1. (seqseries:quiz1)

   (a) Find
   $$\lim_{n\to\infty} \frac{n^2+n+1}{3n^2-n-2}$$

   (b) Find an example of a sequence a_n which is bounded but not convergent.

   Solution:

   (a)
   $$\begin{aligned}\lim_{n\to\infty} \frac{n^2+n+1}{3n^2-n-2} &= \lim_{n\to\infty} \left(\frac{n^2}{n^2}\right) \frac{1+\frac{1}{n}+\frac{1}{n^2}}{3-\frac{1}{n}-\frac{2}{n^2}} \\ &= \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{3-\frac{1}{n}-\frac{2}{n^2}} \\ &= \frac{1}{3}\end{aligned}$$

   (b) An example is given by $a_n = (-1)^n$.

2. (seqseries:quiz2)

   (a) Find
   $$\lim_{n\to\infty} \frac{3n^2+n+1}{n^2-n-2}$$

   (b) Find an example of a sequence a_n which is bounded but not convergent.

   Solution:

   (a)
   $$\begin{aligned}\lim_{n\to\infty} \frac{3n^2+n+1}{n^2-n-2} &= \lim_{n\to\infty} \left(\frac{n^2}{n^2}\right) \frac{3+\frac{1}{n}+\frac{1}{n^2}}{1-\frac{1}{n}-\frac{2}{n^2}} \\ &= \lim_{n\to\infty} \frac{3+\frac{1}{n}+\frac{1}{n^2}}{1-\frac{1}{n}-\frac{2}{n^2}} \\ &= 3\end{aligned}$$

(b) An example is given by $a_n = (-1)^n$.

3. (seqseries:geom2/x)
   If $x > 2$, use the geometric series formula to find $\sum_{n=0}^{\infty} \frac{2^{n+1}}{x^n}$.

   Solution:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2^{n+1}}{x^n} &= 2\sum_{n=0}^{\infty} \frac{2^n}{x^n} \\ &= 2\sum_{n=0}^{\infty} \left(\frac{2}{x}\right)^n \\ &= \frac{2}{1-\frac{2}{x}}\end{aligned}$$

4. (seqseries:pftelescope)
   Let $a_n = \frac{1}{n^2-n}$ and $S_N = \sum_{k=2}^{N} a_n$.

   (a) Use partial fractions to rewrite a_n.

   (b) Use part (a) to write out S_2, S_3 and S_4 explicitly and notice how terms cancel. Generalize this to find a formula for S_N.

   (c) Compute $\sum_{k=2}^{\infty} a_n \,(= \lim_{N\to\infty} S_N)$.

   Solution:

   (a)

$$\begin{aligned}\frac{1}{n^2-n} &= \frac{A}{n-1} + \frac{B}{n} \\ 1 &= An + B(n-1)\end{aligned}$$

   so that $A = 1$ and $B = -1$ and $\frac{1}{n^2-n} = \frac{1}{n-1} - \frac{1}{n}$

(b)

$$\begin{aligned}
S_2 &= \underbrace{1 - \frac{1}{2}}_{a_2} \\
S_3 &= \underbrace{1 - \frac{1}{2}}_{a_2} + \underbrace{\frac{1}{2} - \frac{1}{3}}_{a_3} \\
&= 1 - \frac{1}{3} \\
S_4 &= \underbrace{1 - \frac{1}{2}}_{a_2} + \underbrace{\frac{1}{2} - \frac{1}{3}}_{a_3} + \underbrace{\frac{1}{3} - \frac{1}{4}}_{a_4} \\
&= 1 - \frac{1}{4}
\end{aligned}$$

in general $S_N = 1 - \frac{1}{N}$.

(c) $\lim_{N\to\infty} S_N = 1$.

