Appendix C: Normalized Butterworth and Chebyshev Functions

C.1 NORMALIZED BUTTERWORTH FUNCTION

The normalized Butterworth squared magnitude function is given by

$$|P_n(\omega)|^2 = \frac{1}{1+\varepsilon^2(\omega)^{2n}} \tag{C.1}$$

where n is the order and ε is the specified ripple on the filter passband. The specified ripple in dB is expressed as $\varepsilon_{dB} = 10 \cdot \log_{10}(1+\varepsilon^2)$ dB.

To develop the transfer function $P_n(s)$, we first let $s = j\omega$ and then substitute $\omega^2 = -s^2$ into Equation (C.1) to obtain

$$P_n(s)P_n(-s) = \frac{1}{1+\varepsilon^2(-s^2)^n} \tag{C.2}$$

Equation (C.2) has $2n$ poles, and $P_n(s)$ has n poles on the left-hand half plane (LHHP) on the s-plane, while $P_n(-s)$ has n poles on the right-hand half plane (RHHP) on the s-plane. Solving for poles leads to

$$(-1)^n s^{2n} = -1/\varepsilon^2 \tag{C.3}$$

If n is an odd number, Equation (C.3) becomes

$$s^{2n} = 1/\varepsilon^2$$

and the corresponding poles are solved as

$$p_k = \varepsilon^{-1/n} e^{j\frac{2\pi k}{2n}} = \varepsilon^{-1/n}[\cos(2\pi k/2n) + j\sin(2\pi k/2n)] \tag{C.4}$$

where $k = 0, 1, \cdots, 2n-1$. Thus in phasor form, we have

$$r = \varepsilon^{-1/n}, \quad \text{and} \quad \theta_k = 2\pi k/(2n) \quad \text{for } k = 0, 1, \cdots, 2n-1 \tag{C.5}$$

When n is an even number, it follows that

$$s^{2n} = -1/\varepsilon^2$$

$$p_k = \varepsilon^{-1/n} e^{j\frac{2\pi k+\pi}{2n}} = \varepsilon^{-1/n}[\cos((2\pi k+\pi)/2n) + j\sin((2\pi k+\pi)/2n)] \tag{C.6}$$

where $k = 0, 1, \cdots, 2n - 1$. Similarly, the phasor form is given by

$$r = \varepsilon^{-1/n}, \quad \text{and} \quad \theta_k = (2\pi k + \pi)/(2n) \quad \text{for} \quad k = 0, 1, \cdots, 2n - 1 \tag{C.7}$$

When n is an odd number, we can identify the poles on the LHHP as

$$p_k = -r, \; k = 0 \text{ and}$$

$$p_k = -r\cos(\theta_k) + jr\sin(\theta_k), \; k = 1, \cdots, (n-1)/2 \tag{C.8}$$

Using complex conjugate pairs, we have

$$p_k^* = -r\cos(\theta_k) - jr\sin(\theta_k)$$

Notice that

$$(s - p_k)(s - p_k^*) = s^2 + (2r\cos(\theta_k))s + r^2$$

and from a factor from the real pole $(s + r)$, it follows that

$$P_n(s) = \frac{K}{(s + r)\prod_{k=1}^{(n-1)/2}(s^2 + (2r\cos(\theta_k))s + r^2)} \tag{C.9}$$

and

$$\theta_k = 2\pi k/(2n) \quad \text{for} \quad k = 1, \cdots, (n-1)/2$$

Setting $P_n(0) = 1$ for the unit passband gain leads to

$$K = r^n = 1/\varepsilon$$

When n is an even number, we can identify the poles on the LHHP as

$$p_k = -r\cos(\theta_k) + jr\sin(\theta_k), \; k = 0, 1, \cdots, n/2 - 1 \tag{C.10}$$

Using complex conjugate pairs, we have

$$p_k^* = -r\cos(\theta_k) - jr\sin(\theta_k)$$

The transfer function is given by

$$P_n(s) = \frac{K}{\prod_{k=1}^{n/2}(s^2 + (2r\cos(\theta_k))s + r^2)} \tag{C.11}$$

$$\theta_k = (2\pi k + \pi)/(2n) \quad \text{for} \quad k = 0, 1, \cdots, n/2 - 1$$

Setting $P_n(0) = 1$ for the unit passband gain, we have

$$K = r^n = 1/\varepsilon$$

Let us examine the following examples.

