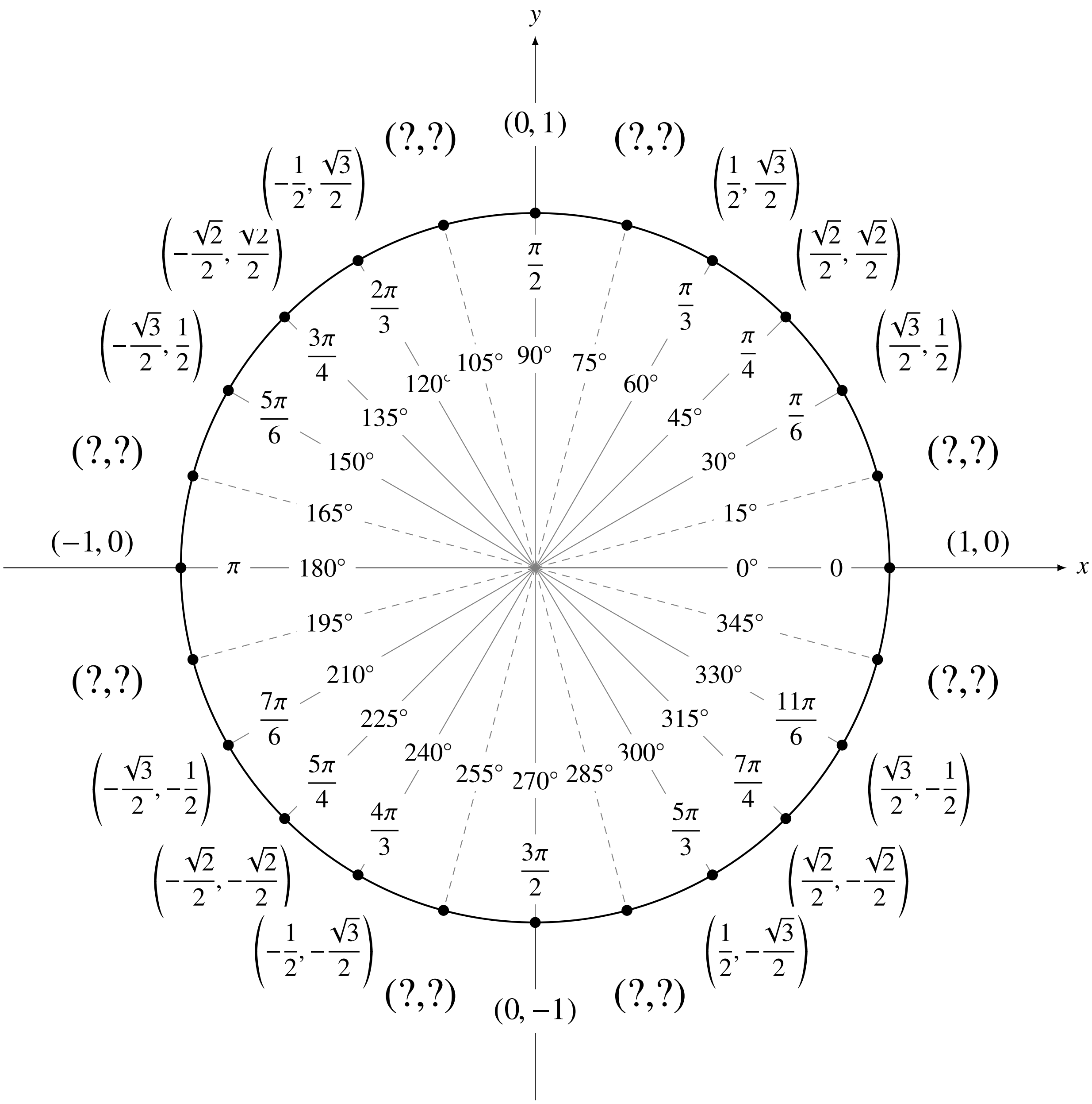
Section 5.3

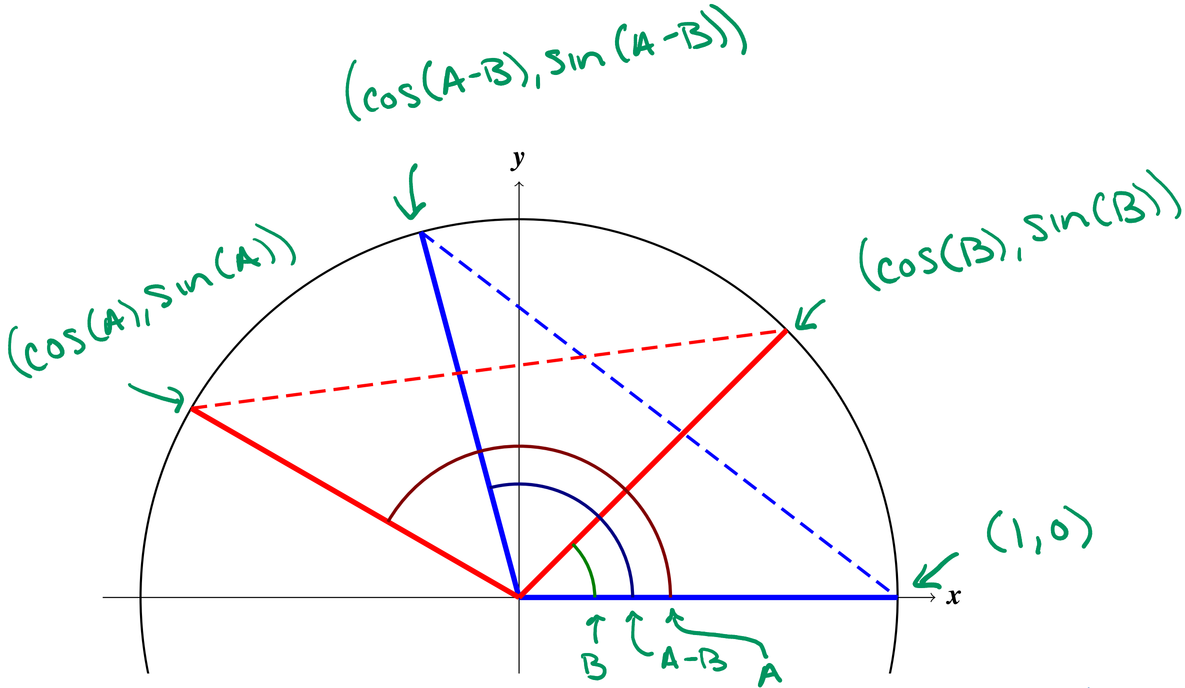
Sum and Difference Formulas for Cosine

One of our goals for this lecture is to figure out a way to *complete* the missing angles in the diagrams. If for no other reason than to satiate a perfectionist's obsession.



The Difference Formula for Cosine

While deriving a difference formula for cosine we will reference the following diagram. Label the respective angles for the green, blue, and red arcs shwon in the diagram. Then label the respective coordinates on the circle. The circle is a unit circle, which simplifies the relationship between trigonometric functions and points on a circle.



Step 1: Draw blue triangle for cos(A-B)

Step 2: Rotate triangle by B degrees.

Both triangles have opposite side with same length.

Step 3: Equate lengths of opposite sides using distance formula.

$$\sqrt{(\cos A-\cos B)^2+(\sin A-\sin B)^2} = \sqrt{(\cos(A-B)-1)^2+\sin(A-B)^2}$$

$$\Rightarrow \underline{\cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B} = \underline{\cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(A-B)}$$

