Partial Fractions

A technique to allow us to solve integrals of rational functions (fractions of polynomials)

Like:

$$\int \frac{x^3-2x+2}{x^2-1}, \quad \int \frac{-x+2}{x^2+1}, \quad \int \frac{x^2+2}{x^2(x^2+1)}$$

Outline:

1. Polynomial Division
2. A simple example of PFs.
3. Integrating fractions of quadratics.
4. General Partial Fractions.

Polynomial Division

Whenever we want to integrate rational function we need to make sure the polynomial in the numerator is smaller than the polynomial in the denominator. We do this with polynomial division.

Ex $\displaystyle\int \frac{x^3+x^2+x+2}{x^2+1}$

Divide:

$$\begin{array}{r l}
 & x + 1 \\
x^2+1 & \overline{)\,x^3+x^2+x+2} \\
 & \underline{x^3 \quad\;\; + x} \\
 & \qquad x^2 + 2 \\
 & \qquad \underline{x^2 + 1} \\
\text{remainder} \longrightarrow & \qquad\quad 1
\end{array}$$

$$\frac{x^3+x^2+x+1}{x^2+1} = x+1+\frac{1}{x^2+1}$$

We get:

$$\int \frac{x^3+x^2+x+2}{x^2+1} = \boxed{\int x+1+\frac{1}{1+x^2} = \frac{1}{2}x^2+x+\arctan(x)+C}$$

A simple example of Partial Fractions:

The idea: seperate the denominator into easy to integrate parts.

Ex $\int \frac{1}{x^2-1}$

We have: $\frac{1}{x^2-1} = \frac{1}{2}\left\{\frac{1}{x-1} - \frac{1}{x+1}\right\}$

Why? $\frac{1}{x-1} - \frac{1}{x+1} = \frac{x+1}{x^2-1} - \frac{x-1}{x^2-1} = \frac{2}{x^2-1}$

Get: $\int \frac{1}{x^2-1} = \frac{1}{2}\int\frac{1}{x-1} - \frac{1}{2}\int\frac{1}{x+1}$

$$\boxed{= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C}$$

This techique works in genal!

Ex $\int \frac{x+1}{x^2-3x+2}$

Note: $x^2-3x+2 = (x-2)(x-1)$.

We can decompose our fraction as:

$$\frac{x+1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

What are A and B? Multiply both sides by $(x-2)(x-1)$, get:

$$x+1 = A(x-1) + B(x-2).$$

Plug in $x=1$ get:

$$2 = A\cdot 0 + B(-1) \Rightarrow B = \frac{-1}{2}.$$

Plug in $x=2$ get:

$$3 = A\cdot(1) + B\cdot 0 \Rightarrow A = 3$$

So:

$$\int \frac{x+1}{x^2-3x+2} = 3\int\frac{1}{x-2} - \frac{1}{2}\int\frac{1}{x-1}$$

$$= \boxed{3\ln|x-2| - \tfrac{1}{2}\ln|x-1| + C.}$$

In general, the idea of partial fractions says if

$Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ Distinct Factors.

$P(x)$ has smaller degree than $Q(x)$

Then we can decompose:

$$\boxed{\frac{P(x)}{(x-a_1)(x-a_2)\cdots(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}}$$

Need to solve for A's then easy to integrate.

Integrating Fractions of Quadratics

How do we integrate $\int \frac{1}{ax^2+bx+c}$?

3 - Possibilities:

Two roots $b^2-4ac>0$	$\frac{1}{ax^2+bx+c} = \frac{1}{a}\frac{1}{(x-r_1)(x-r_2)}$ Use Partial Fractions!
One root $b^2-4ac=0$	$\frac{1}{ax^2+bx+c} = \frac{1}{a}\frac{1}{(x-r)^2}$ Use u-substitution!
No roots $b^2-4ac<0$	$\frac{1}{ax^2+bx+c} = \frac{1}{a}\frac{1}{(x+\alpha)^2+\beta^2}$ Find α & β by completing the square. Use algebra and a u-sub to turn this integral into: $\frac{1}{u^2+1}$

