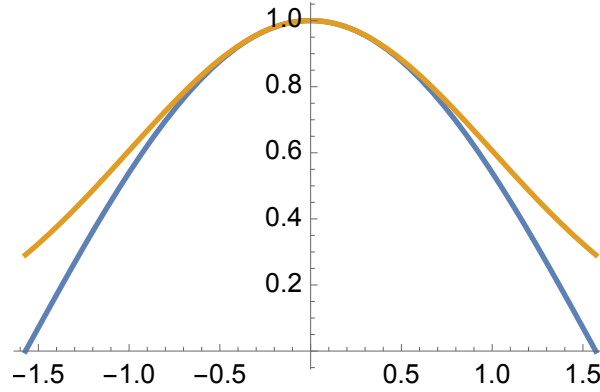


## Cosine Bound (January 2, 2025)

The cosine function is a good lower bound for the standard normal density function near zero.

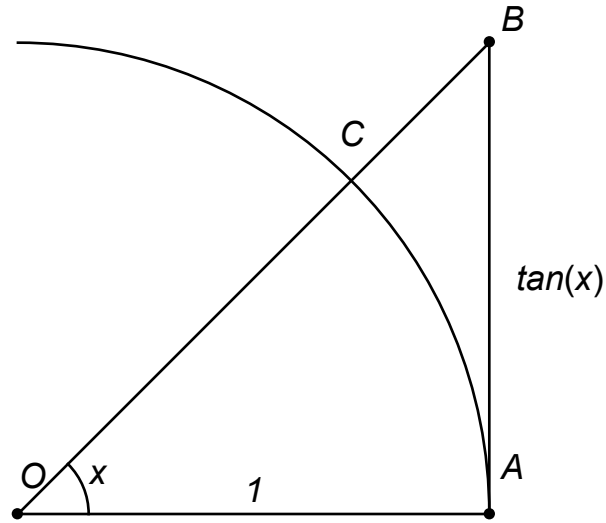


To establish the bound, let  $g(x) = \log \left[ \frac{e^{-x^2/2}}{\cos(x)} \right]$ , so  $g'(x) = \tan(x) - x$ . From Figure 1, the area of  $\triangle OAB$  is larger than that of the circular sector  $OAC$ , so

$$\frac{\tan(x)}{2} \geq \frac{x}{2}$$

Thus, for  $0 \leq x \leq \pi/2$ ,  $g'(x) \geq 0$  and  $g(0) = 0$ , so  $g(x) \geq 0$ . Because  $g$  is an even function, this means  $g(x) \geq 0$  for  $-\pi/2 \leq x \leq \pi/2$ , which is equivalent to  $\cos(x) \leq e^{-x^2/2}$ .

Figure 1:  $\tan(x) \geq x$



This suggests that a good lower bound for textbook standard normal probabilities within one standard deviation of the mean is

$$P(0 \leq Z \leq a) \geq \int_0^a \cos(x) dx = \sin(a) \quad (1)$$

Figure 2 shows the difference between both sides of 1 for  $0 \leq a \leq 1$ .

Figure 2:  $P(0 \leq Z \leq a) - \sin(a)$

