$\sin(A-B) = \sin A \cos B - \cos A \sin B$

Section 5.4

Sum and Difference Formulas for Sine and Tangent

Problem 1. Find the exact values of the following functions.

(a) $\sin 15°$

$$= \sin(45° - 30°) = \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

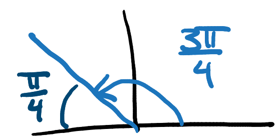
$$\frac{\sqrt{2-\sqrt{3}}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4} \,?$$

Square both sides: $\frac{2-\sqrt{3}}{4} \overset{?}{=} \underbrace{\frac{6-2\sqrt{12}+2}{16}}_{\frac{8-4\sqrt{3}}{16} = \frac{2-\sqrt{3}}{4}}$ ✓

Compare this answer with the result we obtained yesterday: $\sin 15° = \frac{\sqrt{2-\sqrt{3}}}{2}$. Are these two answers the same?

(b) $\tan \frac{11\pi}{12}$

$$\frac{11\pi}{12} = \frac{2\pi + 9\pi}{12} = \frac{\pi}{6} + \frac{3\pi}{4}$$



$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$$\tan\left(\frac{\pi}{6}\right) = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\tan\left(\frac{3\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$$

$$\tan\frac{11\pi}{12} = \frac{\tan\frac{\pi}{6} + \tan\frac{3\pi}{4}}{1 - \tan\frac{\pi}{6}\tan\frac{3\pi}{4}} = \frac{\sqrt{3}/3 + (-1)}{1 - \sqrt{3}/3\,(-1)} \cdot \frac{3}{3} = \frac{\sqrt{3}-3}{\sqrt{3}+3} \cdot \frac{\sqrt{3}-3}{\sqrt{3}-3}$$

$$= \frac{3 - 6\sqrt{3} + 9}{3 - 9} = \frac{12 - 6\sqrt{3}}{-6} = \boxed{\sqrt{3} - 2}$$

(c) $\frac{\tan 100° - \tan 70°}{1 + \tan 100° \tan 70°}$

$$= \tan(100° - 70°) = \tan 30°$$

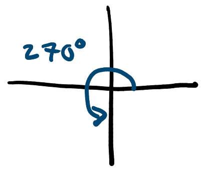
$$= \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \boxed{\frac{\sqrt{3}}{3}}$$

Problem 2. Use the sum and difference formulas to write each function as an expression inolving functions of θ only.

(a) $\sin(\theta - 270°)$

$$\sin(A-B) = \sin A \cos B + \cos A \sin B$$

$$\sin(\theta - 270°) = \sin\theta \cos 270° + \cos\theta \sin 270°$$



$$\cos 270° = 0, \quad \sin 270° = -1$$

$$\Rightarrow \sin(\theta - 270°) = 0 - \cos\theta = -\cos\theta$$

(b) $\tan(\theta + 3\pi)$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(\theta + 3\pi) = \frac{\tan\theta + \tan(3\pi)}{1 - \tan\theta \tan 3\pi}$$

$$\tan 3\pi = \frac{\sin 3\pi}{\cos 3\pi} = \frac{0}{-1} = 0$$

$$= \frac{\tan\theta + 0}{1 - 0} = \tan\theta.$$

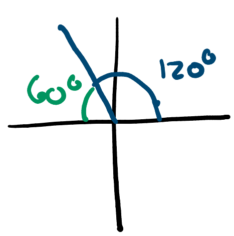
(c) $\sin(120° + \theta)$

$$\sin(A+B) = \sin A \cos B - \cos A \sin B$$

$$\sin(120° + \theta) = \sin 120° \cos\theta - \cos 120° \sin\theta$$

$$= \frac{\sqrt{3}}{2}\cos\theta - \left(\frac{-1}{2}\right)\sin\theta$$

$$= \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta$$



$$\cos 120° = -\cos 60° = -\frac{1}{2}$$

$$\sin 120° = \sin 60° = \frac{\sqrt{3}}{2}$$

Problem 3. Suppose that $\cos A = {-7}/{25}$ with angle A in Quadrant II, and $\sin B = {-3}/{5}$ with angle B in Quadrant IV. Find each of the following:

