Appendix F: Wavelet Analysis and Synthesis Equations

F.1 BASIC PROPERTIES

The inner product of two functions is defined as

$$< x, y > = \int x(t)y(t)dt \tag{F.1}$$

Two functions are orthogonal if

$$< x(t), x(t-k) > = \begin{cases} A & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k \neq 0 \end{cases} \tag{F.2}$$

Two functions are orthonormal if

$$< x(t), x(t-k) > = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \tag{F.3}$$

The signal energy is defined as

$$E = \int x^2(t)dt \tag{F.4}$$

Many wavelet families are designed to be orthonormal:

$$E = \int \psi^2(t)dt = 1 \tag{F.5}$$

$$E = \int \psi_{jk}^2(t)dt = \int [2^{j/2}\psi(2^j t - k)]^2 dt = \int 2^j \psi^2(2^j t - k)dt \tag{F.6}$$

Let $u = 2^j t - k$. Then $du = 2^j dt$. Equation (F.6) becomes

$$E = \int 2^j \psi^2(u) 2^{-j} du = 1 \tag{F.7}$$

Both father and mother wavelets are orthonormal at scale j:

$$\int \phi_{jk}(t)\phi_{jn}(t)dt = \begin{cases} 1 & k = n \\ 0 & \text{otherwise} \end{cases} \tag{F.8}$$

$$\int \psi_{jk}(t)\psi_{jn}(t)dt = \begin{cases} 1 & k = n \\ 0 & \text{otherwise} \end{cases} \tag{F.9}$$

F.2 ANALYSIS EQUATIONS

When a function $f(t)$ is approximated using the scaling functions only at scale $j+1$, it can be expressed as

$$f(t) = \sum_{k=-\infty}^{\infty} c_j(k)2^{j/2}\phi(2^j t - k)$$

Using the inner product,

$$c_j(k) = \langle f(t), \phi_{jk}(t) \rangle = \int f(t)2^{j/2}\phi(2^j t - k)dt \tag{F.10}$$

Note that

$$\phi(t) = \sum_{n=-\infty}^{\infty} \sqrt{2}h_0(n)\phi(2t - n) \tag{F.11}$$

Substituting Equation (F.11) into Equation (F.10) leads to

$$c_j(k) = \langle f(t), \phi_{jk}(t) \rangle = \int f(t)2^{j/2} \sum_{n=-\infty}^{\infty} \sqrt{2}h_0(n)\phi[2(2^j t - k) - n]dt$$

$$c_j(k) = \sum_{n=-\infty}^{\infty} \int f(t)2^{(j+1)/2}h_0(n)\phi(2^{(j+1)}t - 2k - n)dt$$

Let $m = n + 2k$. Interchange of the summation and integral leads to

$$c_j(k) = \sum_{m=-\infty}^{\infty} \int f(t)2^{(j+1)/2}h_0(m - 2k)\phi(2^{(j+1)}t - m)dt$$

$$c_j(k) = \sum_{m=-\infty}^{\infty} \left(\int f(t)\phi_{(j+1)m}(t)dt \right) h_0(m - 2k) \tag{F.12}$$

Using the inner product definition for the DWT coefficient again in (F.12), we achieve

$$c_j(k) = \sum_{m=-\infty}^{\infty} < f(t), \phi_{(j+1)m}(t) > h_0(m-2k) = \sum_{m=-\infty}^{\infty} c_{j+1}(m) h_0(m-2k) \tag{F.13}$$

Similarly, notice that

$$\psi(t) = \sum_{k=-\infty}^{\infty} \sqrt{2} h_1(k) \phi(2t-k)$$

Using the inner product gives

$$d_j(k) = < f(t), \psi_{jk}(t) > = \int f(t) 2^{j/2} \sum_{n=-\infty}^{\infty} \sqrt{2} h_1(n) \phi[2(2^j t - k) - n] dt$$

$$d_j(k) = \sum_{n=-\infty}^{\infty} \int f(t) 2^{(j+1)/2} h_1(n) \phi(2^{(j+1)} t - 2k - n) dt \tag{F.14}$$

