Contents

1 Week 5 Module 4:

1.1 Part 1 Introduction to Filtering

1.1.1 Linear Time-Invariant Systems

LTI System

Linearity and Time Invariance taken together: A Linear Time Invariant System is completely charachterized by its response to the input in particular by its the Impulse Response .

1. Linearity Linearity is expressed by the equivalence

$$\mathfrak{H}\{\alpha\, x_1[n] + \beta\, x_2[n]\} = \alpha\, \mathfrak{H}\{x_1[n]\} + \beta\, \mathfrak{H}\{x_2[n]\} \tag{1}$$

   - Fuzz-Box, example for a none linear device

   (a) **TODO** Add calculation examples

2. Time invariance

   - The system behaves the same way independently of when a it's switched on

$$y[n] = \mathfrak{H}\{x[n]\} \Leftrightarrow \mathfrak{H}\{x[n - n_o]\} = y[n - n_o] \tag{2}$$

   - Wah-Pedal, example of a time variant device

(a) **TODO** Add calculation examples

3. Convolution The impulse response is the output of a filter when the input is the delta function.

$$h[n] = \mathfrak{H}\{\delta[n]\} \tag{3}$$

Impulse Response

Impulse response fully characterize the LTI system!

We can always write

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \tag{4}$$

by linearity and time invariance

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] && (5) \\ &= x[n] * h[n] && (6) \end{aligned}$$

Performing the convolution algorithmically

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

Ingredients

- a sequence x[m]
- a second sequence h[m]

The Recipe

- time-reverse h[m]
- at each step n (from $-\infty\, to\, \infty$):
  - center the time-reversed h[m] in n (i.e. by shift -n)
  - compute the inner product

Furthermore, the convolution can be defined in terms of the inner product between two sequencies.

$$\begin{aligned}(x * y)[n] &= \langle x^*[k], y[n-k] \rangle \\ &= \sum_{n=-\infty}^{\infty} x[k]y[n-k]\end{aligned}$$

1.1.2 Filtering in the Time Domain

For the convolution of two sequencies to exist, the convolution sum must be finite i.e. the both sequencies must be absolutely summable

1. The convolution operator

Linearity	$x[n] * (\alpha \cdot y[n] + \beta \cdot w[n]) = *\alpha \cdot x[n] * y[n] + \beta \cdot x[n] * w[n])$ $w[n] = x[n] * y[n] \iff x[n] * y[n-k] = w[n-k]$
Commutative	$x[n] * y[n] = y[n] * x[n]$
Associative	$(x[n] * y[n]) * w[n] = x[n] * (y[n] * w[n])$

2. Convolution and inner Product

$$\boxed{x[n] * h[n] = \langle\ h^*[n-k], x[k] \rangle}$$

Filtering measures the time-localized similarity between the input sequence and a prototype sequence - the time reversed impulse response.

In general the convolution operator for a signal is defined with repsect to the inner product of its underlaying Hilbert space:

Square Summable Sequence $\ell_2(\mathbb{Z})$	$x[n] * h[n] = \langle\ h^*[n-k], x[k] \rangle$
N-Periodic Sequence	$\tilde{x}[n] * \tilde{y}[n]) \sum_{k=0}^{N-1} \tilde{x}[n-k]\tilde{y}[k]$
Square Integrable Function $L_2([-\pi, \pi])$	$X(e^{j\omega}) * Y(e^{\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\sigma}) \cdot Y(e^{j(\omega-\sigma)})$

3. Properties of the Impulse Response

Causality A system is called causal if its output does not depend on futre values of the input. In practice a causual system is the only type of ”real-time” syste we can actually implement.

