Math 222: Inverse Trig Substitution

February 13, 2016

This document is a summary of what we talked about on Tuesday on the topic of inverse trig substitution. It is organized in three parts: 1. motivation and introduction, 2. basic inverse trig substitution, and 3. variations to trig substitution.

1 Introduction and Motivation

The goal is to integrate functions where you see in them somewhere:

$$\sqrt{x^2+a^2}, \quad \sqrt{a^2-x^2}, \quad \sqrt{x^2+a^2}, \quad \frac{1}{\sqrt{a^2-x^2}}, \quad \frac{1}{\sqrt{x^2-a^2}}, \quad \frac{1}{\sqrt{a^2+x^2}}.$$

Let's first look at the Pythagoras theorem for trig functions and its variation (up to some rearrangements):

$$\begin{aligned}\cos^2\theta &= 1-\sin^2\theta,\\ \tan^2\theta &= \sec^2\theta - 1,\\ \sec^2\theta &= 1+\tan^2\theta.\end{aligned}$$

If we squint our eyes a little, the right-hand sides of the equations above look like $1-x^2$, x^2-1 and $1+x^2$ respectively, and these look awfully like a^2-x^2, x^2-a^2 and a^2+x^2. The idea is to replace x with $a(\text{trig})\theta$ in our function and cross our fingers that we know how to integrate them.

There are alternative ways to integrate certain of those functions above. For those interested, you can find it under UV-substitution in the textbook, or refer to Prof. Gong's lectures. On this note, if we want to completely avoid UV-substitution, then there are certain trig integrals that might be useful to know. For example, one might need to know how to integrate $\sec\theta$, $\sec^3\theta$, etc.

2 Basic Inverse Trig Substitution

Note that $\frac{1}{x^2-a^2}$ and $\frac{1}{a^2-x^2}$ can be integrated using partial fraction decomposition instead, and $\frac{1}{x^2+a^2}$ will end up as some sort of arctangent after some algebra and u-sub.

For now, let's assume that the variable we are integrating is x, and our domain of integration is a subset of $x>0$. Later, we'll go over the two additional steps needed if $x<0$.

Step 0. Are you sure you don't know how to integrate this using any simpler methods? If you do, go use that method instead.

Step 1. Choose your trig identity:

See:	Use	
a^2-x^2	$x=a\sin\theta$	$\mathrm{d}x = a\cos\theta\,\theta$
x^2-a^2	$x=a\sec\theta$	$\mathrm{d}x = a\sec\theta\tan\theta\,\mathrm{d}\theta$
x^2+a^2	$x=a\tan\theta$	$\mathrm{d}x = a\sec^2\theta\,\mathrm{d}\theta$

Step 2. Sub in x and $\mathrm{d}x$ into the integral. Did you remember to sub in $\mathrm{d}x$? Really?

Step 3. Integrate and obtain a function of θ.

Step 4. We need to backsub to replace all the θ's with x's. There are three typical scenarios in how θ will appear, and each will require a slightly different method. (See example below. It'll make more sense, unless you still remember my explanation from Tuesday.)

See:	Use
θ (a naked θ)	Sub in 'arc(trig)$\frac{x}{a}$', where this trig function is the initial trig substitution we used back in step 1.
(trig)θ	Use the initial trig substitution from step 1, and SOH-CAH-TOA to draw a right-angle triangle with angle θ. Then calculate the length of the last side in this triangle using the standard Pythagoras theorem.
(trig)(2θ), etc	Use double-angle and/or sum-difference formulae to reduce this to $\sin\theta$'s and $\cos\theta$'s. Then use the method immediately above this with the right-angle triangle. Warning: $\sin(2\theta) \neq 2\sin\theta$, so do not let me see anyone try to pull out the constant. :)

Step 5. Now we should have a function of x. If it is a definite integral, evaluate using the limits of integrations. If this is an indefinite integral, check to make sure you have your '$+C$'.

What to do when we are integrating where $x < 0$? Add two extra steps to the above! (By the way, Step 4.5 below

Step 0.5. Do a t-sub: $t = -x$ and $\mathrm{d}t = -\mathrm{d}x$.
If this is a definite integral, take the time to change the bounds of integration to in terms of t too.

Step 4.5. If this is a definite integral, go ahead and evaluate according to those new bounds of integrations you found earlier.
If this is an indefinite integral, anywhere you see t, sub in $t = -x$.

An example

Evaluate the integral $I = \displaystyle\int_{-1}^{0} \sqrt{3 - x^2}\,\mathrm{d}x$.

Step 0. Go ahead and ask if we know how to integrate this using another method or not. Nope! We don't.

