Contents

1 Week 3 Module 3:

1.1 Part 1 - Introduction to Fourier Analysis

1.1.1 Introduction to Fourier Analysis

1. The Frequency Domain

   (a) Oscillations are every where

   - A train has got an engine which makes the wheels turn in circular motion
   - Waves, ebb and flow can be modeled as sinusoidal fashion
   - Musical instruments generates sound by vibrating at a certain fundamental frequency
   - Intuitivly: things that don't move in circles can't last
     - bombs
     - rockets
     - human beeings

   (b) Descriptin of the oscillations in the plane

   Period P

   Frequency $f = \frac{1}{P}$

   Ordinate $\sin(ft)$

   Abscissa $\cos(ft)$

   (c) Example Sinusoidal Detectors in our Body:

   cochlea In the inner ear that detects air pressure sinusoids at frequenies from 20 to 20kHz

   retina In the eye to detect electromagnetic sinusoids with frequency 430THz to 790THz. This is the frequency of lights in the visible spectrum

Humans anlayze complex signals (audio, images) in terms of their sinusoidal components

Frequency Domain semms to be as good a the time domain

(d) Fundamental Questions: Can we decompse any signal into sinusoidal elements?

- Yes, Fourier showed us how to do it exactely
- Analysis
  - From time domain to frequency domain
  - Find the contribution of different frequencies
  - Discover "hidden" signal properties
- Synthesis
  - From frequency domain to time domain
  - Create signals with known frequency content
  - Fit signals to specific frequency regions

2. The DFT as a change of basis

- let's start with finite-lenght signals (i.e. vectors in $\mathbb{C}^{\mathbb{N}}$
- **Fourier analysis is a simple change of basis**
- a change of basis is a change of perspective
- a change of perspective can reveal things (if the basis is good)

(a) The Fourier Basis for $\mathbb{C}^N$ **Claim:** the set of N signals in $\mathbb{C}^N$

$$w_k[n] = e^{j\frac{2\pi}{N}nk}\text{with } n, k = 0, 1, ..., N-1$$

is an orthogonal basis in $\mathbb{C}^N$.

- $\mathbb{C}^N$: N different vectors all with lenght N
- k: is the index that indicates different vectors (signals), k = 0,...,N-1
- n: is the index that indicates different elements within the vector, n = 0,..,N-1
- $w_k[n] = e^{j\frac{2\pi}{N}nk}$: Form of the N-Signals
- $\omega = \dfrac{2\pi}{N}nk$: The index of the signal makes up the **Fundamental Frequency** of the complex exponential.

In Vector Notation:

$$\left\{\mathbf{w}^{(k)}\right\}_{k=0,1,..,N-1}$$

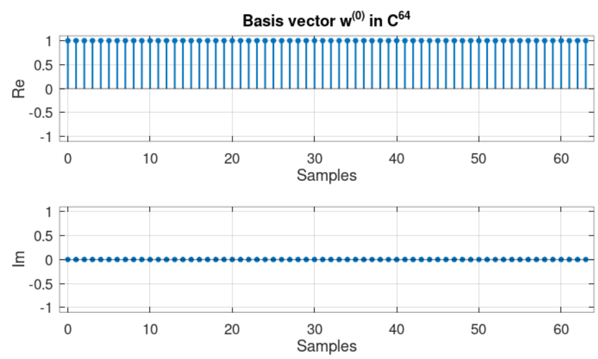
with

$$w_n^{(k)} = e^{j\frac{2\pi}{N}nk}$$

is an orthogonal basis in $\mathbb{C}^N$

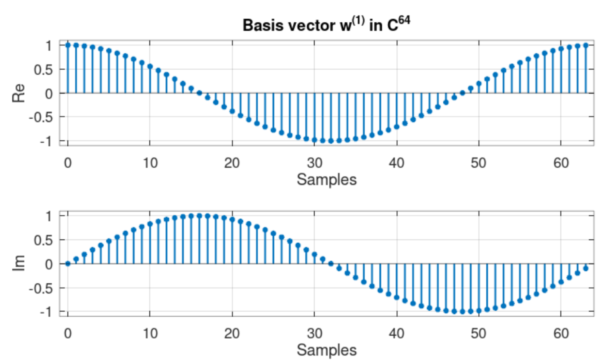
i. Basis vector $\mathbf{w}^{(0)} \in \mathbb{C}^{64}$

