More Applications of Kummer (December 27, 2024)

Kummer's Theorem gives the formula (1) relating $v_p(\gamma_n)$ to $S_p(n)$ and $S_p(2n)$, where $S_p(x)$ is the sum of the base p digits of x. This can be leveraged to give some remarkable conclusions about prime factorizations of very large numbers based on the digits of a much smaller one.

$$v_p(\gamma_n) = \frac{2S_p(n) - S_p(2n)}{p-1} \tag{1}$$

Doubling a number with small digits like 312 is easy because we can just double each digit in its place: $2 \times 312 = 624$. Doubling a number like 473 is not so easy because in doubling the tens place, we wind up with "too many" tens. More generally and precisely, if the base p digits of n are $d_0, d_1, \ldots, d_k$, and $d_i \leq (p-1)/2$ for all i, then the base p digits of $2n$ are $2d_0, 2d_1, \ldots, 2d_k$, which means that $S_p(2n) = 2S_p(n)$ and so $v_p(\gamma_n) = 0$. For example, $27 = 220_5$; none of the base 5 digits of 27 is more than $(5-1)/2$; so it should be the case that $v_5(\gamma_{27}) = 0$ (and it is).

In essence, if the base p representation of n contains all "small" digits, then p does not appear in the prime factorization of γ_n. If $n = 3220102_7$, then γ_n has nearly a quarter million digits, but no appearance of seven in its prime factorization.