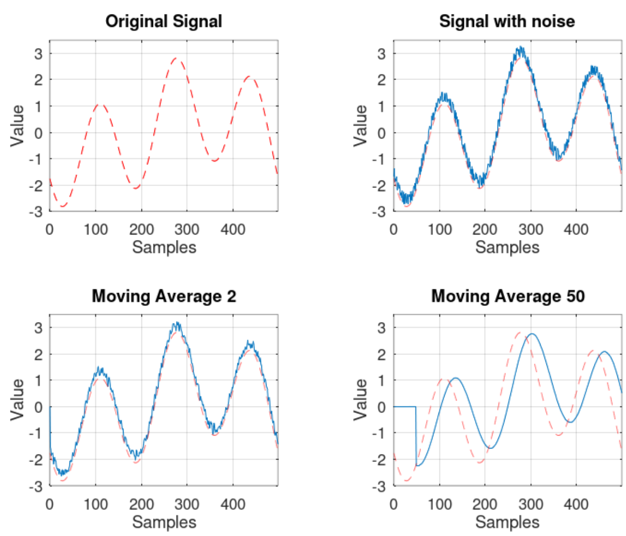
Stability A system is called bounded-input bounded-output stabel (BIBO stable) if its output is bounded for all bounded input sequencies. **FIR** Filter are always stable, since only in the convolution sum only a finite number of terms are involved.

4. Filtering by Example

   (a) FIR Filter: Moving Average Typicale filtering scenario: denoising

   - idea: replace each sample by the local average. Average are useually good to eliminate random variation from which you don’t know mutch about it.
   - for instance: $y[n] = (x[n] + x[n-1])/2$
   - more generally:

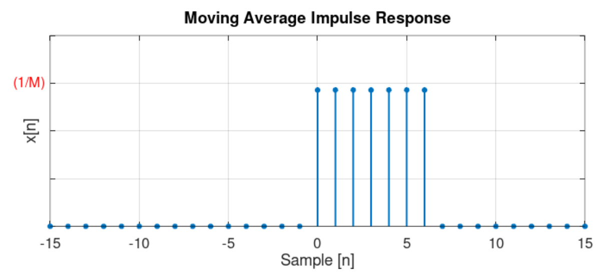
$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$



i. Impulse Response

$$h[n] = \frac{1}{M} \sum_{k=0}^{M-1} \delta[n-k] \begin{cases} \frac{1}{M} & \text{for } 0 \leq n < M \\ 0 & \text{otherwise} \end{cases}$$

```matlab
function [x,n] = ma_impresp(M,n1,n2)
% Generates x(n) = delta(n); 0 <= M
% ------------------------------------------------
% [x,n] = stepseq(n0,n1,n2)
%
  n = [n1:n2]; x = [ (n >= 0) & !((n-M) >= 0) ]./M;
end
```



ii. MA Analysis
   - soomthin effect is proportional to M
   - number of operations and storage also proportional to M

iii. From the MA to first-order recursion

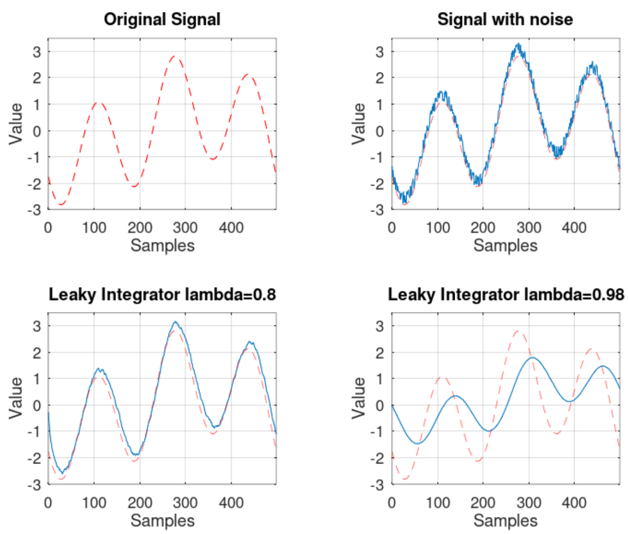
$$
\begin{aligned}
y_{M[n]} &= \sum_{k=0}^{M-1} x[n-k] = x[n]X\sum_{k=1}^{M-1} x[n-k] \\
M_{y_{M[n]}} &= x[n] + (M-1)y_{M-1}[n-1] \\
y_M[n] &= \frac{M-1}{M} y_{M-1}[n-1] + \frac{1}{M}x[n] \\
y_M[n] &= \lambda y_{M-1}[n-1] + (1-\lambda)x[n],\ \lambda = \frac{M-1}{M}
\end{aligned}
$$

