CHAPTER

Digital Signal Processing Systems, Basic Filtering Types, and Digital Filter Realizations

6

CHAPTER OUTLINE

OBJECTIVES:

This chapter illustrates digital filtering operations for a given input sequence; derives transfer functions from the difference equations; analyzes the stability of the linear systems using the z-plane pole-zero plot; and calculates the frequency responses of digital filters. Then the chapter further investigates realizations of the digital filters, and examines spectral effects by filtering speech data using the digital filters.

6.1 THE DIFFERENCE EQUATION AND DIGITAL FILTERING

In this chapter, we begin with developing the filtering concept of digital signal processing (DSP) systems. With the knowledge acquired in Chapter 5, dealing with the z-transform, we will learn how to describe and analyze linear time-invariant systems. We also will become familiar with digital filtering types and their realization structures.

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A DSP system (digital filter) is described in Figure 6.1.

Let $x(n)$ and $y(n)$ be a DSP system's input and output, respectively. We can express the relationship between the input and the output of a DSP system by the following *difference equation*:

$$\begin{aligned} y(n) \;=\; & b_0x(n) + b_1x(n-1) + \cdots + b_Mx(n-M) \\ & - a_1y(n-1) - \cdots - a_Ny(n-N) \end{aligned} \tag{6.1}$$

where b_i, $0 \le i \le M$ and a_j, $1 \le j \le N$, represent the coefficients of the system and n is the time index. Equation (6.1) can also be written as

$$y(n) \;=\; \sum_{i=0}^{M} b_i x(n-i) - \sum_{j=1}^{N} a_j y(n-j) \tag{6.2}$$

From Equations (6.1) and (6.2), we observe that the DSP system output is the weighted summation of the current input value $x(n)$, past values $x(n-1)$, … , $x(n-M)$, and the past output sequence $y(n-1)$, … , $y(n-N)$. The system can be verified as linear, time-invariant, and causal. If the initial conditions are specified, we can compute system output (time response) $y(n)$ recursively. This process is referred to as *digital filtering*. We will illustrate filtering operations in Examples 6.1 and 6.2.

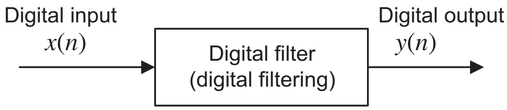

FIGURE 6.1

DSP system with input and output.

EXAMPLE 6.1

Compute the system output

$$y(n) \;=\; 0.5y(n-2) + x(n-1)$$

for the first four samples using the following initial conditions:

a. initial conditions $y(-2) = 1$, $y(-1) = 0$, $x(-1) = -1$, and input $x(n) = (0.5)^n u(n)$
b. zero initial conditions $y(-2) = 0$, $y(-1) = 0$, $x(-1) = 0$, and input $x(n) = (0.5)^n u(n)$

Solution:
According to Equation (6.1), we identify the system coefficients as

$$N = 2, M = 1, \; a_1 = 0, \; a_2 = -0.5, \; b_0 = 0, \quad \text{and} \quad b_1 = 1$$

a. Setting $n = 0$, and using initial conditions, we obtain the input and output as

$$x(0) = (0.5)^0 u(0) = 1$$
$$y(0) \;=\; 0.5y(-2) + x(-1) = 0.5 \cdot 1 + (-1) = -0.5$$

Setting $n = 1$, and using the initial condition $y(-1) = 0$, we achieve

$$x(1) = (0.5)^1 u(1) = 0.5$$
$$y(1) = 0.5y(-1) + x(0) = 0.5 \cdot 0 + 1 = 1.0$$

Similarly, using the past output $y(0) = -0.5$, we get

$$x(2) = (0.5)^2 u(2) = 0.25$$
$$y(2) = 0.5y(0) + x(1) = 0.5 \cdot (-0.5) + 0.5 = 0.25$$

and with $y(1) = 1.0$, we yield

$$x(3) = (0.5)^3 u(3) = 0.125$$
$$y(3) = 0.5y(1) + x(2) = 0.5 \cdot 1 + 0.25 = 0.75$$

....
Clearly, $y(n)$ could be recursively computed for $n > 3$.
b. Setting $n = 0$, we obtain

$$x(0) = (0.5)^0 u(0) = 1$$
$$y(0) = 0.5y(-2) + x(-1) = 0 \cdot 1 + 0 = 0$$

Setting $n = 1$, we achieve

$$x(1) = (0.5)^1 u(1) = 0.5$$
$$y(1) = 0.5y(-1) + x(0) = 0 \cdot 0 + 1 = 1$$

Similarly, with the past output $y(0) = 0$, we determine

$$x(2) = (0.5)^2 u(2) = 0.25$$
$$y(2) = 0.5y(0) + x(1) = 0.5 \cdot 0 + 0.5 = 0.5$$

and with $y(1) = 1$, we obtain

$$x(3) = (0.5)^3 u(3) = 0.125$$
$$y(3) = 0.5y(1) + x(2) = 0.5 \cdot 1 + 0.25 = 0.75$$

....
Clearly, $y(n)$ could be recursively computed for $n > 3$

EXAMPLE 6.2

Given the DSP system

$$y(n) = 2x(n) - 4x(n-1) - 0.5y(n-1) - y(n-2)$$

with initial conditions $y(-2) = 1$, $y(-1) = 0$, $x(-1) = -1$, and the input $x(n) = (0.8)^n u(n)$, compute the system response $y(n)$ for 20 samples using MATLAB.

Solution:

Program 6.1 lists the MATLAB program for computing the system response $y(n)$. The top plot in Figure 6.2 shows the input sequence. The middle plot displays the filtered output using the initial conditions, and the bottom plot shows the filtered output for zero initial conditions. As we can see, the system outputs are different at the beginning, but they approach the same value later.

Program 6.1. MATLAB program for Example 6.2.

```
% Example 6.2
% Compute y(n)=2x(n)-4x(n-1)-0.5y(n-1)-0.5y(n-2)
% Nonzero initial conditions:
% y(-2)=1, y(-1)=0, x(-1) =-1, and x(n)=(0.8)^n*u(n)
%
y = zeros(1,20);      % Set up a vector to store y(n)
y = [ 1 0 y];         % Add initial condition of y(-2) and y(-1)
n=0:1:19;             % Compute time indexes
x=(0.8).^n;           % Compute 20 input samples of x(n)
```

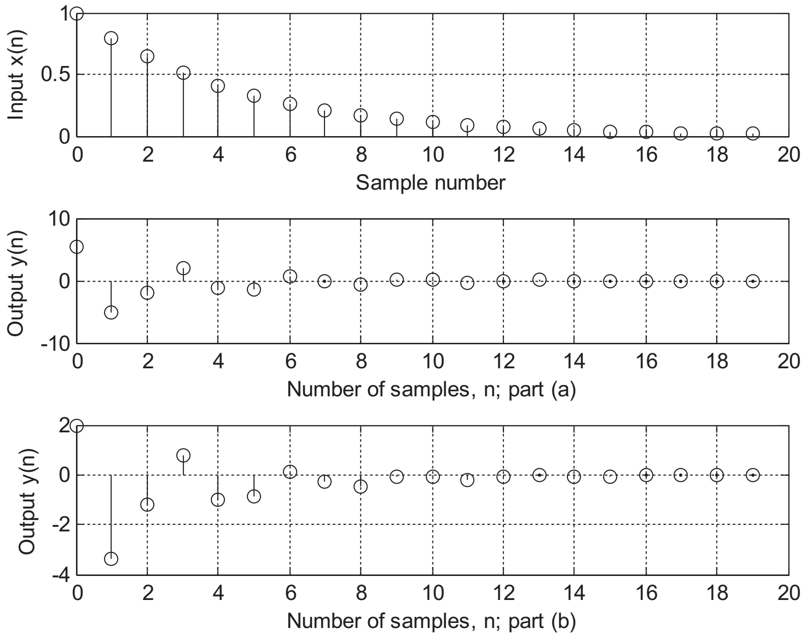

FIGURE 6.2

Plots of the input and system outputs $y(n)$ for Example 6.2.

```
x = [ 0 -1 x];                                        % Add initial conditions of
                                                      % x(-2)=0 and x(-1)=1
for n=1:20
  y(n+2)= 2*x(n+2)-4*x(n+1)-0.5*y(n+1)-0.5*y(n); % Compute 20 outputs of y(n)
end
n=0:1:19;
subplot(3,1,1);stem(n,x(3:22));grid;ylabel('Input x(n)');xlabel('Sample number');
subplot(3,1,2); stem(n,y(3:22)),grid;
xlabel('Number of samples, n; part (a)'); ylabel('Output y(n)');
y(3:22)   %Output y(n)
% Zero initial conditions:
% y(-2)=0, y(-1)=0, x(-1) =0, and x(n)=(0.8)^n
%
y = zeros(1,20);       % Set up a vector to store y(n)
y = [ 0 0 y];          % Add zero initial conditions for y(-2) and y(-1)
n=0:1:19;              % Compute time indexes
x=(0.8).^n;            % Compute 20 input samples of x(n)
x = [ 0 0 x];          % Add zero initial conditions for x(-2) and x(-1)
for n=1:20
y(n+2)= 2*x(n+2)-4*x(n+1)-0.5*y(n+1)-0.5*y(n); % Compute 20 outputs of y(n)
end
n=0:1:19
subplot(3,1,3); stem(n,y(3:22)),grid;
   xlabel('Number of samples, n; part (b)'); ylabel('Output y(n)');
   y(3:22)             %Output y(n)
```

The MATLAB function **filter(),** developed using a direct-form II realization (which will be discussed in a later section), can be used to operate digital filtering, and the syntax is

Zi = filtic(B, A, Yi, Xi)
y = filter(B, A, x, Zi)

where B and A are vectors for the coefficients b_j and a_j whose formats are

$$A = [1 \quad a_1 \quad a_2 \quad \cdots \quad a_N] \quad \text{and} \quad B = [b_0 \quad b_1 \quad b_2 \quad \cdots \quad b_M]$$

and x and y are the input data vector and output data vector, respectively.

Note that the filter function **filtic()** is a MATLAB function which is used to obtain initial states from initial conditions in the difference equation. The initial states are required by the MATLAB filter function **filter()** since it is implemented in a direct-form II. Hence, Z_i contains initial states required for operating MATLAB function **filter(),** that is,

$$Z_i = [w(-1) \quad w(-2) \quad \cdots \ ]$$

which can be recovered by another MATLAB function **filtic()**. X_i and Y_i are initial conditions with the length of the greater of M or N, given by

$$X_i = [x(-1) \quad x(-2) \quad \cdots \ ] \quad \text{and} \quad Y_i = [y(-1) \quad y(-2) \quad \cdots \ ]$$

For zero initial conditions in particular, the syntax is reduced to

y = filter(B, A, x)

Let us verify the filter operation results in Example 6.1 using the MATLAB functions. The MATLAB codes and results for Example 6.1(a) with the nonzero initial conditions are listed as

```
» B = [0 1]; A = [1 0 −0.5];
» x = [1 0.5 0.25 0.125];
» Xi = [−1 0];Yi=[0 1];
» Zi = filtic(B, A, Yi, Xi);
» y = filter(B, A, x, Zi)
y =
− 0.5000 1.0000 0.2500 0.7500
»
```

For the case of zero initial conditions in Example 6.1(b), the MATLAB codes and results are

```
» B = [0 1]; A = [1 0 −0.5];
» x = [1 0.5 0.25 0.125];
» y = filter(B, A, x)
y =
0 1.0000 0.5000 0.7500
»
```

As we expected, the filter outputs match the ones in Example 6.1.

6.2 DIFFERENCE EQUATION AND TRANSFER FUNCTION

To proceed in this section, Equation (6.1) is rewritten as

$$\begin{aligned} y(n) &= b_0x(n) + b_1x(n-1) + \cdots + b_Mx(n-M) \\ &\quad - a_1y(n-1) - \cdots - a_Ny(n-N) \end{aligned}$$

With an assumption that all initial conditions of this system are zero, and with $X(z)$ and $Y(z)$ denoting the z-transforms of $x(n)$ and $y(n)$, respectively, taking the z-transform of Equation (6.1) yields

$$\begin{aligned} Y(z) &= b_0X(z) + b_1X(z)z^{-1} + \cdots + b_MX(z)z^{-M} \\ &\quad - a_1Y(z)z^{-1} - \cdots - a_NY(z)z^{-N} \end{aligned} \tag{6.3}$$

Rearranging Equation (6.3), we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1z^{-1} + \cdots + b_Mz^{-M}}{1 + a_1z^{-1} + \cdots + a_Nz^{-N}} = \frac{B(z)}{A(z)} \tag{6.4}$$

where $H(z)$ is defined as the transfer function with its numerator and denominator polynomials defined below:

$$B(z) = b_0 + b_1z^{-1} + \cdots + b_Mz^{-M} \tag{6.5}$$

$$A(z) = 1 + a_1z^{-1} + \cdots + a_Nz^{-N} \tag{6.6}$$

Clearly the z-transfer function is defined as

$$\text{ratio} = \frac{\text{z-transform of the output}}{\text{z-transform of the input}}$$

In DSP applications, given the difference equation, we can develop the z-transfer function and represent the digital filter in the z-domain as shown in Figure 6.3. Then the stability and frequency response can be examined based on the developed transfer function.

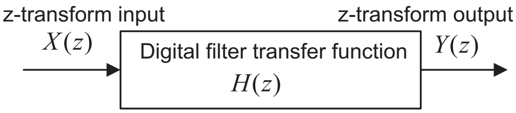

FIGURE 6.3

Digital filter transfer function.

EXAMPLE 6.3

A DSP system is described by the following difference equation:

$$y(n) = x(n) - x(n-2) - 1.3y(n-1) - 0.36y(n-2)$$

Find the transfer function $H(z)$, the denominator polynomial $A(z)$, and the numerator polynomial $B(z)$.

Solution:
Taking the z-transform on both sides of the previous difference equation, we obtain

$$Y(z) = X(z) - X(z)z^{-2} - 1.3Y(z)z^{-1} - 0.36Y(z)z^{-2}$$

Moving the last two terms to the left side of the difference equation and factoring $Y(z)$ on the left side and $X(z)$ on the right side, we obtain

$$Y(z)(1 + 1.3z^{-1} + 0.36z^{-2}) = (1 - z^{-2})X(z)$$

Therefore, the transfer function, which is the ratio of $Y(z)$ over $X(z)$, can be found to be

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}$$

From the derived transfer function $H(z)$, we can obtain the denominator polynomial and numerator polynomial as

$$A(z) = 1 + 1.3z^{-1} + 0.36z^{-2}$$

and

$$B(z) = 1 - z^{-2}$$

The difference equation and its transfer function, as well as the stability issue of the linear time-invariant system, will be discussed in the following sections.

EXAMPLE 6.4

A digital system is described by the following difference equation:

$$y(n) = x(n) - 0.5x(n-1) + 0.36x(n-2)$$

Find the transfer function $H(z)$, the denominator polynomial $A(z)$, and the numerator polynomial $B(z)$.

Solution:
Taking the z-transform on both sides of the previous difference equation, we obtain

$$Y(z) = X(z) - 0.5X(z)z^{-1} + 0.36X(z)z^{-2}$$

Therefore, the transfer function, that is the ratio of $Y(z)$ to $X(z)$, can be found as

$$H(z) = \frac{Y(z)}{X(z)} = 1 - 0.5z^{-1} + 0.36z^{-2}$$

From the derived transfer function $H(z)$, it follows that

$$A(z) = 1$$

$$B(z) = 1 - 0.5z^{-1} + 0.36z^{-2}$$

In DSP applications, the given transfer function of a digital system can be converted into a difference equation for DSP implementation. The following example illustrates the procedure.

