Contents

1 Week 7 Module 5:

1.1 Sampling and Quantization

Interpolation describes the process of building a continuous-time signal $\mathbf{x(t)}$ from a sequence of samples $\mathbf{x[n]}$. In other words, interpolation allows moving from the discrete-time world to the continuous-time world. Interpolation raises two interesting questions:

The first one is how to interpolate between samples?

- In the case, of two samples, this is simple enough and there is there is a straight line that goes between these two samples.
- In the case of three samples, similarly, you have a parabola that goes through these 3 samples.
- If you have many samples, you can try to do the same and go through all samples but you see this is a trickier issue compared to what we have done with two or three samples.

The second question is:

- is there a minimum set of values you need to measure the function at so that you can perfectly reconstruct it.

Later on in the module, we are going to study sampling, i.e. the process of moving from a continuous-time signal to a sequence of samples. In other words, sampling allows moving from the continuous-time world to the

discrete-time world. Suppose we take equally-spaced samples of a function $\mathbf{x(t)}$. The question is when is there a one-to-one relationship between the continuous-time function and its samples, i.e. when do the samples form a unique representation of the continuous-time function? To answer this question, we are going to use all the tools in the toolbox that we have looked at so far:

- Hilbert spaces
- projections
- filtering
- sinc functions
- and so on.

Everything comes together in this module to develop a profound and very useful result, the **sampling theorem**.

Before moving to the heart of the topic, let us briefly review its history. The Shannon sampling theorem has a very interesting history which goes back well before Shannon. Numerical analysts were concerned about interpolating tables of functions and the first one to proove a version of the sampling theorem was Whittaker in England in 1915. Harry Nyquist at Bell Labs came up with the Nyqvist criterion, namely that a function that has a maximum frequency $\mathrm{F0F}_{\mathrm{0F0}}$ could be sampled at $\mathrm{2F02F}_{\mathrm{02F0}}$. In the Soviet Union, Kotelnikov proved a sampling theorem. The son of the first Whittaker further proved results on the sampling theorem. Then Herbert Raabe in Berlin wrote his PhD thesis about a sampling theorem that, wrong time wrong city, he got zero credit for. Denis Gabor worked on a version of the sampling theorem in the mid 1940s. Then Claude Shannon, the inventor of information theory, wrote a beautiful paper that is in the further reading for this class where the Shannon sampling theorem appears in the form that we use today. Last but not least, in 1949 Someya in Japan also proved the sampling theorem. You can see that it's a very varied history, it's a fundamental result where many people independently came up with this result.

1.1.1 The Continous-Time World

1. Introduction The continuous-time world is the world we live in, the physical reality of the world, in contrast with the discrete-time world,

the world inside a computer. We are first going to look at models of the world and compare digital with analog views of the world. Then we are going to study continuous-time signal processing in greater details. Furthermore, we will introduce the last form of Fourier transform we have not yet encountered in this class, the continuous-time Fourier transform.

2. The continous-time paradigm

   Two views of the world

Table 1: Two views of the world 1

Digital World	**Analog World**
arithmetic	calculus
combinatorics	distributions
computer sience	system theory
DSP	electronics

Table 2: Two views of the world 2

Digital World	**Analog World**
countable integer index M	real-valued time t [sec]
sequences $x[n] \in \ell_2(\mathbb{Z})$	function $x(t) \in L_2(\mathbb{R})$
frequency $\omega \in [-\pi, \pi]$	frequency $\Omega \in \mathbb{R}(rad/sec)$
DTFT: $\ell_2(\mathbb{Z}) \to L_2[-\pi, \pi]$	FT: $L_2(\mathbb{R}) \to L_2(\mathbb{R})$

$\ell_2(\mathbb{Z})$	Square-Summable infinite sequences
$L_2([a, b])$	Square-integrable functions over an interval
Sampling	$x(t) \to x[n]$
Interpolation	$x[n] \to x(t)$

3. Continuous-time signal processing

time	real variable t
signal x(t)	complex function of areal variable
finite energy	$x(t) \in L_2(\mathbb{R})$
inner product in $L_2(\mathbb{R})$	$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x^*(t)\, y(t)$

energy $||x(t)||^2 = \langle x(t), x(t) \rangle$

(a) Analog LTI filters

$$\begin{aligned} y(t) &= (x * h)(t) \\ &= \langle^*(t-\tau), x(\tau)\rangle \\ &= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \end{aligned}$$

(b) Fourier analysis

- in discrete time max angular frequency is $\pm\pi$
- in continous time no max frequency: $\Omega \in \mathbb{R}$
- concept is the same:

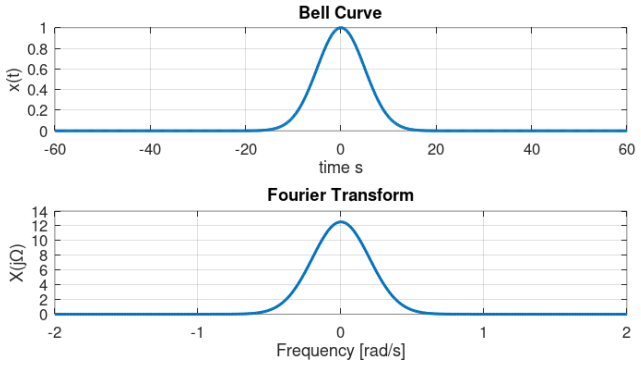
$$X(j\Omega) = \int_{-\infty}^{\infty} e^{-j\Omega\, t}\, dt$$

$$x(t) = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega\, t} dt$$

(c) Real-world frequency

- Ω expresse in rad/s
- $F = \dfrac{\Omega}{2\,\pi}$, expressed in Hertz (1/s)
- period $T = \dfrac{1}{F} = \dfrac{2\,\pi}{\Omega}$

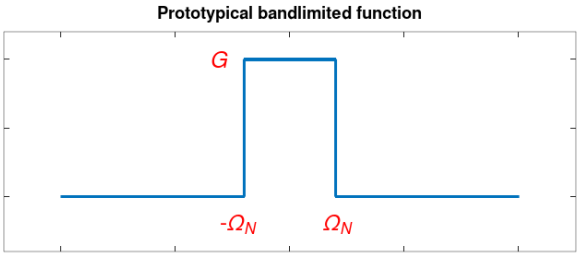
(d) Example



(e) Convolution theorem

$$Y(j\,\Omega) = X(j\Omega)\; H(j\Omega)$$

(f) Prototypical Bandlimited Functions



$$\Phi(j\,\Omega) = G\; rect(\frac{\Omega}{2\;\Omega_N})$$

The time domain function can be determinded by means of its **Inverse Fourier Transform**

$$\begin{aligned}\phi(t) &= \frac{1}{2\;\pi}\int_{-\infty}^{\infty}\Phi(j\Omega)e^{j\Omega\;t}d\Omega \\ &= G\frac{\Omega_N}{\pi}sinc(\frac{\Omega_N}{\pi}\;t)\end{aligned}$$

