Electromechanical Motion Devices, Second Edition
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Chapter 3

DIRECT-CURRENT MACHINES

3.1 INTRODUCTION

The direct-current (dc) machine is not as widely used today as it was in the past. The dc generator has been replaced by solid-state rectifiers that convert alternating current into direct current with provisions to control the magnitude of the dc voltage, and, in drive applications, the dc motor is being replaced by the voltage-controlled, permanent-magnet ac machine (brushless dc motor) and/or the field-orientation-controlled induction motor. Nevertheless, it is still desirable to devote some time to the dc machine in an introductory course since it is still used as a low-power drive motor, especially in automotive applications. Although maintenance and environmental issues hamper the use of dc machines, this device is the only electric machine that inherently produces maximum torque per ampere. Thus, it is a highly efficient device. With the advent of electronic switching devices, there has become a huge effort to control the permanent-magnet ac and induction machines so as to emulate the performance characteristics of the dc motor without the maintenance and environmental problems inherent to the dc machines. This chapter is an attempt to treat dc machines sufficient to introduce the reader to the operating principles of dc machines with focus on the shunt-connected and permanent-magnet dc machines, thereby setting the stage for a comparison of the look-alike operating characteristics with the voltage-controlled, permanent-magnet ac machine and the field-orientation-controlled induction motor.

A simplified method of analysis is presented rather than an analysis wherein commutation is treated in detail. With this type of analytical approach, the dc machine is considered to be the most straightforward to analyze of all electromechanical devices. The dynamic characteristics of the permanent-magnet dc machine are illustrated and the time-domain block diagram and state equations are developed. A brief consideration of a dc converter used as a means of voltage control and thus speed control is presented as one of the final sections of the chapter.

3.2 ELEMENTARY DIRECT-CURRENT MACHINE

It is instructive to discuss the concept of commutation using the elementary machine shown in Fig. 3.2-1 prior to a formal analysis of the performance of a practical dc machine. The two-pole elementary machine is equipped with a field winding wound on the stator poles, a rotor coil ($a-a'$), and a commutator. The commutator is made up of two semicircular copper segments mounted on the shaft at the end of the rotor and insulated from one another as well as from the iron of the rotor. Each terminal of the rotor coil is connected to a copper segment. Stationary carbon brushes ride upon the copper segments, whereby the rotor coil is connected to a stationary circuit by a near frictionless contact.

The voltage equations for the field winding and rotor coil are

$$v_f = r_f i_f + \frac{d\lambda_f}{dt} \tag{3.2-1}$$

$$v_{a-a'} = r_a i_{a-a'} + \frac{d\lambda_{a-a'}}{dt} \tag{3.2-2}$$

where r_f and r_a are the resistance of the field winding and armature coil, respectively. The rotor of a dc machine is commonly referred to as the *armature;* rotor and armature will be used interchangeably. At this point in the analysis it is sufficient to express the flux linkages as

$$\lambda_f = L_{ff} i_f + L_{fa} i_{a-a'} \tag{3.2-3}$$

$$\lambda_{a-a'} = L_{af} i_f + L_{aa} i_{a-a'} \tag{3.2-4}$$

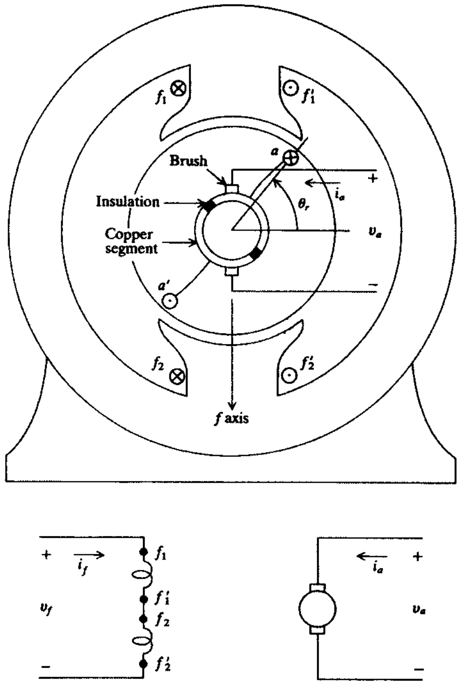

Figure 3.2-1: Elementary two-pole dc machine.

As a first approximation, the mutual inductance between the field winding and an armature coil may be expressed as a sinusoidal function of θ_r as

$$L_{af} = L_{fa} = -L\cos\theta_r \tag{3.2-5}$$

where L is a constant. As the rotor revolves, the action of the commutator is to switch the stationary terminals from one terminal of the rotor coil to the other. For the configuration shown in Fig. 3.2-1, this switching or commutation occurs at $\theta_r = 0,\ \pi,\ 2\pi,\ \ldots$. At the instant of switching, each brush is in contact with both copper segments, whereupon the rotor coil is short-circuited. It is desirable to commutate (short-circuit) the rotor coil at the instant the induced voltage is a minimum. The waveform of the voltage induced in the open-circuited armature coil, during constant-speed operation with a constant field-winding current, may be determined by setting $i_{a-a'} = 0$ and i_f equal to a constant. Substituting (3.2-4) and (3.2-5) into (3.2-2) yields the following expression for the open-circuit voltage of coil $a - a'$ with the field current i_f a constant:

$$v_{a-a'} = \omega_r L I_f \sin\theta_r \tag{3.2-6}$$

where $\omega_r = d\theta_r/dt$ is the rotor speed. The open-circuit coil voltage $v_{a-a'}$ is zero at $\theta_r = 0, \pi, 2\pi, \ldots$, which is the rotor position during commutation. Commutation is illustrated in Fig. 3.2-2. The open-circuit terminal voltage, v_a, corresponding to the rotor positions denoted as θ_{ra}, θ_{rb} ($\theta_{rb} = 0$), and θ_{rc}, are indicated. It is important to note that, during one revolution of the rotor, the assumed positive direction of armature current i_a is down coil side a and out of coil side a' for $0 < \theta_r < \pi$. For $\pi < \theta_r < 2\pi$, positive current is down coil side a and out of coil side a'. In Chaps. 1 and 2, we let positive current flow into the winding denoted without a prime and out the winding denoted with a prime. We will not be able to adhere to this relationship

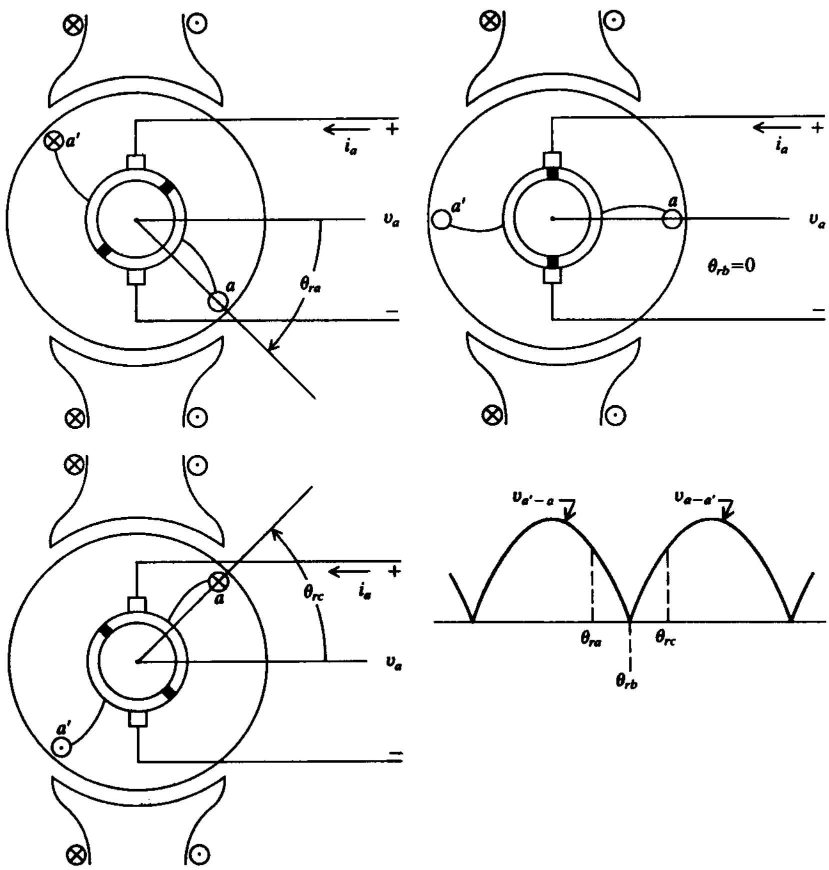

Figure 3.2-2: Commutation of the elementary dc machine.

in the case of the armature windings of a dc machine since commutation is involved. Although the machine shown in Fig. 3.2-1 could be operated as a generator supplying a resistive load, it could not be operated effectively as a motor supplied from a voltage source, owing to the short-circuiting of the armature coil at each commutation. Nevertheless, this impracticable device helps to illustrate two features of commutation; first, commutation takes place when the voltage induced in the rotor winding is ideally zero and, second, commutation can be thought of as a mechanical means of full-wave rectification. We will see another important feature that commutation provides when we consider the torque produced by the dc machine.

A practicable dc machine, with the rotor equipped with an a winding and an A winding, is shown schematically in Fig. 3.2-3. At the rotor position depicted, coils $a_4 - a_4'$ and $A_4 - A_4'$ are being commutated. The bottom brush short-circuits the $a_4 - a_4'$ coil while the top brush short-circuits the $A_4 - A_4'$ coil. Figure 3.2-3 illustrates the instant when the assumed direction of positive current is into the paper in coil sides a_1', A_1; a_2, A_2;..., and out in coil sides a_1', A_1'; a_2', A_2';.... It is instructive to follow the path of current through one of the parallel paths from one brush to the other. For the angular position shown in Fig. 3.2-3, positive current enters the top brush and flows down the rotor via a_1 and back through a_1'; down a_2 and back through a_2'; down a_3 and back through a_3' to the bottom brush. A parallel current path exists through $A_3 - A_3'$, $A_2 - A_2'$, and $A_1 - A_1'$. The open-circuit or induced armature voltage is also shown in Fig. 3.2-3; however, these idealized waveforms require additional explanation. As the rotor advances in the counterclockwise direction, the segment connected to a_1 and A_4 moves from under the top brush, as shown in Fig. 3.2-4. The top brush then rides only on the segment connecting A_3 and A_4'. At the same time, the bottom brush is riding on the segment connecting a_4 and a_3'. With the rotor so positioned, current flows in A_3 and A_4' and out a_4 and a_3'. In other words, current flows down the coil sides in the upper half of the rotor and out of the coil sides in the bottom half. Let us follow the current flow through the parallel paths of the armature windings shown in Fig. 3.2-4. Current now flows through the top brush into A_4' out A_4, into a_1 out a_1', into a_2, out a_2', into a_3 out a_3' to the bottom brush. The parallel path beginning at the top brush is $A_3 - A_3'$, $A_2 - A_2'$, $A_1 - A_1'$, and $a_4' - a_4$ to the bottom brush. The voltage induced in the coils is shown in Figs. 3.2-3 and 3.2-4 for the first parallel path described. It is noted that the induced voltage is plotted only when the coil is in this parallel path.

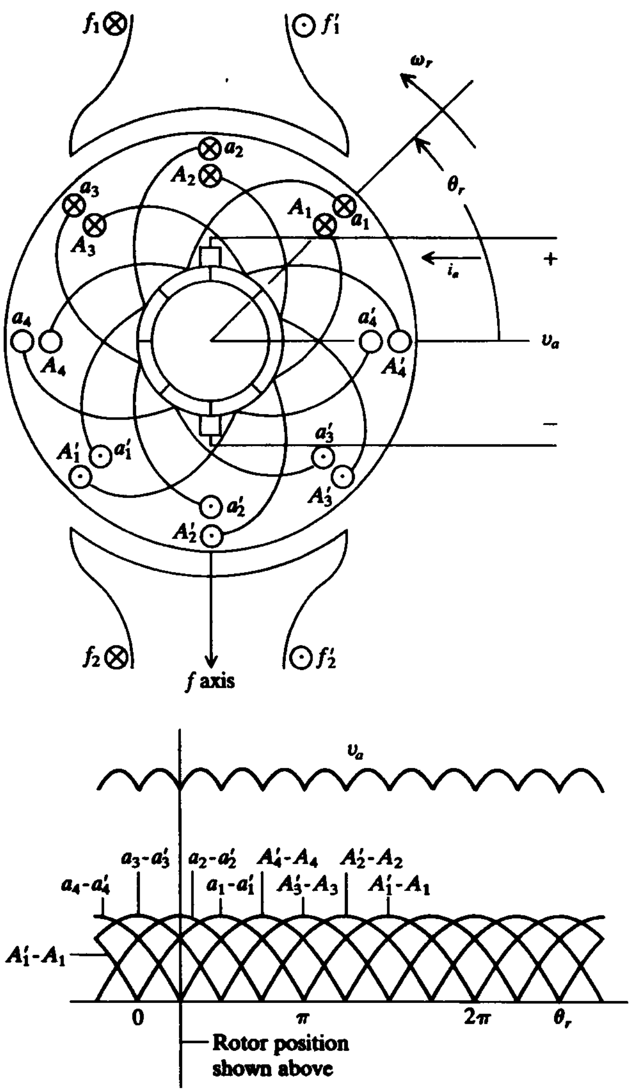

Figure 3.2-3: A dc machine with parallel armature windings.