EXAMPLE 6.5

Convert each of the following transfer functions into its difference equation.

a. $H(z) = \dfrac{z^2 - 1}{z^2 + 1.3z + 0.36}$

b. $H(z) = \dfrac{z^2 - 0.5z + 0.36}{z^2}$

Solution:

a. Dividing the numerator and denominator by z^2 to obtain the transfer function whose numerator and denominator polynomials have the negative power of z, it follows that

$$H(z) = \frac{(z^2 - 1)/z^2}{(z^2 + 1.3z + 0.36)/z^2} = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}$$

We write the transfer function using the ratio of $Y(z)$ to $X(z)$:

$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}$$

Then we have

$$Y(z)(1 + 1.3z^{-1} + 0.36z^{-2}) = X(z)(1 - z^{-2})$$

By distributing $Y(z)$ and $X(z)$, we yield

$$Y(z) + 1.3z^{-1}Y(z) + 0.36z^{-2}Y(z) = X(z) - z^{-2}X(z)$$

Applying the inverse z-transform and using the shift property in Equation (5.3) of Chapter 5, we get

$$y(n) + 1.3y(n-1) + 0.36y(n-2) = x(n) - x(n-2)$$

Writing the output $y(n)$ in terms of inputs and past outputs leads to

$$y(n) = x(n) - x(n-2) - 1.3y(n-1) - 0.36y(n-2)$$

b. Similarly, dividing the numerator and denominator by z^2, we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(z^2 - 0.5z + 0.36)/z^2}{z^2/z^2} = 1 - 0.5z^{-1} + 0.36z^{-2}$$

Thus

$$Y(z) = X(z)(1 - 0.5z^{-1} + 0.36z^{-2})$$

By distributing $X(z)$, we yield

$$Y(z) = X(z) - 0.5z^{-1}X(z) + 0.36z^{-2}X(z)$$

Applying the inverse z-transform with using the shift property in Equation (5.3), we obtain

$$y(n) = x(n) - 0.5x(n-1) + 0.36x(n-2)$$

The transfer function $H(z)$ can be factored into the *pole-zero form:*

$$H(z) = \frac{b_0(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)} \tag{6.7}$$

where the zeros z_i can be found by solving roots of the numerator polynomial, while the poles p_i can be solved for the roots of the denominator polynomial.

EXAMPLE 6.6

Consider the following transfer functions:

$$H(z) = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}$$

Convert it into the pole-zero form.

Solution:

We first multiply the numerator and denominator polynomials by z^2 to achieve the advanced form in which both numerator and denominator polynomials have positive powers of z, that is,

$$H(z) = \frac{(1 - z^{-2})z^2}{(1 + 1.3z^{-1} + 0.36z^{-2})z^2} = \frac{z^2 - 1}{z^2 + 1.3z + 0.36}$$

Letting $z^2 - 1 = 0$, we get $z = 1$ and $z = -1$. Setting $z^2 + 1.3z + 0.36 = 0$ leads to $z = -0.4$ and $z = -0.9$. We then can write numerator and denominator polynomials in the factored form to obtain the pole-zero form:

$$H(z) = \frac{(z - 1)(z + 1)}{(z + 0.4)(z + 0.9)}$$

6.2.1 Impulse Response, Step Response, and System Response

The impulse response $h(n)$ of the DSP system $H(z)$ can be obtained by solving its difference equation using a unit impulse input $\delta(n)$. With the help of the z-transform and noticing that $X(z) = Z\{\delta(\mathrm{n})\}1$, we yield

$$h(n) = Z^{-1}\{H(z)X(z)\} = Z^{-1}\{H(z)\} \tag{6.8}$$

Similarly, for a step input, we can determine step response assuming zero initial conditions. Letting

$$X(z) = Z[u(n)] = \frac{z}{z - 1}$$

the step response can be found as

$$y(n) = Z^{-1}\left\{H(z)\frac{z}{z - 1}\right\} \tag{6.9}$$

Furthermore, the z-transform of the general system response is given by

$$Y(z) = H(z)X(z) \tag{6.10}$$

If we know the transfer function $H(z)$ and z-transform of the input $X(z)$, we are able to determine the system response $y(n)$ by finding the inverse z-transform of the output $Y(z)$:

$$y(n) = Z^{-1}\{Y(z)\} \tag{6.11}$$

EXAMPLE 6.7

Given a transfer function depicting a DSP system

$$H(z) = \frac{z+1}{z-0.5}$$

determine

a. the impulse response $h(n)$,
b. step response $y(n)$, and
c. system response $y(n)$ if the input is given as $x(n) = (0.25)^n u(n)$.

Solution:

a. The transfer function can be rewritten as

$$\frac{H(z)}{z} = \frac{z+1}{z(z-0.5)} = \frac{A}{z} + \frac{B}{z-0.5}$$

where

$$A = \left.\frac{z+1}{(z-0.5)}\right|_{z=0} = -2 \text{ and } B = \left.\frac{z+1}{z}\right|_{z=0.5} = 3$$

Thus we have

$$\frac{H(z)}{z} = \frac{-2}{z} + \frac{3}{z-0.5}$$

and

$$H(z) = \left(-\frac{2}{z} + \frac{3}{z-0.5}\right)z = -2 + \frac{3z}{z-0.5}$$

By taking the inverse z-transform as shown in Equation (6.8), we yield the impulse response

$$h(n) = -2\delta(n) + 3(0.5)^n u(n)$$

b. For the step input $x(n) = u(n)$ and its z-transform $X(z) = \frac{z}{z-1}$, we can determine the z-transform of the step response as

$$Y(z) = H(z)X(z) = \frac{z+1}{z-0.5}\,\frac{z}{z-1}$$

Applying the partial fraction expansion leads to

$$\frac{Y(z)}{z} = \frac{z+1}{(z-0.5)(z-1)} = \frac{A}{z-0.5} + \frac{B}{z-1}$$

where

$$A = \left.\frac{z+1}{z-1}\right|_{z=0.5} = -3 \quad \text{and} \quad B = \left.\frac{z+1}{z-0.5}\right|_{z=1} = 4$$

The z-transform step response is therefore

$$Y(z) = \frac{-3z}{z-0.5} + \frac{4z}{z-1}$$

Applying the inverse z-transform table yields the step response as

$$y(n) = -3(0.5)^n u(n) + 4u(n)$$

c. To determine the system output response, we first find the z-transform of the input $x(n)$,

$$X(z) = Z\{(0.25)^n u(n)\} = \frac{z}{z-0.25}$$

Then $Y(z)$ can be obtained via Equation (6.10), that is,

$$Y(z) = H(z)X(z) = \frac{z+1}{z-0.5} \cdot \frac{z}{z-0.25} = \frac{z(z+1)}{(z-0.5)(z-0.25)}$$

Using the partial fraction expansion, we have

$$\frac{Y(z)}{z} = \frac{(z+1)}{(z-0.5)(z-0.25)} = \left(\frac{A}{z-0.5} + \frac{B}{z-0.25}\right)$$

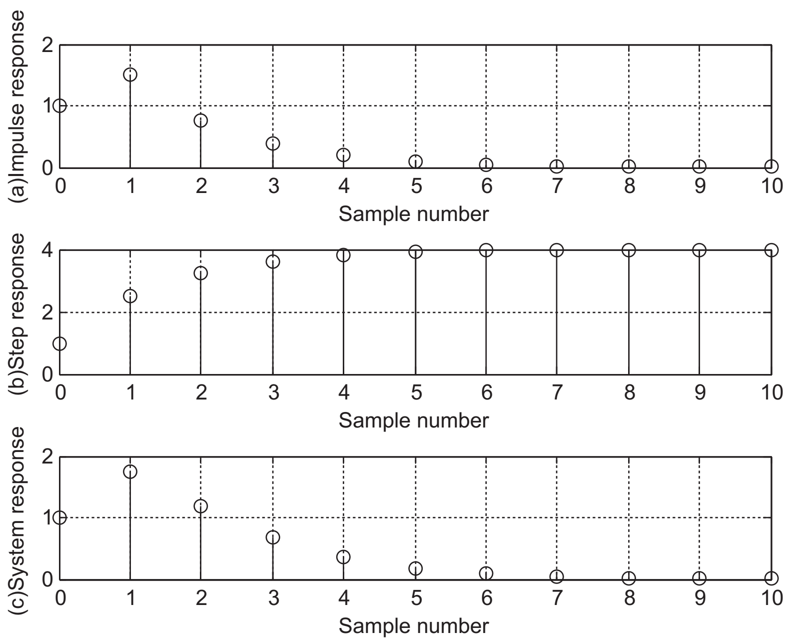

FIGURE 6.4

Impulse, step, and system response in Example 6.7.

$$Y(z) = \left(\frac{6z}{z-0.5} + \frac{-5z}{z-0.25}\right)$$

Using Equation (6.11) and Table 5.1 in Chapter 5, we finally yield

$$y(n) = Z^{-1}\{Y(z)\} = 6(0.5)^n u(n) - 5(0.25)^n u(n)$$

The impulse response for (a), step response for (b), and system response for (c) are each plotted in Figure 6.4.

6.3 THE Z-PLANE POLE-ZERO PLOT AND STABILITY

A very useful tool to analyze digital systems is the z-plane pole-zero plot. This graphical technique allows us to investigate characteristics of the digital system shown in Figure 6.1, including the system stability. In general, a digital transfer function can be written in the pole-zero form as shown in Equation (6.7), and we can plot the poles and zeros on the z-plane. The z-plane is depicted in Figure 6.5 and has the following features:

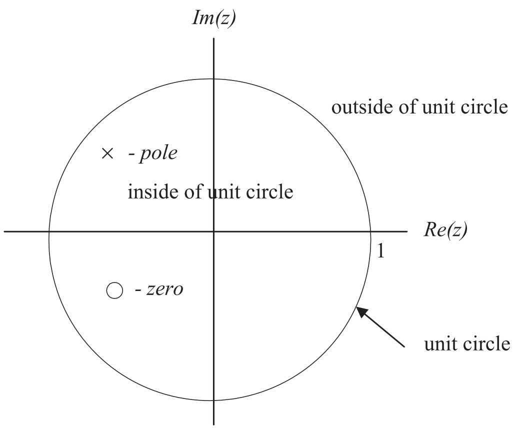

FIGURE 6.5

z-plane and pole-zero plot.

1. The horizontal axis is the real part of the variable z, and the vertical axis represents the imaginary part of the variable z.
2. The z-plane is divided into two parts by a unit circle.
3. Each pole is marked on z-plane using the cross symbol x, while each zero is plotted using the small circle symbol o.

Let's investigate the z-plane pole-zero plot of a digital filter system via the following example.

EXAMPLE 6.8

Given the digital transfer function

$$H(z) = \frac{z^{-1} - 0.5z^{-2}}{1 + 1.2z^{-1} + 0.45z^{-2}}$$

plot poles and zeros.

Solution:

Converting the transfer function to its advanced form by multiplying the numerator and denominator by z^2, it follows that

$$H(z) = \frac{(z^{-1} - 0.5z^{-2})z^2}{(1 + 1.2z^{-1} + 0.45z^{-2})z^2} = \frac{z - 0.5}{z^2 + 1.2z + 0.45}$$

By setting $z^2 + 1.2z + 0.45 = 0$ and $z - 0.5 = 0$, we obtain two poles

$p_1 = -0.6 + j0.3$
$p_2 = p_1^* = -0.6 - j0.3$

and a zero $z_1 = 0.5$, which are plotted on the z-plane shown in Figure 6.6. According to the form of Equation (6.7), we also yield the pole-zero form as

$$H(z) = \frac{z^{-1} - 0.5z^{-2}}{1 + 1.2z^{-1} + 0.45z^{-2}} = \frac{(z - 0.5)}{(z + 0.6 - j0.3)(z + 0.6 + j0.3)}$$

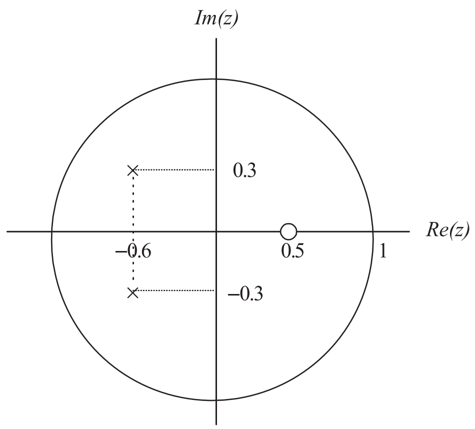

FIGURE 6.6

The z-plane pole-zero plot of Example 6.8.

With zeros and poles plotted on the z-plane, we are able to study system stability. We first establish the relationship between the s-plane in the Laplace domain and the z-plane in the z-transform domain, as illustrated in Figure 6.7.

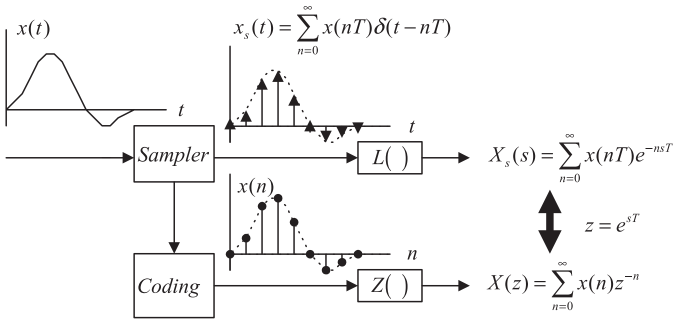

FIGURE 6.7

Relationship between Laplace transform and z-transform.

As shown in Figure 6.7, the sampled signal, which is not quantized, with a sampling period of T is written as

$$x_s(t) = \sum_{n=0}^{\infty} x(nT)\delta(t-nT) = x(0)\delta(t) + x(T)\delta(t-T) + x(2T)\delta(t-2T) + \cdots \tag{6.12}$$

Taking the Laplace transform and using the Laplace shift property as

$$L(\delta(t-nT)) = e^{-nTs} \tag{6.13}$$

leads to

$$X_s(s) = \sum_{n=0}^{\infty} x(nT)e^{-nTs} = x(0)e^{-0\times Ts} + x(T)e^{-Ts} + x(2T)e^{-2Ts} + \cdots \tag{6.14}$$

Compare Equation (6.14) with the definition of a one-sided z-transform of the data sequence $x(n)$ from analog-to-digital conversion (ADC):

$$X(z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n)z^{-n} = x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \cdots \tag{6.15}$$

Clearly, we see the relationship of the sampled system in the Laplace domain and its digital system in the z-transform domain by the following mapping:

$$z = e^{sT} \tag{6.16}$$

Substituting $s = -\alpha \pm j\omega$ into Equation (6.16), it follows that $z = e^{-\alpha T \pm j\omega T}$. In the polar form, we have

$$z = e^{-\alpha T} \angle \pm \omega T \tag{6.17}$$

Equations (6.16) and (6.17) give the following important conclusions.

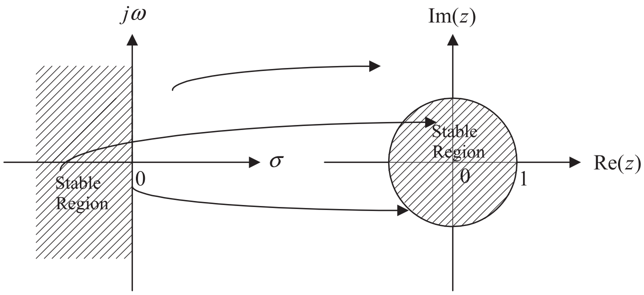

FIGURE 6.8

Mapping between s-plane and z-plane.

If $\alpha > 0$, this means $|z| = e^{-\alpha T} < 1$. Then the left-hand side half plane (LHHP) of the s-plane is mapped to the inside of the unit circle of the z-plane. When $\alpha = 0$, this causes $|z| = e^{-\alpha T} = 1$. Thus the $j\omega$ axis of the s-plane is mapped on the unit circle of the z-plane, as shown in Figure 6.8. Obviously, the right-hand half plane (RHHP) of the s-plane is mapped to the outside of the unit cycle in the z-plane. A stable system means that for a given bounded input, the system output must be bounded. Similar to the analog system, the digital system requires that all poles plotted on the z-plane must be inside the unit circle. We summarize the rules for determining the stability of a DSP system as follows:

1. If the outmost pole(s) of the z-transfer function $H(z)$ describing the DSP system is (are) inside the unit circle on the z-plane pole-zero plot, then the system is stable.
2. If the outmost pole(s) of the z-transfer function $H(z)$ is (are) outside the unit circle on the z-plane pole-zero plot, the system is unstable.
3. If the outmost pole(s) is (are) first-order pole(s) of the z-transfer function $H(z)$ and on the unit circle on the z-plane pole-zero plot, then the system is marginally stable.
4. If the outmost pole(s) is (are) multiple-order pole(s) of the z-transfer function $H(z)$ and on the unit circle on the z-plane pole-zero plot, then the system is unstable.
5. The zeros do not affect the system stability.

