Section 6.1

Inverse Circular Functions

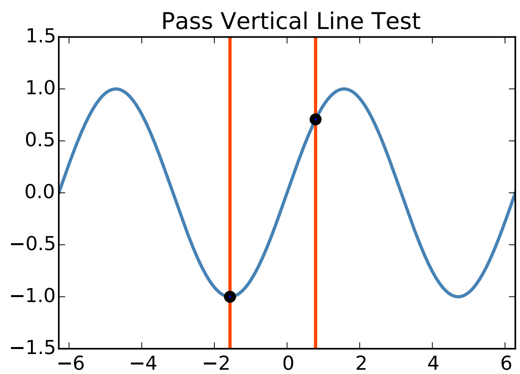
An inverse function computes the input required for a function to give a desired answer. For instance, we can use an inverse function to compute a θ (the input) required for sin to give $1/2$:

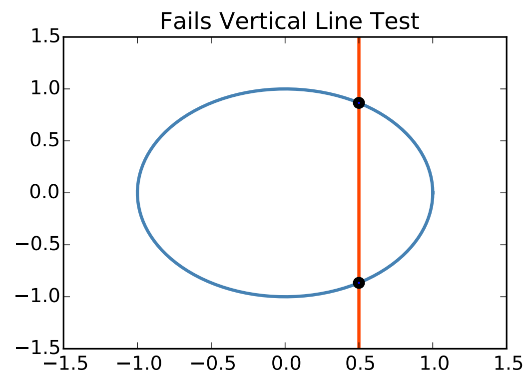
$$\sin\theta = \frac{1}{2} \qquad \Longrightarrow \qquad \theta = \sin^{-1}\frac{1}{2}$$

However, this trigonometric inverse function $\sin^{-1}$ only gives *one* of an infinite number of possible θ. If we plug in $\sin^{-1}\frac{1}{2}$ into a calculator, we will get $\theta = \frac{\pi}{6}$. However, θ could also be $\pi - \frac{\pi}{6}$ or $\frac{\pi}{6} + 2\pi$, and each of these would also give $\sin\theta = \frac{1}{2}$. The calculator needs to restrict the domain of sin to be able to properly provide inverse function values. We will review the concepts of inverse functions here.

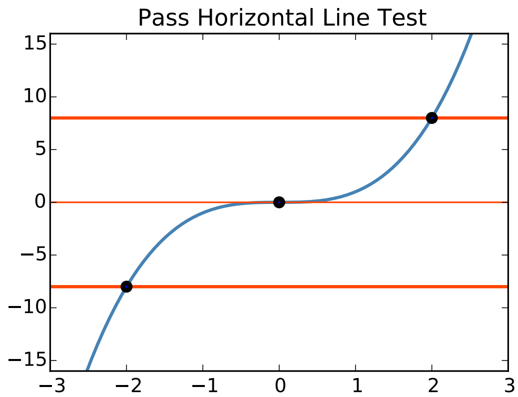
Inverse Functions

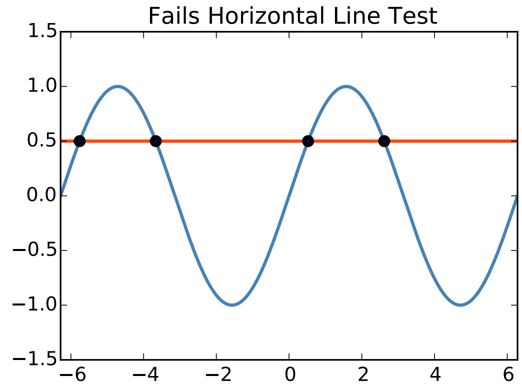
Recall that a function $y = f(x)$ must give *exactly one* value for each given x. If we plot that, that means there is only one point on the graph of $y = f(x)$ for any fixed x-value. The **Vertical Line Test** is a quick graphical way to verify this.