EXAMPLE C.1

Compute the normalized Butterworth transfer function for the following specifications:

Ripple = 3 dB
$n = 2$

Solution:

$n/2 = 1$
$\theta_k = (2\pi \times 0 + \pi)/(2 \times 2) = 0.25\pi$
$\varepsilon^2 = 10^{0.1\times 3} - 1$
$r = 1$ and $K = 1$

Applying Equation (C.11) leads to

$$P_2(s) = \frac{1}{s^2 + 2 \times 1 \times \cos(0.25\pi)s + 1^2} = \frac{1}{s^2 + 1.4141s + 1}$$

EXAMPLE C.2

Compute the normalized Butterworth transfer function for the following specifications:

Ripple = 3 dB
$n = 3$

Solution:

$(n - 1)/2 = 1$
$\varepsilon^2 = 10^{0.1\times 3} - 1$
$r = 1$ and $K = 1$
$\theta_k = (2\pi \times 1)/(2 \times 3) = \pi/3$

From Equation (C.9), we have

$$P_3(s) = \frac{1}{(s+1)(s^2 + 2 \times 1 \times \cos(\pi/3)s + 1^2)}$$
$$= \frac{1}{(s+1)(s^2 + s + 1)}$$

For the unfactored form, we get

$$P_3(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

EXAMPLE C.3

Compute the normalized Butterworth transfer function for the following specifications:

Ripple $= 1.5$ dB
$n = 3$

Solution:

$(n-1)/2 = 1$
$\varepsilon^2 = 10^{0.1\times 1.5} - 1,$
$r = 1.1590$ and $K = 1.5569$
$\theta_k = (2\pi \times 1)/(2\times 3) = \pi/3$

Applying Equation (C.9), we achieve the normalized Butterworth transfer function:

$$P_3(s) = \frac{1}{(s+1.1590)(s^2 + 2\times 1.1590 \times \cos(\pi/3)s + 1.1590^2)}$$
$$= \frac{1}{(s+1)(s^2 + 1.1590s + 1.3433)}$$

For the unfactored form, we obtain

$$P_3(s) = \frac{1.5569}{s^3 + 2.3180s^2 + 2.6866s + 1.5569}$$

C.2 NORMALIZED CHEBYSHEV FUNCTION

Similar to analog Butterworth filter design, the transfer function is derived from the normalized Chebyshev function, and the result is usually listed in a table for design reference. The Chebyshev magnitude response function with an order of n and the normalized cutoff frequency $\omega = 1$ radian per second is given by

$$|B_n(\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2(\omega)}},\ n \geq 1 \tag{C.12}$$

where the function $C_n(\omega)$ is defined as

$$C_n(\omega) = \begin{cases} \cos(n\cos^{-1}(\omega)) & \omega \leq 1 \\ \cos\mathrm{h}(n\cos\mathrm{h}^{-1}(\omega)) & \omega > 1 \end{cases} \tag{C.13}$$

where ε is the ripple specification on the filter passband. Notice that

$$\cos\mathrm{h}^{-1}(x) = \ln(x + \sqrt{x^2 - 1}) \tag{C.14}$$

To develop the transfer function $B_n(s)$, we let $s = j\omega$ and substitute $\omega^2 = -s^2$ into Equation (C.12) to obtain

$$B_n(s)B_n(-s) = \frac{1}{1+\varepsilon^2 C_n^2(s/j)} \tag{C.15}$$

The poles can be found from

$$1+\varepsilon^2 C_n^2(s/j) = 0$$

or

$$C_n(s/j) = \cos\left(n\cos^{-1}(s/j)\right) = \pm\, j1/\varepsilon \tag{C.16}$$

