Quick Review of U/V sub.

Identities needed:

$$\boxed{\begin{array}{lll} U(t) = \frac{1}{2}\left(t + \frac{1}{t}\right), & V(t) = \frac{1}{2}\left(t - \frac{1}{t}\right) & \\ 1 + v^2(t) = u^2(t) & \text{for} & x^2 + a^2 \\ U^2(t) - 1 = V^2(t) & \text{for} & x^2 - a^2 \end{array}}$$

This is an alternative to trig subs:

$$1 + \tan^2(\theta) = \sec^2\theta \quad \text{for} \quad x^2 + a^2$$

$$\sec^2\theta - 1 = \tan^2\theta \quad \text{for} \quad x^2 - a^2$$

More identities:

$$\boxed{t = U(t) + V(t), \quad \frac{1}{t} = U(t) - V(t)}$$

Idea:
- Use U/V substitution like you would w/ trig substitution.
- Get integral of a bunch of t's
- Solve integral
- Use identities:

$$\text{If } x = V(t) \text{ then } \overbrace{U(t) = \sqrt{x^2+1}}^{u^2(t) = \sqrt{v^2(t)+1}}$$

$$\text{If } x = U(t) \text{ then } \underbrace{V(t) = \sqrt{x^2-1}}_{v^2 = \sqrt{u^2-1}}$$

and:

$$\left.\begin{array}{l} t = U(t) + V(t) \\ t = \sqrt{x^2+1} + x \\ t = x + \sqrt{x^2-1} \end{array}\right\} \text{to solve for } t \text{ in terms of } x.$$

Ex $\int \sqrt{1+x^2}\,dx$ $\qquad x = v(t) = \frac{1}{2}\left(t - \frac{1}{t}\right)$

$$= \left\langle \begin{array}{l} x = v(t) \\ dx = \frac{1}{2}\left(1+\frac{1}{t^2}\right)dt \end{array} \right\rangle = \int \sqrt{1+v^2(t)} \cdot \frac{1}{2}\left(1+\frac{1}{t^2}\right)dt$$

$$= \frac{1}{2}\int u(t)\left(1+\frac{1}{t^2}\right)dt$$

$$= \frac{1}{2}\int \frac{1}{2}\left(t+\frac{1}{t}\right)\left(1+\frac{1}{t^2}\right)dt$$

$$= \frac{1}{4}\int t + \frac{1}{t} + \frac{1}{t} + \frac{1}{t^3}\,dt$$

$$= \frac{1}{4}\left\{\frac{1}{2}t^2 + 2\ln|t| - \frac{1}{2t^2}\right\} + C$$

Here: $x = v(t)$ so $u(t) = \sqrt{x^2+1}$

$$t = v(t) + u(t) = x + \sqrt{x^2+1}$$

Get:

$$\boxed{= \frac{1}{8}\left(x+\sqrt{x^2+1}\right)^2 + \frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right| - \frac{1}{8}\frac{1}{(x+\sqrt{x^2+1})^2} + C}$$

New topic: Improper Integrals

Proper Integrals

$\int_a^b f(x)\,dx$

- finite domain.
- f doesn't blow up anywhere.

ie: No vertical asymtopes.

Improper Integrals

Type 1: $\int_a^\infty f(x)\,dx$

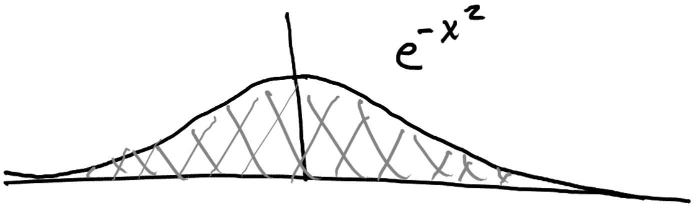
- domain is infinitely large.

Type 2: $\int_a^b f(x)\,dx$

- f has vertical asymtopes in domain.

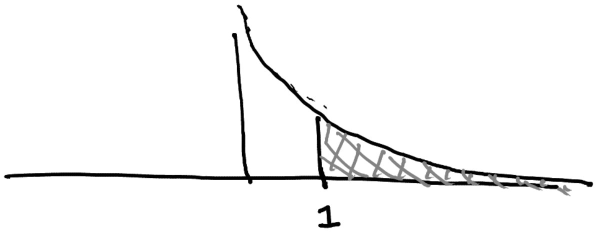
Examples:

- $\int_{-\infty}^{\infty} e^{-x^2}\,dx$



Infinite domain. Improper integral.

