1. (taylorerror:expatapoint)
   Find a bound on the error of the approximation of $e^{\frac{1}{3}}$ by $1+\frac{1}{3}+\frac{1}{3^2 2!}+\frac{1}{3^3(3!)}$.

   Solution: We know that if $f(x)=e^x$ then $f^{(n)}(x)=e^x$. Approximating $e^{\frac{1}{3}}$ by $1+\frac{1}{3}+\frac{1}{3^2 2!}+\frac{1}{3^3(3!)}$ corresponds to approximating e^x by $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}=f(0)+f^{'}(0)x+f^{(2)}(0)\frac{x^2}{2!}+f^{(3)}(0)\frac{x^3}{3!}$. Taylor's theorem then tells us that there is a ξ with $0\leq\xi\leq\frac{1}{3}$

$$e^{\frac{1}{3}}-(1+\frac{1}{3}+\frac{1}{3^2 2!}+\frac{1}{3^3(3!)})=\frac{e^{\xi}}{4!}(\frac{1}{3})^4$$

   Since e^x is an increasing function, we can bound this error by

$$\frac{e^{\xi}}{4!}(\frac{1}{3})^4\leq\frac{e^{\frac{1}{3}}}{4!}(\frac{1}{3})^4$$

   but this is actually not at all helpful. The whole point of this problem was, after all, to estimate $e^{\frac{1}{3}}$! We still know that $e\leq 3<8$, so that $e^{\frac{1}{3}}<8^{\frac{1}{3}}=2$. A final answer is then that

$$e^{\frac{1}{3}}-(1+\frac{1}{3}+\frac{1}{3^2 2!}+\frac{1}{3^3(3!)})<\frac{2}{4!}(\frac{1}{3})^4=\frac{1}{972}\approx .00102$$

2. (taylorerror:sinconvergence)
   Find a bound for $R_n^0\sin(x)$ and use this to show that $T_n^0\sin(x)\to\sin(x)$ for all x as $n\to\infty$.

   Solution: We begin by recalling the Lagrange form of the Taylor remainder term. If $f(x)=\sin(x)$ then

$$R_n^0\sin(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

   Notice that $f^{(n+1)}(x)$ is one of $\sin(x),\cos(x),-\sin(x)$, or $-\cos(x)$. In any of these cases, we know that $|f^{(n+1)}(x)|\leq 1$ for all x. So in particular, we have the bound

$$\begin{aligned}|R_n^0\sin(x)|&=|\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}|\\&\leq\frac{1}{(n+1)!}|x|^{n+1}\end{aligned}$$

Since for any x, $\lim_{n\to\infty} \frac{1}{(n+1)!}|x|^{n+1} = 0$, this shows that for any x, $R_n^0 \sin(x) \to 0$, which implies that $T_n^0 \sin(x) = \sin(x) - R_n^0 \sin(x) \to \sin(x)$.

3. (taylorerror:sin3convergence)
   Find a bound for $R_n^0 \sin(3x)$ and use this to show that $T_n^0 \sin(3x) \to \sin(3x)$ for all x as $n \to \infty$.

   Solution: We begin by recalling the Lagrange form of the Taylor remainder term. If $f(x) = \sin(3x)$ then

$$R_n^0 \sin(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$$

   Notice that $f^{(n+1)}(x)$ is one of $3^{n+1}\sin(3x), 3^{n+1}\cos(3x), -3^{n+1}\sin(3x)$, or $-3^{n+1}\cos(3x)$. In any of these cases, we know that $|f^{(n+1)}(x)| \leq 3^{n+1}$ for all x. So in particular, we have the bound

$$\begin{aligned} |R_n^0 \sin(3x)| &= |\frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}| \\ &\leq \frac{3^n}{(n+1)!}|x|^{n+1} \\ &= \frac{|3x|^{n+1}}{(n+1)!} \end{aligned}$$

   Since for any x, $\lim_{n\to\infty} \frac{1}{(n+1)!}|3x|^{n+1} = 0$, this shows that for any x, $R_n^0 \sin(3x) \to 0$, which implies that $T_n^0 \sin(3x) = \sin(3x) - R_n^0 \sin(3x) \to \sin(3x)$.