Notice that the following facts apply to a stable system (bounded-in/bounded-out [BIBO] stability discussed in Chapter 3):

1. If the input to the system is bounded, then the output of the system will also be bounded, or the impulse response of the system will go to zero in a finite number of steps.
2. An unstable system is one where the output of the system will grow without bound due to any bounded input, initial condition, or noise, or the impulse response will grow without bound.
3. The impulse response of a marginally stable system stays at a constant level or oscillates between two finite values.

Examples illustrating these rules are shown in Figure 6.9.

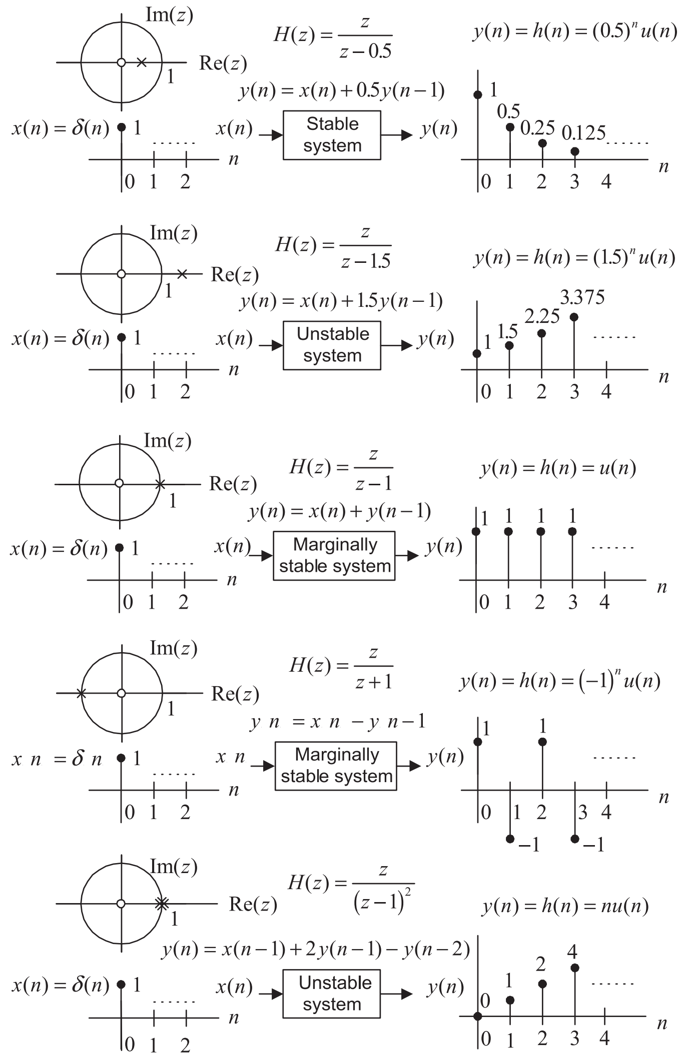

FIGURE 6.9

Stability illustrations.

EXAMPLE 6.9

The following transfer functions describe digital systems.

a. $H(z) = \frac{z+0.5}{(z-0.5)(z^2+z+0.5)}$

b. $H(z) = \frac{z^2+0.25}{(z-0.5)(z^2+3z+2.5)}$

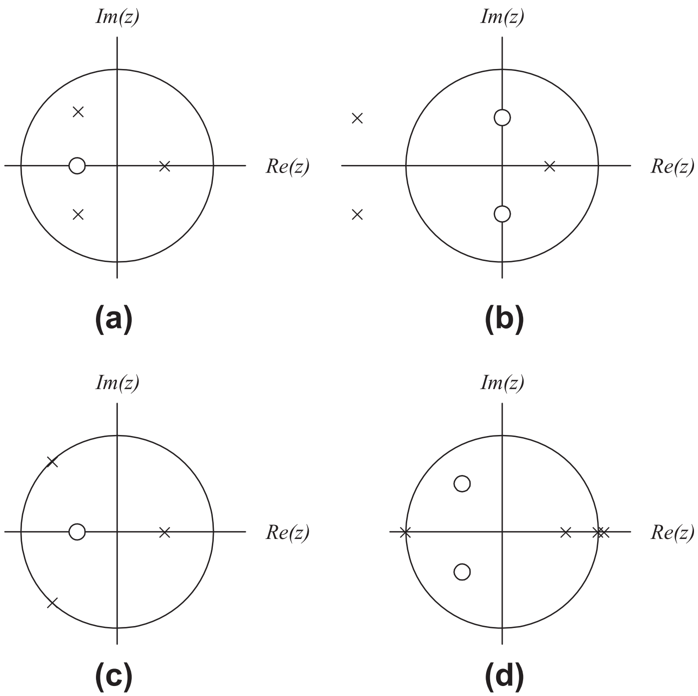

FIGURE 6.10

Pole-zero plots for Example 6.9.

c. $H(z) = \dfrac{z + 0.5}{(z - 0.5)(z^2 + 1.4141z + 1)}$

d. $H(z) = \dfrac{z^2 + z + 0.5}{(z - 1)^2(z + 1)(z - 0.6)}$

For each, sketch the z-plane pole-zero plot and determine the stability status for the digital system.

Solution:

a. A zero is located at $z = -0.5$.

Poles: $z = 0.5$, $|z| = 0.5 < 1$; $z = -0.5 \pm j0.5$,
$|z| = \sqrt{(-0.5)^2 + (\pm 0.5)^2} = 0.707 < 1$.

The plot of poles and a zero is shown in Figure 6.10. Since the outmost poles are inside the unit circle, the system is stable.

b. Zeros are $z = \pm j0.5$.

Poles: $z = 0.5$, $|z| = 0.5 < 1$; $z = -1.5 \pm j0.5$
$|z| = \sqrt{(1.5)^2 + (\pm 0.5)^2} = 1.5811 > 1$.

The plot of poles and zeros is shown in Figure 6.10. Since we have two poles at $z = -1.5 \pm j0.5$ that are outside the unit circle, the system is unstable.

c. A zero is located at $z = -0.5$.

Poles: $z = 0.5$, $|z| = 0.5 < 1$; $z = -0.707 \pm j0.707$,
$|z| = \sqrt{(0.707)^2 + (\pm 0.707)^2} = 1.$

The zero and poles are plotted in Figure 6.10. Since the outmost poles are first order at $z = -0.707 \pm j0.707$ and are on the unit circle, the system is marginally stable.
d. Zeros are $z = -0.5 \pm j0.5$.

Poles: $z = 1$, $|z| = 1$; $z = 1$, $|z| = 1$; $z = -1$, $|z| = 1$;, $z = 0.6$
$|z| = 0.6 < 1$.

The zeros and poles are plotted in Figure 6.10. Since the outmost pole is a multiple order (second order) pole at $z = 1$ and is on the unit circle, the system is unstable.

6.4 DIGITAL FILTER FREQUENCY RESPONSE

From the Laplace transfer function, we can achieve the analog filter steady-state frequency response $H(j\omega)$ by substituting $s = j\omega$ into the transfer function $H(s)$. That is,

$$H(s)|_{s=j\omega} = H(j\omega)$$

Then we can study the magnitude frequency response $|H(j\omega)|$ and phase response $\angle H(j\omega)$. Similarly, in a DSP system, using the mapping in Equation (6.16), we substitute $z = e^{sT}\big|_{s=j\omega} = e^{j\omega T}$ into the z-transfer function $H(z)$ to acquire the digital frequency response, which is converted into the magnitude frequency response $\left|H(e^{j\omega T})\right|$ and phase response $\angle\left|H(e^{j\omega T})\right|$. That is,

$$H(z)|_{z=e^{j\omega T}} = H(e^{j\omega T}) = \left|H(e^{j\omega T})\right| \angle H(e^{j\omega T}) \tag{6.18}$$

Let us introduce a normalized digital frequency in radians in the digital domain:

$$\Omega = \omega T \tag{6.19}$$

Then the digital frequency response in Equation (6.18) becomes

$$H(e^{j\Omega}) = H(z)\big|_{z=e^{j\Omega}} = \left|H(e^{j\Omega})\right| \angle H(e^{j\Omega}) \tag{6.20}$$

The formal derivation for Equation (6.20) can be found in Appendix D.

Now we verify the frequency response via the following simple digital filter. Consider a digital filter with a sinusoidal input of amplitude K (Figure 6.11).

We can determine the system output $y(n)$, which consists of the transient response $y_{tr}(n)$ and the steady-state response $y_{ss}(n)$. We find the z-transform output as

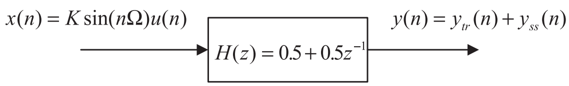

FIGURE 6.11

System transient and steady-state frequency responses.

$$Y(z) = \left(\frac{0.5z + 0.5}{z}\right) \frac{Kz \sin \Omega}{z^2 - 2z \cos \Omega + 1} \tag{6.21}$$

To perform the inverse z-transform to find the system output, we further rewrite Equation (6.21) as

$$\frac{Y(z)}{z} = \left(\frac{0.5z + 0.5}{z}\right) \frac{K \sin \Omega}{(z - e^{j\Omega})(z - e^{-j\Omega})} = \frac{A}{z} + \frac{B}{z - e^{j\Omega}} + \frac{B^*}{z - e^{-j\Omega}}$$

where A, B and the complex conjugate B^* are the constants for the partial fractions. Applying the partial fraction expansion leads to

$$A = 0.5K \sin \Omega$$

$$B = \left.\frac{0.5z + 0.5}{z}\right|_{z=e^{j\Omega}} \frac{K}{2j} = \left|H(e^{j\Omega})\right| e^{j\angle H(e^{j\Omega})} \frac{K}{2j}$$

Notice that the first part of constant B is a complex function, which is obtained by substituting $z = e^{j\Omega}$ into the filter z-transfer function. We can also express the complex function in terms of the polar form:

$$\left.\frac{0.5z + 0.5}{z}\right|_{z=e^{j\Omega}} = 0.5 + 0.5z^{-1}\Big|_{z=e^{j\Omega}} = H(z)\big|_{z=e^{j\Omega}} = H(e^{j\Omega}) = \left|H(e^{j\Omega})\right| e^{j\angle H(e^{j\Omega})}$$

where $H(e^{j\Omega}) = 0.5 + 0.5e^{-j\Omega}$. We call this complex function the steady-state frequency response. Based on the complex conjugate property, we get another residue as

$$B^* = \left|H(e^{j\Omega})\right| e^{-j\angle H(e^{j\Omega})} \frac{K}{-j2}$$

The z-transform system output is then given by

$$Y(z) = A + \frac{Bz}{z - e^{j\Omega}} + \frac{B^* z}{z - e^{-j\Omega}}$$

Taking the inverse z-transform, we achieve the following system transient and steady-state responses:

$$y(n) = \underbrace{0.5K \sin \Omega \delta(n)}_{y_{tr}(n)} + \underbrace{\left|H(e^{j\Omega})\right| e^{j\angle H(e^{j\Omega})} \frac{K}{j2} e^{jn\Omega} u(n) + \left|H(e^{j\Omega})\right| e^{-j\angle H(e^{j\Omega})} \frac{K}{-j2} e^{-jn\Omega} u(n)}_{y_{ss}(n)}$$

Simplifying the response yields the form

$$y(n) = 0.5K \sin \Omega \delta(n) + \left|H(e^{j\Omega})\right| K \frac{e^{jn\Omega + j\angle H(e^{j\Omega})} u(n) - e^{-jn\Omega - j\angle H(e^{j\Omega})} u(n)}{j2}$$

We can further combine the last term using Euler's formula to express the system response as

$$y(n) = \underbrace{0.5K \sin \Omega \delta(n)}_{y_{tr}(n) \text{ will decay to zero after the first sample}} + \underbrace{\left|H(e^{j\Omega})\right| K \sin\left(n\Omega + \angle H(e^{j\Omega})\right) u(n)}_{y_{ss}(n)}$$

Finally, the steady-state response is identified as

$$y_{ss}(n) = K\left|H(e^{j\Omega})\right| \sin(n\Omega + \angle H(e^{j\Omega}))u(n)$$

For this particular filter, the transient response exists for only the first sample in the system response. By substituting $n = 0$ into $y(n)$ and after simplifying algebra, we achieve the response for the first output sample:

$$y(0) = y_{tr}(0) + y_{ss}(0) = 0.5K\sin(\Omega) - 0.5K\sin(\Omega) = 0$$

Note that the first output sample of the transient response cancels the first output sample of the steady-state response, so the combined first output sample has a value of zero for this particular filter. The system response reaches the steady-state response after the first output sample. At this point, we can conclude that

$$\text{Steady-state magnitude frequency response} = \frac{\text{Peak amplitude of steady state response at } \Omega}{\text{Peak amplitude of sinusoidal input at } \Omega}$$

$$= \frac{\left|H(e^{j\Omega})\right|K}{K} = \left|H(e^{j\Omega})\right|$$

$$\text{Steady-state phase frequency response} = \text{Phase difference} = \angle H(e^{j\Omega})$$

Figure 6.12 shows the system response with sinusoidal inputs at $\Omega = 0.25\pi$, $\Omega = 0.5\pi$, and $\Omega = 0.75\pi$, respectively.

Next, we examine the properties of the filter frequency response $H(e^{j\Omega})$. From Euler's identity and the trigonometric identity, we know that

$$\begin{aligned} e^{j(\Omega+k2\pi)} &= \cos(\Omega + k2\pi) + j\sin(\Omega + k2\pi) \\ &= \cos\Omega + j\sin\Omega = e^{j\Omega} \end{aligned}$$

where k is an integer taking values of $k = 0, \pm1, \pm2, \cdots$. Then the frequency response has the following property (assuming all input sequences are real):

1. Periodicity
   a. Frequency response: $H(e^{j\Omega}) = H(e^{j(\Omega+k2\pi)})$
   b. Magnitude frequency response: $\left|H(e^{j\Omega})\right| = \left|H(e^{j(\Omega+k2\pi)})\right|$
   c. Phase response: $\angle H(e^{j\Omega}) = \angle H(e^{j\Omega+k2\pi})$

The second property is given without proof (see proof in Appendix D):

2. Symmetry
   a. Magnitude frequency response: $\left|H(e^{-j\Omega})\right| = \left|H(e^{j\Omega})\right|$
   b. Phase response: $\angle H(e^{-j\Omega}) = -\angle H(e^{j\Omega})$

Since the maximum frequency in a DSP system is the folding frequency, $f_s/2$, where $f_s = 1/T$, and T designates the sampling period, the corresponding maximum normalized frequency of the system frequency can be calculated as

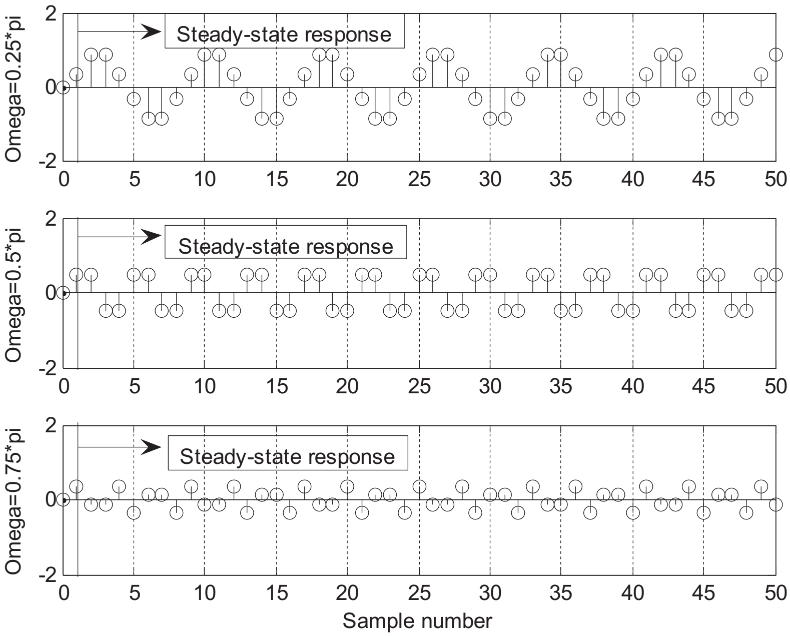

FIGURE 6.12

The digital filter responses to different input sinusoids.

$$\Omega = \omega T = 2\pi \frac{f_s}{2} \times T = \pi \text{ radians} \tag{6.22}$$

The frequency response $H(e^{j\Omega})$ for $|\Omega|\rangle\pi$ consists of the image replicas of $H(e^{j\Omega})$ for $|\Omega| \le \pi$ and will be removed via the reconstruction filter later. Hence, we need to evaluate $H(e^{j\Omega})$ for only the positive normalized frequency range from $\Omega = 0$ to $\Omega = \pi$ radians. The frequency, in Hz, can be determined by

$$f = \frac{\Omega}{2\pi} f_s \tag{6.23}$$

The magnitude frequency response is often expressed in decibels, defined as

$$\left|H(e^{j\Omega})\right|_{dB} = 20\log_{10}\left(\left|H(e^{j\Omega})\right|\right) \tag{6.24}$$

The DSP system stability, magnitude response, and phase response are investigated via the following examples.

