CHAPTER

The z-Transform

5

CHAPTER OUTLINE

OBJECTIVES

This chapter introduces the z-transform and its properties; illustrates how to determine the inverse z-transform using partial fraction expansion; and applies the z-transform to solve linear difference equations.

5.1 DEFINITION

The *z-transform* is a very important tool in describing and analyzing digital systems. It also supports the techniques for digital filter design and frequency analysis of digital signals. We begin with the definition of the z-transform.

The z-transform of a causal sequence $x(n)$, designated by $X(z)$ or $Z(x(n))$, is defined as

$$\begin{aligned} X(z) &= Z(x(n)) = \sum_{n=0}^{\infty} x(n) z^{-n} \\ &= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \cdots \end{aligned} \tag{5.1}$$

where z is the complex variable. Here, the summation taken from $n = 0$ to $n = \infty$ is according to the fact that for most situations, the digital signal $x(n)$ is the causal sequence, that is, $x(n) = 0$ for $n < 0$. Thus, the definition in Equation (5.1) is referred to as a *one-sided z-transform* or a *unilateral transform*. In Equation (5.1), all the values of z that make the summation exist form a *region of convergence* in the z-transform domain, while all other values of z outside the region of convergence will cause the summation to diverge. The region of convergence is defined based on the particular sequence $x(n)$ being applied. Note that we deal with the unilateral z-transform in this book, and hence when

Digital Signal Processing. http://dx.doi.org/10.1016/B978-0-12-415893-1.00005-6
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performing inverse z-transform (which we shall study later), we are restricted to the causal sequence. Now let us study the following typical examples.

EXAMPLE 5.1

Given the sequence

$$x(n) = u(n)$$

find the z-transform of $x(n)$.

Solution:

From the definition of Equation (5.1), the z-transform is given by

$$X(z) = \sum_{n=0}^{\infty} u(n)z^{-n} = \sum_{n=0}^{\infty} \left(z^{-1}\right)^n = 1 + \left(z^{-1}\right) + \left(z^{-1}\right)^2 + \cdots$$

This is an infinite geometric series that converges to

$$X(z) = \frac{z}{z-1}$$

with a condition $|z^{-1}|<1$. Note that for an infinite geometric series, we have $1 + r + r^2 + \cdots = \frac{1}{1-r}$ when $|r|<1$. The region of convergence for all values of$z$ is given as $|z| > 1$.

EXAMPLE 5.2

Consider the exponential sequence

$$x(n) = a^n u(n)$$

and find the z-transform of the sequence $x(n)$.

Solution:

From the definition of the z-transform in Equation (5.1), it follows that

$$X(z) = \sum_{n=0}^{\infty} a^n u(n)z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = 1 + \left(az^{-1}\right) + \left(az^{-1}\right)^2 + \cdots$$

Since this is a geometric series that will converge for $|az^{-1}|<1$, it is further expressed as

$$X(z) = \frac{z}{z-a}, \quad \text{for } |z| > |a|$$

The z-transforms for common sequences are summarized in Table 5.1. Example 5.3 illustrates how to find the z-transform using Table 5.1.

Table 5.1 Table of z-Transform Pairs

Line No.	$x(n),\ n \geq 0$	z-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n)z^{-n}$	
2	$\delta(n)$	1	$\|z\| > 0$
3	$au(n)$	$\dfrac{az}{z-1}$	$\|z\| > 1$
4	$nu(n)$	$\dfrac{z}{(z-1)^2}$	$\|z\| > 1$
5	$n^2 u(n)$	$\dfrac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
6	$a^n u(n)$	$\dfrac{z}{z-a}$	$\|z\| > \|a\|$
7	$e^{-na} u(n)$	$\dfrac{z}{(z-e^{-a})}$	$\|z\| > e^{-a}$
8	$na^n u(n)$	$\dfrac{az}{(z-a)^2}$	$\|z\| > \|a\|$
9	$\sin(an)u(n)$	$\dfrac{z\sin(a)}{z^2 - 2z\cos(a) + 1}$	$\|z\| > 1$
10	$\cos(an)u(n)$	$\dfrac{z[z-\cos(a)]}{z^2 - 2z\cos(a) + 1}$	$\|z\| > 1$
11	$a^n \sin(bn)u(n)$	$\dfrac{[a\sin(b)]z}{z^2 - [2a\cos(b)]z + a^2}$	$\|z\| > \|a\|$
12	$a^n \cos(bn)u(n)$	$\dfrac{z[z - a\cos(b)]}{z^2 - [2a\cos(b)]z + a^2}$	$\|z\| > \|a\|$
13	$e^{-an} \sin(bn)u(n)$	$\dfrac{[e^{-a}\sin(b)]z}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\|z\| > e^{-a}$
14	$e^{-an} \cos(bn)u(n)$	$\dfrac{z[z - e^{-a}\cos(b)]}{z^2 - [2e^{-a}\cos(b)]z + e^{-2a}}$	$\|z\| > e^{-a}$
15	$2\|A\|\|P\|^n \cos(n\theta + \varphi)u(n)$ where P and A are complex constants defined by $P = \|P\|\angle\theta,\ A = \|A\|\angle\varphi$	$\dfrac{Az}{z-P} + \dfrac{A^*z}{z-P^*}$	

EXAMPLE 5.3

Find the z-transform for each of the following sequences:

a. $x(n) = 10u(n)$

b. $x(n) = 10\sin(0.25\pi n)u(n)$

c. $x(n) = (0.5)^n u(n)$
d. $x(n) = (0.5)^n \sin(0.25\pi n)u(n)$
e. $x(n) = e^{-0.1n}\cos(0.25\pi n)u(n)$

