

# Appendix

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Because the methods applied here: Stochastic Dynamic Programming, Iterative Forecasts assessed with Proper Scoring Rules, or Adaptive Management, are not themselves novel, the reader is referred to previous reviews and text books which can provide a thorough introduction to methods, in particular, Marescot *et al.* (2013) and Smith & Walters (1981). However, implementation of these approaches generally depends on numerical methods, for which even precise mathematical formulas or pseudo-code descriptions can be inadequate to specify unambiguously or to properly facilitate replication (Barnes 2010). As such, this appendix includes computer code in the R language, which can also be found in the corresponding R-Markdown document in the paper’s GitHub repository, <https://github.com/cboettig/bad-forecast-good-decision>. While computer code can be more verbose and difficult to read than mathematical formulas presented here, it is also less ambiguous. Though R is already widely used among ecologists, the implementations concern mostly matrix algebra that can be translated to other languages, and the `MDPtoolbox` package used here is also implemented in MatLab (Marescot *et al.* 2013; Chades *et al.* 2017).

Below, I provide carefully annotated code necessary to completely reproduce all of the analysis presented in the main paper. This analysis is run in R (R Core Team 2020) uses `MDPtoolbox` (Chades *et al.* 2017) for solving Markov Decision Processes (MDP) using stochastic dynamic programming functionality, `expm` for matrix exponentials (Goulet *et al.* 2020), and a few custom MDP functions provided by our package, `mdplearning` (Boettiger & Memarzadeh 2020). We will also use `tidyverse` packages for basic manipulation and plotting (Wickham *et al.* 2019). This file is also available as an RMarkdown document (Xie *et al.* 2018) at <https://github.com/cboettig/bad-forecast-good-decision>. The code below generates the data tables required for each plot shown in the main paper. Data are stored as `.csv` tables to be accessible to any relevant software program. Plotting commands used in the main paper are embedded in the paper’s RMarkdown file at <https://github.com/cboettig/bad-forecast-good-decision>.

```
library(tidyverse)
library(MDPtoolbox)
library(expm)
# remotes::install_github("boettiger-lab/mdplearning")
library(mdplearning)
```

## Optimal quotas as a Markov Decision Process (MDP) problem

The question of optimal harvest quotas given a population growth model can be posed as a Markov Decision Process (MDP). Marescot *et al.* (2013) provides an accessible overview of MDP problems and their solutions in conservation context. Reed (1979) provides a formal proof of optimal harvest strategy for stochastic dynamics under the assumption of any concave growth function. The problem can be summarized as follows:

A manager seeks to maximize the sum of the utility derived from such a harvest and such a state,  $U(X_t, H_t)$ , over all time, subject to discount rate  $\delta$ :

$$\sum_{t=0}^{t=\infty} U(X_t, H_t) \delta^t \quad (1)$$

While in principle the utility could reflect many things, including the cost of fishing, market responses to supply and demand, the value of recreational fishing, the intrinsic value fish left in the sea (see Halpern *et al.* 2013), for simplicity we will assume utility is merely a linear function of the harvest quota set by the manager, i.e. a fixed price  $p$  per kilogram of fish harvested:  $U(X_t, H_t) = p \min(H_t, X_t)$  (noting that realized harvest cannot exceed the available stock). As units are already specified in dimensionless quantities in this example, so without loss of generality we will set  $p = 1$ .

To solve this optimization, we must also know the dynamics of  $X_t$ , describing how future stock  $X_{t+1}$  changes from the present state,  $X_t$ , after to harvest  $H_t$ :

$$X_{t+1} = f(X_t, H_t) \quad (2)$$

This problem is already well studied, and it is worth noting that even under the simple utility function which assigns no value to fish that are not harvested, the optimal strategy still seeks to sustain the fish population indefinitely; as Clark (1973) shows for the deterministic function  $f$  and Reed (1979) extended to the stochastic case. As Reed (1979) proves, the optimal strategy can be characterized by a policy of “constant escapement,” in which the manager adjusts the harvest effort each year in an effort to ensure the same number or biomass of fish are left in the sea each year. The optimization problem can then be solved by dynamic programming, using the recursive Bellman equation. This technique, known as stochastic dynamic programming (SDP) is widely used in conservation decision problems Marescot *et al.* (2013).

