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Convex conditions for robust stabilization of uncertain switched systems with guaranteed minimum and mode-dependent dwell-time

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ABSTRACT

Alternative conditions for establishing dwell-time stability properties of linear switched systems are considered. Unlike the hybrid conditions derived in Geromel and Colaneri (2006), the considered ones are affine in the system matrices, allowing then for the consideration of uncertain switched systems with time-varying uncertainties. The low number of decision variables moreover permits to easily derive convex stabilization conditions using a specific class of state-feedback control laws. The resulting conditions are enforced using sum of squares programming which are shown to be less complex numerically that approaches based on piecewise linear functions or looped-functionals previously considered in the literature. The sums of squares conditions are also proven to (1) approximate arbitrarily well the conditions of Geromel and Colaneri (2006); and (2) be invariant with respect to time-scaling, emphasizing that the complexity of the approach does not depend on the size of the dwell-time. Several comparative examples illustrate the efficiency of the approach.

1. Introduction

Switched systems [1–9] are very flexible modeling tools appearing in several fields such as switching control laws [4,10], networked control systems [11], electrical devices/circuits [12,13], and congestion modeling and control in networks [14–16]. When switching between a family of asymptotically stable subsystems holds in a way that is independent of the state of system, stability under minimum and average dwell-times have been shown to be relevant concepts of stability [17,1] for which certain criteria have been proposed. Hybrid conditions, consisting of joint continuous-time and discrete-time conditions, for characterizing minimum dwell-time have been recently proposed in [5] where it is shown that the use of quadratic Lyapunov functions may lead to better results than previous ones. Even more importantly, homogeneous Lyapunov functions have been proved to be able to formulate non-conservative conditions for minimum dwell-time analysis [18,19]. However, extending these important results to uncertain systems, time-varying systems and control design is quite difficult due to the presence of exponential terms that are not applicable to time-varying systems and would create strongly nonconvex terms in the design conditions.

Looped-functionals [20–23], on the other hand, are a particular class of indefinite functionals (i.e. not required to be positive definite) satisfying a looping-condition—a particular boundary algebraic condition. They have been shown to yield stability conditions that are less conservative than those obtained using positive definite Lyapunov functionals; see e.g. [24,21,20]. They have also been shown to provide an alternative framework for dwell-time analysis of switched systems which remains compatible with uncertain switched systems, time-varying subsystems and, potentially, nonlinear switched systems. Tractable conditions for robust stability analysis under mode-dependent dwell-time, a stability concept permitting the instability of the subsystems [23], can be obtained as well using such a framework. However, the structure of the conditions and the large number of decision variables make the derivation of computationally attractive synthesis conditions a hardly possible task.

The approach proposed in this paper is based on clock-dependent Lyapunov functions, a class of Lyapunov functions explicitly depending on the time elapsed since the last discrete-time event (i.e. a clock); see e.g. [9,25–28]. They have been applied to switched systems [9,26,28], sampled-data systems [9,25,27] and impulsive systems [9,27]. The advantages of clock-dependent Lyapunov functions lie in the absence of any exponential term, facilitating the derivation of tractable conditions for establishing the stability of uncertain hybrid systems. The advantages over the use of looped-functionals are a lower computational complexity and

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the possibility of deriving convex conditions for the control of hybrid systems via state-feedback.

The contribution of the paper is manifold. First, alternative minimum dwell-time stability conditions, rigorously shown to be equivalent to those obtained in [5], are provided. The advantage of the proposed conditions lies in their affine dependence in the system matrices, permitting then their extension to uncertain systems with time-varying subsystems, as opposed to the conditions of [5] that are only applicable to LTI subsystems. The price to pay, however, is the characterization of stability with minimum dwell-time using infinite-dimensional convex semidefinite programs, which may be hard to solve when the considered system is of large dimension. A piecewise linear approximation of these conditions have been proposed in [26] and results in a finite-number of linear matrix inequalities. However, as it will be emphasized later, the discretization order often needs to be large in order to obtain accurate results. In contrast, the sum of squares approach [29,30] considered in this paper yields more accurate results while being faster and computationally less expensive than the piecewise-linear approach [26] and the approach based on looped-functionals [23]. It is also proven that, for the class dwell-time conditions we consider, the sum of squares relaxation is asymptotically exact, meaning that by choosing a sufficiently large polynomial order, the conditions based on sum of squares approximate arbitrarily well the conditions of [5]. A result proving the invariance of the sum of squares conditions is also proved and shows that the polynomial order is independent of the minimum dwell-time value and only depends on the matrices of the switched system.