Solution:

a. From Line 3 in Table 5.1, we get

$$X(z) = Z(10u(n)) = \frac{10z}{z-1}$$

b. Line 9 in Table 5.1 leads to

$$X(z) = 10Z(\sin(0.2\pi n)u(n))$$
$$= \frac{10\sin(0.25\pi)z}{z^2 - 2z\cos(0.25\pi) + 1} = \frac{7.07z}{z^2 - 1.414z + 1}$$

c. From Line 6 in Table 5.1, we obtain

$$X(z) = Z((0.5)^n u(n)) = \frac{z}{z-0.5}$$

d. From Line 11 in Table 5.1, it follows that

$$X(z) = Z((0.5)^n \sin(0.25\pi n)u(n)) = \frac{0.5 \times \sin(0.25\pi)z}{z^2 - 2 \times 0.5\cos(0.25\pi)z + 0.5^2}$$
$$= \frac{0.3536z}{z^2 - 0.7071z + 0.25}$$

e. From Line 14 in Table 5.1, it follows that

$$X(z) = Z(e^{-0.1n}\cos(0.25\pi n)u(n)) = \frac{z(z - e^{-0.1}\cos(0.25\pi))}{z^2 - 2e^{-0.1}\cos(0.25\pi)z + e^{-0.2}}$$
$$= \frac{z(z-0.6397)}{z^2 - 1.2794z + 0.8187}$$

5.2 PROPERTIES OF THE Z-TRANSFORM

In this section, we study some important properties of the z-transform. These properties are widely used in deriving the z-transfer functions of difference equations and solving the system output responses of linear digital systems with constant system coefficients, which will be discussed in the next chapter.

Linearity: The z-transform is a linear transformation, which implies

$$Z(ax_1(n) + bx_2(n)) = aZ(x_1(n)) + bZ(x_2(n)) \tag{5.2}$$

where $x_1(n)$ and $x_2(n)$ denote the sampled sequences, while a and b are the arbitrary constants.

EXAMPLE 5.4

Find the z-transform of the sequence defined by

$$x(n) = u(n) - (0.5)^n u(n)$$

Solution:

Applying the linearity of the z-transform discussed above, we have

$$X(z) = Z(x(n)) = Z(u(n)) - Z(0.5^n(n))$$

Using Table 5.1 yields

$$Z(u(n)) = \frac{z}{z-1}$$

and

$$Z(0.5^n u(n)) = \frac{z}{z-0.5}$$

Substituting these results in $X(z)$ leads to the final solution,

$$X(z) = \frac{z}{z-1} - \frac{z}{z-0.5}$$

Shift theorem: Given X(z), the z-transform of a sequence $x(n)$, the z-transform of $x(n-m)$, the time-shifted sequence, is given by

$$Z(x(n-m)) = z^{-m}X(z) \tag{5.3}$$

Note that if $m \geq 0$, then $x(n-m)$ is obtained by right shifting $x(n)$ by m samples. Since the shift theorem plays a very important role in developing the transfer function from a difference equation, we verify the shift theorem for the causal sequence. Note that the shift theorem also works for the noncausal sequence.

Verification: Applying the z-transform to the shifted causal signal $x(n-m)$ leads to

$$\begin{aligned} Z(x(n-m)) &= \sum_{n=0}^{\infty} x(n-m)z^{-n} \\ &= x(-m)z^{-0} + \cdots + x(-1)z^{-(m-1)} + x(0)z^{-m} + x(1)z^{-m-1} + \ldots \end{aligned}$$

Since $x(n)$ is assumed to be a causal sequence, this means that

$$x(-m) = x(-m+1) = \cdots = x(-1) = 0$$

Then we achieve

$$Z(x(n-m)) = x(0)z^{-m} + x(1)z^{-m-1} + x(2)z^{-m-2} + \ldots \tag{5.4}$$

Factoring z^{-m} from Equation (5.4) and applying the definition of z-transform of $X(z)$, we get

$$Z(x(n-m)) = z^{-m}\left(x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots\right) = z^{-m}X(z)$$

EXAMPLE 5.5

Determine the z-transform of the following sequence:

$$y(n) = (0.5)^{(n-5)} \cdot u(n-5)$$

where $u(n-5) = 1$ for $n \geq 5$ and $u(n-5) = 0$ for $n<5$.

Solution:

We first use the shift theorem to obtain

$$Y(z) = Z\left[(0.5)^{n-5}u(n-5)\right] = z^{-5}Z\left[(0.5)^{n}u(n)\right]$$

Using Table 5.1 leads to

$$Y(z) = z^{-5} \cdot \frac{z}{z-0.5} = \frac{z^{-4}}{z-0.5}$$

Convolution: Given two sequences $x_1(n)$ and $x_2(n)$, their convolution can be determined as follows:

$$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(n-k)x_2(k) \tag{5.5}$$

where * designates the linear convolution. In the z-transform domain, we have

$$X(z) = X_1(z)X_2(z) \tag{5.6}$$

Here, $X(z) = Z(x(n))$, $X_1(z) = Z(x_1(n))$, and $X_2(z) = Z(x_2(n))$.

EXAMPLE 5.6

Verify Equation (5.6) using causal sequences $x_1(n)$ and $x_2(n)$.

