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1 Week 3 Module 3:

1.1 Part 1 - Introduction to Fourier Analysis

1.1.1 Introduction to Fourier Analysis

The Frequency Domain

Oscillations are every where

- A train has got an engine which makes the wheels turn in circular motion
- Waves, ebb and flow can be modeled as sinusoidal fashion
- Musical instruments generates sound by vibrating at a certain fundamental frequency
- Intuitivly: things that don't move in circles can't last
 - bombs
 - rockets
 - human beeings

Descriptin of the oscillations in the plane

Period P

Frequency $f = \frac{1}{P}$

Ordinate $\sin(ft)$

Abscissa $\cos(ft)$

Example Sinusoidal Detectors in our Body:

cochlea In the inner ear that detects air pressure sinusoids at frequencies from 20 to 20kHz

retina In the eye to detect electromagnetic sinusoids with frequency 430THz to 790THz. This is the frequency of lights in the visible spectrum

Humans analyze complex signals (audio, images) in terms of their sinusoidal components

Frequency Domain seems to be as good as the time domain

Fundamental Questions: Can we decompose any signal into sinusoidal elements?

- Yes, Fourier showed us how to do it exactly
- Analysis
 - From time domain to frequency domain
 - Find the contribution of different frequencies
 - Discover "hidden" signal properties
- Synthesis
 - From frequency domain to time domain
 - Create signals with known frequency content
 - Fit signals to specific frequency regions

The DFT as a change of basis

- let's start with finite-length signals (i.e. vectors in \mathbb{C}^N)
- **Fourier analysis is a simple change of basis**
- a change of basis is a change of perspective
- a change of perspective can reveal things (if the basis is good)

The Fourier Basis for \mathbb{C}^N **Claim:** the set of N signals in \mathbb{C}^N

$$w_k[n] = e^{j\frac{2\pi}{N}nk} \text{ with } n, k = 0, 1, \dots, N-1$$

is an orthogonal basis in \mathbb{C}^N .

- \mathbb{C}^N : N different vectors all with length N
- k : is the index that indicates different vectors (signals), $k = 0, \dots, N-1$
- n : is the index that indicates different elements within the vector, $n = 0, \dots, N-1$
- $w_k[n] = e^{j\frac{2\pi}{N}nk}$: Form of the N -Signals
- $\omega = \frac{2\pi}{N}nk$: The index of the signal makes up the **Fundamental Frequency** of the complex exponential.

In Vector Notation:

$$\left\{ \mathbf{w}^{(k)} \right\}_{k=0,1,\dots,N-1}$$

with

$$w_n^{(k)} = e^{j\frac{2\pi}{N}nk}$$

is an orthogonal basis in \mathbb{C}^N

1. Basis vector $\mathbf{w}^{(0)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N}0 = 0$$

2. Basis vector $\mathbf{w}^{(1)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N}1 = \frac{2\pi}{N}$$

3. Basis vector $\mathbf{w}^{(2)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N}2$$

4. Basis vector $\mathbf{w}^{(3)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N}3$$

5. Basis vector $\mathbf{w}^{(4)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N}4$$

6. Basis vector $\mathbf{w}^{(5)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N}5$$

7. Basis vector $\mathbf{w}^{(16)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N}2 = \frac{\pi}{2}$$

8. Basis vector $\mathbf{w}^{(20)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N}20$$

9. Basis vector $\mathbf{w}^{(21)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 21$$

10. Basis vector $\mathbf{w}^{(30)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 30$$

11. Basis vector $\mathbf{w}^{(31)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 20$$

The sign alternation is a tell tale sign of a high frequency sinusoid.

12. Basis vector $\mathbf{w}^{(32)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 32 = \frac{2\pi}{64} 32 = \pi$$

At k equal to 32 the highest fundamental frequency is reached.

13. Basis vector $\mathbf{w}^{(33)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 33$$

If we go forward with the index the apparent speed of the point decreases and the direction of the rotation changes from counter clockwise to clockwise.

14. Basis vector $\mathbf{w}^{(34)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 34$$

15. Basis vector $\mathbf{w}^{(60)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 60$$

16. Basis vector $\mathbf{w}^{(62)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 62$$

So here, for instance, if you compare this basis vector to $\mathbf{w}^{(2)}$, you would see that the real part is the same, but the imaginary part has a sign inversion.