EXAMPLE 6.10

Given the digital system

$$y(n) = 0.5x(n) + 0.5x(n-1)$$

with a sampling rate of 8,000 Hz, determine the frequency response.

Solution:

Taking the z-transform on both sides on the difference equation leads to

$$Y(z) = 0.5X(z) + 0.5z^{-1}X(z)$$

Then the transfer function describing the system is easily found to be

$$H(z) = \frac{Y(z)}{X(z)} = 0.5 + 0.5z^{-1}$$

Substituting $z = e^{j\Omega}$, we obtain the frequency response as

$$\begin{aligned} H(e^{j\Omega}) &= 0.5 + 0.5e^{-j\Omega} \\ &= 0.5 + 0.5\cos(\Omega) - j0.5\sin(\Omega). \end{aligned}$$

Therefore, the magnitude frequency response and phase response are given by

$$\left|H(e^{j\Omega})\right| = \sqrt{(0.5 + 0.5\cos(\Omega))^2 + (0.5\sin(\Omega))^2}$$

and

$$\angle H(e^{j\Omega}) = \tan^{-1}\left(\frac{-0.5\sin(\Omega)}{0.5 + 0.5\cos(\Omega)}\right)$$

Several points for the magnitude response and phase response are calculated and shown in Table 6.1. According to the data, we plot the frequency response and phase response of the DSP system in Figure 6.13.

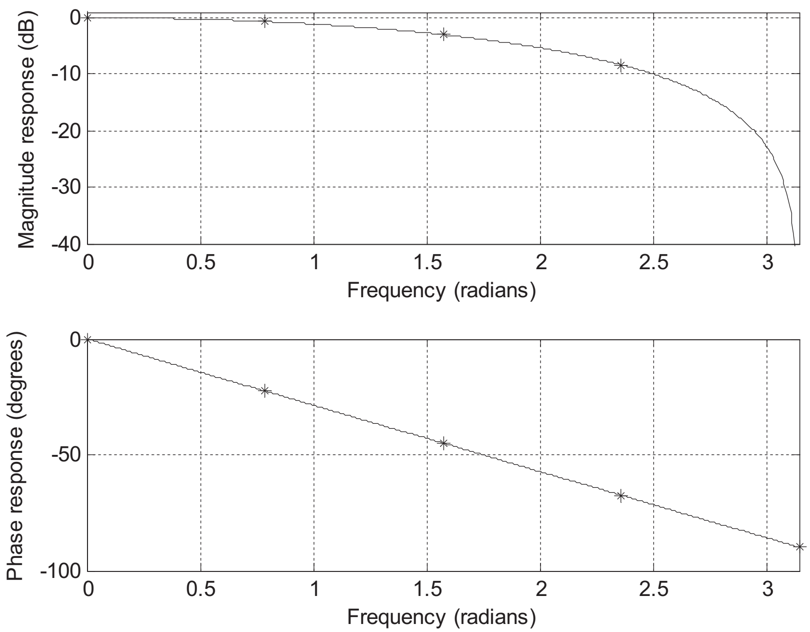

FIGURE 6.13

Frequency responses of the digital filter in Example 6.10.

Table 6.1 Frequency Response Calculations for Example 6.10

Ω (radians)	$f = \frac{\Omega}{2\pi} f_s$ (Hz)	$\lvert H(e^{j\Omega})\rvert$	$\lvert H(e^{j\Omega})\rvert_{dB}$	$\angle H(e^{j\Omega})$
0	0	1.000	0 dB	0^0
0.25π	1,000	0.924	−0.687 dB	-22.5^0
0.50π	2,000	0.707	−3.012 dB	-45.00^0
0.75π	3,000	0.383	−8.336 dB	-67.50^0
1.00π	4,000	0.000	$-\infty$	-90^0

It is observed that when the frequency increases, the magnitude response decreases. The DSP system acts like a digital lowpass filter, and its phase response is linear.

We can also verify the periodicity for $\lvert H(e^{j\Omega})\rvert$ and $\angle H(e^{j\Omega})$ when $\Omega = 0.25\pi + 2\pi$:

$$\begin{aligned}\lvert H(e^{j(0.25\pi+2\pi)})\rvert &= \sqrt{(0.5+0.5\cos(0.25\pi+2\pi))^2 + (0.5\sin(0.25\pi+2\pi))^2} \\ &= 0.924 = \lvert H(e^{j0.25\pi})\rvert\end{aligned}$$

$$\angle H(e^{j(0.25\pi+2\pi)}) = \tan^{-1}\left(\frac{-0.5\sin(0.25\pi+2\pi)}{0.5+0.5\cos(0.25\pi+2\pi)}\right) = -22.5^0 = \angle H(e^{j0.25\pi}).$$

For $\Omega = -0.25\pi$, we can verify the symmetry property as

$$\begin{aligned}\lvert H(e^{-j0.25\pi})\rvert &= \sqrt{(0.5+0.5\cos(-0.25\pi))^2 + (0.5\sin(-0.25\pi))^2} \\ &= 0.924 = \lvert H(e^{j0.25\pi})\rvert\end{aligned}$$

$$\angle H(e^{-j0.25\pi}) = \tan^{-1}\left(\frac{-0.5\sin(-0.25\pi)}{0.5+0.5\cos(-0.25\pi)}\right) = 22.5^0 = -\angle H(e^{j0.25\pi})$$

The properties can be observed in Figure 6.14, where the frequency range is chosen from $\Omega = -2\pi$ to $\Omega = 4\pi$ radians. As shown in the figure, the magnitude and phase responses are periodic with a period of 2π. For a period between $\Omega = -\pi$ to $\Omega = \pi$, the magnitude responses for the portion $\Omega = -\pi$ to $\Omega = 0$ and the portion $\Omega = 0$ to $\Omega = \pi$ are the same, while the phase responses are opposite. Since the magnitude and phase responses calculated for the range from $\Omega = 0$ to $\Omega = \pi$ are sufficient to present frequency response information, this range is only required for generating the frequency response plots.

Again, note that the phase plot shows a sawtooth shape instead of a linear straight line for this particular filter. This is due to the phase wrapping at $\Omega = 2\pi$ radians since $e^{j(\Omega+k2\pi)} = e^{j\Omega}$ is used in the calculation. However, the phase plot shows that the phase is linear in the range from $\Omega = 0$ to $\Omega = \pi$ radians.

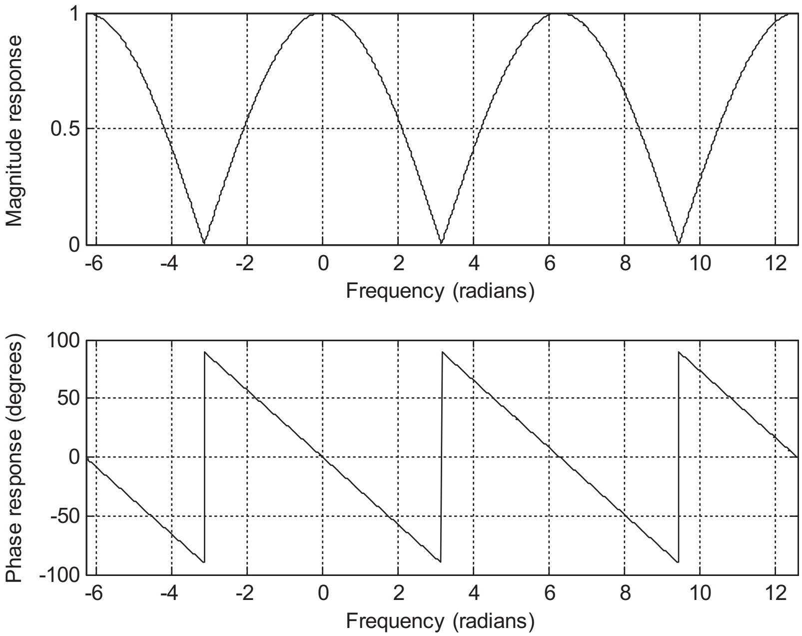

FIGURE 6.14

Periodicity of the magnitude response and phase response in Example 6.10.

EXAMPLE 6.11

Given a digital system

$$y(n) = x(n) - 0.5y(n-1)$$

with a sampling rate of 8,000 Hz, determine the frequency response.

Solution:

Taking the z-transform on both sides of the difference equation leads to

$$Y(z) = X(z) - 0.5z^{-1}Y(z)$$

Then the transfer function describing the system is easily found to be

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+0.5z^{-1}} = \frac{z}{z+0.5}$$

Substituting $z = e^{j\Omega}$, we have the frequency response as

$$H(e^{j\Omega}) = \frac{1}{1+0.5e^{-j\Omega}}$$

$$= \frac{1}{1+0.5\cos(\Omega) - j0.5\sin(\Omega)}$$

Table 6.2 Frequency Response Calculations in Example 6.11

Ω (radians)	$f = \frac{\Omega}{2\pi} f_s$ (Hz)	$\lvert H(e^{j\Omega}) \rvert$	$\lvert H(e^{j\Omega}) \rvert_{dB}$	$\angle H(e^{j\Omega})$
0	0	0.670	−3.479 dB	0^0
0.25π	1,000	0.715	−2.914 dB	14.64^0
0.50π	2,000	0.894	−0.973 dB	26.57^0
0.75π	3,000	1.357	2.652 dB	28.68^0
1.00π	4,000	2.000	6.021 dB	0^0

Therefore, the magnitude frequency response and phase response are given by

$$\left| H\left(e^{j\Omega}\right) \right| = \frac{1}{\sqrt{(1 + 0.5\cos(\Omega))^2 + (0.5\sin(\Omega))^2}}$$

and

$$\angle H(e^{j\Omega}) = -\tan^{-1}\left(\frac{-0.5\sin(\Omega)}{1 + 0.5\cos(\Omega)}\right)$$

Several points for the magnitude response and phase response are calculated and shown in Table 6.2. The magnitude response and phase response of the DSP system are roughly plotted in Figure 6.15 in accordance with the obtained data.

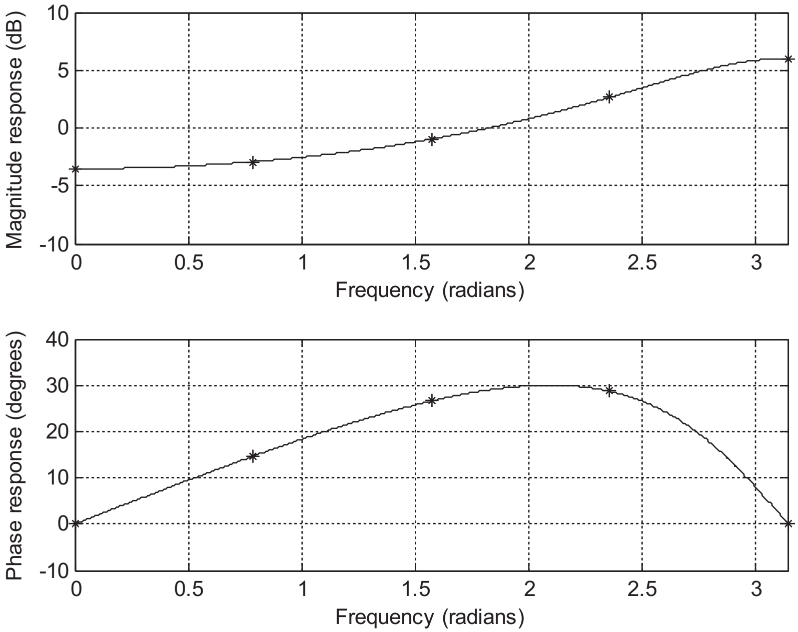

FIGURE 6.15

Frequency responses of the digital filter in Example 6.11.

From Table 6.2 and Figure 6.15, we can see that when the frequency increases, the magnitude response increases. The DSP system actually performs digital highpass filtering.

Notice that if all the coefficients a_i for $i = 0, 1, \cdots, M$ in Equation (6.1) are zeros, Equation (6.2) is reduced to

$$\begin{aligned} y(n) &= \sum_{i=0}^{M} b_i x(n-i) \\ &= b_0 x(n) + b_1 x(n-1) + \cdots + b_K x(n-M) \end{aligned} \tag{6.25}$$

Notice that b_i is the ith impulse response coefficient. Also, since M is a finite positive integer, b_i in this particular case is a finite set, $H(z) = B(z)$; note that the denominator $A(z) = 1$. Such systems are called *finite impulse response* (FIR) systems. If not all a_i in Equation (6.1) are zeros, the impulse response $h(i)$ would consist of an infinite number of coefficients. Such systems are called *infinite impulse response* (IIR) systems. The z-transform of the IIR $h(i)$, in general, is given by

$$H(z) = \frac{B(z)}{A(z)}$$

where $A(z) \neq 1$.

6.5 BASIC TYPES OF FILTERING

The basic filter types can be classified into four categories: *lowpass, highpass, bandpass*, and *bandstop*. Each of them finds a specific application in digital signal processing. One of the objectives in applications may involve the design of digital filters. In general, the filter is designed based on the specifications primarily for the passband, stopband, and transition band of the filter frequency response. The filter passband is the frequency range with the amplitude gain of the filter response being approximately unity. The filter stopband is defined as the frequency range over which the filter magnitude response is attenuated to eliminate the input signal whose frequency components are within that range. The transition band denotes the frequency range between the passband and stopband.

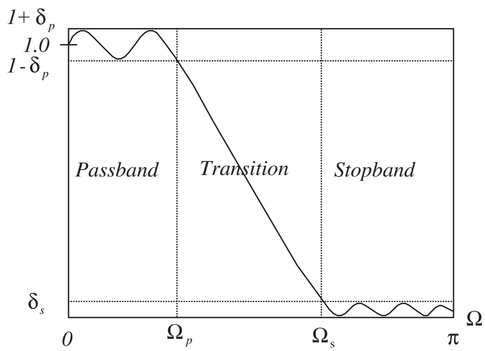

FIGURE 6.16

Magnitude response of the normalized lowpass filter.

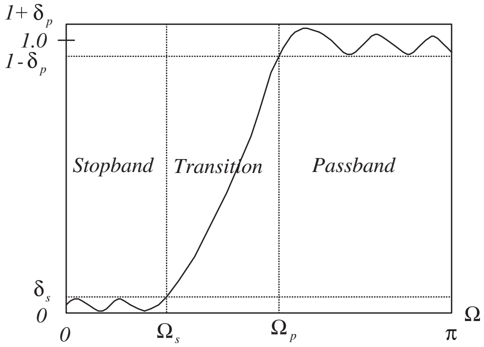

FIGURE 6.17

Magnitude response of the normalized highpass filter.

The design specifications of the lowpass filter are illustrated in Figure 6.16, where the low-frequency components are passed through the filter while the high-frequency components are attenuated. As shown in Figure 6.16, Ω_p and Ω_s are the passband cutoff frequency and the stopband cutoff frequency, respectively; δ_p is the design parameter to specify the ripple (fluctuation) of the frequency response in the passband; and δ_s specifies the ripple of the frequency response in the stopband.

The highpass filter keeps high-frequency components and rejects low-frequency components. The magnitude frequency response for the highpass filter is demonstrated in Figure 6.17.

The bandpass filter attenuates both low- and high-frequency components while keeping the middle-frequency components, as shown in Figure 6.18.

As illustrated in Figure 6.18, Ω_{pL} and Ω_{sL} are the lower passband cutoff frequency and lower stopband cutoff frequency, respectively. Ω_{pH} and Ω_{sH} are the upper passband cutoff frequency and upper stopband cutoff frequency, respectively. δ_p is the design parameter to specify the ripple of the frequency response in the passband, while δ_s specifies the ripple of the frequency response in the stopband.