Let $m = n + 2k$. Interchange of the summation and integral leads to

$$d_j(k) = \sum_{m=-\infty}^{\infty} \int f(t) 2^{(j+1)/2} h_1(m-2k) \phi(2^{(j+1)} t - m) dt$$

$$d_j(k) = \sum_{m=-\infty}^{\infty} \left(\int f(t) \phi_{(j+1)m}(t) dt \right) h_1(m-2k) \tag{F.15}$$

Finally, applying the inner product definition for the wavelet discrete transform (WDT) coefficient, we obtain

$$d_j(k) = \sum_{m=-\infty}^{\infty} < f(t), \phi_{(j+1)m}(t) > h_1(m-2k) = \sum_{m=-\infty}^{\infty} c_{j+1}(m) h_1(m-2k) \tag{F.16}$$

F.2 WAVELET SYNTHESIS EQUATIONS

We begin with

$$f(t) = \sum_{k=-\infty}^{\infty} c_j(k) 2^{j/2} \phi(2^j t - k) + \sum_{k=-\infty}^{\infty} d_j(k) 2^{j/2} \psi(2^j t - k)$$

Taking an inner product using the scaling function at scale level $j+1$ gives

$$c_{j+1}(k) = < f(t), \phi_{(j+1)k}(t) > = \sum_{m=-\infty}^{\infty} c_j(m) 2^{j/2} \int \phi(2^j t - m) \phi_{(j+1)k}(t) dt$$
$$+ \sum_{m=-\infty}^{\infty} d_j(m) 2^{j/2} \int \psi(2^j t - m) \phi_{(j+1)k}(t) dt$$

$$
\begin{aligned}
c_{j+1}(k) &= \sum_{m=-\infty}^{\infty} c_j(m)2^{j/2}\int \sum_{n=-\infty}^{\infty} \sqrt{2}h_0(n)\phi(2^{j+1}t-2m-n)\phi_{(j+1)k}(t)dt \\
&\quad + \sum_{m=-\infty}^{\infty} d_j(m)2^{j/2}\int \sum_{n=-\infty}^{\infty} \sqrt{2}h_1(n)\phi(2^{j+1}t-2m-n)\phi_{(j+1)k}(t)dt
\end{aligned}
\tag{F.17}
$$

Interchange of the summation and integral yields

$$
\begin{aligned}
c_{j+1}(k) &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_j(m)h_0(n)\int 2^{(j+1)/2}\phi(2^{j+1}t-2m-n)\phi_{(j+1)k}(t)dt \\
&\quad + \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} d_j(m)h_1(n)\int 2^{(j+1)/2}\phi(2^{j+1}t-2m-n)\phi_{(j+1)k}(t)dt
\end{aligned}
$$

Using the inner product, we get

$$
\begin{aligned}
c_{j+1}(k) &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} c_j(m)h_0(n) < \phi_{(j+1)(2m+n)}(t), \phi_{(j+1)k}(t) > \\
&\quad + \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} d_j(m)h_1(n) < \phi_{(j+1)(2m+n)}(t), \phi_{(j+1)k}(t) >
\end{aligned}
\tag{F.18}
$$

From the wavelet orthonormal property, we have

$$
< \phi_{(j+1)(2m+n)}(t), \phi_{(j+1)k}(t) > = \begin{cases} 1 & n = k-2m \\ 0 & \text{otherwise} \end{cases}
\tag{F.19}
$$

Substituting Equation (F.19) into Equation (F.18), we finally obtain

$$
c_{j+1}(k) = \sum_{m=-\infty}^{\infty} c_j(m)h_0(k-2m) + \sum_{m=-\infty}^{\infty} d_j(m)h_1(k-2m)
\tag{F.20}
$$