Section 5.5

Double Angle Identities and Sum ↔ Product Identities

You can get double angle formulas from $\cos(A + B)$ by setting both A and B to x. In which case we get:

$$\cos(2x) = \cos(x + x) = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x$$

You can use this same strategy for any of the sum formulas we have established. There are a few variants for the cosine one however, and which one you want to use depends on whether or not you would prefer to have your double angle formula to depend on both $\sin x$ and $\cos x$, or just one of $\sin x$ or $\cos x$. It can be convient if you only know one of the two trigonometric function values.

Double Angle Formulas

$$\cos 2x = \cos^2 x - \sin^2 x \qquad \cos 2x = 1 - 2\sin^2 x$$
$$\cos 2x = 2\cos^2 x - 1 \qquad \sin 2x = 2 \sin x \cos x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

The important one to remember here is the first variant: $\cos 2x = \cos^2 x - \sin^2 x$.

Problem 1. Verify the other two variants of the double angle formula given $\cos 2x = \cos^2 x - \sin^2 x$.

Show $\cos 2x = 2\cos^2 x - 1$

$\cos 2x = \cos^2 x - \sin^2 x$

Pythagoreans $= \cos^2 x - (1 - \cos^2 x)$

$= 2\cos^2 x - 1$

Show $\cos 2x = 1 - 2\sin^2 x$

$\cos 2x = \cos^2 x - \sin^2 x$

Pythagoreans $= (1 - \sin^2 x) - \sin^2 x$

$= 1 - 2\sin^2 x$

Problem 2. Verify the $\sin 2x$ and $\tan 2x$ formulas given above.

$$\sin 2x = \sin(x + x) = \sin x \cos x + \cos x \sin x = \boxed{2 \sin x \cos x}$$

$$\tan 2x = \tan(x + x) = \frac{\tan x + \tan x}{1 - \tan x \tan x} = \boxed{\frac{2\tan x}{1 - \tan^2 x}}$$

Tips for Remembering: A Sanity Check

With all of these formulas you will often find yourself forgetting the order in which the functions appear or whether there should be a plus sign or a minus sign. A sanity check is to verify your fomula for an easy to verify value to see if it holds true.

Example: Suppose you can't remember which one is supposed to be true:

$$\cos(x+y) = \cos x \cos y + \sin x \sin y? \qquad \cos(x+y) = \cos x \cos y - \sin x \sin y?$$

Try pluggin in $\pi/2$ for x and y:

$$\cos\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \cos\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\sin\frac{\pi}{2} \qquad \cos\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \cos\frac{\pi}{2}\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\sin\frac{\pi}{2}$$

$$\cos\pi = 0+1 \quad ✗ \qquad\qquad \cos\pi = 0-1 \quad ✓$$

However, you might have to try a few values to get it right. Suppose instead we tried (my favroite) setting both x and y to 0:

$$\cos(0+0) = \cos 0\cos 0 + \sin 0 \sin 0 \qquad \cos(0+0) = \cos 0\cos 0 - \sin 0 \sin 0$$

$$\cos 0 = 1+0 \quad ✓ \qquad\qquad \cos 0 = 1-0 \quad ✓$$

Then from this information we can't eliminate either formula! Like with most things, this is useful but it won't always work.

Problem 3. Given $\cos\theta = \frac{3}{5}$ and $\sin\theta < 0$, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

know $\cos\theta$ so use: $\cos 2\theta = 2\cos^2\theta - 1$

$$\cos 2\theta = 2\left(\tfrac{3}{5}\right)^2 - 1 = \frac{18}{25} - 1 = \frac{-7}{25}$$

$$\sin 2\theta = 2 \underbrace{\cos\theta}_{3/5} \underbrace{\sin\theta}_{?}$$

$$= 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$= 1 - \left(\tfrac{3}{5}\right)^2$$

$$= 1 - \frac{9}{25} = \frac{16}{25}$$

$\Rightarrow \sin\theta = \pm 4/5.$

$\sin\theta > 0 \Rightarrow \sin\theta = 4/5.$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{24/25}{-7/25}$$

$$\Rightarrow \tan 2\theta = \frac{-24}{7}$$

Problem 4. Verify the identity: $\cos^4 \alpha - \sin^4 \alpha = \cos 2\alpha$.

