The Central Binomial Coefficients (1)

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The numbers $\binom{2n}{n}$ that run vertically down the center of Pascal's Triangle are called the *central binomial coefficients*. I'll use the notation $\gamma_n \doteq \binom{2n}{n}$ for these numbers, which grow quickly with n

n	γ_n
1	2
5	252
10	184756
15	155117520

These numbers are interesting for a number of reasons, one of which is their prime factorizations. A key feature of primes is that if a prime number divides a product, then it must divide one of the factors, so for example the prime 3 divides 36, and no matter how you factor 36, 3 will always divide at least one of the factors.

$$(1)(36)$$
$$(2)(18)$$
$$(3)(12)$$
$$(4)(9)$$
$$(6)(6)$$

This is not the case for non-primes. The number 6 divides 36, but in the facorization $36 = 4 \cdot 9$, 6 divides neither factor.

Any prime that divides $\gamma_n = \dfrac{(2n)!}{(n!)^2}$ must divide the numerator, and any prime that divides $(2n)! = (2n)(2n-1)\cdots 2 \cdot 1$ must divide one of the factors and so cannot be larger than $2n$. This means that while γ_n is a large number even for relatively small n, it can have no "large" prime factors. For example, $\gamma_{50} \approx 10^{29}$ but has no prime factor larger than 100.

Is this unusual? I had my computer randomly select 200 integers between 10^{24} and 10^{34}, and the *smallest* maximum prime factor of the bunch was 113578811. Hmmm...