For a function to have an inverse, we must also insist that each value in the range is obtained *exactly once*. Such a function is called **one-to-one**. This ensures that we can have exactly one answer to the question: what x yields $f(x) = y$? If a function value is obtained more than once, then the answer to this question has more than one answer. Functions can only have one output, which is says there must be exactly one answer. A quick and graphical way to verify this is the **Horizontal Line Test**.





To construct and inverse function, we essentially swap the x and y values. This comes from the relation:

$$f(x) = y \qquad \Longrightarrow \qquad x = f^{-1}(y)$$

We notice that the *input* in $f^{-1}(y)$ was the *output* from $f(x)$. We are essentially swapping the x and y symbols, and this yields the definition:

$$\boldsymbol{f^{-1} = \{\, (y, x) \mid (x, y) \ \text{belongs to } f \,\}}$$

This notation may be a little foriegn, but it says: The graph of f^{-1} is the same graph of f where you just swap the x and y coordinates. This means that the graph of f^{-1} is the same as the graph of $f(x)$ but reflected about the $y = x$ line. But for f^{-1} to be a function, it **must** satisfy the *vertical line test*. Since f^{-1} is just swapping the x and y coordinates of f, this is exactly the same as saying f must satisfy the *horizontal line test*.

Summary

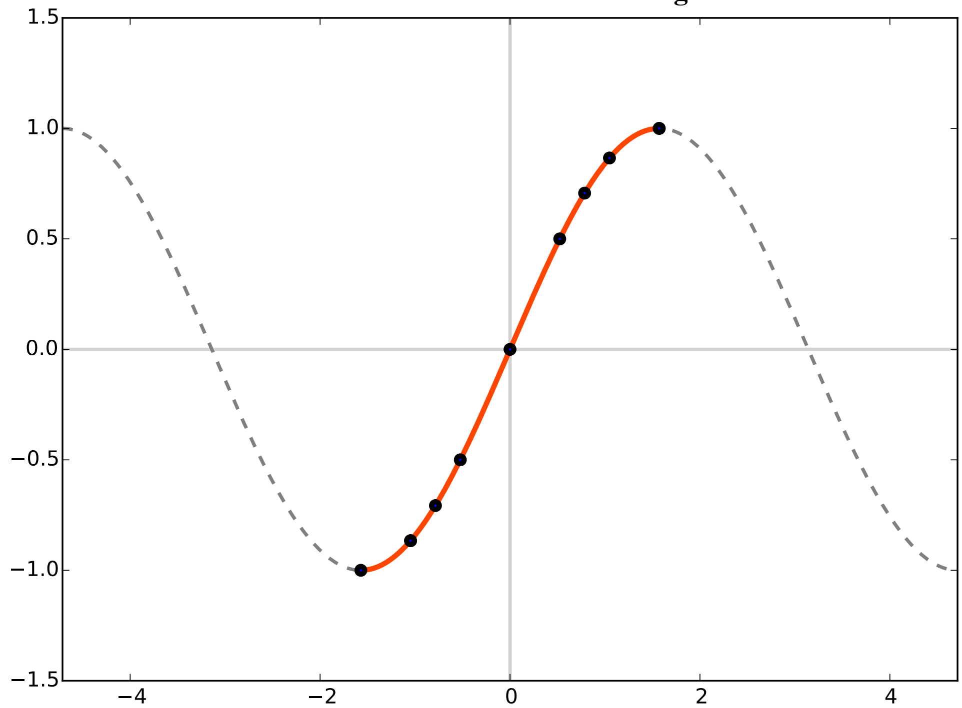
- For **one-to-one** function, each y-value coorsponds to *exactly* one x-value. (Ie, passes the **Horizontal Line Test**)
- If f is one-to-one, then f has an inverse f^{-1}
- The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f
- To solve for an inverse function, swap the x and y symbols (replacing $f(x)$ with y if need be), then solve for y.

Problem 1. Show that the inverse of $f(x) = x^3 - 1$ is $f^{-1}(x) = \sqrt[3]{x+1}$.

