Appendix B: Review of Analog Signal Processing Basics

B.1 FOURIER SERIES AND FOURIER TRANSFORM

Electronics applications require familiarity with some periodic signals such the square wave, rectangular wave, triangular wave, sinusoid, sawtooth wave, and so on. These periodic signals can be analyzed in the frequency domain with the help of the Fourier series expansion. According to Fourier theory, a periodic signal can be represented by a Fourier series that contains the sum of a series of sine and/or cosine functions (harmonics) plus a direct current (DC) term. There are three forms of Fourier series: (1) sine-cosine, (2) amplitude-phase, and (3) complex exponential. We will review each of them individually in the following text. Comprehensive treatments can be found in Ambardar (1999), Soliman and Srinath (1998), and Stanley (2003).

B.1.1 Sine-Cosine Form

The Fourier series expansion of a periodic signal $x(t)$ with a period of T via the sine-cosine form is given by

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \tag{B.1}$$

where $\omega_0 = 2\pi/T_0$ is the fundamental angular frequency in radians per second, while the fundamental frequency in terms of Hz is $f_0 = 1/T_0$. The Fourier coefficients of a_0, a_n, and b_n may be found according to the following integral equations:

$$a_0 = \frac{1}{T_0} \int_{T_0} x(t)dt \tag{B.2}$$

$$a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos(n\omega_0 t)dt \tag{B.3}$$

$$b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin(n\omega_0 t)dt \tag{B.4}$$

Notice that the integral is performed over one period of the signal to be expanded. From Equation (B.1), the signal $x(t)$ consists of a DC term and sums of sine and cosine functions with their corresponding harmonic frequencies. Again, note that $n\omega_0$ is the nth harmonic frequency.

B.1.2 Amplitude-Phase Form

From the sine-cosine form, we notice that there is a sum of two terms with the same frequency. The term in the first sum is $a_n \cos(n\omega_0 t)$ while the other is $b_n \sin(n\omega_0 t)$. We can combine these two terms and modify the sine-cosine form into the amplitude-phase form:

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n) \tag{B.5}$$

The DC term is same as before, that is,

$$A_0 = a_0 \tag{B.6}$$

and the amplitude and phase are given by

$$A_n = \sqrt{a_n^2 + b_n^2} \tag{B.7}$$

$$\phi_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) \tag{B.8}$$

respectively. The amplitude-phase form provides very useful information for spectral analysis. With the calculated amplitude and phase for each harmonic frequency, we can create the spectral plots. One depicts a plot of the amplitude versus its corresponding harmonic frequency (the amplitude spectrum), while the other plot shows each phase versus its harmonic frequency (the phase spectrum). Note that the spectral plots are one-sided, since amplitudes and phases are plotted versus the positive harmonic frequencies. We will illustrate these in Example B.1.

B.1.3 Complex Exponential Form

The complex exponential form is developed based on expanding sine and cosine functions in the sine-cosine form into their exponential expressions using Euler's formula and regrouping these exponential terms. Euler's formula is given by

$$e^{\pm jx} = \cos(x) \pm j \sin(x)$$

which can be written as two separate forms:

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

We will focus on interpretation and application rather than the derivation of this form. Thus the complex exponential form is expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{B.9}$$

where c_n represents the complex Fourier coefficients, which may be found from

$$c_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}dt \tag{B.10}$$

The relationship between the complex Fourier coefficients c_n and the coefficients of the sine-cosine form are

$$c_0 = a_0 \tag{B.11}$$

$$c_n = \frac{a_n - jb_n}{2}, \quad \text{for} \quad n > 0 \tag{B.12}$$

Considering a real signal $x(t)$ ($x(t)$ is not a complex function) in Equation (B.10), c_{-n} is equal to the complex conjugate of c_n, that is, $\bar{c}_n$. It follows that

$$c_{-n} = \bar{c}_n = \frac{a_n + jb_n}{2}, \quad \text{for} \quad n > 0 \tag{B.13}$$

Since c_n is a complex value that can be written in the magnitude-phase form, we obtain

$$c_n = |c_n|\angle\phi_n \tag{B.14}$$

where $|c_n|$ is the magnitude and ϕ_n is the phase of the complex Fourier coefficient. Similar to the magnitude-phase form, we can create the spectral plots for $|c_n|$ and ϕ_n. Since the frequency index n goes from $-\infty$ to ∞, the plots of the resultant spectra are two-sided.

EXAMPLE B.1

Consider the square waveform *x(t)* shown in Figure B.1, where T_0 represents a period. Find the Fourier series expansions in terms of (a) the sine-cosine form, (b) the amplitude-phase form, and (c) the complex exponential form.

Solution:

From Figure B.1, we notice that $T_0 = 1$ second and $A = 10$. The fundamental frequency is

$$f_0 = 1/T_0 = 1 \text{ Hz} \quad \text{or} \quad \omega_0 = 2\pi \times f_0 = 2\pi \text{ rad/sec}$$

a. Using Equations (B.1) to (B.3) yields

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)dt = \frac{1}{1}\int_{-0.25}^{0.25} 10dt = 5$$

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(n\omega_0 t)dt$$

$$= \frac{2}{1}\int_{-0.25}^{0.25} 10\cos(n2\pi t)dt$$

$$= \frac{2}{1}\frac{10\times\sin(n2\pi t)}{n2\pi}\bigg|_{-0.25}^{0.25} = 10\frac{\sin(0.5\pi n)}{0.5\pi n}$$

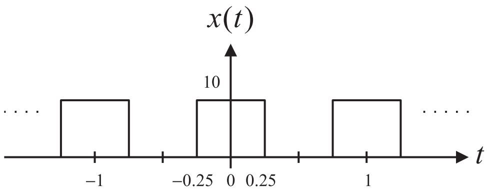


FIGURE B.1

Square waveform in Example B.1.

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\sin(n\omega_0 t)dt$$
$$= \frac{2}{1}\int_{-0.25}^{0.25} 10 \times \sin(n2\pi t)dt$$
$$= \frac{2}{1}\frac{-10\cos(n2\pi t)}{n2\pi}\bigg|_{-0.25}^{0.25} = 0$$

Thus, the Fourier series expansion in terms of the sine-cosine form is written as

$$x(t) = 5 + \sum_{n=1}^{\infty} 10\frac{\sin(0.5\pi n)}{0.5\pi n}\cos(n2\pi t)$$
$$= 5 + \frac{20}{\pi}\cos(2\pi t) - \frac{20}{3\pi}\cos(6\pi t) + \frac{4}{\pi}\cos(10\pi t) - \frac{20}{7\pi}\cos(14\pi t) + \cdots$$

b. Making use of the relations between the sine-cosine form and the amplitude-phase form, we obtain

$$A_0 = a_0 = 5$$
$$A_n = \sqrt{a_n^2 + b_n^2} = |a_n| = 10 \times \left|\frac{\sin(0.5\pi n)}{0.5\pi n}\right|$$

Again, noting that $-\cos(x) = \cos(x + 180^\circ)$, the Fourier series expansion in terms of the amplitude-phase form is

$$x(t) = 5 + \frac{20}{\pi}\cos(2\pi t) + \frac{20}{3\pi}\cos(6\pi t + 180^\circ) + \frac{4}{\pi}\cos(10\pi t) + \frac{20}{7\pi}\cos(14\pi t + 180^\circ) + \cdots$$

c. First let us find the complex Fourier coefficients using the formula, that is,

$$c_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-jn\omega_0 t}dt$$
$$= \frac{1}{1}\int_{-0.25}^{0.25} Ae^{-jn2\pi t}dt$$
$$= 10\times\frac{e^{-jn2\pi t}}{-jn2\pi}\bigg|_{-0.25}^{0.25} = 10 \times \frac{\left(e^{-j0.5\pi n} - e^{j0.5\pi n}\right)}{-jn2\pi}$$

