The Central Binomial Coefficients: Prime Factors between $2n/3$ and $2n$

(September 12, 2024)

In my last post, I made the small observation that the central binomial coefficient

$$\gamma_n \doteq \binom{2n}{n} = \frac{(2n)!}{(n!)^2} \tag{1}$$

has no prime factors larger than $2n$, which is a bit unusual for a number so large. So all the prime factors of γ_n are to be found in $\{1, 2, \ldots, 2n\}$, and most of the primes in this range occur in the prime factorization of γ_n either to the first power or not at all.

Every prime in the range $\{n+1, n+2, \ldots, 2n\}$ divides $(2n)!$ but is too large to divide $n!$, and so must appear in the prime factorization of γ_n. However, if p^2 divides $(2n)!$, then there must be a multiple of p besides p itself in range $\{1, 2 \ldots, 2n\}$. For $p > n$, this is impossible, so every prime in the range $\{n+1, n+2, \ldots, 2n\}$ appears in the prime factorization of γ_n to the first power.

If $\frac{2}{3}n < p \leq n$, then there is exactly one multiple of p in the range $\{1, 2, \ldots, n\}$ and exactly two multiples of p in the range $\{1, 2, \ldots, 2n\}$, so that in the fraction $\frac{(2n)!}{(n!)^2}$, all the p's will cancel and no such prime occurs in the prime factorization of γ_n.

For example, if $n = 50$, then the primes $53, 59, 61, 67, 71, 73, 79, 83, 89, 97$ (the primes between 51 and 100) each occur to the first power in the prime factorization of γ_n, so

$$\gamma_{50} = C(53)(59)(61)(67)(71)(73)(79)(83)(89)(97) \tag{2}$$

Also, the primes $37, 41, 43, 47$ (the primes between $100/3$ and 50) do not divide γ_{50}, so the prime factors of C are to be found among $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$.