$$1 - 2\cos A\cos B - 2\sin A\sin B + 1 = 1 - 2\cos(A-B) + 1$$

$$\Rightarrow \boxed{\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)}$$

Problem 1. Find the exact values for $\cos 15°$ and $\sin 15°$.

$$15° = 45° - 30° \Rightarrow \cos(15°) = \cos(45° - 30°)$$

$$\cos(45° - 30°) = \cos(45°)\cos(30°) + \sin(45°)\sin(30°)$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

$$\sin 15° = \sqrt{1 - \cos^2 15°} = \sqrt{1 - \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2}$$

$$\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2 = \frac{6 + 2\sqrt{6}\sqrt{2} + 2}{16} = \frac{8 + 4\sqrt{3}}{16} = \frac{2+\sqrt{3}}{4}$$

$$1 - \left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2 = \frac{4}{4} - \frac{2+\sqrt{3}}{4} = \frac{2-\sqrt{3}}{4}$$

$$\Rightarrow \sin 15° = \sqrt{\frac{2-\sqrt{3}}{4}} = \boxed{\frac{\sqrt{2-\sqrt{3}}}{2}}$$

Problem 2. Find the exact values for $\cos 75°$ and $\sin 75°$. **Problem 1** should simplify this question quite a bit. Do you see why?