(a) $\sin(A - B)$

$$= \underbrace{\sin A}_{?}\underbrace{\cos B}_{?} + \underbrace{\cos A}_{-7/25}\underbrace{\sin B}_{-3/5}$$

Find $\sin A$ given $\cos A = -7/25$ and A in QII

$$\sin^2 A = 1 - \cos^2 A$$

$$\sin^2 A = 1 - 49/625 = \frac{576}{625}$$

$$\Rightarrow \sin A = \pm\sqrt{\frac{576}{625}} = \pm\frac{24}{25}$$

A in QII $\Rightarrow$ $\sin A > 0$

$$\Rightarrow \sin A = \frac{24}{25}$$

Find $\cos B$ given $\sin B = \frac{-3}{5}$ and B in QIV

$$\cos^2 B = 1 - \sin^2 B$$

$$\cos^2 B = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\Rightarrow \cos B = \pm\frac{4}{5}$$

B in QIV $\Rightarrow$ $\cos B > 0$

$$\Rightarrow \cos B = \frac{4}{5}$$

$$\sin(A-B) = \frac{24}{25}\cdot\frac{4}{5} + \left(\frac{-7}{25}\right)\left(\frac{-3}{5}\right) = \frac{96+21}{125} = \boxed{\frac{117}{125}}$$

(b) $\tan(A - B)$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{24/25}{-7/25} = -\frac{24}{7}$$

$$\tan B = \frac{\sin B}{\cos B} = \frac{-3/5}{4/5} = -\frac{3}{4}$$

$$= \frac{\frac{-24}{7} - \left(\frac{-3}{4}\right)}{1 + \left(\frac{-24}{7}\right)\left(\frac{-3}{4}\right)} \cdot \frac{7\times 4}{7\times 4}$$

$$= \frac{-24\cdot 4 + 3\cdot 7}{28 + 24\cdot 3} = \boxed{\frac{-117}{44}}$$

(c) The quadrant of $A - B$.

$$\underbrace{\sin(A-B)}_{\text{I or II}} > 0 \text{ ; } \underbrace{\tan(A-B)}_{\text{II or IV}} < 0 \Rightarrow \underline{A-B \text{ in QII}}$$

Problem 4. Verify the identity $\tan\left(\frac{\pi}{4}+t\right)+\tan\left(\frac{\pi}{4}-t\right)=\frac{2\sec^2 t}{1-\tan^2 t}$.

$\tan\frac{\pi}{4}=1$

start w/ LHS:

$$\tan\left(\frac{\pi}{4}+t\right)=\frac{\tan\frac{\pi}{4}+\tan t}{1-\tan\frac{\pi}{4}\tan t}=\frac{1+\tan t}{1-\tan t}$$

$$\tan\left(\frac{\pi}{4}-t\right)=\frac{\tan\frac{\pi}{4}-\tan t}{1+\tan\frac{\pi}{4}\tan t}=\frac{1-\tan t}{1+\tan t}$$

$$\tan\left(\frac{\pi}{4}+t\right)+\tan\left(\frac{\pi}{4}-t\right)=\frac{1+\tan t}{1-\tan t}+\frac{1-\tan t}{1+\tan t}$$

Common denominator

$$=\frac{(1+\tan t)^2+(1-\tan t)^2}{(1-\tan t)(1+\tan t)}$$

$$=\frac{1+\cancel{2\tan t}+\tan^2 t+1-\cancel{2\tan t}+\tan^2 t}{1-\tan^2 t}$$

$\leftarrow (1+\tan t)^2$

$\leftarrow (1-\tan t)^2$

$\leftarrow$ diff of squares

$$=\frac{2+2\tan^2 t}{1-\tan^2 t}=\frac{2(1+\tan^2 t)}{1-\tan^2 t}$$

$1+\tan^2 t=\sec^2 t$

$$=\frac{2\sec^2 t}{1-\tan^2 t}\ \checkmark$$

$\uparrow$

RHS