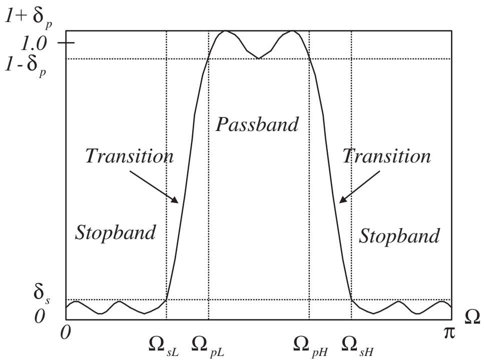

FIGURE 6.18

Magnitude response of the normalized bandpass filter.

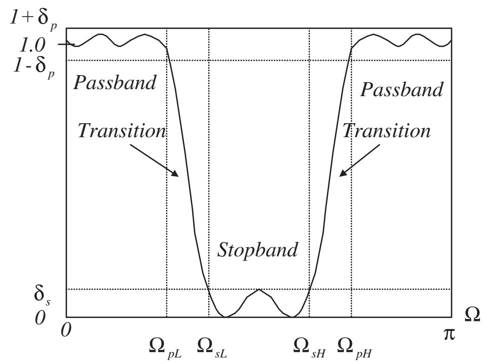

FIGURE 6.19

Magnitude of the normalized bandstop filter.

Finally, the bandstop (band reject or notch) filter shown in Figure 6.19 rejects the middle-frequency components and accepts both the low- and the high-frequency components.

As a matter of fact, all kinds of digital filters are implemented using FIR or IIR systems. Furthermore, the FIR and IIR systems can each be realized by various filter configurations, such as direct forms, cascade forms, and parallel forms. Such topics will be included in the next section.

Given a transfer function, the MATLAB function **freqz()** can be used to determine the frequency response. The syntax is given by

[h, w] = freqz(B, A, N)

where the parameters are defined as follows:

h = an output vector containing frequency response
w = an output vector containing normalized frequency values distributed in the range from 0 to π radians
B = an input vector for numerator coefficients
A = an input vector for denominator coefficients
N = the number of normalized frequency points used for calculating the frequency response

Let's consider Example 6.12.

EXAMPLE 6.12

Consider the following digital transfer function:

a. $H(z) = \dfrac{z}{z - 0.5}$

b. $H(z) = 1 - 0.5z^{-1}$

c. $H(z) = \dfrac{0.5z^2 - 0.32}{z^2 - 0.5z + 0.25}$

d. $H(z) = \dfrac{1 - 0.9z^{-1} + 0.81z^{-2}}{1 - 0.6z^{-1} + 0.36z^{-2}}$

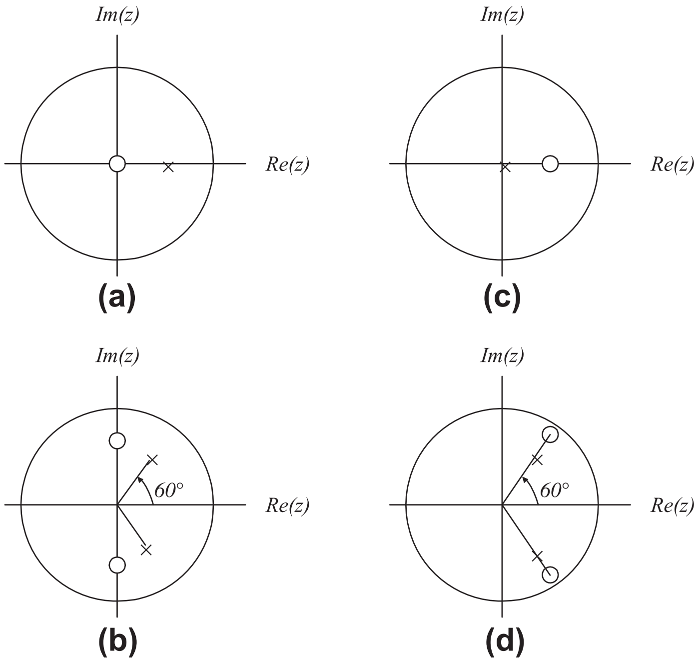

FIGURE 6.20

Pole-zero plots of Example 6.12.

1. Plot the poles and zeros on the z-plane.
2. Use the MATLAB function **freqz()** to plot the magnitude frequency response and phase response for each transfer function.
3. Identify the corresponding filter type (e.g., lowpass, highpass, bandpass, or bandstop).

Solution:

1. The pole-zero plot for each transfer function is demonstrated in Figure 6.20. The transfer functions of (a) and (c) need to be converted into the standard form (delay form) required by the MATLAB function **freqz(),** in which both numerator and denominator polynomials have negative powers of z. Hence, we obtain

$$H(z) = \frac{z}{z-0.5} = \frac{1}{1-0.5z^{-1}}$$

$$H(z) = \frac{0.5z^2-0.32}{z^2-0.5z+0.25} = \frac{0.5-0.32z^{-2}}{1-0.5z^{-1}+0.25z^{-2}}$$

while the transfer functions of (b) and (d) are already in their standard forms (delay forms).
2. The MATLAB program for plotting the magnitude frequency response and the phase response for each case is listed in Program 6.2.
3. From the plots in Figures 6.21A–6.21D of magnitude frequency responses for all cases, we can conclude that case (a) is a low pass filter, (b) is a high pass filter, (c) is a bandpass filter, and (d) is a bandstop (band reject) filter.

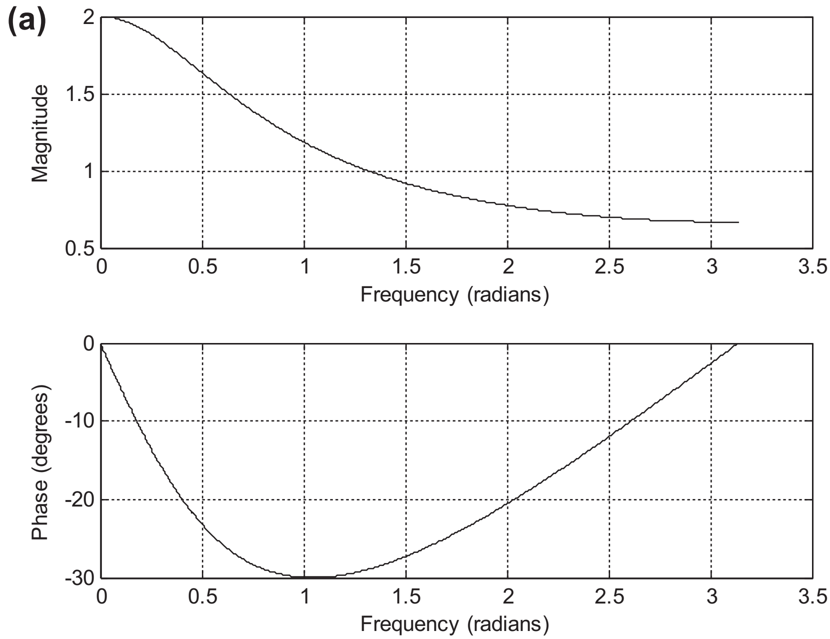

FIGURE 6.21A

Plots of frequency responses for Example 6.12 for (a).

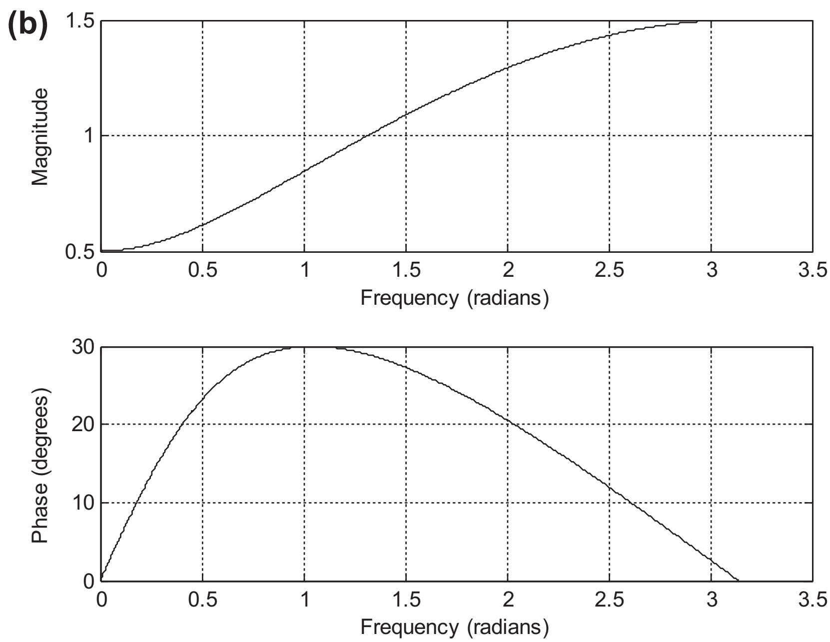

FIGURE 6.21B

Plots of frequency responses for Example 6.12 for (b).

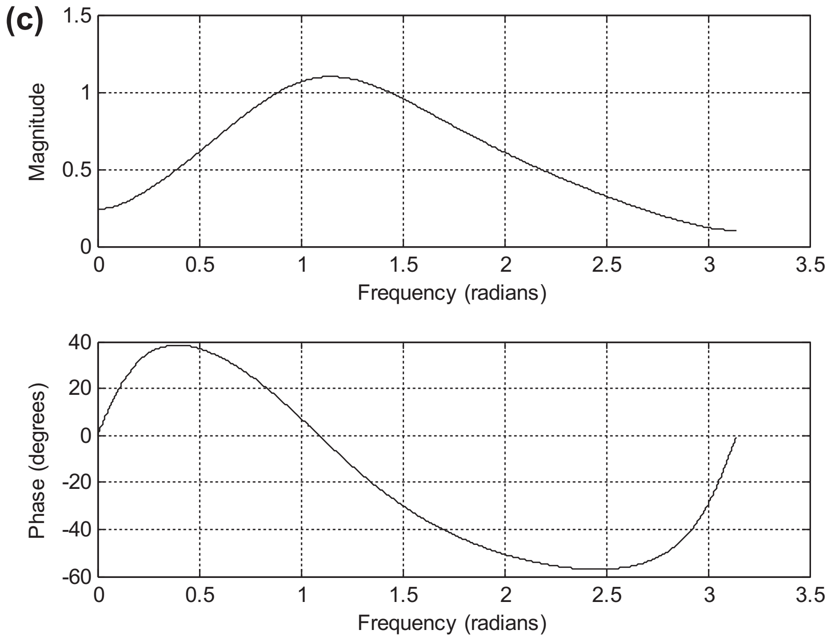

FIGURE 6.21C

Plots of frequency responses for Example 6.12 for (c).

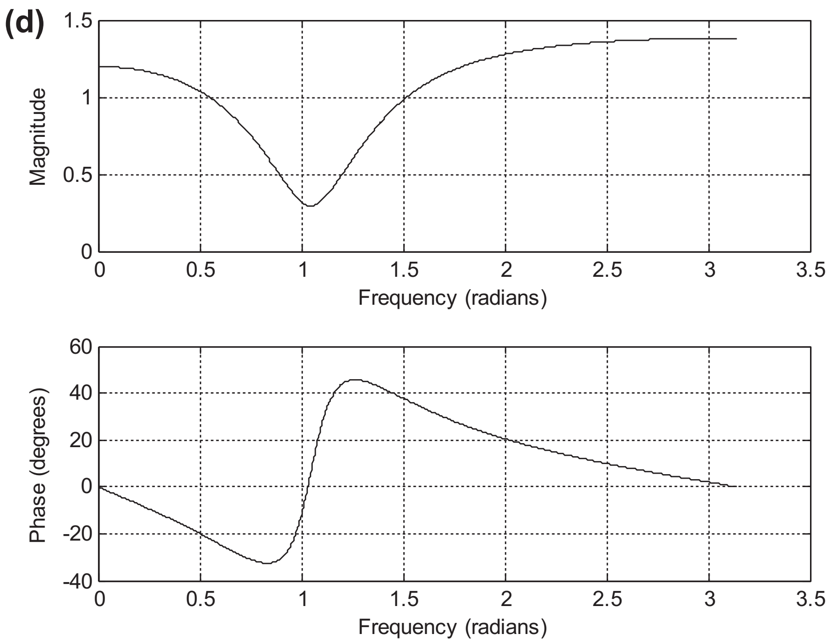

FIGURE 6.21D

Plots of frequency responses for Example 6.12 for (d).

Program 6.2. MATLAB program for Example 6.12.

```
% Example 6.12
% Plot the frequency response and phase response
% Case a
  figure (1)
  [h w] = freqz([1],[1 -0.5],1024);  % Calculate frequency response
  phi=180*unwrap(angle(h))/pi;
  subplot(2,1,1), plot(w,abs(h)),grid; xlabel('Frequency (radians)'),
ylabel('Magnitude')
subplot(2,1,2), plot(w,phi),grid; xlabel('Frequency (radians)'), ylabel('Phase
(degrees)')
  % Case b
  figure (2)
  [h w] = freqz([1 -0.5],[1],1024);  % Calculate frequency response
  phi=180*unwrap(angle(h))/pi;
  subplot(2,1,1), plot(w,abs(h)),grid;xlabel('Frequency (radians)'),
ylabel('Magnitude')
  subplot(2,1,2), plot(w,phi),grid; xlabel('Frequency (radians)'), ylabel('Phase
(degrees)')
  % Case c
  figure (3)
  [h w] = freqz([0.5 0 -0.32],[1 -0.5 0.25],1024); % Calculate frequency response
  phi=180*unwrap(angle(h))/pi;
subplot(2,1,1), plot(w,abs(h)),grid;
xlabel('Frequency (radians)'),ylabel('Magnitude')
subplot(2,1,2), plot(w,phi),grid;
xlabel('Frequency (radians)'), ylabel('Phase (degrees)')
 % Case d
figure (4)
[h w] = freqz([1 -0.9 0.81], [1 -0.6 0.36],1024); % Calculate frequency response
phi=180*unwrap(angle(h))/pi;
subplot(2,1,1), plot(w,abs(h)),grid; xlabel('Frequency (radians)'),
ylabel('Magnitude')
subplot(2,1,2), plot(w,phi),grid; xlabel('Frequency (radians)'), ylabel('Phase
(degrees)')
 %
```

6.6 REALIZATION OF DIGITAL FILTERS

In this section, basic realization methods for digital filters are discussed. Digital filters described by the transfer function $H(z)$ may be generally realized into the following forms:

- Direct-form I
- Direct-form II
- Cascade
- Parallel

(The reader can explore various lattice realizations in the textbook by Stearns and Hush [1990].)

6.6.1 Direct-Form I Realization

As we know, a digital filter transfer function, $H(z)$, is given by

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \tag{6.26}$$

Let $x(n)$ and $y(n)$ be the digital filter input and output, respectively. We can express the relationship in z-transform domain as

$$Y(z) = H(z)X(z) \tag{6.27}$$

where $X(z)$ and $Y(z)$ are the z-transforms of $x(n)$ and $y(n)$, respectively. If we substitute Equation (6.26) into $H(z)$ in Equation (6.27), we have

$$Y(z) = \left(\frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \right) X(z) \tag{6.28}$$

Taking the inverse of the z-transform of Equation (6.28), we yield the relationship between input $x(n)$ and output $y(n)$ in the time domain, as follows:

$$\begin{aligned} y(n) &= b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) \\ &\quad - a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N) \end{aligned} \tag{6.29}$$

This difference equation thus can be implemented by the direct-form I realization shown in Figure 6.22(a). Figure 6.22(b) illustrates the realization of the second-order IIR filter ($M = N = 2$). Note that the notation used in Figures 6.22(a) and (b) are defined in Figure 22(c) and will be applied for discussion of other realizations.

Notice that any of the a_j and b_i can be zero, thus not all the paths need to exist for realization.