17. Basis vector $\mathbf{w}^{(63)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 63$$

18. Basis vector $\mathbf{w}^{(64)} \in \mathbb{C}^{64}$

Fundamental Frequency

$$\omega = \frac{2\pi}{N} 64$$

1.1.2 The Discrete Fourier Transform (DFT)

DFT definition

The Fourier Basis for \mathbb{C}^N in "Signal" Notation

$$w_k[n] = e^{j\frac{2\pi}{N}nk} \text{ with } n, k = 0, \dots, N-1$$

The Fourier Basis in Vector Notation

$$\{\mathbf{w}^{(k)}\}_{k=0,1,\dots,N-1} \text{ with } w_n^{(k)} = e^{j\frac{2\pi}{N}nk}, n = 0, \dots, N-1$$

N Dimension of vector space

k Index for different vectors and goes from $0..N-1$

n Index of element in each vector goes from $0..N-1$

Basis Expansion Vector Notation

1. Analysis Formula

$$X_k = \langle \mathbf{w}^{(k)}, \mathbf{x} \rangle \quad k = 0, \dots, N-1 \quad (3)$$

X_k Coefficient for the new basis. Inner Product of \mathbf{x} with each vector $\mathbf{w}^{(k)}$

\mathbf{x} An arbitrary vector of \mathbb{C}^N

$\mathbf{w}^{(k)}$ New basis

2. Synthesis Formula

$$\mathbf{x} = \frac{1}{N} \sum_{k=0}^{N-1} X_k \mathbf{w}^{(k)} \quad k = 0, \dots, N-1 \quad (4)$$

TODO Change of basis in matrix form

Basis Expansion Signal Notation

- Consider explicitly the operations involved in the transformation
 - This notion is particularly useful if you want to consider the algorithmic nature of the transform
1. Analysis Formula N-point signal in the frequency domain

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j \frac{2\pi}{N} nk}, k = 0, 1, \dots, N-1$$

$X[k]$ Signal vector in the frequency domain

$x[n]$ Signal vector in the (discrete) time domain

Reminder This is the inner Product in explicit form

2. Synthesis Formula N-point signal in the time domain

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j \frac{2\pi}{N} nk}, k = 0, 1, \dots, N-1$$

$X[k]$ Signal vector in the frequency domain

$\frac{1}{N}$ Normalisation coefficient

Reminder This is the inner Product in explicit fashion

Examples of DFT Calculation

DFT of the impulse function

$$x[n] = \delta[n]$$

$$X[k] = \sum_{n=0}^{N-1} \delta[n] e^{-j \frac{2\pi}{N} nk} = 1$$

- The delta contains all frequencies over the range of all possible frequencies

DFT of the unit step

$$x[n] = 1$$

$$X[k] = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}nk} = N\delta[k]$$

DFT Cosine Calculation Problem 1

$$x[n] = 3 \cos(2\pi/16 \times n), \quad x[n] = \mathbb{C}^{64}$$

1. Determine dimension and fundamental frequency of the signal

- Dimension of space $N = 64$
- Fundamental frequency $\omega = \frac{2\pi}{N} = \frac{2\pi}{64}$

All frequencies in the fourier basis will be a multiple of the fundamental frequency ω . With this in mind we can start by expressing our sinuoid as a multiple of the fundamental frequency in space \mathbb{C}^{64} .

2. Express the signal as a multiple of the fundamental frequency in space.

$$\begin{aligned} X[n] &= 3 \cos\left(\frac{2\pi}{16}n\right) \\ &= 3 \cos\left(\frac{2\pi}{64}4n\right) \\ &= \frac{3}{2} \left[e^{j\frac{2\pi}{64}4n} + e^{-j\frac{2\pi}{64}4n} \right], \text{ with Euler: } \cos(\omega) = \frac{e^{j\omega} + e^{-j\omega}}{2} \\ &= \frac{3}{2} \left[e^{j\frac{2\pi}{64}4n} + e^{j\frac{2\pi}{64}60n} \right], \text{ with: } j\frac{2\pi}{64}60n = -j\frac{2\pi}{64}4n + j2\pi n \\ &= \frac{3}{2} \langle w_4[n] + w_{60}[n] \rangle \end{aligned}$$

- $w_4[n]$ Basis vector number 4
- $w_{60}[n]$ Basis vector number 60

Now we don't like this minus. So what we're going to do is exploit the fact that we can always add an integer multiple of 2π to the exponent of the complex exponential. And the point will not change on the complex plane.

