Quadratic Case

Useful formula:

$$\boxed{\int \frac{1}{u^2+\beta^2}\,du = \frac{1}{\beta}\arctan\left(\frac{u}{\beta}\right)+C}$$

Example of why

$\beta = 2$ here.

$$\int \frac{1}{u^2+4}\,du = \frac{1}{4}\int \frac{1}{u^2/4+1}\,du$$

Note: $\frac{u^2}{4} = \frac{u^2}{2^2} = \left(\frac{u}{2}\right)^2$ so:

$$= \frac{1}{4}\int \frac{1}{\left(\frac{u}{2}\right)^2+1}\,du$$

variable sub: $v = \frac{u}{2}$, $dv = \frac{1}{2}du$, $2dv = du$

$$= \frac{2}{4}\int \frac{1}{v^2+1}\,dv = \frac{1}{2}\arctan(v)+C$$

$$\boxed{= \frac{1}{2}\arctan\left(\frac{u}{2}\right)+C}$$

Idea: When given $\int \frac{1}{ax^2+bx+c}\,dx$ where ax^2+bx+c has no roots (ie: can't factor).

Then we:

1. Complete the square.
2. Convert into a $\int \frac{1}{u^2+\beta^2}\,du$ problem.

Ex
$$\int \frac{1}{x^2+2x+3}\,dx = \int \frac{1}{(x+1)^2+2}\,dx$$

Complete the square:

$$x^2+2x+3 = (x+1)^2+2$$

do a u-sub: $u=x+1$, $du=dx$, get:

$$\int \frac{1}{u^2+2}\,du = \frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right)+C$$

our useful formula w/ $\beta=\sqrt{2}$.

$$\boxed{= \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right)+C}$$

Case 2 Have x terms on numerator.

Idea: ① Set up numerator for a u-sub.
② Split up integral.

Ex $\displaystyle\int \frac{x}{x^2+2x+3}\,dx$

Want to do $u=x^2+2x+3$
but then $du=2x+2\,dx$
Need $2x+2$ on numerator!

$$\int \frac{x}{x^2+2x+3}\,dx = \frac{1}{2}\int \frac{2x}{x^2+2x+3}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2-2}{x^2+2x+3}\,dx$$

$$= \frac{1}{2}\int \frac{2x+2}{x^2+2x+3}\,dx + \frac{1}{2}\int \frac{-2}{x^2+2x+3}\,dx$$

First integral:

$$\frac{1}{2}\int \frac{2x+2}{x^2+2x+3}\,dx = \left\langle \begin{matrix} u=x^2+2x+3 \\ du=2x+2\,dx \end{matrix} \right\rangle = \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u|+C$$

$$= \frac{1}{2}\ln|x^2+2x+3|+C.$$

Second Integral:

I did this integral just before. solve after completing square.

$$\frac{1}{2}\int \frac{-2}{x^2+2x+3}\,dx = -\int \frac{1}{x^2+2x+3}\,dx$$

$$= -\frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + C$$

Putting together:

$$\boxed{\int \frac{x}{x^2+2x+3}\,dx = \frac{1}{2}\ln|x^2+2x+3| - \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + C}$$

Ex

$$\int \frac{x+1}{x^2+4x+8}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+4x+8}\,dx$$

first get 2x on top.

Next get 4 on top

$$\to = \frac{1}{2}\int \frac{2x+4-4+2}{x^2+4x+8}\,dx$$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+8}\,dx - \frac{2}{2}\int \frac{1}{x^2+4x+8}\,dx$$

u-sub for this one

complete square for this one.

$$\frac{1}{2}\int \frac{2x+4}{x^2+4x+8}\,dx = \left\langle \begin{matrix} u = x^2+4x+8 \\ du = 2x+4\,dx \end{matrix} \right\rangle$$

$$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+4x+8| + C$$

$$-\int \frac{1}{x^2+4x+8}\,dx.$$

Complete the square:

$$x^2+4x+8 = (x+2)^2+4.$$

$$-\int \frac{1}{x^2+4x+8}\,dx = -\int \frac{1}{(x+2)^2+4}\,dx = \left\langle \begin{array}{l} u=x+2 \\ du=dx \end{array} \right\rangle$$

$$= -\int \frac{1}{u^2+4}\,du = -\frac{1}{2}\arctan\left(\frac{u}{2}\right)+C$$

$$= -\frac{1}{2}\arctan\left(\frac{x+2}{2}\right)+C.$$

Put solutions together:

$$\boxed{\int \frac{x+1}{x^2+4x+8}\,dx = \frac{1}{2}\ln|x^2+4x+8| - \frac{1}{2}\arctan\left(\frac{x+2}{2}\right)+C}$$