Applying Euler's formula yields

$$c_n = 10 \times \frac{\cos 0.5\pi n - j\sin(0.5\pi n) - [\cos(0.5\pi n) + j\sin(0.5\pi n)]}{-jn2\pi} = 5\frac{\sin(0.5\pi n)}{0.5\pi n}$$

Second, using the relationship between the sine-cosine form and the complex exponential form, it follows that

$$c_n = \frac{a_n - jb_n}{2} = \frac{a_n}{2} = 5\frac{\sin(0.5n\pi)}{(0.5n\pi)}$$

Certainly, the result is identical to the one obtained directly from the formula. Note that c_0 cannot be evaluated directly by substituting $n = 0$, since we have the indeterminate term $\frac{0}{0}$. Using L'Hospital's rule, described in Appendix G, leads to

$$c_0 = \lim_{n\to 0} 5\frac{\sin(0.5n\pi)}{(0.5n\pi)} = \lim_{n\to 0} 5\frac{\frac{d(\sin(0.5n\pi))}{dn}}{\frac{d(0.5n\pi)}{dn}}$$

$$= \lim_{n\to 0} 5\frac{0.5\pi\cos(0.5n\pi)}{0.5\pi} = 5$$

Finally, the Fourier expansion in terms of the complex exponential form is shown as follows:

$$x(t) = \cdots + \frac{10}{\pi}e^{-j2\pi t} + 5 + \frac{10}{\pi}e^{j2\pi t} - \frac{10}{3\pi}e^{j6\pi t} + \frac{2}{\pi}e^{j10\pi t} - \frac{10}{7\pi}e^{j14\pi t} + \cdots$$

B.1.4 Spectral Plots

As previously discussed, the magnitude-phase form can provide information to create a one-sided spectral plot. The amplitude spectrum is obtained by plotting A_n versus the harmonic frequency $n\omega_0$, and the phase spectrum is obtained by plotting ϕ_n versus $n\omega_0$, both for $n \geq 0$. Similarly, if the complex exponential form is used, the two-sided amplitude and phase spectral plots of $|c_n|$ and ϕ_n versus $n\omega_0$ for $-\infty < n < \infty$ can be achieved, respectively. We illustrate this by the following example.

EXAMPLE B.2

Based on the solution to Example B.1, plot the one-sided amplitude spectrum and two-sided amplitude spectrum, respectively.

Solution:

Based on the solution for A_n, the one-sided amplitude spectrum is shown in Figure B.2.

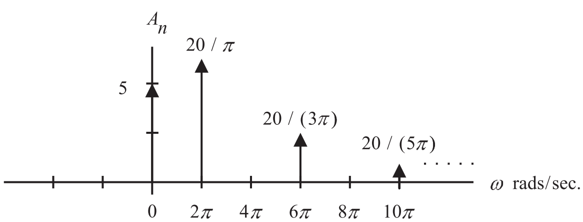


FIGURE B.2

One-sided spectrum of the square waveform in Example B.2.

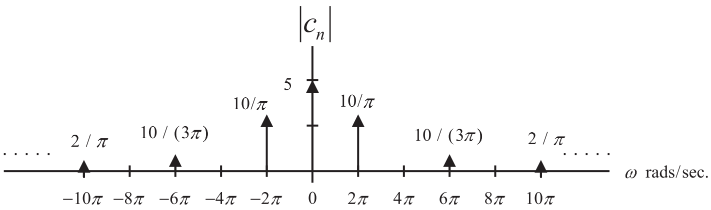


FIGURE B.3

Two-sided spectrum of the square waveform in Example B.2.

According to the solution of the complex exponential form, the two-sided amplitude spectrum is demonstrated in Figure B.3.

A general pulse train $x(t)$ with a period T_0 seconds and a pulse width τ seconds is shown in Figure B.4.

The Fourier series expansions for sine-cosine and complex exponential forms can be derived similarly and are given as follows:

Sine-cosine form:

$$x(t) = \frac{\tau A}{T_0} + \frac{2\tau A}{T_0}\left(\frac{\sin(\omega_0\tau/2)}{(\omega_0\tau/2)}\cos(\omega_0 t) + \frac{\sin(2\omega_0\tau/2)}{(2\omega_0\tau/2)}\cos(2\omega_0 t) + \frac{\sin(3\omega_0\tau/2)}{(3\omega_0\tau/2)}\cos(3\omega_0 t) + \cdots\right) \tag{B.15}$$

Complex exponential form:

$$x(t) = \cdots + \frac{\tau A}{T_0}\frac{\sin(\omega_0\tau/2)}{(\omega_0\tau/2)}e^{-j\omega_0 t} + \frac{\tau A}{T_0} + \frac{\tau A}{T_0}\frac{\sin(\omega_0\tau/2)}{(\omega_0\tau/2)}e^{j\omega_0 t} + \frac{\tau A}{T_0}\frac{\sin(2\omega_0\tau/2)}{(2\omega_0\tau/2)}e^{j2\omega_0 t} + \cdots \tag{B.16}$$

where $\omega_0 = 2\pi f_0 = 2\pi/T_0$ is the fundamental angle frequency of the periodic waveform. The reader can derive the one-sided amplitude spectrum A_n and the two-sided amplitude spectrum $|c_n|$. The expressions for the one-sided amplitude and two-sided amplitude spectra are given by the following:

$$A_0 = \frac{\tau}{T_0}A \tag{B.17}$$

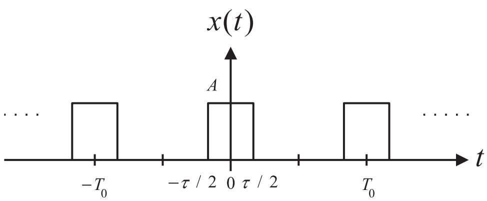


FIGURE B.4

Rectangular waveform (pulse train).

$$A_n = \frac{2\tau}{T_0} A \left| \frac{\sin(n\omega_0 \tau/2)}{(n\omega_0 \tau/2)} \right|, \quad for \quad n = 1, 2, 3 \ldots \tag{B.18}$$

$$|c_n| = \frac{\tau}{T_0} A \left| \frac{\sin(n\omega_0 \tau/2)}{(n\omega_0 \tau/2)} \right|, \quad -\infty < n < \infty \tag{B.19}$$

EXAMPLE B.3

In Figure B.4, if $T_0 = 1$ ms, $\tau = 0.2$ ms, and $A = 10$, use Equations (B.17) to (B.19) to derive the amplitude one-sided spectrum and two-sided spectrum for each of the first four harmonic frequency components.