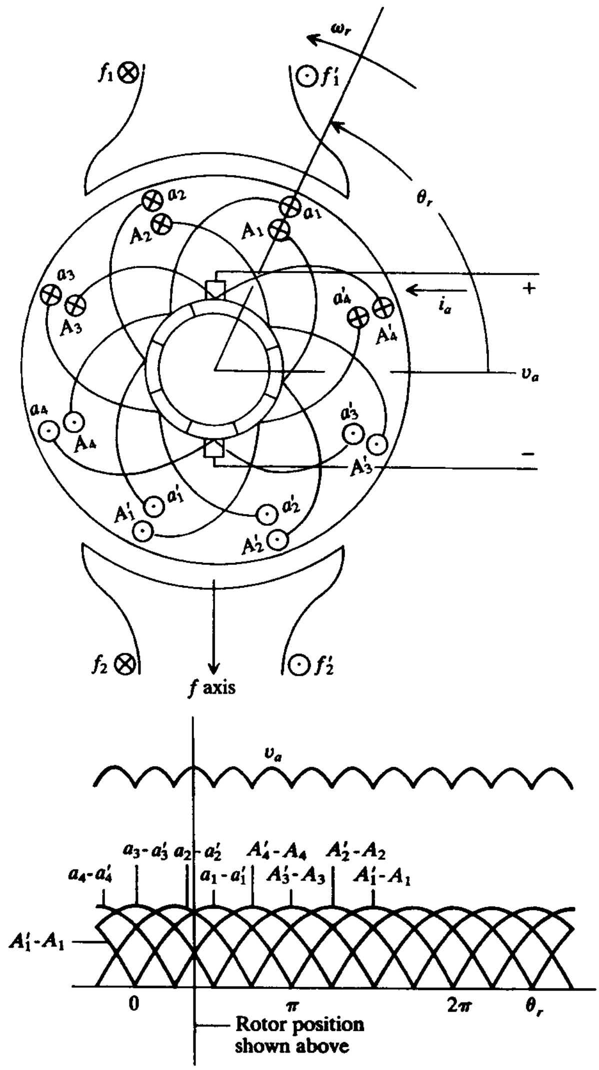

Figure 3.2-4: Same as Fig. 3.2-3 with rotor advanced approximately 22.5° counterclockwise.

In Figs. 3.2-3 and 3.2-4, the parallel windings consist of only four coils. Usually, the number of rotor coils is substantially more than four, thereby reducing the harmonic content of the open-circuit armature voltage. In this case, the rotor coils may be approximated as a uniformly distributed winding, as illustrated in Fig. 3.2-5. Therein the rotor winding is considered as current sheets that are fixed in space due to the action of the commutator and that establish a magnetic axis positioned orthogonal to the magnetic axis of the field winding. We will see the importance of this orthogonal positioning of the magnetic axes when derive the expression of the torque. The brushes are shown positioned on the current sheet for the purpose of depicting commutation. The small angular displacement, denoted by 2γ, designates the region of commutation wherein the coils are short-circuited. However, commutation cannot be visualized from Fig. 3.2-5; one must refer to Figs. 3.2-3 and 3.2-4.

Before proceeding to the development of the equations portraying the operating characteristics of the dc machine, it is instructive to take a brief look at the arrangement of the armature windings and the method of commutation used in many of the low-power, permanent-magnet dc motors. Small dc motors used in low-power control systems are often the permanent-magnet type, wherein a constant field flux is established by a permanent magnet rather than by a current flowing in a field winding.

Three rotor positions of a typical low-power, permanent-magnet dc motor are shown in Fig. 3.2-6. The rotor is turning in the counterclockwise direction and the rotor position advances from left to right. Physically, these devices may be an inch or less in diameter with brushes sometimes as small as a pencil lead. They are mass-produced and relatively inexpensive. Although the brushes actually ride on the outside of the commutator, for convenience they are shown on the inside in Fig. 3.2-6. The armature windings consist of a large number of turns of fine wire; hence, each circle shown in Fig. 3.2-6 represents many conductors. Note that the position of the brushes is shifted approximately 40° relative to a line drawn between the center of the north and south poles. This shift in the brushes was probably determined experimentally by minimizing brush arcing for normal load conditions. Note also that the windings do not span π radians but more like 140°, and there is an odd number of commutator segments.

In Fig. 3.2-6*a*, only winding 4 is being commutated. As the rotor advances from the position shown in Fig. 3.2-6*a*, both windings 4 and 1 are being commutated. In Fig. 3.2-6*b*, winding 1 is being commutated; then

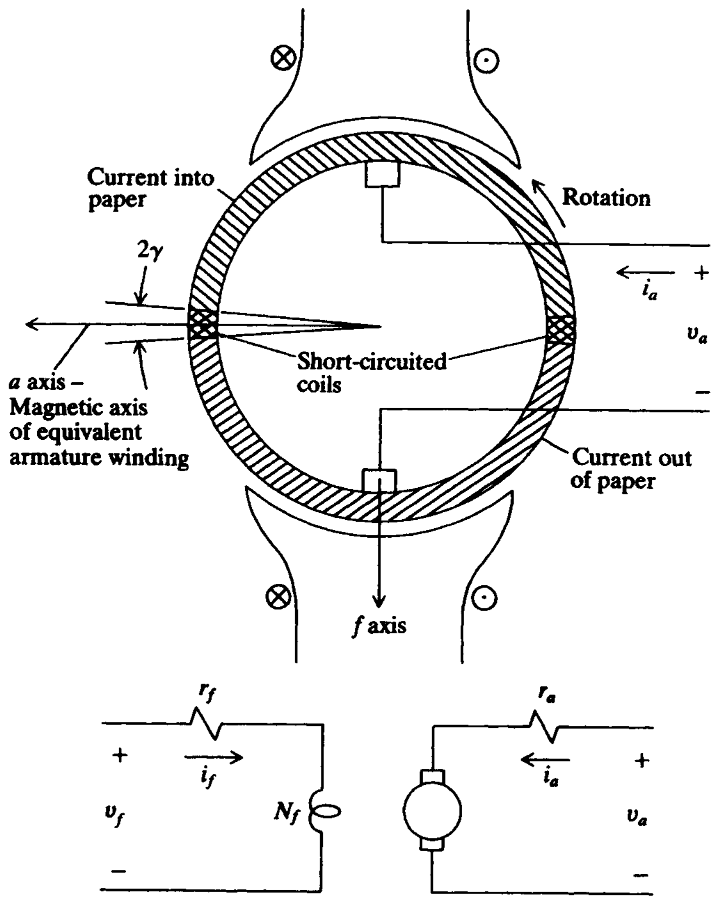

Figure 3.2-5: Idealized dc machine with uniformly distributed rotor winding.

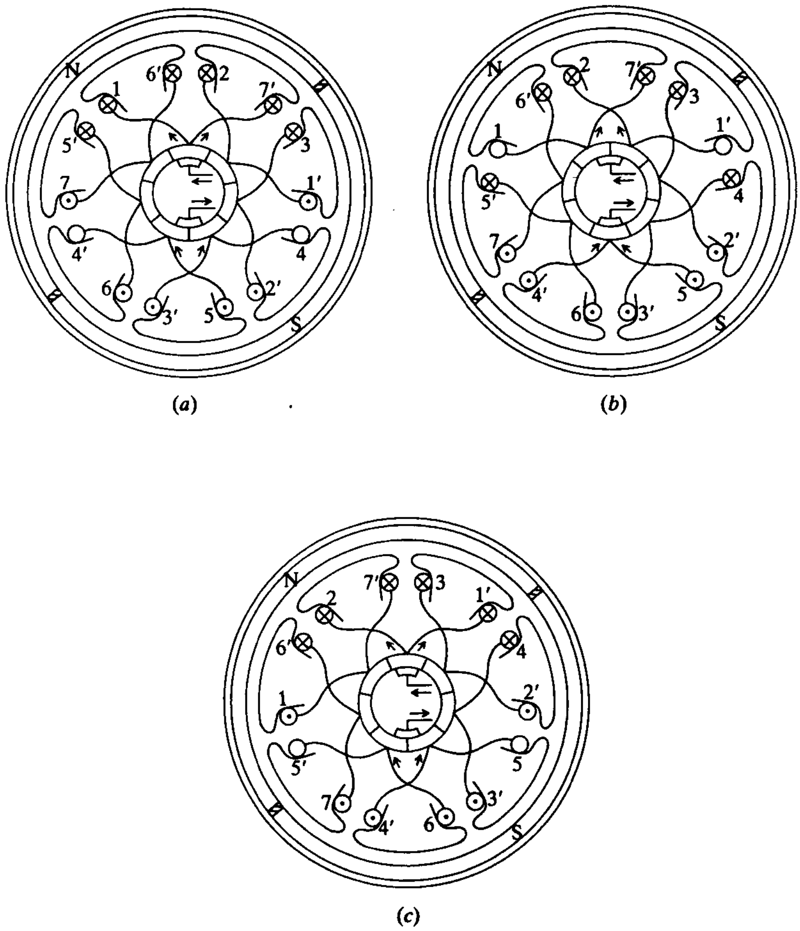

Figure 3.2-6: Commutation of a permanent-magnet dc motor.

windings 1 and 5, and, finally, in Fig. 3.2-6*c*, only winding 5 is being commutated. The windings are being commutated when the induced voltage is nonzero and one would expect some arcing to occur at the brushes. We must realize, however, that the manufacture and sale of these devices is very competitive, and one often must compromise when striving to produce an acceptable motor at the least cost possible. Although we realize that in some cases it may be a rather crude approximation, we will consider the permanent-magnet dc motor as having current sheets on the armature with orthogonal armature and field magnetic axes as shown in Fig. 3.2-5.

For purposes of illustration, a two-pole, general-purpose, shunt-field dc machine is shown in Fig. 3.2-7. A disassembled two-pole, 0.1-hp, 6-V, 12,000-r/min, permanent-magnet dc motor is shown in Fig. 3.2-8. The magnets are made from samarium cobalt and the device is used to drive hand-held battery-operated surgical instruments. Although some of these terms are new to us, they will be defined as we go along.

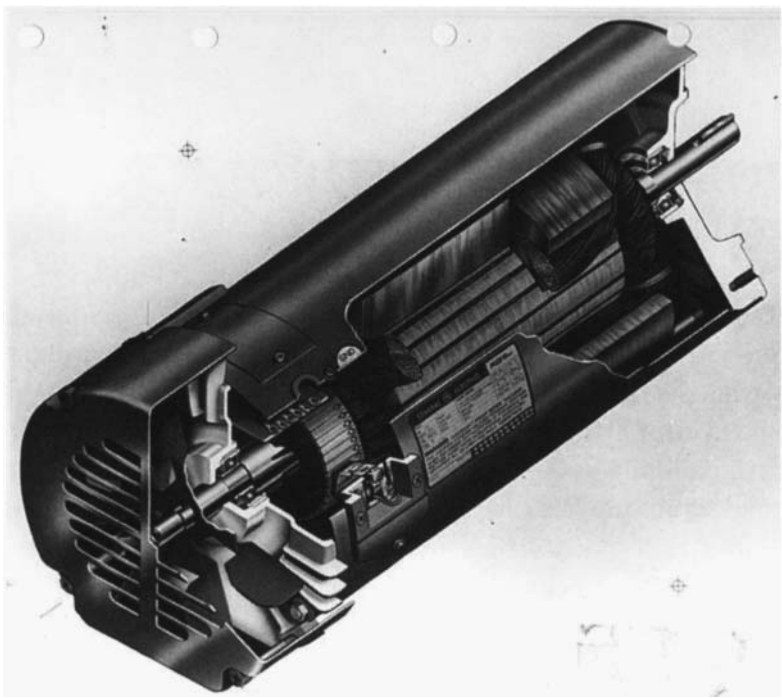

Figure 3.2-7: Cutaway view of two-pole, 3-hp, 180-V, 2500-r/min, shunt-field dc motor. (Courtesy of GE.)

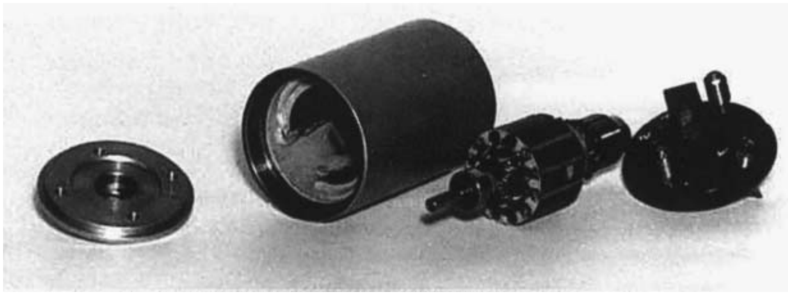

Figure 3.2-8: Two-pole, 0.1-hp, 6-V, 12,000-r/min, permanent-magnet dc motor. (Courtesy of Vick ElectroMech.)

