1. (diffeq:sep1)
   Find a solution to the initial value problem

$$\begin{aligned}\frac{dy}{dx} &= e^y x^3 \\ y(0) &= 0\end{aligned}$$

   Solution: In what follows, the value of the constant of integration may change from line to line.

$$\begin{aligned}\frac{dy}{dx} &= e^y x^3 \\ e^{-y} dy &= x^3 dx \\ \int e^{-y} dy &= \int x^3 dx \\ -e^{-y} &= \frac{1}{4}x^4 + C \\ e^{-y} &= -\frac{1}{4}x^4 + C \\ y &= -\ln(C - \frac{1}{4}x^4)\end{aligned}$$

   Substituting the initial condition $0 = y(0) = -\ln(C)$, we find that $C = 1$ and $y(x) = -\ln(1 - \frac{1}{4}x^4)$

2. (diffeq:sep2)
   Find a solution to the initial value problem

$$\begin{aligned}\frac{dy}{dx} &= (1 + y^2)e^x \\ y(0) &= 0\end{aligned}$$

   Solution:

$$\begin{aligned}\frac{dy}{dx} &= (1 + y^2)e^x \\ \frac{dy}{1 + y^2} &= e^x dx \\ \int \frac{dy}{1 + y^2} &= \int e^x dx \\ \arctan(y) &= e^x + C \\ y &= \tan(e^x + C)\end{aligned}$$

Substituting in the initial condition, we find that $0 = Y(0) = \tan(1 + C)$. A possible choice of C is $C = -1$. Our final answer is then $y(x) = \tan(e^x - 1)$.

3. (diffeq:sep3)
   Find a solution to the initial value problem

$$\begin{aligned} \frac{dy}{dx} &= y\sqrt{y^2-1}\cos(x) \\ y(0) &= 1 \end{aligned}$$

Solution: First, we can observe that one solution to this problem is given by $y(x) = 1$.

We can find another solution by separating variables.

$$\begin{aligned} \frac{dy}{dx} &= y\sqrt{y^2-1}\cos(x) \\ \frac{dy}{y\sqrt{y^2-1}} &= \cos(x)dx \\ \int \frac{dy}{y\sqrt{y^2-1}} &= \int \cos(x)dx \\ \operatorname{arcsec}(y) &= \sin(x) + C \\ y &= \sec(\sin(x) + C) \end{aligned}$$

Substituting in the initial condition $y(0) = 1$ we find that

$$1 = y(0) = \sec(C)$$

So we may take, for example, $C = 0$. Our final solution is then either of $y(x) = 1$ or $y(x) = \sec(\sin(x))$.

4. (diffeq:sep4)
   Find the general solution to the differential equation

$$\frac{dy}{dx} = x^2 + y^2x^2$$

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= x^2 + y^2x^2 \\
\frac{dy}{dx} &= x^2(1+y^2) \\
\frac{dy}{1+y^2} &= x^2 dx \\
\int \frac{dy}{1+y^2} &= \int x^2 dx \\
\arctan(y) &= \frac{x^3}{3} + C \\
y(x) &= \tan\left(\frac{x^3}{3} + C\right)
\end{aligned}$$

.

5. (diffeq:sep5)
   Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{e^y\sqrt{1-x^2}}$$

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{e^y\sqrt{1-x^2}} \\
e^y dy &= \frac{dx}{\sqrt{1-x^2}} \\
\int e^y dy &= \int \frac{dx}{\sqrt{1-x^2}} \\
e^y &= \arcsin(x) + C \\
y &= \ln\left(\arcsin(x) + C\right)
\end{aligned}$$

6. (diffeq:sep6)
   Find the general solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{e^y(1+x^2)}$$

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{e^y(1+x^2)} \\
e^y dy &= \frac{dx}{(1+x^2)} \\
\int e^y dy &= \int \frac{dx}{(1+x^2)} \\
e^y &= \arctan(x) + C \\
y &= \ln\left(\arctan(x) + C\right)
\end{aligned}$$

7. (diffeq:sep7)
   Find a solution to the initial value problem

$$\begin{aligned}
\frac{dy}{dx} &= \sqrt{1-y^2}\sec^2(x) \\
y(0) &= 0
\end{aligned}$$

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \sqrt{1-y^2}\sec^2(x) \\
\int \frac{1}{\sqrt{1-y^2}} dy &= \int \sec^2(x) dx \\
\arcsin(y) &= \tan(x) + C \\
y(x) &= \sin(\tan(x) + C)
\end{aligned}$$

Using the initial condition, we find that

$$0 = \sin(0 + C)$$

so $C = 0$ gives a solution. Our final answer is then $\sin(\tan(x))$.

