First Order Linear Differential Equations

$$\frac{dy}{dx} + a(x)y(x) = K(x)$$

Find $m(x)$ so:

$$m(x)\frac{dy}{dx} + m(x)a(x)y(x) = \frac{d}{dy}\big(m(x)y(x)\big)$$

Solution:

$$\frac{dm}{dx} = a(x)m(x) \Rightarrow m(x) = e^{\int a(x)\,dx}$$

Get:

$$\frac{d}{dx}\big(m(x)y(x)\big) = K(x)m(x)$$

$$\Rightarrow \boxed{\begin{array}{l} y(x) = \frac{1}{m(x)}\left\{\int K(x)m(x)\,dx + C\right\} \\ \text{where: } m(x) = e^{\int a(x)\,dx} \end{array}}$$

Examples

$$\frac{dy}{dx} - 2y = x^2 \qquad y(0) = 0.$$

$a(x) = -2$, $K(x) = x^2$.

$$m(x) = e^{\int -2\,dx} = e^{-2x}$$

Solution:

$$y(x) = \frac{1}{e^{-2x}}\left\{\int x^2 e^{-2x}\,dx + C\right\}$$

IBPs will give:

$$\int x^2 e^{-2x}\,dx = \frac{-1}{4}e^{-2x}(2x^2 + 2x + 1)$$

Get:
$$y(x) = \frac{1}{e^{-2x}} \left\{ \frac{-1}{4} e^{-2x}(2x^2+2x+1) + c \right\}$$

$$\Rightarrow \quad y(x) = \frac{-1}{4}(2x^2+2x+1) + ce^{2x}$$

Using Boundary Condition $y(0)=0$, get:

$$0 = \frac{-1}{4}(0+0+1) + ce^0 = \frac{-1}{4} + c$$

$$\Rightarrow c = 1/4.$$

Final Answer:

$$\boxed{y(x) = \frac{-1}{4}(2x^2+2x+1) + \frac{1}{4}e^{2x}}$$

Ex

$$\frac{dy}{dx} = \cos(x)y + e^{\sin(x)}, \quad y(0) = A.$$

First write in standard form:

$$\frac{dy}{dx} - \cos(x)y(x) = e^{\sin(x)}$$

Here: $a(x) = -\cos(x)$, $k(x) = e^{\sin(x)}$

$$m(x) = e^{\int a(x)} = e^{-\int \cos(x)} = e^{-\sin(x)}.$$

Get:
$$y(x) = \frac{1}{m(x)} \left\{ \int k(x)m(x) + c \right\}$$

$$\Rightarrow y(x) = e^{\sin(x)} \left\{ \int e^{\sin(x)} e^{-\sin(x)} dx + c \right\}$$

$$= e^{\sin(x)} \left\{ \int e^0 dx + c \right\}$$

$$= e^{\sin(x)} \left\{ x + c \right\}$$

Using Boundary condition $y(0)=A$, get:

$$A = e^0\{0+C\} = C. \quad \text{So } C=A.$$

Final Answer:

$$\boxed{y(x) = e^{\sin(x)}\{x+A\}}$$

<u>Ex</u>

$$\cos^2(x)\frac{dy}{dx} = N - y \qquad y(0)=0.$$

Write in standard form:

$$\frac{dy}{dx} = N\sec^2(x) - \sec^2(x)y$$

$$\Rightarrow \quad \frac{dy}{dx} + \sec^2(x)y(x) = N\sec^2(x).$$

Here: $a(x) = \sec^2(x)$, $k(x) = N\sec^2(x)$.

$$m(x) = e^{\int \sec^2(x)} = e^{\tan(x)}$$

Get:

$$y(x) = \frac{1}{m(x)}\left\{\int k(x)m(x)\,dx + C\right\}$$

$$\Rightarrow \quad y(x) = e^{-\tan(x)}\left\{\int N\sec^2(x)e^{\tan(x)}\,dx + C\right\}$$

Solve $\int N\sec^2(x)e^{\tan(x)}\,dx = N\int e^{\tan(x)}\sec^2(x)\,dx.$

$$= \left\langle \begin{matrix} u = \tan(x) \\ du = \sec^2(x)dx \end{matrix} \right\rangle = N\int e^u du = Ne^u du.$$

$$= Ne^{\tan(x)}$$

Get:

$$y(x) = e^{-\tan(x)}\left\{Ne^{\tan(x)} + C\right\}$$

So: $y(x) = N + Ce^{-\tan(x)}$

Using Initial condition $y(0)=0$ get:

$$0 = N + Ce^{-0} = N + C \Rightarrow C = -N.$$

Final Answer:

$$\boxed{y(x) = N - Ne^{-\tan(x)}}$$

Ex $\dfrac{dy}{dx} = -(1-3x^2)y(x)$ $\quad y(1) = 1.$

Note: This is seperable! But we use the integrating factor technique here:

In standard form:

$$\frac{dy}{dx} + (1-3x^2)y(x) = 0$$

Here: $a(x) = 1-3x^2$, $K(x) = 0$.

$$m(x) = e^{\int 1-3x^2 dx} = e^{x-x^3}$$

Get:

$$y(x) = \frac{1}{m(x)}\left\{\int m(x)K(x)dx + C\right\}$$

$$y(x) = e^{-x+x^3}\left\{0 + C\right\} = Ce^{-x+x^3}$$

Using initial condition: $y(1)=1$ get:

$$1 = Ce^{-1+1} = C.$$

Answer: $\boxed{y(x) = e^{-x+x^3}}$

Ex $\quad \frac{dy}{dx} = -(1-3x^2)y^2(x), \quad y(0)=1$

Note: This does not fit the integrating factor form since we have $y^2(x)$ and not $y(x)$! So we must use seperable technique!

Get:
$$\int \frac{1}{y^2}\,dy = \int (3x^2-1)\,dx$$

$$\Rightarrow \quad \frac{-1}{y} = x^3 - x + C$$

$$\Rightarrow \quad y = \frac{-1}{x^3-x+C}$$

Using initial condition $y(0)=1$ get:

$$1 = \frac{-1}{0-0+C} \Rightarrow C=-1$$

Final Answer:

$$\boxed{y(x) = \frac{-1}{x^3-x-1}}$$