6.6.2 Direct-Form II Realization

Considering Equations (6.26) and (6.27) with $N = M$, we can express

$$\begin{aligned} Y(z) &= H(z)X(z) = \frac{B(z)}{A(z)} X(z) = B(z)\left(\frac{X(z)}{A(z)}\right) \\ &= (b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}) \underbrace{\left(\frac{X(z)}{1 + a_1 z^{-1} + \cdots + a_M z^{-M}} \right)}_{W(z)} \end{aligned} \tag{6.30}$$

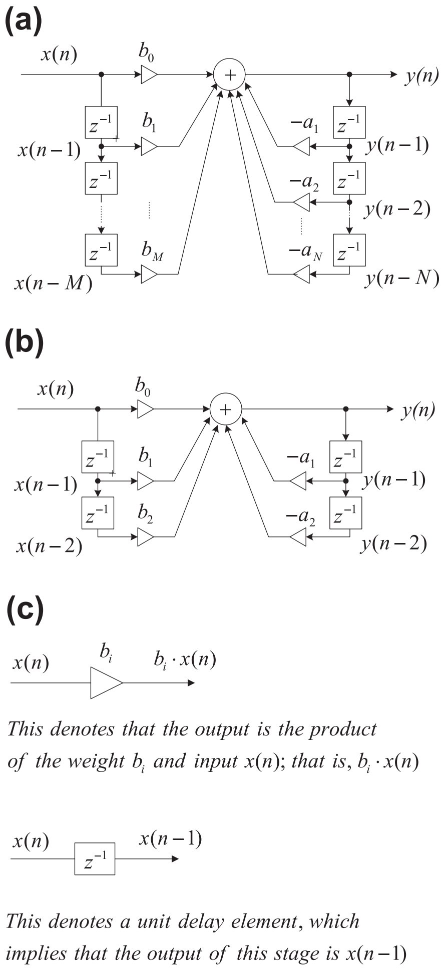

FIGURE 6.22

(a) Direct-form I realization; (b) direct-form I realization with $M = 2$; (c) notation.

Also, if we define a new z-transform function as

$$W(z) = \frac{X(z)}{1 + a_1 z^{-1} + \cdots + a_M z^{-M}} \tag{6.31}$$

we have

$$Y(z) = (b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}) W(z) \tag{6.32}$$

The corresponding difference equations for Equations (6.31) and (6.32), respectively, become

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - \cdots - a_M w(n-M) \tag{6.33}$$

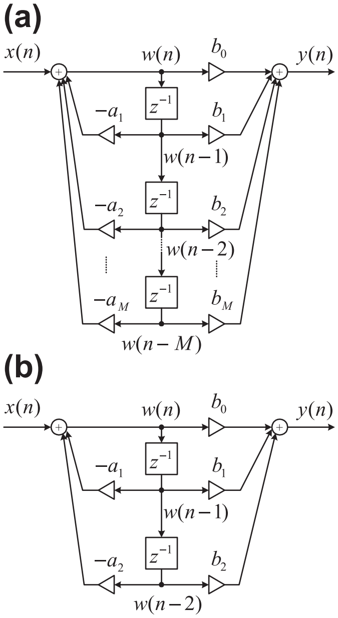

FIGURE 6.23

(a) Direct-form II realization; (b) direct-form II realization with $M = 2$.

and

$$y(n) = b_0 w(n) + b_1 w(n-1) + \ldots + b_M w(n-M) \tag{6.34}$$

Realization of Equations (6.33) and (6.34) produces another direct-form II realization, which is demonstrated in Figure 6.23(a). Again, the corresponding realization of the second-order IIR filter is described in Figure 6.23(b). Note that in Figure 6.23(a), the variables $w(n)$, $w(n-1)$, $w(n-2)$,…, $w(n-M)$ are different from the filter inputs $x(n-1)$, $x(n-2)$, …, $x(n-M)$.

6.6.3 Cascade (Series) Realization

An alternate way to filter realization is to cascade the factorized $H(z)$ in the following form:

$$H(z) = H_1(z) \cdot H_2(z) \cdots H_k(z) \tag{6.35}$$

where $H_k(z)$ is chosen to be the first- or second-order transfer function (section), which is defined by

$$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1}} \tag{6.36}$$

or

$$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}} \tag{6.37}$$

respectively. The block diagram of the cascade, or series, realization is depicted in Figure 6.24.

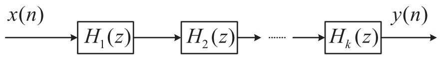

FIGURE 6.24

Cascade realization.

6.6.4 Parallel Realization

Now we convert $H(z)$ into the form

$$H(z) = H_1(z) + H_2(z) + \cdots + H_k(z) \tag{6.38}$$

where $H_k(z)$ is defined as the first- or second-order transfer function (section) given by

$$H_k(z) = \frac{b_{k0}}{1 + a_{k1}z^{-1}} \tag{6.39}$$

or

$$H_k(z) = \frac{b_{k0} + b_{k1}z^{-1}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}} \tag{6.40}$$

respectively. The resulting parallel realization is illustrated in the block diagram in Figure 6.25.

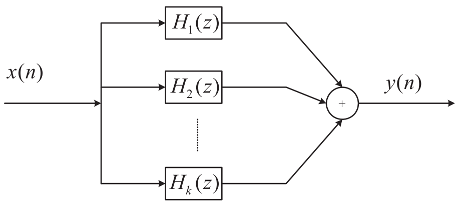

FIGURE 6.25

Parallel realization.

EXAMPLE 6.13

Given a second-order transfer function

$$H(z) = \frac{0.5(1 - z^{-2})}{1 + 1.3z^{-1} + 0.36z^{-2}}$$

perform the filter realizations and write difference equations using the following realizations:

a. direct-form I and direct-form II;
b. cascade form via first-order sections;
c. parallel form via first-order sections.

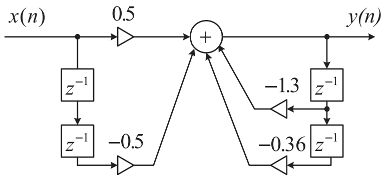

FIGURE 6.26

Direct-form I realization for Example 6.13.

Solution:

a. To perform the filter realizations using direct-form I and direct-form II, we rewrite the given second-order transfer function as

$$H(z) = \frac{0.5 - 0.5z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}$$

and identify that

$$a_1 = 1.3,\ a_2 = 0.36,\ b_0 = 0.5,\ b_1 = 0, \quad \text{and} \quad b_2 = -0.5$$

Based on the realizations in Figure 6.22, we sketch the direct-form I realization in Figure 6.26.
The difference equation for the direct-form I realization is given by

$$y(n) = 0.5x(n) - 0.5x(n-2) - 1.3y(n-1) - 0.36y(n-2)$$

Using the direct-form II realization shown in Figure 6.23, we present the realization in Figure 6.27. The difference equations for the direct-form II realization are expressed as

$$w(n) = x(n) - 1.3w(n-1) - 0.36w(n-2)$$

$$y(n) = 0.5w(n) - 0.5w(n-2)$$

b. To achieve the cascade (series) form realization, we factor $H(z)$ into two first-order sections to yield

$$H(z) = \frac{0.5(1 - z^{-2})}{1 + 1.3z^{-1} + 0.36z^{-2}} = \frac{0.5 - 0.5z^{-1}}{1 + 0.4z^{-1}} \frac{1 + z^{-1}}{1 + 0.9z^{-1}}$$

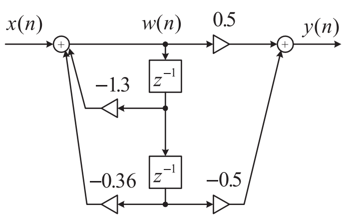

FIGURE 6.27

Direct-form II realization for Example 6.13.

where $H_1(z)$ and $H_2(z)$ are chosen to be

$$H_1(z) = \frac{0.5 - 0.5z^{-1}}{1 + 0.4z^{-1}}$$

$$H_2(z) = \frac{1 + z^{-1}}{1 + 0.9z^{-1}}$$

Notice that the obtained $H_1(z)$ and $H_2(z)$ are not the unique selections for realization. For example, there is another way of choosing $H_1(z) = \dfrac{0.5 - 0.5z^{-1}}{1 + 0.9z^{-1}}$ and $H_2(z) = \dfrac{1 + z^{-1}}{1 + 0.4z^{-1}}$ to yield the same $H(z)$. Using $H_1(z)$ and $H_2(z)$ we have obtained, and with the direct-form II realization, we achieve the cascade form depicted in Figure 6.28.

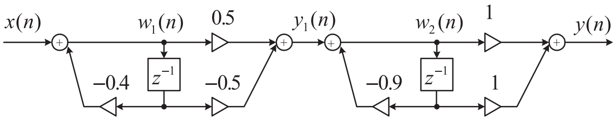

FIGURE 6.28

Cascade realization for Example 6.13.

The difference equations for the direct-form II realization have two cascaded sections, expressed as Section 1:

$$w_1(n) = x(n) - 0.4w(n-1)$$

$$y_1(n) = 0.5w_1(n) - 0.5w_1(n-1)$$

Section 2:

$$w_2(n) = y_1(n) - 0.9w_2(n-1)$$

$$y(n) = w_2(n) + w_2(n-1)$$

c. In order to yield the parallel form of realization, we need to make use of the partial fraction expansion, and we first let

$$\frac{H(z)}{z} = \frac{0.5(z^2-1)}{z(z+0.4)(z+0.9)} = \frac{A}{z} + \frac{B}{z+0.4} + \frac{C}{z+0.9}$$

where

$$A = z\left(\frac{0.5(z^2-1)}{z(z+0.4)(z+0.9)}\right)\bigg|_{z=0} = \frac{0.5(z^2-1)}{(z+0.4)(z+0.9)}\bigg|_{z=0} = -1.39$$

$$B = (z+0.4)\left(\frac{0.5(z^2-1)}{z(z+0.4)(z+0.9)}\right)\bigg|_{z=-0.4} = \frac{0.5(z^2-1)}{z(z+0.9)}\bigg|_{z=-0.4} = 2.1$$

$$C = \left(z+0.9\right)\left(\frac{0.5(z^2-1)}{z(z+0.4)(z+0.9)}\right)\bigg|_{z=-0.9} = \frac{0.5(z^2-1)}{z(z+0.4)}\bigg|_{z=-0.9} = -0.21$$

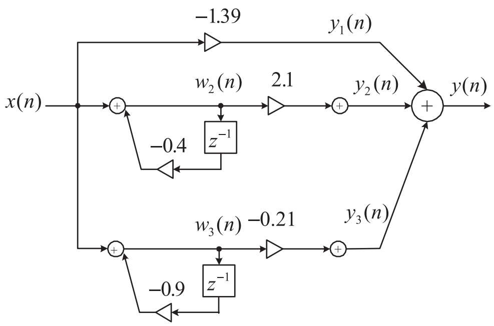

FIGURE 6.29

Parallel realization for Example 6.13.

Therefore

$$H(z) = -1.39 + \frac{2.1z}{z+0.4} + \frac{-0.21z}{z+0.9} = -1.39 + \frac{2.1}{1+0.4z^{-1}} + \frac{-0.21}{1+0.9z^{-1}}$$

Again, using the direct-form II realization for each section, we obtain the parallel realization in Figure 6.29.

The difference equations for the direct-form II realization have three parallel sections, expressed as

$$y_1(n) = -1.39x(n)$$

$$w_2(n) = x(n) - 0.4w_2(n-1)$$

$$y_2(n) = 2.1w_2(n)$$

$$w_3(n) = x(n) - 0.9w_3(n-1)$$

$$y_3(n) = -0.21w_3(n)$$

$$y(n) = y_1(n) + y_2(n) + y_3(n)$$

In practice, the second-order filter module with the direct-form I or direct-form II realization is used. The high-order filter can be factored in the cascade form with the first- or second-order sections. In cases where the first order-filter is required, we can still modify the second-order filter module by setting the corresponding filter coefficients to be zero.

6.7 APPLICATION: SIGNAL ENHANCEMENT AND FILTERING

This section investigates applications of signal enhancement using a pre-emphasis filter and speech filtering using a bandpass filter. Enhancement also includes biomedical signals such as electrocardiogram (ECG) signals.

6.7.1 Pre-Emphasis of Speech

A speech signal may have frequency components that fall off at high frequencies. In some applications such as speech coding, to avoid overlooking the high frequencies, the high frequency components are compensated using pre-emphasis filtering. A simple digital filter used for such compensation is given as

$$y(n) = x(n) - \alpha x(n-1) \tag{6.41}$$

where α is the positive parameter to control the degree of pre-emphasis filtering and usually is chosen to be less than 1. The filter described in Equation (6.41) is essentially a highpass filter. Applying z-transform on both sides of Equation (6.41) and solving for the transfer function, we have

$$H(z) = 1 - \alpha z^{-1} \tag{6.42}$$

The magnitude and phase responses adopting the pre-emphasis parameter $\alpha = 0.9$ and the sampling rate $f_s = 8{,}000$ Hz are plotted in Figure 6.30A using MATLAB.

Figure 6.30B compares the original speech waveform and the pre-emphasized speech using the filter in Equation (6.42). Again, we apply the fast Fourier transform (FFT) to estimate the spectrum of

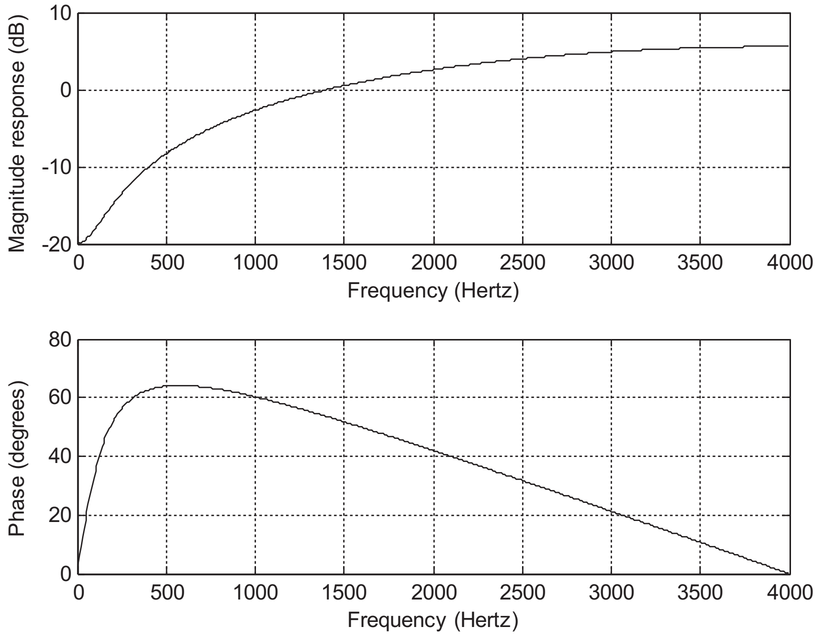

FIGURE 6.30A

Frequency responses of the pre-emphasis filter.

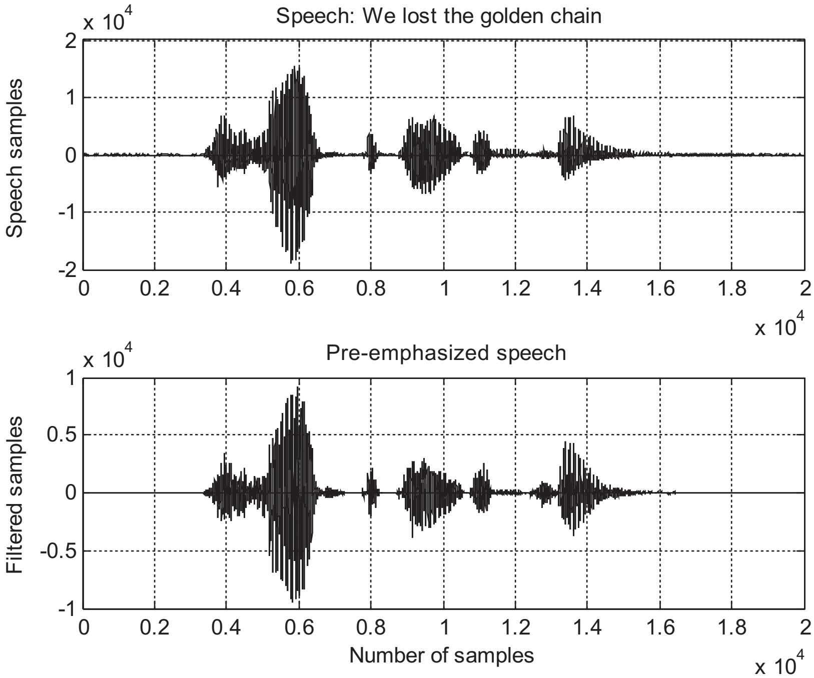

FIGURE 6.30B

Original speech and pre-emphasized speech waveforms.

the original speech and the spectrum of the pre-emphasized speech. The plots are displayed in Figure 6.31.