> **Fundamental Frequency**
>
> $$\omega = \frac{2\pi}{N}0 = 0$$



ii. Basis vector $\mathbf{w}^{(1)} \in \mathbb{C}^{64}$

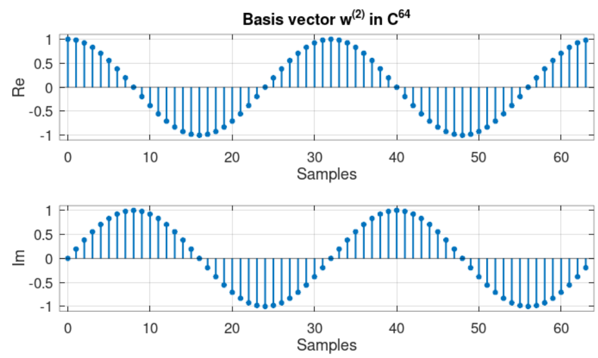
> **Fundamental Frequency**
>
> $$\omega = \frac{2\pi}{N}1 = \frac{2\pi}{N}$$



iii. Basis vector $\mathbf{w}^{(2)} \in \mathbb{C}^{64}$

Fundamental Frequency

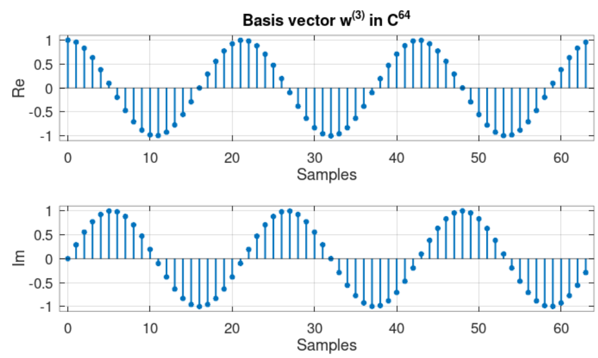
$$\omega = \frac{2\pi}{N}2$$



iv. Basis vector $\mathbf{w}^{(3)} \in \mathbb{C}^{64}$

Fundamental Frequency

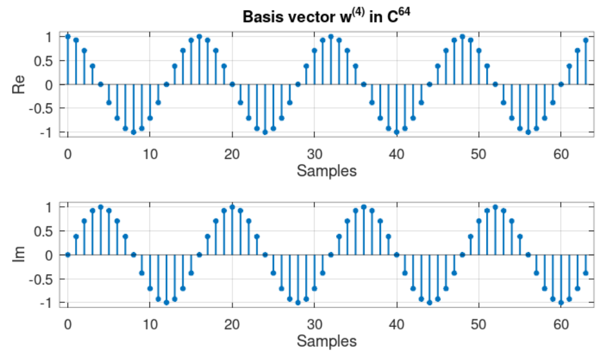
$$\omega = \frac{2\pi}{N}3$$



v. Basis vector $\mathbf{w}^{(4)} \in \mathbb{C}^{64}$

Fundamental Frequency

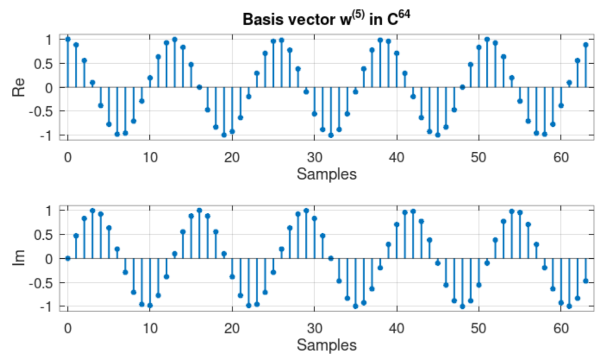
$$\omega = \frac{2\pi}{N}4$$



vi. Basis vector $\mathbf{w}^{(5)} \in \mathbb{C}^{64}$

Fundamental Frequency

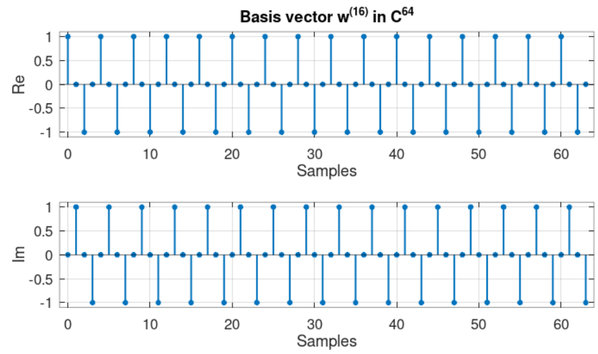
$$\omega = \frac{2\pi}{N} 5$$



vii. Basis vector $\mathbf{w}^{(16)} \in \mathbb{C}^{64}$

Fundamental Frequency

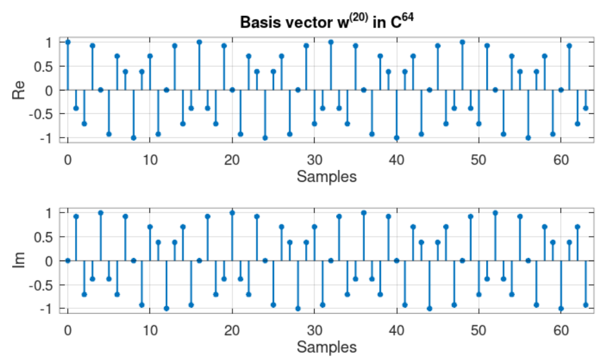
$$\omega = \frac{2\pi}{N} 2 = \frac{\pi}{2}$$



