1. (euler:nonlinear)
   Suppose that p is a function of t, satisfying

$$\begin{aligned} tp' &= a(p^2 + t^2) \\ p(1) &= 2 \end{aligned}$$

   where a is some small unknown constant. Using Euler's method with step size 2, approximate $p(5)$. Your answer will depend on a.

   Solution:

   We know $p(1) = 2$. Plugging into the equation,

$$1 * p'(1) = a(2^2 + 1^2)$$

   so $p'(1) = 5a$. Thus

$$p(3) \approx 2 * p'(1) + p(1) = 2 * 5a + 2 = 10a + 2$$

   We know $p(3) \approx 10a + 2$. Plugging into the equation,

$$3 * p'(3) \approx a((10a + 2)^2 + 9$$

   So $p'(3) \approx \frac{a}{3}(10a + 2)^2 + 3$. Thus

$$p(5) \approx 2 * p'(3) + p(3) = 2 * \left(\frac{a}{3}(10a + 2)^2 + 3\right) + 10a + 2$$

2. (euler:sin(0.3))
   Use Euler's method with step size 0.1 to approximate sin(0.3)

   Solution:

   To apply Euler's method we need a differential equation satisfied by sin, and an initial condition. $y = \sin$ satisfies

$$y'' = -y$$

   As for the initial condition, we don't know the value of $\sin(x)$ for most numbers x so our choice of useful initial conditions is limited - for example, the initial condition $x = 1$, $y = \sin(1)$ is useless to us since we don't know $\sin(1)$. But we do know $y(0) = 0$, $y'(0) = 1$ so let's use that, starting at $x = 0$.

   We have $y(0) = 0$, $y'(0) = 1$. Plugging into $y'' = -y$, we find that $y''(0) = -0 = 0$. Since $y'(0) = 1$ we have

$$y(0.1) \approx 0.1 * y'(0) + y(0) \approx 0.1 * 1 + 0 = 0.1$$

and since $y''(0) \approx 0$ we have

$$y'(0.1) \approx 0.1 * y''(0) + y'(0) \approx 0.1 * 0 + 1 = 1$$

We have $y(0.1) \approx 0.1$, $y'(0.1) \approx 1$. Plugging into $y'' = -y$, we find that $y''(0) \approx -0.1$. Since $y'(0.1) \approx 1$ we have

$$y(0.2) \approx 0.1 * y'(0.1) + y(0.1) \approx 0.1 * 1 + 0.1 = 0.2$$

and since $y''(0) \approx -0.1$ we have

$$y'(0.2) \approx 0.1 * y''(0.1) + y'(0.1) \approx 0.1 * -0.1 + 1 = 0.9$$

We have $y(0.2) \approx 0.2$, $y'(0.2) \approx 0.9$. Sicne we only need to do one more step I won't bother finding $y''(0.2)$. Instead,

$$y(0.3) \approx 0.1 * y'(0.2) + y(0.2) \approx 0.1 * 0.9 + 0.2 = 0.29$$

and our final answer is

$$\sin(0.3) = y(0.3) \approx 0.29$$

One drawback of this approach is that there is no way to estimate your error, but in fact this answer is correct to 2 decimal places.

3. (eulersMethod:Gaussian)
Solve the following initial value problem exactly, then use Euler's method with step size $\Delta x = .1$ to estimate $y(.3)$.

$$\begin{aligned} \frac{dy}{dx} &= -2xy \\ y(0) &= 1 \end{aligned}$$

Solution: This differential equation is separable and the solution is $y(x) = e^{-x^2}$.

To use Euler's method with step size .1, we will iteratively compute estimates to $y(.1), y(.2)$ and $y(.3)$.

