1. (trigsub:nosub)
   Compute $\int t\sqrt{1-t^2}\text{dt}$.

   Solution:

$$\begin{aligned}
\int t\sqrt{1-t^2}dt &= \frac{-1}{2}\int \sqrt{u}du \qquad\qquad u = 1-t^2 \qquad \frac{-1}{2}du = tdt \\
&= \frac{-1}{3}u^{\frac{3}{2}} + C \\
&= \frac{-1}{3}\left(1-t^2\right)^{\frac{3}{2}} + C
\end{aligned}$$

2. (trigsub:sinsub1)
   Compute $\int \sqrt{16-x^2}dx$.

   Solution:

$$\begin{aligned}
\int \sqrt{16-x^2}dx &= \int \left(\sqrt{16-16\sin^2(\theta)}\right)4\cos(\theta)d\theta && x = 4\sin(\theta) \\
&= \int 4\cos(\theta)\sqrt{16(1-\sin^2(\theta))}d\theta && dx = 4\cos(\theta)d\theta \\
&= 16\int \cos^2(\theta)d\theta \\
&= 16\int \frac{1+\cos(2\theta)}{2}d\theta \\
&= 16\left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right) + C \\
&= 8\ \text{arcsin}(\frac{x}{4}) + 4\sin(2\text{arcsin}(\frac{x}{4})) + C \\
&= 8\ \text{arcsin}(\frac{x}{4}) + 8\sin(\text{arcsin}(\frac{x}{4}))\cos(\text{arcsin}(\frac{x}{4})) + C \\
&= 8\ \text{arcsin}(\frac{x}{4}) + 2x\sqrt{1-\left(\frac{x}{4}\right)^2} + C
\end{aligned}$$

3. (trigsub:sinsub2)
   Compute $\int \sqrt{25-x^2}dx$.

Solution:

$$\begin{aligned}
\int \sqrt{25-x^2}dx &= \int \left(\sqrt{25-25\sin^2(\theta)}\right) 5\cos(\theta)d\theta && x = 5\sin(\theta)\\
&= \int 5\cos(\theta)\sqrt{25(1-\sin^2(\theta))}d\theta && dx = 5\cos(\theta)d\theta\\
&= 25\int \cos^2(\theta)d\theta\\
&= 25\int \frac{1+\cos(2\theta)}{2}d\theta\\
&= 25\left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right) + C\\
&= \frac{25}{2}\ \text{arcsin}(\frac{x}{5}) + \frac{25}{4}\sin(2\text{arcsin}(\frac{x}{5})) + C\\
&= \frac{25}{2}\ \text{arcsin}(\frac{x}{5}) + \frac{25}{2}\sin(\text{arcsin}(\frac{x}{5}))\cos(\text{arcsin}(\frac{x}{5})) + C\\
&= \frac{25}{2}\ \text{arcsin}(\frac{x}{5}) + \frac{5}{2}x\sqrt{1-\left(\frac{x}{5}\right)^2} + C
\end{aligned}$$

4. (trigsub:sinsub3)
   Compute $\int \frac{x^3}{\sqrt{4-x^2}}dx$.

   Solution:

$$\begin{aligned}
\int \frac{x^3}{\sqrt{4-x^2}}dx &= \int \frac{8\sin^3(\theta)}{\sqrt{4-4\sin^2(\theta)}}2\cos(\theta)d\theta && x = 2\sin(\theta)\\
&= 8\int \sin^3(\theta)d\theta && dx = 2\cos(\theta)d\theta\\
&= 8\int (1-\cos^2(\theta))\sin(\theta)d\theta && u = \cos(\theta)\\
&= -8\int (1-u^2)du && du = -\sin(\theta)d\theta\\
&= \frac{8}{3}u^3 - 8u + C\\
&= \frac{8}{3}\cos^3(\theta) - 8\cos(\theta) + C\\
&= \frac{8}{3}\cos^3\left(\text{arcsin}\left(\frac{x}{2}\right)\right) - 8\cos\left(\text{arcsin}\left(\frac{x}{2}\right)\right) + C\\
&= \frac{8}{3}\left(1-\left(\frac{x}{2}\right)^2\right)^{\frac{3}{2}} - 8\sqrt{1-\left(\frac{x}{2}\right)^2} + C
\end{aligned}$$

5. (trigsub:sinsub4)
   Compute $\int \frac{x^3}{\sqrt{9-x^2}}dx$.