Solution:

The fundamental frequency is

$$\omega_0 = 2\pi f_0 = 2\pi \times (1/0.001) = 2{,}000\pi \text{ rad/sec}$$

Using Equations (B.17) and (B.18) yields the one-sided spectrum as

$$A_0 = \frac{\tau}{T_0} A = \frac{0.0002}{0.001} \times 10 = 2, \quad \text{for} \quad n = 0, \; n\omega_0 = 0$$

For $n = 1$, $n\omega_0 = 2{,}000\pi$ rad/sec:

$$A_1 = \frac{2 \times 0.0002}{0.001} \times 10 \times \left| \frac{\sin(1 \times 2{,}000\pi \times 0.0002/2)}{(1 \times 2{,}000\pi \times 0.0002/2)} \right| = 4 \frac{\sin(0.2\pi)}{(0.2\pi)} = 3.7420$$

For $n = 2$, $n\omega_0 = 4{,}000\pi$ rad/sec:

$$A_2 = \frac{2 \times 0.0002}{0.001} \times 10 \times \left| \frac{\sin(2 \times 2{,}000\pi \times 0.0002/2)}{(2 \times 2{,}000\pi \times 0.0002/2)} \right| = 4 \frac{\sin(0.4\pi)}{(0.4\pi)} = 3.0273$$

For $n = 3$, $n\omega_0 = 6{,}000\pi$ rad/sec:

$$A_3 = \frac{2 \times 0.0002}{0.001} \times 10 \times \left| \frac{\sin(3 \times 2{,}000\pi \times 0.0002/2)}{(3 \times 2{,}000\pi \times 0.0002/2)} \right| = 4 \frac{\sin(0.6\pi)}{(0.6\pi)} = 2.0182$$

For $n = 4$, $n\omega_0 = 8{,}000\pi$ rad/sec:

$$A_4 = \frac{2 \times 0.0002}{0.001} \times 10 \times \left| \frac{\sin(4 \times 2{,}000\pi \times 0.0002/2)}{(4 \times 2{,}000\pi \times 0.0002/2)} \right| = 4 \frac{\sin(0.8\pi)}{(0.8\pi)} = 0.9355$$

The one-sided amplitude spectrum is plotted in Figure B.5.

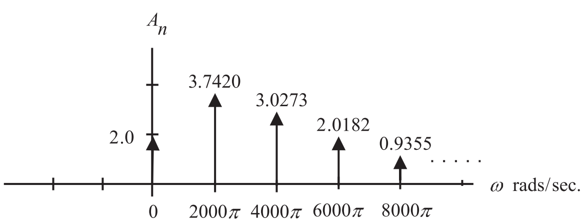


FIGURE B.5

One-sided spectrum in Example B.3.

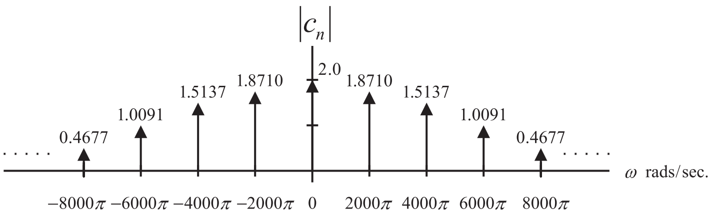


FIGURE B.6

Two-sided spectrum in Example B.3.

Similarly, applying Equation (B.19) leads to

$$|c_0| = \frac{0.0002}{0.001} \times 10 \times \left| \lim_{n \to 0} \frac{\sin(n \times 2{,}000\pi \times 0.0002/2)}{(n \times 2{,}000\pi \times 0.0002/2)} \right| = 2 \times |1| = 2$$

Note: We use the fact that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1.0$ (see L'Hospital's rule in Appendix G).

$$|c_1| = |c_{-1}| = \frac{0.0002}{0.001} \times 10 \times \left| \frac{\sin(1 \times 2{,}000\pi \times 0.0002/2)}{(1 \times 2{,}000\pi \times 0.0002/2)} \right| = 2 \times \left| \frac{\sin(0.2\pi)}{0.2\pi} \right| = 1.8710$$

$$|c_2| = |c_{-2}| = \frac{0.0002}{0.001} \times 10 \times \left| \frac{\sin(2 \times 2{,}000\pi \times 0.0002/2)}{(2 \times 2{,}000\pi \times 0.0002/2)} \right| = 2 \times \left| \frac{\sin(0.4\pi)}{0.4\pi} \right| = 1.5137$$

$$|c_3| = |c_{-3}| = \frac{0.0002}{0.001} \times 10 \times \left| \frac{\sin(3 \times 2{,}000\pi \times 0.0002/2)}{(3 \times 2{,}000\pi \times 0.0002/2)} \right| = 2 \times \left| \frac{\sin(0.6\pi)}{0.6\pi} \right| = 1.0091$$

$$|c_4| = |c_{-4}| = \frac{0.0002}{0.001} \times 10 \times \left| \frac{\sin(4 \times 2{,}000\pi \times 0.0002/2)}{(4 \times 2{,}000\pi \times 0.0002/2)} \right| = 2 \times \left| \frac{\sin(0.8\pi)}{0.8\pi} \right| = 0.4677$$

Figure B.6 shows the two-sided amplitude spectral plot.

The following example illustrates the use of table information to determine the Fourier series expansion of the periodic waveform. Table B.1 consists of the Fourier series expansions for common periodic signals in the sine-cosine form while Table B.2 shows the expansions in the complex exponential form.

Table B.1 Fourier Series Expansions for Some Common Waveform Signals in the Sine-Cosine Form

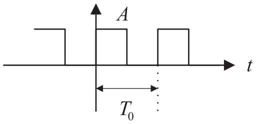
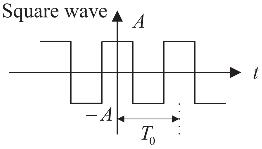
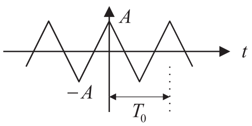
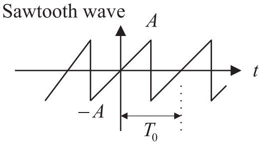
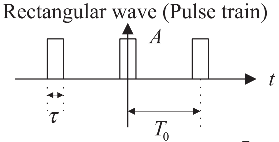
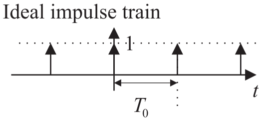
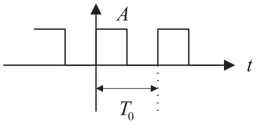
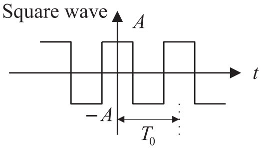
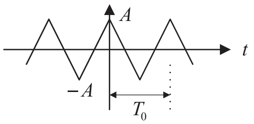
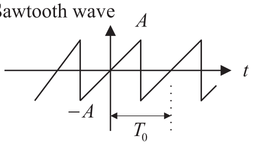
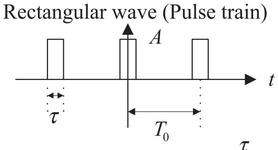
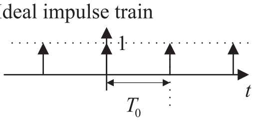
Time Domain Signal $x(t)$	Fourier Series Expansion
Positive square wave 	$x(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \frac{1}{7}\sin 7\omega_0 t + \cdots\right)$
Square wave 	$x(t) = \frac{4A}{\pi}\left(\cos\omega_0 t - \frac{1}{3}\cos 3\omega_0 t + \frac{1}{5}\cos 5\omega_0 t - \frac{1}{7}\cos 7\omega_0 t + \cdots\right)$
Triangular wave 	$x(t) = \frac{8A}{\pi^2}\left(\cos\omega_0 t + \frac{1}{9}\cos 3\omega_0 t + \frac{1}{25}\cos 5\omega_0 t + \frac{1}{49}\cos 7\omega_0 t + \cdots\right)$
Sawtooth wave 	$x(t) = \frac{2A}{\pi}\left(\sin\omega_0 t - \frac{1}{2}\sin 2\omega_0 t + \frac{1}{3}\sin 3\omega_0 t - \frac{1}{4}\sin 4\omega_0 t + \cdots\right)$
Rectangular wave (Pulse train)  Duty cycle $= d = \frac{\tau}{T_0}$	$x(t) = Ad + 2Ad\left(\frac{\sin\pi d}{\pi d}\right)\cos\omega_0 t + 2Ad\left(\frac{\sin 2\pi d}{2\pi d}\right)\cos 2\omega_0 t + 2Ad\left(\frac{\sin 3\pi d}{3\pi d}\right)\cos 3\omega_0 t + \cdots$
Ideal impulse train 	$x(t) = \frac{1}{T_0} + \frac{2}{T_0}(\cos\omega_0 t + \cos 2\omega_0 t + \cos 3\omega_0 t + \cos 4\omega_0 t + \cdots)$