If we introduce a complex variable $v = \alpha + j\beta$ such that

$$v = \alpha + j\beta = \cos^{-1}(s/j) \tag{C.17}$$

we can then write

$$s = j\cos(v) \tag{C.18}$$

Substituting Equation (C.17) into Equation (C.16) and using trigonometric identities, it follows that

$$\begin{aligned} C_n(s/j) &= \cos\left(n\cos^{-1}(s/j)\right) \\ &= \cos(nv) = \cos(n\alpha + jn\beta) \\ &= \cos(n\alpha)\cos\mathrm{h}(n\beta) - j\sin(n\alpha)\sin\mathrm{h}(n\beta) = \pm j1/\varepsilon \end{aligned} \tag{C.19}$$

To solve Equation (C.19), the following conditions must be satisfied:

$$\cos(n\alpha)\cos\mathrm{h}(n\beta) = 0 \tag{C.20}$$

$$-\sin(n\alpha)\sin\mathrm{h}(n\beta) = \pm\, 1/\varepsilon \tag{C.21}$$

Since $\cos\mathrm{h}(n\beta) \geq 1$ in Equation (C.20), we must let

$$\cos(n\alpha) = 0 \tag{C.22}$$

which therefore leads to

$$\alpha_k = (2k+1)\pi/(2n), \; k = 0, 1, 2, \cdots, 2n-1 \tag{C.23}$$

With Equation (C.23), we have $\sin(n\alpha_k) = \pm\, 1$. Then Equation (C.21) becomes

$$\sin\mathrm{h}(n\beta) = 1/\varepsilon \tag{C.24}$$

Solving Equation (C.24) gives

$$\beta = \sin\mathrm{h}^{-1}(1/\varepsilon)/n \tag{C.25}$$

Again from Equation (C.18),

$$s = j\cos(v) = j[\cos(\alpha_k)\cos\mathrm{h}(\beta) - j\sin(\alpha_k)\sin\mathrm{h}(\beta)] \quad \text{for} \quad k = 0, 1, \cdots, 2n-1 \tag{C.26}$$

The poles can be found from Equation (C.26):

$$p_k = \sin(\alpha_k)\sin\mathrm{h}(\beta) + j\cos(\alpha_k)\cos\mathrm{h}(\beta) \quad \text{for} \quad k = 0, 1, \cdots, 2n-1 \tag{C.27}$$

Using Equation (C.27), if n is an odd number, the poles on the left side are

$$p_k = -\sin(\alpha_k)\sin\mathrm{h}(\beta) + j\cos(\alpha_k)\cos\mathrm{h}(\beta),\ k = 0, 1, \cdots, (n-1)/2 - 1 \tag{C.28}$$

Using complex conjugate pairs, we have

$$p_k^* = -\sin(\alpha_k)\sin\mathrm{h}(\beta) - j\cos(\alpha_k)\cos\mathrm{h}(\beta) \tag{C.29}$$

and a real pole

$$p_k = -\sin\mathrm{h}(\beta),\ k = (n-1)/2 \tag{C.30}$$

Notice that

$$(s - p_k)(s - p_k^*) = s^2 + b_k s + c_k \tag{C.31}$$

and from a factor from the real pole $[s + \sin\mathrm{h}(\beta)]$, it follows that

$$B_n(s) = \frac{K}{[s + \sin\mathrm{h}(\beta)]\prod_{k=0}^{(n-1)/2-1}(s^2 + b_k s + c_k)} \tag{C.32}$$

$$b_k = 2\sin(\alpha_k)\sin\mathrm{h}(\beta) \tag{C.33}$$

$$c_k = [\sin(\alpha_k)\sin\mathrm{h}(\beta)]^2 + [\cos(\alpha_k)\cos\mathrm{h}(\beta)]^2 \tag{C.34}$$

where

$$\alpha_k = (2k+1)\pi/(2n) \quad \text{for} \quad k = 0, 1, \cdots, (n-1)/2 - 1 \tag{C.35}$$