SP3.2-1 The peak value of the voltage induced in one coil shown in Fig. 3.2-3 is 1 V. Determine, from Fig. 3.2-3, the maximum and minimum value of v_a. [2.613 V; 2.414 V]

SP3.2-2 Consider Fig. 3.2-3. Indicate the two parallel paths immediately following commutation of $a_3 - a_3'$ and $A_3 - A_3'$. [$A_3' - A_3$, $A_4' - A_4$, $a_1 - a_1'$, and $a_2 - a_2'$; $A_2 - A_2'$, $A_1 - A_1'$, $a_4' - a_4$, and $a_3' - a_3$]

3.3 VOLTAGE AND TORQUE EQUATIONS

It is advantageous to first consider the dc machine with a field and armature winding before turning to the permanent-magnet device exclusively. Although rigorous derivation of the voltage and torque equations is possible, it is rather lengthy and little is gained since these relationships may be deduced. The armature coils revolve in a magnetic field established by a current flowing in the field winding. We have established that voltage is induced in these coils by virtue of this rotation. However, the action of the commutator causes the armature coils to appear as a stationary winding with its magnetic axis orthogonal to the magnetic axis of the field winding. Consequently, voltages are not induced in one winding due to the time rate of change of the current flowing in the other (transformer action). Mindful of these conditions, we can write the field and armature voltage equations in matrix form as

$$\begin{bmatrix} v_f \\ v_a \end{bmatrix} = \begin{bmatrix} r_f + pL_{FF} & 0 \\ \omega_r L_{AF} & r_a + pL_{AA} \end{bmatrix} \begin{bmatrix} i_f \\ i_a \end{bmatrix} \tag{3.3-1}$$

where L_{FF} and L_{AA} are the self-inductances of the field and armature windings, respectively, and p is the short-hand notation for the operator d/dt. The rotor speed is denoted as ω_r, and L_{AF} is the mutual inductance between the field and the rotating armature coils. The above equation suggests the equivalent circuit shown in Fig. 3.3-1. The voltage induced in the armature circuit, $\omega_r L_{AF} i_f$, is commonly referred to as the counter or back emf. It also represents the open-circuit armature voltage.

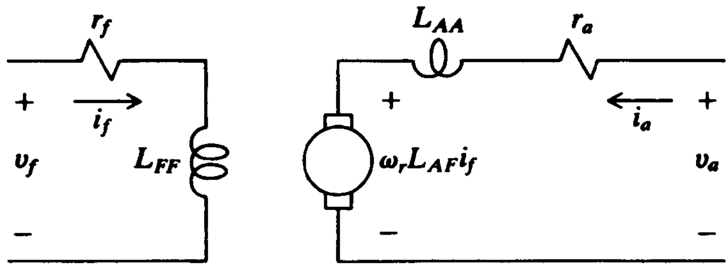

Figure 3.3-1: Equivalent circuit of dc machine.

There are several other forms in which the field and armature voltage equations are often expressed. For example, L_{AF} may also be written as

$$L_{AF} = \frac{N_a N_f}{\Re} \tag{3.3-2}$$

where N_a and N_f are the equivalent turns of the armature and field windings, respectively, and $\Re$ is the reluctance. Thus,

$$L_{AF} i_f = N_a \frac{N_f i_f}{\Re} \tag{3.3-3}$$

If we now replace $N_f i_f/\Re$ with Φ_f, the field flux per pole, then $N_a \Phi_f$ may be substituted for $L_{AF} i_f$ in the armature voltage equation.

Another substitute variable often used is

$$k_v = L_{AF} i_f \tag{3.3-4}$$

We will find that this substitute variable is particularly convenient and frequently used. Even though a permanent-magnet dc machine has no field circuit, the constant field flux produced by the permanent magnet is analogous to a dc machine with a constant k_v.

We can take advantage of previous work to obtain an expression for the electromagnetic torque. In particular, the expression for torque given by (2.8-6) may be used directly to express the torque for the dc machine. If we fix θ_r in Fig. 1.7-4 or Fig. 2.8-1 at $-\frac{1}{2}\pi$, the same relationship exists between the magnetic axes of a dc machine (Fig. 3.2-5) and the magnetic axes of the two-coil machine. Hence, (2.8-6) may be written for the dc machine as

$$T_e = L_{AF} i_f i_a \tag{3.3-5}$$

Here again, the variable k_v is often substituted for $L_{AF} i_f$. In some instances, k_v is multiplied by a factor less than unity when substituted into (3.3-5) so as to approximate the effects of rotational losses. It is interesting that the field winding produces a stationary mmf and, owing to commutation, the armature winding also produces a stationary mmf, which is displaced $\frac{1}{2}\pi$ electrical degrees from the mmf produced by the field winding. It follows then that the interaction of these two mmfs produces the electromagnetic torque and, due to the method of commutation, the mmfs are in quadrature, thereby producing the maximum torque possible for any field and armature currents. In other words, commutation positions the magnet fields stationary and orthogonal, which yields the maximum possible torque, a feature that we will find being emulated by modern-day control of *ac* machines.

The torque and rotor speed are related by

$$T_e = J\frac{d\omega_r}{dt} + B_m \omega_r + T_L \tag{3.3-6}$$

where J is the inertia of the rotor and, in some cases, the connected mechanical load. The units of the inertia are kg $\cdot$ m^2 or J $\cdot$ s^2. A positive electromagnetic torque T_e acts to turn the rotor in the direction of increasing θ_r. The load torque T_L is positive for a torque on the shaft of the rotor, which opposes a positive electromagnetic torque T_e. The constant B_m is a damping coefficient associated with the mechanical rotational system of the machine. It has the units of N $\cdot$ m $\cdot$ s and it is generally small and often neglected.

In the next section, we will focus on the permanent-magnet dc motor; however, it is worthwhile to take a moment to mention that we have established the basis for several types of dc machines. In particular, the machine shown in Fig. 3.3-1 is a separately excited dc machine. If we connect the field winding in parallel with the armature winding, it becomes a shunt-connected dc machine. If the field winding is connected in series with the armature winding, it is a series-connected dc machine. If two windings are used, one in parallel with and another in series with the armature, it is referred to as a compound-connected dc machine. Clearly, this is an overly simplistic description and the reader is referred to [1,2] for a more detailed consideration of these machine types.

SP3.3-1 When a 12-V, permanent-magnet dc motor is driven at 100 rad/s, the open-circuit voltage is 10 V. Calculate k_v. [$k_v = 0.1$ V · s/rad]

SP3.3-2 The armature applied voltage of a dc motor is 240 V; the rotor speed is constant at 50 rad/s. The steady-state armature current is 15 A, the armature resistance is 1 Ω, and $L_{AF} = 1$ H. Calculate the steady-state field current. [I_f= 4.5 A]

SP3.3-3 Calculate the no-load speed ($T_L = 0$) for the permanent-magnet dc motor in SP3.3-1 when rated voltage (12 V) is applied to the armature. [$\omega_r = 120$ rad/s]

SP3.3-4 Multiply the expression for v_a given in (3.3-1) by i_a and identify all terms. [$v_a i_a$ the input power to armature, $i_a^2 r_a$ the armature ohmic loss, $L_{AA} i_a p i_a$ the change of energy stored in L_{AA}, $L_{AF} i_f i_a \omega_r = T_e \omega_r$ the output power]

SP3.3-5 Is SP3.3-4 an alternate method of deriving an expression for torque? Why? [Yes; $L_{AF} i_f i_a$ is the coefficient of $p\theta_r$]

3.4 PERMANENT-MAGNET dc MACHINE

In the case of the permanent-magnet dc machine, $L_{AF} I_f$ is replaced with a constant k_v, whereupon the steady-state armature voltage equation becomes

$$V_a = r_a I_a + k_v \omega_r \tag{3.4-1}$$

If (3.4-1) is solved for I_a and substituted in to (3.3-5) with $L_{AF} I_f$ replaced by k_v, the steady-state torque may be expressed as

$$T_e = \frac{k_v V_a - k_v^2 \omega_r}{r_a} \tag{3.4-2}$$

The steady-state torque-speed characteristic is shown in Fig. 3.4-1.

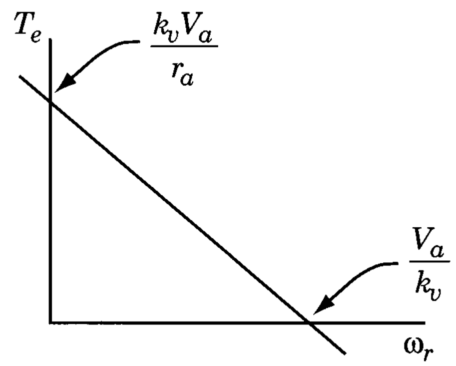

Figure 3.4-1: Steady-state torque-speed characteristic of a permanent-magnet dc machine.

It apparent from Fig.3.4-1, that the stall ($\omega_r = 0$) torque could be made larger for a given armature voltage by reducing r_a. Although the machine may be designed with a smaller armature resistance, there is a problem since, at stall, the steady-state armature current is limited by the armature resistance; hence, for a constant V_a, reducing r_a will result in a larger I_a, which can cause damage to the brushes. On the other hand, increasing the starting torque by reducing r_a causes the torque-speed characteristics to have a steeper slope, which results in a smaller change in speed for a given change in load torque during normal (near rated) operation. If, however, the armature voltage is reduced during the starting period to protect the brushes, the desirable characteristic of a small speed change during load torque variations during normal operation could be achieved. In fact, controlled regulation of the armature voltage is generally employed for large horsepower machines by using a converter; however, low-power, permanent-magnet dc machines are generally supplied from a battery, as in the case of the automobile, and, therefore, a large armature resistance is necessary in order to prevent brush damage during the early part of the starting period. Fortunately, a small speed variation during load torque changes is not required in many applications of the permanent-magnet dc machine; therefore, the steep torque-speed characteristics are not necessary. We will consider the dynamic performance of a permanent-magnet dc motor in the next section. In a later chapter, we will discover that the permanent-magnet ac machine can be controlled to have torque-speed characteristics similar to that of the permanent-magnet dc motor. This device is commonly referred to as a brushless dc machine.

Example 3A. A permanent-magnet dc motor similar to that shown in Fig. 3.2-8 is rated at 6 V with the following parameters: $r_a = 7\,\Omega$, $L_{AA} = 120$ mH, $k_T = 2$ oz·in/A, and $J = 150\,\mu$oz·in·s^2. According to the motor information sheet, the no-load speed is approximately 3350 r/min and the no-load armature current is approximately 0.15 A. Let us attempt to interpret this information.

First, let us convert k_T and J to units that we have been using in this text. In this regard, we will convert the inertia to kg · m^2, which is the same as N · m · s^2. To do this, we must convert ounces to Newtons and inches to meters (Appendix A). Thus,

$$J = \frac{150 \times 10^{-6}}{(3.6)(39.37)} = 1.06 \times 10^{-6}\ \mathrm{kg \cdot m^2} \tag{3A-1}$$

We have not seen k_T before. It is the torque constant and, if expressed in the appropriate units, it is numerically equal to k_v. When k_v is used in the expression for T_e ($T_e = k_v i_a$), it is often referred to as the *torque constant* and denoted k_T. When used in the voltage equation, it is always denoted k_v. Now we must convert oz · in to N · m, whereupon k_T equals our k_v; hence,

$$k_v = \frac{2}{(16)(0.225)(39.37)} = 1.41 \times 10^{-2}\mathrm{N \cdot m/A} = 1.41 \times 10^{-2}\mathrm{V \cdot s/rad} \tag{3A-2}$$

What do we do about the no-load armature current? What does it represent? Well, probably it is a measure of the friction and windage losses. We could neglect it, but we will not. Instead, let us include it as B_m. First, however, we must calculate the no-load speed. We can solve for the no-load rotor speed from the steady-state armature voltage equation (3.4-1):

$$\omega_r = \frac{V_a - r_a I_a}{k_v} = \frac{6 - (7)(0.15)}{1.41 \times 10^{-2}} = 351.1 \text{ rad/s}$$

$$= \frac{(351.1)(60)}{2\pi} = 3353 \text{ r/min} \tag{3A-3}$$

Now at this no-load speed,

$$T_e = k_v i_a = (1.41 \times 10^{-2})(0.15) = 2.12 \times 10^{-3} \text{ N} \cdot \text{m} \tag{3A-4}$$

Since T_L and $J(d\omega_r/dt)$ are zero for this steady-state no-load condition, (3.3-6) tells us that (3A-4) is equal to $B_m\omega_r$; hence,

$$B_m = \frac{2.12 \times 10^{-3}}{\omega_r} = \frac{2.12 \times 10^{-3}}{351.1} = 6.04 \times 10^{-6} \text{ N} \cdot \text{m} \cdot \text{s} \tag{3A-5}$$

Example 3B. The permanent-magnet dc machine described in Example 3A is operating with rated applied armature voltage and a load torque T_L of 0.5 oz · in. Our task is to determine the efficiency where percent eff = (power output/power input)100.