Inverse Sine

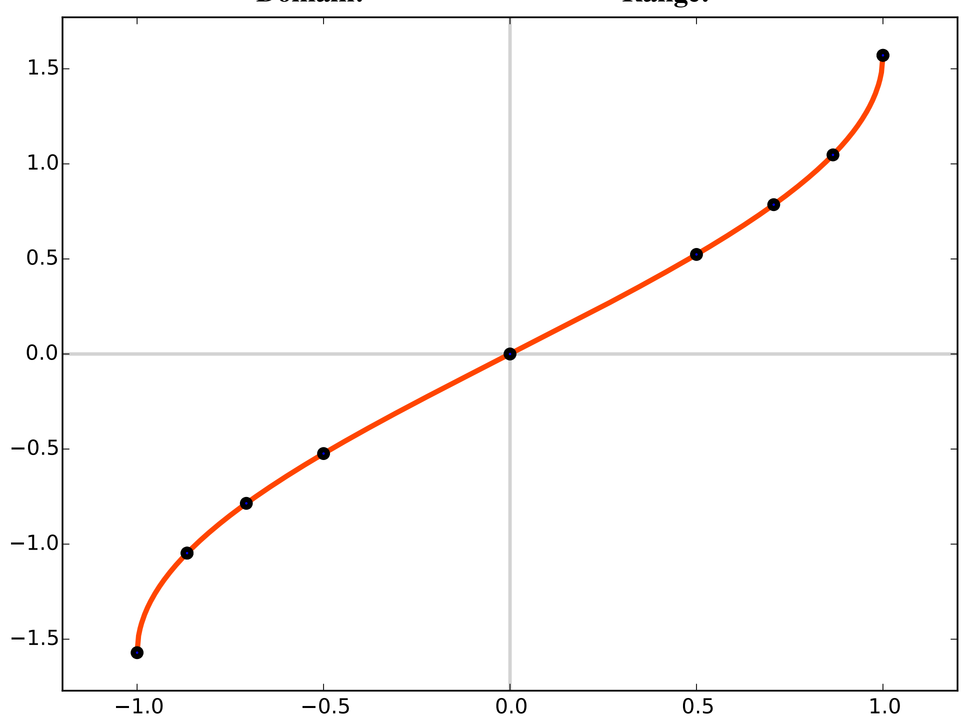
x	$y = \sin x$
$-\pi/2$	-1
$-\pi/3$	$-\sqrt{3}/2$
$-\pi/4$	$-\sqrt{2}/2$
$-\pi/6$	$-1/2$
0	0
$\pi/6$	$1/2$
$\pi/4$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$
$\pi/2$	1

Domain: **Range:**



x	$y = \sin^{-1} x$
-1	$-\pi/2$
$-\sqrt{3}/2$	$-\pi/3$
$-\sqrt{2}/2$	$-\pi/4$
$-1/2$	$-\pi/6$
0	0
$1/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$\sqrt{3}/2$	$\pi/3$
1	$\pi/2$