   Solution:

$$\begin{aligned}
\int \frac{x^3}{\sqrt{9-x^2}}dx &= \int \frac{27\sin^3(\theta)}{\sqrt{9-9\sin^2(\theta)}}3\cos(\theta)d\theta && x = 3\sin(\theta) \\
&= 27\int \sin^3(\theta)d\theta && dx = 3\cos(\theta)d\theta \\
&= 27\int (1-\cos^2(\theta))\sin(\theta)d\theta && u = \cos(\theta) \\
&= -27\int (1-u^2)du && du = -\sin(\theta)d\theta \\
&= 9u^3 - 27u + C \\
&= 9\cos^3(\theta) - 27\cos(\theta) + C \\
&= 9\cos^3\left(\arcsin\left(\frac{x}{3}\right)\right) - 27\cos\left(\arcsin\left(\frac{x}{3}\right)\right) + C \\
&= 9\left(1-\left(\frac{x}{3}\right)^2\right)^{\frac{3}{2}} - 27\sqrt{1-\left(\frac{x}{3}\right)^2} + C
\end{aligned}$$

6. (trigsub:sinsub5)
   Compute $\int e^{4x}\sqrt{1-e^{2x}}dx$.

Solution:

$$\begin{aligned}
\int e^{4x}\sqrt{1-e^{2x}}dx &= \int u^3\sqrt{1-u^2}du && u=e^x\\
&= \int \sin^3(\theta)\sqrt{1-\sin^2(\theta)}\cos(\theta)d\theta && u=\sin(\theta)\\
&= \int \sin^3(\theta)\cos^2(\theta)d\theta\\
&= \int \sin^2(\theta)\cos^2(\theta)\sin(\theta)d\theta\\
&= \int \left(1-\cos^2(\theta)\right)\cos^2(\theta)\sin(\theta)d\theta && v=\cos(\theta)\\
&= -\int \left(1-v^2\right)v^2dv\\
&= \int v^4-v^2dv\\
&= \frac{1}{5}v^5-\frac{1}{3}v^3+C\\
&= \frac{1}{5}\cos^5(\theta)-\frac{1}{3}\cos^3(\theta)+C\\
&= \frac{1}{5}\cos^5\left(\arcsin(e^x)\right)-\frac{1}{3}\cos^3(\arcsin(e^x))+C\\
&= \frac{1}{5}\left(1-e^{2x}\right)^{\frac{5}{2}}-\frac{1}{3}\left(1-e^{2x}\right)^{\frac{3}{2}}+C
\end{aligned}$$

7. (trigsub:sinsub6)
   Compute $\int \frac{e^t}{\sqrt{4-e^{2t}}}dt$

   Solution:

$$\begin{aligned}
\int \frac{e^t}{\sqrt{4-e^{2t}}}dt &= \int \frac{du}{\sqrt{4-u^2}} && u=e^t && du=e^tdt\\
&= \int \frac{2\cos(\theta)d\theta}{\sqrt{4-4\sin^2(\theta)}} && u=2\sin(\theta) && du=2\cos(\theta)d\theta\\
&= \int d\theta\\
&= \theta+C\\
&= \arcsin\left(\frac{u}{2}\right)+C\\
&= \arcsin\left(\frac{e^t}{2}\right)+C
\end{aligned}$$

8. (trigsub:cossub1)
Compute $\int \frac{1}{y\sqrt{1-y^2}}dy$.