It is worth noting that while constant escapement is used in the management of several important fish stocks such as salmon, most marine fisheries are not managed using this optimal solution for a fluctuating population, but instead rely on a ‘constant harvest effort’ (or constant “yield”) policy. In the case of a deterministic model at equilibrium, these policies are identical. Though constant-effort policies are technically sub-optimal for the problem as stated above with stochastic growth, in practice they can be more robust, such as when measurements of the current stock size  $X_t$  are uncertain. These issues are discussed in detail in Memarzadeh & Boettiger (2019). For our purposes, it suffices to note that constant escapement policy is relevant both from a theoretical standpoint as the optimal solution for the problem under consideration, and an applied standpoint as a policy that guides management of many salmon fisheries. The optimal control approach using SDP as illustrated here is used in a wide and growing number of conservation problems.

Bear in mind that in the example presented here, SDP merely serves the role of a convenient and established way for a manager to determine the optimal action, given some model estimation. Whether we use SDP or some other way to determine the optimal policy, given the model, is immaterial to the conclusion.

## Solving MDP via SDP

To solve the decision problem using SDP, we define the state space and the action space on a discrete grid of 240 points. The maximum state is set well above the largest carrying capacity used in the models, which limits the influence of boundary effects introduced by the transformation to a discrete, finite grid. Available harvest actions match the state space, effectively allowing any harvest level to be possible.

```
states <- seq(0,24, length.out = 240)
actions <- states
obs <- states
```

The utility (reward) of an action is set to the (realized) harvest (e.g. a fixed price per unit fish, with no cost applied to harvest effort). Future utility is discounted by fixed factor of 0.99. Classic maximum sustainable yield models (Schaefer 1954) ignore discounting, while modern economic optimization models insist on it, so a small discount conforms to the latter while reasonably approximating the former. The qualitative conclusion is not sensitive to the discounting rate.

```
reward_fn <- function(x,h) pmin(x,h)
discount <- 0.99
```

Alternate reward functions with varying price and cost structures are possible but do not qualitatively impact the conclusions. This reward function is a limiting case of any more complex reward, and corresponds both to classic work that does not model utility explicitly (e.g. MSY theory, Schaefer 1954), as well as explicit assumptions typically made in more recent models (Reed 1979). Moreover, using a simple reward function makes it clear that model that performs best is not doing so merely because of particular features baked into the a carefully chosen reward rule.

## Ecological Model Definitions

The manager chooses between two different logistic growth models, each of which can be thought of as an approximation to the underlying “true” population model:

$$f_i(Y) = Y + Yr_i \left(1 - \frac{Y}{K_i}\right) \xi_i(t) \quad (3)$$

where  $\xi_i(t)$  is a multiplicative log-normal noise term with mean 1 and log-standard-deviation  $\sigma_i$ .

Model 1 has  $r_1 = 2$ ,  $K = 16$ , and  $\sigma_1 = 0.05$ . Model 2 has  $r_2 = 0.5$ ,  $K_2 = 10$ , and  $\sigma_2 = 0.075$ .

Meanwhile, the true population growth rate is simulated using a function with non-linear per-capita growth:

$$f_i(Y) = Y + Y^4 r_i \left(1 - \frac{Y}{K_i}\right) \quad (4)$$

with  $r_3 = 0.002$ ,  $K_3 = 10$  and  $\sigma_3 = 0.05$ .

```
# K is at twice max of f3; 8 * K_3 / 5
f1 <- function(x, h = 0, r = 2, K = 10 * 8 / 5) {
  s <- pmax(x - h, 0)
  s + s * (r * (1 - s / K))
}

f2 <- function(x, h = 0, r = 0.5, K = 10) {
  s <- pmax(x - h, 0)
  s + s * (r * (1 - s / K))
}

# max is at 4 * K / 5
f3 <- function(x, h = 0, r = .002, K = 10) {
  s <- pmax(x - h, 0)
  s + s^4 * r * (1 - s / K)
}

## gather models together, indicate true model
sigma_g <- 0.05
models <- list("1" = f1, "2" = f2, "3" = f3)
model_sigmas <- c(sigma_g, 1.5 * sigma_g, sigma_g)
true_model <- "3"
```

