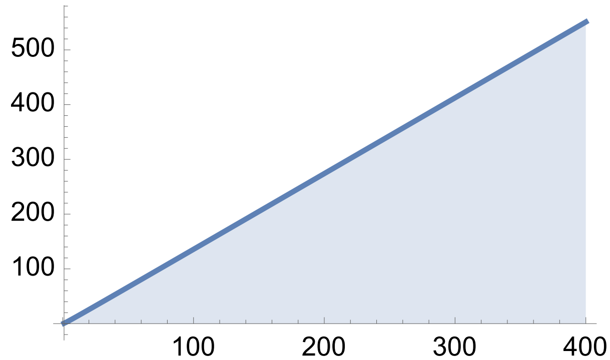
# Bounds for the Central Binomial Coefficients

**(January 9, 2025)**

Just how big are the central binomial coefficients? A plot of $\log(\gamma_n)$ looks linear with slope $\approx 1.4$ suggesting $\gamma_n \approx e^{1.4n}$.




An old problem in *The College Mathematics Journal* [1, 2] is to establish the bounds

$$\frac{2^{2n-1}}{\sqrt{n}} < \gamma_n < \frac{2^{2n-1/2}}{\sqrt{n}} \tag{1}$$

or, equivalently,

$$\frac{4^n}{2\sqrt{n}} < \gamma_n < \frac{4^n}{\sqrt{2n}}$$

An elegant proof of (1) is given by Henry O. Pollak, but a tighter upper bound can be given beginning by expanding $\cos^{2n}(x)$ using the definition $\cos(x) = (e^{ix} + e^{-ix})/2$ and the Binomial Theorem.

$$\cos^{2n}(x) = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^{2n} = \frac{1}{4^n}\sum_{k=0}^{2n}\binom{2n}{k}e^{2i(k-n)x} \tag{2}$$

If $m$ is an even, nonzero integer,

$$\int_{-\pi/2}^{\pi/2} e^{imx}\,dx = \frac{1}{im}\left(e^{im(\pi/2)} - e^{-im(\pi/2)}\right) = 0$$

And of course, if $m = 0$, then

$$\int_{-\pi/2}^{\pi/2} e^{imx}\,dx = \int_{-\pi/2}^{\pi/2} (1)\,dx = \pi$$

So if both sides of (2) are integrated over $[-\pi/2, \pi/2]$, the integral of every term in the sum on the right is zero except for the $k = n$ term, giving

$$\int_{-\pi/2}^{\pi/2} \cos^{2n}(x)\, dx = \frac{\pi}{4^n}\binom{2n}{n} \tag{3}$$

One can bound $\gamma_n$, then, by bounding the cosine function on the interval $[-\pi/2, \pi/2]$, and an excellent such bound is $\cos(x) \leq e^{-x^2/2}$.

$$\begin{aligned}
\gamma_n &\leq \frac{4^n}{\pi} \int_{-\pi/2}^{\pi/2} \cos^{2n}(x)\, dx \\
&\leq \frac{4^n}{\pi} \int_{-\pi/2}^{\pi/2} e^{-nx^2}\, dx \\
&< \frac{4^n}{\pi} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2\cdot\frac{1}{2n}}}\, dx \\
&= \frac{4^n}{\pi}\sqrt{2\pi}\sqrt{\frac{1}{2n}} \\
&= \frac{4^n}{\sqrt{n\pi}}
\end{aligned}$$

The integrand in the third line is a constant multiple of the density function of an $N\left(0, \sqrt{1/(2n)}\right)$ distribution.

# References


[1] Problem 420 proposed by Edward T.H. Wang. *The College Mathematics Journal*, Vol. 21, No. 1.

[2] Pollak, Henry O. and others. Solution to problem 420. *The College Mathematics Journal*, Vol. 22, No. 1.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