No early warning signals for stochastic transitions: insights from large deviation theory

Carl Boettiger* Alan Hastings†

May 29, 2013

In Boettiger & Hastings [1] we demonstrated that conditioning on observing a purely stochastic transition from one stable basin to another could generate time-series trajectories that could be mistaken for an early warning signal of a critical transition (such as might be due to a fold bifurcation [2]), when instead the shift is merely due to chance. While the goal was to highlight a potential danger in mining historical records for patterns showing sudden shifts when seeking to test early warning techniques, Drake [3] draws attention to a potentially more interesting consequence of our analysis. Drake argues that the bias observed could be used to forecast purely stochastic transitions – a task previously thought to be impossible [4]. We feel this interpretation is too generous; as the pattern Drake points to arises in any large deviation, regardless of whether a system is or is not at elevated risk for a transition.

Here we provide a numerical demonstration that the pattern in question for consideration of an early warning signal appears not only before purely stochas-

*Center for Stock Assessment Research, Department of Applied Math and Statistics, University of California, Mail Stop SOE-2, Santa Cruz, CA 95064, USA

†Department of Environmental Science and Policy, University of California Davis, 1 Shields Avenue, Davis, CA 95616 USA

tic transitions (as seen in Reference 1) but during any large deviation. As large deviations can occur even in stochastic systems that have only a single stable point, these patterns cannot be considered indicators of stochastic transitions. We demonstrate this in two scenarios: first using the Allee model of alternative stable states considered in Reference 1, Eqn 2.1 - 2.2 and Figure 2, and then in a simple Ornstein-Uhlenbeck (OU) model which has only a single stable state. Rather than condition on a stochastic transition having occurred (as in Reference 1), we now condition on having merely observed a sufficiently large deviation. It does not matter precisely what "large" deviation is considered, only that the larger the deviation the more replicates or longer simulation times will be needed to sufficiently populate the sample. We pick values such that we get a sample of a few hundred large deviation events in a sample of 20,000 replicates.

The OU model is defined by a stochastic differential equation in which there is only a single optimum whose strength is proportional to the displacement,

$$dX_t = -\alpha X_t dt + \sigma dB_t,$$

where the state X_t oscillates around a stable point (at zero in these arbitrary units), driven by Brownian noise dB_t of intensity σ and restorative force α.

The analysis for each model proceeds exactly as in Reference 1: For each model we generate 20,000 replicate time series. We condition upon only those experiencing a deviation of size L ($X \leq 250$ in the Allee model and $X \leq -4$ in the OU model). For the sequence of observations immediately leading up to the large deviation we compute the warning signals of variance and autocorrelation over a sliding window of half the length of the time-series, and we summarise the increasing or decreasing trend observed in the variance and autocorrelation using Kendall's τ rank correlation coefficient (all following the method for early warning indicators outlined in Reference 5). We repeat this analysis on the entire set of time-series under each model to obtain null distributions for τ statistic.

We find (Figure 1) that τ is significantly skewed towards positive values when conditioning on large deviations in both models. This demonstrates that it is the presence of the large deviation, not the presence of the stochastic transition we condition on in Reference 1, that is responsible for this pattern (just as we claimed without example then).

Observing the bias shown in the figures here depends on having a rapid enough sample frequency to capture the escape trajectory and a long enough trajectory for the statistic to demonstrate an increase over time. Since large deviations due to stochastic forces alone must be fast, so must the accompanying warning signal and management response (which will show up on the time scale of the perturbation). Note that fast relative to the system dynamics may or may not be fast relative to the timescale of management (just as with bifurcation-driven warning signals, Reference 6). The wider null distribution in the OU model results from the sample window being shorter relative to the system timescale.

