Representation Theory

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1 Representations of Finite Groups

1.1 Definitions

Definition. A *representation* of a finite group G on a finite-dimensional complex vector space V is a homomorphism $\rho : G \to GL(V)$. When a vector space V is equipped with a representation ρ of G, we call V a *G-module*.

Definition. The *degree* of a representation $\rho : G \to GL(V)$ is the dimension of V.

Definition. A *G-linear map* is a linear map $\varphi : V \to W$ between representations V and W of G such that

$$\varphi(g \cdot v) = g \cdot \varphi(v) \tag{1}$$

for all $v \in V$. (Diagram.) This can also be called just a map between the representations V and W. We can also define kernel, image, and cokernel of φ which will be G-modules (How?).

Definition. A *subrepresentation* is a vector subspace W of a representation V of G such that W is invariant under G. That is, for all $g \in G$ and for all $w_1 \in W$ there is some $w_2 \in W$ such that

$$g \cdot w_1 = w_2. \tag{2}$$

In other words, for all $g \in G$, $\rho(g)$ restricts to a well defined linear map $\rho(g) : W \to W$.

Definition. A representation V is called *irreducible* if the only invariant subspace is the zero subspace. That is, V has no nontrivial subrepresentations.

Definition. The *direct sum* of representations V and W of G is defined by

$$g \cdot (v \oplus w) = gv \oplus gw. \tag{3}$$

Notice that both V and W are subrepresentations of $V \oplus W$.

Definition. The *tensor product* of representations V and W of G is defined by

$$g \cdot (v \otimes w) = gv \otimes gw. \tag{4}$$

Remark. Note that although the action of G in the direct sum and the tensor product representations "looks the same", it is not true that V and W are subrepresentations of $V \oplus W$. In the case of direct sum, we can identify V as a subspace of $V \otimes W$ as $\{(v, 0_W) : v \in V\}$ and this is G-invariant. In general, there is no such way to identify V as a subspace of $V \otimes W$.

Definition. The *tensor power* representation $V^{\otimes n}$ is defined by the tensor product of V with itself n times. The exterior power $\bigwedge^n(V)$ and symmetric power $\mathrm{Sym}^n(V)$ are subrepresentations of the tensor power. (Definitions?)

Definition. The *dual representation* of a representation V is also a representation on $V^* = \mathrm{Hom}(V, \mathbb{C})$. To define the dual representation, we would like that the representations ρ and ρ^* on V and V^* respect the natural pairing of V and V^* i.e. it preserves the action of V^* on V. Explicitly, we require that for all $v \in V$ and for all $w^* \in V^*$

$$\langle \rho^*(g)(w^*), \rho(g)(v) \rangle = \langle w^*, v \rangle.$$

To satisfy this, we must set $\rho^*(g) = \rho(g^{-1})^T$. Then

$$\langle \rho^*(g)(w^*), \rho(g)(v) \rangle = \langle \rho(g^{-1})^T(w^*), \rho(g)(v) \rangle = \langle w^*, \rho(g^{-1})\rho(g)v \rangle = \langle w^*, v \rangle$$

as desired.

Definition. For representations V and W, we also have the representation $\mathrm{Hom}(V, W)$, the space of linear maps $\varphi : V \to W$ which we can also identify as $\mathrm{Hom}(V, W) = V^* \otimes W$. Let's say we want to find the action on some map φ. From the identification with the tensor product, we can write a rank one map φ as $\varphi = (v^* \otimes w)$ for some $v^* \in V^*$ and some $w \in W$. Note that Using the definition of tensor product representation, we have

$$g \cdot (v^* \otimes w) = g \cdot v^* \otimes g \cdot w.$$

How does this act on a vector $v \in V$? We see

$$(g \cdot v^* \otimes g \cdot w)(v) = (g \cdot w)((g \cdot v^*)()) = (g \cdot w)(v^*)(g^{-1} \cdot v)$$

or in other words, the representation is given by

$$(g\varphi)(v) = g\left(\varphi(g^{-1}v)\right) = g\varphi(g^{-1}v).$$

We have only shown this to hold for rank one maps, but the definition extends to maps of higher rank by linearity. (Diagram, improve explanation.)