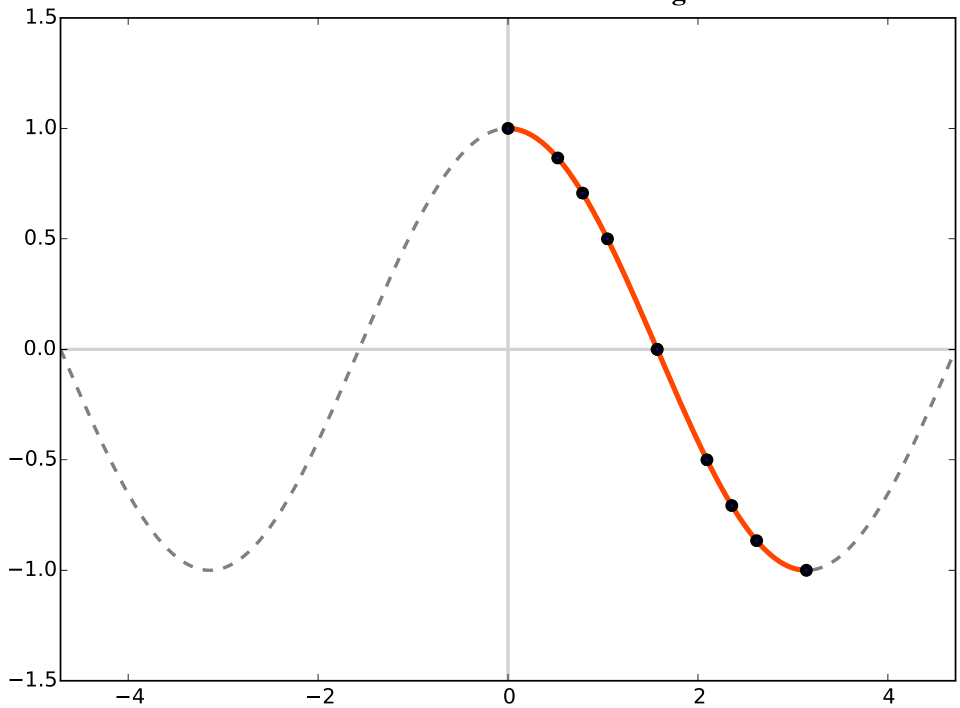
Domain: **Range:**



Inverse Cosine

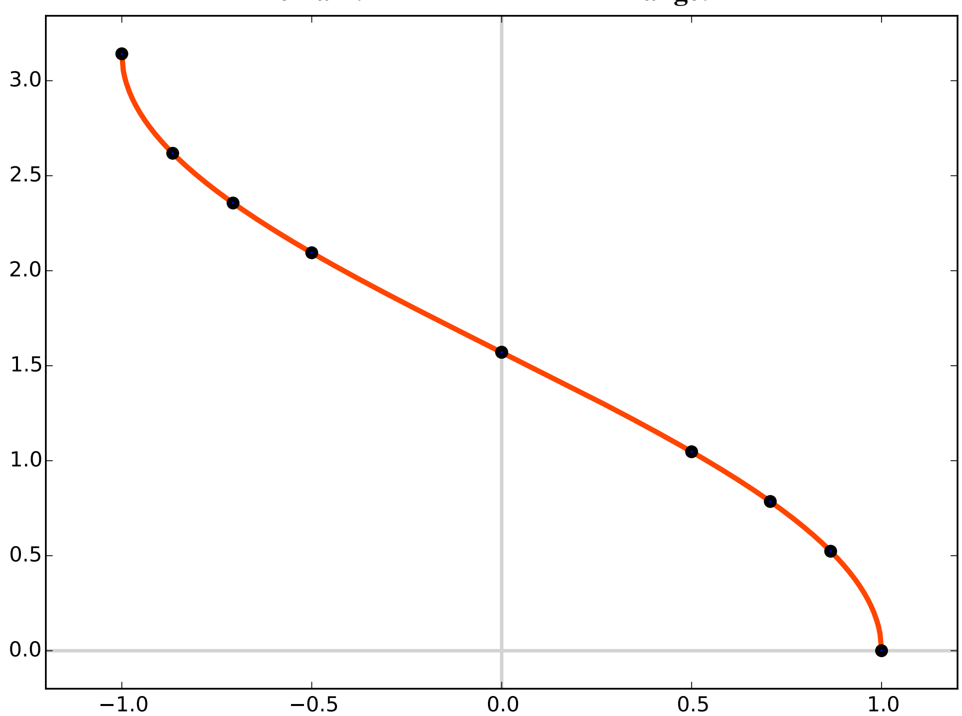
x	$y = \cos x$
0	1
$\pi/6$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$
$\pi/3$	$1/2$
$\pi/2$	0
$2\pi/3$	$-1/2$
$3\pi/4$	$-\sqrt{2}/2$
$5\pi/6$	$-\sqrt{3}/2$
π	-1