- The original signal is now expressed as the sum of two fourier basis vectors

3. Calculate the DFT with the analysis formula

$$X[k] = \langle w_k[n], x[n] \rangle, \text{ with: } k = 0, 1, \dots, N-1$$

$$= \begin{cases} 96 & \text{for } k = 4, 60 \\ 0 & \text{otherwise} \end{cases}$$

- $w_k[n]$ Canonical basis vector number k

DFT Cosine Calculation Problem 2

$$x[n] = 3 \cos(2\pi n/16 + \pi/3), x[n] \in \mathbb{C}^{64}$$

$$X[k] = \begin{cases} 96e^{j\frac{\pi}{3}} & \text{for } k = 4 \\ 96e^{-j\frac{\pi}{3}} & \text{for } k = 96 \\ 0 & \text{otherwise} \end{cases}$$

The calculation of the phase just does not work out of the box with octave.

DFT Cosine Calculation Problem 3

$$x[n] = 3 \cos(2\pi n/10), x[n] \in \mathbb{C}^{64}$$

$$X[k] = \begin{cases} 96e^{j\frac{\pi}{3}} & \text{for } k = 4 \\ 96e^{-j\frac{\pi}{3}} & \text{for } k = 96 \\ 0 & \text{otherwise} \end{cases}$$

Properties of the DFT

Linearity $DFT(\alpha x[n] + \beta y[n]) = \alpha DFT x[n] + \beta DFT y[n]$

Interpreting a DFT Plot

- Frequency coefficients $< \pi[N/2]$ are interpreted as counter clock wise rotation in the plane
- Frequency coefficients $> \pi[N/2]$ are interpreted as clock wise rotation in the plane
- The fastest frequency of the signal in the vector space is at $N/2$



Energy of a Signal



The square magnitude of the k -th DFT coefficient is proportional to the signal's energy at frequency $\omega = (\frac{2\pi}{N})k$

- Energy concentrated on single frequency (counterclockwise and clockwise combine to give real signal)

$$x1[n] = 3 \cos(2 \pi / 16 n), x[n] \in \mathbb{C}^{64}$$

$$x1[n] = u[n] - u[n - 4]$$

- For real signals the DFT is **symmetric** in magnitude
 - $|X[k]| = |X[N - k]|$, for $k = 1, 2, \dots [N/2]$
 - For real signals, magnitude plots need only $[N/2] + 1$ points

1.1.3 The DFT in Practice

TODO DFT Analysis

TODO Mystery Signal revisited

TODO Solar Spots

TODO Daily Temperature (2920 days)

- The recorded signal
- average value (0-th DFT coefficient: 12.3°)
- DFT main peak for $k = 8$, value 6.4°C
- 8 cycles over 29920 days
- $period = \frac{2920}{8} = 365 \text{ days}$
- temperature excursion: $12.3^\circ \pm 12.8^\circ\text{C}$

The fastest positive frequency of a signal is at $\frac{N}{2}$ samples. Since a full revolution of 2π requires N samples, the discrete frequency corresponding with $\frac{N}{2}$ is π .

Labeling Frequency Band Axis

- If "clock" of a System is T_s
 - fastest (positive) frequency is $\omega = \pi$
 - sinusoid at $\omega = \pi$ needs two samples to do a full revolution
 - time between samples: $T_s = \frac{1}{F_s}$ seconds
 - real world period for fastest sinusoid: $2T_s$ seconds
 - real world frequency for fastest sinusoid: $F_s/2$ Hz
- The discrete frequency x of a sinusoid component at peak k can be determined as follows:

$$\frac{x}{k} = \frac{N}{2\pi}, \text{ with } k=0\dots N-1 \quad (5)$$

- The real world frequency of a sinusoid component at peak k can be determined as follows:

$$\begin{aligned} \frac{x}{k} &= \frac{2\pi}{N}, \text{ with } k=0\dots N-1 \\ \frac{f_s}{2} &\rightarrow \pi, f_s \text{ sampling frequency} \\ \frac{x}{k} &= \frac{f_s}{N} \\ x &= \frac{k f_s}{N} \end{aligned}$$

TODO Example: train whistle

Example 1 A DFT analysis of a signal with length $N = 4000$ samples at a frequency $f_s = 44.1kHz$ shows a peak at $k = 500$. What is the corresponding frequency in Hz of this digital frequency in Hz.

