1. (taylor:sinx2)
   Compute the second order Taylor polynomial of $\sin(x^2)$ around 0 and use this to approximate $\sin(\frac{1}{4})$.

   **Solution:** We need to compute two derivatives of $f(x) = \sin(x^2)$.

$$\begin{aligned} f(x) &= \sin(x^2) \\ f'(x) &= \cos(x^2)(2x) \\ f^{(2)}(x) &= 2\cos(x^2) - 4x^2\sin(x^2) \end{aligned}$$

   Now we evaluate the function and its derivatives at zero.

$$\begin{aligned} f(0) &= \sin(0) = 0 \\ f'(0) &= \cos(0)(0) = 0 \\ f^{(2)}(0) &= 2\cos(0) - 4(0)\sin(0) = 2 \end{aligned}$$

   Therefore the degree two Taylor polynomial of $\sin(x^2)$ is $\frac{2}{2!}x^2 = x^2$. Since $\sin\left(\frac{1}{4}\right) = \sin\left(\left(\frac{1}{2}\right)^2\right)$, our approximation is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

2. (taylor:etan)
   Compute the degree two Taylor polynomial of the function $f(x) = e^{\tan(x)}$ around 0. Use this to estimate $e^{\tan(.1)}$.

   **Solution:**

$$\begin{aligned} f(x) &= e^{\tan(x)} \\ f'(x) &= \sec^2(x)e^{\tan(x)} \\ f^{(2)}(x) &= 2\sec^2(x)\tan(x)e^{\tan(x)} + \sec^4(x)e^{\tan(x)} \end{aligned}$$

   We evaluate these at zero.

$$\begin{aligned} f(0) &= e^{\tan(0)} = 1 \\ f'(0) &= \sec^2(0)e^{\tan(0)} = 1 \\ f^{(2)}(0) &= 2\sec^2(0)\tan(0)e^{\tan(0)} + \sec^4(0)e^{\tan(0)} = 1 \end{aligned}$$

   This gives the Taylor polynomial $1 + \frac{1}{1!}x + \frac{1}{2!}x^2 = 1 + x + \frac{1}{2}x^2$. We can then approximate $e^{\tan(.1)} \approx 1 + .1 + \frac{1}{2}(.1)^2 = 1.105$.

3. (taylor:sinexp)
Compute the second order Taylor polynomial of $\sin(e^x - 1)$ around 0 and use this to approximate $\sin(e^{\frac{1}{2}} - 1)$.

**Solution:** We first need to compute two derivatives of $\sin(e^x - 1)$

$$
\begin{aligned}
f(x) &= \sin(e^x - 1) \\
f'(x) &= \cos(e^x - 1)e^x \\
f^{(2)}(x) &= \cos(e^x - 1)e^x - \sin(e^x - 1)e^{2x}
\end{aligned}
$$

We evaluate these at zero.

$$
\begin{aligned}
f(0) &= \sin(0) = 0 \\
f'(0) &= \cos(0)e^0 = 1 \\
f^{(2)}(0) &= \cos(0)e^0 - \sin(0)e^0 = 1
\end{aligned}
$$

Combining these, we find that the second order Taylor polynomial of $\sin(e^x - 1)$ is $x + \frac{1}{2}x^2$. This gives the approximation $\sin(e^{\frac{1}{2}} - 1) \approx \frac{1}{2} + \left(\frac{1}{2}\right)^3 = \frac{5}{8}$.

4. (taylor:polynomial)
Find the second and fourth order Taylor expansions around 1 for the function $f(x) = x^3 + 5x + 1$.

**Solution:** We can first observe that since this function is a third order polynomial, the fourth order Taylor expansion of $f(x)$ is $f(x)$. To find the second order Taylor expansion, we need to differentiate:

$$
\begin{aligned}
f(x) &= x^3 + 5x + 1 \\
f'(x) &= 3x^2 + 5 \\
f^{(2)}(x) &= 6x
\end{aligned}
$$

Now we evaluate at zero.

$$
\begin{aligned}
f(1) &= 7 \\
f'(1) &= 8 \\
f^{(2)}(1) &= 6
\end{aligned}
$$

So the second order Taylor polynomial around 1 is $7+8(x-1)+\frac{6}{2}(x-1)^2 = 7+8(x-1)+3(x-1)^2$.