For the unit passband gain and the filter order as an odd number, we set $B_n(0) = 1$. Then

$$K = \sin\mathrm{h}(\beta) \prod_{k=0}^{(n-1)/2-1} c_k \tag{C.36}$$

$$\beta = \sin\mathrm{h}^{-1}(1/\varepsilon)/n \tag{C.37}$$

$$\sin\mathrm{h}^{-1}(x) = \ln(x + \sqrt{x^2 + 1}) \tag{C.38}$$

Following a similar procedure for when n is even, we have

$$B_n(s) = \frac{K}{\prod_{k=0}^{n/2-1}(s^2 + b_k s + c_k)} \tag{C.39}$$

$$b_k = 2\sin(\alpha_k)\sin\mathrm{h}(\beta) \tag{C.40}$$

$$c_k = [\sin(\alpha_k)\sin h(\beta)]^2 + [\cos(\alpha_k)\cos h(\beta)]^2 \tag{C.41}$$

where

$$\alpha_k = (2k+1)\pi/(2n) \quad \text{for} \quad k = 0, 1, \cdots, \ n/2 - 1 \tag{C.42}$$

For the unit passband gain and the filter order as an even number, we require that $B_n(0) = 1/\sqrt{1+\varepsilon^2}$, so that the maximum magnitude of the ripple on the passband equals 1. Then we have

$$K = \prod_{k=0}^{n/2-1} c_k/\sqrt{1+\varepsilon^2} \tag{C.43}$$

$$\beta = \sin h^{-1}(1/\varepsilon)/n \tag{C.44}$$

$$\sin h^{-1}(x) = \ln(x + \sqrt{x^2+1}) \tag{C.45}$$

Equations (C.32) to (C.45) are applied to compute the normalized Chebyshev transfer function. Now let us look at the following illustrative examples.

EXAMPLE C.4

Compute the normalized Chebyshev transfer function for the following specifications:

Ripple $= 0.5$ dB
$n = 2$

Solution:

$n/2 = 1$

Applying Equations (C.39) to (C.45), we obtain

$\alpha_0 = (2 \times 0 + 1)\pi/(2 \times 2) = 0.25\pi$
$\varepsilon^2 = 10^{0.1 \times 0.5} - 1 = 0.1220,\ 1/\varepsilon = 2.8630$
$\beta = \sin h^{-1}(2.8630)/n = \ln(2.8630 + \sqrt{2.8630^2 + 1})/2 = 0.8871$
$b_0 = 2\sin(0.25\pi)\sin h(0.8871) = 1.4256$
$c_0 = [\sin(0.25\pi)\sin h(0.8871)]^2 + [\cos(0.25\pi)\cos h(0.8871)]^2 = 1.5162$
$K = 1.5162/\sqrt{1+0.1220} = 1.4314$

Finally, the transfer function is derived as

$$B_2(s) = \frac{1.4314}{s^2 + 1.4256s + 1.5162}$$

EXAMPLE C.5

Compute the normalized Chebyshev transfer function for the following specifications:

Ripple = 1 dB
$n = 3$

Solution:

$(n-1)/2 = 1$

Applying Equations (C.32) to (C.38) leads to

$\alpha_0 = (2\times 0+1)\pi/(2\times 3) = \pi/6$
$\varepsilon^2 = 10^{0.1\times 1} - 1 = 0.2589,\ 1/\varepsilon = 1.9653$
$\beta = \sinh^{-1}(1.9653)/n = \ln(1.9653+\sqrt{1.9653^2+1})/3 = 0.4760$
$b_0 = 2\sin(\pi/6)\sinh(0.4760) = 0.4942$
$c_0 = [\sin(\pi/6)\sinh(0.4760)]^2 + [\cos(\pi/6)\cosh(0.4760)]^2 = 0.9942$
$\sinh(\beta) = \sinh(0.4760) = 0.4942$
$K = 0.4942\times 0.9942 = 0.4913$

We can derive the transfer function as

$$B_3(s) = \frac{0.4913}{(s+0.4942)(s^2+0.4942s+0.9942)}$$

Finally, the unfactored form is found to be

$$B_3(s) = \frac{0.4913}{s^3+0.9883s^2+1.2384s+0.4913}$$