1. (littleoh:sindeg3)

   Is it true that $\sin(x) - x + \frac{x^3}{3!} = o(x^4)$?

   Solution: Yes. We know that

   $$\begin{aligned}\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= x - \frac{x^3}{3!} + \underbrace{\frac{x^5}{5!} + o(x^5)}_{o(x^4)}\end{aligned}$$

   A second way of thinking about this problem is to notice that asking whether $\sin(x) - P(x) = o(x^4)$ where $P(x)$ polynomial of degree at most four is the same as asking whether $P(x)$ is the degree four Taylor polynomial of $\sin(x)$. We can compute that for $f(x) = \sin(x)$ we have

   $$\begin{aligned} f(x) &= \sin(x) & f(0) &= 0 \\ f'(x) &= \cos(x) & f'(0) &= 1 \\ f^{(2)}(x) &= -\sin(x) & f^{(2)}(0) &= 0 \\ f^{(3)}(x) &= -\cos(x) & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin(x) & f^{(4)}(0) &= 0 \end{aligned}$$

   Therefore

   $$\begin{aligned} T_4^0 \sin(x) &= 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 \\ &= x - \frac{x^3}{3!} \end{aligned}$$

   so that $\sin(x) - \left(x - \frac{x^3}{3!}\right) = o(x^4)$

2. (littleoh:cosdeg2)

   Is it true that $\cos(x) - 1 + \frac{x^2}{2} = o(x^3)$?

   Solution: Yes. We know that

   $$\begin{aligned}\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\ &= 1 - \frac{x^2}{2} + \underbrace{\frac{x^4}{4!} + o(x^4)}_{o(x^3)}\end{aligned}$$

A second way of thinking about this problem is to notice that asking whether $\cos(x) - P(x) = o(x^3)$ where $P(x)$ polynomial of degree at most three is the same as asking whether $P(x)$ is the degree three Taylor polynomial of $\cos(x)$. We can compute that for $f(x) = \cos(x)$ we have

$$\begin{aligned} f(x) &= \cos(x) & f(0) &= 1 \\ f'(x) &= -\sin(x) & f'(0) &= 0 \\ f^{(2)}(x) &= -\cos(x) & f^{(2)}(0) &= -1 \\ f^{(3)}(x) &= \sin(x) & f^{(2)}(0) &= 0 \end{aligned}$$

so that

$$\begin{aligned} T_3^0 \cos(x) &= 1 + \frac{0}{1!}x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 \\ &= 1 - \frac{x^2}{2} \end{aligned}$$

and therefore $\cos(x) - (1 - \frac{x^2}{2}) = o(x^3)$.

3. (littleoh:exp2deg4)
Is it true that $e^{x^2} = 1 + x^2 + o(x^3)$?

Solution: Yes.

$$\begin{aligned} e^{x^2} &= \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} \\ &= 1 + x^2 + \underbrace{\frac{x^4}{2} + o(x^4)}_{o(x^3)} \end{aligned}$$

4. (littleoh:expsqrt)
Show that

$$e^x - \sqrt{1 + 2x} = o(x)$$

Solution:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad (= 1 + x + o(x))$$

$$\sqrt{1+2x} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (2x)^n \qquad \left(= 1 + \frac{1}{2}(2x) + o(x)\right)$$

From this we can see that

$$\begin{aligned} e^x - \sqrt{1+2x} &= (1 + x + o(x)) - (1 + x + o(x)) \\ &= o(x) \end{aligned}$$