First let us convert oz · in to N · m:

$$T_L = \frac{0.5}{(16)(0.225)(39.37)} = 3.53 \times 10^{-3} \text{ N} \cdot \text{m} \tag{3B-1}$$

In Example 3A, we determined k_v to be 1.41×10^{-2} V · s/rad and B_m to be 6.04×10^{-6} N · m · s.

During steady-state operation, (3.3-6) becomes

$$T_e = B_m\omega_r + T_L \tag{3B-2}$$

From (3.3-5), with $L_{AF}i_f$ replaced by k_v, the steady-state electromagnetic torque is

$$T_e = k_v I_a \tag{3B-3}$$

Substituting (3B-3) into (3B-2) and solving for ω_r yields

$$\omega_r = \frac{k_v}{B_m} I_a - \frac{1}{B_m} T_L \tag{3B-4}$$

Repeating (3.4-1)

$$V_a = r_a I_a + k_v \omega_r \tag{3B-5}$$

Substituting (3B-4) into (3B-5) and solving for I_a yields

$$I_a = \frac{V_a + (k_v/B_m)T_L}{r_a + (k_v^2/B_m)}$$

$$= \frac{6 + [(1.41 \times 10^{-2})/(6.04 \times 10^{-6})](3.53 \times 10^{-3})}{7 + (1.41 \times 10^{-2})^2/(6.04 \times 10^{-6})} = 0.357 \text{ A} \tag{3B-6}$$

From (3B-4),

$$\omega_r = \frac{1.41 \times 10^{-2}}{6.04 \times 10^{-6}} 0.357 - \frac{1}{6.04 \times 10^{-6}}(3.53 \times 10^{-3})$$
$$= 249 \text{ rad/s} \tag{3B-7}$$

The power input is

$$P_{\text{in}} = V_a I_a = (6)(0.357) = 2.14 \text{ W} \tag{3B-8}$$

The power output is

$$P_{\text{out}} = T_L \omega_r = (3.53 \times 10^{-3})(249) = 0.88 \text{ W} \tag{3B-9}$$

The efficiency is

$$\text{Percent eff} = \frac{P_{\text{out}}}{P_{\text{in}}} 100$$
$$= \frac{0.88}{2.14} 100 = 41.1 \text{ percent} \tag{3B-10}$$

This low efficiency is characteristic of low-power dc motors due to the relatively large armature resistance. In this regard, it is interesting to determine the losses due to i^2r, friction, and windage:

$$P_{i^2r} = r_a I_a^2 = (70)(0.357)^2 = 0.89 \text{ W} \tag{3B-11}$$

$$P_{fw} = (B_m \omega_r)\omega_r = (6.04 \times 10^{-6})(249)^2 = 0.37 \text{ W} \tag{3B-12}$$

Let us check our work:

$$P_{\text{in}} = P_{i^2r} + P_{fw} + P_{\text{out}} = 0.89 + 0.37 + 0.88 = 2.14 \text{ W} \tag{3B-13}$$

which is equal to (3B-8).

SP3.4-1 A 12-V, permanent-magnet dc motor has an armature resistance of 12 Ω and $k_v = 0.01$ V · s/rad. Calculate the steady-state stall torque (T_e with $\omega_r = 0$). [$T_e = 0.01$ N · m]
SP3.4-2 Determine T_e in Example 3B. [$T_e = 0.713$ oz · in]

3.5 DYNAMIC CHARACTERISTICS OF A PERMANENT-MAGNET dc MOTOR

Two modes of dynamic operation are of interest; starting from stall and changes in load torque with the machine supplied from a constant-voltage source. The permanent-magnet dc machine considered in Examples 3A and 3B is used to demonstrate these modes of operation. It is important that the reader become familiar with the material in these examples before proceeding.

Dynamic Performance During Starting

In the previous section, it was pointed out that, if the armature resistance is small, damaging armature current could result if rated voltage is applied to the armature terminals when the machine is stalled ($\omega_r = 0$). With the machine at stall, the counter emf is zero; therefore, during the transient starting period, the armature current is opposed only by the voltage drop across the armature resistance ($r_a i_a$) and the armature inductance ($L_{AA} di_a/dt$). We have mentioned and also noted in Examples 3A and 3B that low-power permanent-magnet dc motors are designed with a large armature resistance, making it possible to "direct line" start these devices without damaging brushes or the stator windings. The no-load starting characteristics ($T_L = 0$) of the permanent-magnet dc motor described in Example 3A are shown in Fig. 3.5-1. The armature voltage v_a, the armature current i_a, and the rotor speed ω_r are plotted. Initially, the motor is at stall and, at time zero, 6 V is applied to the armature terminals. The peak transient armature current is limited to approximately 0.55 A due to the voltage drop across the inductance and resistance of the armature and the fact that the rotor is accelerating from stall, thereby developing the voltage $k_v\omega_r$, which also opposes the applied voltage. After about 0.25 s, steady-state operation is achieved with the no-load armature current of 0.15 A. (From Example 3A, $B_m = 6.04 \times 10^{-6}$ N · m · s.) It is noted that the rotor speed is slightly os-

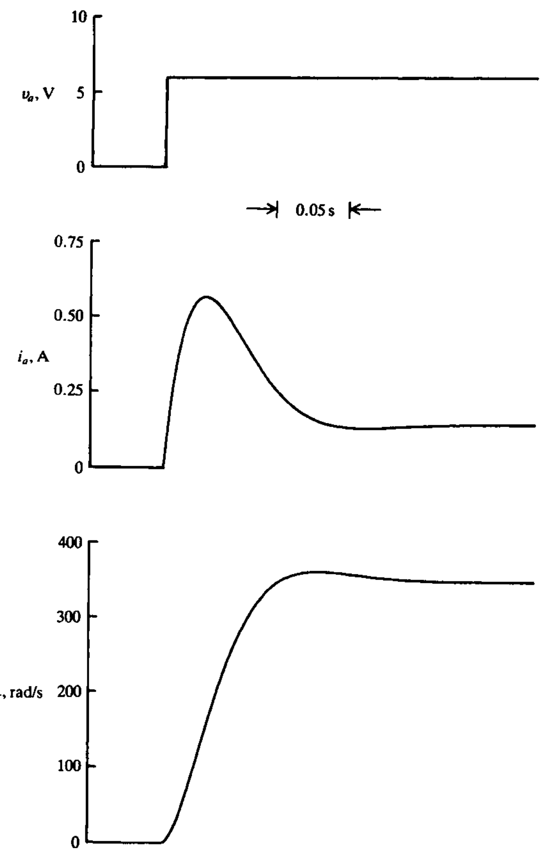

Figure 3.5-1: Starting characteristics of a permanent-magnet dc machine.

cillatory (underdamped), as illustrated by the small overshoot of the final, steady-state value.

Dynamic Performance During Sudden Changes in Load Torque

In Example 3B, we calculated the efficiency of the permanent-magnet dc motor given in Example 3A with a load torque of 0.5 oz·in (3.53×10^{-3}N·m). Let us assume that this load torque was suddenly applied with the motor initially operating at the no-load condition (I_a = 0.15 A). The dynamic characteristics following a step change in load torque T_L from zero to 0.5 oz · in are shown in Fig. 3.5-2. The armature current i_a and the rotor speed ω_r are plotted. Since $T_e = k_v i_a$ and since k_v is constant, T_e differs from i_a by a constant multiplier. It is noted that the system is slightly oscillatory. Also, it is noted that the change in the steady-state rotor speed is quite large.

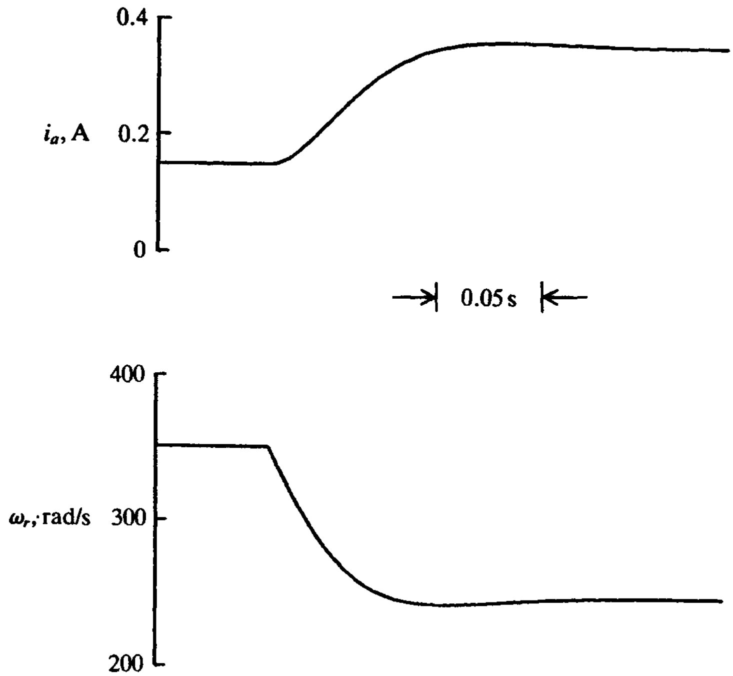

Figure 3.5-2: Dynamic performance of a permanent-magnet dc motor following a sudden increase in load torque from zero to 0.5 oz · in.

From Example 3A or Fig. 3.5-2, we see that no-load speed is 351.1 rad/s. With $T_L = 0.5$ oz · in, the rotor speed is 249 rad/s as calculated in Example 3B and noted from Fig. 3.5-2. There has been approximately a 30 percent decrease in speed for this increase in load torque. As we have mentioned, this is characteristic of a low-power permanent-magnet motor owing to the high armature resistance.

SP3.5-1 Plot the T_e versus ω_r characteristic for the permanent-magnet dc motor given in Example 3A. [A straight line between (12.1×10^{-3}, 0) and (0, 425.5)]

SP3.5-2 Assume that the peak value of i_a in Fig. 3.5-1 is 0.55 A. Calculate $k_v\omega_r$ at peak i_a. [$k_v\omega_r = 2.15\,\mathrm{V}$]

3.6 INTRODUCTION TO CONSTANT-TORQUE AND CONSTANT-POWER OPERATION

If it were not for the frictional losses attributed to the sliding contact between the brushes and commutator segments and the arcing during commutation, the dc machine would be the ideal electromagnetic device. Due to commutation, the stator and rotor magnetic systems are orthogonal, thereby producing the maximum possible torque per ampere (per unit of field strength). We will find that this feature is the goal of many of the control methods used with the permanent-magnet ac machine and the induction machine.

Fortunately, the equations that describe the performance of a dc machine provide an excellent opportunity to introduce these control principles in a straightforward manner. In this section, we will attempt to take advantage of this opportunity and to do this without the complexities of converter switching or the details of the actual controls.

Let us start by repeating the steady-state voltage and torque equations for a permanent-magnet dc machine.

$$V_a = r_a I_a + k_v \omega_r \tag{3.6-1}$$

$$T_e = k_v I_a \tag{3.6-2}$$

Recall that if the device has a field winding rather than a permanent magnet, k_v is replaced by $L_{AF} I_f$. Also, the torque-speed characteristics shown in Fig. 3.4-1 are repeated in Fig. 3.6-1. It is noted that the y- or T_e-intercept is $k_v V_a / r_a$ and the x- or ω_r-intercept is V_a / k_v. The torque-speed characteristics shown in Fig. 3.6-1 are for rated V_a (V_{aR}); however, if we reduce V_a the torque-speed line would shift down toward the origin. Therefore, depending upon the load torque and the value of V_a, steady-state operation is possible without exceeding rated V_a anywhere in the region below the rated voltage torque-speed line. To illustrate this, three load-torque characteristics are shown in Fig. 3.6-1: T_{L1}, T_{L2}, and T_{L3}. It is assumed that the load-torque characteristics are approximated by a linear function of speed. Although this may be an over simplification, it is a convenient approximation.

Three operating points are shown in Fig. 3.6-1. The first is the intersection of the T_{L1} line and the torque-speed plot with $V_a = V_{aR}$. We will assume that rated conditions V_{aR}, I_{aR}, T_{eR}, and ω_{rR} all occur at this operating point.

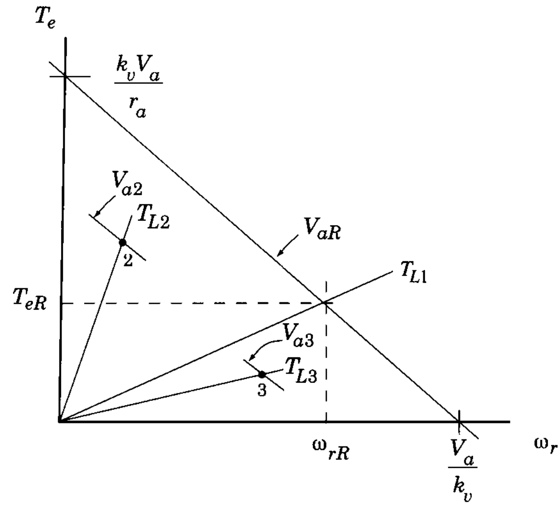

Figure 3.6-1: Torque-speed characteristics with voltage control.

