Arches

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As someone who has taught more calculus than most, my mind spends a lot of time in the plane thinking about curves there, and I have always been fascinated by the subtle (and sometimes not-so-subtle) differences among them. Take the connection of one point on the $x-$axis to another by an "arch." Without being overly rigorous, let's say that an arch is a curve that lies on or above a secant line over any subinterval–we're aiming at concave down–and achieves a maximum value at the midpoint of the two $x-$values.

We may as well position the points at $(-r, 0), (0, r)$ for some $r > 0$ that is half the distance between the two original points, so our arch will pass through the points $(\pm r, 0)$ and $(0, r)$. Any arch will sit in the envelope between the triangle with these vertices and the rectangle $[-r, r] \times [0, r]$. Surely, the curved arch that jumps to mind first is a circular arch, i.e., the graph of the function $y = \sqrt{r^2 - x^2}$. Measuring the area under these arches gives

triangle	$\frac{1}{2}(2r)(r) = r^2$
circle	$\frac{\pi}{2}r^2$
rectangle	$(2r)(r) = 2r^2$

A parabolic arch would have the form $y = c(r-x)(r+x)$, and to pass through the point $(0, r)$ requires $c = \dfrac{1}{r}$, so $y = \dfrac{r^2 - x^2}{r}$. To make a sinusoidal arch, we shift the sine function r units left via $y = \sin(x+r)$, scale it by a factor of r to put the peak at the right height (so now $y = r\sin(x + r)$). Finally, we would like to stretch the graph horizontally to make the point $x = r$ on our sinusoidal arch feel like the point $x = \pi$ on the sine function. Setting $c(r + r) = \pi$ gives $c = \pi/2r$, and so our sinusoidal arch is $\left(r \sin\left(\dfrac{\pi(x+r)}{2r}\right)\right)$.

A little integral calculus shows how these arches compare to the others.

$$\int_{-r}^{r} \frac{r^2 - x^2}{r}\, dx = \frac{4}{3}r^2 \text{ and } \int_{-r}^{r} r\sin\left(\frac{\pi(x+r)}{2r}\right)\, dx = \frac{4}{\pi}r^2$$

So,

triangle	r^2
sinusoidal	$\frac{4}{\pi}r^2$
parabolic	$\frac{4}{3}r^2$
circular	$\frac{\pi}{2}r^2$
rectangle	$2r^2$

The parabolic arch can be generalized to a polynomial arch $y = \dfrac{r^n - x^n}{r^{n-1}}$ for even n. This arch encloses an area of $\dfrac{2n}{n+1}r^2$ and fills out the outer rectangle as n grows.