Solution:

$$\begin{aligned}
\int \frac{1}{y\sqrt{1-y^2}}dy &= \int \frac{-\sin(\theta)d\theta}{\cos(\theta)\sqrt{1-\cos^2(\theta)}} && y=\cos(\theta) \quad dy=-\sin(\theta)d\theta \\
&= \int \frac{-d\theta}{\cos(\theta)} \\
&= -\int \sec(\theta)d\theta \\
&= -\ln|\sec(\theta)+\tan(\theta)|+C \\
&= -\ln|\sec(\arccos(y))+\tan(\arccos(y))|+C \\
&= -\ln|\frac{1}{\cos(\arccos(y))}+\frac{\sin(\arccos(y))}{\cos(\arccos(y))}|+C \\
&= -\ln|\frac{1}{y}+\frac{\sqrt{1-y^2}}{y}|+C
\end{aligned}$$

9. (trigsub:secsub1)
Compute
$$\int t^3\sqrt{t^2-4}dt$$
. **Hints:** you will need to use the facts $\frac{d}{d\theta}\tan(\theta)=\sec^2(\theta)$ and $1+\tan^2(\theta)=\sec^2(\theta)$.

Solution:

$$
\begin{aligned}
\int t^3\sqrt{t^2-4}dt &= \int (2\sec(\theta))^3 \underbrace{\sqrt{4\sec^2(\theta)-4}}_{\mathbf{2}\tan(\theta)} (2\sec(\theta)\tan(\theta))\, d\theta & t = 2\sec(\theta)\\
&= 32\int \tan^2(\theta)\sec^4(\theta)d\theta\\
&= 32\int \tan^2(\theta)\sec^2(\theta)\sec^2(\theta)d\theta\\
&= 32\int \tan^2(\theta)(1+\tan^2(\theta))\sec^2(\theta)d\theta\\
&= 32\int u^2(1+u^2)du & u=\tan(\theta)\\
&= \frac{32}{3}u^3+\frac{32}{5}u^5+C\\
&= \frac{32}{3}\tan^3(\theta)+\frac{32}{5}\tan^5(\theta)+C\\
&= \frac{32}{3}\tan^3\left(\operatorname{arcsec}\left(\frac{t}{2}\right)\right)+\frac{32}{5}\tan^5\left(\operatorname{arcsec}\left(\frac{t}{2}\right)\right)+C\\
&= \frac{32}{3}\left(\left(\frac{t}{2}\right)^2-1\right)^{\frac{3}{2}}+\frac{32}{5}\left(\left(\frac{t}{2}\right)^2-1\right)^{\frac{5}{2}}+C
\end{aligned}
$$

10. (trigsub:lnsin)
    Compute $\int \frac{\ln^5(t)}{t\sqrt{1-\ln^2(t)}}dt$.

Solution:

$$\int \frac{\ln^5(t)}{t\sqrt{1-\ln^2(t)}}dt = \int \frac{u^5}{\sqrt{1-u^2}}du \qquad u = \ln(t) \quad du = \frac{1}{t}dt$$

$$\int \frac{\sin^5(\theta)\cos(\theta)d\theta}{\sqrt{1-\sin^2(\theta)}} u = \sin(\theta) \quad du = \cos(\theta)d\theta$$

$$\begin{aligned}
&= \int \sin^5(\theta)d\theta \\
&= \int \sin^4(\theta)\sin(\theta)d\theta \\
&= \int \left(1-\cos^2(\theta)\right)^2 \sin(\theta)d\theta \\
&= -\int (1-v^2)^2 dv \qquad v = \cos(\theta) \quad dv = -\sin(\theta)d\theta \\
&= \int -v^4 + 2v^2 - 1dv \\
&= \frac{-1}{5}v^5 + \frac{2}{3}v^3 - v + C \\
&= \frac{-1}{5}\cos^5(\theta) + \frac{2}{3}\cos^3(\theta) - \cos(\theta) + C \\
&= \frac{-1}{5}\left(1-\ln^2(t)\right)^{\frac{5}{2}} \\
&+ \frac{2}{3}\left(1-\ln^2(t)\right)^{\frac{3}{2}} - \sqrt{1-\ln^2(t)} + C
\end{aligned}$$

where in the last line we have used the fact that $\theta =$ arcsin(ln(t) and the fact that $\cos(\arcsin(y)) = \sqrt{1-y^2}$.