(b) IIR Filter: The Leaky Integrator
   - when M is large, $y_{M-1}[n] \approx y_M[n]$and $(\lambda \approx 1)$
   - the filter becomes: $y[n] = \lambda y[n-1] + (1-\lambda)x[n]$
   - the filter is now recursive, since it uses its previous output value

```matlab
function y = lky_impresp(a,b,lambda,x)
% Generates x(n) = a^n
% ------------------------------------------------
% [x,n] = lky_impresp(a,b, lambda, x)
```

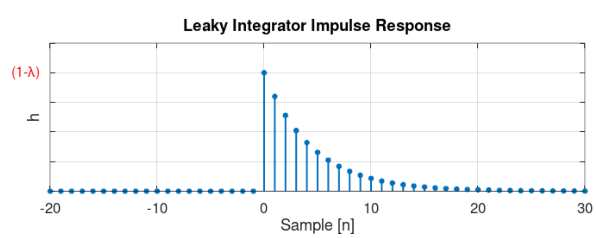
```
% y[n] -lambda y[n-1] = (1-lambda) x[n]
% a = [1, -lambda];
% b = [(1-lambda)];

  b = [1-lambda];
  a = [1, -lambda];
  y = filter(b,a,x);
end
```



i. Impulse Response For the impulse we just need to plug the delta function

$$\begin{aligned} h[n] &= (\lambda y[n-1] + (1-\lambda))\delta[n] \\ &= (1-\lambda)\lambda^n u[n] \end{aligned}$$



The peak at n=0 is $1 - \lambda$.

ii. The leaky integrator why the name

- Discrete Time integrator is a boundless accumulator

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{n} x[k] \\ &= y[n-1] + x[n] \Rightarrow \text{ almost leaky integrator} \end{aligned}$$

To prevent "explosing" we scale the accumulator with λ:

$\lambda y[n-1]$ keep only a fraction λ of the accumulated value so far and forget ("leak") a fraction $\lambda - 1$

$(1-\lambda)x[n]$ add only a fraction $1-\lambda$ of the current value to the accumulator.

So we get the leaky integrator from the accumulator

$$y[n] = \lambda \cdot y[n-1] + (1-\lambda) \cdot x[n] \Rightarrow \text{ almost leaky integrator}$$

1.1.3 Classification of Filters

FIR Finite Impulse Response Filter

- Impulse response has finite support
- only a finite number of samples are involved in the computation of each output
- Example: Moving Average Filter

IIR Infinite Impulse Response Filter

- Impulse response has inifinte support
- a potentially infinite number of samples are involved in the computation of each output sample
- surprisingly, in many cases the computation can still be performed in a finite amount of steps
- Example: The Leaky Integrator

Casual

- impulse response is zero for n $<$ 0

- only past samples are involved in the computation of each output sample
- causul filters can work ”on line” since they only need the past

Noncasual

- impulse response in nonzero for some (or all) n < 0
- can still be implemented in a offline fashing (e.g. image processing)

1.1.4 Filter Stability

FIR Filter

FIR filters are always stable

because their impuls response only contains a finite number of non-zero values, and therefore the sum of their absolute values will always be finite.

1.1.5 Frequency Response

1. References

   - Signal and System for Dummies: Frequency Response

2. Eigensequence If a complex exponential is applied to a LTI filter its response is the DTFT of the impulse response of the LTI filter times the complex exponential.

$$
\begin{aligned}
x[n] &= e^{j\omega_0 n} \\
y[n] &= \mathfrak{H}\{x[n]\} \\
y[n] &= x[n] * h[n] \\
y[n] &= e^{j\omega_0 n} * h[n] \\
y[n] &= H(e^{j\omega_0})e^{j\omega_0 n}
\end{aligned}
$$

   - DTFT of impulse response determinse the frequency characteristic of a filter

- Complex exponential are eignesequences of LTI systems, i.e. linear filters cannot change the frequency of a sinusoid.