The time domain function is up to a scaling, one of these sinc functions. We will normalize this sinc function, so that the area is equal to 2π in the Frequency Domain. Then the inverse continuous time Fourier Transform will have a maxmimum of 1 at the origin.

normalization	$G = \frac{\pi}{\Omega_n}$
totoal bandwith	$\Omega_B = 2\;\Omega_N$
define	$T_s = \frac{2\;\pi}{\Omega_B} = \frac{\pi}{\Omega_N}$

This leads to the normalized prototypical bandlimted function:

Frequency Domain

$$\Phi(j\ \Omega) = \frac{\pi}{\Omega_N}\ rect\left(\frac{\Omega}{2\ \Omega_N}\right)$$

Time Domain

$$\phi(t) = sinc\left(\frac{t}{T_s}\right)$$

4. **TODO** Plot Normalized prototypicale bandlimited function

1.1.2 Interpolation

Main Task $x[n] \Rightarrow x(t)$

Gaps fill the gaps between samples

1. Interpolation requirements
   - decide on T_s
   - make sure $x(nT_s) = x[n]$
   - make sure x(t) is smooth
2. Why smoothness
   - jumps (1st order discontinuities) would require the signal to move "faster than light"
   - 2nd order discontinuities would require infinite acceleration
   - the interpolation should be infinitely differentiable
   - "natural" solution: polynomial interpolation
3. Polynomial interpolation
   - N points $\Rightarrow$ polynomial of degree (N-1)
   - $p(t) = a_0 + a_1 t + a_2 t^2 a... + a_{N-1} t^{(N-1)}$
   - "naive" approach

$$\begin{cases} p(0) & = x[0] \\ p(T_s) & = x[1] \\ p(2T_s) & = x[2] \\ ..... & \\ p((N-1)T_s) & = x[N-1] \end{cases}$$

Without loss of generality:

- consider a symmetric interval $I_N = [-N...N]$
- set $T_s = 1$

$$\begin{cases} p(-N) & = x[-N] \\ p(-N+1) & = x[-N+1] \\ ..... & \\ p(0) & = x[0] \\ p(N) & = x[N] \end{cases}$$

4. Lagrange interpolation The natural solution to this interpolation problem is given by Lagrange interpolation

   - P_N : space of degree-2N polynominals over $\mathrm{I_N}$
   - a basis for $\mathrm{P_N}$ is the family of 2N + 1 Lagrange polynominals

   $$L_n^{(N)}(t) = \prod_{k=-N}^{N} \frac{t-k}{n-k} \text{ for } M = -N,...,N$$

   The formula:

   $$p(t) = \sum_{n=-N}^{N} x[n] L_n^{(N)}(t)$$

   The Lagrange interpolation is the sought-after polynominal interpolation:

   - polynominal of degree 2N through 2N+1 points is unique

- the Lagrangian interpolator satisfies

$$p(N) = x[N] \text{ for } -N \leq M \leq N$$

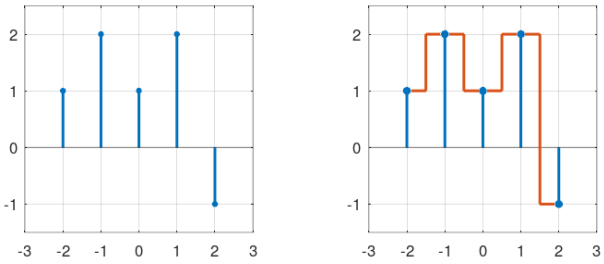
since

$$L_n^{(N)}(N) = \begin{cases} 1 \text{ if } M = N \\ 0 \text{ if } M \neq N \end{cases} \quad -N \leq M, N \leq N$$

key property	maximmally smooth (infinitely many continuous derivatives)
drawback	interpolation "bricks" depend on N

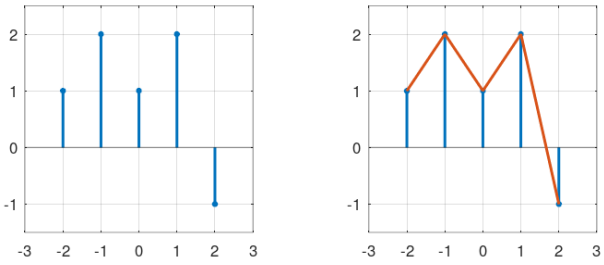
5. Local Interpolation

   (a) Zero-order hold "Box Function"



   - $x(t) = x[t + 0.5]$ for $-N \leq t \leq N$
   - $x(t) = \sum_{n=-N}^{N} x[n]\, rect(t - n)$
   - interpolation kernel: $i_0(t) = rect(t)$
   - $i_0(t)$: zero-ordre hold
   - interpolation support is 1
   - interpolation is not even continous

   (b) First-order piece-wise linear "Hat Function"

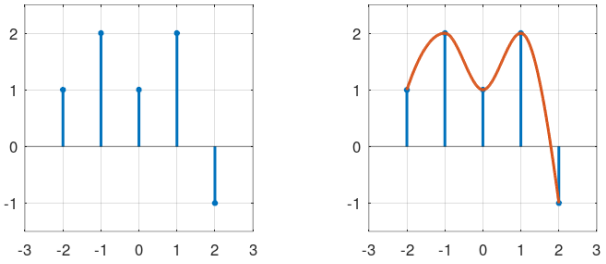


- connect the dots strategy
- $x(t) = \sum_{n=-N}^{N} x[n]\ i_1(t-n)$
- interpolation kernel:

$$i_1(t) = \begin{cases} 1-|t| & |t| \leq 1 \\ 0 & otherwise \end{cases}$$

- interpolation support is 2
- interpolation is continuous but derivative is not

(c) Third-order interpolation



- $x(t) = \sum_{n=-N}^{N} x[n]\ i_3(t-n)$
- interpolation kernel obtained by splicing two cubic polynominals
- interpolation support is 4
- interpolation is continuous up to second derivative

(d) Local Interpolation schemes

$$x(t) = \sum_{n=-N}^{N} x[n]\ ic(t-n)$$

Interpolator's requirements:

- i_c: interpolation kernel
- $i_c(0) = 1$
- $i_c(t) = 0$

Key property	same interpolating function independently of N and of location
drawback	lack of smoothness

6. Sinc interpolation formula

   A remarkable result:

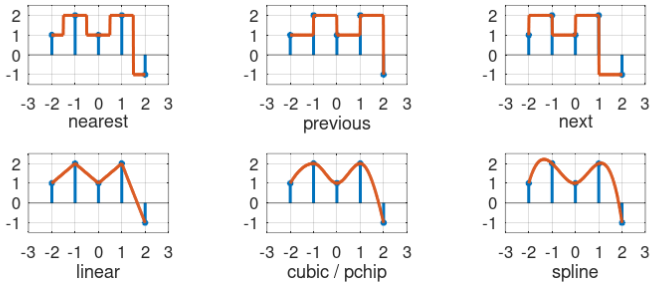
$$\lim_{N\to\infty} L_n^{(N)}(t) = sinc(t-n)$$

   In the limit, local and global interpolation are the same!

$$x(t) = \sum_{n=-N}^{N} x[n] sinc\left(\frac{t-nT_s}{T_s}\right)$$

7. Octave Interpolation Overview

   Octave manual Chapter 29.1 One-dimensional Interpolation



$$x(t) \sum_{n=-N}^{N} x[n] i_c(t-n)$$

1.1.3 Sampling of bandlimited functions

1. The spectrum of interpolated signals

   (a) Sinc interpolation the ingredients:

   - discrete-time signal $x[n]$, $n \in \mathbb{Z}$ (with DTFT $X(e^{j\omega})$)

- interpolation interval T_s
- the sinc function (properly scaled to have zero crossing at multiple of T_s

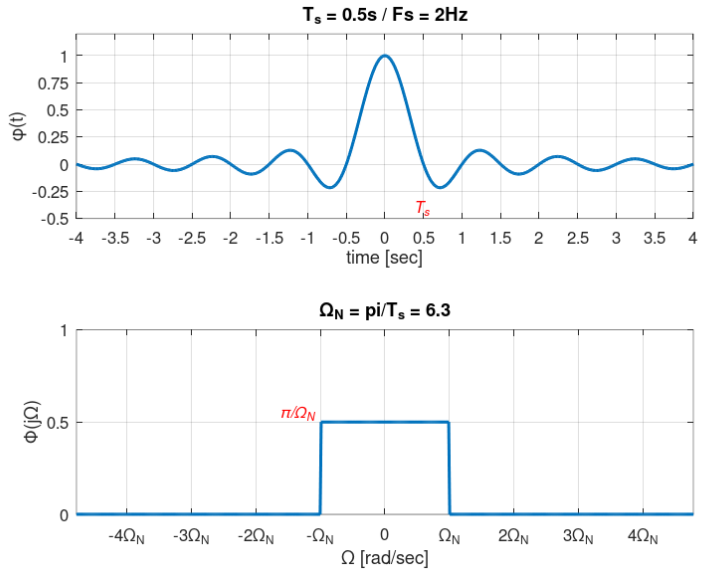
the result

- a smooth, contious-time signal $x(t)$, $t \in \mathbb{R}$

What does the spectrum of x(t) look like?

2. Key Facts about the sinc

$$\phi(t) = sinc\left(\frac{t}{T_s}\right) \quad \longleftrightarrow \quad \Phi(j\Omega) = \frac{\pi}{\Omega_N} rect\left(\frac{\Omega}{2\Omega_N}\right)$$
$$T_s = \frac{\pi}{\Omega_n} \qquad\qquad \Omega_N = \frac{\pi}{T_s}$$



3. Sinc interpolation

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] sinc\left(\frac{t - nT_s}{T_s}\right)$$

1.1.4 Spectral representation (I)

1.1.5 Spectral representation (II)

Let's analyse the formula

$$X(j\Omega) = \sum_{n=-\infty}^{\infty} x[n] \left(\frac{\pi}{\Omega_N}\right) rect\left(\frac{\Omega}{2\Omega_N}\right) e^{-jnT_s\Omega}$$
$$= \left(\frac{\pi}{\Omega_N}\right) rect\left(\frac{\Omega}{2\Omega_N}\right) \sum_{n=-\infty}^{\infty} x[n] e^{-j(\pi/\Omega_N)\Omega\, n}$$

$$= \begin{cases} \left(\frac{\pi}{\Omega_N}\right) X(e^{j\pi(\Omega/\Omega_N)}) & |\Omega| \leq \Omega_N \\ 0 & otherwise \end{cases}$$

The DTFT is periodic and the periodic is

$$\sum_{n=-\infty}^{\infty} x[n] e^{-j(\pi/\Omega_N)\Omega\, n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \text{ with } \omega = 2 \cdot \Omega_N$$

Spectrum of Sinc-Sampling

The spectrum of $x(t)$ is equal to the scaled version of the DTFT of the sequence between $-\Omega_N$ and Ω_N.

1.1.6 Sampling of nonbandlimited functions

1. Raw Sampling Raw sampling is when we don't care about first taking the inner product with the sinc function. So we just take x(t) and every T_s seconds, we take a sample.

   The continous-time complex exponential

Table 3: Aliasing

sampling period	**digital frequency**	$\{\hat{x}\}$
$T_s < \pi/\Omega_0$	$0 < \omega_o < \pi$	$e^{j\Omega_0}$
$\pi/\Omega_0 < T_s < 2\pi/\Omega_0$	$\pi < \omega_0 < 2\pi$	$e^{j\Omega_1}$: $\Omega_1 = \Omega_0 - 2\pi/T_s$
$T_s > 2\pi/\Omega_0$	$\omega_0 > 2\pi$	$e^{j\Omega_2}$: $\Omega_2 = \Omega_0 mod(2\pi/T_s)$

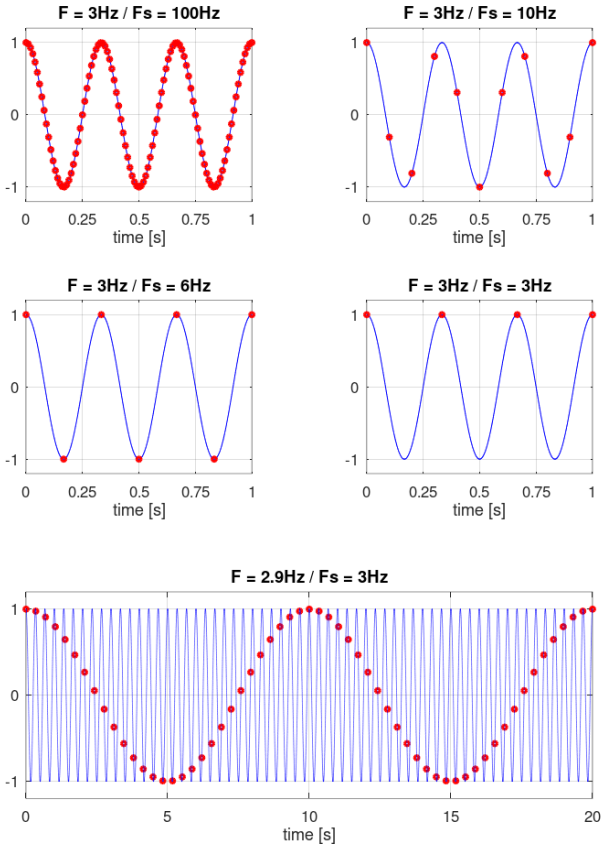
2. Sinusoidal Aliasing

$$x(t) = cos(2_o t)$$
$$x[n] = x(nT_s) = cos(\omega_0 n)$$

with

$$F_s = \frac{1}{T_s}$$
$$\omega_o = 2\pi(\frac{F_0}{F_s})$$

(a) Aliasing: Sampling a Sinusoid



3. Aliasing for arbitrary spectra A contiuous time signal x_c sampled every T_s seconds gives a sequence x[n]. Which is equal to the contious time signals at multiples of the sampling intervals T_s.