One might consider this a corollary of the Prosecutor's Fallacy we originally presented, which demonstrated that examples of sudden transitions historically selected from the literature could be mistaken for positive evidence of early warning signals when they were in fact due to purely stochastic transitions. Here we have seen how any large deviation could be similarly misleading, whether or not it results in a stochastic transition to an alternative stable state. From a classical result of the large deviation theory one can gain considerable intuition about why these chance deviations show much higher variance and autocorrelation than expected from the stationary distribution of a stable point. Though

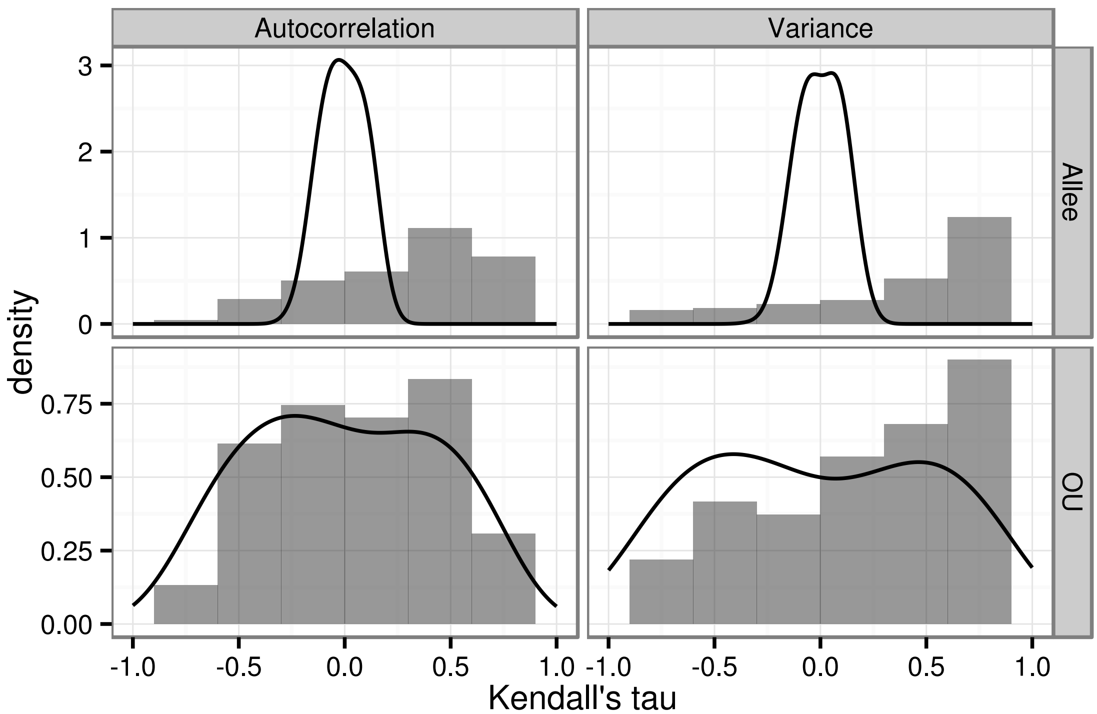

Figure 1: Figure 1. Histogram shows the frequency the correlation statistic τ observed for each warning signal (variance, autocorrelation coefficient) on the large deviation samples from each model. Background distribution of all samples show by smooth line (kernel density estimate). More positive values of tau are supposed to indicate a rising indicator which can be a signal of an approaching transition [2]. The OU model uses $\alpha = 5$, $\sigma = 3.5$, $t \in (0, 10)$, 2000 replicates, 20,000 sample points each. Conditionally selected trajectories experiencing a deviation of at least -4, and analyzed the 1,500 data points prior to the threshold to determine a warning signal (following Reference 5). (Code at: https://raw.github.com/cboettig/earlywarning/resubmission/inst/doc/Figure1.Rmd, data at: https://raw.github.com/cboettig/earlywarning/resubmission/inst/doc/Figure1.csv

large deviations are rare – the time we must wait to observe a deviation of size L in the system above scales as $\exp\left(L^2/\sigma^2\right)$ (the familiar Arrhenius relationship), when these deviations occur they occur very rapidly. The expected time for an excursion to a distant point L that does not again cross the stable point before reaching L scales as $\log(L/\sigma)$, just as a trajectory returning down the gradient of the attractor from L to the stable point (proofs in Reference 7 or Reference 8). While most trajectories in the stationary distribution take steps in each direction with equal probability, these large deviations moving rapidly to the boundary will consequently show the greater autocorrelation. In achieving a much greater deviation than typically observed, these trajectories will also show an increase in variance, as observed. That such trajectories appear to be pulled in the direction of their escape rather than climbing away against a restorative force has led to confusion before. Reference 8 argues how this shows how a "punctuated equilibrium" pattern of stasis followed by rapid change could arise entirely from small steps, and Reference 9 empirically demonstrates this phenomena in the trajectories of local population extinctions.