3. Magnitude and phase

$$\text{if } H(j^{j\omega_0}) = Ae^{j\theta}\text{, then}$$
$$\mathfrak{H}\left\{e^{j\omega_0 n}\right\} = Ae^{j(\omega_0 n+\theta)}$$

amplitude	**A**	**phase shift**	θ
amplification	>1	delay	< 0
attenuation	$0 \leq A < 1$	advancment	> 0

4. The convolution theorem The convolution theorem summerizes this result in

$$\boxed{DTFT\left\{x[n] * h[n]\right\} = X(e^{j\omega})H(e^{j\omega})}$$

5. Frequency response The DTFT of the impulse response is called the frequency response

$$\boxed{H(e^{j\omega}) = DTFT\left\{h[n]\right\}}$$

magnitude	$\mid H(e^{j\omega} \mid$	**phase**
amplification	> 1	overall shape and
attenuation	< 1	phase changes

6. Example of Frequency Response: Moving Average Filter The difference equation from M-point averager is

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

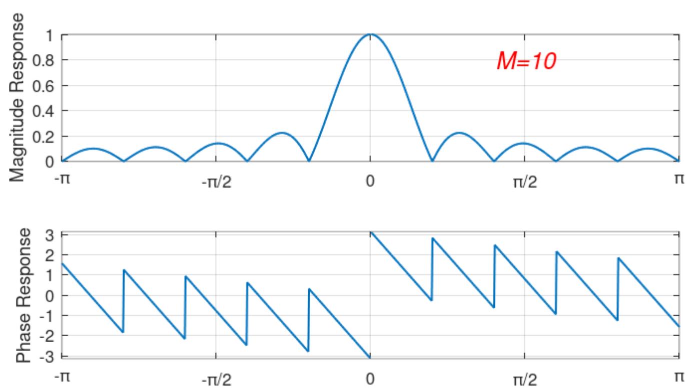
The Frequency response of the moving average filter

$$H(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} e^{-j\omega k} = \frac{1}{M}\sum_{k=0}^{M-1}\left(e^{-j\omega}\right)^k$$
$$= \frac{1}{M}\frac{(1-e^{-j\omega M})}{(1-e^{j\omega})}$$

- The frequency response is composed of a linear term $e^{-j\omega\frac{M-1}{2}}$ and $\pm\pi$ due to the sign changes of $\dfrac{sin(\frac{\omega}{2}M)}{sin(\frac{\omega}{2}M)}$

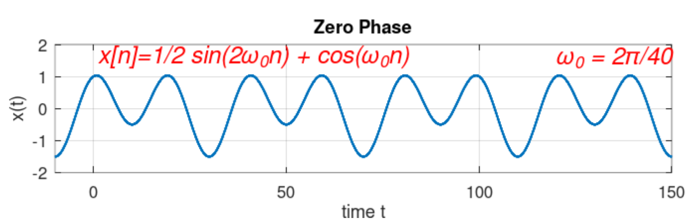
The Magnetute response of the moving average filter

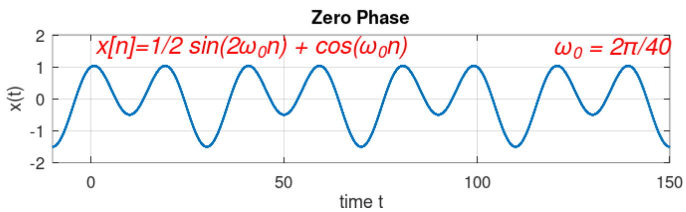
$$H(e^{j\omega}) = \frac{1}{M}\left|\frac{sin(\frac{\omega}{2}M)}{sin(\frac{\omega}{2})}\right|$$



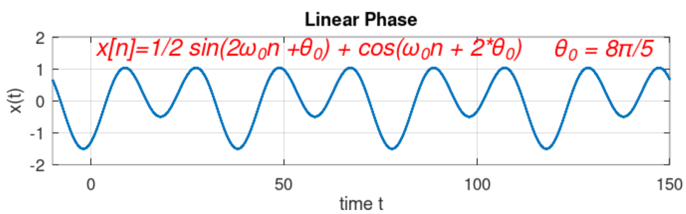
7. Phase and signal shape To understand the effects of the phase on a signal is to distinguihs three different cases

