1. (improper:cauchym)
   Compute $\int_{-\infty}^{0} \frac{x}{1+x^2}dx$ and $\int_{0}^{\infty} \frac{x}{1+x^2}dx$. What does this say about $\int_{-\infty}^{\infty} \frac{xdx}{1+x^2}$?
   Solution:

$$\begin{aligned}
\int_{-\infty}^{0} \frac{x}{1+x^2}dx &= \lim_{A\to-\infty} \int_{A}^{0} \frac{xdx}{1+x^2} \\
&= \lim_{A\to-\infty} \left[\frac{1}{2}\ln(1+x^2)\right]_A^0 \\
&= \lim_{A\to-\infty} \ln(1) - \ln(1+A^2) \\
&= -\infty \\
\int_{0}^{\infty} \frac{x}{1+x^2}dx &= \lim_{B\to\infty} \int_{0}^{B} \frac{xdx}{1+x^2} \\
&= \lim_{B\to\infty} \left[\frac{1}{2}\ln(1+x^2)\right]_0^B \\
&= \lim_{B\to\infty} \ln(1+B^2) - \ln(1) \\
&= \infty
\end{aligned}$$

   Notice that this problem shows that $\int_{-\infty}^{\infty} \frac{x}{1+x^2}dx$ does not exist.

2. (improper:ln)
   Compute $\int_0^1 \ln(t)dt$.
   Solution:

$$\begin{aligned}
\int_0^1 \ln(t)dt &= \lim_{A\downarrow 0} \int_A^1 \ln(t)dt \\
&= \lim_{A\downarrow 0} \int_A^1 \underbrace{\ln(t)}_{\text{F(t)}} \underbrace{dt}_{\text{G'(t)dt}} \\
&= \lim_{A\downarrow 0} \left[\underbrace{t}_{\text{G(t)}} \underbrace{\ln(t)}_{\text{F(t)}}\right]_A^1 - \int_A^1 \underbrace{t}_{\text{G(t)}} \underbrace{\frac{1}{t}dt}_{\text{F'(t)dt}} \\
&= \lim_{A\downarrow 0} 1\ln(1) - A\ln(A) - 1 + A \\
&= -1
\end{aligned}$$

   since $\lim_{A\downarrow 0} A\ln(A) = 0$.

3. (improper:pf1)
   Compute $\int_{10}^{\infty} \frac{dx}{x^2-4}$

   Solution:

$$\begin{aligned}
\int_{10}^{\infty} \frac{dx}{x^2-4} &= \lim_{L\to\infty} \int_{10}^{L} \frac{dx}{x^2-4} \\
&= \lim_{L\to\infty} \int_{10}^{L} \frac{dx}{(x-2)(x+2)} \\
&= \lim_{L\to\infty} \int_{10}^{L} \frac{1}{4(x-2)} - \frac{1}{4(x+2)} dx \\
&= \lim_{L\to\infty} \left[\frac{1}{4} \ln|x-2| - \frac{1}{4} \ln|x+2| \right]_{10}^{L} \\
&= \lim_{L\to\infty} \left[\frac{1}{4} \ln|\frac{x-2}{x+2}| \right]_{10}^{L} \\
&= \underbrace{\lim_{L\to\infty} \frac{1}{4} \ln|\frac{L-2}{L+2}|}_{0} - \frac{1}{4} \ln|\frac{8}{12}| \\
&= \frac{1}{4} \ln\left(\frac{3}{2}\right)
\end{aligned}$$

4. (improper:pf2)
   Show that $\int_{1}^{\infty} \frac{dx}{x^2-4}$ is not a finite number. What answer do you get if you forget that the integrand has an asymptote at 2 and fail to split the integral up there?