Domain: **Range:**



x	$y = \cos^{-1} x$
-1	π
$-\sqrt{3}/2$	$5\pi/6$
$-\sqrt{2}/2$	$3\pi/4$
$-1/2$	$2\pi/3$
0	$\pi/2$
$1/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$\sqrt{3}/2$	$\pi/6$
1	0

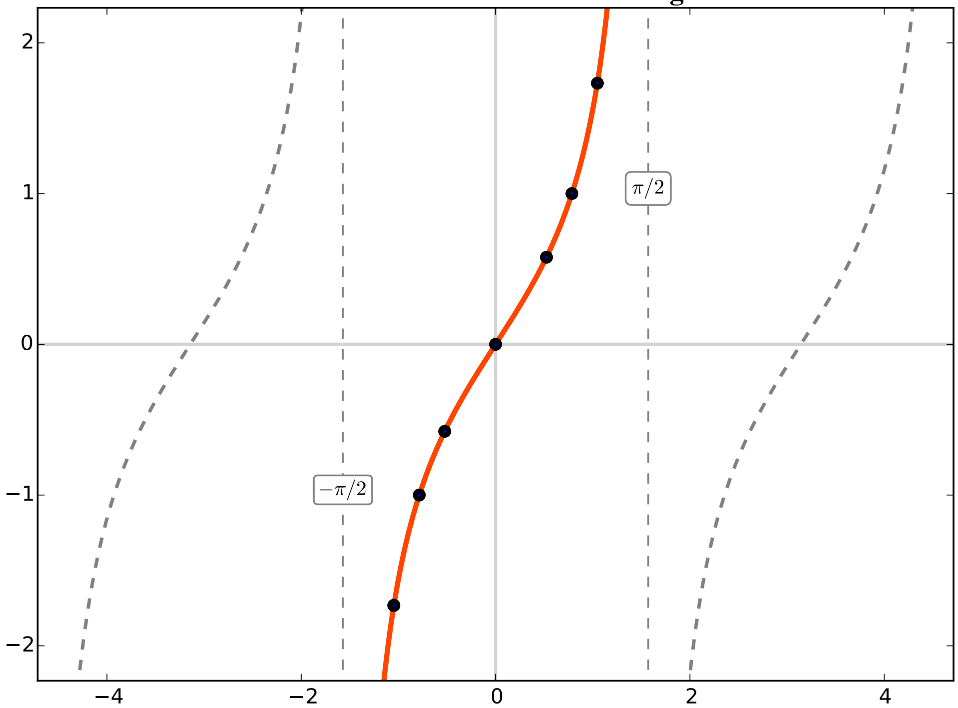
Domain: **Range:**



Inverse Sine

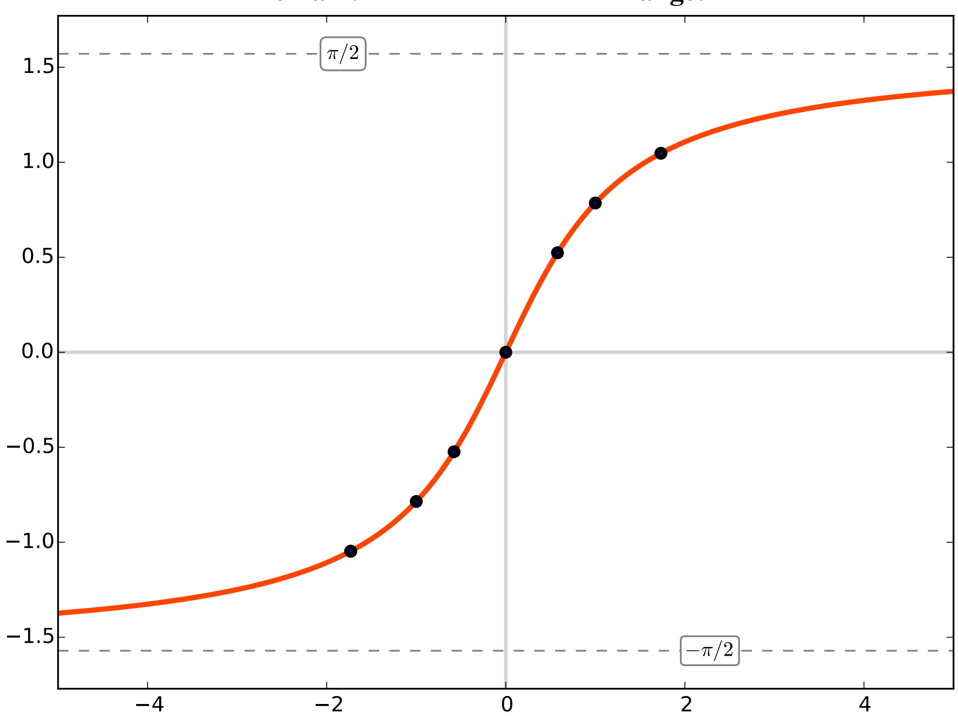
x	$y = \tan x$
$-\pi/2$	Und
$-\pi/3$	$-\sqrt{3}$
$-\pi/4$	-1
$-\pi/6$	$-\sqrt{3}/3$
0	0
$\pi/6$	$\sqrt{3}/3$
$\pi/4$	1
$\pi/3$	$\sqrt{3}$
$\pi/2$	Und