TABLE B.2 Fourier Series Expansions for Some Common Waveform Signals in the Complex Exponential Form

Time Domain Signal $x(t)$	Fourier Series Expansion
Positive square wave 	$x(t) = \left(\cdots - \frac{A}{j3\pi}e^{-j3\omega_0 t} - \frac{A}{j\pi}e^{-j\omega_0 t} + \frac{A}{2} + \frac{A}{j\pi}e^{j\omega_0 t} + \frac{A}{j3\pi}e^{j3\omega_0 t} + \frac{A}{j5\pi}e^{j5\omega_0 t} + \cdots\right)$
Square wave 	$x(t) = \frac{2A}{\pi}\left(\cdots + \frac{1}{5}e^{-j5\omega_0 t} - \frac{1}{3}e^{-j3\omega_0 t} + e^{-j\omega_0 t} + e^{j\omega_0 t} - \frac{1}{3}e^{j3\omega_0 t} + \frac{1}{5}e^{j5\omega_0 t} - \cdots\right)$
Triangular wave 	$x(t) = \frac{4A}{\pi^2}\left(\cdots + \frac{1}{25}e^{-j5\omega_0 t} + \frac{1}{9}e^{-j3\omega_0 t} + e^{-j\omega_0 t} + e^{j\omega_0 t} + \frac{1}{9}e^{j3\omega_0 t} + \frac{1}{3}e^{j5\omega_0 t} + \cdots\right)$
Sawtooth wave 	$x(t) = \frac{A}{j\pi}\left(\cdots - \frac{1}{3}e^{-j3\omega_0 t} + \frac{1}{2}e^{-j2\omega_0 t} - e^{-j\omega_0 t} + e^{j\omega_0 t} - \frac{1}{2}e^{j2\omega_0 t} + \frac{1}{3}e^{j3\omega_0 t} + \cdots\right)$
Rectangular wave (Pulse train)  Duty cycle $= d = \frac{\tau}{T_0}$	$x(t) = \cdots + Ad\left(\frac{\sin \pi d}{\pi d}\right)e^{-j\omega_0 t} + Ad\left(\frac{\sin \pi d}{\pi d}\right)e^{j\omega_0 t} + Ad\left(\frac{\sin 2\pi d}{2\pi d}\right)e^{j2\omega_0 t} + Ad\left(\frac{\sin 3\pi d}{3\pi d}\right)e^{j3\omega_0 t} + \cdots$
Ideal impulse train 	$x(t) = \frac{1}{T_0}(\cdots + e^{-j3\omega_0 t} + e^{-j2\omega_0 t} + e^{-j\omega_0 t} + 1 + e^{j\omega_0 t} + e^{j2\omega_0 t} + e^{j3\omega_0 t} + \cdots)$

EXAMPLE B.4

In the sawtooth waveform shown in Table B.1 and reprinted in Figure B.7, if $T_0 = 1$ ms and $A = 10$, use the formula in the table to determine the Fourier series expansion in a magnitude-phase form, and determine the frequency f_3 and amplitude value of A_3 for the third harmonic. Write the Fourier series expansion in a complex exponential form also, and determine $|c_3|$ and $|c_{-3}|$ for the third harmonic.

Solution:

a. Based on the information in Table B.1, we have

$$x(t) = \frac{2A}{\pi}\left(\sin \omega_0 t - \frac{1}{2}\sin 2\omega_0 t + \frac{1}{3}\sin 3\omega_0 t - \frac{1}{4}\sin 4\omega_0 t + \cdots\right)$$

Since $T_0 = 1$ ms, the fundamental frequency is

$$f_0 = 1/T_0 = 1{,}000 \text{ Hz}, \quad \text{and} \quad \omega_0 = 2\pi f_0 = 2{,}000\pi \text{ rad/sec}$$

Then, the expansion is determined as

$$x(t) = \frac{2\times 10}{\pi}\left(\sin 2{,}000\pi t - \frac{1}{2}\sin 4{,}000\pi t + \frac{1}{3}\sin 6{,}000\pi t - \frac{1}{4}\sin 8{,}000\pi t + \cdots\right)$$

Using the trigonometric identities

$$\sin x = \cos(x - 90^\circ) \text{ and } -\sin x = \cos(x + 90^\circ)$$

and simple algebra, we finally obtain

$$x(t) = \frac{20}{\pi}\cos(2{,}000\pi t - 90^\circ) + \frac{10}{\pi}\cos(4{,}000\pi t + 90^\circ)$$
$$+\frac{20}{3\pi}\cos(6{,}000\pi t - 90^\circ) + \frac{5}{\pi}\cos(8{,}000\pi t + 90^\circ) + \cdots$$

From the magnitude-phase form, we then determine f_3 and A_3 as follows:

$$f_3 = 3\times f_0 = 3{,}000 \text{ Hz}, \quad \text{and} \quad A_3 = \frac{20}{3\pi} = 2.1221$$

b. From Table B.2, the complex exponential form is

$$x(t) = \frac{10}{j\pi}\left(\cdots - \frac{1}{3}e^{-j6{,}000\pi t} + \frac{1}{2}e^{-j4{,}000\pi t} - e^{-j2{,}000\pi t} + e^{j2{,}000\pi t} - \frac{1}{2}e^{j4{,}000\pi t} + \frac{1}{3}e^{j6{,}000\pi t} + \cdots\right)$$

From the expression, we have

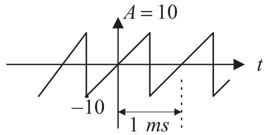


FIGURE B.7

Sawtooth waveform for Example B.4.

$$|c_3| = \left|\frac{10}{j\pi} \times \frac{1}{3}\right| = \left|\frac{1.061}{j}\right| = 1.061 \text{ and}$$

$$|c_{-3}| = \left|-\frac{10}{j\pi} \times \frac{1}{3}\right| = \left|-\frac{1.061}{j}\right| = 1.061$$

B.1.5 FOURIER TRANSFORM

The Fourier transform is a mathematical function that provides frequency spectral analysis for a nonperiodic signal. The Fourier transform pair is defined as

Fourier transform:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{B.20}$$

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \tag{B.21}$$

where $x(t)$ is a nonperiodic signal and $X(\omega)$ is a two-sided continuous spectrum versus the continuous frequency variable ω, where $-\infty < \omega < \infty$. Again, the spectrum is a complex function that can be further written as

$$X(\omega) = |X(\omega)|\angle\phi(\omega) \tag{B.22}$$

where $|X(\omega)|$ is the continuous amplitude spectrum, while $\angle\phi(\omega)$ designates the continuous phase spectrum.

EXAMPLE B.5

Let $x(t)$ be a single rectangular pulse, shown in Figure B.8, where the pulse width is $\tau = 0.5$ second. Find its Fourier transform and sketch the amplitude spectrum.

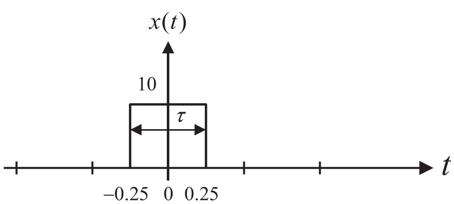


FIGURE B.8

Rectangular pulse in Example B.5.