4. (taylorerror:exp2convergence)
   Find a bound on $R_n^0 e^{2x}$ and use this to show that for every x, $T_n^0 e^{2x} \to e^{2x}$ as $n \to \infty$.

   Solution: We begin by recalling the Lagrange form of the Taylor remainder term. If $f(x) = e^{2x}$ then

$$R_n^0 e^{2x} = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$$

   If we recall that $f^{(n)}(x) = 2^n e^{2x}$ we can see that $|f^{(n+1)}(\xi)| \leq 2^{n+1} e^{2|x|}$ for any ξ between 0 and x. Then $|R_n^0 e^{2x}| \leq \frac{2^{n+1} e^{2|x|}}{(n+1)!}|x|^{n+1} = \frac{e^{2|x|}}{(n+1)!}|2x|^{n+1}$. Notice that $e^{2|x|}$ is a number that does not depend on n, so it suffices to observe that $\lim_{n\to\infty} \frac{|2x|^{n+1}}{(n+1)!} = 0$ for all real numbers x. Therefore $\lim_{n\to\infty} R_n^0 e^{2x} = 0$ and so $\lim_{n\to\infty} T_n^0 e^{2x} = e^{2x}$.

5. (taylorerror:sincosconvergence)
   Find a bound on $R_n^0\,(\sin(x) + \cos(x))$ and use this to show that $T_n^0\,(\sin(x) + \cos(x))$ converges to $\sin(x) + \cos(x)$ as $n \to \infty$.

   Solution: We begin by recalling the Lagrange form of the Taylor remainder term. If $f(x) = \sin(x) + \cos(x)$ then

   $$R_n^0 \sin(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$$

   Notice that $f^{(n+1)}(x)$ is of the form $\pm \sin(x) \pm \cos(x)$, so that for all x we know that $|f^{(n+1)}(x)| \leq 2$.

   $$\begin{aligned} |R_n^0 \sin(x) + \cos(x)| &= |\frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}| \\ &\leq \frac{2}{(n+1)!}|x|^{n+1} \\ &= \frac{2|x|^{n+1}}{(n+1)!} \end{aligned}$$

   since $\frac{2|x|^{n+1}}{(n+1)!} \to 0$ for any x, this shows that $R_n^0\,(\sin(x) + \cos(x)) \to 0$ and therefore that $T_n^0\,(\sin(x) + \cos(x)) \to \sin(x) + \cos(x)$.

6. (taylorerror:cosgoodenough)
   Find a bound on $|R_n \cos(x)|_{x=1}|$ and use this information to find a decimal approximation of $\cos(1)$ with an error of at most .1.

   Solution: Recall that $\cos(x) - T_n \cos(x) = R_n \cos(x)$ by definition, so if $|R_n \cos(x)_{x=1}|$ is less than $\frac{1}{10}$, then $\cos(1)$ is within two decimal digits of $T_n \cos(x)|_{x=1}$. If $f(x) = \cos(x)$ then $f^{n+1}(x)$ is $\pm \cos(x)$ or $\pm \sin(x)$. In any case, we have $|f^{n+1}(x)| \leq 1$. It follows then that

   $$|R_n \cos(x)|_{x=1}| \leq \frac{1(1)^{n+1}}{(n+1)!} = \frac{1}{(n+1)!}$$

   so if n is sufficiently large that $\frac{1}{(n+1)!} \leq \frac{1}{10}$, then $|R_n \cos(x)|_{x=1}| \leq \frac{1}{10}$. This is true, for example, for $n = 3$, since $(3 + 1)! = 24$. Our approximation is then $T_3 \cos(x)|_{x=1} = 1 - \frac{1}{2} = .5$.