   - zero phase: the spectrum is real: $\angle H(e^{jw}) = 0$

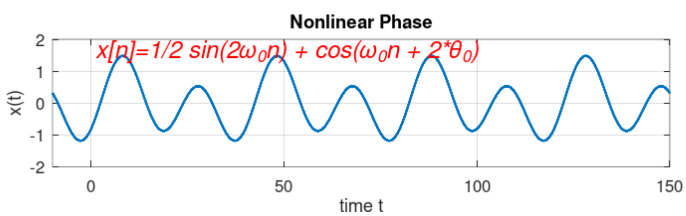




- linear phase: the phase is proportional to the frequency via a real factor, d: $\angle H(e^{jw}) = d\omega$ the phase is proportional to the frequency of the sinusoid. The net effect is a shift of the signal if the phase component is porportional to the frequency.



- non linear phase: which covers all the other properties now the shape of the signal in the time domain changes.



Spectrum

The spectrum of all three signals x[n] remains exactely the same in magnitude.

8. Linear Phase

$$y[n] = x[n - d]$$
$$Y(e^{j\omega}) = e^{-j\omega d}\, X(e^{j\omega})$$
$$H(e^{j\omega}) = e^{-j\omega d} \Rightarrow linearphaseterm$$

9. Moving Average is linear Phase

$$
\begin{aligned}
H(e^{j\omega}) &= A(e^{j\omega})e^{-j\omega d} \\
&\Rightarrow A(e^{j\omega})\text{: pure real term} \\
&\Rightarrow e^{-j\omega d}\text{: pure phase term} \\
&= \frac{1}{M}\frac{sin(\frac{\omega}{2}M)}{sin(\frac{\omega}{2}M)}e^{-j\omega\frac{M-1}{2}} \Rightarrow \frac{M-1}{2} = d
\end{aligned}
$$

10. Example of Frequency Response: Leaky Integrator The Frequency response of the leaky integrator

$$H(e^{j\omega}) = \frac{1-\lambda}{1-\lambda e^{j\omega}}$$

Finding the magnitude and phaser requires a little algebra

From Complex Algebra

$$\frac{1}{a+jb} = \frac{1-jb}{a^2+b^2}$$

So that if $x = \dfrac{1}{a+jb}$

$$
\begin{aligned}
|x|^2 &= \frac{1}{a^2+b^2} \\
\angle x &= tan^{-1}\left[-\frac{a}{b}\right]
\end{aligned}
$$

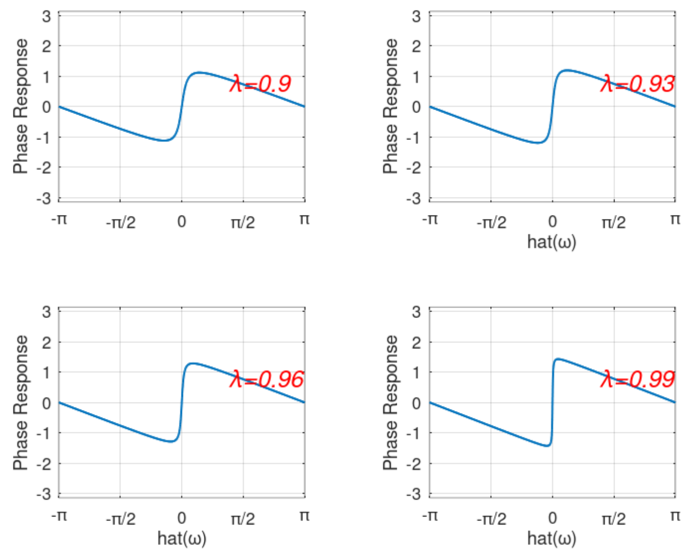
$$H(e^{j\omega}) = \frac{1-\lambda}{(1-\lambda cos\omega) - jsin\omega}$$