First, we need an estimate for $y(.1) \approx y(0) + \frac{dy}{dx}(0)\Delta x$, so we need to compute $\frac{dy}{dx}(0)$.

$$\frac{dy}{dx}(0) = -2(0)y(0) = 0$$

so we estimate that

$$\begin{aligned} y(.1) &\approx y(0) + \frac{dy}{dx}(0)(.1) \\ &= 1 + (0)(.5) = 1 \end{aligned}$$

Now we repeat the procedure above to find an approximation for $y(.2)$

$$y(.2) \approx y(.1) + \frac{dy}{dx}(.1)\Delta x$$

To compute $\frac{dy}{dx}(.1)$, we recall that we have the differential equation $\frac{dy}{dx} = -2xy$, so $\frac{dy}{dx}(.1) = -2(.1)y(.5)$. We have estimated that $y(.1) = 1$, so we use that in our computation, and get $\frac{dy}{dx}(.5) \approx -2(.1)(1) = -.2$. This gives us the approximation

$$\begin{aligned} y(.2) &\approx y(.1) + \frac{dy}{dx}(.1)\Delta x \\ &\approx 1 + (-.2)(.1) = .98 \end{aligned}$$

Finally, we compute an approximation to $y(.3) \approx y(.2) + \frac{dy}{dx}(.2)\Delta x$. We first compute

$$\begin{aligned} \frac{dy}{dx}(.2) &= -2(.2)y(.2) \\ &\approx -2(.2)(.98) \\ &= -.392 \end{aligned}$$

Then our approximation to $y(.3)$ is

$$\begin{aligned} y(.3) &\approx y(.2) + \frac{dy}{dx}(.2)\Delta x \\ &\approx .98 + (-.392)(.1) \\ &= .9408 \end{aligned}$$

As a comment, the true value of $e^{-(.3)^2}$ is about .914.

4. (eulersMethod:linear1)
Solve the following initial value problem exactly, then use Euler's method with a step size of $\Delta x = .1$ to approximate $y(1.3)$.

$$\begin{aligned} \frac{dy}{dx} &= 1 + y \\ y(1) &= 0 \end{aligned}$$

Solution: This equation is first order linear, so to solve it exactly, we rewrite it as

$$\frac{dy}{dx} - y = 1$$

The integrating factor is $m(x) = e^{-x}$, which converts this equation to

$$\begin{aligned}
\underbrace{e^{-x}\frac{dy}{dx} - e^{-x}y}_{\frac{d}{dx}(e^{-x}y)} &= e^{-x} \\
e^{-x}y &= -e^{-x} + C \\
y &= -1 + Ce^{x}
\end{aligned}$$

Susbtituting in the initial condition gives the solution $y(x) = -1 + e^{x-1}$.

We can now use Euler's method to approximate $y(.3)$. We first compute

$$y(1.1) \approx y(1) + \frac{dy}{dx}(1)\Delta x$$

From the differential equation $\frac{dy}{dx} = 1+y$ and initial condition $y(1) = 0$ we find that $\frac{dy}{dx}(1) = 1 + y(1) = 1$. Therefore, we approximate

$$\begin{aligned}
y(1.1) &\approx y(1) + \frac{dy}{dx}(1)\Delta x \\
&= 0 + (1)(.1) \\
&= .1
\end{aligned}$$

Continuing as in the previous step,

$$y(1.2) \approx y(1.1) + \frac{dy}{dx}(1.1)\Delta x$$

where $\frac{dy}{dx}(1.1) = 1 + y(1.1) \approx 1 + .1 = 1.1$, so that

$$\begin{aligned}
y(1.2) &\approx y(1.1) + \frac{dy}{dx}(1.1)\Delta x \\
&\approx .1 + (1.1)(.1) \\
&= .21
\end{aligned}$$

Finally, we approximate

$$y(1.3) \approx y(1.2) + \frac{dy}{dx}(1.2)\Delta x$$

where $\frac{dy}{dx}(1.2) = 1 + y(1.2) \approx 1 + .21 = 1.21$ so that

$$\begin{aligned} y(1.3) &\approx y(1.2) + \frac{dy}{dx}(1.2)\Delta x \\ &\approx .21 + (1.21)(.1) \\ &= .331 \end{aligned}$$