On a discrete grid of possible states and actions, we can define the growth rate of a given state  $X_t$  subject to harvest  $H_t$ ,  $f(X_t, H_t)$  as set of matrices. Each matrix  $i$  gives the transition probabilities for any current state to any future state, given that action  $i$  is taken.

```
transition_matrices <- function(f, states, actions, sigma_g) {
  n_s <- length(states)
  n_a <- length(actions)
  transition <- array(0, dim = c(n_s, n_s, n_a))
```

```

for (k in 1:n_s) {
  for (i in 1:n_a) {
    nextpop <- f(states[k], actions[i])
    if (nextpop <= 0) {
      transition[k, , i] <- c(1, rep(0, n_s - 1))
    } else if (sigma_g > 0) {
      x <- dlnorm(states, log(nextpop), sdlog = sigma_g)
      if (sum(x) == 0) { ## nextpop is computationally zero
        transition[k, , i] <- c(1, rep(0, n_s - 1))
      } else {
        x <- x / sum(x) # normalize evenly
        transition[k, , i] <- x
      }
    }
  }
}
transition
}

```

88 This follows the standard setup for standard stochastic dynamic programming, see Marescot *et al.* (2013).  
 89 Having defined a function to compute the transition matrix, we can use it to create matrices corresponding  
 90 to each of the three models:

```

transitions <- lapply(seq_along(models), function(i) {
  transition_matrices(models[[i]], states, actions, model_sigmas[[i]])
})
names(transitions) <- c("1", "2", "3")

```

91 Likewise, a corresponding matrix defining the rewards associated with each state  $X$  and each harvest action  
 92  $H$  can also be defined.

```

## Compute reward matrix (shared across all models)
n_s <- length(states)
n_a <- length(actions)
reward <- array(0, dim = c(n_s, n_a))
for (k in 1:n_s) {
  for (i in 1:n_a) {
    reward[k, i] <- reward_fn(states[k], actions[i])
  }
}

```

## 93 Optimal control solutions

94 We use value iteration to solve the stochastic dynamic program (Marescot *et al.* 2013; Chades *et al.* 2017) for  
 95 each model. This determines the optimal harvest policy for each possible state, given each model. Because  
 96 this step is the most computationally intensive routine, we cache the results using memosization conditioned  
 97 on the transition matrices (Wickham *et al.* 2017). Running this code with alternate transition matrices  
 98 automatically invalidates that cache, reducing the risk of loading spurious results.

```

mdp <- memoise::memoise(mdp_value_iteration,
  cache = memoise::cache_filesystem("cache/")
)

policies <-
  map_dfr(transitions,

```

```

function(P) {
  soln <- mdp(P, reward,
    discount = discount,
    epsilon = 0.01, max_iter = 2000, V0 = rep(0, dim(P)[[1]]))
  escapement <- states - actions[soln$policy]
  tibble(states, policy = soln$policy, escapement)
},
.id = "model"
)

write_csv(policies, "../data/policies.csv")

```

## 99 Simulations and step-ahead forecasts (Fig 1, 2A)

100 We simulate fishing dynamics under the optimal policy for each model, using a simple helper function from  
 101 the `mdplearning` package. Because growth dynamics are stochastic, we perform 100 simulations of each  
 102 model from identical starting condition to ensure results are not the result of chance alone.

```

library(mdplearning)
Tmax <- 100
x0 <- which.min(abs(states - 6))
reps <- 100
set.seed(12345)

## Simulate each policy reps times, with `3` as the true model:
simulate_policy <- function(i, policy) {
  mdp_planning(transitions[[true_model]], reward, discount,
    policy = policy, x0 = x0, Tmax = Tmax
  ) %>%
  select(value, state_index = state, time, action_index = action) %>%
  mutate(state = states[state_index])
}

sims <-
  map_dfr(names(transitions),
    function(m) {
      policy <- policies %>%
        filter(model == m) %>%
        pull(policy)
      map_dfr(1:reps, simulate_policy, policy = policy, .id = "reps")
    },
    .id = "model"
  )

write_csv(sims, "../data/sims.csv")