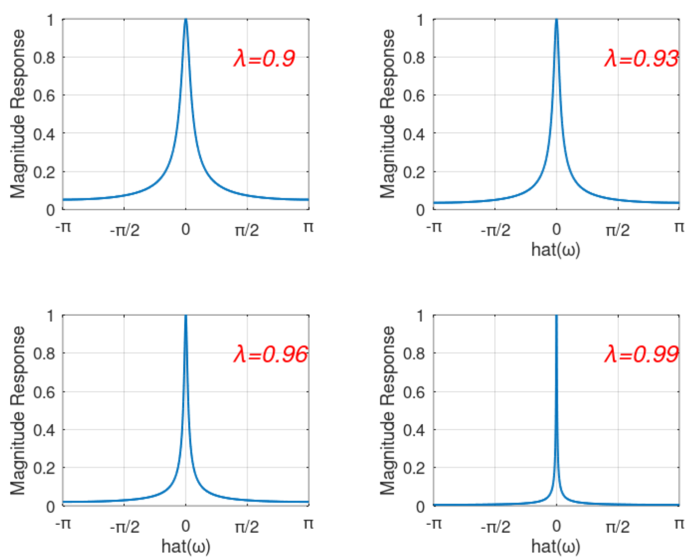
so that:

$$|H(e^{j\omega})|^2 = \frac{(1-\lambda)^2}{1-2\lambda cos\omega+\lambda^2}$$

$$\angle H(e^{j\omega}) = tan^{-1}\left[\frac{\lambda sin\omega}{1-\lambda cos\omega}\right]$$

The phase is nonlinear in this case





11. **TODO** Example of Frequency Response: Karplus Strong Algorithm

$$y[n] = \alpha y[n-M] + x[n]$$

The Karplus-Strong algorithm is initialized with a finite support signal x of support M. And then we use a feedback loop with a delay of M taps. To qproduce multiple copies of the original finite support signal, scaled by an exponentially decaying factor alpha.

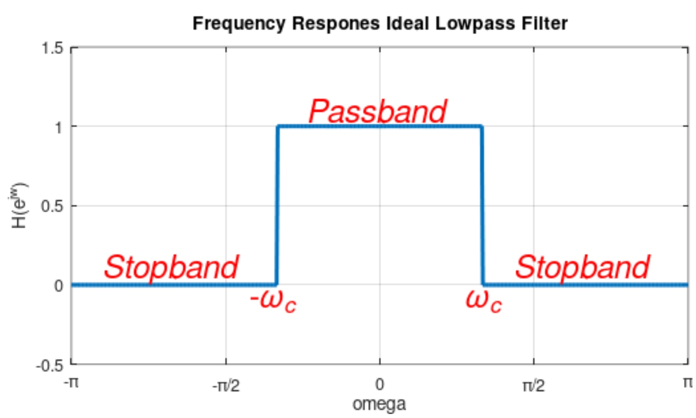
(a) With Sawtooth Wave

$$\tilde{X}(j\omega)W(j\omega) = e^{-j\omega}\left(\frac{M+1}{M-1}\right)\frac{1-e^{-j(M-1)\omega}}{(1-e^{j\omega})^2} - \frac{1-e^{-j(M+1)\omega}}{(1-e^{j\omega})^2}$$

$$X(j\omega)W(j\omega) = \frac{1}{1-\alpha e^{-j\omega M}}$$

1.1.6 Ideal Filters

1. The ideal lowpass filter frequency response



2. Ideal lowpass filter impulse response

   - Lets low frequencies go through
   - Attenuates i.e. kills high frequencies

Cut off Frequency ω_c - the frequency response transitions from 1 to zero

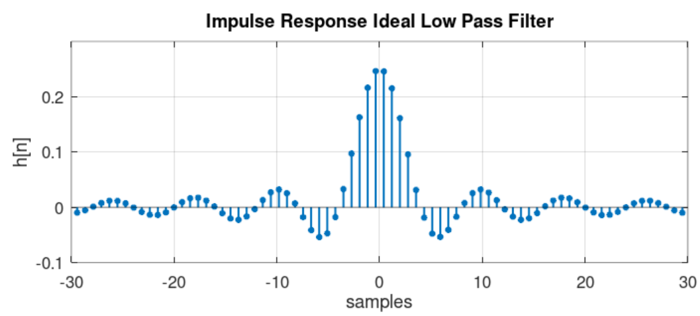
Passband $\omega_b = 2\omega_c$

$$H(e^{j\omega}) = \begin{cases} 1 & \text{for } |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

- perfectly flat passband
- infinite attenuation in stopband
- zero-phase (no delay)

