More Sums via the Polylogarithm (July 15, 2025)

Differentiating $\mathrm{Li}_0(z)$ twice, we get

$$\mathrm{Li}_0''(z) = \frac{d}{dz}\left[\frac{1}{z}\mathrm{Li}_{-1}(z)\right] = -\frac{1}{z^2}\mathrm{Li}_{-1}(z) + \frac{1}{z^2}\mathrm{Li}_{-2}(z)$$

Because we have closed form formulas for $\mathrm{Li}_0(z)$ and $\mathrm{Li}_{-1}(z)$, this allows us to compute such a formula for $\mathrm{Li}_{-2}(z)$, which is

$$\mathrm{Li}_{-2}(z) = \sum_{n=1}^{\infty} n^2 z^n = -\frac{z(z+1)}{(z-1)^3} \tag{1}$$

And again, taking $z = 1/2, 1/3, 1/4$, we can compute some remarkable sums.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^2}{2^n} &= 6 \\ \sum_{n=1}^{\infty} \frac{n^2}{3^n} &= \frac{3}{2} \\ \sum_{n=1}^{\infty} \frac{n^2}{4^n} &= \frac{20}{27} \end{aligned}$$

Again, it is the case that if $z = a/(a+1)$, then substituting into the right side of (1) gives

$$\sum_{n=1}^{\infty} n^2 \left(\frac{a}{a+1}\right)^n = a(a+1)(2a+1)$$

and again we get some sums with perhaps unexpected integer values

$$\sum_{n=1}^{\infty} n^2 \left(\frac{2}{3}\right)^n = 30, \sum_{n=1}^{\infty} n^2 \left(\frac{6}{7}\right)^n = 546, \text{ etc.}$$