Domain: **Range:**



x	$y = \tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
0	0
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

Domain: **Range:**



Practicing With Known Values

Problem 2. Find the **exact** inverse trig values for the following.

(a) $\sin^{-1}\left(\frac{1}{2}\right)$

(b) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

(c) $\arccos\left(-\frac{1}{2}\right)$

(d) $\arccos(2)$

(e) $\tan^{-1}(1)$

(f) $\arcsin\left(-\frac{\sqrt{1}}{2}\right)$

(g) $\arctan\left(-\sqrt{3}\right)$

(h) $\sin^{-1}\left(\sqrt{3}\right)$

Problem 3. Give the degree measure of θ if it exists.

(a) $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

(b) $\theta = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

(c) $\theta = \cos^{-1}(1)$

(d) $\theta = \arcsin\left(-\frac{\sqrt{1}}{2}\right)$

Problem 4. For each given inverse trig function, write out what quadrant the resulting angle should yield. Then use a calculator to find an approximate value of each inverse trig function. Your answer should be in **radians**.

(a) $\arcsin 0.779$

(b) $\sin^{-1}(-0.345)$

(c) $\arctan(-3.141592)$

(d) $\cos^{-1}(-0.12345)$

(e) $\sin^{-1}(-0.10101)$

(f) $\arctan(2.7182)$

Problem 5. For each inverse trig function given in the previous problem, find an approximate value of each angle measured in **degrees**.

(a) $\theta = \arcsin 0.779$

(b) $\theta = \sin^{-1}(-0.345)$

(c) $\theta = \arctan(-3.141592)$

(d) $\theta = \cos^{-1}(-0.12345)$

(e) $\theta = \sin^{-1}(-0.10101)$

(f) $\theta = \arctan(2.7182)$

Problem 6. By sketching an appropriate triangle, evaluate each expression given below without using a calculator.

(a) $\cos\left(\sin^{-1}\frac{2}{3}\right)$

(b) $\sin\left(\arctan\left(-\frac{3}{4}\right)\right)$

Problem 7. It may be necessary to use trigonometric identites before proceeding. Evaluate each expression given below without using a calculator.

(a) $\sin\left(\arctan\frac{4}{3} - \arccos\frac{12}{13}\right)$

(b) $\cos\left(2\arccos\frac{3}{5}\right)$

We have found a procedure that allows us to calculate exact values when given an inverse trig function inside of a trig function. This means that we will likely be able to calculate a general formula.

Problem 8. Write each expression as an algebraic expression in u.

(a) $\sin\left(\cos^{-1} u\right)$

(b) $\cos\left(2 \arccos \dfrac{u}{u^2+1}\right)$