In conclusion, we heartily agree with the need for a decision-theoretic approach to early warning signal questions [10]. Central to a decision-theoretic approach is enumerating alternative scenarios that are possible given the observed data. We have highlighted how purely stochastic transitions and large deviations are such possibilities. The challenge of sufficient or unique early warning indicators is not limited to stochastic shifts, but includes the more typical critical transitions. For instance, rising variance or autocorrelation patterns typical of fold bifurcations can be observed in more benign bifurcations or smooth transitions [11]. Early warning signals may offer a promising technique that will one day allow us to avoid seemingly unpredictable catastrophes – but we must not lose sight of just how difficult are the challenges involved. A key step here and for early warning indicators more generally is to understand these other circumstances in which they can arise, that we may then develop ways to eliminate those possibilities. Though we may never be able to detect purely stochastic transitions, perhaps these approaches in this discussion may lead to more unique and sufficient indicators for true critical transitions.

The authors acknowledge the generous support of NSF grant to AH and helpful comments from TA Perkins, as well as the reviews from an anonymous reviewer and P. Ditlevsen.

1 Boettiger, C. & Hastings, A. 2012 Early warning signals and the prosecutor's fallacy. *Proceedings of the Royal Society B: Biological Sciences* (doi:10.1098/rspb.2012.2085)

2 Scheffer, M. et al. 2009 Early-warning signals for critical transitions. *Nature* **461**, 53–9.

3 Drake, J. M. 2013 Early warning signals of stochastic switching. *Proceedings of The Royal Society B* **in press**.

4 Ditlevsen, P. D. & Johnsen, S. J. 2010 Tipping points: Early warning and wishful thinking. *Geophysical Research Letters* **37**, 2–5. (doi:10.1029/2010GL044486)

5 Dakos, V., Scheffer, M., van Nes, E. H., Brovkin, V., Petoukhov, V. & Held, H. 2008 Slowing down as an early warning signal for abrupt climate change. *Proceedings of the National Academy of Sciences* **105**, 14308–12. (doi:10.1073/pnas.0802430105)

6 Hughes, T. P., Linares, C., Dakos, V., van de Leemput, I. a & van Nes, E. H. 2013 Living dangerously on borrowed time during slow, unrecognized regime shifts. *Trends in ecology & evolution* **28**, 149–55. (doi:10.1016/j.tree.2012.08.022)

7 Ludwig, D. 1981 Escape from Domains of Attraction for Sytstems Perturbed by Noise. In *Nonlinear Phenomena in Physics and Biology* (eds R. H. Enns B. L. Jones R. M. Miura & S. S. Rangnekar), pp. 549–566. Boston, MA: Springer New York.

8 Lande, R. 1985 Expected time for random genetic drift of a population

between stable phenotypic states. *Proceedings of the National Academy of Sciences* **82**, 7641–7645.

9 Drake, J. M. & Griffen, B. D. 2009 Speed of expansion and extinction in experimental populations. *Ecology letters* **12**, 772–8. (doi:10.1111/j.1461-0248.2009.01325.x)

10 Boettiger, C. & Hastings, A. 2012 Quantifying limits to detection of early warning for critical transitions. *Journal of The Royal Society Interface* **9**, 2527–2539. (doi:10.1098/rsif.2012.0125)

11 Kfi, S., Dakos, V., Scheffer, M., Van Nes, E. H. & Rietkerk, M. 2012 Early warning signals also precede non-catastrophic transitions. *Oikos* (doi:10.1111/j.1600-0706.2012.20838.x)