The Central Binomial Coefficients Are All Even

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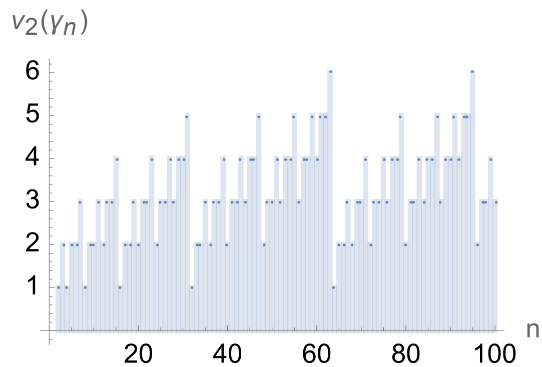
This is the latest in a series of posts about the prime factorizations of the Central Binomial Coefficients, the numbers $\gamma_n = \binom{2n}{n}$. So far, we have noted that if a prime p is such that either $2n/3 < p \leq n$ or $p > 2n$, then p does not appear in the prime factorization of γ_n, and if $n < p \leq 2n$, then p appears to the first power in the prime factorization of γ_n. This leaves the contribution of the primes between 2 and $2n/3$ undetermined. A tiny step forward can be taken using very basic properties of Pascal's Triangle.

The binomial coefficient $\binom{r}{k}$ is the number in row r and column k in Pascal's famous triangle (in the cases of both rows and columns, the count begins at 0). The rows are symmetric, that is $\binom{r}{k} = \binom{r}{r-k}$, which means that $\binom{2n-1}{n-1} = \binom{2n-1}{n}$. Also, every number in the triangle is the sum of the two numbers above it, a fact known as Pascal's Identity. Applying this to the central binomial coefficients gives:

$$\binom{2n}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n} = (2)\binom{2n-1}{n}$$

This means γ_n is always even, so 2 *always* appears in its prime factorization, and the question becomes what power of 2 appears in this prime factorization.

The largest power of a prime p that divides an integer n is called the $p-$adic valuation of n and symbolized $v_p(n)$. Below is a graph of $v_2(\gamma_n)$ for the first 100 values of n



The function is not periodic but does have a repetitive structure.