# Partial Fractions

Last time:

- Studied how to integrate $\int \frac{P(x)}{Q(x)}dx$.
- Step 1 Polynomial division! (if necessary).
- Learned if $Q(x)$ can be factored into distinct linear factors

$$Q(x)=(x-r_1)(x-r_2)\cdots(x-r_n)$$

Then we can simplify by:

$$\frac{P(x)}{(x-r_1)(x-r_2)\cdots(x-r_n)} = \frac{A_1}{x-r_1}+\frac{A_2}{x-r_2}+\cdots+\frac{A_n}{x-r_n}$$

- Studied how to solve the quadratic case:

$$\int \frac{1}{ax^2+bx+c}$$

Had 3 cases:

1. Two roots $\Rightarrow$ $a(x-r_1)(x-r_2) \Rightarrow$ PFs
2. One root $\Rightarrow$ $a(x-r)^2 \Rightarrow$ u-sub to get $\frac{1}{u^2}$ case.
3. No roots $\Rightarrow$ complete square, use:

$$\int \frac{1}{u^2+\beta^2} = \frac{1}{\beta}\arctan\left(\frac{u}{\beta}\right)+C.$$

Today we will discuss Partial Fractions in it's full generality.

We will start by writing out the statement of Partial Fractions in its various cases, and give examples.

Important note to keep in mind:

> For partial Fractions to be useful, we must be able to factor the denominator!

No worries though, the denominator will usually be given to you factored.

## Case 1 Linear Factors (Not necessarily distinct)

$$Q(x) = (x-r_1)^{k_1}(x-r_2)^{k_2}\cdots(x-r_n)^{k_n}$$

When we have repeated roots then partial fractions gives more terms! One for each repeated root.

**Ex**

repeated root

extra term.

$$\frac{1}{(x-1)(x-2)^2} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

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$$\frac{1}{x^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

two repeated roots

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$$\frac{5x^2+3x+1}{x(x-1)(x+2)^3} = \frac{A}{x} + \frac{B}{x-1} + \underbrace{\frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3}}_{\text{three roots so three terms}}$$

Note this is a proper function.

three roots so three terms

Ex (cont.) need polynomial division first!

$$\frac{x^3+x^2+1}{x^2(x+1)} = 1 + \frac{1}{x^2(x+1)}$$

Now partial fractions.

$$= 1 + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

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## Case 2 Quadratic Factors (w/ no roots)

Now numerator of terms are linear factors!

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Ex

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Linear term now.

no roots can't be factored further.

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$$\frac{3x^2+2x+1}{(x+1)^2(x^2+2x+5)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+2x+5}$$

no roots!

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$$\frac{x^3+3x^2+6x+1}{x(x^2+2x+6)} = x + \frac{x+1}{x(x^2+2x+6)}$$

Divide first!

$$= x + \frac{A}{x} + \frac{Bx+C}{x^2+2x+6}$$

## Case 3 Quadratic terms (no roots, repeated terms)

Again, new term for each repeated factor:

**Ex**

new term.

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

no roots

repeated factor

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$$\frac{1}{x^2(x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

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$$\frac{x^2+x+1}{x(x^2+2x+5)^3} = \frac{A}{x} + \frac{Bx+C}{x^2+2x+5} + \frac{Dx+E}{(x^2+2x+5)^2} + \frac{Fx+G}{(x^2+2x+5)^3}$$

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Hopefully this pattern is clear now.

Question Now that I have PF decomposition How do I solve for all of these constants?

Two Methods:

1. Heaviside Method
2. Equating Coefficients Method.