Solution:

Taking the z-transform of Equation (5.5) leads to

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x_1(n-k)x_2(k)z^{-n}$$

This expression can be further modified to

$$X(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x_2(k)z^{-k}x_1(n-k)z^{-(n-k)}$$

Now interchanging the order of the previous summation gives

$$X(z) = \sum_{k=0}^{\infty} x_2(k) z^{-k} \sum_{n=0}^{\infty} x_1(n-k) z^{-(n-k)}$$

Now, let $m = n - k$:

$$X(z) = \sum_{k=0}^{\infty} x_2(k) z^{-k} \sum_{m=0}^{\infty} x_1(m) z^{-m}$$

By the definition of Equation (5.1), it follows that

$$X(z) = X_2(z) X_1(z) = X_1(z) X_2(z)$$

EXAMPLE 5.7

Consider two sequences,

$$x_1(n) = 3\delta(n) + 2\delta(n-1)$$

$$x_2(n) = 2\delta(n) - \delta(n-1)$$

a. Find the z-transform of the convolution:

$$X(z) = Z(x_1(n) * x_2(n))$$

b. Determine the convolution sum using the z-transform:

$$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(k) x_2(n-k)$$

Solution:

a. Applying the z-transform for $x_1(n)$ and $x_2(n)$, respectively, it follows that

$$X_1(z) = 3 + 2z^{-1}$$

$$X_2(z) = 2 - z^{-1}$$

Using the convolution property, we have

$$\begin{aligned} X(z) &= X_1(z) X_2(z) = \left(3 + 2z^{-1}\right)\left(2 - z^{-1}\right) \\ &= 6 + z^{-1} - 2z^{-2} \end{aligned}$$

b. Applying the inverse z-transform and using the shift theorem and Line 1 of Table 5.1 leads to

$$x(n) = Z^{-1}\left(6 + z^{-1} - 2z^{-2}\right) = 6\delta(n) + \delta(n-1) - 2\delta(n-2)$$

Table 5.2 z-Transform Properties

Property	Time Domain	z-Transform
Linearity	$ax_1(n) + bx_2(n)$	$aZ(x_1(n)) + bZ(x_2(n))$
Shift theorem	$x(n-m)$	$z^{-m}X(z)$
Linear convolution	$x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(n-k)x_2(k)$	$X_1(z)X_2(z)$

The properties of the z-transform discussed in this section are listed in Table 5.2.

5.3 INVERSE Z-TRANSFORM

The z-transform of the sequence $x(n)$ and the inverse z-transform for the function $X(z)$ are defined as, respectively

$$X(z) = Z(x(n)) \tag{5.7}$$

and

$$x(n) = Z^{-1}(X(z)) \tag{5.8}$$

where $Z(\ )$ is the z-transform operator, and $Z^{-1}(\ )$ is the inverse z-transform operator.
The inverse of the z-transform may be obtained by at least three methods:

1. partial fraction expansion and lookup table;
2. power series expansion;
3. residue method.

The first method is widely utilized, and it is assumed that the reader is well familiar with the partial fraction expansion method in learning Laplace transform. Therefore, we concentrate on the first method in this book. As for the power series expansion and residue methods, the interested reader is referred to the textbook by Oppenheim and Schafer (1975). The key idea of the partial fraction expansion is that if $X(z)$ is a proper rational function of z, we can expand it to a sum of the first-order factors or higher-order factors using the partial fraction expansion that can be inverted by inspecting the z-transform table. The partial fraction expansion method is illustrated via the following examples. (For simple z-transform functions, we can directly find the inverse z-transform using Table 5.1.)

EXAMPLE 5.8

Find the inverse z-transform for each of the following functions:

a. $X(z) = 2 + \dfrac{4z}{z-1} - \dfrac{z}{z-0.5}$

b. $X(z) = \dfrac{5z}{(z-1)^2} - \dfrac{2z}{(z-0.5)^2}$

c. $X(z) = \dfrac{10z}{z^2 - z + 1}$

d. $X(z) = \dfrac{z^{-4}}{z-1} + z^{-6} + \dfrac{z^{-3}}{z+0.5}$

Solution:

a. $x(n) = 2Z^{-1}(1) + 4Z^{-1}\left(\dfrac{z}{z-1}\right) - Z^{-1}\left(\dfrac{z}{z-0.5}\right)$

From Table 5.1, we have

$$x(n) = 2\delta(n) + 4u(n) - (0.5)^n u(n)$$

b. $x(n) = Z^{-1}\left(\dfrac{5z}{(z-1)^2}\right) - Z^{-1}\left(\dfrac{2z}{(z-0.5)^2}\right) = 5Z^{-1}\left(\dfrac{z}{(z-1)^2}\right) - \dfrac{2}{0.5}Z^{-1}\left(\dfrac{0.5z}{(z-0.5)^2}\right)$

Then

$$x(n) = 5nu(n) - 4n(0.5)^n u(n)$$

c. $X(z) = \dfrac{10z}{z^2 - z + 1} = \left(\dfrac{10}{\sin(a)}\right)\dfrac{\sin(a)z}{z^2 - 2z\cos(a) + 1}$

By coefficient matching, we have

$$-2\cos(a) = -1$$

Hence, $\cos(a) = 0.5$, and $a = 60^0$. Substituting $a = 60^0$ into the sine function leads to

$$\sin(a) = \sin(60^0) = 0.866$$

Finally, we have

$$x(n) = \frac{10}{\sin(a)}Z^{-1}\left(\frac{\sin(a)z}{z^2 - 2z\cos(a) + 1}\right) = \frac{10}{0.866}\sin(n\cdot 60^0) = 11.547\sin(n\cdot 60^0)$$

d.

$$x(n) = Z^{-1}\left(z^{-5}\frac{z}{z-1}\right) + Z^{-1}\left(z^{-6}\cdot 1\right) + Z^{-1}\left(z^{-4}\frac{z}{z+0.5}\right)$$

Using Table 5.1 and the shift property, we get

$$x(n) = u(n-5) + \delta(n-6) + (-0.5)^{n-4}u(n-4)$$

Now, we are ready to deal with the inverse z-transform using the partial fraction expansion and lookup table. The general procedure is as follows:

1. Eliminate the negative powers of z for the z-transform function $X(z)$.
2. Determine the rational function $X(z)/z$ (assuming it is proper), and apply the partial fraction expansion to the determined rational function $X(z)/z$ using the formula in Table 5.3.
3. Multiply the expanded function $X(z)/z$ by z on both sides of the equation to obtain $X(z)$.
4. Apply the inverse z-transform using Table 5.1.

