$p-$adic valuations and digit sums

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Previous posts have discussed the prime factorizations of the central binomial coefficients, $\gamma_n = \binom{2n}{n}$, arriving at the question of the $p-$adic valuation of $\gamma_n, v_p(\gamma_n)$, for relatively small primes ($< 2n/3$). There is an interesting relationship between $p-$adic valuations and digit sums. We typically write numbers in base ten, so the symbol 352 abbreviates the number

$$3 \times 10^2 + 5 \times 10^1 + 2 \times 10^0$$

Any positive integer p can be used as a base. Given a positive integer n, the base p digits of n are numbers $d_0, d_1, \ldots, d_m$ with $0 \leq d_i < p$ such that

$$n = d_0 p^0 + d_1 p^1 + \cdots d_m p^m$$

For example

$$352 = 2 \times 5^3 + 4 \times 5^2 + 0 \times 5^1 + 2 \times 5^0$$

so the base 5 digits of the number 352 are $d_0 = 2, d_1 = 0, d_2 = 4, d_3 = 2$. We'll use the notation $S_p(n)$ to express the sum of the base p digits of n, so $S_5(352) = 2 + 4 + 0 + 2 = 8$ and $S_{10}(352) = 3 + 5 + 2 = 10$.

If $v_p(k) = l > 0$, then the base p representation of k has zeroes in the rightmost l places in the same way that a number divisible by 1000 ends in three zeroes (in base ten). That is, $k = d_l p^l + d_{l+1} p^{l+1} + \cdots + d_m p^m$. This means:

$$\begin{aligned} k - 1 &= (p^l - 1) + (d_l - 1)p^l + d_{l+1} p^{l+1} + \cdots + d_m p^m \\ &= \sum_{i=0}^{l-1} (p-1)p^i + (d_l - 1)p^l + d_{l+1} p^{l+1} + \cdots + d_m p^m \end{aligned}$$

So if we add up the digits above

$$S_p(k-1) = (p-1)l + S_p(k) - 1$$

or

$$v_p(k) = \frac{1 + S_p(k-1) - S_p(k)}{p-1} \tag{1}$$

If $v_p(k) = 0$, then the ones digit of k (in base p) is not zero and so the base p digits of $k-1$ are identical to those of k except the ones digit is one less, meaning $S_p(k-1) = S_p(k) - 1$, and (1) still holds (both sides of the equation are zero).

What is interesting about this result is that it gives a very concrete relationship between two somewhat distinct aspects of a number–its $p-$adic valuation and its base p digit sum.