Appendix D: Sinusoidal Steady-State Response of Digital Filters

D.1 SINUSOIDAL STEADY-STATE RESPONSE

Analysis of the sinusoidal steady-state response of digital filters will lead to the development of the magnitude and phase responses of digital filters. Let us look at the following digital filter with a digital transfer function $H(z)$ and a complex sinusoidal input

$$x(n) = Ve^{j(\Omega n+\varphi_x)} \tag{D.1}$$

where $\Omega = \omega T$ is the normalized digital frequency, while T is the sampling period and $y(n)$ denotes the digital output, as shown in Figure D.1.

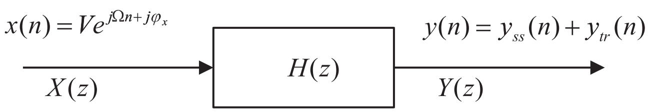

FIGURE D.1

Steady-state response of the digital filter.

The z-transform output from the digital filter is then given by

$$Y(z) = H(z)X(z) \tag{D.2}$$

Since $X(z) = \frac{Ve^{j\varphi_x}z}{z - e^{j\Omega}}$, we have

$$Y(z) = \frac{Ve^{j\varphi_x}z}{z - e^{j\Omega}}H(z) \tag{D.3}$$

Based on the partial fraction expansion, $Y(z)/z$ can be expanded to the following form:

$$\frac{Y(z)}{z} = \frac{Ve^{j\varphi_x}}{z - e^{j\Omega}}H(z) = \frac{R}{z - e^{j\Omega}} + \text{sum of the rest of partial fractions} \tag{D.4}$$

Multiplying the factor $(z - e^{j\Omega})$ on both sides of Equation (D.4) yields

$$Ve^{j\phi_x}H(z) = R + (z - e^{j\Omega})(\text{sum of the rest of partial fractions}) \tag{D.5}$$

Substituting $z = e^{j\Omega}$, we get the residue as

$$R = Ve^{j\phi_x}H(e^{j\Omega})$$

Then substituting $R = Ve^{j\phi_x}H(e^{j\Omega})$ back into Equation (D.4) results in

$$\frac{Y(z)}{z} = \frac{Ve^{j\phi_x}H(e^{j\Omega})}{z - e^{j\Omega}} + \text{sum of the rest of partial fractions} \tag{D.6}$$

and multiplying z on both sides of Equation (D.6) leads to

$$Y(z) = \frac{Ve^{j\phi_x}H(e^{j\Omega})z}{z - e^{j\Omega}} + z \times \text{sum of the rest of partial fractions} \tag{D.7}$$

Taking the inverse z-transform leads to two parts of the solution:

$$y(n) = Ve^{j\phi_x}H(e^{j\Omega})e^{j\Omega n} + Z^{-1}(\text{z} \times \text{sum of the rest of partial fractions}) \tag{D.8}$$

From Equation (D.8), we have the steady-state response

$$y_{ss}(n) = Ve^{j\phi_x}H(e^{j\Omega})e^{j\Omega n} \tag{D.9}$$

and the transient response

$$y_{tr}(n) = Z^{-1}(\text{z} \times \text{sum of the rest of partial fractions}) \tag{D.10}$$

Note that since the digital filter is a stable system, and the locations of its poles must be inside the unit circle on the z-plane, the transient response will settle to zero eventually. To develop the filter magnitude and phase responses, we write the digital steady-state response as

$$y_{ss}(n) = V\left|H(e^{j\Omega})\right|e^{j\Omega + j\phi_x + \angle H(e^{j\Omega})} \tag{D.11}$$

Comparing Equation (D.11) and Equation (D.1), it follows that

$$\begin{aligned} \text{Magnitude response} &= \frac{\text{Amplitude of the steady-state output}}{\text{Amplitude of the sinusoidal input}} \\ &= \frac{V\left|H(e^{j\Omega})\right|}{V} = \left|H(e^{j\Omega})\right| \end{aligned} \tag{D.12}$$

$$\text{Phase response} = \frac{e^{j\phi_x + j\angle H(e^{j\Omega})}}{e^{j\phi_x}} = e^{j\angle H(e^{j\Omega})} = \angle H(e^{j\Omega}) \tag{D.13}$$

Thus we conclude that

$$\text{Frequency response: } H(e^{j\Omega}) = H(z)|_{z=e^{j\Omega}} \tag{D.14}$$