viii. Basis vector $\mathbf{w}^{(20)} \in \mathbb{C}^{64}$

Fundamental Frequency

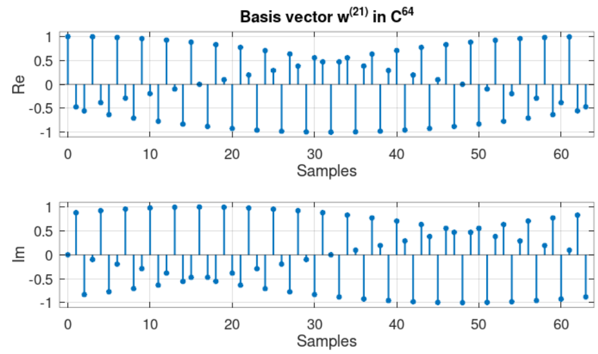
$$\omega = \frac{2\pi}{N} 20$$



ix. Basis vector $\mathbf{w}^{(21)} \in \mathbb{C}^{64}$

Fundamental Frequency

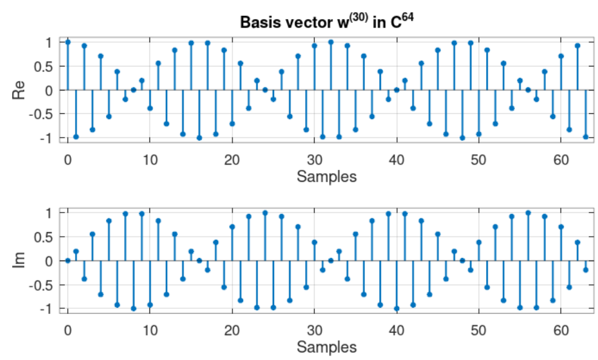
$$\omega = \frac{2\pi}{N} 21$$



x. Basis vector $\mathbf{w}^{(30)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 30$$

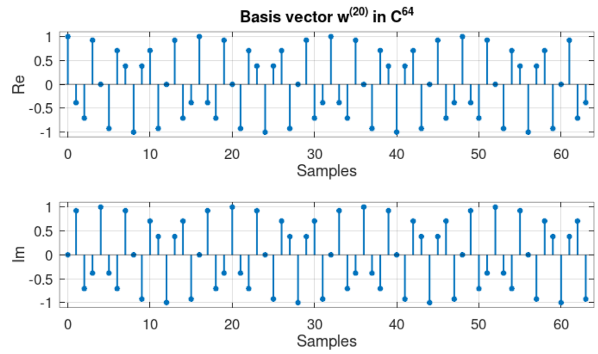


xi. Basis vector $\mathbf{w}^{(31)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 20$$

The sign alternation is a tell tale sign of a high frequency sinusoid.

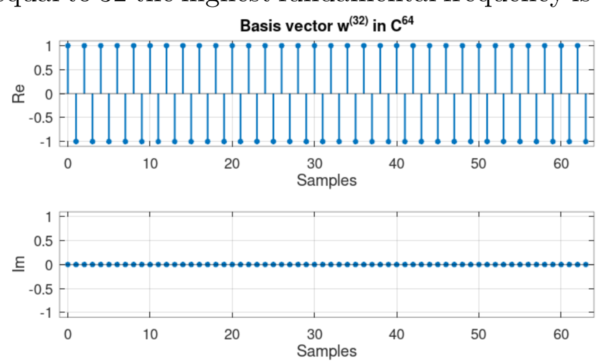


xii. Basis vector $\mathbf{w}^{(32)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N}32 = \frac{2\pi}{64}32 = pi$$

At k equal to 32 the highest fundamental frequency is reached.

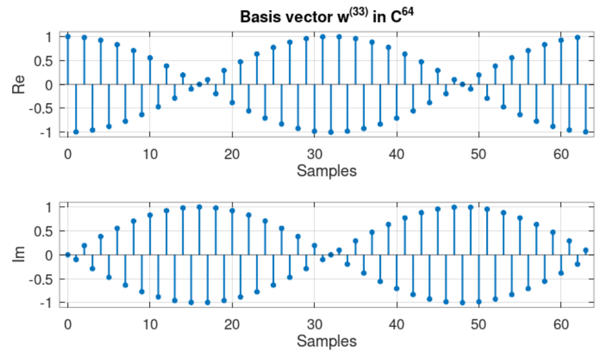


xiii. Basis vector $\mathbf{w}^{(33)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N}33$$

