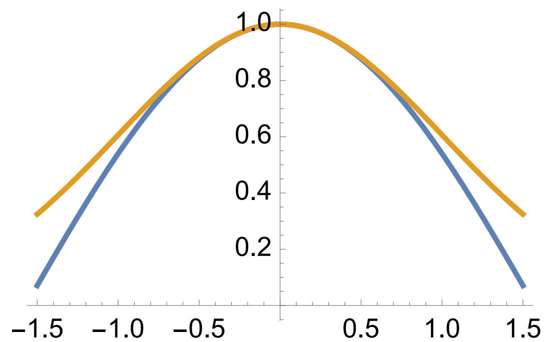
Cosine Bound (January 2, 2025)

The cosine function is a good lower bound for the standard normal density function near zero.

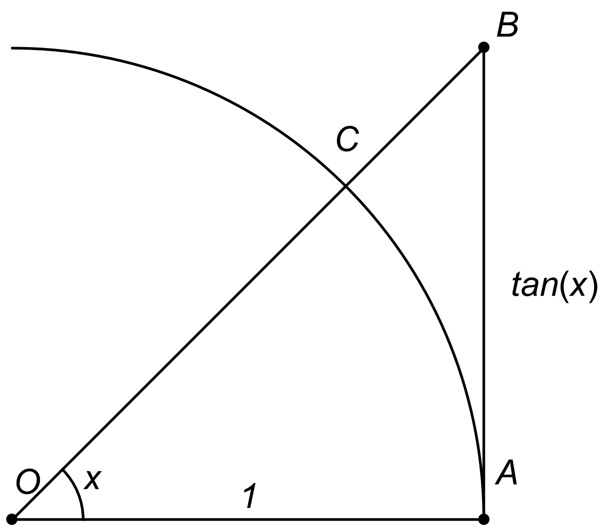


To establish the bound, let $g(x) = \log\left[\frac{e^{-x^2/2}}{\cos(x)}\right]$, so $g'(x) = \tan(x) - x$. From Figure 1, the area of $\triangle OAB$ is larger than that of the circular sector OAC, so

$$\frac{\tan(x)}{2} \geq \frac{x}{2}$$

Thus, for $0 \leq x \leq \pi/2$, $g'(x) \geq 0$ and $g(0) = 0$, so $g(x) \geq 0$. Because g is an even function, this means $g(x) \geq 0$ for $-\pi/2 \leq x \leq \pi/2$, which is equivalent to $\cos(x) \leq e^{-x^2/2}$.

Figure 1: $\tan(x) \geq x$



This suggests that a good lower bound for textbook standard normal probabilities within one standard deviation of the mean is

$$P(0 \leq Z \leq a) \geq \int_0^a \frac{1}{\sqrt{2\pi}} \cos(x)\, dx = \frac{\sin(a)}{\sqrt{2\pi}} \tag{1}$$

Figure 2 shows the difference between both sides of 1 for $0 \leq a \leq 1$.

Figure 2: $P(0 \leq Z \leq a) - \sin(a)/\sqrt{2\pi}$