   - $x_c(t) \Rightarrow x[n] = x_c(nT_s)$

   In Fourier Transform domain we have a spectra of the continuous time signal $X_c(j\Omega)$. And at the output we have a discrete time Fourier Transform of the sequence $X(j\omega)$. What is that going to be in genaral? And how is it going to be related to the input spectrum?

   - $X(j\Omega) \Rightarrow X(j\omega) =?$

   The key idea:

   - pick T_s and set $\Omega_N = \pi/T_s$
   - pick $\Omega_= < \Omega_N$

$$
\begin{aligned}
e^{j\Omega_0 t} &\rightarrow e^{j\Omega_0 T_s n} \\
e^{j(\Omega_0+2\Omega_N)t} &\rightarrow e^{j(\Omega_0+2\Omega_N)T_s n}, \text{ add } 2\Omega_N \\
e^{j(\Omega_0+2\Omega_N)t} &\rightarrow e^{j(\Omega_0 T_s n+2\Omega_N T_s n)}, \text{ expand this product} \\
e^{j(\Omega_0+2\Omega_N)t} &\rightarrow e^{j(\Omega_0 T_s n+\frac{2\pi}{T_s} T_s n)} \\
e^{j(\Omega_0+2\Omega_N)t} &\rightarrow e^{j(\Omega_0 T_s n+2)}, \ e^{j2} \text{ is equal to one} \\
e^{j(\Omega_0+2\Omega_N)t} &\rightarrow e^{j\Omega_0 T_s n}, \text{ the same discrete time sequence as before}
\end{aligned}
$$

So we do not see the higher frequency complex exponential, it simply looks like the lower frequency exponential Ω_0.

> So in general, if we have two frequencies sampled, the higher frequency is aliased back onto the lower frequency and we simply see the sum of these two.

(a) Spectrum of raw-sampled signals

- start with the inverse Fourier Transform

$$x[n] = x_c(nT_s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_c(j\Omega)e^{j\Omega MT_s}d\Omega$$

- frequencies $2\Omega_N$ apart will be aliased, so split the integration interval

$$x[n] = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{(2k-1)\Omega_N}^{(2k+1)\Omega_N} X_c(j\Omega)e^{j\Omega MT_s}d\Omega$$

- with a change of variable and using $e^{j(\Omega+2k\Omega_N)T_sM} = e^{j\Omega T_sM}$

$$x[n] = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\Omega T_sM}^{\Omega T_sM} X_c(j(\Omega-2k\Omega_N)e^{j\Omega MT_s}d\Omega$$

- interchange summation and integral

$$x[n] = \frac{1}{2\pi}\int_{-\Omega T_sM}^{\Omega T_sM}\left[\sum_{k=-\infty}^{\infty} X_c(j(\Omega-2k\Omega_N)\right]e^{j\Omega MT_s}d\Omega$$

- periodization of the spectrum; define

$$X_c(j\Omega) = \sum_{k=-\infty}^{\infty} X_c(j(\Omega-2k\Omega_N)$$

- so that

$$x[n] = \frac{1}{2\pi}\int_{-\Omega T_sM}^{\Omega T_sM} X_x(j\Omega)e^{j\Omega MT_s}d\Omega$$

- set $\omega = \Omega T_s$

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{T_s}X_c(j\frac{\omega}{T_s})e^{j\omega M}d\omega \\ &= IDTFT\ \frac{1}{T_s}X_c(j\frac{\omega}{T_s}) \\ X(e^{j\omega}) &= \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s}\right) \end{aligned}$$

$$X(e^{j\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s}\right)$$

(b) **TODO** Example: signal bandlimited to Ω_0 and $\Omega_N > \Omega_0$

(c) **TODO** Example: signal bandlimited to Ω_0 and $\Omega_N = \Omega_0$

(d) **TODO** Example: signal bandlimited to Ω_0 and $\Omega_N < \Omega_0$

(e) **TODO** Example: non-bandlimited signal

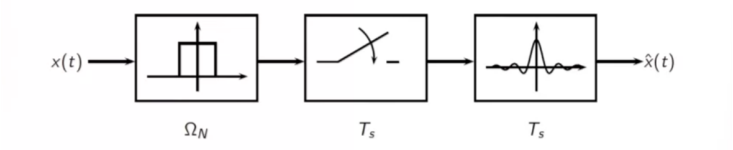
4. Sampling strategies

   given a sampling period T_s

   - if the signal is bandlimited to π/T_s or less, raw sampling is fine i.e. equivalent to sinc sampling up to scaling factor T_s.
   - if the signal is not bandlimited, two choices:
     - bandlimit via lowpass filter in the *continuous-time domain* before sampling i.e. sinc sampling
     - or raw sample the signal an incur aliasing
   - aliasing sounds horrible, so usualle we choose to bandlimit in continuous time

   (a) Sinc Sampling and Interpolation

$$\hat{X}[n] = \langle sinc\left(\frac{t - nT_s}{T_s}\right), x(t)\rangle = (sinc_{Ts} * x)(nT_s)$$
$$\hat{X}[n] = \sum_n x[n]\ sinc\left(\frac{t - nT_s}{T_s}\right)$$



1.1.7 Quantization

1. Stochastic signal processing

   (a) Terminology (from W.Smith)

   Mean, Average

   $$\mu = \frac{1}{N}\sum_{i=0}^{N-1} x_i = (x_0 + x_1 + x_2 + ... + x_{N-1})/N$$

   In electronics, the mean is commonly called the **DC** (direct current) value. Likewise, **AC** (alternationg current) refers to how the signal fluctuates around the mean value. For simple repetitive waveform, its excursion can be described by its peak-to peak value. If the signal has a random nature, a more generalized method must be used.

   Standard Deviation

   $$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=0}^{N-1}(x_i - \mu)^2)} = \sqrt{(x_0 - \mu)^2 + (x_1 - \mu)^2 + ... + (x_{N-1} - \mu)}$$

   $|x_i - \mu|$ describes how far the i^{th} sample **deviates** (differs) from the mean. The **average deviation** of a signal is found by summing the deviations of all the individual samples, and then dividing by the number of samples N. We take the absolute value of each deviation before summation; otherwise the positive and the negative termss would average to zero.

