - x_j)Q_{i,j-1}(x)}{x_j - x_i} \tag{3}$$

Proof. Let $s(x)$ denote the right-hand side of (3). Because of uniqueness, we simply wish to show that $s(x_k) = f_k$. For $k = i+1, \ldots, j-1$, $Q_{i+1,j}(x_k) = f_k = Q_{i,j-1}(x_k)$, and hence

$$s(x_k) = \frac{(x_k - x_i)Q_{i+1,j}(x_k) - (x_k - x_j)Q_{i,j-1}(x_k)}{x_j - x_i} = f_k.$$

We also have that $Q_{i+1,j}(x_j) = f_j$ and $Q_{i,j-1}(x_i) = f_i$, and hence

$$s(x_i) = Q_{i,j-1}(x_i) = f_i \ \text{ and } \ s(x_j) = Q_{i+1,j}(x_j) = f_j.$$

□

Comment: this can be used as the basis for constructing interpolating polynomials. In books: may find topics such as the Newton form and divided differences.

Generalisation: **Hermite interpolating polynomial** matches function data and derivative data. Can also be constructed in terms of $L_{n,k}$.