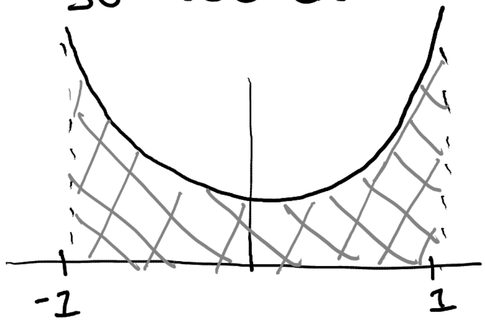
- $\int_{1}^{\infty} \frac{1}{x^2}\,dx$



Infinite domain. Improper Integral

Also has vertical asymtope but not in our domain so we don't care.

- $\int_{-1}^{1} \frac{1}{1-x^2}\,dx$



Two vertical asymtopes. Improper Integral.

How to find vertical asymtopes?

Normally we have either:

(a) $f(x) = \frac{\text{something}}{g(x)}$ where $g(x)=0$.

ie: $f(x) = \frac{1}{1-x^2}$ Vert. Asy. @ $x = \pm 1$.

$f(x) = \frac{1}{\sin(x)}$ vert. Asy. @ $x = \pm k\pi$.

(b) $f(x)$ has $\text{Log}(0)$ in it:

$f(x) = \cos(x)\log(x)$, vert asy. @ $x=0$.

$f(x) = \log(1-x^2)$, vert asy @ $x = \pm 1$.

How to calculate Improper Integrals?

Infinite Domain Case

$$\int_a^\infty f(x)\,dx = \lim_{M\to\infty} \int_a^M f(x)\,dx.$$

- First calculate $\int_a^M f(x)\,dx$.
- Then take limit as $M \to \infty$.

Ex $$\int_1^\infty \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \int_1^M \frac{1}{1+x^2}\,dx$$

$$= \lim_{M\to\infty} \arctan(M) - \arctan(1)$$

$$= \pi/2 - \pi/4 = \boxed{\pi/4}$$

Ex $$\int_0^\infty \cos(x)\,dx = \lim_{M\to\infty} \int_0^M \cos(x)\,dx$$

$$= \lim_{M\to\infty} \sin(M) - \underbrace{\sin(0)}_{0}$$

$\boxed{\text{Does not exist!}}$

Ex $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{M\to\infty} \int_1^M \frac{1}{x^2}\,dx$$

$$= \lim_{M\to\infty} \left.\frac{-1}{x}\right|_{x=1}^{x=M} = \lim_{M\to\infty} \frac{-1}{M} - \frac{-1}{1}$$

$$= 0 + 1 = \boxed{1}$$

Doubly Infinite Integrals

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^\infty f(x)\,dx + \int_{-\infty}^{0} f(x)\,dx$$

$$= \left\{\lim_{M\to\infty} \int_0^M f(x)\,dx\right\} + \left\{\lim_{M\to\infty} \int_{-M}^{0} f(x)\,dx\right\}$$

- First break integral into two peices.
- Calculate each limit seperately.
- Add the limits.

Ex $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_0^\infty \frac{1}{1+x^2}\,dx + \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$$

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \int_0^M \frac{1}{1+x^2}\,dx$$

$$= \lim_{M\to\infty} \arctan(M) - \overset{0}{\arctan(0)} = \pi/2$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{M\to\infty} \int_{-M}^{0} \frac{1}{1+x^2}\,dx$$

$$= \lim_{M\to\infty} \arctan(0) - \arctan(-M) = \pi/2.$$

So: $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$

<u>Example</u> $$\int_{-\infty}^{\infty} e^{-|x|}\,dx$$

Split Integral:

$$\int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{0}^{\infty} e^{-x}\,dx + \int_{-\infty}^{0} e^{x}\,dx$$

$$\int_0^{\infty} e^{-x}\,dx = \lim_{M\to\infty} \int_0^{M} e^{-x}\,dx$$

$$= \lim_{M\to\infty} \left\{ -e^{-M} + e^{0} \right\} = 0 + 1 = 1.$$

$$\int_{-\infty}^{0} e^{x}\,dx = \lim_{M\to\infty} \int_{-M}^{0} e^{x}\,dx$$

$$= \lim_{M\to\infty} \left\{ e^{0} - e^{-M} \right\} = 1 - 0 = 1.$$

So: $$\int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{0}^{\infty} e^{-x}\,dx + \int_{-\infty}^{0} e^{x}\,dx = 1 + 1 = \boxed{2}$$