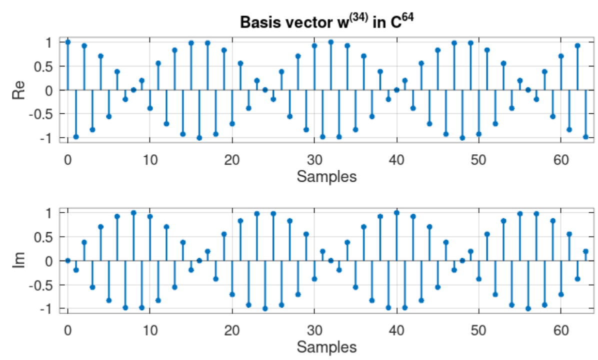
If we go forward with the index the apparent speed of the point decreases and the directon of the rotation changes from counter clockwise to clockwise.



xiv. Basis vector $\mathbf{w}^{(34)} \in \mathbb{C}^{64}$

Fundamental Frequency

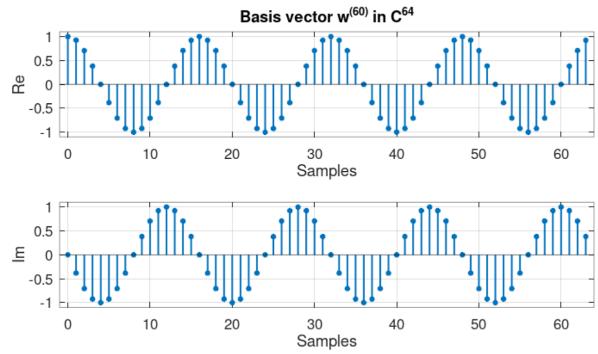
$$\omega = \frac{2\pi}{N} 34$$



xv. Basis vector $\mathbf{w}^{(60)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 60$$

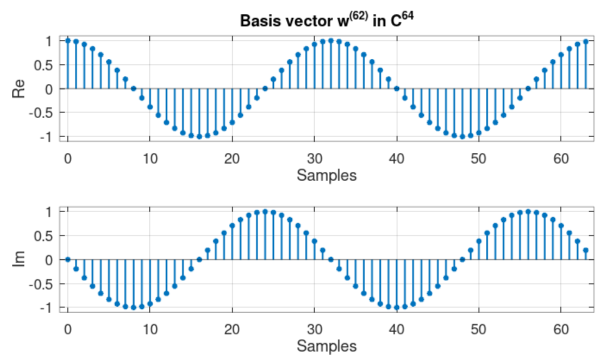


xvi. Basis vector $\mathbf{w}^{(62)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 62$$

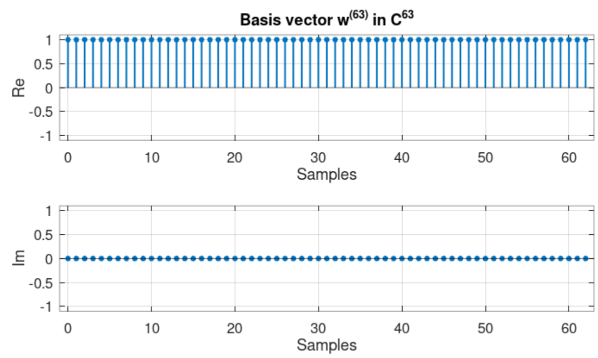
So here, for instance, if you compare this basis vector to $\mathbf{w}^{(2)}$, you would see that the real part is the same, but the imaginary part has a sign inversion.



xvii. Basis vector $\mathbf{w}^{(63)} \in \mathbb{C}^{64}$

Fundamental Frequency

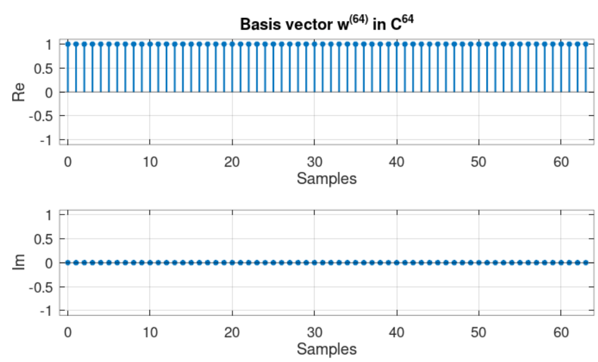
$$\omega = \frac{2\pi}{N} 63$$



xviii. Basis vector $\mathbf{w}^{(64)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 64$$



1.1.2 The Discrete Fourier Transform (DFT)

1. DFT definition

   (a) The Fourier Basis for $\mathbb{C}^N$ in "Signal" Notation

   $$w_k[n] = e^{j\frac{2\pi}{N}nk} \text{with } n, k = 0, 1, ..., N-1 \tag{1}$$

   (b) The Fourier Basis in Vector Notation

$$\{\mathbf{w}^{(k)}\}_{k=0,1\ldots N-1} \text{with } w_n^{(k)} = e^{j\frac{2\pi}{N}nk},\ n = 0, 1, \ldots N-1 \quad (2)$$

