# Trig Substitution Part 2

## Useful Identities (need to memorize)

$1 - \sin^2\theta = \cos^2\theta$ for $a^2 - x^2$

$1 + \tan^2\theta = \sec^2\theta$ for $a^2 + x^2$

$\sec^2\theta - 1 = \tan^2\theta$ for $x^2 - a^2$

Recall, we ultimately are trying to learn how to integrate things with radicals:

$$\sqrt{ax^2+bx+c}$$

**Idea:** Complete the Square to convert the problem into one of the following:

$$\sqrt{a^2-b^2x^2}, \quad \sqrt{a^2+b^2x^2}, \quad \sqrt{b^2x^2-a^2}$$

Then do a little algebra/substitution to turn into:

$$\sqrt{a^2-x^2} \quad \text{or} \quad \sqrt{a^2+x^2} \quad \text{or} \quad \sqrt{x^2-a^2}$$

Then use respective trig substitution.

**Ex** $\displaystyle\int \sqrt{x^2+2x+2}\,dx \overset{\text{complete square.}}{=} \int \sqrt{(x+1)^2+1}\,dx$

$$= \left\langle \begin{matrix} u = x+1 \\ du = dx \end{matrix} \right\rangle = \int \sqrt{u^2+1}\,du$$

Skip this Problem till last... too long.

Have $\sqrt{u^2+1}$ term, so use $1+\tan^2\theta = \sec^2\theta$.

$$= \left\langle \begin{matrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{matrix} \right\rangle = \int \sqrt{1+\tan^2\theta}\, \sec^2\theta\, d\theta$$

$$= \int \sec^3\theta\, d\theta.$$

This is a bit of a tricky integral.
Try integration by parts. Rember: ask yourself, what do I know how to integrate.

$$\int \underbrace{\sec\theta}_{g} \cdot \underbrace{\sec^2\theta}_{f'}\, d\theta = \tan\theta\sec\theta - \int \tan^2\theta\sec\theta\, d\theta$$

$$f'(\theta) = \sec^2\theta \rightarrow f(\theta) = \tan\theta$$
$$g(\theta) = \sec\theta \rightarrow g'(\theta) = \tan\theta\sec\theta$$

Now what? Get rid of $\tan\theta$ w/ identity:

$$\tan^2\theta = \sec^2\theta - 1.$$

$$\int \sec^3\theta\, d\theta = \tan\theta\sec\theta - \int (\sec^2\theta - 1)\sec\theta\, d\theta$$
$$\underline{\int \sec^3\theta\, d\theta} = \tan\theta\sec\theta \underline{- \int \sec^3\theta\, d\theta} + \int \sec\theta\, d\theta$$

Solve for $\int \sec^3\theta\, d\theta$

$$2\int \sec^3\theta\, d\theta = \tan\theta\sec\theta + \int \sec\theta\, d\theta$$

$$\Rightarrow \int \sec^3\theta\, d\theta = \frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\int \sec\theta\, d\theta$$

We know $\int \sec\theta\, d\theta$ from Calc 1:

$$\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta|$$

So: $\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|.$

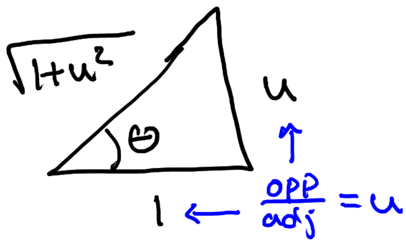
We have:

$$\int \sqrt{(x+1)^2+1}\, dx = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

Recall: $u = \tan\theta \Rightarrow \theta = \arctan(u).$

$\tan\theta = \frac{\text{opp}}{\text{adj}}$ u

$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{1+u^2}}{1}$




$$\int \sqrt{(x+1)^2+1}\, dx = \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln\left|\sqrt{1+u^2}+u\right| + C$$

$$= \frac{1}{2}(x+1)\sqrt{1+(x+1)^2} + \frac{1}{2}\ln\left|\sqrt{1+(1+x)^2} + (1+x)\right| + C$$

Ex $\int \frac{1}{\sqrt{3-2x-x^2}}\, dx$

Complete Square:

$$3-2x-x^2 = -\{x^2+2x-3\}$$

$$= -\{(x+1)^2-4\} = 4-(x+1)^2$$

$$\int \frac{1}{\sqrt{4-(x+1)^2}}\, dx = \left\langle \begin{matrix} u = x+1 \\ du = dx \end{matrix} \right\rangle = \int \frac{1}{\sqrt{4-u^2}}\, du$$

$\sqrt{4-u^2}$ term, use: $4-4\sin^2\theta = 4\cos^2\theta$.

$$= \left\langle \begin{matrix} u = 2\sin\theta \\ du = 2\cos\theta\, d\theta \end{matrix} \right\rangle = \int \frac{2\cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}}$$

$$= \int \frac{2\cos\theta\, d\theta}{2\cos\theta} = \int d\theta = \theta + C.$$