Table 5.3 Partial Fraction(s) and Formulas for Constant(s)

Partial fraction with the first-order real pole: $\frac{R}{z-p}$ $\quad R = (z-p)\frac{X(z)}{z}\Big|_{z=p}$

Partial fraction with the first-order complex poles: $\frac{Az}{(z-P)} + \frac{A^* z}{(z-P^*)}$ $\quad A = (z-P)\frac{X(z)}{z}\Big|_{z=P}$

P^* = complex conjugate of P A^* = complex conjugate of A

Partial fraction with mth-order real poles:

$\frac{R_m}{(z-p)} + \frac{R_{m-1}}{(z-p)^2} + \cdots + \frac{R_1}{(z-p)^m}$ $\quad R_k = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left((z-p)^m \frac{X(z)}{z}\right)\Big|_{z=p}$

The partial fraction format and the formulas for calculating the constants are listed in Table 5.3.

Example 5.9 considers the situation of the z-transform function having first-order poles.

EXAMPLE 5.9

Find the inverse of the following z-transform:

$$X(z) = \frac{1}{(1-z^{-1})(1-0.5z^{-1})}$$

Solution:

Eliminating the negative power of z by multiplying the numerator and denominator by z^2 yields

$$X(z) = \frac{z^2}{z^2(1-z^{-1})(1-0.5z^{-1})}$$

$$= \frac{z^2}{(z-1)(z-0.5)}$$

Dividing both sides by z leads to

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}$$

Again, we write

$$\frac{X(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}$$

where A and B are constants found using the formula in Table 5.3, that is,

$$A = (z-1)\frac{X(z)}{z}\Big|_{z=1} = \frac{z}{(z-0.5)}\Big|_{z=1} = 2$$

$$B = (z - 0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z}{(z-1)}\bigg|_{z=0.5} = -1$$

Thus

$$\frac{X(z)}{z} = \frac{2}{(z-1)} + \frac{-1}{(z-0.5)}$$

Multiplying z on both sides gives

$$X(z) = \frac{2z}{(z-1)} + \frac{-z}{(z-0.5)}$$

Using Table 5.1 with the z-transform pairs, it follows that

$$x(n) = 2u(n) - (0.5)^n u(n)$$

Tabulating this solution in terms of integer values of n, we obtain the results in Table 5.4.

Table 5.4 Determined Sequence in Example 5.9

n	0	1	2	3	4	...	∞
$x(n)$	1.0	1.5	1.75	1.875	1.9375	...	2.0

The following example considers the case where $X(z)$ has first-order complex poles.

EXAMPLE 5.10

Find $y(n)$ if

$$Y(z) = \frac{z^2(z+1)}{(z-1)(z^2 - z + 0.5)}$$

Solution:
Dividing $Y(z)$ by z, we have

$$\frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)(z^2 - z + 0.5)}$$

Applying the partial fraction expansion leads to

$$\frac{Y(z)}{z} = \frac{B}{z-1} + \frac{A}{(z-0.5-j0.5)} + \frac{A^*}{(z-0.5+j0.5)}$$

We first find B :

$$B = (z-1)\frac{Y(z)}{z}\bigg|_{z=1} = \frac{z(z+1)}{(z^2-z+0.5)}\bigg|_{z=1} = \frac{1\times(1+1)}{(1^2-1+0.5)} = 4$$

Notice that A and A^* are a complex conjugate pair. We determine A as follows:

$$A = (z-0.5-j0.5)\frac{Y(z)}{z}\bigg|_{z=0.5+j0.5} = \frac{z(z+1)}{(z-1)(z-0.5+j0.5)}\bigg|_{z=0.5+j0.5}$$

$$= \frac{(0.5+j0.5)(0.5+j0.5+1)}{(0.5+j0.5-1)(0.5+j0.5-0.5+j0.5)} = \frac{(0.5+j0.5)(1.5+j0.5)}{(-0.5+j0.5)j}$$

Using the polar form, we get

$$A = \frac{(0.707\angle 45^0)(1.58114\angle 18.43^0)}{(0.707\angle 135^0)(1\angle 90^0)} = 1.58114\angle -161.57^0$$

$$A^* = \overline{A} = 1.58114\angle 161.57^0$$

Assume that a first-order complex pole takes the form

$$P = 0.5+0.5j = |P|\angle\theta = 0.707\angle 45^0 \text{ and } P^* = |P|\angle -\theta = 0.707\angle -45^0$$

We have

$$Y(z) = \frac{4z}{z-1} + \frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)}$$

Applying the inverse z-transform from Line 15 in Table 5.1 leads to

$$y(n) = 4Z^{-1}\left(\frac{z}{z-1}\right) + Z^{-1}\left(\frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)}\right)$$

Using the previous formula, the inversion and subsequent simplification yield

$$\begin{aligned} y(n) &= 4u(n) + 2|A|(|P|)^n\cos(n\theta+\varphi)u(n) \\ &= 4u(n) + 3.1623(0.7071)^n\cos(45^0 n - 161.57^0)u(n) \end{aligned}$$

The situation dealing with real repeated poles is presented next.