5. (seqseries:serieslist)
Determine whether the following series converge:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

(b) $\sum_{n=1}^{\infty} \frac{e^n}{n^3}$

(c) $\sum_{n=3}^{\infty} \frac{1}{n^3+n-1}$

(d) $\sum_{n=1}^{\infty} \left(\frac{n^3}{n!}\right)^n$

(e) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$

(f) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

(g) $\sum_{n=1}^{\infty} e^{-(\ln(n))^2}$ (Hints: $a^{bc} = (a^b)^c$ and $e^{-\ln(n)} = \frac{1}{n}$)

Solution:

(a) We can use the integral test for this. Since $\int_1^{\infty} \frac{1}{x^3}dx < \infty$ we just have to check that $\frac{1}{x^3}$ is a positive decreasing function. It is clearly positive for $x > 0$, so that is not an issue. To check that it is decreasing, take a derivative. $\frac{d}{dx}\frac{1}{x^3} = \frac{-3}{x^4} < 0$. Since the derivative is negative, the function is decreasing.

(b) This sum diverges. We can see this by applying the n^{th} term test:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{e^n}{n^3} = \infty$$

This limit would have to be zero for the sum to have any hope of converging.

(c) We can do this with a limit comparison test. Call $b_n = \frac{1}{n^3}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3}{n^3+n-1} = 1$$

Since the limit is 1, both sequences are positive, and $\sum_{n=0}^{\infty} \frac{1}{n^3} < \infty$, it follows that $\sum_{n=0}^{\infty} \frac{1}{n^3+n-1}$ converges.

(d) This sum converges. We can see this with the root test.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^3}{n!} = 0$$

so the series converges.

(e) This series converges for all x and we can see this with the ratio test.

$$\begin{aligned}\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{\left(\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}\right)}{\left(\frac{x^{2n+1}}{(2n+1)!}\right)} \\ &= \lim_{n\to\infty} \frac{x^2}{(2n+3)(2n+2)} = 0\end{aligned}$$

so this sum converges for all x.

(f) We can solve this with the integral test once we check that the function $\frac{1}{x\ln(x)}$ is positive and decreasing on $(2,\infty)$. It is clearly positive, so we just need to check that it is decreasing.

$$\frac{d}{dx}\frac{1}{x\ln(x)} = -\frac{\ln(x)+1}{(x\ln(x))^2} < 0$$

for $x \in (2,\infty)$. We can now compare to the integral

$$\begin{aligned}\int_3^{\infty} \frac{1}{x\ln(x)}dx &= \lim_{b\to\infty} \int_3^b \frac{1}{x\ln(x)}dx \\ &= \lim_{b\to\infty} \int_{x=3}^{x=b} \frac{1}{u}du \qquad\qquad u = \ln(x) \quad du = \frac{1}{x}dx \\ &= \lim_{b\to\infty} [\ln|u|]_{x=3}^{x=b} \\ &= \lim_{b\to\infty} [\ln|\ln(x)|]_3^b = \infty\end{aligned}$$

so the sum diverges.

(g) This sum converges, which we can see by direct comparison to $\frac{1}{n^3}$ (or any power of n that converges). To see this, observe that $e^{-(\ln(n))^2} = \left(e^{-\ln(n)}\right)^{\ln(n)} = \frac{1}{n^{\ln(n)}}$ and for n large, $e^{-(\ln(n))^2}$ will be strictly less than $\frac{1}{n^3}$, since $\ln(n) \to \infty$.

6. (seqseries:serieslist2)
Determine whether the following series converge. If the series depends on x, determine for which values of x it converges.

(a) $\sum_{n=0}^{\infty} e^{-nx}$
(b) $\sum_{n=1}^{\infty} \frac{1}{n^6+5n}$
(c) $\sum_{n=1}^{\infty} \frac{n!}{e^n}$
(d) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$
(e) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

Solution:

(a) Notice that $e^{-nx} = \left(\frac{1}{e^x}\right)^n$ and we know that this sum converges if and only if $\left|\frac{1}{e^x}\right| < 1$ which is if and only if $e^x > 1$. So this sum converges exactly for $x > 0$.

