Worksheet 11

Lagrange's Error Estimation

Problem 1. Find the 4th degree Taylor polynomial of $\sin x$ (which we often write as $T_4 \sin x$). Estimate the error $|\sin x - T_4 \sin x|$ for $|x| < 1$.

Recall: $|\sin x - T_4 \sin x| = |R_4 \sin x|$

Lagrange: $R_4 \sin x = \dfrac{f^{(5)}(c)}{5!} x^5$ ← $0 < c < x$ or $x < c < 0$ depending if $x > 0$ or $x < 0$

$f^{(5)}(c) = \cos(c) \Rightarrow |R_4 \sin x| = \left| \dfrac{\cos(c)}{5!} x^5 \right|$ since $|x|<1$

$\leq \dfrac{1}{5!} |x^5| \leq \dfrac{1}{5!} \cdot 1^5 \leq \dfrac{1}{5!}$

Conclusion:

* $|\sin x - T_4 \sin x| \leq \dfrac{1}{5!}$ for $|x| < 1$ *

* $T_4 \sin x = x - \frac{1}{6} x^3$ *

Problem 2. Calculate the 5th degree Taylor polynomial $T_5\, e^x$ to find an approximation for e. Estimate the error of your approximation.

$T_5 e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!}$

$x=1$

$\Rightarrow e^1 \sim 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$

Error?

$R_5 e^x = \dfrac{f^{(6)}(c)}{6!} x^6$ ← $f^{(6)}(x) = e^x$ here.

$\Rightarrow |R_5 e^x| = \left| \dfrac{e^c}{6!} x^6 \right|$ ← $0 < c < x$ $x=1$ here

$\Rightarrow |R_5 e^1| = \left| \dfrac{e^c}{6!} \cdot 1 \right| \leq \dfrac{e^1}{6!}$ ← $c=1$ worst case scenario.

$\leq \boxed{\dfrac{3}{6!}}$ ← let's take for granted $e<3$.

error bound

Problem 3. Recall that $\sqrt{225} = 15$. Calculate as many terms as you need of the taylor polynomial for $\sqrt{225 - x}$ to find an approximation of $\sqrt{222}$ with an error less than $\frac{1}{1000}$.

Try a few terms, see if it works:

$f(x) = \sqrt{225-x}$ $\quad f(0) = 15$ $\quad$ Try:

$f'(x) = -\frac{1}{2}(225-x)^{-1/2}$ $\quad f'(0) = \frac{-1}{30}$ $\quad T_1\sqrt{225-x} = 15 - \frac{1}{30}x$

$f''(x) = \frac{-1}{4}(225-x)^{-3/2}$

$$R_1\sqrt{225-x} = \frac{f''(c)}{2!}x^2$$

$$\Rightarrow \left|R_1\sqrt{225-x}\right| = \frac{1}{2!\cdot 4}\frac{1}{(225-c)^{3/2}}x^2$$

$x=3$ here.

Know: $0 < c < x$

worst case when $c=3$

$$\Rightarrow \left|R_1\sqrt{225-3}\right| \leq \frac{1}{8}\frac{1}{(\sqrt{222})^3}3^2$$

not good!

use: $14 < \sqrt{222}$

$$\Rightarrow \left|R_1\sqrt{225-3}\right| \leq \frac{9}{8}\frac{1}{(14)^3} < \frac{1}{1000}$$

✓

Conclusion:

$\sqrt{222} \sim 15 - \frac{1}{30}\cdot 3$

$\Rightarrow \sqrt{222} \sim 15 - \frac{1}{10}$

w/ error $< \frac{1}{1000}$

Problem 4. A commonly used approximation is $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ for small x. How small must x be for this approximation to be accurate to within 1% error? For simplicity, assume that $x > 0$.

Challenge Problem.
give it a try!

Problem 5. How many terms of the Taylor series for $\ln(1+x)$ do you need to approximate $\ln\left(\frac{3}{2}\right)$ with an error less than $\frac{1}{100}$?

Need to approximate error of $T_n \ln(1+x)$ w/ $x = \frac{1}{2}$.

$$\text{Error} = R_n \ln(1+x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}.$$

Calculate: $f^{(n+1)}(c) = (-1)^{n+2} n! (1+c)^{-n-1}$

$f^{(0)}: \ln(1+x)$

$f': (1+x)^{-1}$

$f'': -(1+x)^{-2}$

$f''': 2(1+x)^{-3}$

$f^{(4)}: -2\cdot 3(1+x)^{-4}$

$f^{(n)}: (-1)^{n+1}(n-1)!(1+x)^{-n}$

$$R_n \ln(1+x) = \frac{(-1)^{n+2}\cancel{n!}}{(n+1)\cancel{!}(1+c)^{n+1}} x^{n+1}$$

Know: $0 < c < x \Rightarrow 0 < c < \frac{1}{2}$

$x = \frac{1}{2}$ here.

$$\Rightarrow |R_n \ln(1+x)| = \left|\frac{(-1)^{n+2}}{(n+1)(1+c)^{n+1}} x^{n+1}\right|$$

$$\leq \frac{1}{n+1} x^{n+1}$$

@ $x = \frac{1}{2}$

$$|R_n \ln(1+\tfrac{1}{2})| \leq \frac{1}{n+1}\left(\frac{1}{2}\right)^{n+1}$$

need $< \frac{1}{100}$

$\boxed{\text{taking } n = 6 \text{ works!}}$

Problem 6. How many terms of the Taylor series for $\ln(1+x)$ do you need to approximation $\ln 2$ with an error less than $\frac{1}{100}$?

Need to approximate error of $T_n \ln(1+x)$ at $x = 1$.

Last Problem concluded:

$$|R_n \ln(1+x)| = \left|\frac{(-1)^{n+2}}{(n+1)(1+c)^{n+1}} x^{n+1}\right|$$

$0 < c < x$ $x=1$

$0 < c < 1$

$$\Rightarrow |R_n \ln(1+x)| \leq \frac{1}{n+1}|x^{n+1}|$$

So at $x = 1$, get:

$$|R_n \ln(1+1)| \leq \frac{1}{n+1}\cdot 1^{n+1} = \frac{1}{n+1}$$

need $< \frac{1}{100}$

$\boxed{\text{So we need } n = 99 \text{ the first 100 terms!}}$