EXAMPLE 5.11

Find $x(n)$ if

$$X(z) = \frac{z^2}{(z-1)(z-0.5)^2}$$

Solution:

Dividing both sides of the previous z-transform by z yields

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)^2} = \frac{A}{z-1} + \frac{B}{z-0.5} + \frac{C}{(z-0.5)^2}$$

where

$$A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)^2}\bigg|_{z=1} = 4$$

Using the formulas for mth-order real poles in Table 5.3, where $m = 2$ and $p = 0.5$, to determine B and C yields

$$B = R_2 = \frac{1}{(2-1)!}\frac{d}{dz}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

$$= \frac{d}{dz}\left(\frac{z}{z-1}\right)\bigg|_{z=0.5} = \frac{-1}{(z-1)^2}\bigg|_{z=0.5} = -4$$

$$C = R_1 = \frac{1}{(1-1)!}\frac{d^0}{dz^0}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

$$= \frac{z}{z-1}\bigg|_{z=0.5} = -1$$

Then

$$X(z) = \frac{4z}{z-1} + \frac{-4z}{z-0.5} + \frac{-1z}{(z-0.5)^2} \tag{5.9}$$

The inverse z-transform for each term on the right-hand side of Equation (5.9) can be obtained using the result listed in Table 5.1, that is,

$$Z^{-1}\left\{\frac{z}{z-1}\right\} = u(n)$$

$$Z^{-1}\left\{\frac{z}{z-0.5}\right\} = (0.5)^n u(n)$$

$$Z^{-1}\left\{\frac{z}{(z-0.5)^2}\right\} = 2n(0.5)^n u(n)$$

From these results, it follows that

$$x(n) = 4u(n) - 4(0.5)^n u(n) - 2n(0.5)^n u(n)$$

5.3.1 Partial Fraction Expansion Using MATLAB

The MATLAB function **residue()** can be applied to perform the partial fraction expansion of a z-transform function $X(z)/z$. The syntax is given as

$$[\mathbf{R},\mathbf{P},\mathbf{K}] = \mathbf{residue}(\mathbf{B},\mathbf{A})$$

Here, B and A are the vectors consisting of coefficients for the numerator and denominator polynomials, $B(z)$ and $A(z)$, respectively. Notice that $B(z)$ and $A(z)$ are the polynomials with increasing positive powers of z.

$$\frac{B(z)}{A(z)} = \frac{b_0z^M + b_1z^{M-1} + b_2z^{M-2} + \cdots + b_M}{z^N + a_1z^{N-1} + a_2z^{-2} + \cdots + a_N}$$

The function returns the residues in vector R, corresponding poles in vector P, and polynomial coefficients (if any) in vector K. The expansion format is shown as

$$\frac{B(z)}{A(z)} = \frac{r_1}{z-p_1} + \frac{r_2}{z-p_2} + \cdots + k_0 + k_1z^{-1} + \cdots$$

For a pole p_j of multiplicity m, the partial fraction includes the following terms:

$$\frac{B(z)}{A(z)} = \cdots + \frac{r_j}{z-p_j} + \frac{r_{j+1}}{(z-p_j)^2} + \cdots + \frac{r_{j+m}}{(z-p_j)^m} + \cdots + k_0 + k_1z^{-1} + \cdots$$

EXAMPLE 5.12

Find the partial expansion for each of the following z-transform functions:

a. $X(z) = \dfrac{1}{(1-z^{-1})(1-0.5z^{-1})}$

b. $Y(z) = \dfrac{z^2(z+1)}{(z-1)(z^2-z+0.5)}$

c. $X(z) = \dfrac{z^2}{(z-1)(z-0.5)^2}$

Solution:

a. From MATLAB, we can show the denominator polynomial as

```
» conv([1 −1],[1 −0.5])
D =
1.0000 −1.5000 0.5000
```

This leads to

$$X(z) = \frac{1}{(1-z^{-1})(1-0.5z^{-1})} = \frac{1}{1-1.5z^{-1}+0.5^{-2}} = \frac{z^2}{z^2-1.5z+0.5}$$

and

$$\frac{X(z)}{z} = \frac{z}{z^2-1.5z+0.5}$$

From MATLAB, we have

```
» [R,P,K]=residue([1 0],[1 -1.5 0.5])
R =
2
-1
P =
1.0000
0.5000
K =
[]
»
```

Then the expansion is written as

$$X(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

b. From the MATLAB entry

```
» N=conv([1 0 0],[1 1])
N =
1 1 0 0
» D=conv([1 -1], [1 -1 0.5])
D =
1.0000 -2.0000 1.5000 -0.5000
```

we get

$$Y(z) = \frac{z^2(z+1)}{(z-1)(z^2-z+0.5)} = \frac{z^3+z^2}{z^3-2z^2+1.5z-0.5}$$

and

$$\frac{Y(z)}{z} = \frac{z^2+z}{z^3-2z^2+1.5z-0.5}$$

Using the MATLAB residue function yields

```
» [R,P,K]=residue([1 1 0],[1 -2 1.5 -0.5])
R =
4.0000
-1.5000 - 0.5000i
-1.5000 + 0.5000i
P =
1.0000
0.5000 + 0.5000i
0.5000 - 0.5000i
K =
[]
»
```

