The Ratio of Area to Circumference is Half the Radius

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Pretending that we know some integral calculus but not how to measure circles, we know that the area of the first quadrant of a circle with radius r centered at the origin is

$$A = \int_0^r \sqrt{r^2 - x^2}\, dx$$

and its length is

$$L = \int_0^r \sqrt{1 + \left(\frac{d}{dx}\left(\sqrt{r^2 - x^2}\right)\right)^2}\, dx = \int_0^r \frac{r}{\sqrt{r^2 - x^2}}\, dx$$

This theorem connects these two integrals:

Theorem 1. *For* $n = 2, 3, \ldots,$ *and* $r > 0$,

$$\int_0^r \frac{r^n}{\sqrt{r^n - x^n}}\, dx = \left(1 + \frac{2}{n}\right) \int_0^r \sqrt{r^n - x^n}\, dx \tag{1}$$

Proof. Observing that

$$\begin{aligned}
\int_0^r \sqrt{r^n - x^n}\, dx &= \int_0^r \frac{r^n - x^n}{\sqrt{r^n - x^n}}\, dx \\
&= \int_0^r \frac{r^n}{\sqrt{r^n - x^n}}\, dx - \int_0^r \frac{x^n}{\sqrt{r^n - x^n}}\, dx \\
&= \int_0^r \frac{r^n}{\sqrt{1 - x^n}}\, dx + \int_0^r \frac{2x}{n} \cdot \frac{d}{dx}\sqrt{r^n - x^n}\, dx
\end{aligned}$$

Applying integration by parts to the second integral on the right with $u = 2x/n$ and $dv = \dfrac{d}{dx}\sqrt{r^n - x^n}\, dx$ gives

$$\begin{aligned}
\int_0^r \frac{2x}{n} \cdot \frac{d}{dx}\sqrt{r^n - x^n}\, dx &= \left[\frac{2x}{n}\sqrt{r^n - x^n}\right]_0^r - \frac{2}{n}\int_0^r \sqrt{r^n - x^n}\, dx \\
&= -\frac{2}{n}\int_0^r \sqrt{r^n - x^n}\, dx
\end{aligned}$$

and (1) follows. □

Taking $n = 2$ gives

$$rL = \left(1 + \frac{2}{2}\right) A \text{ or } \frac{A}{L} = \frac{r}{2}$$