

No early warning signals for stochastic transitions: insights from large deviation theory

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In Boettiger & Hastings [1] we demonstrated that conditioning on observing a purely stochastic transition from one stable basin to another could generate time-series trajectories that could be mistaken for an early warning signal of a critical transition (such as might be due to a fold bifurcation [2]), when instead the shift is merely due to chance. While the goal was to highlight a potential danger in mining historical records for patterns showing sudden shifts when seeking to test early warning techniques, Drake [3] draws attention to a potentially more interesting consequence of our analysis. Drake argues that the bias observed could be used to forecast purely stochastic transitions – a task previously thought to be impossible [4]. We feel this interpretation is too generous. The pattern Drake points to arises in any large deviation, regardless of whether a system is or is not at elevated risk for a transition. We illustrate this pattern in systems with and without bistability, demonstrating that early warning signals do not exist for purely stochastic transitions.

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19 Here we provide a numerical demonstration that the pattern in question for
 20 consideration of an early warning signal appears not only before purely stochas-
 21 tic transitions (as seen in Reference 1) but during any large deviation. As large
 22 deviations can occur even in stochastic systems that have only a single stable
 23 point, these patterns cannot be considered indicators of stochastic transitions.
 24 We demonstrate this in two scenarios: first using the Allee model of alterna-
 25 tive stable states considered in Reference 1, Eqn 2.1 - 2.2 and Figure 2, and
 26 then in a simple Ornstein-Uhlenbeck (OU) model which has only a single stable
 27 state. Rather than condition on a stochastic transition having occurred (as in
 28 Reference 1), we now condition on having merely observed a sufficiently large
 29 deviation. It does not matter precisely what “large” deviation is considered,
 30 only that the larger the deviation the more replicates or longer simulation times
 31 will be needed to sufficiently populate the sample. We pick values such that
 32 we get a sample of a few hundred large deviation events in a sample of 20,000
 33 replicates.

34 The OU model is defined by a stochastic differential equation in which there
 35 is only a single optimum whose strength is proportional to the displacement,

$$dX_t = -\alpha X_t dt + \sigma dB_t,$$

36 where the state X_t oscillates around a stable point (at zero in these arbitrary
 37 units), driven by Brownian noise dB_t of intensity σ and restorative force α .

38 The analysis for each model proceeds exactly as in Reference 1: For each
 39 model we generate 20,000 replicate time series. We condition upon only those
 40 experiencing a deviation of size L ($X \leq 250$ in the Allee model and $X \leq -4$ in
 41 the OU model). For the sequence of observations immediately leading up to the
 42 large deviation we compute the warning signals of variance and autocorrelation
 43 over a sliding window of half the length of the time-series, and we summarise

44 the increasing or decreasing trend observed in the variance and autocorrelation
45 using Kendall's τ rank correlation coefficient (all following the method for early
46 warning indicators outlined in Reference 5). We repeat this analysis on the
47 entire set of time-series under each model to obtain null distributions for τ
48 statistic.

49 We find (Figure 1) that τ is significantly skewed towards positive values when
50 conditioning on large deviations in both models. This demonstrates that it is
51 the presence of the large deviation, not the presence of the stochastic transition
52 we condition on in Reference 1, that is responsible for this pattern (just as we
53 claimed without example then).

54 Observing the bias shown in the figures here depends on having a rapid
55 enough sample frequency to capture the escape trajectory and a long enough
56 trajectory for the statistic to demonstrate an increase over time. Since large
57 deviations due to stochastic forces alone must be fast, so must the accompanying
58 warning signal and management response (which will show up on the time scale
59 of the perturbation). Note that fast relative to the system dynamics may or may
60 not be fast relative to the timescale of management (just as with bifurcation-
61 driven warning signals, Reference 6). The wider null distribution in the OU
62 model results from the sample window being shorter relative to the system
63 timescale.

64 One might consider this a corollary of the Prosecutor's Fallacy we originally
65 presented, which demonstrated that examples of sudden transitions historically
66 selected from the literature could be mistaken for positive evidence of early warn-
67 ing signals when they were in fact due to purely stochastic transitions. Here
68 we have seen how any large deviation could be similarly misleading, whether or
69 not it results in a stochastic transition to an alternative stable state. From a
70 classical result of the large deviation theory one can gain considerable intuition