Since $H(e^{j\Omega}) = |H(e^{j\Omega})|\angle H(e^{j\Omega})$

$$\text{Magnitude response: } \left|H(e^{j\Omega})\right| \tag{D.15}$$

$$\text{Phase response: } \angle H(e^{j\Omega}) \tag{D.16}$$

D.2 Properties of the Sinusoidal Steady-State Response

From Euler's identity and trigonometric identity, we know that

$$\begin{aligned} e^{j(\Omega+k2\pi)} &= \cos(\Omega+k2\pi)+j\sin(\Omega+k2\pi) \\ &= \cos\Omega + j\sin\Omega = e^{j\Omega} \end{aligned} \tag{D.17}$$

where k is an integer taking values of $k = 0, \pm 1, \pm 2, \cdots$. Then

$$\text{Frequency response: } H(e^{j\Omega}) = H(e^{j(\Omega+k2\pi)}) \tag{D.18}$$

$$\text{Magnitude frequency response: } \left|H(e^{j\Omega})\right| = \left|H(e^{j(\Omega+k2\pi)})\right| \tag{D.19}$$

$$\text{Phase response: } \angle H(e^{j\Omega}) = \angle H(e^{j\Omega+2k\pi}) \tag{D.20}$$

Clearly, the frequency response is periodic, with a period of 2π. Next, let us develop the symmetric properties. Since the transfer function is written as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \tag{D.21}$$

substituting $z = e^{j\Omega}$ into Equation (D.21) yields

$$H(e^{j\Omega}) = \frac{b_0 + b_1 e^{-j\Omega} + \cdots + b_M e^{-jM\Omega}}{1 + a_1 e^{-j\Omega} + \cdots + a_N e^{-jN\Omega}} \tag{D.22}$$

Using Euler's identity, $e^{-j\Omega} = \cos\Omega - j\sin\Omega$, we have

$$H(e^{j\Omega}) = \frac{(b_0 + b_1\cos\Omega + \cdots + b_M\cos M\Omega) - j(b_1\sin\Omega + \cdots + b_M\sin M\Omega)}{(1 + a_1\cos\Omega + \cdots + a_N\cos N\Omega) - j(a_1\sin\Omega + \cdots + a_N\sin N\Omega)} \tag{D.23}$$

Similarly,

$$H(e^{-j\Omega}) = \frac{(b_0 + b_1\cos\Omega + \cdots + b_M\cos M\Omega) + j(b_1\sin\Omega + \cdots + b_M\sin M\Omega)}{(1 + a_1\cos\Omega + \cdots + a_N\cos N\Omega) + j(a_1\sin\Omega + \cdots + a_N\sin N\Omega)} \tag{D.24}$$

Then the magnitude response and phase response can be expressed as

$$\left|H(e^{j\Omega})\right| = \frac{\sqrt{(b_0 + b_1\cos\Omega + \cdots + b_M\cos M\Omega)^2 + (b_1\sin\Omega + \cdots + b_M\sin M\Omega)^2}}{\sqrt{(1 + a_1\cos\Omega + \cdots + a_N\cos N\Omega)^2 + (a_1\sin\Omega + \cdots + a_N\sin N\Omega)^2}} \tag{D.25}$$

$$\begin{aligned} \angle H(e^{j\Omega}) &= \tan^{-1}\left(\frac{-(b_1\sin\Omega + \cdots + b_M\sin M\Omega)}{b_0 + b_1\cos\Omega + \cdots + b_M\cos M\Omega}\right) \\ &\quad - \tan^{-1}\left(\frac{-(a_1\sin\Omega + \cdots + a_N\sin N\Omega)}{1 + a_1\cos\Omega + \cdots + a_N\cos N\Omega}\right) \end{aligned} \tag{D.26}$$

Based on Equation (D.24), we also obtain the magnitude and phase response for $H(e^{-j\Omega})$ as

$$\left|H(e^{-j\Omega})\right| = \frac{\sqrt{(b_0 + b_1\cos\Omega + \cdots + b_M\cos M\Omega)^2 + (b_1\sin\Omega + \cdots + b_M\sin M\Omega)^2}}{\sqrt{(1 + a_1\cos\Omega + \cdots + a_N\cos N\Omega)^2 + (a_1\sin\Omega + \cdots + a_N\sin N\Omega)^2}} \tag{D.27}$$

$$\begin{aligned}\angle H(e^{-j\Omega}) = {} & \tan^{-1}\left(\frac{b_1\sin\Omega + \cdots + b_M\sin M\Omega}{b_0 + b_1\cos\Omega + \cdots + b_M\cos M\Omega}\right) \\ & - \tan^{-1}\left(\frac{a_1\sin\Omega + \cdots + a_N\sin N\Omega}{1 + a_1\cos\Omega + \cdots + a_N\cos N\Omega}\right)\end{aligned} \tag{D.28}$$

Comparing Equation (D.25) with (D.27), and Equation (D.26) with (D.28), respectively, we obtain the symmetric properties as

$$\left|H(e^{-j\Omega})\right| = \left|H(e^{j\Omega})\right| \tag{D.29}$$

$$\angle H(e^{-j\Omega}) = -\angle H(e^{j\Omega}) \tag{D.30}$$