5. (taylor:intexp2)
   Find the second order Taylor polynomial around 0 for $f(x) = \int_0^x e^{-t^2}\,dt$ and use this to estimate $f(.1)$.

   **Solution:**

$$\begin{aligned} f(x) &= \int_0^x e^{-t^2}\,dt \\ f'(x) &= e^{-x^2} \\ f^{(2)}(x) &= -2xe^{-x^2} \end{aligned}$$

   so that

$$\begin{aligned} f(0) &= \int_0^0 e^{-t^2}\,dt = 0 \\ f'(0) &= e^{-0^2} = 1 \\ f^{(2)}(0) &= -2(0)e^{-0^2} = 0 \end{aligned}$$

   so that the degree 2 Taylor polynomial for $f(x)$ is $x$. Our estimate for $f(.1)$ is therefore .1.

6. (taylor:intexpsin)
   Find the first order Taylor polynomial for the function $f(x) = \int_0^{\sin(x)} e^{-t^3}\,dt$ and use this to find an approximation for $f(\frac{1}{2})$.

   **Solution:**

$$\begin{aligned} f(x) &= \int_0^{\sin(x)} e^{-t^3}\,dt \\ f'(x) &= e^{-\sin^3(x)}\cos(x) \end{aligned}$$

   so that

$$\begin{aligned} f(0) &= \int_0^{\sin(0)} e^{-t^3}\,dt = \int_0^0 e^{-t^3}\,dt = 0 \\ f'(0) &= e^{-\sin^3(0)}\cos(0) = 1 \end{aligned}$$

so the first order Taylor polynomial for $f(x)$ is given by $x$ and our approximation for $f(\frac{1}{2})$ is $\frac{1}{2}$.

7. (taylor:intcomp)
   Find the second order Taylor polynomial of $\cos(x)$ around 0 then integrate this polynomial. Additionally, find the third order Taylor polynomial of $\sin(x)$ around 0 . Recall that $\int \cos(x)dx = \sin(x) + C$ and compare your answer to the previously computed Taylor polynomial for the integral of $\cos(x)$.

   **Solution:** We begin by calling $f(x) = \cos(x)$ and $g(x) = \sin(x)$. We need to compute two derivatives of $f$ and three derivatives of $g$.

$$\begin{aligned} f(x) &= \cos(x) \\ f'(x) &= -\sin(x) \\ f^{(2)}(x) &= -\cos(x) \\ g(x) &= \sin(x) \\ g'(x) &= \cos(x) \\ g^{(2)}(x) &= -\sin(x) \\ g^{(3)}(x) &= -\cos(x) \end{aligned}$$

   so that

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f^{(2)}(0) &= -1 \\ g(0) &= 0 \\ g'(0) &= 1 \\ g^{(2)}(0) &= 0 \\ g^{(3)}(0) &= -1 \end{aligned}$$

   The degree two Taylor polynomial of $\cos(x)$ around 0 is then $1 - x^2$ and the integral of this is $C + x - \frac{x^3}{3}$. Similarly, we find that the degree three Taylor polynomial of $\sin(x)$ is $x - \frac{x^3}{3}$. For $C = 0$, these agree– for nice functions, we can exchange the operation of taking Taylor polynomials and integration.

8. (taylor:arctanseries)
Find the Taylor series around 0 for $\arctan(x)$, $T^0_\infty \arctan(x)$.