```

103 Using the transition matrices directly, we can examine what each model would have forecast the future stock  
 104 size to be in the following year when no fishing occurs (note that for each model, we use the transition matrix  
 105 that corresponds to 'no fishing,' `model[[state_index, ,1]]`) (Fig 1a, main text).

106 The transition matrices give the full (discretized) probability distribution, from which we can easily calculate  
 107 both the expected value and the 95% confidence interval.

```

stepahead_unfished <- sims
stepahead_unfished$state_index <- rep(sims$state_index[sims$model == "1"], 3)

stepahead_unfished <- stepahead_unfished %>%
  filter(model != "3") %>%
  mutate(next_state = dplyr::lead(state_index), model = as.integer(model)) %>%
  rowwise() %>%
  mutate(
    expected = transitions[[model]][state_index, , 1] %*% states,
    var = transitions[[model]][state_index, , 1] %*% states^2 - expected^2,
    low = states[max(which(cumsum(transitions[[model]][state_index, , 1]) < 0.025))],
    high = states[min(which(cumsum(transitions[[model]][state_index, , 1]) > 0.975))],
    true = states[next_state]
  )

```

108 We also look at the forecast each model makes when implementing the corresponding optimal harvest:

```

stepahead_fished <- sims %>%
  filter(model != "3") %>%
  mutate(next_state = dplyr::lead(state_index), model = as.integer(model)) %>%
  rowwise() %>%
  mutate(
    prob = transitions[[model]][state_index, next_state, action_index],
    expected = transitions[[model]][state_index, , action_index] %*% states,
    var = transitions[[model]][state_index, , action_index] %*% states^2 - expected^2,
    low = states[max(which(cumsum(transitions[[model]][state_index, , action_index]) < 0.025))],
    high = states[min(which(cumsum(transitions[[model]][state_index, , action_index]) > 0.975))],
    true = states[next_state]
  ) %>%
  select(time, model, true, expected, low, high, var, prob, reps)

```

## 109 Proper scores (Fig 1)

110 It is straight forward to apply the proper scoring formula of Gneiting & Raftery (2007) based on the first two  
 111 moments of the distribution to score the respective forecasts under both the un-fished and actively managed  
 112 scenario for each model:

```

# Gneiting & Raftery (2007), eq27
scoring_fn <- function(x, mu, sigma) {
  -(mu - x)^2 / sigma^2 - log(sigma)
}

stepahead_unfished <- stepahead_unfished %>%
  mutate(
    sd = sqrt(var),
    score = scoring_fn(expected, true, sd)
  )

stepahead_fished <- stepahead_fished %>%
  mutate(
    sd = sqrt(var),
    score = scoring_fn(expected, true, sd)
  )

```

113 We store the replicate predictions under each model at each timestep, along with the scores, in a  
 114 `predictions.csv` table for plotting.

```
predictions <-
  stepahead_unfished %>%
  select(time, model, reps, expected, low, high, true, score) %>%
  mutate(scenario = "A_unfished") %>%
  bind_rows(stepahead_fished %>%
    select(time, model, reps, expected, low, high, true, score) %>%
    mutate(scenario = "B_fished")) %>%
  mutate(model = as.character(model))

write_csv(predictions, "../data/predictions.csv")
```

## 115 Economic value tabulation (Fig 2B)

116 To plot the economic value over time, we must sum up the discounted values at each time step, and then  
 117 average over replicate simulations of each model:

```
## Net Present Value accumulates over time
npv_df <- sims %>%
  group_by(model, reps) %>%
  mutate(npv = cumsum(value * discount^time)) %>%
  group_by(time, model) %>%
  summarise(mean_npv = mean(npv), .groups = "drop") %>%
  arrange(model, time)

write_csv(npv_df, "../data/npv_df.csv")
```