11. (trigsub:lnsec)
Compute $\int \frac{1}{t\ln(t)\sqrt{\ln^2(t)-1}}dt$.

Solution:

$$\begin{aligned}
\int \frac{1}{t\ln(t)\sqrt{\ln^2(t)-1}}dt &= \int \frac{1}{u\sqrt{u^2-1}}du && u=\ln(t) \quad du=\frac{1}{t}dt \\
&= \int \frac{\sec(\theta)\tan(\theta)}{\sec(\theta)\sqrt{\sec^2(\theta)-1}}d\theta && u=\sec(\theta) \quad du=\sec(\theta)\tan(\theta)d\theta \\
&= \int d\theta \\
&= \theta + C \\
&= \operatorname{arcsec}(u) + C \\
&= \operatorname{arcsec}(\ln(t)) + C
\end{aligned}$$

12. (trigsub:exp)
    Compute $\int \frac{dx}{\sqrt{1-e^{2x}}}$.

    Solution:

$$\begin{aligned}
\int \frac{dx}{\sqrt{1-e^{2x}}} &= \int \frac{du}{u\sqrt{1-u^2}} && u=e^x \quad \frac{du}{u}=dx \\
&= \int \frac{\cos(\theta)}{\sin(\theta)\sqrt{1-\sin^2(\theta)}} && u=\sin(\theta) \quad du=\cos(\theta)d\theta \\
&= \int \csc(\theta)d\theta \\
&= -\ln|\csc(\theta)+\cot(\theta)| + C \\
&= -\ln|\csc(\arcsin(u))+\cot(\arcsin(u))| + C \\
&= -\ln|\frac{1}{u}+\frac{\sqrt{1-u^2}}{u}| + C \\
&= \ln|\frac{e^x}{1+\sqrt{1-e^{2x}}}| + C \\
&= x - \ln(1+\sqrt{1-e^{2x}}) + C
\end{aligned}$$

13. (trigsub:arctansub)
    Compute $\int \frac{1}{(1+x^2)\sqrt{\arctan^2(x)-1}}dx$.

Solution:

$$
\begin{aligned}
&\int \frac{1}{(1+x^2)\sqrt{\arctan^2(x)-1}}dx \\
&=\int \frac{1}{\sqrt{u^2-1}}du && u=\arctan(x) \quad du=\frac{1}{1+x^2}dx \\
&=\int \frac{\sec(\theta)\tan(\theta)}{\sqrt{\sec^2(\theta)-1}}d\theta && u=\sec(\theta) \quad du=\sec(\theta)\tan(\theta)d\theta \\
&=\int \sec(\theta)d\theta \\
&=\ln|\sec(\theta)+\tan(\theta)|+C \\
&=\ln|u+\sqrt{u^2-1}|+C && \text{arcsec}(u)=\theta \\
&=\ln|\arctan(x)+\sqrt{\arctan^2(x)-1}|+C
\end{aligned}
$$

14. (trigsub:sqrtsub)
Compute $\int \frac{1}{\sqrt{x}+x^{\frac{3}{2}}}dx$.

Solution:

$$
\begin{aligned}
\int \frac{1}{\sqrt{x}+x^{\frac{3}{2}}} &= \int \frac{2udu}{u(1+u^2)} \qquad u=\sqrt{x} \quad \underbrace{du=\frac{1}{2\sqrt{x}}dx}_{\text{2udu = dx}} \\
&= 2\int \frac{du}{1+u^2} \\
&= 2\arctan(u) \\
&= 2\arctan(\sqrt{x})
\end{aligned}
$$