Outline: The structure of the paper is as follows: in Section 2 preliminary definitions and results are given. Section 3 is devoted to minimum dwell-time stability analysis whereas Section 4 addresses stability under mode-dependent dwell-time. Results on the stabilization under minimum and mode-dependent dwell-time are derived in Section 5.

Notations: The sets of symmetric and positive definite matrices of dimension n are denoted by $\mathbb{S}^n$ and $\mathbb{S}^n_{\succ 0}$ respectively. Given two symmetric real matrices A and B, the inequalities $A \succ (\succeq) B$ mean that $A-B$ is positive (semi)definite. For any square matrix M, we define $\mathrm{Sym}[M] = M + M^{\intercal}$.

2. Preliminaries

2.1. System definition

From now on, the following class of linear switched system

$$\begin{aligned}\dot{x}(t) &= A_{\sigma(t)}x(t)\\ x(t_0) &= x_0\end{aligned} \tag{1}$$

are considered where $x, x_0 \in \mathbb{R}^n$ are the state of the system and the initial condition, respectively. The switching signal σ is defined as a left-continuous piecewise constant function $\sigma : [0,\infty) \to \{1,\ldots,N\}$. At some point, the matrices A_i of the subsystems will be uncertain and/or time-varying, this will be explicitly stated when this is the case. We also assume that the sequence of switching instants $\{t_1, t_2, \ldots\}$ is increasing and does not admit any accumulation point. Consequently, any Zeno motion is excluded.

2.2. Stability with periodic switching times

We start with a stability result under periodic switching that allows us to state the main ideas in a simple context. By periodic switching, it is meant here that switching times are periodic, i.e. $t_{k+1} = t_k + \bar{T}$, for some $\bar{T} > 0$. Note, however, that the sequence of subsystems is not necessarily periodic and, thus, periodic systems theory does not apply here. The following result will be shown to be directly involved in the derivation of the results on minimum dwell-time stability in the next section.

Theorem 1 (*Stability with Periodic Switching Times*)**.** *The following statements are equivalent:*

(a) *The quadratic form $V(x(t),\sigma(t)) = x(t)^{\intercal}P_{\sigma(t)}x(t)$, $P_i \in \mathbb{S}^n_{\succ 0}$, $i = 1,\ldots,N$, is a discrete-time Lyapunov function for the switched system (1) with $\bar{T}$-periodic switching times in the sense that the inequality*

$$V(x(t_{k+1}),\sigma(t_{k+1})) - V(x(t_k),\sigma(t_k)) \le -\mu\|x(t_k)\|_2^2 \tag{2}$$

holds for some $\mu > 0$, all $x(t_k) \in \mathbb{R}^n$ and all $k \in \mathbb{N}$.

(b) *There exist matrices $P_i \in \mathbb{S}^n_{\succ 0}$, $i = 1,\ldots,N$ such that the LMIs*

$$e^{A_i^{\intercal}\bar{T}}P_ie^{A_i\bar{T}} - P_j \prec 0 \tag{3}$$

hold for all $i,j = 1,\ldots,N$, $i \neq j$.

(c) *There exist differentiable matrix functions $R_i : [0,\bar{T}] \mapsto \mathbb{R}^n$, $R_i(0) \in \mathbb{S}^n_{\succ 0}$, $i = 1,\ldots,N$, and a scalar $\varepsilon > 0$ such that the LMIs*

$$A_i^{\intercal}R_i(\tau) + R_i(\tau)A_i - \dot{R}_i(\tau) \preceq 0 \tag{4}$$

and

$$R_i(\bar{T}) - R_j(0) + \varepsilon I \preceq 0 \tag{5}$$

hold for all $\tau \in [0,\bar{T}]$ and all $i,j = 1,\ldots,N$, $i \neq j$.