## 118 Adaptive Management

119 Passive adaptive management using a Bayesian learning scheme still learns the wrong model. Adaptive  
 120 management for MDP problems is particularly computationally intensive, and not directly supported by  
 121 the `MDPtoolbox` implementation. Nevertheless, the principle involves only a relatively straightforward, if not  
 122 particularly efficient extension to the task of solving an MDP problem for a single model shown above. Given  
 123 a set of transition probability matrices  $P_i$  and a belief probability  $b_i$  that the  $i$ th model is correct, it is simple  
 124 to define the transition matrices over possible models by the laws of conditional probability,  $P = \sum_i P_i b_i$ ,  
 125 given  $\sum_i b_i = 1$ . After each subsequent action and then new observation  $x_j$ , the  $b_i$  probabilities for each  
 126 model are updated according to Bayes law: if  $b_i = P(M_i)$  is the prior probability that the model  $M_i$  is  
 127 correct, then the posterior probability that model  $M_i$  is correct given observation  $x_j$  is:

$$b'_i = P(M_i|x_j) = P(M_i)P(x_j|M_i) = b_i P(x_j|M_i)$$

128 Note that because the transition matrices are now altered after every action and observation, it is necessary to  
 129 repeat the complete SDP calculation to determine the optimal action given the updated uncertainty. While  
 130 straight forward, this is what makes adaptive management for MDP problems particularly computationally  
 131 intensive and thus poorly suited for handling a large space of candidate models. Nevertheless, our example  
 132 of two models is most feasible, and even the larger example of 36 models is not unreasonable. Here I make  
 133 use of the `mdplearning` R package, which provides a straight-forward implementation of this algorithm.

```
adaptive_management <- memoise::memoise(mdp_learning, cache = memoise::cache_filesystem("cache/"))
am1 <- adaptive_management(transitions[1:2], reward, discount,
```

```

model_prior = c(0.99, 0.01), x0 = x0,
Tmax = 50, true_transition = transitions[[3]],
epsilon = 0.001, max_iter = 2000
)

```

134 I also compare the adaptive management solution to an approach which still integrates uncertainty over  
135 both models, but treats the resulting policy as fixed for the duration of management, rather than adaptive  
136 *learning* by updating probabilities. This approach is sometimes referred to as uncertainty “planning,” as it  
137 accounts for the same initial uncertainty over models but does not attempt to learn. For consistency, I use  
138 the same model prior as in the learning case, which heavily favors model 1 to begin with. Not surprisingly,  
139 the resulting policy is nearly identical to that under model 1 alone.

```

# Compute policy using SDP over both models, given fixed prior beliefs
policy_planning <- memoise::memoise(mdp_compute_policy, cache = memoise::cache_filesystem("cache/"))
policy <- policy_planning(transitions[1:2],
  reward = reward,
  discount = discount,
  model_prior = c(0.99, 0.01),
  Tmax = 50,
  epsilon = 0.001,
  max_iter = 2000
)

# Simulate under true model using fixed planning policy
non_am <- mdp_planning(transitions[[3]],
  reward = reward,
  discount = discount,
  model_prior = c(0.99, 0.01),
  x0 = x0,
  a0 = 1,
  policy = policy$policy,
  Tmax = 50
) %>%
mutate(
  belief = 0.99, # belief in model 1 is fixed
  method = "planning"
)

```

140 We combine results of the two methods and store the resulting `data.frame` of stock size, quota, and belief  
141 in Model 1 over time as `am.csv` for later plotting.

```

am <- am1$df %>%
  mutate(
    belief = am1$posterior$V1,
    method = "learning"
  ) %>%
  bind_rows(non_am) %>%
  mutate(
    stock = states[state],
    quota = actions[action]
  ) %>%
  select(time, stock, quota, belief, method)

write_csv(am, "../data/am.csv")

