Improper Integrals

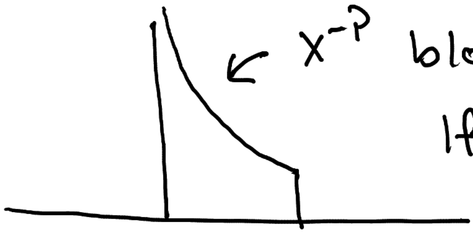
p-test

$$\int_0^1 x^{-p}\,dx = \begin{cases} +\infty & \text{if } p \geq 1 \\ \text{finite} & \text{if } p < 1 \end{cases}$$

$$\int_1^\infty x^{-p}\,dx = \begin{cases} +\infty & \text{if } p \leq 1 \\ \text{finite} & \text{if } p > 1 \end{cases}$$

How to think about this?

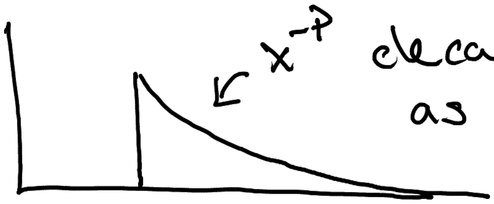
For $\boxed{\int_0^1 x^{-p}\,dx,}$ we worry about how fast x^{-p} blows-up at $x = 0$.



x^{-p} blows up for $p > 0$.
If $p \leq 0$, no blow-up.

p-test says: If $p \geq 1$ x^{-p} blows up too fast and we get infinite area.

For $\boxed{\int_1^\infty x^{-p}\,dx,}$ we worry about x^{-p} going to zero fast enough.



x^{-p} decays to zero as $x \to \infty$ if $p > 0$.

p-test says: If $p > 1$, x^{-p} decays fast enough to give us finite area.

How I rember the p-test

p=1 is the critical transition power for both cases.

$\boxed{\int_0^1 x^{-p}\,dx}$ - $\boxed{\text{Things that blow up faster than } x^{-1} \text{ is } +\infty.}$

ex: $\frac{1}{x^2}$ blows-up faster than $\frac{1}{x}$ so $\int_0^1 \frac{1}{x^2}dx = +\infty$.

ex: $\frac{1}{\sqrt{x}}$ blows-up slower than $\frac{1}{x}$ so $\int_0^1 \frac{1}{\sqrt{x}}dx$ is finite.

$\boxed{\int_1^\infty x^{-p}\,dx}$ - $\boxed{\text{Things that go to zero faster than } \frac{1}{x} \text{ is finite.}}$

ex: $\frac{1}{x^2}$ decays to zero faster than $\frac{1}{x}$ so $\int_1^\infty \frac{1}{x^2}dx$ is finite.

ex: $\frac{1}{\sqrt{x}}$ decays to zero slower than $\frac{1}{x}$ so $\int_1^\infty \frac{1}{\sqrt{x}}dx = +\infty$.

Estimating Improper Integrals

Recall, we have 4 possible answers for improper integrals:

$$\int_1^\infty f(x)\,dx = \begin{cases} +\infty & \text{diverges to } +\infty \\ -\infty & \text{diverges to } -\infty \\ \text{DNC} & \text{does not converge.} \\ \text{finite.} & \end{cases}$$

We will find techiques that will answer:

$\boxed{\text{Question 1}}$ Is $\int_1^\infty f(x)\,dx$ $\pm\infty$, DNC, or finite?

$\boxed{\text{Question 2}}$ If $\int_1^\infty f(x)\,dx$ is finite, can I find some kind of crude estimate if I can't solve explicitly?

First technique. Comparison Test.

Idea: Smaller things have smaller area.
Bigger things have bigger area.

Statement:

1. $$|f(x)| \le g(x) \Rightarrow \left|\int_1^\infty f(x)\,dx\right| \le \int_1^\infty g(x)\,dx$$

Conclusion:

If $|f(x)| \le g(x)$ and $\int_1^\infty g(x)\,dx$ is finite then: $\int_1^\infty f(x)\,dx$ is finite.

2. $$|g(x)| \le f(x). \Rightarrow \left|\int_1^\infty g(x)\,dx\right| \le \int_1^\infty f(x)\,dx$$

Conclusion:

If $|g(x)| \le f(x)$ and $\int_1^\infty g(x)\,dx = +\infty$ Then $\int_1^\infty f(x)\,dx = +\infty$.

Note: To use the comparison test, you kind of need to guess if the integral is finite of $+\infty$, then find an appropriate comparison.

Ex $\displaystyle\int_1^\infty \frac{1}{x^2+3x+1}\,dx$

We have:

$$\frac{1}{x^2+3x+1} \le \frac{1}{x^2} \quad \text{for } x>0.$$

We know:

$\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is finite by p-test.

So: $\displaystyle\int_1^\infty \frac{1}{x^2+3x+1}\,dx \le \int_1^\infty \frac{1}{x^2}\,dx$ is finite.

Ex $$\int_1^\infty \frac{2-\sin(3x)}{x}\,dx.$$

Note: $1 \le 2-\sin(3x) \le 3.$

So: $$\frac{1}{x} \le \frac{2-\sin(3x)}{x}$$

p-test says $\displaystyle\int_1^\infty \frac{1}{x}\,dx = +\infty.$

Comparison test says: $\displaystyle\int_1^\infty \frac{2-\sin(3x)}{x}\,dx = +\infty.$

Ex $\int_0^\infty \frac{u^2}{(u^2+1)^2}\,du$

$$\frac{u^2}{(u^2+1)^2} \sim \frac{u^2}{u^4} = \frac{1}{u^2} \text{ as } u \to \infty$$

So I think it converges.

Have:

$$\frac{u^2}{(u^2+1)^2} = \frac{u^2}{u^4+2u^2+1} \leq \frac{u^2}{u^4} = \frac{1}{u^2} \text{ if } u \geq 1.$$

$\int_1^\infty \frac{1}{u^2}\,du$ converges by p-test.

So $\int_1^\infty \frac{u^2}{(u^2+1)^2}\,du$ converges.

What about $\int_0^1 \frac{u^2}{(u^2+1)^2}\,du$?

No blow-up here, so this is finite, hence:

$$\boxed{\int_0^\infty \frac{u^2}{(u^2+1)^2}\,du \text{ converges}}$$