Appendix E: Finite Impulse Response Filter Design Equations by the Frequency Sampling Design Method

Recall in Section 7.5 in Chapter 7 on the "Frequency Sampling Design Method" that we obtained

$$h(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)W_N^{-kn} \tag{E.1}$$

where $h(n)$, $0 \le n \le N-1$, is the causal impulse response that approximates the finite impulse response (FIR) filter, $H(k)$, $0 \le k \le N-1$, represents the corresponding coefficients of the discrete Fourier transform (DFT), and $W_N = e^{-j\frac{2\pi}{N}}$. We further write the DFT coefficients, $H(k)$, $0 \le k \le N-1$, in polar form:

$$H(k) = H_k e^{j\varphi_k},\ 0 \le k \le N-1 \tag{E.2}$$

where H_k and φ_k are the kth magnitude and the phase angle, respectively. The frequency response of the FIR filter is expressed as

$$H(e^{j\Omega}) = \sum_{n=0}^{N-1} h(n)e^{-jn\Omega} \tag{E.3}$$

Substituting (E.1) into (E.3) yields

$$H(e^{j\Omega}) = \sum_{n=0}^{N-1}\frac{1}{N}\sum_{k=0}^{N-1} H(k)W_N^{-kn}e^{-j\Omega n} \tag{E.4}$$

Interchanging the order of the summation in Equation (E.4) leads to

$$H(e^{j\Omega}) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)\sum_{n=0}^{N-1}\left(W_N^{-k}e^{-j\Omega}\right)^n \tag{E.5}$$

Since $W_N^{-k}e^{-j\Omega} = \left(e^{-j2\pi/N}\right)^{-k}e^{-j\Omega} = e^{-(j\Omega-2\pi k/N)}$ and using the identity $\sum_{n=0}^{N-1} r^n = 1 + r + r^2 + \cdots + r^{N-1} = \frac{1-r^N}{1-r}$, we can write the second summation in Equation (E.5) as

$$\sum_{n=0}^{N-1}(W_N^{-k}e^{-j\Omega})^n = \frac{1-e^{-j(\Omega-2\pi k/N)N}}{1-e^{-j(\Omega-2\pi k/N)}} \tag{E.6}$$

Using the Euler formula, Equation (E.6) becomes

$$\begin{aligned}\sum_{n=0}^{N-1}(W_N^{-k}e^{-j\Omega})^n &= \frac{e^{-jN(\Omega-2\pi k/N)/2}\left(e^{jN(\Omega-2\pi k/N)/2}-e^{-jN(\Omega-2\pi k/N)/2}\right)/2j}{e^{-j(\Omega-2\pi k/N)/2}\left(e^{j(\Omega-2\pi k/N)/2}-e^{-j(\Omega-2\pi k/N)/2}\right)/2j} \\ &= \frac{e^{-jN(\Omega-2\pi k/N)/2}\sin[N(\Omega-2\pi k/N)/2]}{e^{-j(\Omega-2\pi k/N)/2}\sin[(\Omega-2\pi k/N)/2]} \end{aligned} \tag{E.7}$$

Substituting Equation (E.7) into Equation (E.5) leads to

$$H(e^{j\Omega}) = \frac{1}{N}e^{-j(N-1)\Omega/2}\sum_{k=0}^{N-1}H(k)e^{j(N-1)k\pi/N}\frac{\sin[N(\Omega-2\pi k/N)/2]}{\sin[(\Omega-2\pi k/N)/2)]} \tag{E.8}$$

Let $\Omega = \Omega_m = \frac{2\pi m}{N}$, and substitute it into Equation (E.8) to get

$$H(e^{j\Omega_m}) = \frac{1}{N}e^{-j(N-1)2\pi m/(2N)}\sum_{k=0}^{N-1}H(k)e^{j(N-1)k\pi/N}\frac{\sin[N(2\pi m/N-2\pi k/N)/2]}{\sin[(2\pi m/N-2\pi k/N)/2)]} \tag{E.9}$$

Clearly, when $m \neq k$, the last term of the summation in Equation (E.9) becomes

$$\frac{\sin[N(2\pi m/N-2\pi k/N)/2]}{\sin[(2\pi m/N-2\pi k/N)/2)]} = \frac{\sin(\pi(m-k))}{\sin(\pi(m-k)/N)} = \frac{0}{\sin(\pi(m-k)/N)} = 0$$

When $m = k$, using L'Hospital's rule we have

$$\frac{\sin[N(2\pi m/N-2\pi k/N)/2]}{\sin[(2\pi m/N-2\pi k/N)/2)]} = \frac{\sin(N\pi(m-k)/N)}{\sin(\pi(m-k)/N)} = \lim_{x\to 0}\frac{\sin(Nx)}{\sin(x)} = N$$

Then Equation (E.9) is simplified to

$$H(e^{j\Omega_k}) = \frac{1}{N}e^{-j(N-1)\pi k/N}H(k)e^{j(N-1)k\pi/N}N = H(k)$$

that is,

$$H(e^{j\Omega_k}) = H(k),\ 0 \le k \le N-1 \tag{E.10}$$

where $\Omega_k = \frac{2\pi k}{N}$, corresponding to the kth DFT frequency component. The fact is that if we specify the desired frequency response, $H(\Omega_k)$, $0 \le k \le N-1$, at the equally spaced sampling frequency determined by $\Omega_k = \frac{2\pi k}{N}$, they are actually the DFT coefficients; that is, $H(k)$, $0 \le k \le N-1$, via Equation (E.10). Furthermore, the inverse of the DFT calculated using (E.10) will give the desired impulse response, $h(n)$, $0 \le n \le N-1$.