```

142 We can also compare these results to the net economic value achieved under Model 1 alone:



```

# tabular comparisons
npv <- npv_df %>%
  group_by(model) %>%
  summarize(npv = max(mean_npv))

am_npv <- sum(am$value * discount^am$time)
am_economics_percent <- round(am_npv / npv[[1, "npv"]] * 100)

mean_state <- sims %>%
  group_by(model) %>%
  summarize(state = mean(state))
am_ecology_percent <- round(mean(states[am$state]) / mean_state[[1, "state"]] * 100)

```

143 Under adaptive management, the manager realizes only 0% of the economic value that would be achieved  
 144 under Model 1 alone, and only NaN% of the spawning stock biomass that would have been achieved under  
 145 Model 1 alone.

## 146 Adaptive management over many models

147 We repeat the analysis above over a grid of  $r$  and  $K$  values.

```

# compare over grid set of candidate r and K
rs <- c(0.01, 0.1, 0.2, 0.3, 0.4, 0.5, 2)
Ks <- c(0.5, 1, 1.5, 4, 10, 16)
sigma <- 0.1
n_models <- length(rs) * length(Ks)
pars <- expand.grid(rs, Ks)
names(pars) <- c("r", "K")
pars$sigma <- 0.1
closure <- function(r, K) {
  function(x, h = 0) {
    s <- pmax(x - h, 0)
    s + s * (r * (1 - s / K))
  }
}
many_models <- lapply(1:n_models, function(i) closure(pars[i, "r"], pars[i, "K"]))

# transition matrices
many_transitions <- lapply(seq_along(many_models), function(i) {
  transition_matrices(many_models[[i]], states, actions, sigma)
})
names(many_transitions) <- as.character(seq_along(many_models))
true_transition <- transition_matrices(f3, states, actions, sigma_g)

## Once again, initial belief favors "model 1"
model_prior <- c(rep(.01, n_models - 1) / (n_models - 1), .99)

```

148 With the transition matrices defined for all 42 models, we perform adaptive management as before:

```

adaptive_management <- memoise::memoise(mdp_learning, cache = memoise::cache_filesystem("cache/"))
am_many <- adaptive_management(many_transitions,
  reward = reward,
  discount = discount,
  model_prior = model_prior,
  x0 = x0,

```

```

Tmax = 20,
true_transition = true_transition,
epsilon = 0.001,
max_iter = 2000
)

```

149 As before, we can compare the adaptive learning case to the policy which accounts for the initial uncertainty  
150 over all 42 models, but does not seek to learn.

```

policy_many <- policy_planning(many_transitions,
  reward = reward, discount = discount,
  model_prior = model_prior, Tmax = 20,
  epsilon = 0.001, max_iter = 2000
)

non_am_many <- mdp_planning(true_transition,
  reward = reward,
  discount = discount,
  model_prior = model_prior,
  x0 = x0,
  a0 = 1,
  policy = policy_many$policy,
  Tmax = 20
) %>%
mutate(
  belief = 0.99, # belief in model 1 is fixed
  method = "planning"
)

```

151 Again we record the posterior belief in the ‘model 1’ parameter values, and stack the data frames for learning  
152 and planning-only cases into a single file used to generate figure 3B.

```

model1 <- dim(am_many$posterior)[2]
am_multi <- am_many$df %>%
  mutate(
    belief = am_many$posterior[, model1],
    method = "learning"
  ) %>%
  bind_rows(non_am_many) %>%
  mutate(
    stock = states[state],
    quota = actions[action]
  ) %>%
  select(time, belief, stock, quota, method)

write_csv(am_multi, "../data/am_multi.csv")