**Solution:**

$$\begin{aligned}
\arctan(x) &= \int \underbrace{\frac{1}{1+x^2}}_{\frac{1}{1-(-x^2)}} dx \\
&= \int \sum_{n=0}^{\infty} (-x^2)^n dx \\
&= \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx \\
&= \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx \\
&= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}
\end{aligned}$$

Now we need to solve for $C$, which we can do by observing that $\arctan(0) = 0$, so $C = 0$ and $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$.

9. (taylor:cosh2series)
Find the Taylor series around zero for $\cosh(2x) = \frac{1}{2}\left(e^{2x} + e^{-2x}\right)$.

**Solution:**

$$\begin{aligned}
\frac{1}{2}\left(e^{2x} + e^{-2x}\right) &= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} + \frac{(-1)^n 2^n x^n}{n!}\right) \\
&= \frac{1}{2}\sum_{n=0}^{\infty} \frac{1+(-1)^n}{n!} 2^n x^n
\end{aligned}$$

We can observe that $1 + (-1)^n = 0$ if $n$ is odd and 2 if $n$ is even. We therefore only need to sum over the even positive integers $n = 2k$

$$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{2}{(2k)!} 2^{2k} x^{2k}$$
$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!} 2^{2k} x^{2k}$$

10. (taylor:sinhx2series)
Find the Taylor series around zero for $\sinh(x^2) = \frac{1}{2}\left(e^{x^2} - e^{-x^2}\right)$.

**Solution:**

$$\frac{1}{2}\left(e^{x^2} - e^{-x^2}\right) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{1}{n!}(x^2)^n - \sum_{n=0}^{\infty} \frac{1}{n!}(-x^2)^n\right)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{n!}(x^2)^n - \frac{1}{n!}(-1)^n(x^2)^n\right)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty} \frac{1-(-1)^n}{n!}(x^2)^n$$

Since $1-(-1)^n = 0$ if $n$ is even and 2 if $n$ is odd, we only need to sum over the odd positive integers $n = 2k+1$.

$$= \frac{1}{2}\sum_{k=0}^{\infty} \frac{2}{(2k+1)!}(x^2)^{2k+1}$$
$$= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2(2k+1)}$$

11. (taylor:exp/1-x)
Find the degree two Taylor polynomial around 0 of $\frac{e^x}{1-x}$ without computing any derivatives.

**Solution:** Recall that $e^x = 1 + x + \frac{x^2}{2} + o(x^2)$ and $\frac{1}{1-x} = 1 + x + x^2 +$

$o(x^2)$. Then

$$\begin{aligned}
\frac{e^x}{1-x} &= \left(1+x+\frac{x^2}{2}+o(x^2)\right)\left(1+x+x^2+o(x^2)\right)\\
&= \underbrace{1\left(1+x+x^2+o(x^2)\right)}_{\text{term 1}} + \underbrace{x\left(1+x+x^2+o(x^2)\right)}_{\text{term 2}}\\
&+ \underbrace{\frac{x^2}{2}\left(1+x+x^2+o(x^2)\right)}_{\text{term 3}} + \underbrace{o(x^2)\left(1+x+x^2+o(x^2)\right)}_{\text{term 4}}\\
&= \underbrace{1+x+x^2+o(x^2)}_{\text{term 1}} + \underbrace{x+x^2+o(x^2)}_{\text{term 2}} + \underbrace{\frac{x^2}{2}+o(x^2)}_{\text{term 3}} + \underbrace{o(x^2)}_{\text{term 4}}\\
&= 1+2x+\frac{5}{2}x^2+o(x^2)
\end{aligned}$$

Therefore $T_2^0 \frac{e^x}{1-x} = 1+2x+\frac{5}{2}x^2$.

12. (taylor:ex3)
Find the degree nine Taylor polynomial around zero for $e^{x^3}$ without computing any derivatives.

**Solution:** We know the full Taylor series for $e^x$ is given by

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

and therefore we know that

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{1}{n!}(x^3)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{3n}.$$

We can recover the degree nine Taylor polynomial by observing that

$$\sum_{n=0}^{\infty} \frac{1}{n!}(x^3)^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^{3n} = 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6} + o(x^9)$$

from which it follows that $T_9^0 e^{x^3} = 1 + x^3 + \frac{x^6}{2} + \frac{x^9}{6}$.