Solution:
Applying Equation (B.21) and using Euler's formula, we have

$$X(\omega) = \int_{-\infty}^{\infty} x\left(t\right) e^{-j\omega t} dt = \int_{-0.25}^{0.25} 10 e^{-j\omega} dt$$

$$= 10 \frac{e^{-j\omega t}}{-j\omega}\bigg|_{-0.25}^{0.25} = 10 \times \frac{\left(e^{-j0.25\omega} - e^{j0.25\omega}\right)}{-j\omega}$$

$$= 10 \times \frac{\cos(0.25\omega) - j\sin(0.25\omega) - [\cos(0.25\omega) + j\sin(0.25\omega)]}{-j\omega}$$

$$= 5 \frac{\sin(0.25\omega)}{0.25\omega}$$

where the amplitude spectrum is expressed as

$$|X(\omega)| = 5 \times \left|\frac{\sin(0.25\omega)}{0.25\omega}\right|$$

Using $\omega = 2\pi f$, we can express the spectrum in terms of Hz as

$$|X(f)| = 5 \times \left|\frac{\sin(0.5\pi f)}{0.5\pi f}\right|$$

The amplitude spectrum is shown in Figure B.9. Note that the first null point is at $\omega = 2\pi/0.5 = 4\pi$ rad/sec, and the spectrum is symmetric.

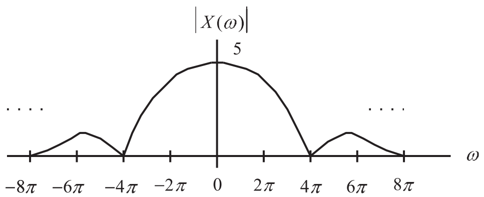


FIGURE B.9

Amplitude spectrum for Example B.5.

EXAMPLE B.6

Let $x(t)$ be an exponential function given by

$$x(t) = 10e^{-2t}u(t) = \begin{cases} 10e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Find its Fourier transform.

Solution:

According to the definition of the Fourier transform,

$$X(\omega) = \int_0^{\infty} 10e^{-2t}u(t)e^{-j\omega t}dt = \int_0^{\infty} 10e^{-(2+j\omega)t}dt$$

$$= \left.\frac{10e^{-(2+j\omega)t}}{-(2+j\omega)}\right|_0^{\infty} = \frac{10}{2+j\omega}$$

$$X(\omega) = \frac{10}{\sqrt{2^2+\omega^2}}\angle -\tan^{-1}\left(\frac{\omega}{2}\right)$$

Using $\omega = 2\pi f$, we get

$$X(f) = \frac{10}{2+j2\pi f} = \frac{10}{\sqrt{2^2+(2\pi f)^2}}\angle -\tan^{-1}(\pi f)$$

The Fourier transforms for some common signals are listed in Table B.3. Some useful properties of the Fourier transform are summarized in Table B.4.

EXAMPLE B.7

Find the Fourier transforms of the following functions:

a. $x(t) = \delta(t)$, where $\delta(t)$ is an impulse function defined by

$$\delta(t) = \begin{cases} \neq 0 & t = 0 \\ 0 & elsewhere \end{cases}$$

with a property given as

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = f(\tau)$$

b. $x(t) = \delta(t-\tau)$

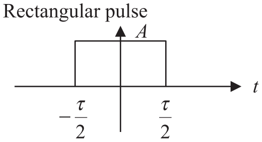
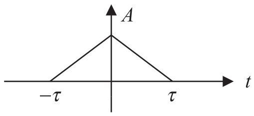
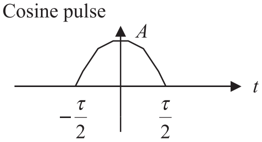
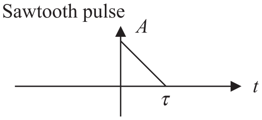
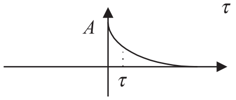
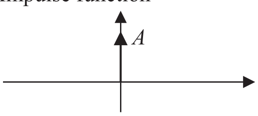
Solution:

a. We first use the Fourier transform definition and then apply the delta function property,

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = \left.e^{-j\omega t}\right|_{t=0} = 1$$

b. Similar to (a), we obtain

Table B.3 Fourier Transforms for Some Common Signals

Time Domain Signal $x(t)$	**Fourier Spectrum $X(f)$**
Rectangular pulse 	$X(f) = A\tau \frac{\sin \pi f \tau}{\pi f \tau}$
Triangular pulse 	$X(f) = A\tau \left(\frac{\sin \pi f \tau}{\pi f \tau}\right)^2$
Cosine pulse 	$X(f) = \frac{2A\tau}{\pi}\left(\frac{\cos \pi f \tau}{1 - 4f^2\tau^2}\right)$
Sawtooth pulse 	$X(f) = \frac{jA}{2\pi f}\left(\frac{\sin \pi f \tau}{\pi f \tau} e^{-j\pi f \tau} - 1\right)$
Exponential function $\alpha = \frac{1}{\tau}$ 	$X(f) = \frac{A}{\alpha + j2\pi f}$
Impulse function 	$X(f) = A$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t-\tau)e^{-j\omega t}dt = e^{-j\omega t}\Big|_{t=\tau} = e^{-j\omega\tau}$$

Example B.8 shows how to use the table information to determine the Fourier transform of a nonperiodic signal.

Table B.4 Properties of the Fourier Transform

Line	Time Function	Fourier Transform
1	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(f) + \beta X_2(f)$
2	$\frac{dx(t)}{dt}$	$j2\pi f X(f)$
3	$\int_{-\infty}^{t} x(t)dt$	$\frac{X(f)}{j2\pi f}$
4	$x(t-\tau)$	$e^{-j2\pi f\tau}X(f)$
5	$e^{j2\pi f_0 t}x(t)$	$X(f-f_0)$
6	$x(at)$	$\frac{1}{a}X\left(\frac{f}{a}\right)$

EXAMPLE B.8

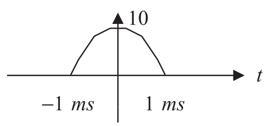


FIGURE B.10

Cosine pulse in Example B.8.

Use Table B.3 to determine the Fourier transform for the cosine pulse in Figure B.10.

Solution:
According to the graph, we can identify

$$\frac{\tau}{2} = 1\ ms, \text{ and } A = 1$$

τ is given by

$$\tau = 2 \times 1\ ms = 0.002 \text{ second}$$

Applying the formula from Table B.3 gives

$$X(f) = \frac{2 \times 10 \times 0.002}{\pi}\left(\frac{\cos \pi f 0.002}{1 - 4f^2 0.002^2}\right) = \frac{0.04}{\pi}\left(\frac{\cos 0.002\pi f}{1 - 4 \times 0.002^2 f^2}\right)$$

B.2 LAPLACE TRANSFORM

In this section, we will review Laplace transform and its applications.

B.2.1 Laplace Transform and Its Table

The Laplace transform plays an important role in the analysis of continuous signals and systems. We define the Laplace transform pairs as

$$X(s) = L\{x(t)\} = \int_{0}^{\infty} x(t)e^{-st}dt \tag{B.23}$$

$$x(t) = L^{-1}\{X(s)\} = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} X(s)e^{st}ds \tag{B.24}$$

Notice that the symbol $L\{\}$ denotes the forward Laplace operation, while the symbol $L^{-1}\{\}$ indicates the inverse Laplace operation. Some common Laplace transform pairs are listed in Table B.5.