```

## 153 Posteriors

154 With 42 possible models instead of two, the probability weight (belief) in Model 1 does not fully capture the  
155 evolution of the belief probabilities over time. To better illustrate how these beliefs evolve over the course  
156 of the simulation, I plot the marginal probabilities over values of  $r$  and  $K$  across the models over time. Note  
157 that Model 1 values,  $r = 2$ ,  $K = 16$ , are quickly disfavored.

```

posterior <-
data.frame(model = as.character(1:n_models), t(am_many$posterior)) %>%
left_join(mutate(pars, model = as.character(1:n_models))) %>%
pivot_longer(starts_with("X"),
  values_to = "probability",
  names_to = "time"
) %>%
mutate(time = as.integer(as.factor(time)))

r_marginals <- posterior %>%
  group_by(time, r) %>%
  summarise(prob = sum(probability), .groups = "drop")

K_marginals <- posterior %>%
  group_by(time, K) %>%
  summarise(prob = sum(probability), .groups = "drop")

```

158 From the table of marginal probabilities, we generate the corresponding plots

```

belief_r <- r_marginals %>%
  mutate(r = as.character(r)) %>%
  ggplot(aes(time, prob, group = r, col = r)) +
  geom_line() +
  geom_point()

belief_K <- K_marginals %>%
  mutate(K = as.character(K)) %>%
  ggplot(aes(time, prob, group = K, col = K)) +
  geom_line() +
  geom_point()

belief_r + belief_K

```

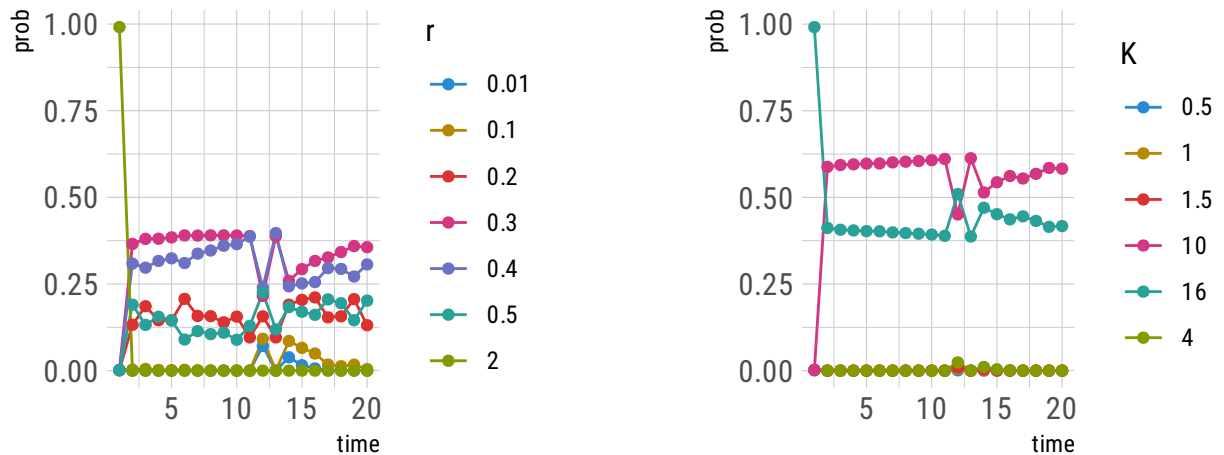


Figure 1: Fig S1: Marginal probabilities of parameters  $r$  and  $K$  over the course of adaptive management over all 42 models.

159 For most of the simulation, the most probable  $r$  value is around 0.3, and the most probable  $K$  value around  
160 1.5.

## Growth models

To plot growth curves of individual models (Fig 4a), we evaluate

$$\Delta x = x_{t+1} - x_t = f(x_t) - x_t$$

for each model for all possible states  $x_t$ .

```
model_curves <-  
  map_dfc(models, function(f) f(states) - states) %>%  
  mutate(state = states) %>%  
  pivot_longer(names(models), "model")  
  
write_csv(model_curves, "../data/model_curves.csv")
```

The resulting model curves are shown in the main paper, Fig 4A.

