Formulas from Legendre and Kummer

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Prime factorizations of products concatenate with products of powers of the same base simplifying according to the rules of exponents. For example, the prime factorization of 9450 is $2 \cdot 3^3 \cdot 5^2 \cdot 7$ and the prime factorization of 7938 is $2 \cdot 3^4 \cdot 7^2$, so the prime factorization of (9450)(7938) is

$$\left(2 \cdot 3^3 \cdot 5^2 \cdot 7\right)\left(2 \cdot 3^4 \cdot 7^2\right) = 2^2 \cdot 3^7 \cdot 5^2 \cdot 7^3$$

A fancy way of expressing this is to say, for example, that $v_3\left((9450)(7938)\right) = v_3(9450) + v_3(7938)$–the $p-$adic valuation function has the log-like property that

$$v_p(ab) = v_p(a) + v_p(b)$$

Consequently

$$v_p(n!) = v_p(n(n-1)\cdots 2 \cdot 1) = \sum_{k=1}^{n} v_p(k)$$

Using the relationship between $p-$adic valuations and digit sums discussed here, we can re-express this:

$$\begin{aligned} v_p(n!) &= \sum_{k=1}^{n} \frac{1 + S_p(k-1) - S_p(k)}{p-1} \\ &= \frac{1}{p-1}\left(n + S_p(0) - S_p(n)\right) \\ &= \frac{n - S_p(n)}{p-1} \end{aligned}$$

This result was proved by Legendre in 1830.

In a previous post, we considered the $2-$adic valuations of the central binomial coefficients $\gamma_n = \binom{2n}{n}$, that is, the numbers $v_2(\gamma_n)$. Playing around with this function, one is struck by the observation that $v_2\left(\gamma_n\right) = 1$ if and only if $n = 2^k$ for some nonnegative integer k. This is a consequence of Kummer's Theorem, which says that if $S_p(x)$ is the sum of the base p digits of x, then

$$v_p\left(\binom{r}{k}\right) = \frac{S_p(k) + S_p(r-k) - S_p(r)}{p-1} \tag{1}$$

Because of the log-like nature of the $p-$adic valuation,

$$v_p\left(\binom{r}{k}\right) = v_p\left(\frac{r!}{k!(r-k)!}\right) = v_p(r!) - v_p(k!) - v_p((r-k)!)$$

Using Legendre's Formula to replace the terms on the right and simplifying gives (1), and applying (1) to the central binomial coefficients we get

$$v_p(\gamma_n) = \frac{2S_p(n) - S_p(2n)}{p-1} \tag{2}$$

Multiplying a number by 10 shifts its base ten digits one place to the right but does not change its digit sum. More generally, multiplying a number by p does not change its base p digit sum. So in particular, $S_2(2n) = S_2(n)$, and in the case $p = 2$, (2) becomes the remarkable formula

$$v_2(\gamma_n) = S_2(n)$$

That is, the power of 2 that appears in the prime factorization of $\binom{2n}{n}$ is just the sum of the base two digits of n, also called the Hamming Weight of n.