(d) *There exist differentiable matrix functions $S_i : [0,\bar{T}] \mapsto \mathbb{S}^n$, $S_i(\bar{T}) \in \mathbb{S}^n_{\succ 0}$, $i = 1,\ldots,N$, and a scalar $\varepsilon > 0$ such that the LMIs*

$$A_i^{\intercal}S_i(\tau) + S_i(\tau)A_i + \dot{S}_i(\tau) \preceq 0 \tag{6}$$

and

$$S_i(0) - S_j(\bar{T}) + \varepsilon I \preceq 0 \tag{7}$$

hold for all $\tau \in [0,\bar{T}]$ and all $i,j = 1,\ldots,N$, $i \neq j$.

Proof. *Proof of* (a) $\Leftrightarrow$ (b): Assume $\sigma(t_k) = j$ and $\sigma(t_k+\tau) = i$, $\tau \in (0,\bar{T}]$. Then, we have

$$\begin{aligned}&V(x(t_{k+1}),\sigma(t_{k+1})) - V(x(t_k),\sigma(t_k))\\ &\quad = x(t_k)^{\intercal}\left[e^{A_i^{\intercal}\bar{T}}P_ie^{A_i\bar{T}} - P_j\right]x(t_k)\end{aligned} \tag{8}$$

and there exists $\mu > 0$ such that (2) holds if and only if (3) holds. The proof is complete.

Proof of (c) $\Rightarrow$ (b): Assume (c) holds. Solving (4) for $R_i(\tau)$ yields [31]

$$R_i(\tau) \succeq e^{A_i^{\intercal}\tau}R_i(0)e^{A_i\tau} \tag{9}$$

and thus

$$e^{A_i^{\intercal}\bar{T}}R_i(0)e^{A_i\bar{T}} - R_i(\bar{T}) \preceq 0. \tag{10}$$

From (5), we have that $R_i(\bar{T}) \preceq R_j(0) - \varepsilon I$ and therefore, combining this with (10), we obtain

$$e^{A_i^{\intercal}\bar{T}}R_i(0)e^{A_i\bar{T}} - R_j(0) + \varepsilon I \preceq 0 \tag{11}$$

which implies in turn that (3) holds with $P_i = R_i(0)$.

Proof of (d) $\Rightarrow$ (b): The proof follows the same lines as the one above and is omitted. Note that, in this case, (3) holds with $P_i = S_i(\bar{T})$.

Proof of (b) $\Rightarrow$ (c): The idea of the proof is to show that, when there exists matrices $P_i \in \mathbb{S}^n_{\succ 0}$, $i = 1,\ldots,N$, such that the LMI (3) holds, then the LMIs (4)–(5) hold with the matrix-valued functions $R_i(\tau) = R_i^*(\tau)$ with $R_i^*(\tau) = e^{A_i^{\intercal}\tau}P_ie^{A_i\tau}$.

Computing then the derivative of $R_i^*(\tau)$ with respect to τ and noting that $A_i^{\intercal}e^{A_i^{\intercal}\tau}P_ie^{A_i\tau} = A_i^{\intercal}R_i^*(\tau)$, we obtain that $-\dot{R}^*(\tau) + A_i^{\intercal}R_i^*(\tau) + R_i^*(\tau)A_i = 0$ for all $\tau \in [0,\bar{T}]$; hence (4) holds. Noting now that the condition (3) is equivalent to saying that there exists a sufficiently small $\varepsilon > 0$ such that $e^{A_i^\intercal \bar{T}} P_i e^{A_i \bar{T}} - P_j \preceq -\varepsilon I$ allows us to conclude that, after substitution of $R_i^*(\tau)$ in (5), we have

$$R_i^*(\bar{T}) - R_j^*(0) + \varepsilon I = e^{A_i^\intercal \bar{T}} P_i e^{A_i \bar{T}} - P_j + \varepsilon I \preceq 0. \tag{12}$$

The proof is complete.