## Growth models in the many-models case

The corresponding figure for the case of 42 models.

```
final_belief <- posteriors %>%  
  filter(time == max(time)) %>%  
  select(model, belief = probability) %>%  
  mutate(type = "candidate") %>%  
  bind_rows(data.frame(belief = 1, model = "true", type = "true"))  
  
model_set <- c(f3, many_models)  
names(model_set) <- c("true", 1:length(many_models))  
d <-  
  map_dfc(model_set, function(f) f(states) - states) %>%  
  mutate(state = states) %>%  
  pivot_longer(names(model_set), "model") %>%  
  left_join(final_belief)  
  
d %>%  
  ggplot(aes(state, value, group = model, col = type, alpha = belief)) +  
  geom_line(lwd = 1) +  
  coord_cartesian(ylim = c(-5, 8), xlim = c(0, 16)) +  
  ylab(bquote(f(x) - x)) +  
  xlab("x")
```

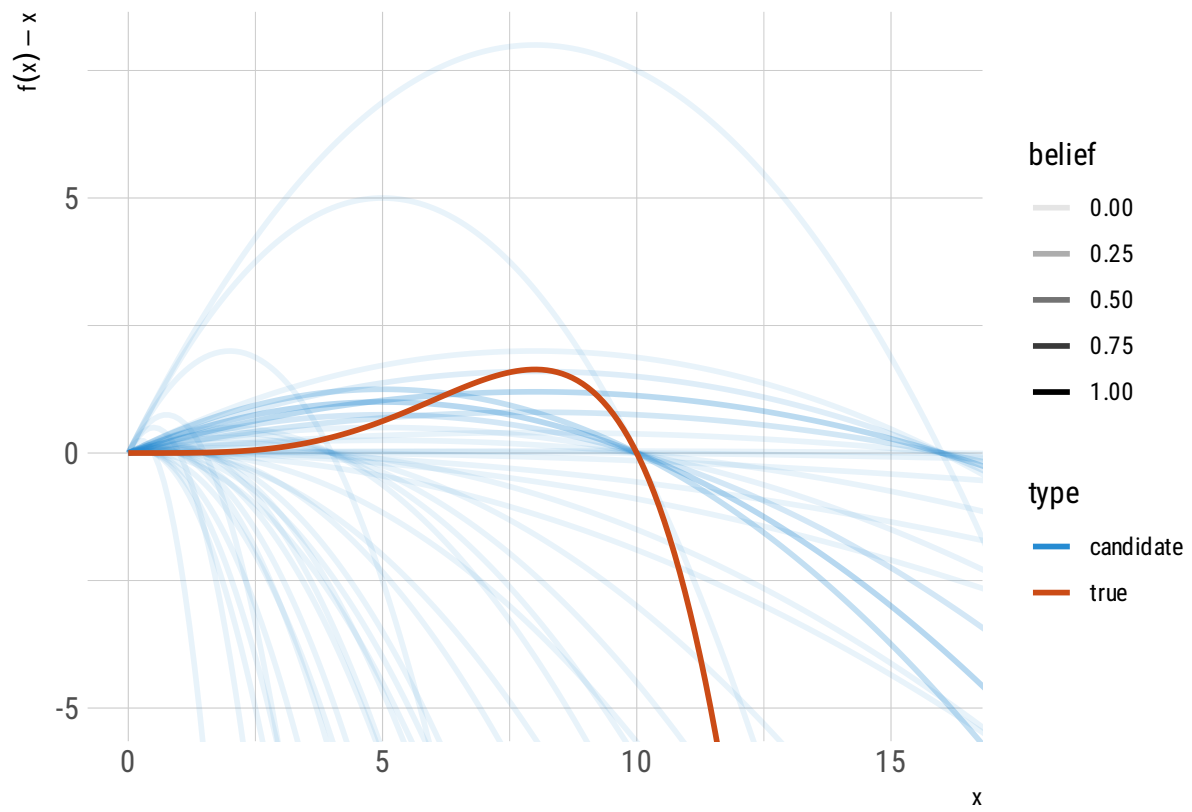


Figure 2: growth curves for 42 candidate models, shaded by belief at timestep 20, relative to the true model. The curve with parameter values corresponding to ‘model 1’ are recognized in the highest arc, with belief weight of less than 1%.

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