   Solution: You can use partial fractions to compute that

$$\int \frac{dx}{x^2-4} = \frac{1}{4} \ln|\frac{x-2}{x+2}| + C$$

   If we write

$$\int_{1}^{\infty} \frac{dx}{x^2-4} = \int_{1}^{2} \frac{dx}{x^2-4} + \int_{2}^{3} \frac{dx}{x^2-4} + \int_{3}^{\infty} \frac{dx}{x^2-4}$$

it suffices to show that one of these integrals is infinite. For example

$$\begin{aligned}
\int_1^2 \frac{dx}{x^2-4} &= \lim_{A\uparrow 2}\int_1^2 \frac{dx}{x^2-4} \\
&= \lim_{A\uparrow 2}\left[\frac{1}{4}\ln|\frac{x-2}{x+2}|\right]_1^A \\
&= \lim_{A\uparrow 2}\frac{1}{4}\ln|\frac{A-2}{A+2}| - \frac{1}{4}\ln|\frac{1}{2}| \\
&= -\infty
\end{aligned}$$

Now we will see what happens if we forget that the integrand has an asymptote at 2.

$$\lim_{A\to\infty}\frac{1}{4}\ln|\frac{A-2}{A+2}| - \frac{1}{4}\ln|\frac{-1}{3}| = \frac{1}{4}\ln(3)$$

5. (improper:pf3)
Compute $\int_1^\infty \frac{dx}{1+e^{2x}}$

Solution:

$$\begin{aligned}
\int_1^\infty \frac{dx}{1+e^{2x}} &= \lim_{L\to\infty}\int_1^L \frac{dx}{1+e^{2x}} \\
&= \lim_{L\to\infty}\int_{x=1}^{x=L}\frac{1}{u(1+u^2)}du && u=e^x \quad \frac{du}{u}=dx \\
&= \lim_{L\to\infty}\int_{x=1}^{x=L}\frac{1}{u}-\frac{u}{1+u^2}du \\
&= \lim_{L\to\infty}[\ln|u|]_{x=1}^{x=L} - \left[\frac{1}{2}\ln|1+u^2|\right]_{x=1}^{x=L} \\
&= \lim_{L\to\infty}\left[\underbrace{\ln(e^x)}_{\text{x}}\right]_1^L - \left[\frac{1}{2}\ln|1+e^{2x}|\right]_1^L \\
&= \lim_{L\to\infty} L-1-\frac{1}{2}\ln|1+e^{2L}| + \frac{1}{2}\ln(1+e^2) \\
&= \lim_{L\to\infty} L-1-\underbrace{\frac{1}{2}\ln(e^{2L})}_{\text{L}} - \frac{1}{2}\ln(e^{-2L}+1) + \frac{1}{2}\ln(1+e^2) \\
&= \frac{1}{2}\ln(1+e^2)-1
\end{aligned}$$

6. (improper:pf4)
   Compute $\int_{10}^{\infty} \frac{dx}{x^2-9}$.

   Solution:

$$\begin{aligned}
\int_{10}^{\infty} \frac{dx}{x^2-9} &= \lim_{L\to\infty} \int_{10}^{L} \frac{dx}{x^2-9} \\
&= \lim_{L\to\infty} \int_{10}^{L} \frac{1}{9(x-3)} - \frac{1}{9(x+3)} dx \\
&= \lim_{L\to\infty} \left[\frac{1}{9}\ln(x-3) - \frac{1}{9}\ln(x+3)\right]_{10}^{L} \\
&= \lim_{L\to\infty} \left[\frac{1}{9}\ln\left(\frac{x-3}{x+3}\right)\right]_{10}^{L} \\
&= \underbrace{\lim_{L\to\infty} \frac{1}{9}\ln\left(\frac{L-3}{L+3}\right)}_{0} - \frac{1}{9}\ln\left(\frac{7}{13}\right) \\
&= \frac{1}{9}\ln\left(\frac{13}{7}\right)
\end{aligned}$$

7. (improper:trig1)
   Compute $\int_{1}^{2} \frac{dt}{t\sqrt{t^2-1}}$.

   Solution:

$$\begin{aligned}
\int_{1}^{2} \frac{dt}{t\sqrt{t^2-1}} &= \lim_{B\to 1} \int_{B}^{2} \frac{dt}{t\sqrt{t^2-1}} \\
&= \lim_{B\to 1} \operatorname{arcsec}(2) - \operatorname{arcsec}(B) \\
&= \operatorname{arcsec}(2)
\end{aligned}$$

8. (improper:trig2)
   Compute $\int_{1}^{2} \frac{dt}{\sqrt{4-t^2}}$.

