Contents

1 Week 4 Module 3: Part 2 - Advanced Fourier Analysise

1.1 Discrete Fourier Series DFS

Discrete Fourier Series

DFS = DFT with periodicity explicit $\tilde{X}[k] = DFS\{x[n]\}$

- The DFS maps an N-Periodic signal onto an N-Periodic sequence of Fourier coeffients
- The inverse DFS maps $\mathrm{n}_{\mathrm{periodic}}$ sequence of Fourier coeffiencts a set onto an N-periodic signal
- DFS is an extension of the DFT for periodic sequencies
- A circular time-shift is an natural extension of a shift fo finite length signals.

1.1.1 Finite-length time shifts revisted

- The DFS helps us understand how to define time shifts for finite-lenght signals.

test

For an N-periodic sequence $\tilde{x}[n]$

$$\tilde{x}[n-M] \text{ is well-defined for all } M \in \mathbb{N}$$
$$DFS\{\tilde{x}[n-M]\} = \boxed{e^{-j\frac{2\pi}{N}Mk}}\tilde{X}[k] \text{ delay factor}$$
$$IDFS\left\{\boxed{e^{-j\frac{2\pi}{N}Mk}}\tilde{X}[k]\right\} = \tilde{x}[n-M] \text{ delay factor}$$

For an N-length signal $x[n]$

$$\tilde{x}[n-M] \text{ not well-defined for all } M \in \mathbb{N}$$
$$build\ \tilde{x}[n] = x[n\ mod\ N] \Rightarrow\ \tilde{X}[k] = X[k]$$
$$IDFT\left\{\boxed{e^{-j\frac{2\pi}{N}Mk}}X[k]\right\} = IDFS\left\{\boxed{e^{-j\frac{2\pi}{N}Mk}}\tilde{X}[k]\right\} = \tilde{x}[n-M] = x[(n-M)\ mod\ N]$$

Periodicity

Shifts for finite-length signals are ”naturally” circular

Analysis Formula for a N-Periodic Signal in the frequency domain

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n]e^{-j\frac{2\pi}{N}nk},\ k \in \mathbb{Z} \tag{1}$$

$X[k]$ Signal vector in the frequency domain

$x[n]$ Signal vector in the (discrete) time domain

Reminder This is the inner Product in explicite form

Synthesis Formula for a N-Periodic Signal in the time domain

$$\tilde{x}[n] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{X}[k]e^{j\frac{2\pi}{N}nk},\ k \in \mathbb{Z} \tag{2}$$

$x[n]$ Signal vector in the (discrete) time domain

$X[k]$ Signal vector in the frequency domain

$\frac{1}{N}$ Normalisation coeficent

Reminder This is the inner Product in explicite fashion

1.2 The Discret-Time Fourier Transform (DTFT)

1.2.1 Overview Fourier Transform

- N-Point finite-length siganls: DFT
- N-Point periodic signals: DFS
- Infinite length (non periodic) signals: DTFT

1.2.2 Karplus Strong revisted and the DTFT

Plotting the DTFT

- Frequencies go from $-\pi$ to π
- Positive frequencies are on the right hand side of the x-axis
- Negative frequencies are on the left hand side of the x-axis
- Low frequencies are centered around 0
- High frequnecies will be on the extreme of the bound

1.2.3 Formal Definition of the DTFT

- $x[n] \in \ell_2(\mathbb{Z})$, the space of square summable infinity sequneces
- define the function of $\omega \in \mathbb{R}$

$$F(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}, \text{ with } \omega = \frac{2\pi}{N} \text{ and } N \to \infty$$

- inversion (when $F(\omega)$ exists):

$$x[n] = \frac{1}{2\;\pi} \int_{-\pi}^{\pi} F(e^{j\omega}); e^{j\;\omega\;n} \mathrm{d}\omega, \text{ with } n \in \mathbb{Z}$$

1.2.4 Properties of the DTFT

linearity	$DTFT\{\alpha x[n] + \beta y[n]\} = \alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
timeshift	$DTFT\{x[n-M]\} = e^{-j\omega M}\ X(e^{j\omega})$
modulation	$DTFT\{e^{-j\omega_0 M}\ x[n]\} = X(e^{j\ (\omega-\omega_0)})$
time reversal	$DTFT\{x[-n]\} = X(e^{-jw})$
conjugation	$DTFT\{x^*[n]\} = X^* X(e^{-j\ \omega})$

1.2.5 Some particular cases

- if $x[n]$ is symmetric, the DTFT is symmetric: $x[n] = x[-n] \iff X(e^{j\omega}) = X(e^{-j\omega})$
- if $x[n]$ is real, the DTFT is Hemitian-symmetric: $x[n] = x^*[n] \iff X(e^{j\omega}) = X^*(e^{-j\omega})$
- if $x[n]$ is real, the magnitude of th eDTFT is symmetric $x[n] \in \mathbb{R} \implies |X(e^{j\omega})| = |X(e^{-j\omega})|$
- if $x[n]$ is real and symmetric, $X(e^{j\omega})$ is also real and symmetric

1.3 TODO Sinusoidal Modulation