However, notice that $-1 < x < 0$, so we first need to do a substitution. Let $t = -x$ and $\mathrm{d}t = -\mathrm{d}x$. So

$$I = \int_{t=1}^{0} \sqrt{3 - t^2}(-\mathrm{d}t) = \int_{t=0}^{1} \sqrt{3 - t^2}\,\mathrm{d}t.$$

This puts us back into the positive part of the number line. Yay for no absolute values!

Steps 1,2,3. We'll consider the indefinite integral J right now. Substituting $t = \sqrt{3}\sin\theta$, and so $\mathrm{d}t = \sqrt{3}\cos\theta\,\mathrm{d}\theta$:

$$\begin{aligned}
J &= \int \sqrt{3 - (3\sin^2\theta)} \cdot \sqrt{3}\cos\theta\,\mathrm{d}\theta \\
&= \sqrt{3}\sqrt{3}\int \sqrt{1 - \sin^2\theta} \cdot \cos\theta\,\mathrm{d}\theta \\
&= 3\int \cos\theta \cdot \cos\theta\,\mathrm{d}\theta \\
&= 3\int \cos^2\theta\,\mathrm{d}\theta
\end{aligned}$$

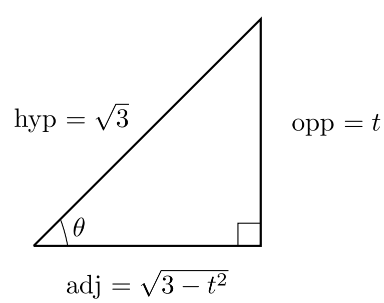
Double Angle!

$$\begin{aligned}
&= 3\int \frac{1+\cos(2\theta)}{2}\,\mathrm{d}\theta \\
&= \frac{3}{2}\int 1 + \cos(2\theta)\,\mathrm{d}\theta \\
&= \frac{3}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C.
\end{aligned}$$

Step 1. Have to decide what to do with those θ's. We see a lonely θ standing around, but also a $\sin(2\theta)$. Remember what we started with:

$$\begin{aligned}
t = \sqrt{3}\sin\theta \quad &\Leftrightarrow \quad \frac{t}{\sqrt{3}} = \sin\theta = \frac{\text{opp}}{\text{hyp}} \\
&\Leftrightarrow \quad \arcsin\left(\frac{t}{\sqrt{3}}\right) = \theta.
\end{aligned}$$

Let's draw our right-angle triangle:



Back to our integral:

$$\begin{aligned}
J &= \frac{3}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right] + C \\
&= \frac{3}{2}\arcsin\left(\frac{t}{\sqrt{3}}\right) + \frac{3}{2}\frac{1}{2}\left(2\sin\theta\cos\theta\right) + C \\
&= \frac{3}{2}\arcsin\left(\frac{t}{\sqrt{3}}\right) + \frac{3}{2}\underbrace{\left(\frac{t}{\sqrt{3}}\right)}_{\sin\theta}\underbrace{\left(\frac{\sqrt{3-t^2}}{\sqrt{3}}\right)}_{\cos\theta} + C \\
&= \frac{3}{2}\arcsin\left(\frac{t}{\sqrt{3}}\right) + \frac{1}{2}t\sqrt{3-t^2} + C.
\end{aligned}$$

Step 4.5-ish. We have a definite integral. Remember that we are integrating the function $\sqrt{3-t^2}$ from $t=0$ to $t=1$, so let's finish the problem.

$$
\begin{aligned}
I = J\Big|_{t=0}^{1} &= \left[\frac{3}{2}\arcsin\frac{1}{\sqrt{3}} + \frac{1}{2}\sqrt{2}\right] - \left[\frac{3}{2}\arcsin(0) + 0\right] \\
&= \frac{3}{2}\arcsin\frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{2}.
\end{aligned}
$$

3 Variations of inverse trig sub

I won't dwell on this for too long. Essentially, the variations involve trying to integrate the functions where you see in them

$$\sqrt{ax^2+bx+c}, \qquad \frac{1}{\sqrt{ax^2+bx+c}}.$$

The first thing to do is to complete the squares. There are two scenarios:

$$ax^2+bx+c = A(x-B)^2, \quad \text{or} \quad ax^2+bc+c = A(x-B)^2 + C,$$

where C might be positive or negative. In the first case, the square root and the square cancels (let's just hope A is positive), and we end up with a rational function; this we have learned to integrate. In the second case, do a u-sub with $u = x - B$, and we should end up in one of the more basic inverse trig sub.

With that, happy trig-subbing!