Solution:

$$\begin{aligned}
\int_1^2 \frac{dt}{\sqrt{4-t^2}} &= \lim_{L\to 2} \int_1^L \frac{dt}{\sqrt{4-t^2}} \\
&= \lim_{L\to 2} \Big[\arcsin(\frac{t}{2}) \Big]_1^L \\
&= \lim_{L\to 2} \arcsin(\frac{L}{2}) - \arcsin(\frac{1}{2}) \\
&= \frac{\pi}{2} - \frac{\pi}{6} \\
&= \frac{\pi}{3}
\end{aligned}$$

9. (improper:trig3)
Compute $\int_1^3 \frac{dt}{\sqrt{9-t^2}}$.

Solution:

$$\begin{aligned}
\int_1^3 \frac{dt}{\sqrt{9-t^2}} &= \lim_{L\to 3} \int_1^3 \frac{dt}{\sqrt{9-t^2}} \\
&= \lim_{L\to 3} \Big[\arcsin(\frac{t}{3}) \Big]_1^L \\
&= \lim_{L\to 3} \arcsin(\frac{L}{3}) - \arcsin(\frac{1}{3}) \\
&= \frac{\pi}{2} - \arcsin(\frac{1}{3})
\end{aligned}$$

10. (improper:partial1)
Compute $\int_1^\infty \frac{3}{x^2+3x} dx$.

Solution: First, recall that $\frac{3}{x^2+3x} = \frac{3}{x(x)}$. By partial fractions, we can find that

$$\frac{3}{x^2+3x} = \frac{1}{x} - \frac{1}{x+3}$$

so that

$$\begin{aligned}\int_1^\infty \frac{3}{x^2+3x}dx &= \lim_{b\to\infty}\int_1^b \frac{3}{x^2+3x}dx\\ &= \lim_{b\to\infty}\int_1^b \frac{1}{x}-\frac{1}{x+3}dx\\ &= \lim_{b\to\infty}[\ln|x|-\ln|x+3|]_1^b\\ &= \lim_{b\to\infty}\left[\ln\left|\frac{x}{x+3}\right|\right]_1^b\\ &= \lim_{b\to\infty}\ln\left|\frac{b}{b+3}\right|-\ln\left(\frac{1}{4}\right)\\ &= \ln(4)\end{aligned}$$

11. (improper:exp)
Find $\int_0^\infty \frac{1}{e^t-1}dt$.

Solution: First notice that the endpoints 0 and ∞ are both improper, so we need to break the integral up.

$$\int_0^\infty \frac{1}{e^t-1}dt = \lim_{a\downarrow 0}\int_a^1 \frac{1}{e^t-1}dt + \lim_{b\to\infty}\int_1^b \frac{1}{e^t-1}dt$$

We compute the antiderivative

$$\begin{aligned}\int \frac{1}{e^t-1}dt &= \int \frac{1}{u(u-1)}du \qquad\qquad u=e^t \quad \frac{1}{u}du=dt\\ &= \int \frac{1}{u-1}du - \int\frac{1}{u}du\\ &= \ln|u-1|-\ln|u|+C\\ &= \ln|\frac{u-1}{u}|+C\\ &= \ln|1-\frac{1}{e^t}|+C\end{aligned}$$

and therefore

$$\begin{aligned}\lim_{a\downarrow 0}\int_a^1 \frac{1}{e^t-1}dt + \lim_{b\to\infty}\int_1^b \frac{1}{e^t-1}dt &= \lim_{a\downarrow 0}\left[\ln|1-\frac{1}{e^t}|\right]_a^1 + \lim_{b\to\infty}\left[\ln|1-\frac{1}{e^t}|\right]_1^b\\ &= \lim_{a\downarrow 0}\ln|1-\frac{1}{e}| - \underbrace{\ln|1-\frac{1}{e^a}|}_{\text{Limit DNE}} + \underbrace{\lim_{b\to\infty}\ln|1-\frac{1}{e^b}|}_{\text{Limit is 0}} - \ln|1-\frac{1}{e}|\end{aligned}$$

so the integral does not exist.