Then the expansion is shown below

$$X(z) = \frac{Bz}{z-p_1} + \frac{Az}{z-p} + \frac{A^*z}{z-p^*}$$

$$\text{where } B = 4,\ p_1 = 1,$$

$$A = -1.5 - 0.5j, \quad p = 0.5 + 0.5j,$$

$$A^* = -1.5 + 0.5j, \quad \text{and } p = 0.5 - 0.5j$$

c. Similarly, if we use

```
» D=conv(conv([1 -1], [1 -0.5]),[1 -0.5])
D =
1.0000 -2.0000 1.2500 -0.2500
```

then

$$X(z) = \frac{z^2}{(z-1)(z-0.5)^2} = \frac{z^2}{z^3 - 2z^2 + 1.25z - 0.25}$$

and we yield

$$\frac{X(z)}{z} = \frac{z}{z^3 - 2z^2 + 1.25z - 0.25}$$

From MATLAB, we obtain

```
» [R,P,K]=residue([1 0],[1 -2 1.25 -0.25])
R =
4.0000
-4.0000
-1.0000
P =
1.0000
0.5000
0.5000
K =
[]
»
```

Using the previous results leads to

$$X(z) = \frac{4z}{z-1} - \frac{4z}{z-0.5} - \frac{z}{(z-0.5)^2}$$

5.4 SOLUTION OF DIFFERENCE EQUATIONS USING THE Z-TRANSFORM

To solve a difference equation with initial conditions, we have to deal with time-shifted sequences such as $y(n-1), y(n-2), \ldots, y(n-m)$, and so on. Let us examine the z-transform of these terms. Using the definition of the z-transform, we have

$$\begin{aligned} Z(y(n-1)) &= \sum_{n=0}^{\infty} y(n-1)z^{-n} \\ &= y(-1) + y(0)z^{-1} + y(1)z^{-2} + \cdots \\ &= y(-1) + z^{-1}\left(y(0) + y(1)z^{-1} + y(2)z^{-2} + \cdots\right) \end{aligned}$$

It holds that

$$Z(y(n-1)) = y(-1) + z^{-1}Y(z) \tag{5.10}$$

Similarly, we have

$$\begin{aligned} Z(y(n-2)) &= \sum_{n=0}^{\infty} y(n-2)z^{-n} \\ &= y(-2) + y(-1)z^{-1} + y(0)z^{-2} + y(1)z^{-3} + \cdots \\ &= y(-2) + y(-1)z^{-1} + z^{-2}\left(y(0) + y(1)z^{-1} + y(2)z^{-2} + \cdots\right) \end{aligned}$$

$$Z(y(n-2)) = y(-2) + y(-1)z^{-1} + z^{-2}Y(z) \tag{5.11}$$

$$Z(y(n-m)) = y(-m) + y(-m+1)z^{-1} + \cdots + y(-1)z^{-(m-1)} + z^{-m}Y(z) \tag{5.12}$$

where $y(-m), y(-m+1), \ldots, y(-1)$ are the initial conditions. If all initial conditions are considered to be zero, that is,

$$y(-m) = y(-m+1) = \cdots y(-1) = 0 \tag{5.13}$$

then Equation (5.12) becomes

$$Z(y(n-m)) = z^{-m}Y(z) \tag{5.14}$$

which is the same as the shift theorem in Equation (5.3).
The following two examples serve as illustrations of applying the z-transform to find the solutions of difference equations. The procedure is as follows:

1. Apply the z-transform to the difference equation.
2. Substitute the initial conditions.
3. Solve for the difference equation in the z-transform domain.
4. Find the solution in the time domain by applying the inverse z-transform.

EXAMPLE 5.13

A digital signal processing (DSP) system is described by the difference equation

$$y(n) - 0.5y(n-1) = 5(0.2)^n u(n)$$

Determine the solution when the initial condition is given by $y(-1) = 1$.

Solution:

Applying the z-transform on both sides of the difference equation and using Equation (5.12), we have

$$Y(z) - 0.5(y(-1) + z^{-1}Y(z)) = 5Z(0.2^n u(n))$$

Substituting the initial condition and $Z(0.2^n u(n)) = Z(0.2^n u(n)) = z/(z-0.2)$, we achieve

$$Y(z) - 0.5(1 + z^{-1}Y(z)) = 5z/(z-0.2)$$

Simplification leads to

$$Y(z) - 0.5z^{-1}Y(z) = 0.5 + 5z/(z-0.2)$$

Factoring out $Y(z)$ and combining the right-hand side of the equation, it follows that

$$Y(z)(1 - 0.5z^{-1}) = (5.5z - 0.1)/(z-0.2)$$

Then we obtain

$$Y(z) = \frac{(5.5z-0.1)}{(1-0.5z^{-1})(z-0.2)} = \frac{z(5.5z-0.1)}{(z-0.5)(z-0.2)}$$

Using the partial fraction expansion method leads to

$$\frac{Y(z)}{z} = \frac{5.5z-0.1}{(z-0.5)(z-0.2)} = \frac{A}{z-0.5} + \frac{B}{z-0.2}$$

where

$$A = (z-0.5)\frac{Y(z)}{z}\bigg|_{z=0.5} = \frac{5.5z-0.1}{z-0.2}\bigg|_{z=0.5} = \frac{5.5\times 0.5-0.1}{0.5-0.2} = 8.8333$$

$$B = (z-0.2)\frac{Y(z)}{z}\bigg|_{z=0.2} = \frac{5.5z-0.1}{z-0.5}\bigg|_{z=0.2} = \frac{5.5\times 0.2-0.1}{0.2-0.5} = -3.3333$$

Thus

$$Y(z) = \frac{8.8333z}{(z-0.5)} + \frac{-3.3333z}{(z-0.2)}$$

which gives the solution as

$$y(n) = 8.3333(0.5)^n u(n) - 3.3333(0.2)^n u(n)$$

EXAMPLE 5.14

A relaxed (zero initial conditions) DSP system is described by a difference equation

$$y(n) + 0.1y(n-1) - 0.2y(n-2) = x(n) + x(n-1)$$

a. Determine the impulse response $y(n)$ due to the impulse sequence $x(n) = \delta(n)$.
b. Determine the system response $y(n)$ due to the unit step function excitation, where $u(n) = 1$ for $n \geq 0$.