The second operating point occurs at T_{L2} and V_{a2}, where $V_{a2} < V_{aR}$. The third operating point is at the intersection of the T_{L3} load line with the torque-speed plot with V_{a3}, where $V_{aR} > V_{a1} > V_{a3}$. Therefore, the region below the torque-speed plot with $V_a = V_{aR}$ is the region of the operation without exceeding V_{aR}. This is assuming that we have a means of reducing the voltage applied to the armature, which we will cover in a later section.

Constant-Torque Operation

There is something we are overlooking. What about rated armature current? Well, we know from (3.6-2), that I_{aR} occurs at T_{eR} and we see from Fig. 3.6-1 that operating point T_{L2} with V_{a2} occurs at more than twice rated torque (armature current). This operating point is not practical due to overheating and, moreover, the commutator may not be able to handle a continuous current of this magnitude. If the region of operation is controlled so that rated V_{aR} and I_{aR} are not exceeded, then the upper limit of the operating region is T_{eR}, shown by the dashed line in Fig. 3.6-1. This region or envelope of operation is shown again in Fig. 3.6-2 with solid-line boundaries. For speeds less than ω_{rR}, the maximum torque is determined by the rated current I_{aR}, whereas for speeds greater than ω_{rR}, the maximum torque is determined by the rated voltage V_{aR}. Also shown are three load-torque lines $\frac{5}{3}T_{L1}$, T_{L1}, and $\frac{2}{3}T_{L1}$. This figure is the same as the third trace in Fig. 3.6-3, where machine

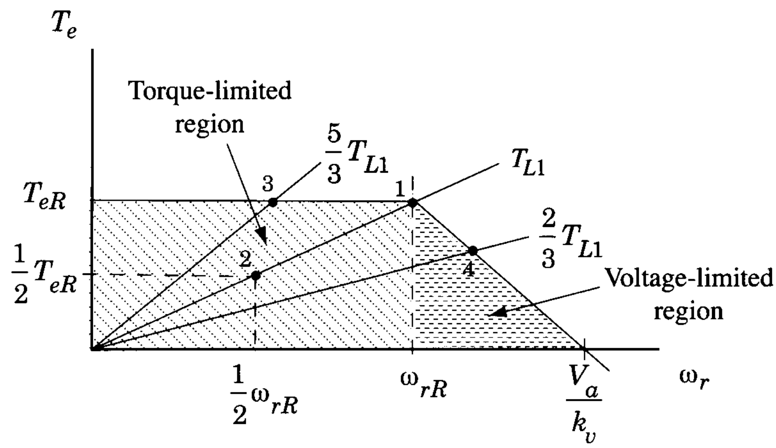

Figure 3.6-2: Torque control of a permanent-magnet dc machine for $\omega_r \leq \omega_{rR}$.

variables I_a, V_a, P_{in} and output power P_o are also plotted. The plots shown in Fig. 3.6-3 depict the values of the armature voltage V_a and current I_a necessary to produce the boundary or envelope of operation shown in Fig. 3.6-2. However, it is important to emphasize that the region of operation is defined by the envelope of the torque-versus-speed characteristics as shown by the shaded area in Fig. 3.6-2.

We have noticed the dots and numbers, which we assume represent operating points, in Figs. 3.6-2 and 3.6-3, and we have been waiting for an explanation. Depending on the type of control being implemented, the commanded values could be one or all of the desired values of armature voltage, torque (armature current), and/or rated speed. Also, depending upon the type of control, the sensed or measured machine variables could include armature voltage and/or current and rotor speed.

The schematic block diagram shown in Fig. 3.6-4 is sufficient to illustrate torque control for the points of operation shown in Figs. 3.6-2 and 3.6-3. As mentioned, three load-torque characteristics are shown in Figs. 3.6-2 and 3.6-3. The T_{L1} load-torque line intersects rated conditions. This operating point is indicated as "1" on the plots shown in Figs. 3.6-2 and 3.6-3. This would be the point of operation for this load-torque characteristic with commanded $T_e^* = T_{eR}$. Is it clear that operating point 1 would also be the normal (uncontrolled) intersection of the T_{L1} load-torque character-

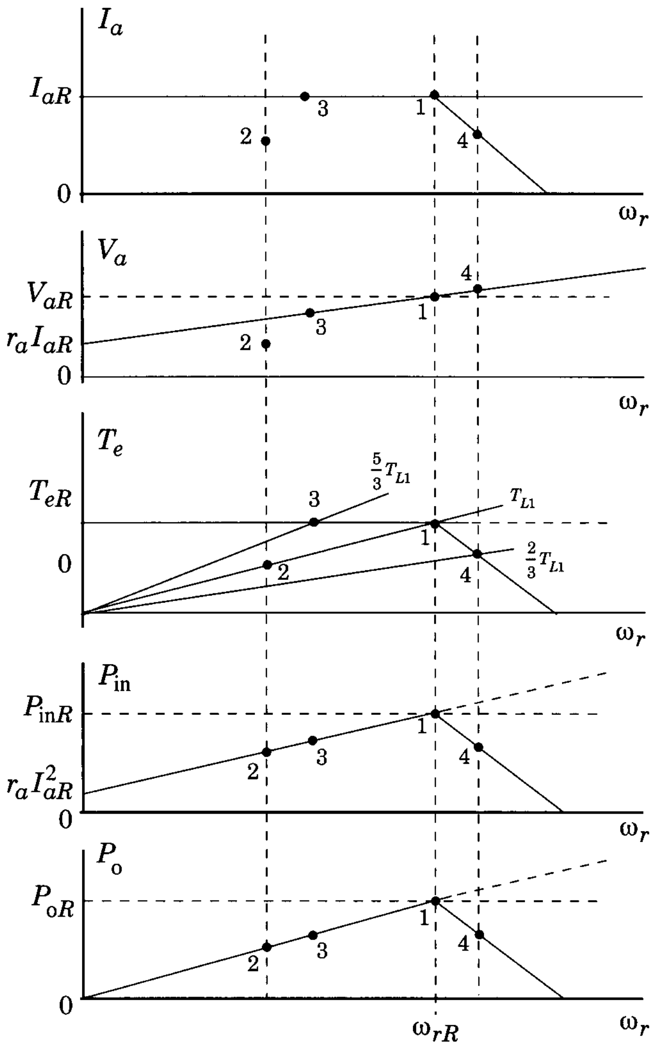

Figure 3.6-3: Machine variables for the controlled operating region depicted in Fig. 3.6-2.

istics and the torque-speed plot with V_{aR} applied to the terminals? Let us now command T_e^* to be $\frac{1}{2}T_{eR}$ with the same load-torque characteristics T_{L1}. Steady-state operation would occur at point 2. Since a linear load-torque characteristic is assumed, the rotor speed would be $\frac{1}{2}\omega_{rR}$. This operating point is indicated by "2" on the plots of the machine variables in Fig. 3.6-3. The machine is operating well within the operating envelope. If we assume that the load-torque characteristic is changed to $\frac{5}{3}T_{L1}$ and $T_e^* = T_{eR}$. Point 3 in Fig. 3.6-2 would be the controlled operating point. We see that the current limit for T_{eR}^* would be exceeded at higher rotor speeds. This operating point is shown as "3" on the plots in Fig. 3.6-3. Assume that the load-torque characteristic is changed to $\frac{2}{3}T_{L1}$ with $T_e^* = T_{eR}$. The steady-state operating point would be at "4." In this case, T_{eR} cannot be achieved without exceeding V_{aR}. We have reached the voltage limit and the control would be unable to maintain T_{eR} and with $V_a = V_{aR}$ point 4 would be the normal, uncontrolled intersection of the rated torque-speed characteristic. This operating point is also shown as "4" on the plots of machine variables in Fig. 3.6-3.

Constant-Power Operation

Before leaving this introductory discussion of controlled operation of a dc machine, let us consider one addition type of control referred to as *constant power* or *field weakening.* Although this control has limited application in the case of dc machines, the concept of this type of control, which is used in permanent-magnet ac machines, is readily explained for a dc machine. For this purpose, let us express the steady-state voltage and torque equations for the dc machine with a field winding:

$$V_a = I_a r_a + L_{AF} I_f \omega_r \tag{3.6-3}$$

$$V_f = I_f r_f \tag{3.6-4}$$

$$T_e = L_{AF} I_f I_a \tag{3.6-5}$$

The discussion presented thus far in this section applies not only to permanent-magnet dc machines but also to a dc machine with a field winding with the field current held constant. In fact, if $L_{AF}I_f$ is made equal to k_v there would be no difference; however, it is difficult to explain field weakening with a permanent-magnet dc machine. Therefore, it is convenient to change to a machine with a field winding for discussion purposes and let the rated field current I_{fR} occur at ω_{rR}.

The reason behind field weakening is to extend the speed range of the machine by reducing the back emf ($L_{AF}I_f\omega_r$) in (3.6-3), thereby allowing V_{aR} to maintain I_{aR} at rotor speeds higher than ω_{rR}. It must be assumed that higher rotor speeds can be achieved without mechanical damage.

Field weakening is illustrated in Figs. 3.6-5 and 3.6-6. The characteristics for operation below ω_{rR} are those discussed previously and shown in Figs. 3.6-2 and 3.6-3. Field weakening occurs at rotor speeds above ω_{rR}, wherein the field current is decreased as the rotor speed increases. In particular, it will be assumed that for $\omega_r > \omega_{rR}$, the field current for the boundary where $P_o^* = P_{oR}$ will be varied in accordance with

$$I_f = \frac{P_{oR}}{L_{AF}\omega_r I_{aR}} \qquad \text{for } \omega_r > \omega_{rR} \tag{3.6-6}$$

where

$$\begin{aligned} P_{oR} &= T_{eR}\omega_{rR} \\ &= L_{AF}I_{fR}I_{aR}\omega_{rR} \end{aligned} \tag{3.6-7}$$

The upper boundary of the region of operation (Fig. 3.6-5) during the field-weakening mode is obtained with $V_a = V_{aR}$ and $I_a = I_{aR}$. Thus, the input power and output power are constant at P_{inR} and P_{oR}, respectively.

Three load-torque characteristics are shown in Figs. 3.6-5 and 3.6-6: T_{L1}, $\frac{2}{3}T_{L1}$, and $\frac{1}{3}T_{L1}$. The load-torque line labeled T_{L1} is the same as used in previous figures and is indicated as operating point 1 in Figs. 3.6-5 and 3.6-6. Operating point 2 is for $\frac{2}{3}T_{L1}$ and operating point 3 for $\frac{1}{3}T_{L1}$. The schematic block diagram shown in Fig. 3.6-7 is a control method that could be used to achieve constant-power control.

The controlled-torque mode operation illustrated in Figs. 3.6-2 and 3.6-3 is often referred to as the constant-torque mode, whereas the field-weakening mode illustrated in Figs. 3.6-5 and 3.6-6 is often called the constant-power mode. One might confuse the boundary or envelope as a plot of the only operating points. Indeed, the boundaries shown in Figs. 3.6-5 and 3.6-6 form the envelope of an array of possible operating points within the envelope. It is interesting that the upper boundary of T_e during the constant-power (field-weakening) mode varies as I_f, (3.6-7), since $I_a = I_{aR}$.

When operating in the constant-power mode, it would be customary to command another operating condition in addition to P_o. If, for example, $P_o^* = \frac{1}{2}P_{oR}$ and $I_a^* = I_{aR}$, then V_f would be controlled so that I_f would be

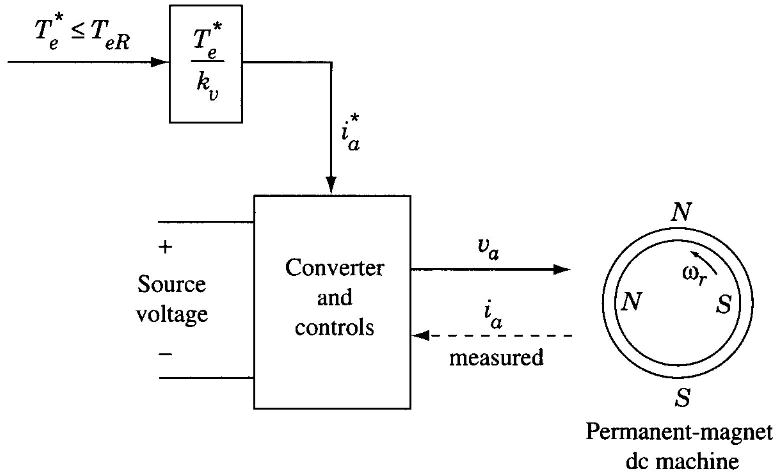

Figure 3.6-4: Torque control of a permanent-magnet dc machine for $\omega_r \le \omega_{rR}$.