(b) We can do this by limit comparison. We know that $\sum_{n=1}^{\infty} \frac{1}{n^6}$ converges, so we just need to find the limit of the ratio of the summands in these two series.

$$\begin{aligned}\lim_{n\to\infty} \frac{\frac{1}{n^6}}{\frac{1}{n^6+5n}} &= \lim_{n\to\infty} \frac{n^6+5n}{n^6} \\ &= 1.\end{aligned}$$

Since this limit is a positive finite number and $\sum_{n=1}^{\infty} \frac{1}{n^6}$ converges, we know that $\sum_{n=1}^{\infty} \frac{1}{n^6+5n}$ converges.

(c) We can do this with the term test. Recall that $\sum_{n=0}^{\infty} a_n$ does not converges if $\lim_{n\to\infty} a_n \neq 0$ or does not exist. But $\lim_{n\to\infty} \frac{n!}{e^n} = \infty$, so this sum cannot converge.

(d) This problem can be solved with the integral test. Notice that $\frac{\ln(x)}{x} \geq 0$ for $x \geq 1$ and that $\frac{d}{dx}\frac{ln(x)}{x} = \frac{1-x\ln(x)}{x^2} < 0$ so long as $1 - x\ln(x) < 0$, which is true for sufficiently large x. For example $\ln(x) > 1$ for $x > e$ and clearly $x > 1$ for $x > e$, so on the interval

$(3, \infty)$ the function $\frac{\ln(x)}{x}$ is decreasing. We can then apply the integral test.

$$\begin{aligned}
\int_3^\infty \frac{\ln(x)}{x} dx &= \lim_{b\to\infty} \int_3^b \frac{\ln(x)}{x} dx \\
&= \lim_{b\to\infty} \int_{\ln(3)}^{\ln(b)} u du \\
&= \lim_{b\to\infty} \frac{1}{2} u^2 |_{\ln(3)}^{\ln(b)} \\
&= \infty.
\end{aligned}$$

From this, we see that $\sum_{n=1}^\infty \frac{\ln(n)}{n}$ does not converge.

(e) This question may have been a little tricky. Although it looks like a good candidate for the root test, the terms here are genuinely not comparable to a geometric series (which is what the root test is checking for). Instead, we can solve this with the term test if we recall that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$. From this, we see that $\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1} \neq 0$, so the series cannot converge.

7. (seqseries:quiz3)
Determine whether or not the series $\sum_{n=1}^\infty ne^{-n}$ converges.

Solution: There are many ways to solve this problem. The easiest is probably the root test, so long as you know that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. With this, we find that

$$\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{ne^{-n}} &= \lim_{n\to\infty} \sqrt[n]{n} e^{-1} \\
&= e^{-1} < 1
\end{aligned}$$

so the series converges. The ratio test also works.

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n+1)e^{-(n+1)}}{ne^{-n}} &= \lim_{n\to\infty} \frac{n+1}{n} e^{-1} \\
&= e^{-1} < 1
\end{aligned}$$

so again the series converges. The integral test can also be used to solve this problem.

8. (seqseries:quiz4)
Determine whether or not the series $\sum_{n=1}^\infty ne^{-2n}$ converges.

Solution: There are many ways to solve this problem. The easiest is probably the root test, so long as you know that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. With this, we find that

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{ne^{-2n}} &= \lim_{n\to\infty} \sqrt[n]{n}e^{-2} \\ &= e^{-2} < 1\end{aligned}$$

so the series converges. The ratio test also works.

$$\begin{aligned}\lim_{n\to\infty} \frac{(n+1)e^{-2(n+1)}}{ne^{-2n}} &= \lim_{n\to\infty} \frac{n+1}{n}e^{-2} \\ &= e^{-2} < 1\end{aligned}$$

so again the series converges. The integral test can also be used to solve this problem.