Calculation of the impulse response from the frequency response of an ideal low pass filter. Impulse Respones

$$
\begin{aligned}
h[n] &= IDFT\{H(e^{j\omega})\} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega\, n}d\omega \\
&= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega\, n}d\omega \\
&= \frac{1}{\pi\; n}\frac{e^{j\omega_c\, n} - e^{-j\omega_c\, n}}{2j} \\
&= \frac{sin(\omega_c\; n)}{\pi\; n}
\end{aligned}
$$



- from Mathworks
- The impulse response has infinite support to the right and to the left
- Independant of how the convolution is computed, it will always take an inifintie number of operations.
- The impulse response decays slowly in time $\left(\frac{1}{n}\right)$, we need a lot of samples for a good approximation.

(a) Impulse Response: From normlized Algorithm to Octave Implementation

$$
\begin{aligned}
\frac{sin(\omega_c\, n)}{\pi\, n} &= \frac{\omega_c}{\pi}.\cdot sinc(n\frac{\omega_c}{\pi});\\
&= \frac{\frac{\pi}{c}}{\pi}.\cdot sinc(n\frac{\frac{\pi}{c}}{\pi});\\
&= \frac{1}{c}.\cdot sinc(n\frac{1}{c});\\
&= \frac{1}{c}.\cdot sinc(\frac{n}{c});
\end{aligned}
$$

(b) The sinc-rect pair:

$$
rect(x) = \begin{cases} 1 & |x| \le \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}
$$

$$
sinc(x) = \begin{cases} \frac{sin(\pi\, x)}{\pi\, x} & x \neq 0 \\ 1 & x = 0 \end{cases}
$$

- rect is the indicator function from $-\frac{1}{2}$ to $\frac{1}{2}$

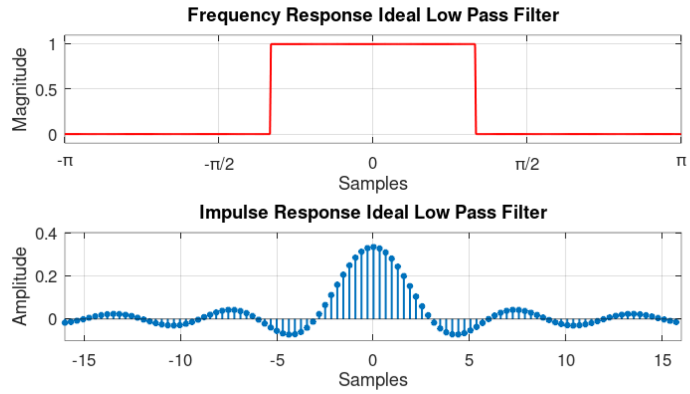
(c) Canonical form of the ideal low pass filter The sinct-rect pair can be written in canonical form as follow: $~$

$$
\boxed{H(e^{j\,\omega}) = rect\left(\frac{\omega}{2\,\omega_c}\right)} \overset{DTFT}{\longleftrightarrow} \boxed{\frac{\omega_c}{\pi}\, sinc\left(\frac{\omega_c}{\pi}\, n\right) = h[n]}
$$

- The Impulse response is normalized by $\frac{\omega_c}{\pi}$

3. Example

Calculation of the impulse- and frequency response for an ideal low pass filter with $\frac{\pi}{3}$

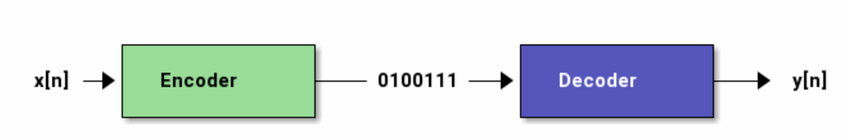


4. **TODO** Ideal filters derived from the ideal low pass filter

5. **TODO** Demodulation revisted

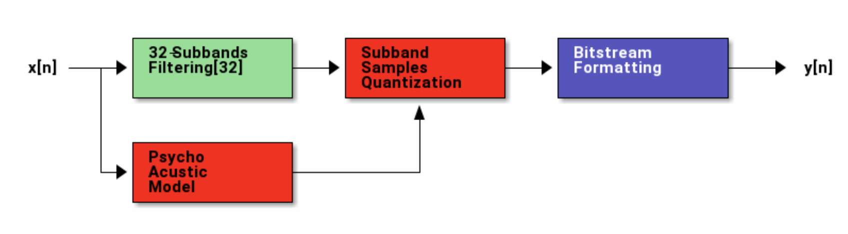
1.1.7 MP3 Encoder

- **Goal:** Reduce number of bits to represent original signal x[n]
- MP3: Motion Picture Expert G3roup