   The **standard deviation** is similar to the average deviation, except the averaging is done with power instead of amplitude.

   The standard deviation is a measure of how far the signal fluctuates from the mean.

   Variance

   $$\sigma^2 = \frac{1}{N-1}\sum_{i=0}^{N-1}(x_i - \mu)^2)$$

The variance represents the power of signal fluctuation from the mean.

RMS Root Mean Square The standard deviation measures only the AC portion of a signal, while rms value measures both the AC and DC components. If a signal has no DC component, its rms value is identical to its standard deviation.

SNR Signal to Noise Ratio

$$snr = \frac{mean}{standarddeviation} = \frac{\mu}{\sigma} = \frac{\frac{1}{N}\sum_{i=0}^{N-1} x_i}{\sqrt{\frac{1}{N-1}\sum_{i=0}^{N-1}(x_i - \mu)^2)}}$$

CV Coefficent Variation

$$CV = \frac{standard\ deviation}{mean} \times 100 = \frac{\sqrt{\frac{1}{N-1}\sum_{i=0}^{N-1}(x_i - \mu)^2)}}{\frac{1}{N}\sum_{i=0}^{N-1} x_i} \times 100$$

Probability Density Function PDF The probability density function is a measure of the likelihood of a particular value occuring in some function.

- PDF values are never negative $f(x) \geq 0$
- The sum of all the PDF values is one: $\int_{-\infty}^{\infty} f(x)dx = 1$
- Mean: $\mu_x = \int_{-\infty}^{\infty} x \cdot f(x)dx$
- Variance: $\mu_x^2 = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f(x)dx = \int_{-\infty}^{\infty} x^2 \cdot p(x)dx - \mu_x^2$

(b) **TODO** Deterministic vs. stochastic

(c) A simple discrete-time random signal generator

For each new sample, toss a fair coin:

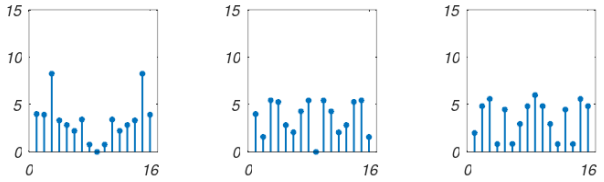
$$x[n] = \begin{cases} +1 & \text{if the outcome of the n-th toss is head} \\ -1 & \text{if the outcome of the n-th toss is tail} \end{cases}$$

- each sample is independet from all others
- each sample value has 50% probability

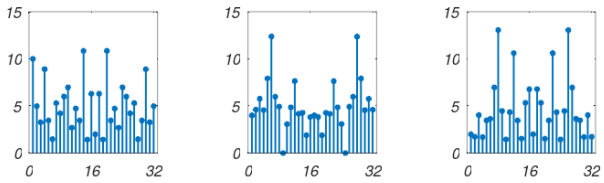
- every time we turn on the generator we obtain a different *realization* of the signal
- we know the "mechanics" behind each instance
- but how can we analyze a random signal?

(d) Spectral Properties

- let's try with the DFT of a finite set of random samples



- every time it's a different
- try with more data



- no clear pattern

(e) Averaging

- when faced with random data an intuitive response is to take "averages"
- in probability theory the average is across realizations and it's called Expectation
- Expectation for the coin-toss signal

$$\text{E[x[n]]} = \text{-1} \times P[\text{n-th toss is tail}] + 1 \times P[\text{n-th toss is head}] = 0$$

(f) so the average value for each sample is zero….

(g) as a consequence, averaging the DFT will not work

(h) $E[x[n]] = 0$

(i) however the signal "moves", so its energy over power must be nonzero

2. **TODO** Averaging the DFT

3. Energy and power

   - the coin-toss signal has infinite energy

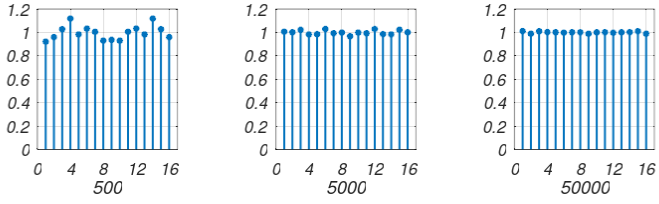
$$E_x = \sum_{k=-\infty}^{\infty} |x[n]|^2 = \lim_{N\to\infty} = \infty$$

   - however it has finite power over any interval:

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = 1$$

4. Averaging the DFT's square magnitude, normalized

   - pick an interval length N
   - pick an number of iterations M
   - run the signal generator M times and obtain M N-point realizations
   - compute the DFT of each realizations
   - average their square magnitude divided by N



5. Power spectral density

Power Spectral Density

$$P[k] = E\left[|X_N[k]|^2 / N\right]$$

- it looks very much as if P[k] = 1
- if $|X_N[k]|^2$ tends to the *energy* distribution in frequnec….
- … $|X_N[k]|^2 / N$ tends to the *power* distribution (aka **density**) in frequency

PSD

The frequency-domain representation for stochastic processes is the power spectral density: $P[k] = \frac{1}{N}|X_N[k]|^2$

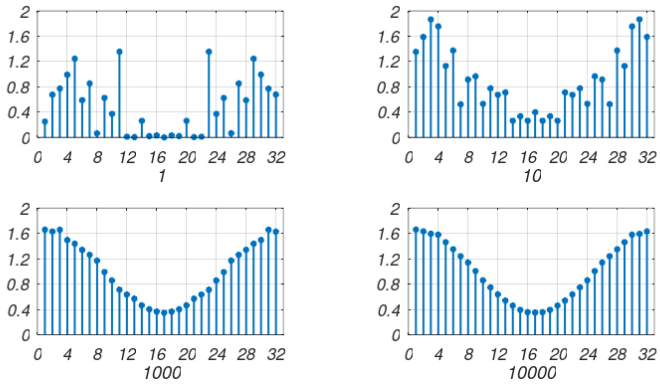
6. Power spectral density: Intuition

- P[k] = 1 means that the power is equally distributed over all frequencies
- i.e. we cannot predict the signal moves ”slowly” or ”super-fast”
- this is because each sample is independent of each other: we could have a realization of all ones or a realization in which the sign changes every other sample or anything in between.