Solution:

a. Applying the z-transform on both sides of the difference equation and using Equation (5.3) or Equation (5.14), we obtain

$$Y(z) + 0.1Y(z)z^{-1} - 0.2Y(z)z^{-2} = X(z) + X(z)z^{-1} \quad (5.15)$$

Factoring out Y(z) on the left side and substituting $X(z) = Z(\delta(n)) = 1$ to the right side in the Equation (5.15) we get

$$Y(z)(1 + 0.1z^{-1} - 0.2z^{-2}) = 1(1 + z^{-1})$$

Then Y(z) can be expressed as

$$Y(z) = \frac{1 + z^{-1}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

To obtain the impulse response, which is the inverse z-transform of the transfer function, we multiply the numerator and denominator by z^2.
Thus

$$Y(z) = \frac{z^2 + z}{z^2 + 0.1z - 0.2} = \frac{z(z+1)}{(z-0.4)(z+0.5)}$$

Using the partial fraction expansion method leads to

$$\frac{Y(z)}{z} = \frac{z+1}{(z-0.4)(z+0.5)} = \frac{A}{z-0.4} + \frac{B}{z+0.5}$$

where

$$A = (z-0.4)\frac{Y(z)}{z}\bigg|_{z=0.4} = \frac{z+1}{z+0.5}\bigg|_{z=0.4} = \frac{0.4+1}{0.4+0.5} = 1.5556$$

$$B = (z+0.5)\frac{Y(z)}{z}\bigg|_{z=-0.5} = \frac{z+1}{z-0.4}\bigg|_{z=-0.5} = \frac{-0.5+1}{-0.5-0.4} = -0.5556$$

Thus

$$Y(z) = \frac{1.5556z}{(z-0.4)} + \frac{-0.5556z}{(z+0.5)}$$

which gives the impulse response

$$y(n) = 1.5556(0.4)^n u(n) - 0.5556(-0.5)^n u(n)$$

b. To obtain the response due to a unit step function, the input sequence is set to be

$$x(n) = u(n)$$

and the corresponding z-transform is given by

$$X(z) = \frac{z}{z-1}$$

Notice that

$$Y(z) + 0.1Y(z)z^{-1} - 0.2Y(z)z^{-2} = X(z) + X(z)z^{-1}$$

Then the z-transform of the output sequence $y(n)$ can be obtained as

$$Y(z) = \left(\frac{z}{z-1}\right)\left(\frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}}\right) = \frac{z^2(z+1)}{(z-1)(z-0.4)(z+0.5)}$$

Using the partial fraction expansion method as before gives

$$Y(z) = \frac{2.2222z}{z-1} + \frac{-1.0370z}{z-0.4} + \frac{-0.1852z}{z+0.5}$$

and the system response is found by using Table 5.1:

$$y(n) = 2.2222u(n) - 1.0370(0.4)^n u(n) - 0.1852(-0.5)^n u(n)$$

5.5 SUMMARY

1. The one-sided (unilateral) z-transform, which can be used to transform the causal sequence to the z-transform domain, was defined.
2. The lookup table of the z-transform determines the z-transform for a simple causal sequence, or the causal sequence from a simple z-transform function.
3. The important properties of the z-transform, such as linearity, the shift theorem, and convolution were introduced. The shift theorem can be used to solve a difference equation. The z-transform of a digital convolution of two digital sequences is equal to the product of their z-transforms.
4. Methods to determine the inverse of the z-transform, such as partial fraction expansion, invert the complicated z-transform function, which can have first-order real poles, multiple-order real poles, and first-order complex poles assuming that the z-transform function is proper. The MATLAB techniques to determine the inverse were introduced.
5. The z-transform can be applied to solve linear difference equations with nonzero initial conditions and zero initial conditions.

5.6 PROBLEMS

5.1. Find the z-transform for each of the following sequences:

a. $x(n) = 4u(n)$

b. $x(n) = (-0.7)^n u(n)$

c. $x(n) = 4e^{-2n}u(n)$

d. $x(n) = 4(0.8)^n \cos(0.1\pi n)u(n)$

e. $x(n) = 4e^{-3n}\sin(0.1\pi n)u(n)$

5.2. Using the properties of the z-transform, find the z-transform for each of the following

a. $x(n) = u(n) + (0.5)^n u(n)$

b. $x(n) = e^{-3(n-4)}\cos(0.1\pi(n-4))u(n-4)$, where $u(n-4) = 1$ for $n \geq 4$ while $u(n-4) = 0$ for $n<4$

5.3. Find the z-transform for each of the following sequences:

a. $x(n) = 3u(n-4)$

b. $x(n) = 2(-0.2)^n u(n)$

c. $x(n) = 5e^{-2(n-3)}u(n-3)$

d. $x(n) = 6(0.6)^n \cos(0.2\pi n)u(n)$

e. $x(n) = 4e^{-3(n-1)}\sin(0.2\pi(n-1))u(n-1)$.