$u = 2\sin\theta \Rightarrow \quad \theta = \arcsin\left(\frac{u}{2}\right).$

$$\int \frac{1}{\sqrt{3-2x-x^2}}\, dx = \arcsin\left(\frac{u}{2}\right) + C$$

$$\boxed{= \arcsin\left(\frac{x+1}{2}\right) + C}$$

**Ex** $\displaystyle\int \frac{dx}{\sqrt{x^2+2x+10}}$

Complete Square: $x^2+2x+10 = (x+1)^2+9$

$$\int \frac{1}{\sqrt{x^2+2x+10}}\, dx = \int \frac{1}{\sqrt{(x+1)^2+9}}\, dx$$

$$= \left\langle \begin{matrix} u = x+1 \\ du = dx \end{matrix} \right\rangle = \int \frac{1}{\sqrt{u^2+9}}\, dx$$

Use formula: $9+9\tan^2\theta = 9\sec^2\theta$.

$$= \left\langle \begin{matrix} u = 3\tan\theta \\ du = 3\sec\theta\, d\theta \end{matrix} \right\rangle = \int \frac{1}{\sqrt{9+9\tan^2\theta}}\, d\theta$$

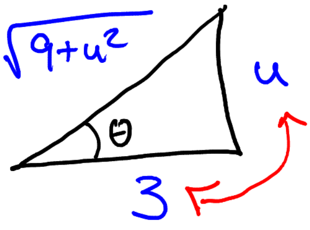
$$= \int \frac{1}{3\sec\theta}\, d\theta$$

$$= \frac{1}{3}\int \cos\theta \, d\theta = \frac{1}{3}\sin\theta + C$$

$$u = 3\tan\theta \Rightarrow \theta = \arctan\left(\frac{u}{3}\right)$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \leftarrow u/3$$

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \boxed{\frac{u}{\sqrt{9+u^2}}}$$




$$\frac{u}{3} = \frac{\text{opp}}{\text{adj}}$$

So:
$$\int \frac{1}{\sqrt{x^2+2x+10}}\,dx = \frac{1}{3}\sin\theta + C$$
$$= \frac{1}{3}\frac{u}{\sqrt{9+u^2}} + C$$
$$\boxed{= \frac{1}{3}\frac{x+1}{\sqrt{(x+1)^2+9}} + C}$$

**Ex**
$$\int \frac{1}{\sqrt{2x-3x^2}}\,dx$$

Complete Square:
$$2x - 3x^2 = -3\left\{x^2 - \frac{2}{3}x\right\}$$
$$= -3\left\{\left(x-\frac{1}{3}\right)^2 - \frac{1}{9}\right\}$$
$$= \boxed{\frac{1}{3} - 3\left(x-\frac{1}{3}\right)^2}$$

So:
$$\int \frac{1}{\sqrt{2x-3x^2}}\,dx = \int \frac{1}{\sqrt{1/3 - 3(x-1/3)^2}}\,dx$$

$$= \left\langle \begin{matrix} u = x - 1/3 \\ du = dx \end{matrix} \right\rangle = \int \frac{1}{\sqrt{1/3 - 3u^2}} du.$$

Factor out 3:

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{1/9 - u^2}} du$$

Have: $1/9 - u^2$ term. use: $1/9 - 1/9 \sin^2\theta = 1/9 \cos^2\theta$

$$= \left\langle \begin{matrix} u = 1/3 \sin\theta \\ du = 1/3 \cos\theta \, d\theta \end{matrix} \right\rangle = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{1/9 \cos^2\theta}} \cdot \frac{1}{3} \cos\theta \, d\theta$$

$$= \frac{1}{\sqrt{3}} \int \frac{1/3 \cos\theta}{1/3 \cos\theta} d\theta = \frac{1}{\sqrt{3}} \theta + C.$$

Recall: $u = 1/3 \sin\theta \Rightarrow \theta = \arcsin(3u)$.

Get:

$$\int \frac{1}{\sqrt{2x - 3x^2}} dx = \frac{1}{\sqrt{3}} \theta + C$$

$$= \frac{1}{\sqrt{3}} \arcsin(3u) + C$$

$$\boxed{= \frac{1}{\sqrt{3}} \arcsin(3x - 1) + C}$$

**Ex** $\int \frac{1}{x^2 + 2x + 2} dx$

Note, we discussed how to solve this in PFs. But we will show different technique here.

$$x^2 + 2x + 2 = (x+1)^2 + 1$$

$$\int \frac{1}{x^2+2x+2}\,dx = \int \frac{1}{(x+1)^2+1}\,dx$$

$$= \left\langle \begin{array}{l} u = x+1 \\ du = dx \end{array} \right\rangle = \int \frac{1}{u^2+1}\,du.$$

Have $u^2+1$ term. Use: $1+\tan^2\theta = \sec^2\theta$.

$$= \left\langle \begin{array}{l} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{array} \right\rangle = \int \frac{1}{\sec^2\theta} \sec^2\theta\, d\theta$$

$$= \int d\theta = \theta + C.$$

Recall: $u = \tan\theta \Rightarrow \theta = \arctan(u)$

So:

$$\int \frac{1}{x^2+2x+2}\,dx = \theta + C$$

$$= \arctan(u) + C$$

$$\boxed{= \arctan(x+1) + C}$$

Quick Review

1. Given $\sqrt{ax^2+bx+c}$, complete square.
2. Get: $a^2-x^2$, $a^2+x^2$ or $x^2-a^2$.
3. Use appropriate trig substitution.