8. (diffeq:fol1)
   Find the general solution to the differential equation (for $x \neq 0$)

$$x\frac{dy}{dx} = -y + x$$

Solution: We rewrite the equation as

$$x\frac{dy}{dx} + y = x$$

and observe that this equation is already in the form

$$\frac{d(xy)}{dx} = x$$

which is separable. We solve

$$\begin{aligned}
\frac{d(xy)}{dx} &= x \\
\int d(xy) &= \int x dx \\
xy &= \frac{1}{2}x^2 + C \\
y(x) &= \frac{1}{2}x + \frac{C}{x}
\end{aligned}$$

9. (diffeq:fol2)
Find the general solution to the differential equation

$$\frac{1}{2x}\frac{dy}{dx} = y + e^{x^2}$$

Solution: We begin by writing the problem in standard form as

$$\frac{dy}{dx} - 2xy = 2xe^{x^2}$$

The integrating factor for this problem is $m(x) = e^{\int -2x dx} = e^{-x^2}$. If we multiply through by e^{-x^2}, then the equation becomes separable and we can find the general solution directly directly.

$$\begin{aligned}
e^{-x^2}\frac{dy}{dx} - 2xe^{-x^2}y &= 2x \\
\frac{d(e^{-x^2}y)}{dx} &= 2x \\
\int d(e^{-x^2}y) &= \int 2x dx \\
e^{-x^2}y &= x^2 + C \\
y(x) &= x^2e^{x^2} + Ce^{x^2}
\end{aligned}$$

10. (diffeq:fol3)
Find a solution to the initial value problem

$$x\frac{dy}{dx} + 2y = \frac{\cos(x)}{x}$$
$$y(\pi) = 1$$

Solution: We being by writing the differential equation in standard form as

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos(x)}{x^2}$$

The integrating factor for this problem is $m(x) = e^{\int \frac{2}{x}dx} = e^{2\ln(x)} = x^2$. Multiplying through by x^2 converts this problem to

$$\begin{aligned} x^2\frac{dy}{dx} + 2xy &= \cos(x) \\ \frac{d(x^2y)}{dx} &= \cos(x) \\ \int d(x^2y) &= \int \cos(x)dx \\ x^2y &= \sin(x) + C \\ y(x) &= \frac{\sin(x)}{x^2} + \frac{C}{x^2} \end{aligned}$$

We substitute in the initial condition $y(\pi) = 1$ to find that

$$y(\pi) = \underbrace{\frac{\sin(\pi)}{\pi^2}}_{0} + \frac{C}{\pi^2}$$

so $C = \pi^2$ and $y(x) = \frac{\sin(x)}{x^2} + \frac{\pi^2}{x^2}$.

11. (diffeq:fol4)
Find a solution to the initial value problem

$$\cos(x)\frac{dy}{dx} = 1 - \sin(x)y$$
$$y(0) = 1$$

Solution: We begin by writing the equation in standard form

$$\frac{dy}{dx} + \tan(x)y = \sec(x)$$

The integrating factor for this problem is $m(x) = e^{\int \tan(x)dx} = e^{-\ln(\cos(x))} = \sec(x)$. Multiplying through by $m(x)$ makes this equation separable.

$$\begin{aligned}
\frac{dy}{dx} + \tan(x)y &= \sec(x) \\
\sec(x)\frac{dy}{dx} + \sec(x)\tan(x)y &= \sec^2(x) \\
\frac{d(\sec(x)y)}{dx} &= \sec^2(x) \\
\int d(\sec(x)y) &= \int \sec^2(x)dx \\
\sec(x)y &= \tan(x) + C \\
y &= \sin(x) + C\cos(x)
\end{aligned}$$

Substituting in $y(0) = 1$ we find that $C = 1$ and $y(x) = \sin(x) + \cos(x)$.