Proof of (c) $\Leftrightarrow$ (d): Assume (d) holds for some $S_i(\tau)$, it is immediate to see that $R_i(\tau) := S_i(\bar{T} - \tau)$ solves (4) and (5). Reverting the argument proves the equivalence. $\diamondsuit$

The advantages of the conditions of statements (c) and (d) over conditions of statement (b) are multiple. First of all, the conditions are convex in the system matrices A_i, allowing then for an immediate extension to the uncertain case. Secondly, the low number of decision matrices tends to suggest the possibility of deriving tractable synthesis conditions. Finally, conditions of statements (c) and (d) are more appealing from a computational perspective than the conditions obtained from looped-functionals [23] since the number of decision variables and constraints is smaller.

A peculiarity of the approach is that, by virtue of (9), one has to only impose $R_i(0)$ to be positive definite to obtain a positive definite $R(\tau)$ for $\tau \in [0, \bar{T}]$. The same remark holds for $S_i(\tau)$ and $S_i(\bar{T})$. This dramatically reduces the number of constraints that have to be considered.

2.3. Computational considerations

The compensation for these interesting properties is the consideration of infinite-dimensional feasibility problems which may be hard to solve. Fortunately, recently developed polynomial programming techniques [32], such as sum of squares programming [29], provide an adapted framework for solving such problems by restricting the matrix functions $R(\tau)$ and $S(\tau)$ to matrix-valued polynomial functions. The package SOSTOOLS [33] together with the semidefinite programming solver SeDuMi [34] supply the necessary material for solving such problems. Another approach consists of considering the Lyapunov matrices to be piecewise linear as in [26]. However, it will be shown later that this approach suffers from a high numerical complexity compared to the proposed one. Finally, it is very important to stress that while the necessity of the conditions is lost when approximating the matrix functions $R(\tau)$ or $S(\tau)$ by polynomials, we can still prove that the original LMI conditions can approached original arbitrarily well by choosing polynomials of sufficiently large order.

2.3.1. Sum of squares program

The following proposition states the sum of squares program associated with the conditions of Statement (c) of Theorem 1.

Proposition 2. *Let $\mu, \nu, \bar{T} > 0$ be given. Assume that the sum of squares program*

$$\begin{array}{rl} \textit{Find polynomials} & R_i, \Gamma_i : [0 : \bar{T}] \to \mathbb{S}^n, \\ \textit{such that} & R_i(\bar{T}) - \nu I_n \succeq 0 \\ & \Gamma_i(\tau) \text{ is SOS} \\ & \dot{R}_i(\tau) - A_i^\intercal R_i(\tau) - R_i(\tau) A_i \\ & -\Gamma_i(\tau)\tau(\bar{T} - \tau) \text{ is SOS} \\ & -S_i(0) + S_j(\bar{T}) - \mu I \succeq 0 \end{array} \tag{13}$$

where $i, j = 1, \ldots, N$, $i \neq j$, is feasible. Then, the conditions of Statement (c) *of Theorem 1 hold.*

The role of the Γ_i's is the introduction of the constraint that $\tau \in [0, \bar{T}]$ in the program in a way that is theoretically justified by Putinar's Positivstellensatz [35] in a way akin to the S-procedure (although way more general).

We prove now that we can approximate arbitrarily well the conditions of Statement (c).

Theorem 3 (*Asymptotic Exactness*)**.** *Let $\bar{T}_{th}$ be the minimal value for $\bar{T}$ that can be computed using Theorem 1,* (b)*. Then, for any $\varepsilon > 0$, there exist an integer $d = d(\epsilon) \in \mathbb{N}$ such that the above SOS program is feasible with $\deg(R_i) = \deg(\Gamma_i) = d$ and $\bar{T} = \bar{T}_{th} + \varepsilon$.*

Proof. The key idea is to show that we can approximate as closely as desired the optimal function

$$R_i^*(\tau) = e^{A_i^\intercal \tau} R_i^*(0) e^{A_i \tau} \tag{14}$$

with a polynomial that is computed with the above SOS program. To prove this, we will need the following result that we adapt to our setup:

Theorem 4 (*[36,32]*)**.** *Let $\mathcal{T}_{\bar{T}} := \left\{\tau \in \mathbb{R} : g(\tau) := \tau(\bar{T} - \tau) \geq 0\right\}$ be given and let $Q : \mathbb{R} \to \mathbb{S}^n$. Then, $Q(\tau) \succeq 0$ over $\mathcal{T}_{\bar{T}}$ if and only if there exists a sum of squares matrix $S(\tau) \in \mathbb{S}^n$ such that the matrix $Q(\tau) + S(\tau)g(\tau)$ is SOS.*

It is also important to note that since the matrices are univariate polynomial matrices, then we have that $Q(\tau) + S(\tau)g(\tau)$ is SOS if and only if it is positive semidefinite for all $\tau \in \mathbb{R}$; see e.g. [37]. This means that the SOS conditions (first and second SOS conditions in (13)), encode exactly the condition that $\dot{R}_i(\tau) + A_i^\intercal R_i(\tau) + R_i(\tau)A_i$ is a negative semidefinite polynomial over $\tau \in [0, \bar{T}]$. Now, the only restriction is that $R_i(\tau)$ is a polynomial matrix while the optimal value $R_i^*(\tau)$ is exponential. Noting that the Stone–Weierstrass theorem also applies to matrix polynomials (note also that given a continuously differentiable function, we can approximate arbitrarily well both the function and its derivative using polynomials), we can conclude that matrix polynomials can approximate any continuous matrix-valued function over $[0, \bar{T}]$ arbitrarily well. This implies, in turn, that we can approximate arbitrarily well the optimal solution $R_i^*(\tau)$ and, correspondingly, that we can approach arbitrarily well the minimal minimum dwell-time $\bar{T}_{th}$. This completes the proof. $\diamondsuit$

2.3.2. Invariance with respect to time-scaling

A particularity of the conditions stated in Theorem 1 lies in the property of invariance with respect to time-scaling. This is stated in the following result:

Proposition 5. *Let the matrices $M_i \in \mathbb{R}^{n\times n}$, $P_i \in \mathbb{S}^n_{\succ 0}$ and the positive scalars $c, T > 0$ be given. Then, the following statements are equivalent:*

(a) *The LMIs (3) hold with $A_i = M_i$ and $\bar{T} = T$.*
(b) *The LMIs (3) hold with $A_i = cM_i$ and $\bar{T} = T/c$.*

Proof. The proof follows from simple substitutions. $\diamondsuit$

Interestingly, the invariance property is also verified for the infinite-dimensional conditions proposed in Theorem 1.

Proposition 6. *Let the positive scalars $c, T > 0$, the matrices $M_i \in \mathbb{R}^{n\times n}$ and the matrix-valued functions $Q_i : [0, T] \to \mathbb{S}^n$, $Q_i(0) \succ 0$ be given. Then, the following statements are equivalent:*

(a) *The LMIs (4) and (5) holds for all $\tau \in [0, \bar{T}]$ with $R_i(\tau) = Q_i(\tau)$, $A_i = M_i$, $\bar{T} = T$.*
(b) *The LMIs (4) and (5) holds for all $\tau \in [0, \bar{T}]$ with $R_i(\tau) = Q_i(\tau/c)$, $A_i = cM_i$, $\bar{T} = T/c$.*

The above results demonstrate that the polynomial orders remain unchanged when time is rescaled. In this regard, the computational complexity for the N-uplet $(A_1, \ldots, A_n)$ is the same as for the N-uplet $(cA_1, \ldots, cA_N)$, $c > 0$. This establishes that the computational complexity is independent of the length of the dwell-time and only depends on the matrices of the system. This is a rather convenient property as dwell-time values may be large. Scaling can also been performed in order to obtain matrices having coefficients of reasonable magnitude, improving then the numerical properties of the approach. Note also that the same remark applies to the piecewise linear approximation in [26] as it is simply a particular case of the general result obtained in Theorem 1.

Table 1
Upper bounds on the minimal dwell-time of Systems (48)–(50) determined using Theorem 8 for different degrees for the polynomial functions R_i. The number of variables is given in brackets as {primal/dual}.