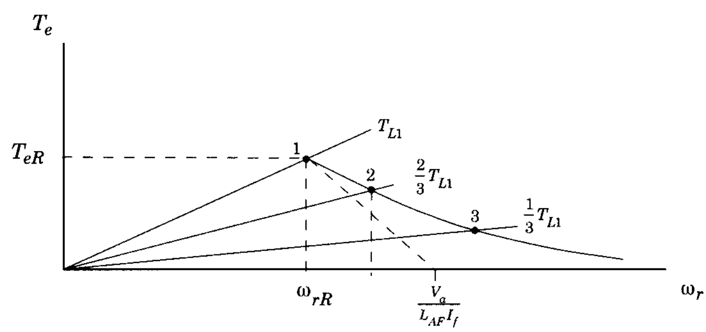

Figure 3.6-5: Controlled operating region with constant power (field weakening) added.

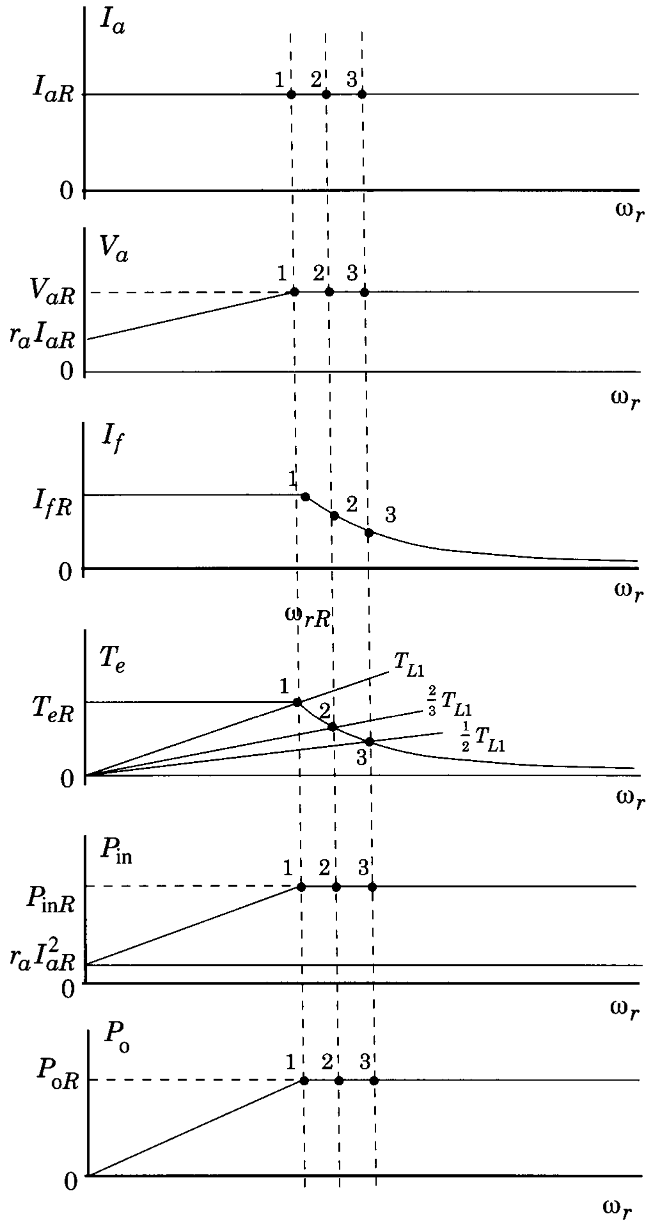

Figure 3.6-6: Machine variables for controlled operation depicted in Fig. 3.6-5.

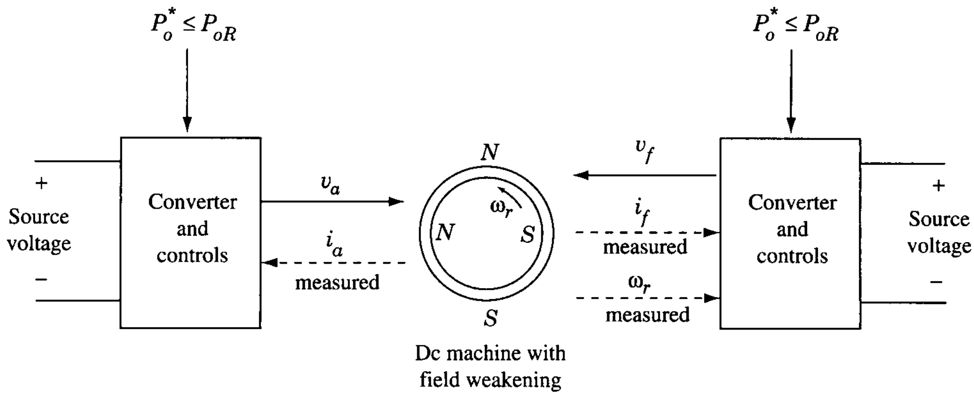

Figure 3.6-7: Power output control (field weakening) for dc machine with field winding and $\omega_r > \omega_{rR}$.

$$I_f = \frac{\omega_{rR}}{\omega_r} I_{fR} \tag{3.6-8}$$

The armature voltage would be regulated to

$$V_a = r_a I_a^* + \frac{P_o^*}{I_a^*} \tag{3.6-9}$$

wherein $I_a^* = I_{aR}$ and $P_o^* = \frac{1}{2}P_{oR}$.

SP3.6-1 If in Fig. 3.6-1 the load-torque characteristic is T_{L3} and $V_a = V_{aR}$, define the operating point. [Intersection of T_{L3} and the rated V_a torque-speed curve]

SP3.6-2 What would happen to the rotor speed if, during field weakening, the field current accidentally became zero? Why? [ω_r would accelerate to machine destruction; field flux would be only the residue]

SP3.6-3 For the operating points shown in Fig. 3.6-1, identify those that would fall within the constant-torque operating envelope. [All but operating point 2 in Fig. 3.6-1]

SP3.6-4 Verify (3.6-8) and (3.6-9).

3.7 TIME-DOMAIN BLOCK DIAGRAM AND STATE EQUATIONS FOR THE PERMANENT-MAGNET dc MACHINE

Although the analysis of control systems is not our intent, it is worthwhile to set the stage for this type of analysis by means of a first look at time-domain block diagrams and state equations. Block diagrams, which portray the interconnection of the system equations, are used extensively in control system analysis and design. Arranging the equations of a permanent-magnet dc machine into a block diagram representation is straightforward. The armature voltage equation, (3.3-1), and the relationship between torque and rotor speed, (3.3-6), may be written as

$$v_a = r_a(1 + \tau_a p)i_a + k_v\omega_r \tag{3.7-1}$$

$$T_e - T_L = (B_m + Jp)\omega_r \tag{3.7-2}$$

where the armature time constant is $\tau_a = L_{AA}/r_a$. Here, p denotes d/dt and $1/p$ denotes integration. Solving (3.7-1) for i_a, and (3.7-2) for ω_r yields

$$i_a = \frac{1/r_a}{\tau_a p + 1}(v_a - k_v\omega_r) \tag{3.7-3}$$

$$\omega_r = \frac{1}{Jp + B_m}(T_e - T_L) \tag{3.7-4}$$

A few comments are in order regarding these expressions. In (3.7-3), we see that $(v_a - k_v\omega_r)$ is multiplied by the operator $(1/r_a)/(\tau_a p + 1)$ to obtain the armature current i_a. The fact that we are multiplying the voltage by an operator to obtain current is in no way indicative of the procedure that we might actually use to calculate the current i_a given the voltage $(v_a - k_v\omega_r)$. We are simply expressing the dynamic relationship between voltage and current in a form convenient for drawing block diagrams. However, to calculate i_a, we may prefer to express (3.7-3) in its equivalent form (3.7-1) and solve the given first-order differential equation using standard techniques. The operator $(1/r_a)/(\tau_a p + 1)$ in (3.7-3) may also be interpreted as a transfer function relating the voltage and current. Those of you who are familiar with

Laplace transform methods are likely accustomed to seeing transfer functions expressed in terms of the Laplace operator s instead of the differentiation operator p. In fact, the same transfer functions are obtained by using Laplace transform theory with p replaced by s.

The time-domain block diagram portraying (3.7-3) and (3.7-4) with $T_e = k_v i_a$ is shown in Fig. 3.7-1. This diagram consists of a set of linear blocks, wherein the relationship between the input and corresponding output variable is depicted in transfer function form.

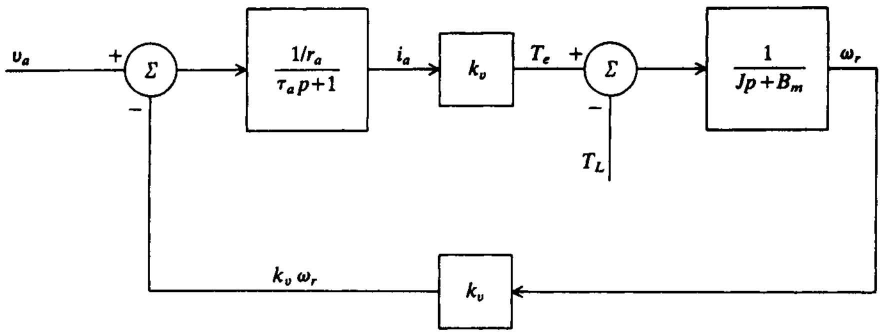

Figure 3.7-1: Time-domain block diagram of a permanent-magnet dc machine.

The so-called state equations of a system represent the formulation of the state variables into a matrix form convenient for computer implementation for linear systems. The state variables of a system are defined as a minimal set of variables such that knowledge of these variables at any initial time t_0 plus information on the input excitation subsequently applied is sufficient to determine the state of the system at any time $t > t_0$ [2]. In the case of the permanent-magnet dc machines, the armature current i_a, the rotor speed ω_r, and the rotor position θ_r are the state variables. However, since θ_r can be established from ω_r by using

$$\frac{d\theta_r}{dt} = \omega_r \tag{3.7-5}$$

and since θ_r is considered a state variable only when the shaft position is a controlled variable, we will omit θ_r from consideration in this development.

Solving the armature voltage equation, (3.7-1), for di_a/dt yields

$$\frac{di_a}{dt} = -\frac{r_a}{L_{AA}}i_a - \frac{k_v}{L_{AA}}\omega_r + \frac{1}{L_{AA}}v_a \tag{3.7-6}$$

From (3.7-2), with $k_v i_a$ substituted for T_e yields

$$\frac{d\omega_r}{dt} = -\frac{B_m}{J}\omega_r + \frac{k_v}{J}i_a - \frac{1}{J}T_L \tag{3.7-7}$$

The system is described by a set of linear differential equations. In matrix form, the state equations become

$$p\begin{bmatrix} i_a \\ \omega_r \end{bmatrix} = \begin{bmatrix} -\dfrac{r_a}{L_{AA}} & -\dfrac{k_v}{L_{AA}} \\ \dfrac{k_v}{J} & -\dfrac{B_m}{J} \end{bmatrix}\begin{bmatrix} i_a \\ \omega_r \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L_{AA}} & 0 \\ 0 & -\dfrac{1}{J} \end{bmatrix}\begin{bmatrix} v_a \\ T_L \end{bmatrix} \tag{3.7-8}$$

The form in which the state equations are expressed in (3.7-8) is called the fundamental form. In particular, the previous matrix equation may be expressed symbolically as

$$p\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{3.7-9}$$

which is called the fundamental form, where $\mathbf{x}$ is the state vector (column matrix of state variables), and $\mathbf{u}$ is the input vector (column matrix of inputs to the system). We see that (3.7-8) and (3.7-9) are identical in form. Methods of solving equations of the fundamental form given by (3.7-9) are well known. Consequently, it is used extensively in control system analysis. If we were to consider a dc machine with a field winding as shown in Fig. 3.3-1, the field current would be a state variable. In this case, the state equations become nonlinear due to the product of state variables in $L_{AF}i_f i_a$ and $L_{AF}i_f\omega_r$.

Example 3C. Once the permanent-magnet dc motor is portrayed in block diagram form (Fig. 3.6-1), it is often advantageous, for control design purposes, to express transfer functions between state and input variables. Our task is to derive transfer functions between the state variables (i_a and ω_r) and the input variables (v_a and T_L) for the permanent-magnet dc machine. Repeating (3.7-3) and (3.7-4)

$$i_a = \frac{1/r_a}{\tau_a p + 1}(v_a - k_v\omega_r) \tag{3C-1}$$

$$\omega_r = \frac{1}{Jp + B_m}(k_v i_a - T_L) \tag{3C-2}$$

If (3C-1) is substituted into (3C-2), we obtain, after considerable work,

$$\omega_r = \frac{(1/k_v\tau_a\tau_m)v_a - (1/J)(p + 1/\tau_a)T_L}{p^2 + (1/\tau_a + B_m/J)p + (1/\tau_a)(1/\tau_m + B_m/J)} \tag{3C-3}$$

where a new time constant has been introduced. The inertia time constant, which is what τ_m is called, is defined as

$$\tau_m = \frac{Jr_a}{k_v^2} \tag{3C-4}$$

The transfer function between ω_r and v_a may be obtained from (3C-3) by setting T_L equal to zero in (3C-3) and dividing both sides by v_a. Similarly, the transfer function between ω_r and T_L is obtained by setting v_a to zero and dividing by T_L. To calculate ω_r given v_a and T_L, we note that p is d/dt and p^2 is d^2/dt^2 and, if we multiply each side of (3C-3) by the denominator of the right-side of the equation, we would have a second-order differential equation in terms of the state variable ω_r.