Use Cofunction Identities!

$$\cos 75° = \sin(90° - 75°) = \sin 15°$$

$$\sin 75° = \cos(90° - 75°) = \cos 15°$$

$$\Rightarrow \cos 75° = \boxed{\frac{\sqrt{2-\sqrt{3}}}{2}}$$

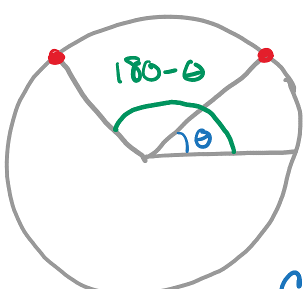
$$\sin 75° = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}}$$

Problem 3. Calculate the exact value of $\cos \frac{5\pi}{12}$. Is this one that we already know how to calculate? Or do we need to apply what we've learned today? It might be helpful to reference the diagram on the front page to guide you.

$$\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$$

$$\begin{aligned}
\Rightarrow \cos\left(\frac{5\pi}{12}\right) &= \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \\
&= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right). \\
&= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \\
&= \boxed{\frac{\sqrt{6} - \sqrt{2}}{4}}
\end{aligned}$$

Problem 4. Draw out a circle and conjecture a formula to simplify $\cos(180^\circ - \theta)$. Verify if your formula is correct or not by applying the difference formula for cosine.



Conjecture: $\cos(180^\circ - \theta) = -\cos(\theta)$.

Verify:

$$\begin{aligned}
\cos(180^\circ - \theta) &= \cos(180^\circ)\cos(\theta) + \sin(180^\circ)\sin(\theta) \\
&= -1 \cdot \cos\theta + 0 \cdot \sin\theta \\
&= -\cos\theta. \checkmark
\end{aligned}$$

Problem 5. Suppose that you know $\sin s = \frac{3}{5}$ and $\cos t = -\frac{12}{13}$, where both s and t are in quadrant II. Find $\cos(s+t)$.

Sum formula says:

$$\cos(s+t) = \overset{?}{\cos(s)}\,\overset{\text{Known}}{\cos(t)} - \overset{\text{Known}}{\sin(s)}\,\overset{?}{\sin(t)}.$$

$$\cos(t) = -\frac{12}{13}$$

in quad II so $\sin > 0$.

$$\Rightarrow \sin(t) = \pm\sqrt{1-\left(\tfrac{-12}{13}\right)^2} \qquad 1-\left(\tfrac{-12}{13}\right)^2 = 1-\frac{144}{169} = \frac{25}{169}$$

$$\Rightarrow \sin(t) = +\sqrt{25/169} = \frac{5}{13}$$

$$\sin(s) = \frac{3}{5}$$

in quad II so $\cos < 0$

$$\Rightarrow \cos(s) = \pm\sqrt{1-\left(\tfrac{3}{5}\right)^2} \qquad 1-\frac{9}{25} = \frac{16}{25}$$

$$= -\sqrt{16/25} = \frac{-4}{5}.$$

$$\cos(s+t) = \left(\frac{-4}{5}\right)\left(\frac{-12}{13}\right) - \left(\frac{3}{5}\right)\left(\frac{5}{13}\right)$$

$$= \frac{48}{65} - \frac{15}{65} = \boxed{\frac{33}{65}}$$

Problem 6. In an household United States electrical outlet an alternating current of is provided with a voltage of 115-volts, provided a 60Hz. An equation to model this voltage over time is given by

$$V(t) = 163 \sin \omega t$$

(a) It is a regulation that the frequency of the voltage provided from the outlet is 60 Hz. Many clocks that are plugged into an outlet depend on this by counting the periods of the alternating current to determine how much time as passed. What must the angular speed ω be to provide an alternating current of 60 Hz?