Case 3 Vertical Asymptotes

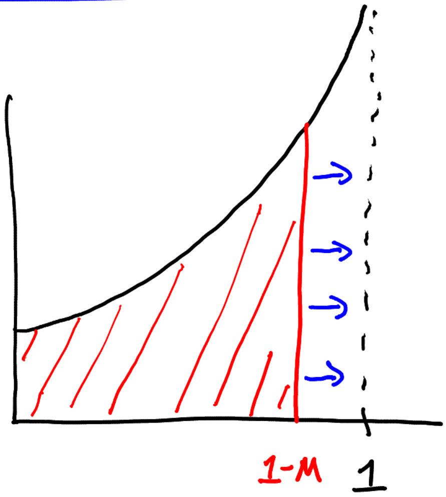
Suppose, say, f has vertical asym. @ $x=1$.

Then we do:

$$\boxed{\int_0^1 f(x)\,dx = \lim_{M\to 0^+} \int_0^{1-M} f(x)\,dx}$$

Ex $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

This has vert. asym. @ $x=1$.



We avoid asym. by subtracting a small number from 1; $1-M$.
Then take limit as $M\to 0^+$.

Note: It's important that $M\to 0^+$, because if M was negative then we would integrate into the Right Hand Side of asymtope!

Now our calculation:

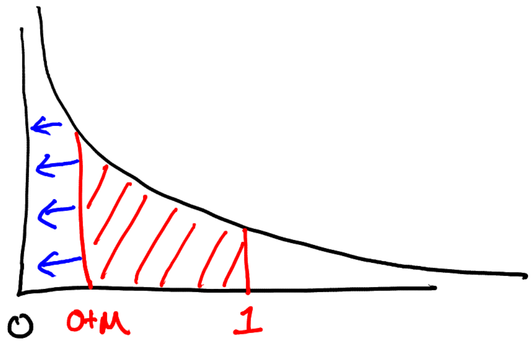
$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{M\to 0^+} \int_0^{1-M} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{M\to 0^+} \arcsin(1-M) - \arcsin(0)$$

$$= \pi/2 - 0 = \boxed{\pi/2}$$

Ex $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ $\Big\{$ vertical asym. at $x=0$.

$\Big\{$ Avoid asym. at $x=0$ by adding μ. To stay on RHS of asym. $\Big\}$



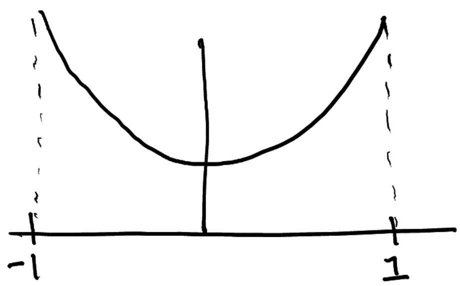
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\mu\to 0^+} \int_\mu^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{\mu\to 0^+} \left\{ 2\sqrt{1} - 2\sqrt{\mu} \right\}$$

$$= 2 - 0 = \boxed{2}$$

Ex $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$ $\Big\{$ vertical asym. at $x=\pm 1$.

Two asymptotes!
Split into two integrals to handle each seperately.



$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx = \int_{-1}^0 \frac{1}{\sqrt{1-x^2}}\,dx + \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int_{-1}^0 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\mu\to 0^+} \int_{-1+\mu}^0 \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \lim_{\mu\to 0^+} \left\{ \arcsin(0) - \arcsin(-1+\mu) \right\} = \boxed{\pi/2}$$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{M\to 0^+} \int_0^{1-M} \frac{1}{\sqrt{1-x^2}}\,dx = \boxed{\pi/2}$$

So:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}$$