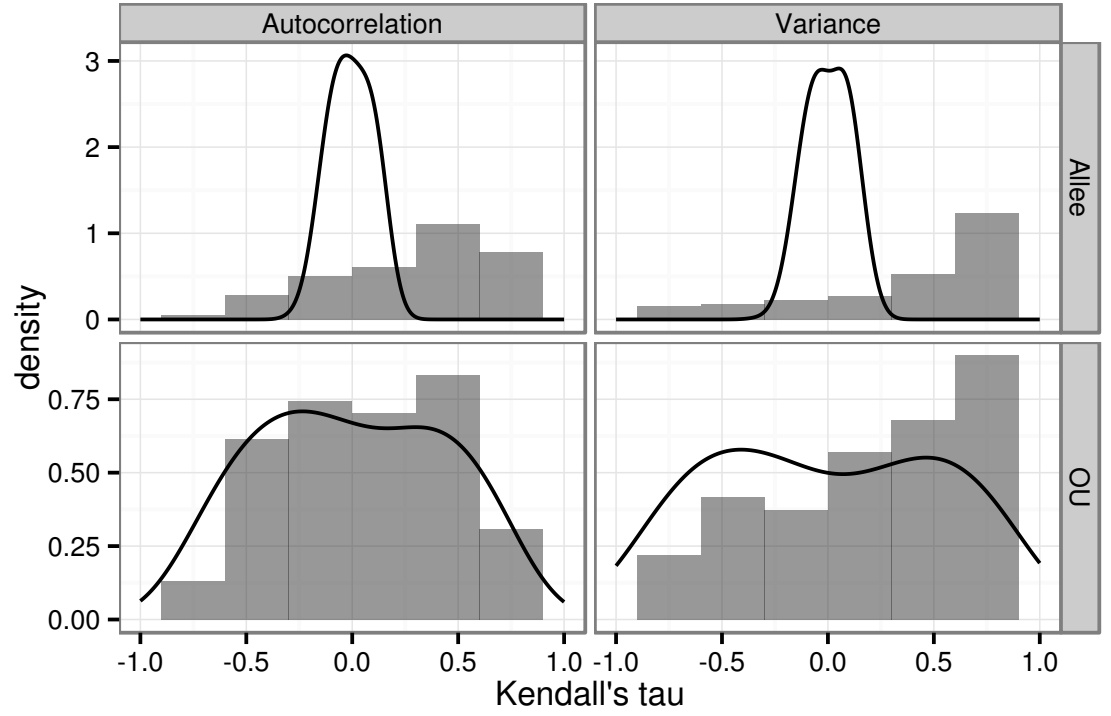


Figure 1: Figure 1. Histogram shows the frequency the correlation statistic τ observed for each warning signal (variance, autocorrelation coefficient) on the large deviation samples from each model. Background distribution of all samples show by smooth line (kernel density estimate). More positive values of tau are supposed to indicate a rising indicator which can be a signal of an approaching transition [2]. The OU model uses $\alpha = 5$, $\sigma = 3.5$, $t \in (0, 10)$, 2000 replicates, 20,000 sample points each. Conditionally selected trajectories experiencing a deviation of at least -4, and analyzed the 1,500 data points prior to the threshold to determine a warning signal (following Reference 5). (Code at: <https://raw.githubusercontent.com/cboettig/earlywarning/resubmission/inst/doc/Figure1.Rmd>, data at: <https://raw.githubusercontent.com/cboettig/earlywarning/resubmission/inst/doc/Figure1.csv>)

71 about why these chance deviations show much higher variance and autocorrela-
 72 tion than expected from the stationary distribution of a stable point. Though
 73 large deviations are rare – the time we must wait to observe a deviation of size
 74 L in the system above scales as $\exp(L^2/\sigma^2)$ (the familiar Arrhenius relation-
 75 ship), when these deviations occur they occur very rapidly. The expected time
 76 for an excursion to a distant point L that does not again cross the stable point
 77 before reaching L scales as $\log(L/\sigma)$, just as a trajectory returning down the
 78 gradient of the attractor from L to the stable point (proofs in Reference 7 or
 79 Reference 8). While most trajectories in the stationary distribution take steps
 80 in each direction with equal probability, these large deviations moving rapidly to
 81 the boundary will consequently show the greater autocorrelation. In achieving
 82 a much greater deviation than typically observed, these trajectories will also
 83 show an increase in variance, as observed. That such trajectories appear to
 84 be pulled in the direction of their escape rather than climbing away against a
 85 restorative force has led to confusion before. Reference 8 argues how this shows
 86 how a “punctuated equilibrium” pattern of stasis followed by rapid change could
 87 arise entirely from small steps, and Reference 9 empirically demonstrates this
 88 phenomenon in the trajectories of local population extinctions.

89 In conclusion, we heartily agree with the need for a decision-theoretic ap-
 90 proach to early warning signal questions [10]. Central to a decision-theoretic
 91 approach is enumerating alternative scenarios that are possible given the ob-
 92 served data. We have highlighted how purely stochastic transitions and large
 93 deviations are such possibilities. The challenge of sufficient or unique early
 94 warning indicators is not limited to stochastic shifts, but includes the more
 95 typical critical transitions. For instance, rising variance or autocorrelation pat-
 96 terns typical of fold bifurcations can be observed in more benign bifurcations or
 97 smooth transitions [11]. Early warning signals may offer a promising technique

that will one day allow us to avoid seemingly unpredictable catastrophes – but we must not lose sight of just how difficult are the challenges involved. A key step here and for early warning indicators more generally is to understand these other circumstances in which they can arise, that we may then develop ways to eliminate those possibilities. Though we may never be able to detect purely stochastic transitions, perhaps these approaches in this discussion may lead to more unique and sufficient indicators for true critical transitions.

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