60 Hz $\Rightarrow$ Period $= \frac{1}{60}$ sec.

$\Rightarrow \frac{2\pi}{\omega} = \frac{1}{60} \Rightarrow \omega = 120\pi.$

(b) Determine a phase-shift ϕ such that the graph of $163\cos(\omega t - \phi) = 163 \sin \omega t$.

Use cofunction identity!

$\sin(x) = \cos\left(\frac{\pi}{2} - x\right).$ since cos is odd

$163\cos(\omega t - \phi) = 163\cos(\phi - \omega t)$

set $\phi = \frac{\pi}{2}$

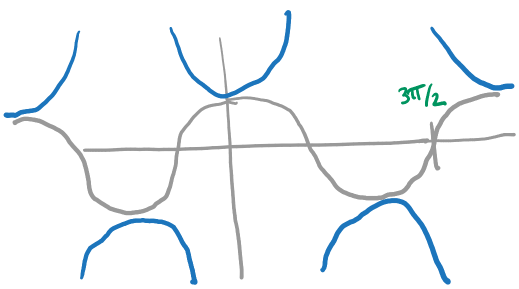
$163\cos\left(\frac{\pi}{2} - \omega t\right) = 163\sin(\omega t).$ ✓

Problem 7. Sketch the graph of $\sec\left(\frac{3\pi}{2} - x\right)$ and conjecture a formula to simply the expression. Prove that your conjecture is correct.

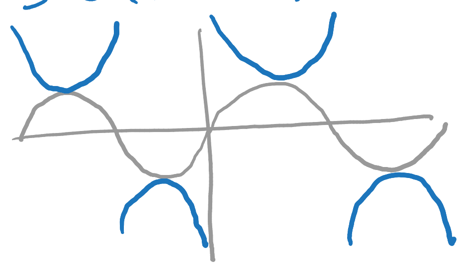
$$\sec\left(\frac{3\pi}{2} - x\right) = \underbrace{\sec\left[-\left(x - \frac{3\pi}{2}\right)\right]}$$

(1) $\sec x \to$ (2) $\sec\left(x - \frac{3\pi}{2}\right)$
$\to$ (3) $\sec\left[-\left(x - \frac{3\pi}{2}\right)\right]$

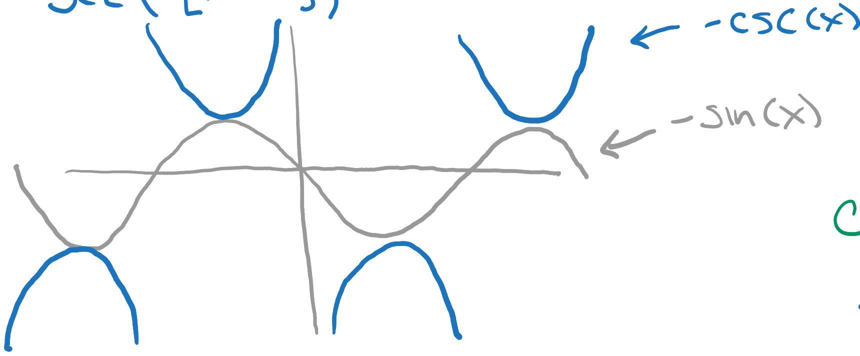
$\sec x$



$\sec\left(x - \frac{3\pi}{2}\right)$



$\sec\left(-\left[x - \frac{3\pi}{2}\right]\right)$



Conjecture:

$$\sec\left(-\left[x - \frac{3\pi}{2}\right]\right) = -\csc(x).$$

Verification:

$$\sec\left(\frac{3\pi}{2} - x\right) = \frac{1}{\cos\left(\frac{3\pi}{2} - x\right)}$$

$$= \frac{1}{\cos\left(\frac{3\pi}{2}\right)\cos(x) + \sin\left(\frac{3\pi}{2}\right)\sin(x)}$$

$$= \frac{1}{0 \cdot \cos(x) + (-1)\sin(x)} = -\frac{1}{\sin(x)}$$

$$\boxed{= -\csc(x)}$$