N N Dimension of vector space

k Index for different vectors and goes from 0..N-1

n Index of element in each vector goes from 0...N-1

(c) Basis Expansion Vector Notation

i. Analysis Formula

$$X_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle \text{ k = 0,...N-1} \quad (3)$$

X_k Coefficent for the new basis. Inner Product of **x** with each vector $\mathbf{w}^{(k)}$

x An arbitrary vector of $\mathbb{C}^N$

$\mathbf{w}^{(k)}$ New basis

ii. Synthesis Formula

$$\mathbf{x} = \frac{1}{N} \sum_{k=0}^{N-1} X_k \mathbf{w}^{(k)} \text{ k = 0,...N-1} \quad (4)$$

(d) **TODO** Change of basis in matrix form

(e) Basis Expansion Signal Notation

- Consider explicitely the operations involved in the transformation
- This notion is particulary useful if you want to consider the algorithmic nature of the transform

i. Analysis Formula N-point signal in the frequency domain

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk},\ k = 0, 1, ..N-1$$

$X[k]$ Signal vector in the frequency domain
$x[n]$ Signal vector in the (discrete) time domain
Reminder This is the inner Product in explicite form

ii. Synthesis Formula N-point signal in the time domain

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]e^{j\frac{2\pi}{N}nk},\ k = 0, 1, ..N-1$$

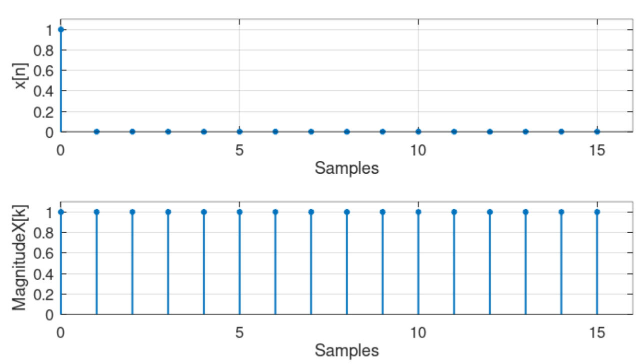
$X[k]$ Signal vector in the frequency domain
$\frac{1}{N}$ Normalisation coeficent
Reminder This is the inner Product in explicite fashion

2. Examples of DFT Calculation

(a) DFT of the impulse function

$$x[n] = \delta[n]$$

$$X[k] = \sum_{n=0}^{N-1} \delta[n]e^{-j\frac{2\pi}{N}nk} = 1$$

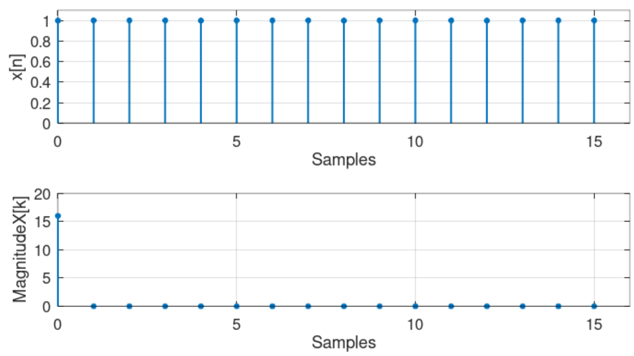


- The delata contains all frequencies over the range of all possible frequencies

(b) DFT of the unit step

$$x[n] = 1$$

$$X[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk} = N\delta[k]$$



(c) DFT Cosine Calculation Problem 1

$$x[n] = 3\cos(2\pi/16 \times n),\ x[n] = \mathbb{C}^{64}$$

i. Determine dimension and fundamental frequency of the signal

- Dimension of space N = 64
- Fundamental frequency $\omega = \dfrac{2\pi}{N} = \dfrac{2\pi}{64}$

  All frequencies in the fourier basis will be a multiple of the fundamental frequency ω. With this in mind we can start by expressing our sinuoid as a muiltiple of the fundamental frequenncy in space $\mathbb{C}^{64}$.

ii. Express the signal as a multiple of the fundamental frequency in space.

$$\begin{aligned}
X[n] &= 3\cos(\frac{2\pi}{16}n) \\
&= 3\cos(\frac{2\pi}{64}4n) \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{-j\frac{2\pi}{64}4n}\right], \text{ with Euler: } cos(\omega) = \frac{e^{j\omega}+e^{-j\omega}}{2} \\
&= \frac{3}{2}\left[e^{j\frac{2\pi}{64}4n} + e^{j\frac{2\pi}{64}60n}\right], \text{ with: } j\frac{2\pi}{64}60n = -j\frac{2\pi}{64}4n + j2\pi n \\
&= \frac{3}{2}\langle w_4[n] + w_{60}[n]\rangle
\end{aligned}$$