Exercise. Verify that in general the vector space of G-linear maps between two representations V and W of G is just the subspace $\mathrm{Hom}(V, W)^G$ of elements of $\mathrm{Hom}(V, W)$ fixed under the action of G. This is often denoted $\mathrm{Hom}_G(V, W)$.

Solution. We seek to show that $\operatorname{Hom}(V, W)^G$ consists of exactly the G-linear maps from V to W.

First let φ be a G-linear map from V to W. We seek to show that φ is fixed under the action of G. Recall that $(g \cdot \varphi)(v) = g\varphi(g^{-1}v)$. Since φ is G-linear we can take the g inside and obtain $g\varphi(g^{-1}v) = \varphi(gg^{-1}v) = \varphi(v)$. Thus, the map φ is fixed by the action of G.

Conversely, let φ be a map which is fixed under the action of G. That is, $g \cdot \varphi = \varphi$ or more explicitly, for all $v \in V$ we have $(g \cdot \varphi)(v) = \varphi(v)$ i.e. $g\varphi(g^{-1}v) = \varphi(v)$. We see immediately that $\varphi(g^{-1}v) = g^{-1}\varphi(g)$ so φ is G-linear. ■

Definition. Let X be a finite set and G acts on the left on X by a homomorphism $G \to \operatorname{Aut}(X)$. Then there is an associated *permutation representation*. Let V be the vector space with basis $\{e_x : x \in X\}$. Then the permutation representation of G on V by $ge_x = e_{gx}$ and extend by linearity.

Definition. The *regular representation* denoted R_G of G is given by the permutation representation of G on itself with the action given by the group product. That is, G acts on the space V with basis $\{e_g : g \in G\}$ by $ge_h = e_{gh}$ extended linearly to all of V. Equivalently, one can define R_G as the space of complex-valued functions on G where G acts on a function α by $(g\alpha)(h) = \alpha(g^{-1}h)$.

1.2 Complete Reducibility; Schur's Lemma

We have seen many ways of making new representations from smaller representations. The simplest of these is the direct sum representations. Our goal is to classify all (complex, finite-dimensional) representations of a given group. This is a hard problem as there are an infinite number of them. To make this problem easier to approach, we will show that every representation decomposes into a direct sum of a finite set of smaller representations which do not decompose further i.e. indecomposable representations.

Before we can move on to classifying these indecomposable representations, we have several tasks to complete. First, we must show that every representation does in fact decompose as a direct sum of indecomposable representations. Then, we would like to show that such a decomposition is unique.

Proposition. *Let V be a representation of a finite group G and let W be an invariant subspace of V. Then there is a complementary subspace W' of W such that $V = W \oplus W'$.*

Proof. To proceed, we will define a (positive-definite) Hermitian inner product H on V that is preserved for each $g \in G$. Let H_0 be an arbitrary Hermitian inner product on V. Then define

$$H(v, w) = \sum_{h \in G} H_0(hv, hw).$$

This is preserved by any $g \in G$ since every term in $H(v, w)$ will have a corresponding equal term in $H(gv, gw)$. Then the perpendicular subspace $W^{\perp}$ is complementary to W in V. Further, W' is also G-invariant. Suppose toward a contradiction that W' is not G-invariant. Then there is some $u \in W'$ and some $g \in G$ such that $gu \notin W'$. Then $gu = w + w'$ for some $w \in W$ and some $w' \in W'$ where $w \neq 0_W$. Then $g^{-1}gu = g^{-1}w + g^{-1}w'$ which implies $u - w' = g^{-1}w$. Since $u, w' \in W'$ this implies $g^{-1}w$ is in W' which is a contradiction since W is G-invariant. Thus, W' is also G-invariant.