13. (taylor:calcplusoh)
Compute the degree seven Taylor polynomial around zero for $\frac{4x^3}{(1-x^4)^2}$.
Hint: You should not differentiate this function.

**Solution:**

$$\begin{aligned}
\frac{4x^3}{(1-x^4)^2} &= \frac{d}{dx}\frac{1}{1-x^4} \\
&= \frac{d}{dx}\left(1+\sum_{n=1}^{\infty}(x^4)^n\right) \\
&= 0+\sum_{n=1}^{\infty}\frac{d}{dx}x^{4n} \\
&= \sum_{n=1}^{\infty}(4n)x^{4n-1} \\
&= 4x^3+8x^7+o(x^7)
\end{aligned}$$

Therefore $T_7^0\frac{4x^3}{(1-x^4)2} = 4x^3+8x^7$.

14. (taylor:14expminus)
Find $T_{14}^0 e^{x^6}-\frac{1}{1-x^5}$.
**Solution:**

$$\begin{aligned}
e^{x^6} &= 1+x^6+\frac{x^{12}}{2}+\frac{x^{18}}{3!}+o(x^{18}) \\
&= 1+x^6+\frac{1}{2}x^{12}+o(x^{14}) \\
\frac{1}{1-x^5} &= 1+x^5+x^{10}+x^{15}+o(x^{15}) \\
&= 1+x^5+x^{10}+o(x^{14})
\end{aligned}$$

so that

$$\begin{aligned}
e^{x^6}-\frac{1}{1-x^5} &= \left(1+x^6+\frac{1}{2}x^{12}+o(x^{14})\right)-\left(1+x^5+x^{10}+o(x^{14})\right) \\
&= -x^5+x^6-x^{10}+\frac{1}{2}x^{12}+o(x^{14})
\end{aligned}$$

so that $T_{14}^0\left(e^{x^6}-\frac{1}{1-x^5}\right) = -x^5+x^6-x^{10}+\frac{1}{2}x^{12}$.

15. (taylor:expplusseries)
Find

$$T_{\infty}^0 x\left(e^x-\frac{1}{1-x}\right)$$

**Solution:**

$$\begin{aligned}
T^0_\infty x\left(e^x - \frac{1}{1-x}\right) &= T^0_\infty x\left(T^0_\infty e^x - T^0_\infty \frac{1}{1-x}\right) \\
&= x\left(\sum_{n=0}^{\infty} \frac{1}{n!}x^n - \sum_{n=0}^{\infty} x^n\right) \\
&= x\left(\sum_{n=0}^{\infty} (\frac{1}{n!} - 1)x^n\right) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{n!} - 1\right) x^{n+1}
\end{aligned}$$

noticing that the first two terms in this sum are zero, we can rewrite this as $\sum_{n=2}^{\infty} \left(\frac{1}{n!} - 1\right) x^{n+1}$.

16. (taylor:seriesrational)
Find the Taylor series around 0 ($T^0_\infty$) of the function $f(x) = \frac{10x^4}{(1-x^5)^2}$

**Solution:**

$$\begin{aligned}
T_\infty \frac{10x^4}{(1-x^5)^2} &= T_\infty 2 \frac{d}{dx} \frac{1}{1-x^5} \\
&= 2\frac{d}{dx} T_\infty \frac{1}{1-x^5} \\
&= 2\frac{d}{dx} \sum_{n=0}^{\infty} (x^5)^n \\
&= 2\frac{d}{dx} \left(1 + \sum_{n=1}^{\infty} x^{5n}\right) \\
&= 2\sum_{n=1}^{\infty} \frac{d}{dx} x^{5n} \\
&= 2\sum_{n=1}^{\infty} (5n) x^{5n-1}
\end{aligned}$$