5.4. Using the properties of the z-transform, find the z-transform for each of the following sequences:

a. $x(n) = -2u(n) - (0.75)^n u(n)$

b. $x(n) = e^{-2(n-3)}\sin(0.2\pi(n-3))u(n-3)$, where $u(n-3) = 1$ for $n \geq 3$ while $u(n-3) = 0$ for $n < 3$

5.5. Given two sequences

$$x_1(n) = 5\delta(n) - 2\delta(n-2) \text{ and } x_2(n) = 3\delta(n-3)$$

a. determine the z-transform of the convolution of the two sequences using the convolution property of z-transform

$$X(z) = X_1(z)X_2(z)$$

b. determine the convolution by the inverse z-transform

$$x(n) = Z^{-1}(X_1(z)X_2(z))$$

from the result in (a).

5.6. Using Table 5.1 and the z-transform properties, find the inverse z-transform for each of the following functions:

a. $X(z) = 4 - \dfrac{10z}{z-1} - \dfrac{z}{z+0.5}$

b. $X(z) = \dfrac{-5z}{(z-1)} + \dfrac{10z}{(z-1)^2} + \dfrac{2z}{(z-0.8)^2}$

c. $X(z) = \dfrac{z}{z^2 + 1.2z + 1}$

d. $X(z) = \dfrac{4z^{-4}}{z-1} + \dfrac{z^{-1}}{(z-1)^2} + z^{-8} + \dfrac{z^{-5}}{z-0.5}$

5.7. Given two sequences

$$x_1(n) = -2\delta(n) + 5\delta(n-2) \text{ and } x_2(n) = 4\delta(n-4)$$

a. determine the z-transform of convolution of the two sequences using the convolution property of z-transform

$$X(z) = X_1(z)X_2(z)$$

b. determine the convolution by the inverse z-transform

$$x(n) = Z^{-1}(X_1(z)X_2(z))$$

from the result in (a).

5.8. Using Table 5.1 and z-transform properties, find the inverse z-transform for each of the following functions:

a. $X(z) = 5 - \dfrac{7z}{z+1} - \dfrac{3z}{z-0.5}$

b. $X(z) = \dfrac{-3z}{(z-0.5)} + \dfrac{8z}{(z-0.8)} + \dfrac{2z}{(z-0.8)^2}$

c. $X(z) = \dfrac{3z}{z^2 + 1.414z + 1}$

d. $X(z) = \dfrac{5z^{-5}}{z-1} - \dfrac{z^{-2}}{(z-1)^2} + z^{-10} + \dfrac{z^{-3}}{z-0.75}$

5.9. Using the partial fraction expansion method, find the inverse of the following z-transforms:

a. $X(z) = \dfrac{1}{z^2 - 0.3z - 0.24}$

b. $X(z) = \dfrac{z}{(z-0.2)(z+0.4)}$

c. $X(z) = \dfrac{z}{(z+0.2)(z^2 - z + 0.5)}$

d. $X(z) = \dfrac{z(z+0.5)}{(z-0.1)^2(z-0.6)}$

5.10. A system is described by the difference equation

$$y(n) + 0.5y(n-1) = 2(0.8)^n u(n)$$

Determine the solution when the initial condition is $y(-1) = 2$.

5.11. Using the partial fraction expansion method, find the inverse of the following z-transforms:

a. $X(z) = \dfrac{1}{z^2 + 0.2z + 0.2}$

b. $X(z) = \dfrac{z}{(z+0.3)(z-0.5)}$

c. $X(z) = \dfrac{5z}{(z-0.75)(z^2-z+0.5)}$

d. $X(z) = \dfrac{2z(z-0.4)}{(z-0.2)^2(z+0.8)}$

5.12. A system is described by the difference equation

$$y(n) + 0.2y(n-1) = 4(0.3)^n u(n)$$

Determine the solution when the initial condition is $y(-1) = 1$.

5.13. A system is described by the difference equation

$$y(n) - 0.5y(n-1) + 0.06y(n-2) = (0.4)^{n-1}u(n-1)$$

Determine the solution when the initial conditions are $y(-1) = 1$, and $y(-2) = 2$.

5.14. Given the following difference equation with the input–output relationship of a certain initially relaxed system (all initial conditions are zero),

$$y(n) - 0.7y(n-1) + 0.1y(n-2) = x(n) + x(n-1)$$

a. find the impulse response sequence $y(n)$ due to the impulse sequence $\delta(n)$;

b. find the output response of the system when the unit step function $u(n)$ is applied.

5.15. A system is described by the difference equation

$$y(n) - 0.6y(n-1) + 0.08y(n-2) = (0.5)^{n-1}u(n-1)$$

Determine the solution when the initial conditions are $y(-1) = 2$, and $y(-2) = 1$.

5.16. Given the following difference equation with the input–output relationship of a certain initially relaxed system (all initial conditions are zero),

$$y(n) - 0.6y(n-1) + 0.25y(n-2) = x(n) + x(n-1)$$

a. find the impulse response sequence $y(n)$ due to the impulse sequence $\delta(n)$;

b. find the output response of the system when the unit step function $u(n)$ is applied.

5.17. Given the following difference equation with the input–output relationship of a certain initially relaxed DSP system (all initial conditions are zero),

$$y(n) - 0.4y(n-1) + 0.29y(n-2) = x(n) + 0.5x(n-1)$$

a. find the impulse response sequence $y(n)$ due to the impulse sequence $\delta(n)$;

b. find the output response of the system when the unit step function $u(n)$ is applied.

5.18. Given the following difference equation with the input–output relationship of a certain initially relaxed DSP system (all initial conditions are zero),

$$y(n) - 0.2y(n-1) + 0.17y(n-2) = x(n) + 0.3x(n-1)$$

a. find the impulse response sequence $y(n)$ due to the impulse sequence $\delta(n)$;

b. find the output response of the system when the unit step function $u(n)$ is applied.