diff of squares

$$\cos^4\alpha - \sin^4\alpha = (\underbrace{\cos^2\alpha + \sin^2\alpha}_{=1 \text{ by pyth}})(\underbrace{\cos^2\alpha - \sin^2\alpha}_{\text{double angle form.}})$$

$$= 1 \cdot \cos 2\alpha = \cos 2\alpha.$$

Problem 5. Simplify the expressions:

(a) $\cos^2 7x - \sin^2 7x$

$$= \cos(2 \cdot 7x) = \cos(14x)$$

(b) $2\cos^2 5x - 1$

$$= \cos(2 \cdot 5x) = \cos(10x)$$

Like double angle formula. ← but triple angle formula

Problem 6. Write $\cos 3x$ in terms of $\cos x$.

$$\cos(3x) = \cos(2x+x)$$
$$= \cos(2x)\cos x - \sin 2x \sin x$$
$$= (2\cos^2 x - 1)\cos x - (2\cos x \underbrace{\sin x)\sin x}_{\sin^2 x = 1-\cos^2 x}$$
$$= 2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x)$$
$$= 4\cos^3 x - \cos x - 2\cos x$$
$$= 4\cos^3 x - 3\cos x$$

Product ↔ Sum Identities

These formulas are a little trick to remember, but can be done so by playing around with adding and subtracting what you think should work and hope for the best. I won't expect you to **remember** these on an exam, but you will be exected on how to **use** the identities.

Product ↔ Sum Identities

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$
$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$
$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Problem 7. Writeout $\sin 2\theta - \sin 4\theta$ as a product of two functions.

$$\sin 2\theta - \sin 4\theta = 2\cos\left(\frac{2\theta+4\theta}{2}\right)\sin\left(\frac{2\theta-4\theta}{2}\right)$$
$$= 2\cos(3\theta)\sin(-\theta)$$
$$= -2\cos(3\theta)\sin(\theta).$$

Application. When a muscian tunes a guitar, they try to match the pitch given from two different strings by either tightening or loosening on of them. When pitches on differ only by a small amount, then muscian hears a single pitch with a faint oscillation in the volume. They can often listen to the oscillation and try to minimizing how fast the oscillation occurs as a tool for tuning their instrument. The **Sum-to-Product** formulas describes this phenomena.

Notes. In the standard tuning the second string (the second lowest-pitch) string should be tuned at 110.00 Hz when plucked. Suppose for a moment that the thrid string is slightly out of tuned at when plucked you get 105 Hz. We can calculate the periods of these sound waves from their frequency:

$$110.00\text{Hz} = \frac{1}{110}\text{ Period} \qquad\qquad 105\text{ Hz} = \frac{1}{105}\text{ Period}$$

We can then use this period to calculate a model for these sound-waves in the form $s(t) = \cos(\omega t)$. We get:

$$\text{Second string: } \cos\left(\frac{2\pi}{\frac{1}{110}}t\right) = \cos(220\pi t) \qquad\qquad \text{Third string: } \cos\left(\frac{2\pi}{\frac{1}{105}}t\right) = \cos(210\pi t)$$

When these two strings are played at the same time you get a sound-wave that can be graphed by the equation:

$$s(t) = \cos 220\pi t + \cos 210\pi t$$

The sum-to-product identites then applies with $A = 220\pi t$ and $B = 210\pi t$ and we get:

$$s(t) = 2\cos\left(\frac{220\pi t + 210\pi t}{2}\right)\cos\left(\frac{220\pi t - 210\pi t}{2}\right)$$

$$s(t) = 2\cos(215\pi t)\cos(5\pi t)$$

You can graph this equation in the same way we graphed damnped oscillation. Notice how you have a fast oscillating wave [this comes from $\cos(215\pi t)$] that "lives" inside a slowly oscillating wave [this comes from $\cos(5\pi t)$].