How to solve something like this:

$$\int \frac{e^{6x}}{e^{4x}-1}\,dx \ ?$$

Notice: $e^{4x} = (e^x)^4$. Make a u-sub:

$$u = e^x, \quad du = e^x\,dx$$

Get:

$$\int \frac{e^{6x}}{e^{4x}-1}\,dx = \int \frac{(e^x)^6}{(e^x)^4-1}\,dx$$

$$= \int \frac{(e^x)^5 \; e^x dx}{(e^x)^4-1}$$

$$= \int \frac{u^5}{u^4-1}\,du$$

This is a polynomial function we ran into in class. We quickly review:

first divide, get:

$$\frac{u^5}{u^4-1} = u + \frac{u}{u^4-1}$$

Now factor denominator

$$u^4-1 = (u^2-1)(u^2+1) = (u-1)(u+1)(u^2+1)$$

difference of squares

Use Partial Fractions:

$$\frac{u}{(u-1)(u+1)(u^2+1)} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{Cx+D}{u^2+1}$$

Mult. by least common denominator:

$$u = A(u+1)(u^2+1) + B(u-1)(u^2+1) + ((x+D))(u-1)(u+1)$$

Expand polynomial and equate coefficients:

$$u = A(u^3+u^2+u+1) + B(u^3-u^2+u-1) + C(u^3-u) + D(u^2-1)$$

$$u = (A+B+C)u^3 + (A-B+D)u^2 + (A+B-C)u + (A-B-D)$$

Equating coefficients gives:

$$A+B+C=0$$
$$A-B+D=0$$
$$A+B-C=1$$
$$A-B-D=0$$

Use substitution to solve for unknowns:

Eq 4 gives: $A=B+D$

Sub into Eq 2: $(B+D)-B+D=0 \Rightarrow \boxed{D=0}$

So $A=B$ now.

Plug into eq 1 & eq 3 get:

$$\left.\begin{array}{l} B+B+C=0 \\ B+B-C=1 \end{array}\right\} \quad \boxed{\begin{array}{l} B=\frac{1}{4}=A \\ C=\frac{-1}{2} \end{array}}$$

Plug coefficients back in:

$$\frac{u}{u^4-1} = \frac{1}{4}\frac{1}{u+1} + \frac{1}{4}\frac{1}{u-1} + \frac{\frac{-1}{2}u}{u^2+1}$$

Now integrate:

$$\int \frac{1}{4}\frac{1}{u+1}\,du = \frac{1}{4}\ln|u+1| + C$$

$$\int \frac{1}{4}\frac{1}{u-1}\,du = \frac{1}{4}\ln|u-1| + C$$

$$\int \frac{\frac{-1}{2}u}{u^2+1}\,du = \frac{-1}{2}\int \frac{u}{u^2+1} = \frac{-1}{4}\int \frac{2u}{u^2+1}\,du$$

Let $v=u^2+1$, $dv=2u$, get:

$$\frac{-1}{4}\int \frac{dv}{v} = \frac{-1}{4}\ln|v| + C$$

$$= \frac{-1}{4}\ln|u^2+1| + C$$

Get:

$$\int \frac{u}{u^4-1}\,du = \frac{1}{4}\ln|u-1| + \frac{1}{4}\ln|u+1| - \frac{1}{4}\ln|u^2+1| + C$$

Put everything together get:

$$\int \frac{u^5}{u^4-1}\,du = \frac{1}{2}u^2 + \frac{1}{4}\ln|u-1| + \frac{1}{4}\ln|u+1| - \frac{1}{4}\ln|u^2+1| + C$$

Don't forget we did a u-sub!

$$\boxed{\int \frac{e^{6x}}{e^{4x}-1}\,dx = \frac{1}{2}e^{2x} + \frac{1}{4}\ln|e^x-1| + \frac{1}{4}\ln|e^x+1| - \frac{1}{4}\ln|e^{2x}+1| + C}$$