Appendix G: Some Useful Mathematical Formulas

Form of a complex number:
Rectangular form:

$$a + jb, \text{ where } j = \sqrt{-1} \tag{G.1}$$

Polar form:

$$Ae^{j\theta} \tag{G.2}$$

Euler formula:

$$e^{\pm jx} = \cos x \pm j \sin x \tag{G.3}$$

Conversion from the polar form to the rectangular form:

$$Ae^{j\theta} = A\cos\theta + jA\sin\theta = a + jb \tag{G.4}$$

where $a = A\cos\theta$, and $b = A\sin\theta$.
Conversion from the rectangular form to the polar form:

$$a + jb = Ae^{j\theta} \tag{G.5}$$

where $A = \sqrt{a^2 + b^2}$. We usually specify the principal value of the angle such that $-180^\circ < \theta \le 180^\circ$. The angle value can be determined as

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \quad \text{if} \quad a \ge 0$$

(that is, the complex number is in the first or fourth quadrant in the rectangular coordinate system);

$$\theta = 180^\circ + \tan^{-1}\left(\frac{b}{a}\right) \quad \text{if} \quad a < 0 \quad \text{and} \quad b \ge 0$$

(that is, the complex number is in the second quadrant in the rectangular coordinate system); and

$$\theta = -180^\circ + \tan^{-1}\left(\frac{b}{a}\right) \quad \text{if} \quad a < 0 \quad \text{and} \quad b \le 0$$

(that is, the complex number is in the third quadrant in the rectangular coordinate system). Note that

$$\theta\ radian = \frac{\theta\ degree}{180^{\circ}} \times \pi$$

$$\theta\ degree = \frac{\theta\ radian}{\pi} \times 180^{\circ}$$

Complex numbers:

$$e^{\pm j\pi/2} = \pm j \tag{G.6}$$

$$e^{\pm j2n\pi} = 1 \tag{G.7}$$

$$e^{\pm j(2n+1)\pi} = -1 \tag{G.8}$$

Complex conjugate of $a + jb$:

$$(a + jb)^* = con\,j(a + jb) = a - jb \tag{G.9}$$

Complex conjugate of $Ae^{j\theta}$:

$$\left(Ae^{j\theta}\right)^* = con\,j\left(Ae^{j\theta}\right) = Ae^{-j\theta} \tag{G.10}$$

Complex number addition and subtraction:

$$(a_1 + jb_1) \pm (a_2 + jb_2) = (a_1 \pm a_2) + j(b_1 \pm b_2) \tag{G.11}$$

Complex number multiplication:
Rectangular form:

$$(a_1 + jb_1) \times (a_2 + jb_2) = a_1a_2 - b_1b_2 + j(a_1b_2 + a_2b_1) \tag{G.12}$$

$$\left(a + jb\right)\cdot con\,j\left(a + jb\right) = \left(a + jb\right)\left(a - jb\right) = a^2 + b^2 \tag{G.13}$$

Polar form:

$$A_1e^{j\theta_1}A_2e^{j\theta_2} = A_1A_2e^{j(\theta_1+\theta_2)} \tag{G.14}$$

Complex number division:
Rectangular form:

$$\begin{aligned} \frac{a_1 + jb_1}{a_2 + jb_2} &= \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)} \\ &= \frac{(a_1a_2 + b_1b_2) + j(a_2b_1 - a_1b_2)}{(a_2)^2 + (b_2)^2} \end{aligned} \tag{G.15}$$

Polar form:

$$\frac{A_1 e^{j\theta_1}}{A_2 e^{j\theta_2}} = \left(\frac{A_1}{A_2}\right) e^{j(\theta_1-\theta_2)} \tag{G.16}$$

Trigonometric identities:

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \tag{G.17}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2} \tag{G.18}$$

$$\sin\left(x \pm 90^\circ\right) = \pm \cos x \tag{G.19}$$

$$\cos\left(x \pm 90^\circ\right) = \mp \sin x \tag{G.20}$$

$$\sin x \cos x = \frac{1}{2} \sin 2x \tag{G.21}$$

$$\sin^2 x + \cos^2 x = 1 \tag{G.22}$$

$$\sin^2 x = \frac{1}{2}\left(1 - \cos 2x\right) \tag{G.23}$$

$$\cos^2 x = \frac{1}{2}\left(1 + \cos 2x\right) \tag{G.24}$$

$$\sin (x \pm y) = \sin x \cos y \pm \cos x \sin y \tag{G.25}$$

$$\cos (x \pm y) = \cos x \cos y \mp \sin x \sin y \tag{G.26}$$

$$\sin x \cos y = \frac{1}{2}(\sin (x + y) + \sin (x - y)) \tag{G.27}$$

$$\sin x \sin y = \frac{1}{2}(\cos (x - y) - \cos (x + y)) \tag{G.28}$$

$$\cos x \cos y = \frac{1}{2}(\cos (x - y) + \cos (x + y)) \tag{G.29}$$

Series of exponentials:

$$\sum_{k=0}^{N-1} a^k = \frac{1-a^N}{1-a},\ a \neq 1 \tag{G.30}$$

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a},\ |a| < 1 \tag{G.31}$$

$$\sum_{k=0}^{\infty} k a^k = \frac{1}{(1-a)^2},\ |a| < 1 \tag{G.32}$$

$$\sum_{k=0}^{N-1} e^{\left(j\frac{2\pi nk}{N}\right)} = \begin{cases} 0 & 1 \leq n \leq N-1 \\ N & n = 0, N \end{cases} \tag{G.33}$$

L'Hospital's rule:

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ results in the undetermined form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \tag{G.34}$$

where $f'(x) = \frac{df(x)}{dx}$ and $g'(x) = \frac{dg(x)}{dx}$.

Solution of the quadratic equation:

For a quadratic equation expressed as

$$ax^2 + bx + c = 0 \tag{G.35}$$

the solution is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{G.36}$$

Solution of simultaneous equations:

Simultaneous linear equations are listed below:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{G.37}$$

The solution is given by Cramer's rule, that is

$$x_1 = \frac{D_1}{D},\ x_2 = \frac{D_2}{D}, \cdots, x_n = \frac{D_n}{D} \tag{G.38}$$

where $D, D_1, D_2, \ldots, D_n$ are the $n \times n$ determinants. Each is defined below:

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{G.39}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} \tag{G.40}$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{vmatrix} \tag{G.41}$$

$$\cdots$$

$$D_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix} \tag{G.42}$$

$$D = (-1)^{1+1} a_{11} M_{11} + (-1)^{1+2} a_{12} M_{12} + \cdots (-1)^{1+n} a_{1n} M_{1n} \tag{G.43}$$

where M_{ij} is an $(n-1) \times (n-1)$ determinant obtained from D by crossing out the ith row and jth column. D can also be expanded by any row or column. As an example, using the second column,

$$D = (-1)^{1+2} a_{12} M_{12} + (-1)^{2+2} a_{22} M_{12} + \cdots (-1)^{n+2} a_{n2} M_{n2} \tag{G.44}$$

2×2 determinant:

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21} \tag{G.45}$$

3 × 3 determinant:

$$
\begin{aligned}
D &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})
\end{aligned} \tag{G.46}
$$

Solution for two simultaneous linear equations:

$$
\begin{aligned}
ax + by &= e \\
cx + dy &= f
\end{aligned} \tag{G.47}
$$

The solution is given by

$$
x = \frac{D_1}{D} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - bf}{ad - bc} \tag{G.48}
$$

$$
y = \frac{D_2}{D} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ec}{ad - bc} \tag{G.49}
$$