To devise the design procedure, we substitute Equation (E.2) in Equation (E.8) to obtain

$$H(e^{j\Omega}) = \frac{1}{N} e^{-j(N-1)\Omega/2} \sum_{k=0}^{N-1} H_k e^{j\varphi_k + j(N-1)k\pi/N} \frac{\sin[N(\Omega - 2\pi k/N)/2]}{\sin[(\Omega - 2\pi k/N)/2)]} \quad \text{(E.11)}$$

It is required that the frequency response of the designed FIR filter expressed in Equation (E.11) be linear phase. This can easily be accomplished by setting

$$\varphi_k + (N-1)k\pi/N = 0, \; 0 \le k \le N-1 \quad \text{(E.12)}$$

in Equation (E.11) so that the summation part becomes a real value, thus resulting in the linear phase of $H(e^{j\Omega})$, since only one complex term, $e^{-j(N-1)\Omega/2}$, is left, which presents the constant time delay of the transfer function. Second, the sequence $h(n)$ must be real. To proceed, let $N = 2M + 1$, and due to the properties of DFT coefficients, we have

$$\overline{H}(k) = H(N-k), \;\; 1 \le k \le M \quad \text{(E.13)}$$

where the bar indicates complex conjugate. Note the fact that

$$\overline{W}_N^{-k} = W_N^{-(N-k)}, \;\; 1 \le k \le M \quad \text{(E.14)}$$

From Equation (E.1), we write

$$h(n) = \frac{1}{N}\left(H(0) + \sum_{k=1}^{M} H(k) W_N^{-kn} + \sum_{k=M+1}^{2M} H(k) W_N^{-kn} \right) \quad \text{(E.15)}$$

Equation (E.15) is equivalent to

$$h(n) = \frac{1}{N}\left(H(0) + \sum_{k=1}^{M} H(k) W_N^{-kn} + \sum_{k=1}^{M} H(N-k) W_N^{-(N-k)n} \right)$$

Using Equations (E.13) and (E.14) in the last summation term leads to

$$\begin{aligned} h(n) &= \frac{1}{N}\left(H(0) + \sum_{k=1}^{M} H(k) W_N^{-kn} + \sum_{k=1}^{M} \overline{H}(k) \overline{W}_N^{-kn} \right) \\ &= \frac{1}{2M+1}\left(H(0) + \sum_{k=1}^{M} (H(k) W_N^{-kn} + \overline{H}(k) \overline{W}_N^{-kn}) \right) \end{aligned}$$

Combining the last two summation terms, we achieve

$$h(n) = \frac{1}{2M+1}\left\{ H(0) + 2\mathrm{Re}\left(\sum_{k=1}^{M} H(k) W_N^{-kn} \right) \right\}, \;\; 0 \le n \le N-1 \quad \text{(E.16)}$$

Solving Equation (E.12) gives

$$\varphi_k = -(N-1)k\pi/N, \;\; 0 \le k \le N-1 \quad \text{(E.17)}$$

Again, note that Equation (E.13) is equivalent to

$$H_k e^{-j\varphi_k} = H_{N-k} e^{j\varphi_{N-k}}, \quad 1 \le k \le M \tag{E.18}$$

Substituting (E.17) in (E.18) yields

$$H_k e^{j(N-1)k\pi/N} = H_{N-k} e^{-j(N-1)(N-k)\pi/N}, \quad 1 \le \mathrm{k} \le \mathrm{M} \tag{E.19}$$

Simplification of Equation (E.19) leads to the following result:

$$H_k = H_{N-k} e^{-j(N-1)\pi} = (-1)^{N-1} H_{N-k}, \quad 1 \le k \le M \tag{E.20}$$

Since we constrain the filter length to be $N = 2M + 1$, Equation (E.20) can be further reduced to

$$H_k = (-1)^{2M} H_{2M+1-k} = H_{2M+1-k}, \quad 1 \le k \le M \tag{E.21}$$

Finally, by substituting (E.21) and (E.17) into (E.16), we obtain a very simple design equation:

$$h(n) = \frac{1}{2M+1} \left\{ H_0 + 2 \sum_{k=1}^{M} H_k \cos\left(\frac{2\pi k(n-M)}{2M+1} \right) \right\}, \quad 0 \le n \le 2M \tag{E.22}$$

Thus the design procedure is simply summarized as follows: Given the filter length, $2M + 1$, and the specified frequency response, H_k at $\Omega_k = \dfrac{2\pi k}{(2M+1)}$ for $k = 0, 1, \cdots, M$, FIR filter coefficients can be calculated via Equation (E.22).