7. Filtering a random process

- let’s filter the random process with a 2-point Moving Average filter
- y[n] = (x[n] + x[n-1])/2
- what is the power spectral density
- pick an interval length N
- pick an number of iterations M
- run the signal generator M times and obtain M N-point realizations

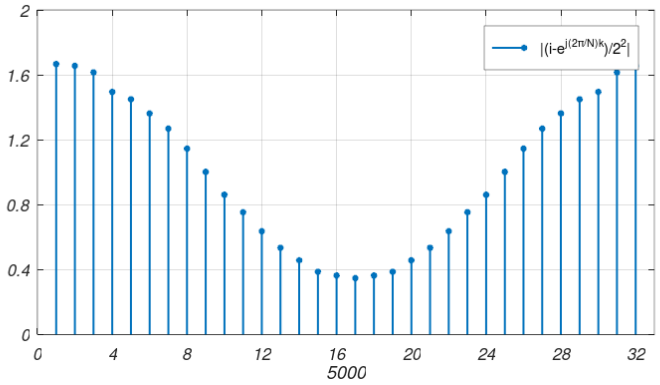
- filter all M-realization
- compute the DFT of each filtered realizations
- average their square magnitude divided by N



The frequency response of the moving average filter

$$H(e^{j\omega}) = \frac{1 - e^{j\omega}}{2}$$

wheraes the ploted shape is nothing but the square magnitude of the moving average filter evaluated at the DFT point $\frac{2\pi}{N}k$



8. Stochaistic signal processing

- a stochastic process is characterized by its power spectral density (PSD)
- it can be shown (see text book) that the PSD is

  $$P_x(e^{j\omega}) = DTFT\{r_x[n]\}$$

  where

  $$r_x[n] = E\left[x[k]x[n+k]\right]$$

  is the autocorrelation of the proess
- for a filtered stochastic process $y[n] = H\{x[n]\}$, it is:

  $$P_x(e^{j\omega}) = |H(e^{j\omega})|^2 P_x(e^{j\omega})$$

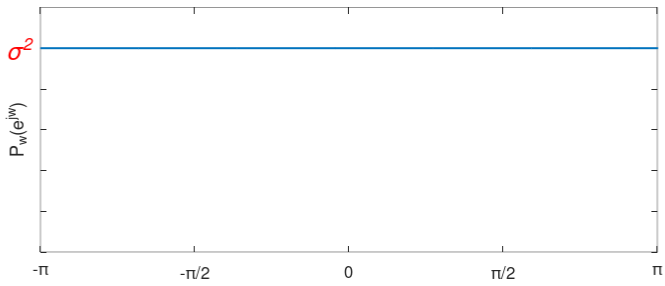
  In Words

  The power spectral density of the output is equal to the power spectral density of the input times the frequency response in magnitude square.

  - filters designed for deterministic signals work (in magnitude) in the stochaistic case
  - we lose the concept of phase since we don't know the shape of a realization in advance.

9. **TODO** Noise
10. White noise
    - white indicates uncorrelated samples
    - $r_w[n] = \sigma^2\delta[n]$: The autocorrleation is zero except at zero where it will take the value of the variant
    - $P_w(e^{j\omega} = \sigma^2$: The power spectral density is the constant σ^2 wher σ is the variance of the stochastic signal.

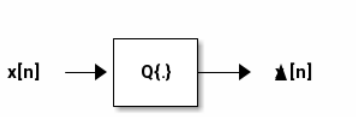
Graphically the power spectral density of a white signal couldn't be any simpler.



- the PSD is independent of the probability distribution of the single samples (depends only on the variance)
- distribution is important to estimate bounds for the signal
- very often a Gaussioan distribution models the experimental data the best
- **AWGN**: additive white Gaussian noise

Quantization

1. Quantization schemes
   - digital devices can only deal with integers (b bits per sample)
   - we need to map the range of a signal onto a finite set of values
   - irreverible loss of information $\longrightarrow$ Quantization Noise



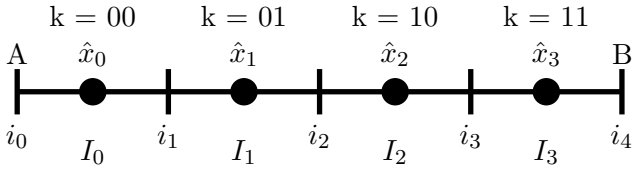
Several factors at play:

- storage budget (bits per sample)
- storage scheme (fixed point, floating point)
- properties of the input (input $\in \mathbb{C} \rightarrow$ output $\in \mathbb{N}$

2. Scalar quantization

   The simplest quantizer:

   - each sample is encoded individually (hence scalar)
   - each sample is quantized independently (memoryless quantization)
   - each sample is encoded using R bits



   - what are the optimal interval boundries I_k ?
   - what are the optimal quantization values $\hat{x}_k$?

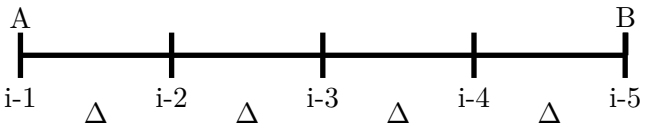
3. Quantization Error

$$e[n] = Q\{x[n]\} - x[n] = \hat{x}[n] - x[n]$$

   - model x[n] as a stochastic process
   - model error as a white noise sequence
     - error samples are uncorrelated
     - all error samples have the same distribution
   - we need statistics of the input to study the error

   (a) Uniform quantization

   - simple but very general case
   - range is split into 2^R equal intervals of width $\Delta = (B-A)2^{-R}$
   - $f_x(\tau)$: PDF of the input
   - **With a Bit-Rate R** of 2 bits is a region split into 4 equally spaced intervals



   Mean Square Error is the variance of the error signal

$$\begin{aligned}\sigma_e^2 &= E\left[|Q\,x[n] - x[n]|^2\right] \\ &= \int_A^B f_x(\tau)(Q\{\tau\} - \tau)^2 d\tau \\ &= \sum_{k=0}^{2^R-1} \int_{I_k} f_x(\tau)(\hat{x}_k - \tau)^2 d\tau\end{aligned}$$

- $Q\{\tau\} - \tau$: Error function

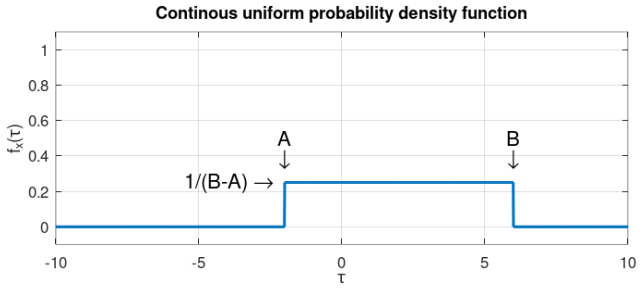
error depends on the probability density of the input. The calculation is done in the following subsections here we already subsitutes the results.

$$\begin{aligned}\sigma_e^2 &= \sum_{k=0}^{2^R-1} \int_{A+k\Delta}^{A+k\Delta+k} \frac{(A + k\Delta + \Delta/2 - \tau)^2}{B - A} d\tau \\ &= 2^R \int_0^{\Delta} \frac{(\Delta/2 - \tau)^2}{B - A} d\tau \\ &= \frac{\Delta^2}{12} \text{ with } \Delta = B - A2^R\end{aligned}$$

