Worksheet 12

Little Oh and Taylor Series Manipulations

Problem 1. Show that $\sqrt{1+x^2} + \sqrt{1-x^2} = \cancel{1}\,2 + o(x^3)$.

Calc $T_2\sqrt{1+x}$

$f(x) = \sqrt{1+x}$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$f''(x) = \frac{-1}{4}(1+x)^{-3/2}$

$f(0) = 1$

$f'(0) = 1/2$

$f''(0) = \frac{-1}{4}$

$\sqrt{1+x} = T_2\sqrt{1+x} + o(x^2)$

$\Rightarrow \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o(x^2)$

sub x^2: $\sqrt{1+x^2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4)$

sub $-x^2$: $\sqrt{1-x^2} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 + o(x^4)$

$\Rightarrow \sqrt{1+x^2} + \sqrt{1-x^2} = 1 + \cancel{\frac{1}{2}x^2} - \frac{1}{8}x^4 + o(x^4) + 1 - \cancel{\frac{1}{2}x^2} - \frac{1}{8}x^4 + o(x^4)$

$= 1 - \frac{1}{4}x^4 + o(x^4) = 2 + o(x^3) + o(x^4) = \underline{2 + o(x^3)}$ ✓

Problem 2. For which k is it true that $\sqrt{1+3x^4} = 1 + o(x^k)$?

Recall: $\sqrt{1+x} = 1 + \frac{1}{2}x + o(x)$

sub $3x^4$

$\Rightarrow \sqrt{1+3x^4} = 1 + \frac{1}{2}(3x^4) + o(x^4)$

$= 1 + \underbrace{\frac{3}{2}x^4 + o(x^4)}$

$= o(x^k)$ for $\underline{k = 0, 1, 2, 3}$.

Problem 3. Calculate $T_3\cos(x)$, and estimate the error $|\cos(x) - T_3\cos(x)|$. Use this to show that the following is true for any x:

$$1 - \frac{x^2}{2} - \frac{x^4}{24} \le \cos(x) \le 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$T_3\cos(x) = 1 - \frac{1}{2}x^2$

$|\cos(x) - T_3\cos(x)| = |R_3\cos(x)|$

$R_3\cos(x) = \frac{\cos(c)}{4!}x^4$

$\Rightarrow |R_3\cos(x)| \le \frac{1}{4!}x^4$

$\Rightarrow |\cos(x) - T_3\cos(x)| \le \frac{1}{4!}x^4$

$\Rightarrow \frac{-1}{4!}x^4 \le \cos(x) - \{1 - \frac{1}{2}x^2\} \le \frac{1}{4!}x^4$

$\Rightarrow 1 - \frac{1}{2}x^2 - \frac{1}{4!}x^4 \le \cos(x) \le 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4$ ✓

Problem 4. Calculate $T_{11}f(x)$ for $f(x) = x^2\sin(4x^3)$, and use this to find the derivatives $f^{(8)}(0)$, $f^{(10)}(0)$, and $f^{(11)}(0)$.

$$\sin(x) = x - \tfrac{1}{6}x^3 + o(x^3)$$

$o(x^9)$

$$\Rightarrow \sin(4x^3) = 4x^3 - \tfrac{1}{6}(4x^3)^3 + o((x^3)^3)$$

$$= 4x^3 - \frac{4^3}{6}x^9 + o(x^9)$$

$x^2 \cdot o(x^9) = o(x^{11})$

$$\Rightarrow x^2\sin(4x^3) = 4x^5 - \frac{4^3}{6}x^{11} + o(x^{11})$$

$$* \Rightarrow T_{11}\{x^2\sin(4x^3)\} = 4x^5 - \frac{4^3}{6}x^{11} \; *$$

$$f^{(8)}(0) = 0, \quad f^{(10)}(0) = 0, \quad \frac{f^{(11)}(0)}{11!} = -\frac{4^3}{6} \Rightarrow \boxed{f^{(11)}(0) = -11!\,\frac{4^3}{6}}$$

Problem 5. Calculate $T_8f(x)$ for $f(x) = (1+x^2)e^{-x^4}$, and use this to find the derivatives $f^{(6)}(0)$, $f^{(7)}(0)$, and $f^{(8)}(0)$.

$$e^x = 1 + x + \tfrac{1}{2}x^2 + o(x^2)$$

$$\Rightarrow e^{-x^4} = 1 - x^4 + \tfrac{1}{2}x^8 + o(x^8)$$

$$\Rightarrow (1+x^2)e^{-x^4} = (1+x^2)(1 - x^4 + \tfrac{1}{2}x^8 + o(x^8))$$

$x^2 o(x^8) = o(x^{10})$

$$= 1 - x^4 + \tfrac{1}{2}x^8 + o(x^8)$$
$$+ x^2 - x^6 + \tfrac{1}{2}x^{10} + o(x^{10})$$
$$= \underbrace{1 + x^2 - x^4 - x^6 + \tfrac{1}{2}x^8}_{T_8\{(1+x^2)e^{-x^4}\}} + o(x^8).$$

$$\frac{f^{(6)}(0)}{6!} = -1 \Rightarrow f^{(6)}(0) = -6!$$

$$f^{(7)}(0) = 0.$$

$$\frac{f^{(8)}(0)}{8!} = \frac{1}{2} \Rightarrow f^{(8)}(0) = \tfrac{1}{2}8!$$

Problem 6. Calculate $T_4f(x)$ for $f(x) = \sin(3x)e^{x^2}$ without calculating any derivatives.

$$e^x = 1 + x + \tfrac{1}{2}x^2 + o(x^2)$$

$$\Rightarrow e^{x^2} = 1 + x^2 + \tfrac{1}{2}x^4 + o(x^4)$$

$\frac{9}{2}$

$$\sin(x) = x - \tfrac{1}{6}x^3 + o(x^4) \Rightarrow \sin(3x) = 3x - \frac{3^3}{6}x^3 + o(x^4)$$

$$\Rightarrow \sin(3x)e^{x^2} = \{3x - \tfrac{9}{2}x^3 + o(x^4)\}\{1 + x^2 + \tfrac{1}{2}x^4 + o(x^4)\}$$

We will "drop" terms powers ≥ 5 and put them all into on $o(x^4)$ at end

$$= 3x + 3x^3 + o(x^4)$$
$$- \tfrac{9}{2}x^3 + o(x^4)$$
$$= \underbrace{3x - \tfrac{3}{2}x^3}_{} + o(x^4)$$

$$\Rightarrow \boxed{T_4\{\sin(3x)e^{x^2}\} = 3x - \tfrac{3}{2}x^3}$$