5. (eulersMethod:sep1)
Find an exact solution to the following initial value problem, then use Euler's method with step size $\Delta x = .1$ to estimate $y(.2)$.

$$\frac{dy}{dx} = -xy + 2x$$

Solution:

6. (eulersMethod:q1)
Find an exact solution to the following initial value problem, then use Euler's method with step size $\Delta x = .1$ to estimate $y(.2)$

$$\begin{aligned} \frac{dy}{dx} &= 2xy + x \\ y(0) &= 0 \end{aligned}$$

Solution: We begin by putting the differential equation into standard form

$$\frac{dy}{dx} - 2xy = x$$

The integrating factor for this problem is $m(x) = e^{\int -2x dx} = e^{-x^2}$.

Multiplication turns this into

$$\underbrace{e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y}_{\frac{d}{dx}e^{-x^2}y} = xe^{-x^2}$$

$$\begin{aligned} e^{-x^2}y &= \int xe^{-x^2}dx \\ &= \frac{-1}{2}e^{-x^2} + C \\ y(x) &= -\frac{1}{2} + Ce^{x^2} \end{aligned}$$

Substituting in the initial condition $y(0) = 0$ gives that $y(x) = \frac{1}{2}e^{x^2} - \frac{1}{2}$.

To approximate $y(.2)$ we first need an approximation for $y(.1)$.

$$y(.1) \approx y(0) + \frac{dy}{dx}(0)\Delta x$$

where $\frac{dy}{dx}(0) = 2(0)y(0) + 0 = 0$. So we have $\frac{dy}{dx}(0) \approx y(0) + 0(.1) = 0$. We now have

$$y(.2) \approx y(.1) + \frac{dy}{dx}(.1)\Delta x$$

where $\frac{dy}{dx}(.1) = 2(.1)y(.1) + .1 = 2(.1)(0) + .1 = .1$. Then, we have

$$\begin{aligned} y(.2) &\approx y(.1) + \frac{dy}{dx}(.1)\Delta x \\ &\approx 0 + .1(.1) \\ &= .01 \end{aligned}$$

7. (eulersMethod:q2)
Find an explicit solution to the following initial value problem, then use Euler's method with step size $\Delta x = .1$ to estimate $y(.2)$.

$$\begin{aligned} \frac{dy}{dx} &= 2xy + 2x \\ y(0) &= 0 \end{aligned}$$

Solution: We begin by observing that this equation is separable

$$\begin{aligned}
\frac{dy}{dx} &= 2xy + 2x \\
&= 2x(1+y) \\
\int \frac{dy}{1+y} &= \int 2x dx \\
\ln|1+y| &= x^2 + C \\
1 + y &= \underbrace{\pm e^C}_{B \neq 0} e^{x^2} \\
y &= Be^{x^2} - 1
\end{aligned}$$

Substituting in the initial condition $y(0) = 0$ we find that $B = 1$ and $y(x) = e^{x^2} - 1$.

For Euler's method, we first estimate $y(.1)$

$$y(.1) \approx y(0) + \frac{dy}{dx}(0)\Delta x$$

where $\frac{dy}{dx}(0) = 2(0)y(0) + 2(0) = 0$. Then our approximation is

$$y(.1) \approx 0 + 0(.1) = 0$$

Now we approximate $y(.2)$.

$$y(.2) \approx y(.1)\frac{dy}{dx}(.1)\Delta x$$

where $\frac{dy}{dx}(.1) = 2(.1)y(.1) + 2(.1) \approx 2(.1)(0) + 2(.1) = .2$. Then, we have the approximation

$$\begin{aligned}
y(.2) &\approx y(.1) + \frac{dy}{dx}(.1)\Delta x \\
&\approx 0 + (.2)(.1) \\
&= .02
\end{aligned}$$