TABLE B.5 Laplace Transform Table

Line	Time Function $x(t)$	Laplace Transform $X(s) = L(x(t))$
1	$\delta(t)$	1
2	1 or $u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$e^{-at}u(t)$	$\frac{1}{s+a}$
5	$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$
6	$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$
7	$\sin(\omega t+\theta)u(t)$	$\frac{s\sin(\theta)+\omega\cos(\theta)}{s^2+\omega^2}$
8	$e^{-at}\sin(\omega t)u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
9	$e^{-at}\cos(\omega t)u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

(*continued*)

TABLE B.5 Laplace Transform Table *(continued)*

Line	Time Function $x(t)$	Laplace Transform $X(s) = L(x(t))$
10	$\left(A\cos(\omega t) + \frac{B - aA}{\omega}\sin(\omega t)\right)e^{-at}u(t)$	$\frac{As + B}{(s+a)^2 + \omega^2}$
11a	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
11b	$\frac{1}{(n-1)!}t^{n-1}u(t)$	$\frac{1}{s^n}$
12a	$e^{-at}t^n u(t)$	$\frac{n!}{(s+a)^{n+1}}$
12b	$\frac{1}{(n-1)!}e^{-at}t^{n-1}u(t)$	$\frac{1}{(s+a)^n}$
13	$(2\mathrm{Real}(A)\cos(\omega t) - 2\mathrm{Imag}(A)\sin(\omega t))e^{-\alpha t}u(t)$	$\frac{A}{s+\alpha-j\omega} + \frac{A^*}{s+\alpha+j\omega}$
14	$\frac{dx(t)}{dt}$	$sX(s) - x(0^-)$
15	$\int_0^t x(t)dt$	$\frac{X(s)}{s}$
16	$x(t-a)u(t-a)$	$e^{-as}X(s)$
17	$e^{-at}x(t)u(t)$	$X(s+a)$

In Example B.9, we examine the Laplace transform in light of its definition.

EXAMPLE B.9

Derive the Laplace transform of the unit step function.

Solution:

By the definition in Equation (B.23),

$$X(s) = \int_0^\infty u(t)e^{-st}dt$$

$$= \int_0^\infty e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{e^{-\infty}}{-s} - \frac{e^0}{-s} = \frac{1}{s}$$

The answer is consistent with the result listed in Table B.5. Now we use the results in Table B.5 to find the Laplace transform of a function.

EXAMPLE B.10

Perform the Laplace transform for each of the following functions.

a. $x(t) = 5\sin(2t)u(t)$
b. $x(t) = 5e^{-3t}\cos(2t)u(t)$

Solution:

a. Using line 5 in Table B.5 and noting that $\omega = 2$, the Laplace transform immediately follows:

$$X(s) = 5L\{2\sin(2t)u(t)\}$$
$$= \frac{5 \times 2}{s^2 + 2^2} = \frac{10}{s^2 + 4}$$

b. Applying line 9 in Table B.5 with $\omega = 2$ and $a = 3$ yields

$$X(s) = 5L\{e^{-3t}\cos(2t)u(t)\}$$
$$= \frac{5(s+3)}{(s+3)^2 + 2^2} = \frac{5(s+3)}{(s+3)^2 + 4}$$

B.2.2 Solving Differential Equations Using the Laplace Transform

One of the important applications of the Laplace transform is to solve differential equations. Using the differential property in Table B.5, we can transform a differential equation from the time domain to the Laplace domain. This will change the differential equation into an algebraic equation, and we then solve the algebraic equation. Finally, the inverse Laplace operation is processed to yield the time domain solution.

EXAMPLE B.11

Solve the following differential equation using the Laplace transform:

$$\frac{dy(t)}{dt} + 10y(t) = x(t) \text{ with an initial condition } y(0) = 0,$$

where the input $x(t) = 5u(t)$.

Solution:
Applying the Laplace transform on both sides of the differential equation and using the differential property (line 14 in Table B.5), we get

$$sY(s) - y(0) + 10Y(s) = X(s)$$

Note that

$$X(s) = L\{5u(t)\} = \frac{5}{s}$$

Substituting the initial condition yields

$$Y(s) = \frac{5}{s(s+10)}$$

Then we use a partial fraction expansion by writing

$$Y(s) = \frac{A}{s} + \frac{B}{s+10}$$

where

$$A = sY(s)|_{s=0} = \left.\frac{5}{s+10}\right|_{s=0} = 0.5$$

and

$$B = (s+10)Y(s)|_{s=-10} = \left.\frac{5}{s}\right|_{s=-10} = -0.5$$

Hence,

$$Y(s) = \frac{0.5}{s} - \frac{0.5}{s+10}$$

$$y(t) = L^{-1}\left\{\frac{0.5}{s}\right\} - L^{-1}\left\{\frac{0.5}{s+10}\right\}$$

Finally, applying the inverse of the Laplace transform leads to using the results listed in Table B.5, and we obtain the time domain solution as

$$y(t) = 0.5u(t) - 0.5e^{-10t}u(t)$$

B.2.3 Transfer Function

A linear analog system can be described using the Laplace transfer function. The transfer function relating the input and output of the linear system is depicted as

$$Y(s) = H(s)X(s) \tag{B.25}$$

where $X(s)$ and $Y(s)$ are the system input and response (output), respectively, in the Laplace domain, and the transfer function is defined as a ratio of the Laplace response of the system to the Laplace input given by

$$H(s) = \frac{Y(s)}{X(s)} \tag{B.26}$$

The transfer function will allow us to study the system behavior. Considering an impulse function as the input to a linear system, that is, $x(t) = \delta(t)$, whose Laplace transform is $X(s) = 1$, we then find the system output due to the impulse function to be

$$Y(s) = H(s)X(s) = H(s) \tag{B.27}$$

Therefore, the response in the time domain $y(t)$ is called the impulse response of the system and can be expressed as

$$h(t) = L^{-1}\{H(s)\} \tag{B.28}$$

The analog impulse response can be sampled and transformed to obtain a digital filter transfer function. This topic is covered in Chapter 8.

EXAMPLE B.12

Consider a linear system described by the differential equation shown in Example B.11. $x(t)$ and $y(t)$ designate the system input and system output, respectively. Derive the transfer function and the impulse response of the system.

Solution:

Taking the Laplace transform on both sides of the differential equation yields

$$L\left\{\frac{dy(t)}{dt}\right\} + L\{10y(t)\} = L\{x(t)\}$$

Applying the differential property and substituting the initial condition, we have

$$Y(s)(s+10) = X(s)$$

Thus, the transfer function is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+10}$$

The impulse response can be found by taking the inverse Laplace transform as

$$h(t) = L^{-1}\left\{\frac{1}{s+10}\right\} = e^{-10t}u(t)$$

B.3 POLES, ZEROS, STABILITY, CONVOLUTION, AND SINUSOIDAL STEADY-STATE RESPONSE

This section is a review of analog system analysis.

B.3.1 Poles, Zeros, and Stability

To study system behavior, the transfer function is written in a general form given by

$$H(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0} \tag{B.29}$$

It is a ratio of the numerator polynomial of degree m to the denominator polynomial of degree n. The numerator polynomial is expressed as

$$N(s) = b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0 \tag{B.30}$$

while the denominator polynomial is given by

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0 \tag{B.31}$$

Again, the roots of $N(s)$ are called zeros, while the roots of $D(s)$ are called poles of the transfer function $H(s)$. Notice that zeros and poles could be real numbers or complex numbers.

Given a system transfer function, the poles and zeros can be found. Further, a pole-zero plot could be created on the s-plane. With the pole-zero plot, the stability of the system is determined by the following rules:

1. The linear system is stable if the rightmost pole(s) is/are on the left-hand half plane (LHHP) on the s-plane.
2. The linear system is marginally stable if the rightmost pole(s) is/are simple-order (first-order) on the $j\omega$ axis, including the origin on the s-plane.
3. The linear system is unstable if the rightmost pole(s) is/are on the right-hand half plane (RHHP) of the s-plane or if the rightmost pole(s) is/are multiple-order on the $j\omega$ axis on the s-plane.
4. Zeros do not affect system stability.