12. (diffeq:fol5)
Find a solution to the initial value problem

$$\begin{aligned}
x\frac{dy}{dx} + 2y &= -\frac{\sin(x)}{x} \\
y(\frac{\pi}{2}) &= 1
\end{aligned}$$

Solution: We being by writing the differential equation in standard form as

$$\frac{dy}{dx} + \frac{2}{x}y = -\frac{\sin(x)}{x^2}$$

The integrating factor for this problem is $m(x) = e^{\int \frac{2}{x}dx} = e^{2\ln(x)} =$

x^2. Multiplying through by x^2 converts this problem to

$$\begin{aligned} x^2\frac{dy}{dx} + 2xy &= -\sin(x) \\ \frac{d(x^2y)}{dx} &= -\sin(x) \\ \int d(x^2y) &= -\int \sin(x)dx \\ x^2y &= \cos(x) + C \\ y(x) &= \frac{\cos(x)}{x^2} + \frac{C}{x^2} \end{aligned}$$

Susbtituting in the initial condition, we find that

$$1 = y(\frac{\pi}{2}) = \underbrace{\frac{\cos(\frac{\pi}{2})}{(\frac{\pi}{2})^2}}_{0} + \frac{C}{(\frac{\pi}{2})^2}$$

so that $y(x) = \frac{\cos(x)}{x^2} + \frac{\pi^2}{4}\frac{1}{x^2}$.

13. (diffeq:twoBranches)
Find a solution to the initial value problem

$$\begin{aligned} \frac{dy}{dx} &= (y-1)\frac{1}{x} \\ y(-1) &= 0 \end{aligned}$$

Solution:

$$\frac{dy}{dx} = (y-1)\frac{1}{x}$$

$$\begin{aligned} \frac{1}{y-1}\frac{dy}{dx} &= \frac{1}{x} \\ \frac{1}{y-1}dy &= \frac{1}{x}dx \\ \int \frac{1}{y-1}dy &= \int \frac{1}{x}dx \\ \ln|y-1| &= ln|x| + c \\ y-1 &= \pm|x|e^c \\ y &= 1 \pm |x|e^c \end{aligned}$$

We are working near -1, so $|x| = -x$. Plugging in $y(-1) = 0$,

$$0 = 1 \pm e^c \underbrace{(-(-1))}_{|-1|}$$

e^c is always positive, so we must have

$$0 = y(-1) = 1 - e^c$$

Thus $1 = e^c$ and we get as our final answer

$$\begin{aligned} y(x) &= 1 - e^c(-x) \\ y(x) &= 1 + x \end{aligned}$$

14. (diffeq:fol6)
Find the general solution to the differential equation

$$\cos(x)\frac{dy}{dx} = y + \sin(x) + 1$$

where we assume that $\frac{-\pi}{2} < x < \frac{\pi}{2}$.

Solution: We begin by writing the differential equation in standard form

$$\begin{aligned} \cos(x)\frac{dy}{dx} &= y + \sin(x) + 1 \\ \frac{dy}{dx} - \sec(x)y &= \tan(x) + \sec(x) \end{aligned}$$

The integrating factor is $m(x) = e^{\int -\sec(x)dx} = e^{-\ln|\sec(x)+\tan(x)|}$. Recalling that we assumed $-\frac{\pi}{2} < x < \frac{\pi}{2}$, this is $\frac{1}{\sec(x)+\tan(x)}$. Multiplying through, we find that

$$\begin{aligned} \frac{d}{dx}\left(\frac{y}{\sec(x)+\tan(x)}\right) &= 1 \\ \int d\left(\frac{y}{\sec(x)+\tan(x)}\right) &= \int dx \\ \frac{y}{\sec(x)+\tan(x)} &= x + C \\ y(x) &= x\left(\sec(x)+\tan(x)\right) + C\left(\sec(x)+\tan(x)\right) \end{aligned}$$

15. (diffeq:fol7)

Find the general solution to the differential equation

$$\frac{dy}{dx} + \frac{1}{x^2-1}y = \frac{3}{2}\sqrt{1+x}$$

where we assume that $x > 1$.

Solution: The equation is already in standard form, so we can solve for the integrating factor $m(x) = e^{\int \frac{1}{x^2-1}dx} = e^{\ln\sqrt{|\frac{x-1}{x+1}|}} = \sqrt{\frac{x-1}{x+1}}$ for $x > 1$. Multiplying through, we find that

$$\begin{aligned}
\sqrt{\frac{x-1}{x+1}}\frac{dy}{dx} + \frac{1}{x^2-1}\sqrt{\frac{x-1}{x+1}}y &= \frac{3}{2}\sqrt{\frac{x-1}{x+1}}\sqrt{1+x} \\
\frac{d}{dx}y\sqrt{\frac{x-1}{x+1}} &= \frac{3}{2}\sqrt{x-1} \\
\int d\left(y\sqrt{\frac{x-1}{x+1}}\right) &= \frac{3}{2}\int \sqrt{x-1}dx \\
y\sqrt{\frac{x-1}{x+1}} &= (x-1)^{\frac{3}{2}} + C \\
y(x) &= (x-1)\sqrt{x+1} + C\sqrt{\frac{x+1}{x-1}}
\end{aligned}$$