		System (48)	System (49)	System (50)
Theorem 8, (c)	$\deg(R_i)=2$	3.6769 {146/48}	0.6796 {146/48}	2.0302 {324/96}
	$\deg(R_i)=4$	2.9281 {254/60}	0.6226 {254/60}	1.9193 {564/120}
	$\deg(R_i)=6$	2.9048 {394/72}	0.6222 {394/72}	1.9167 {876/144}
Th. 1, [26]	$K=20$	3.0962 {504/126}	0.6400 {504/126}	2.0565 {1134/252}
Th. 1, [26]	$K=50$	2.9593 {1224/306}	0.6309 {1224/306}	1.9717 {2754/612}
Th. 1, [26]	$K=80$	2.9218 {1944/486}	0.6277 {1944/486}	1.9511 {4374/972}
Th. 4, [23]	$\deg(Z_i)=2$	3.6310 {1078/236}	0.6222 {1078/236}	1.9176 {2412/504}
Th. 4, [23]	$\deg(Z_i)=4$	2.9147 {2026/320}	–	1.9167 {4536/684}
Th. 4, [23]	$\deg(Z_i)=6$	2.7545 {3262/404}	–	1.9135 {7308/864}
Theorem 8, (b)	–	2.7508 {24/6}	0.6222 {24/6}	1.9134 {54/12}

Table 2
Comparison of the computation times (in seconds) of the different methods in the format "solving time/preprocessing time". The number of variables in indicated in the format "primal/dual".

	System (48)	System (49)	System (50)	No. vars.
Theorem 8, (c), $\deg(R_i)=6$	0.35/0.98	0.34/0.98	–	394/72
	–	–	0.36/1.90	876/144
Th. 1, [26], $K=80$	6.22/26.74	4.91/25.95	–	1944/486
	–	–	17.26/31.41	4374/972
Th. 4, [23], $\deg(Z_i)=6$	0.63/3.02	–	–	3262/404
Th. 4, [23], $\deg(Z_i)=2$	–	0.41/1.67	–	1078/236
Th. 4, [23], $\deg(Z_i)=6$	–	–	1.83/9.04	7308/864

2.3.3. *Comparison with Euler-based discretization*

An approximate solution to the LMI (4) can be obtained by using the Euler-method with discretization step $h>0$ as

$$R_i((k+1)h) \succeq R_i(kh) - h(A_i^{\intercal}R_i(kh) + R_i(kh)A_i) \tag{15}$$

for $k=0,\ldots,\phi$, $\phi := \bar{T}/h$. In such a case, the LMI (4) is only solved pointwise and may fail to be feasible between consecutive discretization points, unless the matrix-valued function $R_i(\cdot)$ is assumed to be piecewise-linear as in [26]. Regarding, the complexity of the approach for large $\bar{T}$, the complexity remains unchanged as for the piecewise-linear and the proposed sum of squares approaches, due to the property of invariance with respect to time-scaling. Indeed, if we consider $c\bar{T}$, $c>0$, then we can also consider the discretization step ch. The computational complexity of the Euler-based discretization approach is similar to the one of the piecewise-linear approximation approach and is thus higher than the one based on sum of squares approach; see Tables 1 and 2.

3. Stability of switched systems under minimum dwell-time

In this section, a minimum dwell-time stability result is recalled first. Then, new formulations for stability under minimum dwell-time are provided and extended to the uncertain case. In what follows, we shall consider the family of sequence of switching times

$$\mathbb{I}_\eta := \{\{t_1,t_2,\ldots\} : T_k := t_{k+1}-t_k \in [\eta,+\infty),\ k\in\mathbb{N}\}, \tag{16}$$

which contains sequences satisfying the minimum dwell-time η.

3.1. *A preliminary result*

The following result consists of an *equivalent* reformulation of the minimum dwell-time stability result of [5], but reproved according to some ideas taken from [21–23].