To be particularly thorough, we would need to argue that not only does V decompose as a vector space as $W \oplus W'$ but also that $V = W \oplus W'$ as a representation. This follows immediately from the fact that both W and W' inherit their representation structure from V. □

Corollary. *Any representation is the direct sum of irreducible representations.*

Proposition. *If V and W are irreducible representations of G and φ is a G-module homomorphism, then*

1. *Either φ is an isomorphism or $\varphi = 0$,*
2. *If $V = W$, then $\varphi = \lambda I$ for some $\lambda \in \mathbb{C}$, where I is the identity map.*

Proof. To prove the first statement, notice that $\mathrm{Ker}(\varphi)$ is G-invariant. Explicitly, since φ is G-linear, we have that for any $v \in \mathrm{Ker}(\varphi)$, $\varphi(gv) = g\varphi(v) = g0 = 0$. Similarly, $\mathrm{Im}(\varphi)$ is G-invariant. Let $w \in \mathrm{Im}(\varphi)$. Then there is some $v \in V$ such that $\varphi(v) = w$. Then $gw = g\varphi(v) = \varphi(gv) \in \mathrm{Im}(\varphi)$. If $\varphi \neq 0$ and is not an isomorphism, then either $\mathrm{Ker}(\varphi)$ or $\mathrm{Im}(\varphi)$ is nontrivial, which gives a subrepresentation of V or W, respectively.

For the second claim, since $\mathbb{C}$ is algebraically closed, it follows that φ must have an eigenvalue. In particular, this implies that $\varphi - \lambda I$ has nonzero kernel. Thus, from the first statement, we must have that $\varphi - \lambda I$ is the zero map which implies that $\varphi = \lambda I$. □

Proposition. *For any representation V of a finite group G, there is a decomposition*

$$V = V_1^{\oplus a_i} \oplus \cdots \oplus V_k^{\oplus a_k},$$

where the V_i are distinct irreducible representations. Further, the decomposition of V into a direct sum of the k factors is unique, as are the V_i that occur and their multiplicities a_i.

Proof. Suppose that V has another decomposition

$$V = W_1^{\oplus b_1} \oplus \cdots \oplus W_m^{\oplus b_m}.$$

Consider the identity map I_i on V restricted to one of the V_i. Since each V_i and each W_j is irreducible, then for each W_j either $\mathrm{Im}(I_i) = W_j$ or $\mathrm{Im}(I_i) \cap W_j = \{0\}$. Since I_i is not the zero map, it follows that there is some W_j for which $\mathrm{Im}(I_i) = W_j$ which implies $V_i \cong W_j$ and thus $a_i = b_j$. Hence, the decomposition is unique. □

Now, we can see that if we are able to classify all the irreducible representations of a group, we will in principle have successfully classified all possible finite dimensional representations. However, we will still have to develop techniques for how to decompose a given representation into irreducible representations.

1.3 Examples: Abelian Groups; S_3

First we will consider irreducible representations of abelian groups. If G is abelian and V is an irreducible representation of G, then each $g \in V$ is a G-linear map which implies that $gv = \lambda v$. Thus, every subspace of V is fixed, so V must be 1-dimensional.

We now consider the simplest nontrivial abelian group S_3. Immediately, there are two 1-dimensional representations. The first is the trivial representation where $gv = v$. The other is the alternating representation where $gv = \mathrm{sgn}(g)v = \pm v$. Further, there is a representation on $\mathbb{C}^3$ given by permutation of the basis. That is, if $\{e_1, e_2, e_3\}$ is a basis for $\mathbb{C}^3$ then $ge_i = e_{g(i)}$. However, we see that this is not irreducible. There is an invariant subspace spanned by $(1,1,1)$ which is isomorphic to the trivial representation. Then the complementary subspace is given by

$$V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + z_2 + z_3 = 0\}.$$

This has basis $\{(1,1,-2),(1,-2,1)\}$. Clearly this is irreducible since no subspace is left fixed by G. This is called the *standard representation*.