From Figure 6.31, we can conclude that the filter does its job to boost the high-frequency components and attenuate the low-frequency components. We can also try this filter with different values of α to examine the degree of the pre-emphasis filtering of the digitally recorded speech. The MATLAB list is in Program 6.3.

Program 6.3. MATLAB program for pre-emphasis of speech.

```
% MATLAB program for Figures 6.30 and 6.31
close all;clear all
fs=8000;                        % Sampling rate
alpha =0.9;                     % Degree of pre-emphasis
figure(1);
freqz([1 -alpha],1,512,fs);     % Calculate and display frequency response
load speech.dat
figure(2);
y=filter([1 -alpha],1,speech);  % Filtering speech
subplot(2,1,1),plot(speech);grid;
ylabel('Speech samples')
```

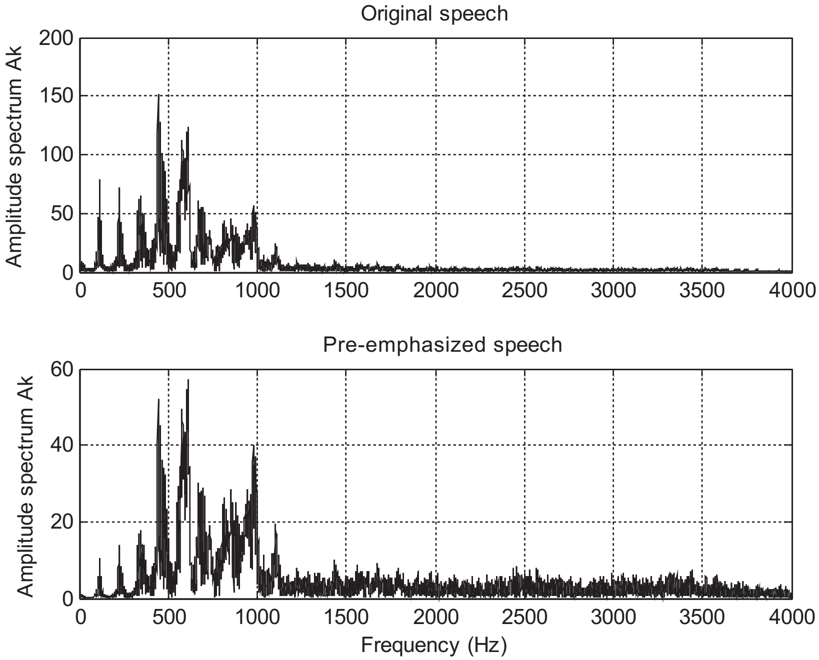

FIGURE 6.31

Amplitude spectral plots for the original speech and pre-emphasized speech.

```
title('Speech: We lost the golden chain.')
subplot(2,1,2),plot(y);grid
ylabel('Filtered samples')
xlabel('Number of samples');
title('Preemphasized speech.')
figure(3);
N=length(speech);                        % Length of speech
Axk=abs(fft(speech.*hamming(N)'))/N;     % Two-sided spectrum of speech
Ayk=abs(fft(y.*hamming(N)'))/N;          % Two-sided spectrum of pre-emphasized speech
f=[0:N/2]*fs/N;
Axk(2:N)=2*Axk(2:N);                     % Get one-sided spectrum of speech
Ayk(2:N)=2*Ayk(2:N);                     % Get one-sided spectrum of filtered speech
subplot(2,1,1),plot(f,Axk(1:N/2+1));grid
ylabel('Amplitude spectrum Ak')
title('Original speech');
subplot(2,1,2),plot(f,Ayk(1:N/2+1));grid
ylabel('Amplitude spectrum Ak')
xlabel('Frequency (Hz)');
title('Preemphasized speech');
%
```

6.7.2 Bandpass Filtering of Speech

Bandpass filtering plays an important role in DSP applications. It can be used to pass the signals according to the specified frequency passband and reject the frequency other than the passband specification. Then the filtered signal can be further used for the signal feature extraction. Filtering can also be applied to perform applications such as noise reduction, frequency boosting, digital audio equalizing, and digital crossover, among others.

Let us consider the following digital fourth-order bandpass Butterworth filter with a lower cutoff frequency of 1,000 Hz, an upper cutoff frequency of 1,400 Hz (that is, the bandwidth is 400 Hz), and a sampling rate of 8,000 Hz:

$$H(z) = \frac{0.0201 - 0.0402z^{-2} + 0.0201z^{-4}}{1 - 2.1192z^{-1} + 2.6952z^{-2} - 1.6924z^{-3} + 0.6414z^{-4}} \tag{6.43}$$

Converting the z-transfer function into the DSP difference equation yields

$$\begin{aligned} y(n) = {} & 0.0201x(n) - 0.0402x(n-2) + 0.0201x(n-4) \\ & +2.1192y(n-1) - 2.6952y(n-2) + 1.6924y(n-3) - 0.6414y(n-4) \end{aligned} \tag{6.44}$$

The filter frequency responses are computed and plotted in Figure 6.32A with MATLAB. Figure 6.32B shows the original speech and filtered speech, while Figure 6.32C displays the spectral plots for the original speech and filtered speech.

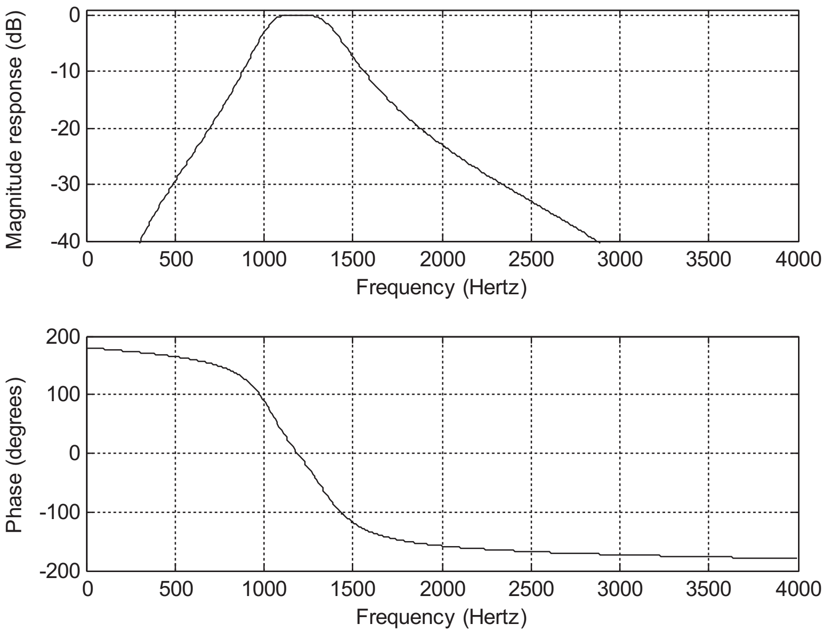

FIGURE 6.32A

Frequency responses of the designed bandpass filter.

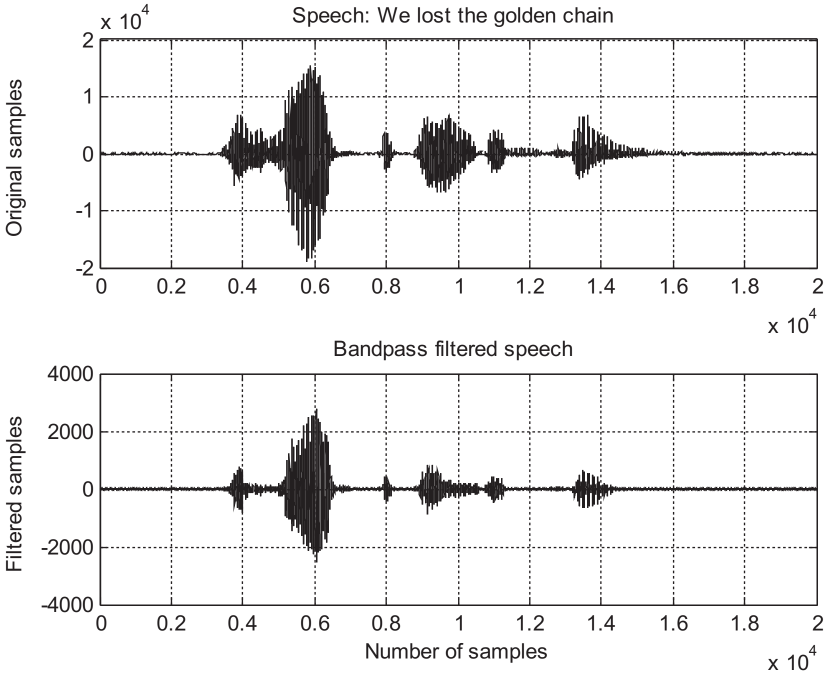

FIGURE 6.32B

Plots of the original speech and filtered speech.

As shown in Figure 6.32C, the designed bandpass filter significantly reduces low-frequency components, which are less than 1,000 Hz, and the high-frequency components above 1,400 Hz, while letting the signals with the frequencies ranging from 1,000 Hz to 1,400 Hz pass through the filter. Similarly, we can design and implement other types of filters, such as lowpass, highpass, bandpass, and band reject (bandstop) to filter the signals and examine the performance of their designs. MATLAB implementation details are given in Program 6.4.

Program 6.4. MATLAB program for bandpass filtering of speech.

```
fs=8000;                                % Sampling rate
freqz([0.0201 0.00 -0.0402 0 0.0201],[1 -2.1192 2.6952 -1.6924 0.6414],512,fs);
axis([0 fs/2 -40 1]);                   % Frequency response of bandpass filter
figure
load speech.dat
y=filter([0.0201 0.00 -0.0402 0.0201],[1 -2.1192 2.6952 -1.6924 0.6414],speech);
subplot(2,1,1),plot(speech); grid;   % Filtering speech
ylabel('Origibal Samples')
title('Speech: We lost the golden chain.')
subplot(2,1,2),plot(y);grid
xlabel('Number of Samples');ylabel('Filtered Samples')
```

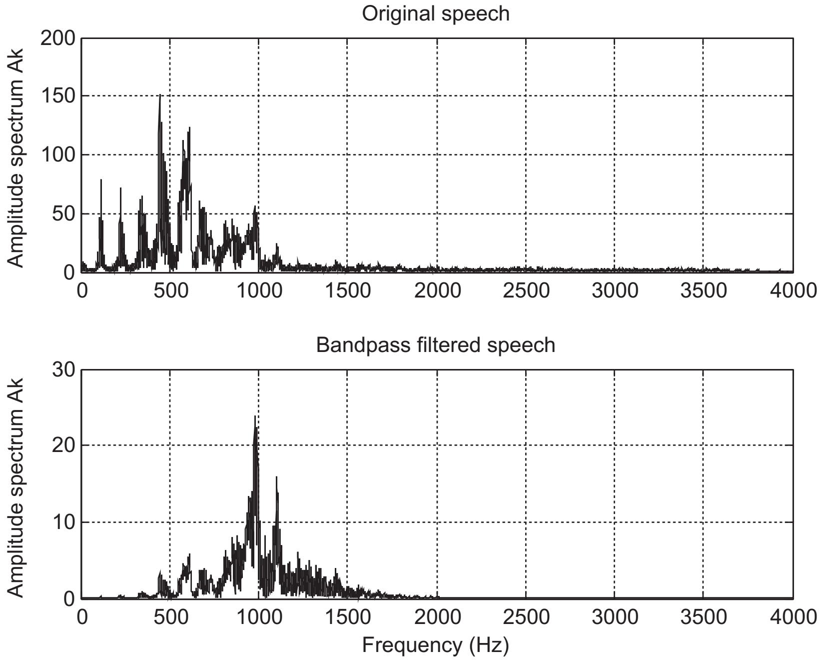

FIGURE 6.32C

Amplitude spectra of the original speech and bandpass filtered speech.

```
title('Bandpass filtered speech.')
figure
N=length(speech);
Axk=abs(fft(speech.*hamming(N)'))/N;        % One-sided spectrum of speech
Ayk=abs(fft(y.*hamming(N)'))/N;             % One-sided spectrum of filtered speech
f=[0:N/2]*fs/N;
Axk(2:N)=2*Axk(2:N);Ayk(2:N)=2*Ayk(2:N);    % One-sided spectra
subplot(2,1,1),plot(f,Axk(1:N/2+1));grid
ylabel('Amplitude spectrum Ak')
title('Original speech');
subplot(2,1,2),plot(f,Ayk(1:N/2+1));grid
ylabel('Amplitude spectrum Ak');xlabel('Frequency (Hz)');
title('Bandpass filtered speech');
```

6.7.3 Enhancement of ECG Signal Using Notch Filtering

A notch filter is a bandstop filter with a very narrow bandwidth. It can be applied to enhance an ECG signal that is corrupted during the data acquisition stage, where the signal is exposed to 60-Hz interference induced from the power line. Let us consider the following digital second-order notch

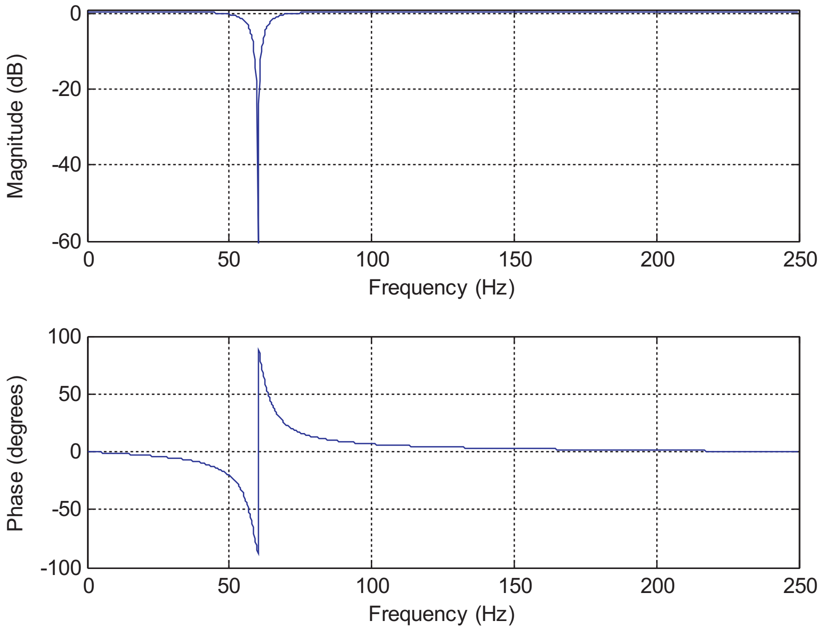

FIGURE 6.33

Notch filter frequency responses.

filter with a notch frequency of 60 Hz where the digital system has a sampling frequency of 500 Hz. We obtain a notch filter (details can be found in Chapter 8) as follows:

$$H(z) = \frac{1 - 1.4579z^{-1} + z^{-2}}{1 - 1.3850z^{-1} + 0.9025z^{-2}} \tag{6.45}$$

The DSP difference equation is expressed as

$$y(n) = x(n) - 1.4579x(n-1) + x(n-2) + 1.3850y(n-1) - 0.9025y(n-2) \tag{6.46}$$

The frequency responses are computed and plotted in Figure 6.33. Comparisons of the raw ECG signal corrupted by 60-Hz interference with the enhanced ECG signal for both the time domain and frequency domain are displayed in Figures 6.34 and 6.35, respectively. As we can see, the notch filter completely removes the 60-Hz interference.

6.8 SUMMARY

1. The digital filter (DSP system) is represented by a difference equation, which is linear and time invariant.
2. The filter output depends on the filter current input, past input(s), and past output(s) in general. Given arbitrary inputs and nonzero or zero initial conditions, operating the difference equation can generate the filter output recursively.