Ex $\int \frac{1}{x^2+2x+5}$

x^2+2x+5 has no roots, so complete square:

$$\underbrace{x^2+2x}_{(x+1)^2-1}+5 = (x+1)^2+4$$

Useful formula! $\int \frac{1}{x^2+b^2} = \frac{1}{b}\arctan\left(\frac{x}{b}\right)+C$

$$\frac{1}{(x+1)^2+4} = \frac{1}{4}\frac{1}{\left(\frac{x+1}{2}\right)^2+1}$$

So: $\int \frac{1}{x^2+2x+5} = \frac{1}{4}\int \frac{1}{\left(\frac{x+1}{2}\right)^2+1} = \boxed{\frac{1}{2}\arctan\left(\frac{x+1}{2}\right)+C}$

Ex $\displaystyle\int \frac{1}{2x^2-4x+2}$

$2x^2-4x+2$ has one root: $r=1$.

Get: $2x^2-4x+2 = 2(x-1)^2$.

$$\int \frac{1}{2x^2-4x+2} = \frac{1}{2}\int \frac{1}{(x-1)^2} = \boxed{\frac{-1}{x-1} + C}$$

More examples w/ Quadratics

Ex $\displaystyle\int \frac{x^2+2x+2}{x^2-1}$

First do polynomial division:

$$\begin{array}{r|l} & \ \ 1 \\ \hline x^2+2x+2 & x^2-1 \\ & x^2+2x+2 \\ \hline & -2x-3 \end{array}$$

Get:

$$\frac{x^2+2x+2}{x^2-1} = 1 - \frac{2x+3}{x^2-1}$$

$$\int \frac{x^2+2x+2}{x^2-1}dx = \int 1 - \frac{2x+3}{x^2-1}dx = x - \int \frac{2x+3}{x^2-1}$$

Now evaluate this second integral:

Here $x^2-1=(x+1)(x-1)$ so use partial fractions:

$$\frac{2x+3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$\Rightarrow \quad 2x+3 = A(x+1) + B(x-1)$$

Plug in $x=-1$:

$$1 = A\cdot 0 + B(-2) \Rightarrow B = \frac{-1}{2}$$

Plug in $x=1$:

$$5 = 2A + B\cdot 0 \Rightarrow A = \frac{5}{2}$$

Get:

$$\int \frac{2x+3}{x^2-1} = \frac{5}{2}\int \frac{1}{x-1} - \frac{1}{2}\int \frac{1}{x+1}$$

$$= \frac{5}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

Plug back in and get:

$$\int \frac{x^2+2x+2}{x^2-1} = \boxed{x + \frac{5}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C}$$

Ex

$$\int \frac{2x+3}{x^2+9}$$

Here we don't use Partial Fractions since x^2+9 does not have roots.

Split the integral:

$$\int \frac{2x+3}{x^2+9} = 2\underbrace{\int \frac{x}{x^2+9}}_{u\text{-sub}} + 3\underbrace{\int \frac{1}{x^2+9}}_{\text{arctan}}$$

For the first integral let $u=x^2+1$.

Get:

$$2\int \frac{x}{x^2+9}\,dx = \int \frac{1}{u}\,du = \ln|x^2+9| + C$$

$u=x^2+9$
$du = 2x\,dx$

For the second Integral:

$$3\int \frac{1}{x^2+9}\,dx = \frac{3}{9}\int \frac{1}{(x/3)^2+1}\,dx = \arctan\left(\frac{x}{3}\right)+C$$

Add these solutions together:

$$\int \frac{2x+3}{x^2+9}\,dx = \boxed{\ln|x^2+9| + \arctan\left(\frac{x}{3}\right) + C}$$

Observation of Previous calculations:

Given $\displaystyle\int \frac{Ax+B}{x^2+bx+c}$ like last two examples, we still first look at roots of denominator.

2-Roots $\Rightarrow$ Partial Fractions, call it good. (this is first example)

No Roots $\Rightarrow$ Complete square, and do u-sub to make it look like:

$$\frac{Au+B}{u^2+1}$$

Then split into two integrals.

One root? $\Rightarrow$ Will talk more about with general PFs next time.