0.1 Overview

Let

$$\xi_1 = (-1, 1, 0) \text{ and } \xi_2 = (0, -1, 1),$$

and let X_n^θ be a discrete time Markov chain on $\mathbb{Z}^3_{\geq 0}$ with the following transition probabilities:

$$p_{\vec{x},\vec{x}+\xi_1} = \frac{\theta xy}{\theta xy + y} \tag{1}$$

$$p_{\vec{x},\vec{x}+\xi_2} = 1 - p_1(x, y, z), \tag{2}$$

Assume that $X_0^\theta = (100, 5, 0)$ and $\theta = 0.05$. Let $f(X^\theta) = \left(X_1^\theta\right)_{100}$; that is the first component of the process after 100 steps. We estimate $\frac{d}{d\theta}E\left[f(X^\theta)\right]$ using a number of different techniques.

0.2 Finite Difference Method

In this problem, we employ a centered finite difference method, using an estimator

$$\hat{\mu}_k^{\theta,h} = \frac{f(X_k^{\theta+h/2}) - f(X_k^{\theta-h/2})}{h}.$$

In an attempt to reduce the variance, we employ the common random variables technique; in which we will use the same sequence of uniform random variables when generating relizations $X_k^{\theta+h/2}$ and $X_k^{\theta-h/2}$.

For the finite difference method we employed the niave algorithm. That is, for a given value h, we sample the random variable:

$$Y^{\theta,h} := \frac{f(X^{\theta+h}) - f(X^\theta)}{h}.$$

Here we estimated the expected value $\hat{Y}^{\theta,h} = \frac{1}{n}\sum^n Y_k^{\theta,h}$, and the sample variance $\sqrt{\sigma}$. We *did not*, however, use

$$Y^{\theta,h} := h^2 \sum^{h^{-2}} \frac{f(X^{\theta+h}) - f(X^\theta)}{h}$$

for our estimator. During our trials, we did, and the sample variance remained roughly constant throughout different choices of h. We decided not to use these estimators for our table to illustrate the additional work required when estimating for small h. We took 100,000 samples.

h	$\hat{Y}^{\theta,h}$	σ^2	Confidence
0.01	-268.59	236102.86	3.01
0.005	-286.76	969545.56	6.10
0.001	-290.61	24351035.83	30.59
0.0005	-349.46	97367597.71	61.16

0.3 Likelihood Ratio Method

For likelihood ratio method, we took $n = 100,000$ samples.

$\hat{Y}^{\theta}$	σ^2	Confidence
-331.29	5640541.81	46.55

While emplying a control variate with the weight function obtained using the Likelihood Ratio method, we had a dramatic reduction in variance.

$\hat{Y}^{\theta}$	σ^2	Confidence
-302.56	190379.28	8.55