Quantization Error

$$\sigma_e^2 = \frac{\Delta^2}{12} \text{ with } \Delta = \frac{B - A}{2^R}$$

(b) Uniform quantization of uniform input



(c) Uniform-input hypothesis

$$f_x(\tau) = \frac{1}{B - A}$$

$$\sigma^2 = \sum_{k=0}^{2^R-1} \int_{I_k} \frac{(\hat{x} - \tau)^2}{B - A} d\tau$$

(d) Find the optimal quantization point by minimazing the error

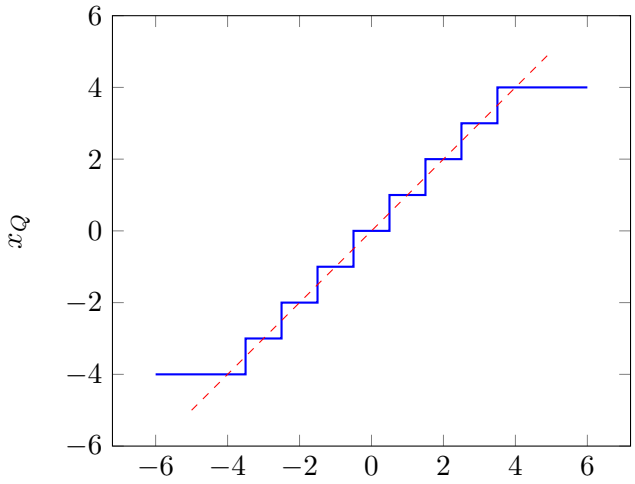
$$\begin{aligned}
\frac{\partial \sigma_e^2}{\partial \hat{x}_m} &= \frac{\partial}{\partial \hat{x}_m} \sum_{k=0}^{2^R-1} \int_{I_k} \frac{(\hat{x}_k - \tau)^2}{B - A} d\tau \\
&= \int_{I_m} \frac{2(\hat{x}_m - \tau)^2}{B - A} d\tau \\
&= \frac{2(\hat{x}_m - \tau)^2}{B - A} \Big|_{A+m\Delta}^{A+m\Delta+\Delta}
\end{aligned}$$

Minimizing the error:

$$\frac{\partial \sigma_e^2}{\partial \hat{x}_m} = 0 \text{ for } \hat{x}_m = A + m\Delta + \frac{\Delta}{2}$$

optimal quantization point is the interval's midpoint, for all intervals

Quantization Characteristic



The quantiziser associates each quantization interval to its midpoint.

4. Error Analysis

Error Energy

$$\sigma_e^2 = \frac{\Delta^2}{12} \text{ with } \Delta = \frac{B-A}{2^R}$$

Signal Energy

$$\sigma_x^2 = \frac{(B-A)^2}{12}$$

Signal to Noise Ratio

$$SNR = 2^{2R}$$

Signal to Noise Ratio in db

$$SNR_{db} = 10_{10}2^{2R} \approx 6R\ db$$

The 6db/bit rule of thumb

- a compact disk has 16 bits/sample:

$$maxSNR_{db} = \approx 6R\ db = 6 \cdot 16\ db = 96\ db$$

- a DVD has 24 bits/sample:

$$maxSNR_{db} = \approx 6R\ db = 6 \cdot 24\ db = 144\ db$$

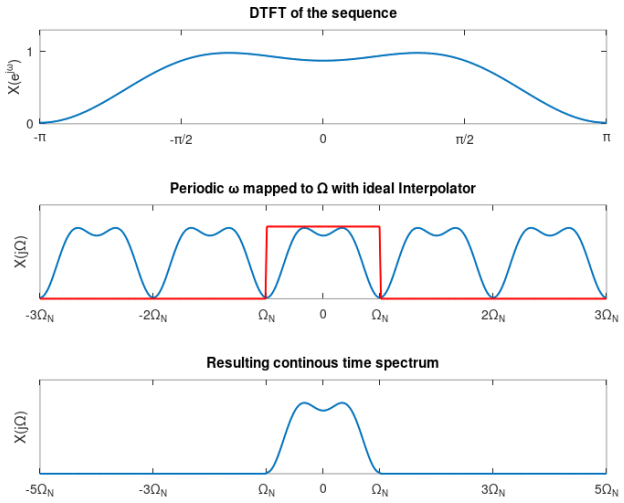
1.1.8 Practical interpolation and sampling

1. Time Domain to Discrete
2. Discrete to Time Domain

ideally	x[n] $\Rightarrow$ (x(t)	in practice
$x(t) = \sum_{-\infty}^{\infty} x[n] sinc\left(\frac{t-nT_s}{T_s}\right)$		$x(t) = \sum_{n=-\infty}^{\infty} x[n]\, i\left(\frac{t-nT_s}{T_s}\right)$
$X(j\Omega) = \frac{\pi}{\Omega_N} X(e^{j\pi\Omega/\Omega_n} rect\left(\frac{\Omega}{\Omega_N}\right))$		$X(j\Omega) = \frac{\pi}{\Omega_N} I\left(j\pi\frac{\Omega}{\Omega_N}\right) X(e^{j\pi\frac{\Omega}{\Omega_N}})$