- $w_4[n]$ Basis vector number 4
- $w_{60}[n]$ Basis vector number 60

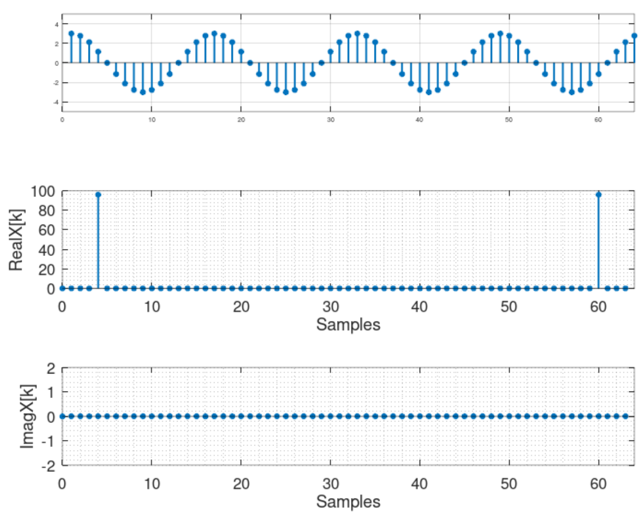
Now we don't like this minus. So what we're going to do is exploit the fact that we can always add an integer multiple of 2pi to the exponent of the complex exponential. And the point will not change on the complex plane.

- **The original signal is now expressed as the sum of two fourier basis vectors**

iii. Calculate the DFT with the analysis forumla

$$
\begin{aligned}
X[k] &= \langle w_k[n], x[n] \rangle, \text{ with: } k = 0, 1, ..N-1 \\
&= \begin{cases} 96 & \text{for } k = 4, 60 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

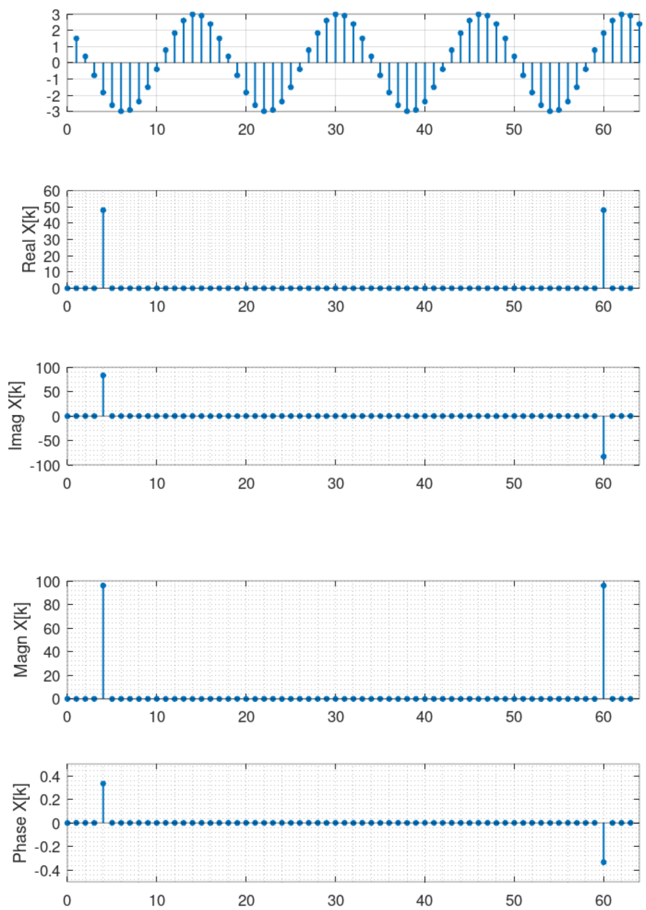
- $w_k[n]$ Canonical basis vector number k



(d) DFT Cosine Calculation Problem 2

$$x[n] = 3\ cos(2\ pi/16\ n + pi/3),\ x[n] \in \mathbb{C}^{64}$$

$$X[k] = \begin{cases} 96e^{j\frac{\pi}{3}} & \text{for } k = 4 \\ 96e^{-j\frac{\pi}{3}} & \text{for } k = 96 \\ 0 & \text{otherwise} \end{cases}$$

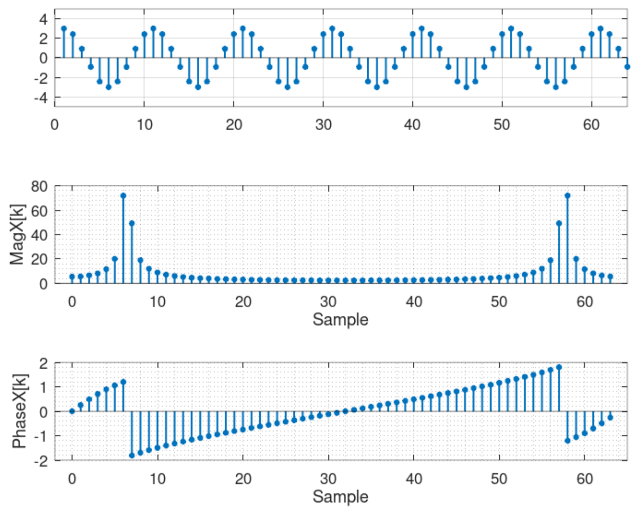


The calcution of the phase just does not work out of the box with octave.