Ex $\displaystyle\int \frac{3x+2}{5x^2+10x+30}\,dx = \frac{1}{5}\int \frac{3x+2}{x^2+2x+6}\,dx$

get $2x$ on top. $\longrightarrow$ $\displaystyle = \frac{1}{5}\,\frac{3}{2}\int \frac{2x+4/3}{x^2+2x+6}\,dx$

get 2 on top. $\longrightarrow$ $\displaystyle = \frac{3}{10}\int \frac{2x+2-2+4/3}{x^2+2x+6}\,dx$

$$= \frac{3}{10}\int \frac{2x+2}{x^2+2x+6}\,dx + \frac{3}{10}\int \frac{-8/3}{x^2+2x+6}\,dx.$$

First integral:

$$\frac{3}{10}\int \frac{2x+2}{x^2+2x+6}\,dx = \left\langle \begin{array}{l} u=x^2+2x+6 \\ du=2x+2\,dx \end{array} \right\rangle$$

$$= \frac{3}{10}\int \frac{du}{u} = \frac{3}{10}\ln|u|+C$$

$$= \frac{3}{10}\ln|x^2+2x+6|+C$$

Second integral:

$$\frac{-4}{5}\int \frac{1}{x^2+2x+6}\,dx = \frac{-4}{5}\int \frac{1}{(x+1)^2+5}\,dx$$

$$x^2+2x+6 = (x+1)^2+5$$

$$= \left\langle \begin{matrix} u = x+1 \\ du = dx \end{matrix} \right\rangle$$

$$= \frac{-4}{5}\int \frac{du}{u^2+5} = \frac{-4}{5}\cdot\frac{1}{\sqrt{5}}\arctan\left(\frac{u}{\sqrt{5}}\right)+C$$

$$= \frac{-4}{5\sqrt{5}}\arctan\left(\frac{x+1}{\sqrt{5}}\right)+C.$$

Final answer:

$$\boxed{\int \frac{3x+2}{5x^2+20x+30}\,dx = \frac{3}{10}\ln|x^2+4x+2| - \frac{4}{5\sqrt{5}}\arctan\left(\frac{x+1}{\sqrt{5}}\right)+C}$$

Partial Fractions

1. Polynomial division (if necessary)
2. Factor denominator of polynomial.
3. Apply PF formula.
4. Solve for unknowns:
   Heaviside or Equating Coefficients.
5. Integrate terms.

Ex $\int \frac{x^5}{x^2-1}\,dx. = \int x^3+x+\frac{x}{x^2-1}\,dx.$

Poly division:

$$\begin{array}{r|l} & x^3 + x \\ \hline x^2-1 & x^5 \\ & -(x^5-x^3) \\ \hline & x^3 \\ & -(x^3-x) \\ \hline & x \end{array}$$

$$\frac{x^5}{x^2-1} = x^3+x+\frac{x}{x^2-1}$$

Denominator: $x^2-1 = (x+1)(x-1)$.

Partial fractions:

$$\frac{x}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$x = A(x-1) + B(x+1)$$

Solve Coefficients w/ Heaviside:

$@\ x=1, \quad 1 = A\cdot(0) + B\cdot 2 \Rightarrow B = \frac{1}{2}.$

$@\ x=-1, \quad -1 = A\cdot(-2) + B\cdot 0 \Rightarrow A = \frac{1}{2}.$

So:

$$\frac{x}{x^2-1} = \frac{1}{2}\frac{1}{x+1} + \frac{1}{2}\frac{1}{x-1}.$$

So:

$$\int \frac{x^5}{x^2-1}\,dx = \int x^3 + x + \frac{1}{2}\frac{1}{x+1} + \frac{1}{2}\frac{1}{x-1}\,dx$$

$$\boxed{= \frac{1}{4}x^4 + \frac{1}{2}x^2 + \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C.}$$

<u>Note:</u> We did another example in class:

$\int \frac{x^5}{x^4-1}\,dx$. See the document

"example - e^x Partial Fraction"

For the solution...