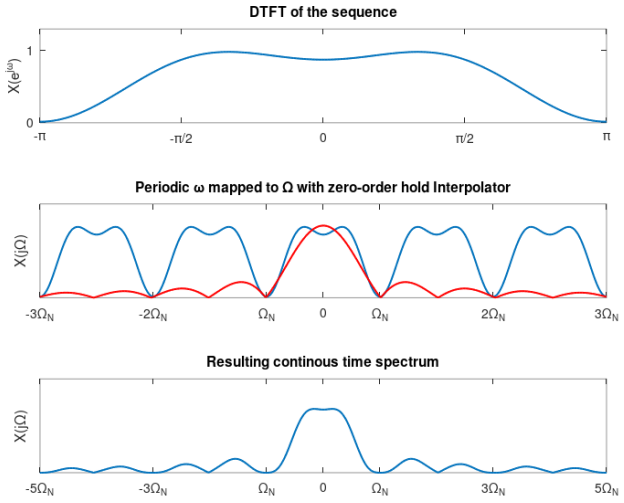
(a) Ideal Interpolator

- Frequency Domain: Rect
- Time Domain: sinc



(b) Zero-Order Hold Interpolator

- Frequency Domain: sinc
- Time Domain: rect



(c) First-Order Interpolator

- Frequency Domain: $sinc^2$
- Time Domain: Triangle

```
N = 64                              # Size of the sequence
q = 5;                              # divisior for Omega_N
Omega_N = pi/q;                     # Nyquiste Frequency
omega = -pi:1/N:pi;                 # Axis Discrete Time Frequency Domain
Omega = pi.*omega./Omega_N;         # Axis Continous Time Frequency Domain - pha

XD = 0.43*(sin(omega+pi/2) + 0.5*cos(2*omega+pi)) + 0.66;      # DTFT{x[n]}
XT = (0.43*(sin(Omega+pi/2) + 0.5*cos(2*Omega+pi)) + 0.66);    # FT{x(t)} =>
# I = (((omega + Omega_N) >= 0) - ((omega - Omega_N) >= 0));     # rect:   i
# I = abs(sinc(omega*pi/2));                                     # sinc:   Z
I = abs(sinc(omega*pi/2).^2)                                   # sinc^2: Firs

figure( 1, "visible", "off" )      # Do not open the graphic window in org

subplot(3,1,1)
plot(omega, XD, "linewidth", 2 );
axis([-pi,pi,0,1.3]);
set(gca, "fontsize", 20)
set(gca,'XTick',-pi:pi/2:pi)
set(gca,'XTickLabel',{'-\pi','-\pi/2','0','\pi/2','\pi'})
set(gca, 'YTick', 0:1);
ylabel('X(e^j^\omega)');
grid off;
title('DTFT of the sequence')

subplot(3,1,2)
plot(omega, XT, "linewidth", 2);
hold;
plot(omega, I, "linewidth", 2, "color", "red" );
axis([-pi,pi,0,1.3]);
set(gca, "fontsize", 20)
set(gca,'XTick',[-q*Omega_N:2*Omega_N:-Omega_N, 0, Omega_N:2*Omega_N:q*Omega
set(gca,'XTickLabel',{'-3\Omega_N','-2\Omega_N', '\Omega_N','0','\Omega_N',
set(gca, 'YTick', 2);
                                    #set(gca, 'YTickLabel',{'0', '1'})
```

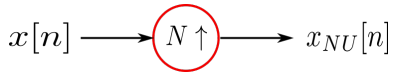
```
ylabel('X(j\Omega)');
grid off;
title('Periodic \omega mapped to \Omega with first-order Interpolator')

subplot(3,1,3)
plot(omega, XT.*I, "linewidth", 2 );
axis([-pi,pi,0,1.3]);
set(gca, "fontsize", 20)
set(gca,'XTick',[-q*Omega_N:2*Omega_N:-Omega_N, 0, Omega_N:2*Omega_N:q*Omega
set(gca,'XTickLabel',{'-5\Omega_N','-3\Omega_N', '\Omega_N','0','\Omega_N',
set(gca, 'YTick', 2);
ylabel('X(j\Omega)');
title('Resulting continous time spectrum')
grid off


print -dpng "-S800,600" ./image/w7_first_order_interpolation_01.png;
ans = "./image/w7_first_order_interpolation_01.png";
```

1.1.9 Multirate signal Processing

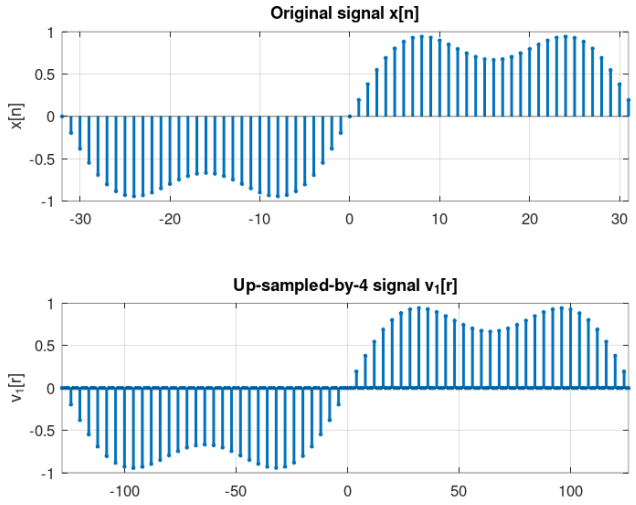
Book: Multirate Filtering for Digital Signal Processing Matlab Applications

1. Upsampling

$$x_{NU}[n] = \begin{cases} x[k] \text{ for } n = kN,\ k \in \mathbb{Z} \\ 0 \text{ otherwise} \end{cases}$$



   (a) **TODO** Plot time domain upsampling example



Spectral Representation

$$X_{NU}(e^{j\omega}) = X(e^{j\omega N})$$

The spectrum of the upsampled sequence is equal to the spectrum of the original sequence contracted by a factor N.

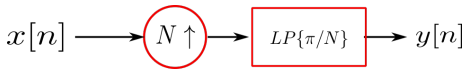
(b) **TODO** Plot frequency domain upsampling example

(c) What we don't like

- in the time domain: zeors between nonzero samples are not "natural"
- in the frequency domain: extra replicas of the spectrum; can we get rid of them?

(d) The ideal digital interpolator

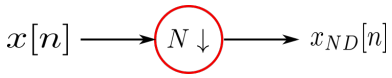
$$x_{ND}[n] = x[nN]$$



with a discrete time ideal low pass filter with cut off frequency π/N

2. Downsampler

   Every $N-1$ input sample is discareded resp. only every N-th sample is used.



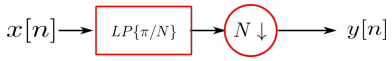
   (a) **TODO** Plot time domain downsampling example

   Spectral Representation

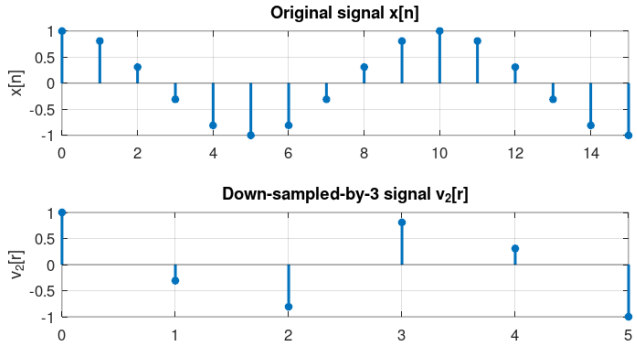
$$X_{ND}(e^{j\omega}) = \frac{1}{N}\sum_{m=0}^{N-1} X(e^{j\frac{\omega-2\pi m}{N}})$$

   There's 1/N normalization factor in front, that multiplies the sum of capital N copies of the original spectrum, where each copy has been shifted by a multiple of 2 pi/N, and where the frequency axis has been stretched out by a factor of capital N. So the interval -pi/N, pi/N becomes the interval -pi, pi.

   (b) **TODO** ideal interpolator



   (c) **TODO** Plot frequency domain downsampling example



3. **TODO** Sampling Rate Change