EXAMPLE B.13

Determine whether each of the following transfer functions is stable, marginally stable, and unstable:

a. $H(s) = \dfrac{s+1}{(s+1.5)(s^2+2s+5)}$

b. $H(s) = \dfrac{(s+1)}{(s+2)(s^2+4)}$

c. $H(s) = \dfrac{s+1}{(s-1)(s^2+2s+5)}$

Solution:

a. A zero is found at $s = -1$. The poles are calculated as $s = -1.5$, $s = -1 + j2$, $s = -1 - j2$. The pole-zero plot is shown in Figure B.11A. Since all the poles are located on the LHHP, the system is stable.

b. A zero is found at $s = -1$. The poles are calculated as $s = -2$, $s = j2$, $s = -j2$. The pole-zero plot is shown in Figure B.11B. Since the first-order poles $s = \pm j2$ are located on the $j\omega$ axis, the system is marginally stable.

c. A zero is found at $s = -1$. The poles are calculated as $s = 1$, $s = -1 + j2$, $s = -1 - j2$. The pole-zero plot is shown in Figure B.11C. Since there is a pole $s = 1$ located on the RHHP, the system is unstable.

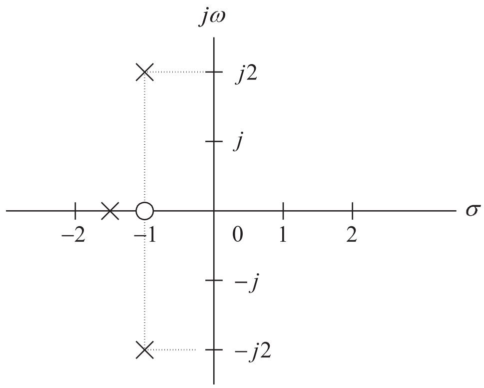


FIGURE B.11A

Pole-zero plot for (a).

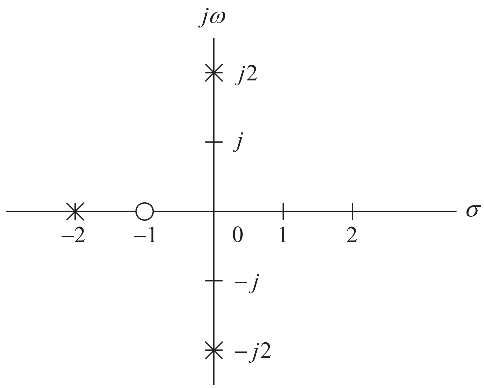


FIGURE B.11B

Pole-zero plot for (b).

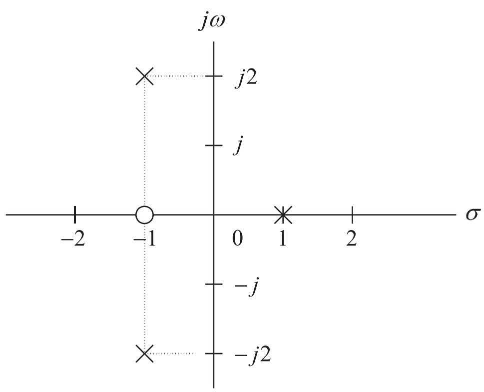


FIGURE B.11C

Pole-zero plot for (c).

B.3.2 Convolution

As we discussed before, the input and output relationship of a linear system in the Laplace domain is

$$Y(s) = H(s)X(s) \tag{B.32}$$

It is apparent that in the Laplace domain, the system output is the product of the Laplace input and transfer function. But in the time domain, the system output is given as

$$y(t) = h(t) * x(t) \tag{B.33}$$

where $*$ denotes linear convolution of the system impulse response $h(t)$ and the system input $x(t)$. The linear convolution is further expressed as

$$y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau \tag{B.34}$$

EXAMPLE B.14

As you have seen in Examples B.11 and B.12, for a linear system, the impulse response and the input are given, respectively, by

$$h(t) = e^{-10t}u(t) \quad \text{and} \quad x(t) = 5u(t)$$

Determine the system response $y(t)$ using the convolution method.

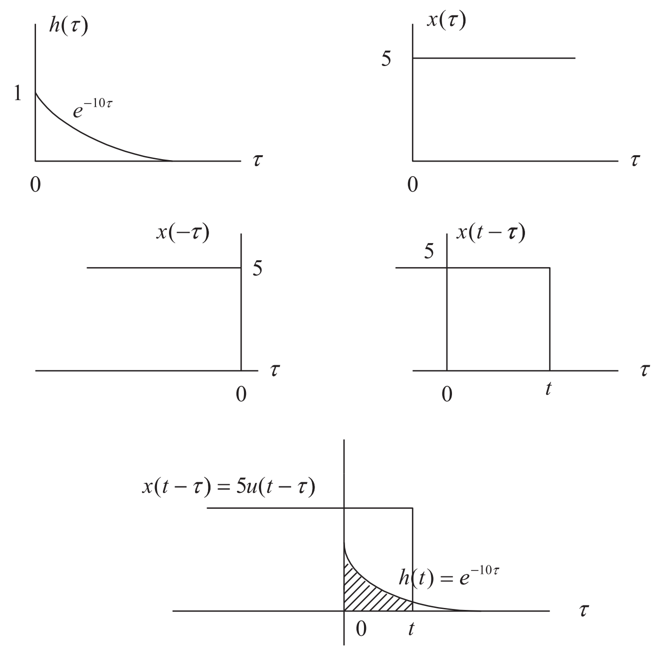


FIGURE B.12

Convolution illustration for Example B.14.

Solution:

Two signals $h(\tau)$ and $x(\tau)$ that are involved in the convolution integration are displayed in Figure B.12. To evaluate the convolution, the time-reversed signal $x(-\tau)$ and the shifted signal $x(t-\tau)$ are also plotted for reference. Figure B.12 shows an overlap of $h(\tau)$ and $x(t-\tau)$. According to the overlapped (shaded) area, the lower limit and the upper limit of the convolution integral are determined to be 0 and t, respectively. Hence,

$$y(t) = \int_0^t e^{-10\tau} \cdot 5 d\tau = \frac{5}{-10} e^{-10\tau}\Big|_0^t$$
$$= -0.5e^{-10t} - \left(-0.5e^{-10\times 0}\right)$$

Finally, the system response is found to be

$$y(t) = 0.5u(t) - 0.5e^{-10t}u(t)$$

The solution is the same as that obtained using the Laplace transform method described in Example B.11.

B.3.3 Sinusoidal Steady-State Response

For linear analog systems, if the input to a system is a sinusoid of radian frequency ω, the steady-state response of the system will also be a sinusoid of the same frequency. Therefore, the transfer function, which provides the relationship between a sinusoidal input and a sinusoidal output, is called the steady-state transfer function. The steady-state transfer function is obtained from the Laplace transfer function by substituting $s = j\omega$, as shown in the following:

$$H(j\omega) = H(s)|_{s=j\omega} \tag{B.35}$$

Thus we have a system relationship in a sinusoidal steady state as

$$Y(j\omega) = H(j\omega)X(j\omega) \tag{B.36}$$

Since $H(j\omega)$ is a complex function, we may write it in the phasor form:

$$H(j\omega) = A(\omega)\angle\beta(\omega) \tag{B.37}$$

where the quantity $A(\omega)$ is the amplitude response of the system defined as

$$A(\omega) = |H(j\omega)| \tag{B.38}$$

and the phase angle $\beta(\omega)$ is the phase response of the system. The following example is presented to illustrate the application.

EXAMPLE B.15

Consider a linear system described by the differential equation shown in Example B.12, where $x(t)$ and $y(t)$ designate the system input and system output, respectively. The transfer function has been derived as

$$H(s) = \frac{10}{s+10}$$

a. Derive the steady-state transfer function.
b. Derive the amplitude response and phase response.
c. If the input is given as a sinusoid, that is, $x(t) = 5\sin(10t + 30^{\circ})u(t)$, find the steady-state response $y_{ss}(t)$.

