Worksheet 10

Taylor Polynomials

Definition: Taylor Polynomials

The **Taylor polynomial** of a function $y = f(x)$ of degree n at a point a is the polynomial:

$$T_n^a f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^(n)(a)}{n!}(x-a)^n$$

The **Taylor series** of a function $y = f(x)$ centered at $a = 0$ is the infinite sum:

$$T_\infty f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^(n)(0)}{n!}x^n + \cdots$$

Calculate the derivatives of the following functions and look for a pattern. Use this pattern to calculate the *Taylor series* for the following functions.

Problem 1. e^x

Problem 2. $\sin x$

Problem 3. $\sqrt{1+x}$

Fundamental Taylor Series Formulas

Write out the formulas for the following Taylor series. You will be expected to know how to calculate these by the definitions, but also memorized.

$$T_\infty\, e^x =$$

$$T_\infty\, \sin x =$$

$$T_\infty\, \cos x =$$

$$T_\infty\, \frac{1}{1-x} =$$

$$T_\infty\, \ln(1-x) =$$

Taylor Series by Substitution

Suppose you know the Taylor series for $f(x)$, then you can calculate the Taylor series for $f(3x)$ or $f(x^2)$, etc, by simply substituting $3x$ or x^2 into your Taylor series.

For example, we know:

$$T_\infty e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

So to find the Taylor series for e^{3x} and e^{-x^2} is a simple as a substitution:

$$\begin{aligned} T_\infty e^{3x} &= 1 + (3x) + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \cdots + \frac{(3x)^n}{n!} + \cdots \\ &= 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \cdots + \frac{3^n x^n}{n!} + \cdots \end{aligned}$$

$$\begin{aligned} T_\infty e^{-x^2} &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots \\ &= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots \end{aligned}$$

Use the substitution method to find a Taylor series for the following problems.

Problem 4. $\dfrac{1}{1-x^2}$

Problem 5. $\ln(1-2x^3)$

Problem 6. $\sin(2x^4)$

Calculating Taylor Series by Multiplying by x^n

If you know the Taylor series for $f(x)$ then the Taylor series for $x^n f(x)$ is obtained by multiplying $T_\infty f(x)$ by x^n. Use this technique for the following.

Problem 7. $\dfrac{3}{2-x}$. Hint: $\dfrac{3}{2-x} = \dfrac{3}{2} \cdot \dfrac{1}{1-x/2}$.

Problem 8. $\frac{x}{2-x^2}$

Problem 9. $x\ln(2+2x)$

Adding Taylor Series

Calculate the Taylor series for the following.

Problem 10. $\sinh(x)$. Recall: $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$.

Problem 11. $\dfrac{1}{x^2 - 3x + 2}$. Hint: PFD!

Problem 12. $\dfrac{1 + x}{1 - x}$.