- **Lossy Compression:** $x[n] \neq y[n]$
- Put noise where not perceptible by human ear
- **Example:** Raw Storage Consumption DVD
  - Sample Rate: 48kHz
  - Bits per Sample: 16
  - Bit Rate: $\frac{48000\text{samples}}{second} \frac{16bits}{samples} = 768kbits/s$
  - Duration: 60s
  - Mono Raw Data Storage Usage: $60s \times 76.8kbits/s = 46Mbit = 5.8MByte$

- Stereo Raw Data Storage Usage: $2 \times 5.8MBytes = 12MBytes$
- MP3 Compressed Storage Usage: $1.5MBytes$



- Clever Quantiziation Scheme: Number of bits allocated to each sub-band is dependent on the perceptual importance of each sub-band with respect to overall quality of the audio wave-form
- Masking Effect of the human auditory system.

1. Psycho Acoustic Model, How it Works
   - The psycho acoustic model is not part of the mp3 standard
   - calculate the minimum number of bits that we need to quantize each of the 32 subband filter outputs, so that the perceptual distortion is as little as possible

   step 1 Use FFT to estimate the energy of the signal in each sub-band

   step 2 Distinguish beween tonal (sinusoid like) and non-tonal (nois-like) compnent

   step 3 Determine indicidual masking effect of tonal and non-tonal component in each critical band

   step 4 Determine the total masking effect by summing the individual contirbution

   step 5 Map this total effect to the 32 subbands

   step 6 Determine bit alloction by allocating priority bits to sub-bands with lowest singal-to-mask ratio

2. Subband Filter

$$h_i[n] = h[n]cos\left(\frac{pi}{64}(2i+1)(n-16)\right)$$

1.1.8 Programing Assignment 1

```
import matplotlib
import numpy as np
matplotlib.use('Agg')
import matplotlib.pyplot as plt


def scaled_fft_db(x):
    """ ASSIGNMENT 1:
        Module 4 Part 1:
        Apply a hanning window to len(x[n]) = 512
    """

    N = len(x)              # number of samples
    n = np.arange(N)        # time vector
    # a) Compute a 512-point Hann window and use it to weigh the input data.
    sine_sqr = np.sin((np.pi*n)/(N-1))**2    # sin(x)^2 = 1/2*(1 - cos(2x))
    c = np.sqrt(511/np.sum(sine_sqr))
    w = c/2 * (1 - np.cos((2 * np.pi * n)/(N - 1)))
    # b) Compute the DFT of the weighed input, take the magnitude in dBs and
    #    normalize so that the maximum value is 96dB.
    y = w * x
    Y = np.fft.fft(y) / N
    # c) Return the first 257 values of the normalized spectrum
    Y = Y[0: np.int(N/2+1)]
    # Take the magnitude of X
    Y_mag = np.abs(Y)
    nonzero_magY = np.where(Y_mag != 0)[0]

    # Convert the magnitudes to dB
    Y_db = -100 * np.ones_like(Y_mag)    # Set the default dB to -100
    Y_db[nonzero_magY] = 20*np.log10(Y_mag[nonzero_magY])  # Compute the dB for non

    # Rescale to amx of 96 dB
    max_db = np.amax(Y_db)
    Y_db = 96 - max_db + Y_db

    return Y_db
```

```python
def test():
    N = 512
    n = np.arange(N)
    x = np.cos(2*np.pi*n/10)

    # Y = scaled_fft_db(x)
    Y = scaled_fft_db(x)

    fig=plt.figure(figsize=(6,3))
    plt.semilogy(abs(Y))
    plt.grid(True)

    fig.tight_layout()
    plt.savefig('image/python-matplot-fig-04.png')
    return 'image/python-matplot-fig-04.png' # return filename to org-mode

return test()
```