How can we tell if these three given representations in fact constitute all possible irreducible representations? Let W be an arbitrary representation of S_3. First, we will investigate the action of the abelian subgroup $\mathbb{Z}_3 \subset S_3$ generated by a 3-cycle τ. Then every vector $w \in W$ is an eigenvector of τ since $\tau^3 w = ew = w$ (Check your logic here.). In particular, this means that W is spanned by eigenevectors v_i of τ with eigenvalues which are powers of a primitive cube root of unity $\omega = e^{2\pi i/3}$. Thus we have the decomposition

$$W = \bigoplus V_i$$

where $V_i = \mathbb{C}v_i$ and $\tau v_i = \omega^{\alpha_i} v_i$.

Next we wish to investigate how σ acts on the V_i where sigma is a transposition, so σ, τ generate S_3 with relation $\sigma\tau\sigma = \tau^2$. What happens if we act σ on $\tau(v_i)$? Using the relation, we have

$$\tau\sigma v_i = \sigma\tau^2 v_i = \sigma\omega^{2\alpha_i} v_i = \omega^{2\alpha_i}\sigma v_i.$$

Thus, if v_i is an eigenvector of τ with eigenvalue ω^{α_i} then σv is also an eigenvector of τ with eigenvalue $\omega^{2\alpha_i}$.

Now, given an eigenvector of τ, we can determine which representation it belongs to by looking at the action of σ. Suppose v has is an eigenvector of τ with eigenvalue $\omega^{\alpha_i} \neq 1$. Then σv is also an eigenvector of τ with eigenvalue $\omega^{2\alpha_i}$. Thus, $v, \sigma v$ span a two dimensional subspace of W invariant under S_3. This is isomorphic to the standard representation.

Now suppose v is an eigenvector of τ with eigenvalue 1. Then, either σv is dependent on v or independent. If it is dependent, either $\sigma v = v$ in which case v spans an invariant subspace isomorphic to the trivial representation, or $\sigma v = -v$ in which case v spans an invariant subspace isomorphic to the alternating representation. If v and σv are independent, then $v + \sigma v$ spans the trivial representation and $v - \sigma v$ spans the alternating representation.

Thus, we have shown that any representation of S_3 can be constructed from the trivial, alternating, and standard representations, and that we can give explicitly the multiplicities of each representation by examining the eigenvectors and eigenvalues under σ and τ. In particular, W decomposes into trivial rep U, alternating rep U', and standard rep V as

$$W = U^{\oplus a} \oplus U'^{\oplus v} \oplus V^{\oplus c}$$

where c is the number of independent eigenvectors of τ with eigenvalue ω, $a + c$ is the number of independent eigenvectors of σ with eigenvalue 1, and $b + c$ is the number of independent eigenvectors of σ with eigenvalue -1. (See Exercise 1.10 for more detail.)

Proposition. *The number of irreducible representations is less than or equal to the number of normal subgroups of G, since for any irreducible representation ρ of G, we have that $\mathrm{Ker}(\rho)$ is a normal subgroup of G.*

2 Character Theory

2.1 Characters

Thus, we have seen that knowing the eigenvalues and eigenvectors of a representation gives a lot of information to use towards classifying the representation. This leads us to the idea of character theory, which will make this process a bit more systematic (and technical).

Note that if we know the sum of the k-th powers (for all k, not just a particular k) of eigenvalues of an element $g \in G$ then we know the eigenvalues (For more info, see Newton's identities.) Thus we have the following definition.

Definition. Let V be a representation of a finite group G. Then the *character* of V is the complex-valued function on group elements defined by

$$\chi_V(g) = \mathrm{Tr}(g|_V),$$

the trace of g on V.

Notice that χ_V is constant on conjugacy classes of by the invariance of trace under cyclic permutations of the arguments. That is,

$$\chi_V(g) = \mathrm{Tr}(hgh^{-1}) = \mathrm{Tr}(hh^{-1}g) = \mathrm{Tr}(g).$$

Thus, the character is a *class function* (defined in the previous chapter). In particular, notice that $\chi_V(e) = \dim(V)$.