Solution:

a. By substituting $s = j\omega$ into the transfer function in terms of a suitable form, we get the steady-state transfer function as

$$H(j\omega) = \frac{1}{\frac{s}{10}+1} = \frac{1}{\frac{j\omega}{10}+1}$$

b. The amplitude response and phase response are found to be

$$A(\omega) = \frac{1}{\sqrt{\left(\frac{\omega}{10}\right)^2 + 1}}$$

$$\beta(\omega) = \angle -\tan^{-1}\left(\frac{\omega}{10}\right)$$

c. When $\omega = 10$ rad/sec, the input sinusoid can be written in terms of the phasor form as

$$X(j10) = 5\angle 30^{\circ}$$

For the amplitude and phase of the steady-state transfer function at $\omega = 10$, we have

$$A(10) = \frac{1}{\sqrt{\left(\frac{10}{10}\right)^2 + 1}} = 0.7071$$

$$\beta(10) = -\tan^{-1}\left(\frac{10}{10}\right) = -45^{\circ}$$

Hence, we yield

$$H(j10) = 0.7071\angle -45^{\circ}$$

Using Equation (B.36), the system output in phasor form is obtained as

$$Y(j10) = H(j10)X(j10) = (1.4141\angle -45^{\circ})(5\angle 30^{\circ})$$

$$Y(j10) = 3.5355\angle -15^{\circ}$$

Converting the output in phasor form back to the time domain results in the steady-state system output:

$$y_{ss}(t) = 3.5355\sin(10t - 15^{\circ})u(t)$$

B.4 PROBLEMS

B.1. Develop equations for the amplitude spectra, that is, A_n (one-sided) and $|c_n|$ (two-sided), of the pulse train $x(t)$ displayed in Figure B.13, where $\tau = 10$ μsec.

a. Plot and label the one-sided amplitude spectrum up to 4 harmonic frequencies including DC.

b. Plot and label the two-sided amplitude spectrum up to 4 harmonic frequencies including DC.

B.2. In the waveform shown in Figure B.14, $T_0 = 1$ ms and $A = 10$. Use the formula in Table B.1 to write a Fourier series expansion in magnitude-phase form. Determine the frequency f_3 and amplitude value of A_3 for the third harmonic.

B.3. In the waveform shown in Figure B.15, $T_0 = 1$ ms, $\tau = 0.2$ ms, and $A = 10$.

a. Use the formula in Table B.1 to write a Fourier series expansion in magnitude-phase form.

b. Determine the frequency f_2 and amplitude value of A_2 for the second harmonic.

B.4. Find the Fourier transform $X(\omega)$ and sketch the amplitude spectrum for the rectangular pulse $x(t)$ displayed in Figure B.16.

B.5. Use Table B.3 to determine the Fourier transform for the pulse in Figure B.17.

B.6. Use Table B.3 to determine the Fourier transform for the pulse in Figure B.18.

B.7. Determine the Laplace transform $X(s)$ for each of the following time domain functions using the Laplace transform in Table B.5.

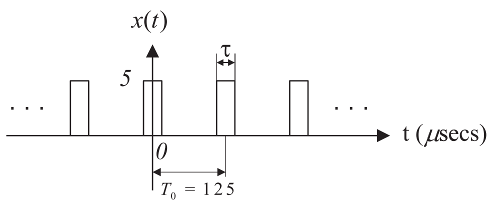


FIGURE B.13

Pulse train in Problem B.1.

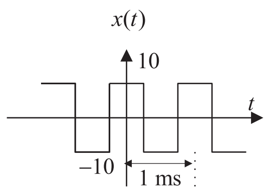


FIGURE B.14

Square wave in Problem B.2.

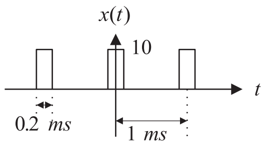


FIGURE B.15

Rectangular wave in Problem B.3.

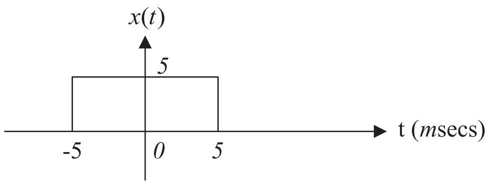


FIGURE B.16

Rectangular pulse in Problem B.4.

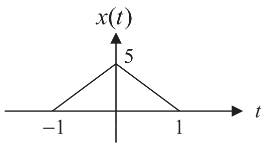


FIGURE B.17

Triangular pulse in Problem B.5.

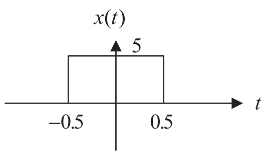


FIGURE B.18

Rectangular pulse in Problem B.6.

a. $x(t) = 10\delta(t)$

b. $x(t) = -100tu(t)$

c. $x(t) = 10e^{-2t}u(t)$

d. $x(t) = 2u(t-5)$

e. $x(t) = 10\cos(3t)u(t)$

f. $x(t) = 10\sin(2t + 45^{\circ})u(t)$

g. $x(t) = 3e^{-2t}\cos(3t)u(t)$

h. $x(t) = 10t^5u(t)$

B.8. Determine the inverse transform of the analog signal $x(t)$ for each of the following functions using Table B.5 and partial fraction expansion.

a. $X(s) = \dfrac{10}{s+2}$

b. $X(s) = \dfrac{100}{(s+2)(s+3)}$

c. $X(s) = \dfrac{100s}{s^2+7s+10}$

d. $X(s) = \dfrac{25}{s^2+4s+29}$

B.9. Solve the following differential equation using the Laplace transform method:

$$2\frac{dx(t)}{dt} + 3x(t) = 15u(t) \text{ with } x(0) = 0$$

a. Determine $X(s)$.

b. Determine the continuous signal $x(t)$ by taking the inverse Laplace transform of $X(s)$.

B.10. Solve the following differential equation using the Laplace transform method:

$$\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t) = 10u(t) \text{ with } x'(0) = 0 \text{ and } x(0) = 0$$

a. Determine $X(s)$.

b. Determine $x(t)$ by taking the inverse Laplace transform of $X(s)$.

B.11. Determine the locations of all finite zeros and poles in the following functions. In each case, make an s-plane plot of the poles and zeros, and determine whether the given transfer function is stable, unstable, or marginally stable.

a. $H(s) = \dfrac{(s-3)}{(s^2+4s+4)}$

b. $H(s) = \dfrac{s(s^2+5)}{(s^2+9)(s^2+2s+4)}$

c. $H(s) = \dfrac{(s^2+1)(s+1)}{s(s^2+7s-8)(s+3)(s+4)}$

B.12. Given the transfer function of a system

$$H(s) = \frac{5}{s+5}$$

and the input $x(t) = u(t)$,

a. determine the system impulse response $h(t)$;

b. determine the system Laplace output based on $Y(s) = H(s)X(s)$;

c. determine the system response $y(t)$ in the time domain by taking the inverse Laplace transform of $Y(s)$.

B.13. Given the transfer function of a system

$$H(s) = \frac{5}{s+5}$$

a. determine the steady-state transfer function;

b. determine the amplitude response and phase response in terms of the frequency ω;

c. determine the steady-state response of the system output $y_{ss}(t)$ in time domain using the results that you obtained in (b), given an input to the system of $x(t) = 5\sin(2t)u(t)$.

B.14. Given the transfer function of a system

$$H(s) = \frac{5}{s+5}$$

and the input $x(t) = u(t)$, determine the system output $y(t)$ using the convolution method; that is, $y(t) = h(t) * x(t)$.