16. (diffeq:fol8)

Find a solution to the initial value problem

$$\begin{aligned}
x^2\frac{dy}{dx} - 2xy &= x^4\cos(x) \\
y(\pi) &= 1
\end{aligned}$$

Solution: We begin by putting the differential equation into standard form

$$\begin{aligned}
x^2\frac{dy}{dx} - 2xy &= x^4\cos(x) \\
\frac{dy}{dx} - \frac{2}{x}y &= x^2\cos(x)
\end{aligned}$$

The integrating factor is then $m(x) = e^{-2\int \frac{1}{x}dx} = e^{-\ln|x|} = \frac{1}{x^2}$. We multiply through to find that

$$\begin{aligned}
\frac{dy}{dx} - \frac{2}{x}y &= x^2 \cos(x) \\
\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y &= \cos(x) \\
\frac{d}{dx}\left(\frac{1}{x^2}y\right) &= \cos(x) \\
\int d\left(\frac{1}{x^2}y\right) &= \int \cos(x)dx \\
\frac{1}{x^2}y &= \sin(x) + C \\
y(x) &= x^2\sin(x) + Cx^2
\end{aligned}$$

Substituting in the initial condition, we find that

$$1 = y(\pi) = \pi^2 \sin(\pi) + C\pi^2 = C\pi^2$$

so that $C = \frac{1}{\pi^2}$ and $y(x) = x^2\sin(x) + \frac{x^2}{\pi^2}$.

17. (diffeq:fol9)
Find a solution to the initial value problem

$$\begin{aligned}
(1+x^2)\arctan(x)\frac{dy}{dx} &= (1+x^2)e^x - y \\
y(\tan(1)) &= e^{\tan(1)}
\end{aligned}$$

Solution: We begin by putting the equation into standard form

$$\frac{dy}{dx} + \frac{1}{(1+x^2)\arctan(x)}y = \frac{e^x}{\arctan(x)}$$

Notice that

$$\begin{aligned}
\int \frac{1}{\arctan(x)(1+x^2)}dx &= \int \frac{1}{u}du \qquad\qquad u = \arctan(x) \quad du = \frac{1}{1+x^2}dx \\
&= \ln|u| + C \\
&= \ln|\arctan(x)| + C
\end{aligned}$$

We are working near $\tan(1) > 0$ so we may assume that $\arctan(x) > 0$.

$$
\begin{aligned}
m(x) &= e^{\int \frac{1}{\arctan(x)(1+x^2)} dx} \\
&= e^{\ln(\arctan(x))} \\
&= \arctan(x)
\end{aligned}
$$

Multiplying through by the integrating factor, we find that

$$
\begin{aligned}
\frac{d}{dx}\left(\arctan(x) y\right) &= e^x \\
\int d\left(\arctan(x) y\right) &= \int e^x dx \\
\arctan(x) y &= e^x + C \\
y(x) &= \frac{e^x}{\arctan(x)} + \frac{C}{\arctan(x)}
\end{aligned}
$$

Substituting in the initial condition

$$
\begin{aligned}
e^{\tan(1)} = y(\tan(1)) &= \frac{e^{\tan(1)}}{\arctan(\tan(1))} + \frac{C}{\arctan(\tan(1))} \\
&= e^{\tan(1)} + C
\end{aligned}
$$

so $C = 0$ and $y(x) = \frac{e^x}{\arctan(x)}$.

18. (diffeq:fol10)
Find a particular solution to the differential equation

$$
\begin{aligned}
\frac{1+x^3}{3x^2}\frac{dy}{dx} &= 1 - y(x) \\
y(1) &= 2
\end{aligned}
$$

Solution: We first put the equation into standard form

$$
\frac{dy}{dx} + \frac{3x^2}{1+x^3} y = \frac{3x^2}{1+x^3}
$$

The integrating factor for this problem is $m(x) = (1 + x^3)$ and the solution is

$$
\begin{aligned}
y(x) &= \frac{1}{1+x^3}\left(\int 3x^2 dx\right) \\
&= \frac{1}{1+x^3}\left(x^3 + C\right)
\end{aligned}
$$

The initial condition $y(1) = 2$ gives that $\frac{1+C}{2} = 2$, so $C = 3$ and $y(x) = \frac{3+x^3}{1+x^3}$.