(e) DFT Cosine Calculation Problem 3

$$x[n] = 3\,cos(2\,pi/10\,n),\; x[n] \in \mathbb{C}^{64}$$

$$X[k] = \begin{cases} 96e^{j\frac{\pi}{3}} & \text{for } k = 4 \\ 96e^{-j\frac{\pi}{3}} & \text{for } k = 96 \\ 0 & \text{otherwise} \end{cases}$$



3. Properties of the DFT

   Linearity $DFT\alpha x[n] + \beta y[n] = DFT\alpha x[n] + DFT\beta y[n]$

4. Interpreting a DFT Plot

   - Frequency coefficence $< \pi[0...N/2]$ are interpreted as counter clock wise rotation in the plane
   - Frequency coefficence $> \pi[N/2...N-1]$ are interpreted as clock wise rotation in the plane
   - The fastest frequency of the signal in the vector space is at N/2

Energy of a Signal

The square magnitude of the k-th DFT coefficent is proportional to the signal's energy at frequency $\omega = (\frac{2\pi}{N})k$

- Energy concentrated on single frequency (counterclockwise and clockwise combine to give real signal)

$$x1[n] = 3\ cos(2\ pi/16\ n),\ x[n] \in \mathbb{C}^{64}$$
$$x1[n] = u[n] - u[n-4]$$

- For real signals the DFT is symmetric in magnitude
  - $|X[k]| = |X[N-k]|$, for $k = 1, 2, ...[N/2]$
  - For real signals, magnitude plots need only $[N/2] + 1$ points

1.1.3 The DFT in Practice

1. **TODO** DFT Analysis
   (a) **TODO** Mystery Signal revisted
   (b) **TODO** Solar Spots
   (c) **TODO** Daily Temeperature (2920 days)
   - The recorded signal
   - average value (0-th DFT coefficient: 12.3°
   - DFT main peak for $k = 8$, value 6.4°C
   - 8 cycles over 29920 days
   - $period = \frac{2920}{8} = 365 days$
   - temperature exursion: 12.3° +/- 12.8°C

   The fastest positive frequency of a singnal is at $fracN2$ samples. Since a full revolution of 2 π requires N samples, the discrete frequency corresponding with $\frac{N}{2}$) is π.

   (d) Labeling Frequency Band Axis
   - If "clock" of a System is T_s
     - fastest (positive) frequency is $\omega = \pi$
     - sinosoid at $\omega = \pi$ needs two samples to do a full revolution

- time between samples: $T_s = \dfrac{1}{F_s}$ seconds
- real world period for fastest sinosoid: $2T_s$ seconds
- reald world frequency for fastest sinosoid: $F_s/2$ Hz

- The discrete frequency x of a sinusoid compenent at peak k can be determined as follows:

$$\frac{x}{k} = \frac{N}{2\pi}, \text{ with k=0...N-1} \tag{5}$$

- The real world frequency of a siusoid compenent at peak k can be determined as follows:

$$\begin{aligned}
\frac{x}{k} &= \frac{2\pi}{N}, \text{ with k=0...N-1} \\
\frac{f_s}{2} &=\rightarrow \pi,\ f_s \text{ sampling frequency} \\
\frac{x}{k} &= \frac{f_s}{N} \\
x &= \frac{kf_s}{N}
\end{aligned}$$

(e) **TODO** Example: train whistle

(f) Example 1 A DFT analysis of a signal with length $N = 4000$ samples at a frequency $fs = 44.1kHz$ shows a peak at $k = 500$. What is the corresponding frequency in Hz of this digital frequency in Hz.

