Bounding Normal Tails

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If Z is a standard normal random variable, the tail probability $P(Z > a)$ is given by

$$\int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

A crude bound for this can be had by observing

$$\int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \le \int_a^\infty \frac{1}{\sqrt{2\pi}} \left(\frac{x}{a}\right) e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi} a e^{a^2/2}} \tag{1}$$

The middle integral can be evaluated explicitly via the substitution $u = x^2/2$.

The approximation error–the difference between the two sides of (1), and the relative error–the approximation error divided by the correct value, i.e., the left side of (1)–both appear to behave well.