The characteristic or force-free equation for this linear system is obtained by setting the denominator equal to zero. It is of the general form

$$p^2 + 2\alpha p + \omega_n^2 = 0 \tag{3C-5}$$

We are aware that α is the exponential damping coefficient and ω_n is the undamped natural frequency. The damping factor is defined as

$$\zeta = \frac{\alpha}{\omega_n} \tag{3C-6}$$

Let us denote b_1 and b_2 as the negative values of the roots of this second-order equation,

$$b_1, b_2 = \zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{3C-7}$$

If $\zeta > 1$, the roots are real and the natural response consists of two exponential terms with negative real exponents. When $\zeta > 1$, the roots are a conjugate complex pair and the natural response consists of an exponentially decaying sinusoid.

Now, the transfer function relationship between i_a and the input variables v_a and T_L may be obtained by substituting (3C-2) into (3C-1). After some work, we obtain

$$i_a = \frac{(1/\tau_a r_a)(p + B_m J)v_a + (1/k_v \tau_a \tau_m)T_L}{p^2 + (1/\tau_a + B_m/J)p + (1/\tau_a)(1/\tau_m + B_m/J)} \tag{3C-8}$$

SP3.7-1 Express the transfer function relationship between θ_r and the input variables v_a and T_L. [$\theta_r = (1/p)$ (3C-3)]
SP3.7-2 A permanent-magnet dc machine is operating without load torque ($T_L = 0$) and $B_m = 0$. Express the transfer function between i_a and v_a. [(3C-8) with T_L and B_m both zero]

3.8 AN INTRODUCTION TO VOLTAGE CONTROL

Although the dc machine is not used as extensively as in the past, the dc drive still plays a role in some drive applications and a brief look at a method of voltage control is appropriate. Our focus will be on the permanent-magnet dc machine supplied from a two-quadrant dc converter. Steady-state and dynamic performance are illustrated. An average value model is developed and a time-domain block diagram is given. This section is not a prerequisite for material in later chapters.

Since the dc converters used in dc drive systems are often called choppers, we will use dc converter and chopper interchangeably. In this section, we will analyze the operation and establish the average-value model for a two-quadrant chopper drive. A two-quadrant dc converter is depicted in Fig. 3.8-1. Each switch $S1$ or $S2$ is a transistor. It is assumed to be ideal; that is, if $S1$ or $S2$ is closed, current is allowed to flow in the direction of the arrow; current is not permitted to flow opposite to the arrow. If $S1$ or $S2$ is open, current is not allowed to flow in either direction regardless of the voltage across the switch. If $S1$ or $S2$ is closed and the current is positive, the voltage drop across the switch is assumed to be zero. Similarly, each diode $D1$ or $D2$ is ideal. Therefore, if the diode current i_{D1} or i_{D2} is greater than zero, the voltage across the diode is zero. The diode current can never be less than zero.

A voltage control scheme that is often used in dc drives is shown in Fig. 3.8-2. As illustrated, a ramp generator provides a sawtooth waveform of period T that ramps from zero to one. This ramp is compared to k, which is referred to as the duty cycle control signal. As the name implies, k is often the output variable of an open- or closed-loop control. The switches

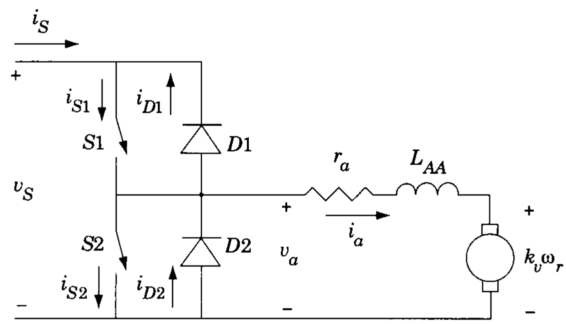

Figure 3.8-1: Two-quadrant chopper drive.

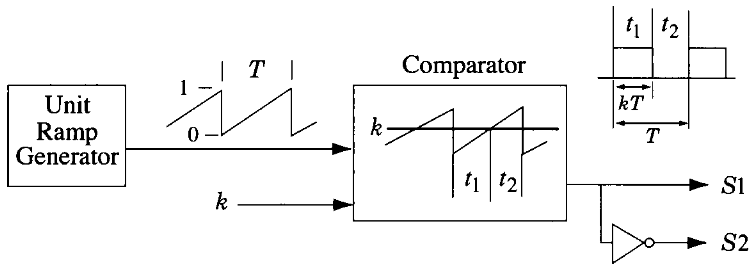

Figure 3.8-2: Pulse-width modulation voltage control.

are controlled by the output of the comparator. The duty cycle control signal may vary between zero and one ($0 \leq k \leq 1$). From Fig. 3.8-2, we see that whenever k is greater than the ramp signal, the logic output of the comparator is high. This corresponds to the time interval t_1 in Fig. 3.8-2. Now, since the ramp signal (sawtooth waveform) varies between zero and one, and since $0 \leq k \leq 1$, we can relate k, T, and t_1 as

$$t_1 = kT \tag{3.8-1}$$

which may be written

$$t_1 = \frac{k}{f_s} \tag{3.8-2}$$

where f_s is the switching or chopping frequency $\left(f_s = \frac{1}{T}\right)$.

When k is less than the ramp signal, the logic output is low. This corresponds to the time interval t_2. Thus, since

$$t_1 + t_2 = T \tag{3.8-3}$$

We can write

$$t_2 = (1-k)T = (1-k)\frac{1}{f_s} \tag{3.8-4}$$

It follows that if k is fixed at one, $S1$ is always closed ($S2$ is always open) and if k is fixed at zero, $S1$ is always open ($S2$ is always closed).

Waveforms of the converter variables during steady-state operation are shown in Fig. 3.8-3. Therein, the switching period T is large relative to the armature time constant τ_a for the purpose of depicting the transient of the armature current. Normally, the switching period is much smaller than the armature time constant and the switching segments of i_a are nearly sawtooth in shape. This is portrayed later in this section. With a two-quadrant chopper, the armature voltage cannot be negative ($v_a \geq 0$); however, the armature current can be positive or negative. That is, I_1 and I_2 (Fig. 3.8-3) can both be positive, or I_1 can be negative and I_2 positive, or I_1 and I_2 can both be negative. In Fig. 3.8-3, I_1 is negative and I_2 is positive and the average value of i_a is positive. Since the average value of v_a is positive, the mode of operation depicted is motor action if ω_r is positive (ccw).

In a two-quadrant chopper, the switching logic is generated from the duty cycle k, as shown in Fig. 3.8-2. When the comparator output signal is high, $S1$ is closed and $S2$ is open (interval A in Fig. 3.8-3); when it is low, $S1$ is open and $S2$ is closed (interval B in Fig. 3.8-3). When $S1$ is closed, current will flow either through $S1$ or $D1$; when $S2$ is closed current will flow either through $S2$ or $D2$. There is a practical consideration that must be mentioned. Electronic switches have finite turn-off and turn-on times. The turn-off time is generally longer than the turn-on time. Therefore, the switching logic must be arranged so that the turn-on signal is delayed in order to prevent short-circuiting the source, causing "shoot through." Although the delay is very short, it must be considered in the design; however, it does not make our analysis invalid, wherein we will assume instant-on, instant-off operation.

It is important to discuss the mode of operation depicted in Fig. 3.8-3. During interval A, $S1$ is closed and $S2$ is open and, at the start of interval A, $i_a = I_1$, which is negative. Since $S2$ is open, a negative i_a (I_1) can only flow through $D1$. It is important to note that $-i_{D1}$ and $-i_{S2}$ are plotted

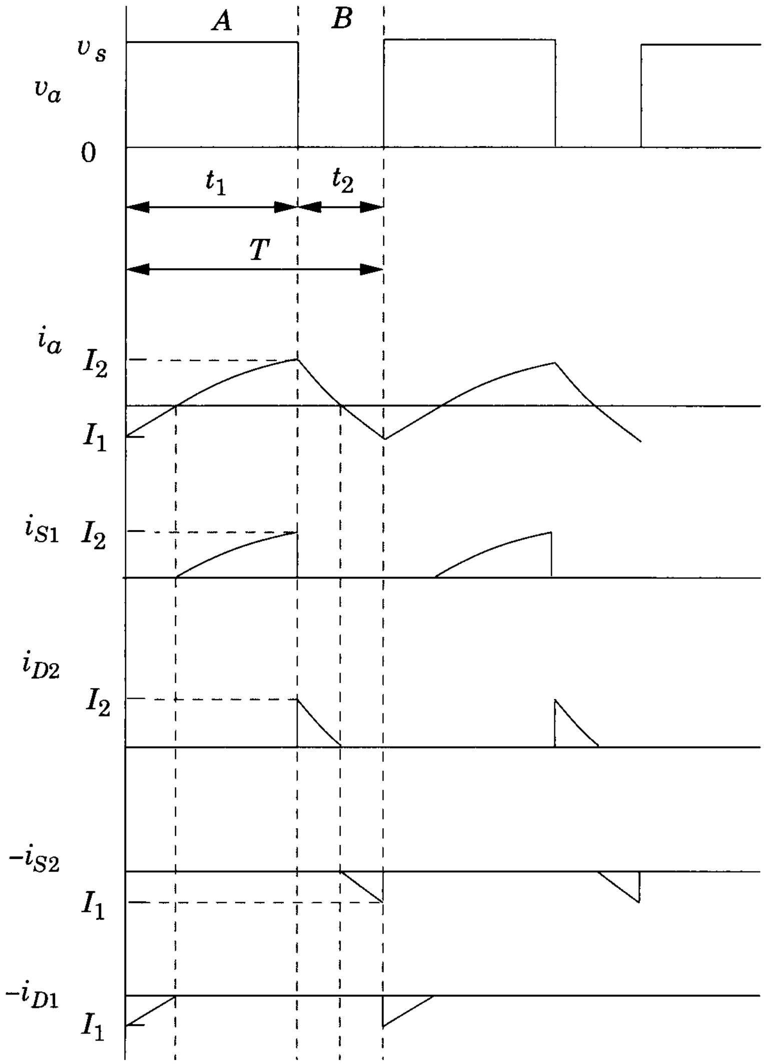

Figure 3.8-3: Steady-state operation of a two-quadrant dc converter drive.

in Fig. 3.8-3 to allow ready comparison with the waveform of i_a, since they are opposite to positive i_a. Let us go back to the start of interval A. How did i_a become negative? Well, during the interval B in the preceding period $S2$ was closed with $S1$ open. With $S2$ closed, the armature terminals are short-circuited and the counter emf has driven i_a negative. Therefore, when $S1$ is closed and $S2$ is opened at the start of interval A, the source voltage has to contend with this negative I_1. We see from Fig. 3.8-3 that the average value of i_a is slightly positive; therefore, v_S is larger than the counter emf and at the start of interval A when v_S is applied to the machine the armature current begins to increase toward zero from the negative value of I_1. Once i_a reaches zero, the diode D_1 blocks the current flow. That is, i_{D1} cannot become negative (cannot conduct positive i_a); however, $S1$ has been closed since the start of interval A and since i_{S1} can only be positive, $S1$ is ready to carry the positive i_a. The armature current, which is now i_{S1}, continues to increase until the end of interval A (I_2).

At the beginning of interval B, $S1$ is opened and $S2$ is closed; however, $S2$ cannot conduct a positive armature current. Therefore, the positive current (I_2) is diverted to diode $D2$ which is short-circuiting the armature terminals. Now, the counter emf has the positive current (I_2) with which to contend. It is clear that if the armature terminals were permanently short-circuited, the counter-emf would drive i_a negative. At the start of interval B, the counter-emf begins to do just that; however, when i_a becomes zero, diode $D2$ blocks i_{D2} and the negative armature current is picked up by $S2$, which has been closed since the beginning of interval B, waiting to be called upon to conduct a negative armature current. This continues until the end of interval B, whereupon we are back to where we started.

It is apparent that if the mode of operation is such that I_1 and I_2 are both positive, then the machine is acting as a motor with a substantial load torque if ω_r is positive (ccw). In this mode, either $S1$ or $D2$ will carry current during a switching period T. If both I_1 and I_2 are negative, the machine is operating as a generator, delivering power to the source if ω_r is positive (ccw). In this case, either $S2$ or $D1$ will carry current during a switching period.