- Solution

$$\begin{aligned}
\frac{x}{k} &= \frac{2\ \pi}{N} \\
x &\rightarrow \frac{2\ \pi\ k}{N} \\
\frac{f_s}{2} &\rightarrow \pi \\
x &= \frac{k}{N} f_s && = 55125.5
\end{aligned}$$

(g) Example 2 Calculation of the corresponding frequency vector for a signal for which its spectrum is analysed with the fourier transform

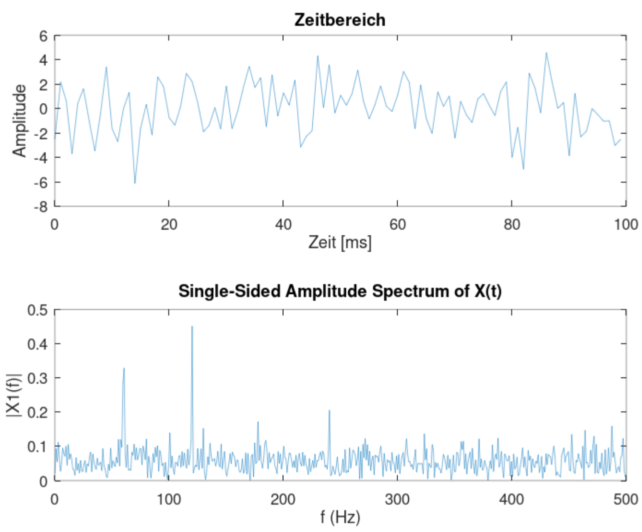
- Sampling Period: $T_s = 1/1000s$
- Sampling Frequency: $f_s = 1/T = 1000Hz$
- Vector Length $N = 2^1 0 = 1024$

$$\frac{X}{k} = \sum_{n=1}^{N} x[n] e^{-j2\pi(k-1)(\frac{n-1}{N})}$$
$$f(k) = \frac{k-1}{NT}, \text{ corresponding Frequency in Hz}$$

- StackOverflow

```
clear all;
close all;
N  = 1024;   # vector length
Fs = 1000;   # Sample Frequency Fs = 1000Hz
Ts = 1/Fs;   # Sampling Period  Ts = 0.001s
f1 = 60;     # 50Hz
f2 = 120;    # 120Hz

n  = 0:Ts:(N-1)*Ts;                        # time vector
x  = sin(2*pi*f1*n) + sin(2*pi*f2*n);      # a sinusoid signal
xr = x + 2*randn(size(n));                 # a noisy signal

X  =fft(xr);                               # FFT
X2 = 1/N*abs(X);                           # FFT magnitude full buffer length
F2 = Fs*(0:(N-1))/N;                       # Frequency vector full buffer lengt

X1 = X2(1:N/2+1)/2;                        # FFT magniture half buffer lenght
X1(2:end-1) = 2*X1(2:end-1);               # Arranged values
F1 = Fs*(0:(N/2))/N;                       # Frequency vector half buffer lengt

figure( 1, "visible", "off" )

subplot(2,1,1)
plot(Fs*n(1:100),xr(1:100));
title('Zeitbereich')
ylabel('Amplitude');
xlabel('Zeit [ms]')
set(gca, "fontsize", 24);
```

```
subplot(2,1,2)
plot(F1,X1)
title('Single-Sided Amplitude Spectrum of X(t)')
xlabel('f (Hz)')
ylabel('|X1(f)|')
set(gca, "fontsize", 24);


## subplot(2,1,3);
## plot(F2,X2);
## title('Two-Sided Amplitude Spectrum of X(t)')
## ylabel('|X2(f)|')
## xlabel('Frequenz [Hertz]')
## set(gca, "fontsize", 24);

                                          # Org-Mode specific setting
print -dpng "-S800,600" ./image/eth-example.png;
ans = "./image/eth-example.png";
```



2. **TODO** DFT Example Analysis of Musical Instruments

   - The fundamental note is the first peak in the spectrum

- The relative size of the harmonics gives the timber or the charachter of an instrument

3. **TODO** DFT Synthesis
4. **TODO** DFT Example - Tide Prediction in Venice
5. **TODO** DFT Example - MP3 Compression
    - MP3 compression approx. factor 20 or more
    - Compression introduces nois from approximation error
    - Noise Shaping : Error shaped as the song in the Fourier domain.
    - Perceptual Compression inclueds the human hearing system properties intto compression algorithm
6. **TODO** Signal of the Day: The first man-made signal from outer space

$$f = \frac{\omega f_s}{2\pi}$$

    - A multiplication in time domain corresponds to a convolution in frequency domain

1.1.4 The Short-Time Fourier Transform STFT

- STFT is a clever way of using DFT
- Spectrogram, is a graphical way to represent the STFT data

1. The short-time Fourier transform
    - DTMF Dual-Tone Multi Frequency dialing
    - Time representation obfuscates frequency
    - Frequency representation obfuscates time

$$x[m;k] = \sum_{n=0}^{L-1} x[m+n] e^{-j\frac{2\pi}{L}nk}$$

    - **m** Starting point of the localiced DFT
    - **k** Is the DFT index
2. **TODO** The spectrogram

- color-code the magnitued: dark is small, white is large
- use $10log_{10}(|X[m,k]|$ to see better (powr in dBs)
- plot spectral slices one after another

3. **TODO** Time-frequency tiling

4. STFT Example