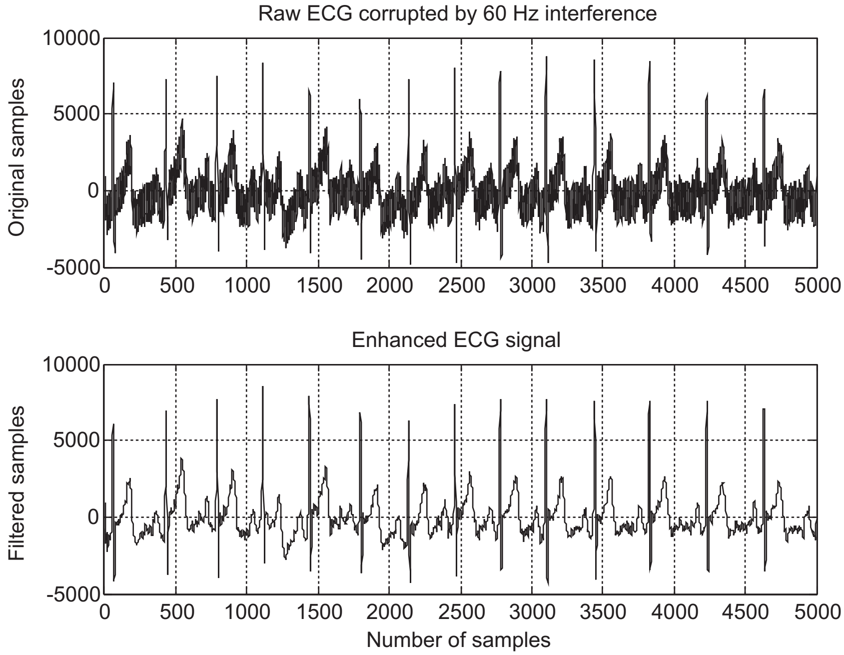

FIGURE 6.34

The corrupted ECG signal and the enhanced ECG signal.

3. System responses such as the impulse response and step response can be determined analytically using the z-transform.
4. The transfer function can be obtained by applying z-transform to the difference equation to determine the ratio of the output z-transform over the input z-transform. A digital filter (DSP system) can be represented by its transfer function.
5. System stability can be studied using a very useful tool, a z-plane pole-zero plot.
6. The frequency response of the DSP system was developed and illustrated to investigate the magnitude and phase responses. In addition, the FIR (finite impulse response) and IIR (infinite impulse response) systems were defined.
7. Digital filters and their specifications, such as lowpass, highpass, bandpass, and bandstop, were reviewed.
8. A digital filter can be realized using standard realization methods such as direct-form I; direct-form II; cascade, or series form; and parallel form.
9. Digital processing of speech using the pre-emphasis filter and bandpass filter was investigated to study spectral effects of the processed digital speech. The pre-emphasis filter boosts the high-frequency components, while bandpass filtering keeps the midband frequency components and rejects other lower- and upper-band frequency components.

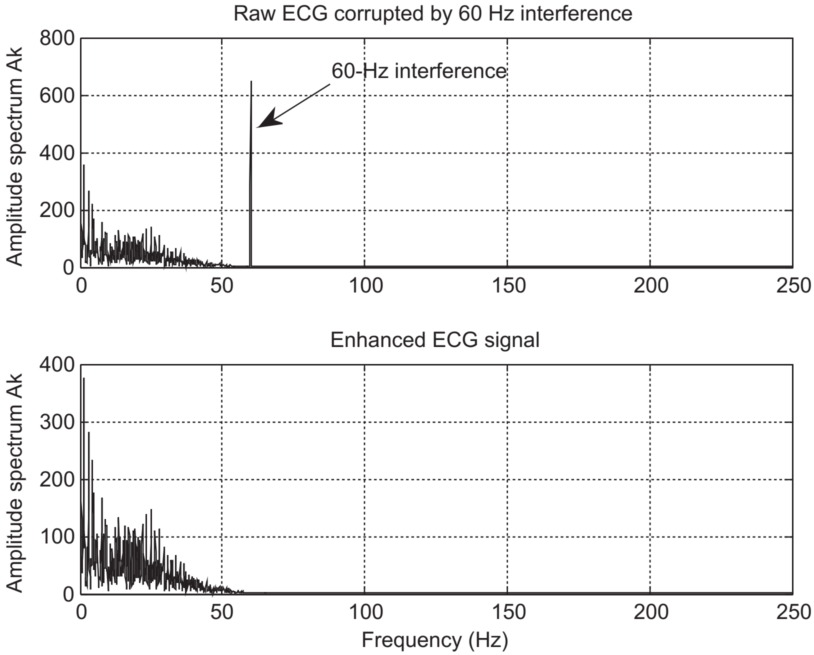

FIGURE 6.35

The corrupted ECG signal spectrum and the enhanced ECG signal spectrum.

6.9 PROBLEMS

6.1. Given the difference equation

$$y(n) = x(n) - 0.5y(n-1)$$

a. calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input $x(n) = (0.5)^n u(n)$ and initial condition $y(-1) = 1$;

b. calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input $x(n) = (0.5)^n u(n)$ and zero initial condition $y(-1) = 0$.

6.2. Given the difference equation

$$y(n) = 0.5x(n-1) + 0.6y(n-1)$$

a. calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input $x(n) = (0.5)^n u(n)$ and initial conditions $x(-1) = -1$, and $y(-1) = 1$;

b. calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input $x(n) = (0.5)^n u(n)$ and zero initial conditions $x(-1) = 0$, and $y(-1) = 0$.

6.3. Given the difference equation

$$y(n) = x(n-1) - 0.75y(n-1) - 0.125y(n-2)$$

a. calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input $x(n) = (0.5)^n u(n)$ and initial conditions: $x(-1) = -1$, $y(-2) = 2$, and $y(-1) = 1$;

b. calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input $x(n) = (0.5)^n u(n)$ and zero initial conditions: $x(-1) = 0$, $y(-2) = 0$, and $y(-1) = 0$.

6.4. Given the difference equation

$$y(n) = 0.5x(n) + 0.5x(n-1)$$

a. find $H(z)$;

b. determine the impulse response $y(n)$ if the input $x(n) = 4\delta(n)$;

c. determine the step response $y(n)$ if the input $x(n) = 10u(n)$.

6.5. Given the difference equation,

$$y(n) = x(n) - 0.5y(n-1)$$

a. find $H(z)$;

b. determine the impulse response $y(n)$ if the input $x(n) = \delta(n)$;

c. determine the step response $y(n)$ if the input $x(n) = u(n)$.

6.6. A digital system is described by the following difference equation:

$$y(n) = x(n) - 0.25x(n-2) - 1.1y(n-1) - 0.28y(n-2)$$

Find the transfer function $H(z)$, the denominator polynomial $A(z)$, and the numerator polynomial $B(z)$.

6.7. A digital system is described by the following difference equation:

$$y(n) = 0.5x(n) + 0.5x(n-1) - 0.6y(n-2)$$

Find the transfer function $H(z)$, the denominator polynomial $A(z)$, and the numerator polynomial $B(z)$.

6.8. A digital system is described by the following difference equation:

$$y(n) = 0.25x(n-2) + 0.5y(n-1) - 0.2y(n-2)$$

Find the transfer function $H(z)$, the denominator polynomial $A(z)$, and the numerator polynomial $B(z)$.

6.9. A digital system is described by the following difference equation:

$$y(n) = x(n) - 0.3x(n-1) + 0.28x(n-2)$$

Find the transfer function $H(z)$, the denominator polynomial $A(z)$, and the numerator polynomial $B(z)$.

6.10. Convert each of the following transfer functions into difference equations:

a. $H(z) = 0.5 + 0.5z^{-1}$

b. $H(z) = \dfrac{1}{1 - 0.3z^{-1}}$

6.11. Convert each of the following transfer functions into difference equations:

a. $H(z) = 0.1 + 0.2z^{-1} + 0.3z^{-2}$

b. $H(z) = \dfrac{0.5 - 0.5z^{-2}}{1 - 0.3z^{-1} + 0.8z^{-2}}$

6.12. Convert each of the following transfer functions into difference equations:

a. $H(z) = \dfrac{z^2 - 0.25}{z^2 + 1.1z + 0.18}$

b. $H(z) = \dfrac{z^2 - 0.1z + 0.3}{z^3}$

6.13. Convert each of the following transfer functions into pole-zero form:

a. $H(z) = \dfrac{z^2 + 2z + 1}{z^2 + 5z + 6}$

b $H(z) = \dfrac{1 - 0.16z^{-2}}{1 + 0.7z^{-1} + 0.1z^{-2}}$

c $H(z) = \dfrac{z^2 + 4z + 5}{z^3 + 2z^2 + 6z}$

6.14. A transfer function depicting a discrete-time system is given by

$$H(z) = \frac{10(z+1)}{(z+0.75)}$$

a. Determine the impulse response $h(n)$ and step response.

b. Determine the system response $y(n)$ if the input is $x(n) = (0.25)^n u(n)$.

6.15. Given each of the following transfer functions that describe digital systems, sketch the z-plane pole-zero plot and determine the stability for each digital system.

a. $H(z) = \dfrac{z - 0.5}{(z + 0.25)(z^2 + z + 0.8)}$

b. $H(z) = \dfrac{z^2 + 0.25}{(z - 0.5)(z^2 + 4z + 7)}$

c. $H(z) = \dfrac{z + 0.95}{(z + 0.2)(z^2 + 1.414z + 1)}$

d. $H(z) = \dfrac{z^2 + z + 0.25}{(z - 1)(z + 1)^2(z - 0.36)}$

6.16. Given the digital system

$$y(n) = 0.5x(n) + 0.5x(n-2)$$

with a sampling rate of 8,000 Hz,

a. determine the frequency response;

b. calculate and plot the magnitude and phase frequency responses;

c. determine the filter type based the magnitude frequency response.

6.17. Given the digital system,

$$y(n) = 0.5x(n-1) + 0.5x(n-2)$$

with a sampling rate of 8,000 Hz,

a. determine the frequency response;

b. calculate and plot the magnitude and phase frequency responses;

c. determine the filter type based the magnitude frequency response.

6.18. For the digital system

$$y(n) = 0.5x(n) + 0.5y(n-1)$$

with a sampling rate of 8,000 Hz,

a. determine the frequency response;

b. calculate and plot the magnitude and phase frequency responses;

c. determine the filter type based the magnitude frequency response.

6.19. For the digital system

$$y(n) = x(n) - 0.5y(n-2)$$

with a sampling rate of 8,000 Hz,

a. determine the frequency response;

b. calculate and plot the magnitude and phase frequency responses;

c. determine the filter type based the magnitude frequency response.

6.20. Given the difference equation

$$y(n) = x(n) - 2 \cdot \cos(\alpha)x(n-1) + x(n-2) + 2\gamma \cdot \cos(\alpha)y(n-1) - \gamma^2 y(n-2)$$

where $\gamma = 0.8$ and $\alpha = 60^0$,

a. find the transfer function $H(z)$;

b. plot the poles and zeros on the z-plan with the unit circle;

c. determine the stability of the system from the pole-zero plot;

d. calculate the amplitude (magnitude) response of $H(z)$;

e. calculate the phase response of $H(z)$.

6.21. For the difference equations

a. $y(n) = 0.5x(n) + 0.5x(n-1)$

b $y(n) = 0.5x(n) - 0.5x(n-1)$

c. $y(n) = 0.5x(n) + 0.5x(n-2)$

d. $y(n) = 0.5x(n) - 0.5x(n-2)$

1. find $H(z)$;

2. calculate the magnitude response;

3. specify the filtering type based on the calculated magnitude response.

6.22. Given an IIR system expressed as

$$y(n) = 0.5x(n) + 0.2y(n-1), \; y(-1) = 0$$

a. find $H(z)$;

b. find the system response $y(n)$ due to the input $x(n) = (0.5)^n u(n)$.

6.23. Given the IIR system

$$y(n) = 0.5x(n) - 0.7y(n-1) - 0.1y(n-2)$$

with zero initial conditions,

a. find $H(z)$;

b. find the unit step response.

6.24. Given the first-order IIR system

$$H(z) = \frac{1 + 2z^{-1}}{1 - 0.5z^{-1}}$$

realize $H(z)$ and develop the difference equations using the following forms:

a. direct-form I;

b. direct-form II.

6.25. Given the filter

$$H(z) = \frac{1 - 0.9z^{-1} - 0.1z^{-2}}{1 + 0.3z^{-1} - 0.04z^{-2}}$$

realize $H(z)$ and develop difference equations using the following forms:

a. direct-form I;

b. direct-form II;

c. cascade (series) form via the first-order sections;

d. parallel form via the first-order sections.

6.26. Given the pre-emphasis filters:

$$H(z) = 1 - 0.5z^{-1}$$

$$H(z) = 1 - 0.7z^{-1}$$

$$H(z) = 1 - 0.9z^{-1}$$

a. write the difference equation for each;

b. determine which emphasizes high frequency components most.

6.9.1 MATLAB Problems

6.27. Given a filter

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.5z^{-1} + 0.25z^{-2}}$$

use MATLAB to plot

a. its magnitude frequency response;

b. its phase response.

6.28. Given a difference equation

$$y(n) = x(n-1) - 0.75y(n-1) - 0.125y(n-2)$$

a. use the MATLAB functions **filter()** and **filtic()** to calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input of $x(n) = (0.5)^n u(n)$ and initial conditions $x(-1) = -1$, $y(-2) = 2$, and $y(-1) = 1$;

b. use the MATLAB function **filter()** to calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input of $x(n) = (0.5)^n u(n)$ and zero initial conditions $x(-1) = 0$, $y(-2) = 0$, and $y(-1) = 0$.

6.29. Given a filter

$$H(z) = \frac{1 - z^{-1} + z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

a. plot the magnitude frequency response and phase response using MATLAB;

b. specify the type of filtering;

c. find the difference equation;

d. perform filtering, that is, calculate $y(n)$ for the first 1,000 samples for each of the following inputs and plot the filter outputs using MATLAB, assuming that all initial conditions are zeros and the sampling rate is 8,000 Hz:

$$x(n) = \cos\left(\pi \cdot 10^3 \frac{n}{8{,}000}\right)$$

$$x(n) = \cos\left(\frac{8}{3}\pi \cdot 10^3 \frac{n}{8{,}000}\right)$$

$$x(n) = \cos\left(6\pi \cdot 10^3 \frac{n}{8{,}000}\right)$$

e. repeat (d) using the MATLAB function **filter().**

6.30. Repeat (d) in Problem 6.29 using direct-form II structure.

6.9.2 MATLAB Projects

6.31. Sound effects of pre-emphasis filtering:

A pre-emphasis filter is shown in Figure 6.36 with a selective parameter $0 \le \alpha < 1$, which controls the degree of pre-emphasis filtering. Assuming the system has a sampling rate of 8,000 Hz, plot the frequency responses for $\alpha = 0$, $\alpha = 0.4$, $\alpha = 0.8$, $\alpha = 0.95$, $\alpha = 0.99$, respectively. For each case, apply the pre-emphasis filter to the given speech ("speech.dat") and discuss the sound effects.

6.32. Echo generation (sound regeneration):

Echo is the repetition of sound due to sound wave reflection from the objects. It can easily be generated using an FIR filter such as that in Figure 6.37, where $|\alpha| < 1$ is an attenuation

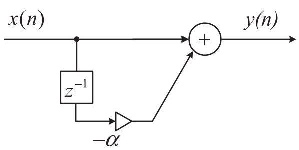

FIGURE 6.36

A pre-emphasis filter.

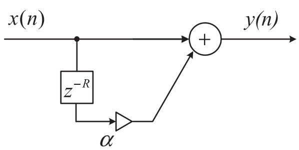

FIGURE 6.37

A single echo generator using an FIR filter.

factor and R the delay of the echo. The echo signal is generated by the sum of a delayed version of sound with the attenuation of α and the nondelayed version.

However, a single echo generator may not be useful, so a multiple-echo generator using an IIR filter is usually applied, as shown in Figure 6.38.

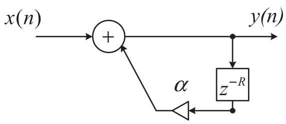

FIGURE 6.38

A multiple-echo generator using an IIR filter.

a. Assuming the system has a sampling rate of 8,000 Hz, plot the IIR filter frequency responses for the following cases: $\alpha = 0.5$ and $R = 1$; $\alpha = 0.6$ and $R = 4$; $\alpha = 0.7$ and $R = 10$, and characterize the frequency responses.

b. Implement the multiple-echo generator using the following code:

```
y=filter([1], [1 zeros(1, R-1) alpha], x)
```

Following that, evaluate the sound effects of the speech file ("speech.dat") for the following cases: $\alpha = 0.5$ and $R = 500$ (62.5 ms); $\alpha = 0.7$ and $R = 1000$ (125 ms); $\alpha = 0.5$, $R = 2000$ (250 ms); and $\alpha = 0.5$, $R = 4000$ (500 ms).