It is instructive to derive expressions for I_1 and I_2. The period T in Fig. 3.8-3 is divided into interval A and interval B. During interval A, $v_a = v_S$, $i_a = i_S$, and $i_D = 0$. For interval A,

$$L_{AA}\frac{di_a}{dt} + r_a i_a = v_S - k_v\omega_r \tag{3.8-5}$$

If v_S and ω_r are assumed constant during this interval, the solution of (3.8-5) may be expressed in the form

$$i_a(t) = i_{a,ss} + i_{a,tr} \tag{3.8-6}$$

where $i_{a,ss}$ is the steady-state current that would flow if the given interval were to last indefinitely. This current can be calculated by assuming $di_a/dt = 0$, whereupon from (3.8-7) we obtain

$$i_{a,ss} = \frac{v_s - k_v\omega_r}{r_a} \tag{3.8-7}$$

The transient component ($i_{a,tr}$) of (3.8-6) is the solution of the homogeneous or force-free equation

$$L_{AA}\frac{di_a}{dt} + r_a i_a = 0 \tag{3.8-8}$$

Therefore,

$$i_{a,tr} = Ke^{-t/\tau_a} \tag{3.8-9}$$

where $\tau_a = L_{AA}/r_a$. Thus, during interval A, the armature current may be expressed as

$$i_a = \frac{1}{r_a}(v_s - k_v\omega_r) + Ke^{-t/\tau_a} \tag{3.8-10}$$

At $t = 0$ for interval A, $i_a(0) = I_1$ (Fig. 3.8-3); thus

$$I_1 = \frac{1}{r_a}(v_s - k_v\omega_r) + K \tag{3.8-11}$$

Solving for K and substituting the result into (3.8-10) yields the following expression for i_a:

$$i_a = I_1 e^{-t/\tau_a} + \frac{(v_s - k_v\omega_r)}{r_a}(1 - e^{-t/\tau_a}) \tag{3.8-12}$$

At $t = t_1$ or kT, $i_a = I_2$; from (3.8-12) we obtain

$$I_2 = I_1 e^{-kT/\tau_a} + \frac{(v_s - k_v\omega_r)}{r_a}(1 - e^{-kT/\tau_a}) \tag{3.8-13}$$

Equation (3.8-13) relates I_2, the current at the end of interval A, to I_1, the current at the beginning of interval A.

During interval B, we have

$$L_{AA}\frac{di_a}{dt} + r_a i_a = -k_v\omega_r \tag{3.8-14}$$

Solving for the steady-state current yields

$$i_{a,ss} = -\frac{k_v\omega_r}{r_a} \tag{3.8-15}$$

and $i_{a,tr}$ is still (3.8-9). Thus,

$$i_a = -\frac{k_v\omega_r}{r_a} + Ke^{-t/\tau_a} \tag{3.8-16}$$

For convenience of analysis, we will define a "new" time zero at the beginning of interval B. Thus, at this new $t = 0$, we have $i_a = I_2$ and

$$I_2 = -\frac{k_v\omega_r}{r_a} + K \tag{3.8-17}$$

Therefore, during interval B, with $t = 0$ at the start of interval B, we have

$$i_a = I_2 e^{-t/\tau_a} - \frac{k_v\omega_r}{r_a}(1 - e^{-t/\tau_a}) \tag{3.8-18}$$

Equation (3.8-12) defines the current during interval A, assuming that the initial current, I_1, for this interval is known, whereas (3.8-18) defines the current during interval B, assuming the initial current, I_2, for interval B is known. How can we establish these currents? Well, the initial current during interval B is the final current in interval A. That is, I_2 is calculated from (3.8-12) by setting $t = kT$, giving an expression for I_2 in terms of I_1 (3.8-13). But what determines the value of I_1? At the end of interval B, when $t = t_2$ in (3.8-18), the current i_a must return to I_1 for steady-state operation. Now, from (3.8-4), $t_2 = (1 - k)T$. In other words, $i_a = I_1$ when t in (3.8-18) is $(1 - k)T$. Thus,

$$I_1 = I_2 e^{-(1-k)T/\tau_a} - \frac{k_v\omega_r}{r_a}(1 - e^{-(1-k)T/\tau_a}) \tag{3.8-19}$$

Equations (3.8-13) and (3.8-19) can be used to solve for I_1 and I_2 in terms of v_s, k, T, ω_r, and the machine parameters. In particular, with some work, we can write

$$I_1 = \frac{v_s}{r_a}\left[\frac{e^{-T/\tau_a}(e^{kT/\tau_a} - 1)}{1 - e^{-T/\tau_a}}\right] - \frac{k_v\omega_r}{r_a} \tag{3.8-20}$$

$$I_2 = \frac{v_s}{r_a}\left[\frac{1-e^{-kT/\tau_a}}{1-e^{-T/\tau_a}}\right] - \frac{k_v\omega_r}{r_a} \tag{3.8-21}$$

Average-Value Time-Domain Block Diagram

The average-value time-domain block diagram for the two-quadrant chopper drive system is shown in Fig. 3.8-4. From Fig. 3.8-3, the average armature voltage may be determined as

$$\bar{v}_a = \frac{1}{T}\left[\int_0^{t_1} v_s\, d\xi + \int_{t_1}^{T} 0\, d\xi\right] \tag{3.8-22}$$

Since $t_1 = kT$, the average armature voltage becomes

$$\bar{v}_a = kv_s \tag{3.8-23}$$

In Fig. 3.8-4, the bars over the variables denote average values. Other than depiction of (3.8-23) and the bar notation, the block diagram shown in Fig. 3.8-4 is the same as that shown in Fig. 3.6-1.

The starting characteristics of a permanent-magnet dc machine with a two-quadrant chopper drive are depicted in Fig. 3.8-5. The machine parameters are identical to those in Fig. 3.5-1. Here, the switching frequency f_s is set to 200 Hz and the source voltage to 10 V. Typically, the switching frequency is much higher, generally greater than 20 kHz. The frequency was selected to illustrate the dynamics introduced by the converter. Even at this low switching frequency, the switching period T is much less than the armature time constant τ_a. Thus, the armature current essentially consists of piecewise linear segments about an average response. In Fig. 3.8-5, the duty

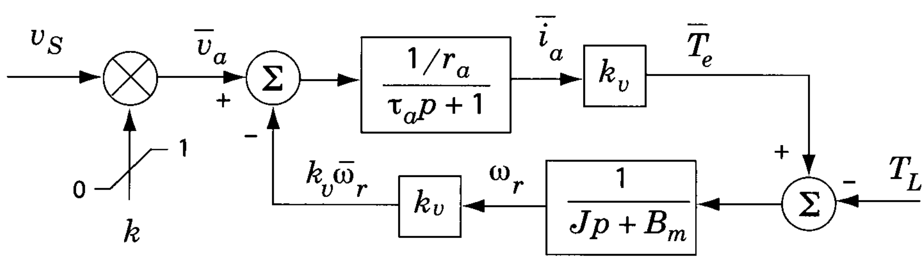

Figure 3.8-4: Average-value model of two-quadrant dc converter drive.

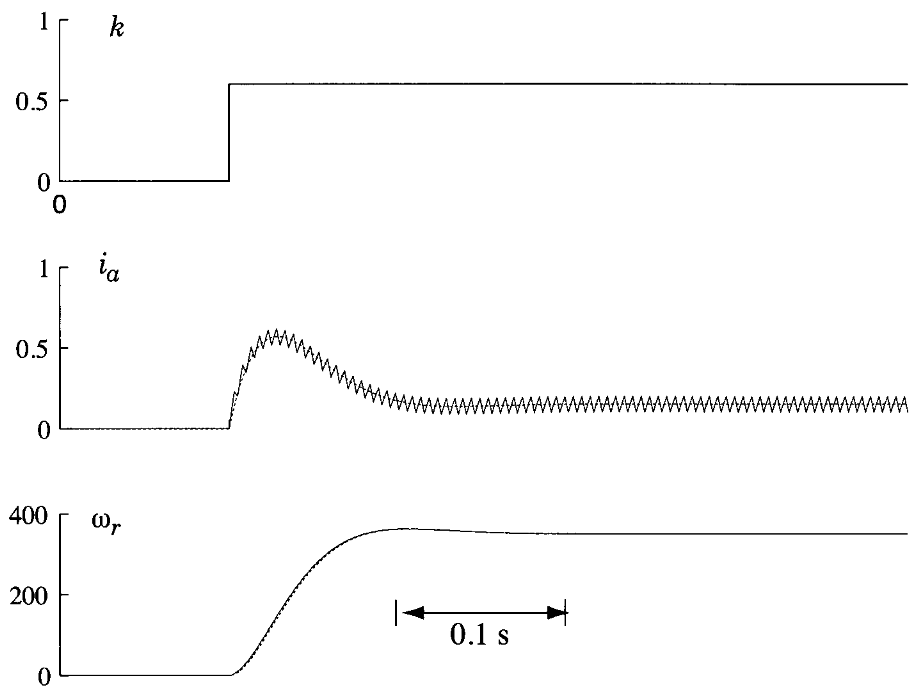

Figure 3.8-5: Starting characteristics of a permanent-magnet dc machine with a two-quadrant dc/dc converter drive.

cycle is stepped from 0 to 0.6, corresponding to a step increase in average applied voltage from 0 to 6 V. The start-up response established using the average-value model is superimposed for purposes of comparison. As shown, the only salient difference between the two responses is the "sawtooth" behavior of the armature current due to converter switching. The rotor speeds are indistinguishable.

3.9 RECAPPING

The dc machine is unique in that it exerts a torque on the rotating member as a result of the interaction of two stationary, orthogonal magnetic systems. One is produced by current flowing in the windings or a permanent magnet on the stationary member, and the other is caused by the current flowing in the windings of the rotating member. Direct current machines are being replaced by the converter-controlled permanent-magnet ac machine, which by modern control techniques is made to emulate the maximum torque-per-ampere characteristics of a dc machine. The equations that describe the dc machine offer a straightforward means of describing the modes of operation of this look-alike machine. Therefore, we are taking advantage of this to describe constant-torque and constant-power controlled modes of operation common to permanent-magnet ac machines. Since the dynamic characteristics of this device can be described by linear differential equations, we have also tried to set the stage for those who are interested in control system analysis by formulating the state equations of a permanent-magnet dc motor in fundamental form. Also, transfer functions have been developed with the control-systems student in mind. In a later chapter, we will consider the permanent-magnet ac machine operated as a brushless dc motor, which is rapidly replacing the permanent-magnet dc motor. The equations that describe the operation of these two devices are very similar. Hence, much of the material presented here for the permanent-magnet dc motor can be applied directly to the analysis of the brushless dc motor. In the last section of this chapter, a brief analysis of a converter used in dc-drive systems has been presented. An average-value block diagram for this converter supplying a permanent-magnet dc machine was derived.

3.10 REFERENCES

[1] P. C. Krause, O. Wasynczuk, S. D. Sudhoff, *Analysis of Electric Machinery and Drive Systems*, IEEE Press, 2002.

[2] S. J. Chapman, *Electric Machinery Fundamentals*, 3^{rd} Edition, McGraw-Hill Book Company, New York, 1999.

3.11 PROBLEMS

1. A permanent-magnet dc motor has the following parameters: $r_a = 8\,\Omega$ and $k_v = 0.01$ V $\cdot$ s/rad. The shaft load torque is approximated as $T_L = K\omega_r$, where $K = 5 \times 10^{-6}$ N $\cdot$ m $\cdot$ s. The applied voltage is 6 V and $B_m = 0$. Calculate the steady-state rotor speed ω_r in rad/s.

2. A permanent-magnet dc motor is driven by a mechanical source at 3820 r/min. The measured open-circuit armature voltage is 7 V. The mechanical source is disconnected, and a 12-V electric source is connected to the armature. With zero-load torque, $I_a = 0.1$ A and $\omega_r = 650$ rad/s. Calculate k_v, B_m, and r_a.

3. Modify the state equations given by (3.7-8) for a permanent-magnet dc machine to include θ_r as a state variable.

* 4. Write the state equations in fundamental form for the coupled circuits considered in Section 1.5. Start with (1.5-37) and use λ_1 and λ_2' as state variables. Relate currents and flux linkages by (1.5-34) and (1.5-35).

5. Develop the time-domain block diagram for the coupled circuits considered in preceding problem.

6. The parameters of a permanent-magnet dc machine are $r_a = 6\,\Omega$ and $k_v = 2 \times 10^{-2}\,$V $\cdot$ s/rad. V_a can be varied from zero to 10 volts. The device is to be operated in the constant-torque mode with $T_e^* = 4 \times 10^{-3}\,$N $\cdot$ m. (a) Determine V_a for $\omega_r = 0$. (b) Determine maximum ω_r range of the constant-torque mode of operation.

7. A dc machine is rated 5 hp with $V_f = 240$ V, $V_a = 240$ V, $r_a = 0.63\,\Omega$, and $L_{AF} = 1.8$ H. At rated conditions $\omega_r = 127.7$ rad/s and the total resistance of the field circuit is $240\,\Omega$. (a) Calculate the rated armature

current. The machine is to be operated in the constant-power mode beyond rated speed. (b) Calculate I_f at four times rated speed. (c) Calculate the boundary value of T_e for (b).

8. The machine in Prob. 7 is operating at $2\omega_{rR}$ in the field-weakening mode. $V_a = V_{aR}$ and $I_a = I_{aR}$. The load torque is reduced by one-half. It is desirable to maintain the steady-state rotor speed at $\omega_r = 2\omega_{rR}$. (a) Calculate I_a if $V_a = V_{aR}$. (b) Calculate V_a if $I_a = I_{aR}$.