The Polylogarithm (June 25, 2025)

The polylogarithm function is $\mathrm{Li}_s(z) = \displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^s}$. It can be shown using the ratio test that this series converges for any s and $|z| < 1$. For some values of s, we can determine a closed form expression for $\mathrm{Li}_s(z)$, which will allow for easy calculation of the sums of some infinite series.

When $s = 0$, this sum is a geometric series:

$$\mathrm{Li}_0(z) = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z}$$

This is a start, but what allows us really make some headway is the relationship yielded by differentiating $\mathrm{Li}_s(z)$ and assuming that we can move the derivative inside the sum:

$$\mathrm{Li}_s'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n^{s-1}} = \frac{1}{z}\mathrm{Li}_{s-1}(z)$$

With $s = 0$ this equation becomes $\dfrac{1}{z}\mathrm{Li}_{-1}(z) = \mathrm{Li}_0'(z)$. Since we have a closed form expression for $\mathrm{Li}_0(z)$, this gives us a closed form expression for $\mathrm{Li}_{-1}(z)$, namely

$$\mathrm{Li}_{-1}(z) = z\mathrm{Li}_o'(z) = \frac{z}{(z-1)^2} \tag{1}$$

Recalling that $\mathrm{Li}_{-1}(z) = \displaystyle\sum_{n=1}^{\infty} nz^n$, equation (1) allows the evaluation of a number of interesting sums. Taking $z = 1/2, 1/3, 1/4$, we see that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n}{2^n} &= 2 \\
\sum_{n=1}^{\infty} \frac{n}{3^n} &= \frac{3}{4} \\
\sum_{n=1}^{\infty} \frac{n}{4^n} &= \frac{4}{9}
\end{aligned}$$

If z is a positive rational number, so $z = a/b$ for positive integers a and b, then the right side of (1) becomes $\dfrac{ab}{(a-b)^2}$. I am struck by the fact that if z has the form $a/(a+1)$, then the sum $\displaystyle\sum_{n=1}^{\infty} n \left(\frac{a}{a+1}\right)^n$ has an integer value.

$$\sum_{n=1}^{\infty} n \left(\frac{4}{5}\right)^n = 20, \sum_{n=1}^{\infty} n \left(\frac{8}{9}\right)^n = 72, \text{ etc.}$$