- Solution

$$\begin{aligned}\frac{x}{k} &= \frac{2\pi}{N} \\ x &\rightarrow \frac{2\pi k}{N} \\ \frac{f_s}{2} &\rightarrow \pi \\ x &= \frac{k}{N} f_s = 55125.5\end{aligned}$$

Example 2 Calculation of the corresponding frequency vector for a signal for which its spectrum is analysed with the fourier transform

- Sampling Period: $T_s = 1/1000s$
- Sampling Frequency: $f_s = 1/T = 1000Hz$
- Vector Length $N = 2^{10} = 1024$

$$\begin{aligned}\frac{X}{k} &= \sum_{n=1}^N x[n] e^{-j2\pi(k-1)(\frac{n-1}{N})} \\ f(k) &= \frac{k-1}{NT}, \text{ corresponding Frequency in Hz}\end{aligned}$$

- [StackOverflow](#)

```
clear all;
close all;
N = 1024;    # vector length
Fs = 1000;   # Sample Frequency Fs = 1000Hz
Ts = 1/Fs;   # Sampling Period Ts = 0.001s
f1 = 60;     # 50Hz
f2 = 120;    # 120Hz

n = 0:Ts:(N-1)*Ts;           # time vector
x = sin(2*pi*f1*n) + sin(2*pi*f2*n); # a sinusoid signal
xr = x + 2*randn(size(n));    # a noisy signal

X = fft(xr);                  # FFT
X2 = 1/N*abs(X);              # FFT magnitude full buffer length
```

```

F2 = Fs*(0:(N-1))/N;                                # Frequency vector full buffer length

X1 = X2(1:N/2+1)/2;                                  # FFT magnitude half buffer length
X1(2:end-1) = 2*X1(2:end-1);                         # Arranged values
F1 = Fs*(0:(N/2))/N;                                  # Frequency vector half buffer length

figure( 1, "visible", "off" )

subplot(2,1,1)
plot(Fs*n(1:100),xr(1:100));
title('Zeitbereich')
ylabel('Amplitude');
xlabel('Zeit [ms]')
set(gca, "fontsize", 24);

subplot(2,1,2)
plot(F1,X1)
title('Single-Sided Amplitude Spectrum of X(t)')
xlabel('f (Hz)')
ylabel('|X1(f)|')
set(gca, "fontsize", 24);

## subplot(2,1,3);
## plot(F2,X2);
## title('Two-Sided Amplitude Spectrum of X(t)')
## ylabel('|X2(f)|')
## xlabel('Frequenz [Hertz]')
## set(gca, "fontsize", 24);

# Org-Mode specific setting
print -dpng "-S800,600" ./image/eth-example.png;
ans = "./image/eth-example.png";

```

TODO DFT Example Analysis of Musical Instruments

- The fundamental note is the [first peak](#) in the spectrum
- The relative size of the harmonics gives the timber or the character of an instrument

TODO DFT Synthesis

TODO DFT Example - Tide Prediction in Venice

TODO DFT Example - MP3 Compression

- MP3 compression approx. factor 20 or more
- Compression introduces noise from approximation error
- [Noise Shaping](#) : Error shaped as the song in the Fourier domain.
- [Perceptual Compression](#) includes the human hearing system properties into compression algorithm

TODO Signal of the Day: The first man-made signal from outer space

$$f = \frac{\omega f_s}{2\pi}$$

- A [multiplication](#) in time domain corresponds to a [convolution](#) in frequency domain

1.1.4 The Short-Time Fourier Transform STFT

- STFT is a clever way of using DFT
- Spectrogram, is a graphical way to represent the STFT data

The short-time Fourier transform

- DTMF Dual-Tone Multi Frequency dialing
- Time representation obfuscates frequency
- Frequency representation obfuscates time

$$x[m; k] = \sum_{n=0}^{L-1} x[m+n] e^{-j \frac{2\pi}{L} nk}$$

- **m** Starting point of the localized DFT
- **k** Is the DFT index

TODO The spectrogram

- color-code the magnitude: dark is small, white is large
- use $10\log_{10}(|X[m, k]|)$ to see better (power in dBs)
- plot spectral slices one after another

TODO Time-frequency tiling

STFT Example