Lemma 7 (*Minimum Dwell-Time*). *The following statements are equivalent:*

(a) *The quadratic form $V(x(t),\sigma(t)) = x(t)^{\intercal}P_{\sigma(t)}x(t)$, $P_i\in\mathbb{S}^n_{\succ 0}$, $i=1,\ldots,N$, is a Lyapunov function for the system (1) in the sense that*

$$\dot{V}(x(t),i) \le -\mu\|x(t)\|_2^2,\quad t\in(t_k,t_{k+1}),\ i=1,\ldots,N \tag{17}$$

and

$$V(x(t_{k+1}),\sigma(t_{k+1})) - V_j(x(t_k),\sigma(t_k)) \le -\zeta\|x(t_k)\|_2^2 \tag{18}$$

hold for some $\mu,\zeta>0$, all $x(t),x(t_k)\in\mathbb{R}^n$ and any sequence $\{t_k\}_{k\in\mathbb{N}}\in\mathbb{I}_{\bar{T}}$.

(b) *There exist $P_i\in\mathbb{S}^n_{\succ 0}$, $i=1,\ldots,N$, such that the LMIs*

$$A_i^{\intercal}P_i + P_iA_i \prec 0 \tag{19}$$

and

$$e^{A_i^{\intercal}\theta}P_ie^{A_i\theta} - P_j \prec 0 \tag{20}$$

hold for all $i,j=1,\ldots,N$, $i\ne j$ and all $\theta\ge\bar{T}$.

(c) *There exist $P_i\in\mathbb{S}^n_{\succ 0}$, $i=1,\ldots,N$, such that the LMIs*

$$A_i^{\intercal}P_i + P_iA_i \prec 0 \tag{21}$$

and

$$e^{A_i^{\intercal}\bar{T}}P_ie^{A_i\bar{T}} - P_j \prec 0 \tag{22}$$

hold for all $i,j=1,\ldots,N$, $i\ne j$.

Moreover, when one of the above statements holds, the switched system (1) is asymptotically stable for any sequence of switching instants in $\mathbb{I}_{\bar{T}}$. △

Proof. *Proof of* (a) ⇔ (b). Assume first that (17) holds. This then implies that

$$x(t)^{\intercal}[A_iP_i + P_iA_i]x(t) \le -\mu\|x(t)\|_2^2$$

for all $x(t)\in\mathbb{R}^n$, $t\ne t_k$, $k\in\mathbb{N}$, which is equivalent to stating that (19) holds. The proof that (18) implies (20) follows the same lines. Reverting the arguments proves that (b) ⇒ (a).

Proof of (b) ⇒ (c). Immediate.

Proof of (c) $\Rightarrow$ (b). Let us consider that (21) and (22) hold. A Taylor expansion of

$$\mathcal{L}_i(\theta) := e^{A_i^\intercal\theta}P_ie^{A_i\theta}$$

around $\theta = \theta_0 \geq \bar{T}$ yields

$$\mathcal{L}_i(\theta_0+\delta) := e^{A_i^\intercal\theta_0}P_ie^{A_i\theta_0} + \delta e^{A_i^\intercal\theta_0}\mathrm{Sym}[A_i^\intercal P_i]e^{A_i\theta_0} + o(\delta) \tag{23}$$

where $o(\cdot)$ is the Landau small-o notation. Hence, we have

$$\mathcal{L}_i(\theta_0+\delta) - \mathcal{L}_i(\theta_0) = \delta e^{A_i^\intercal\theta_0}\mathrm{Sym}[A_i^\intercal P_i]e^{A_i\theta_0} + o(\delta). \tag{24}$$

Since (21) holds, the right-hand side is negative definite for all $\theta_0 \geq \bar{T}$, therefore we have

$$e^{A_i^\intercal(\bar{T}+\delta)}P_ie^{A_i(\bar{T}+\delta)} \preceq e^{A_i^\intercal\bar{T}}P_ie^{A_i\bar{T}} \tag{25}$$

for all $\delta \geq 0$ and thus

$$e^{A_i^\intercal\theta}P_ie^{A_i\theta} - P_j \preceq e^{A_i^\intercal\bar{T}}P_ie^{A_i\bar{T}} - P_j \prec 0 \tag{26}$$

holds for all $\theta \geq \bar{T}$. The proof is complete. $\diamondsuit$

3.2. Nominal stability under minimum dwell-time

The theorem below addresses the case of minimum dwell-time stability for systems without uncertainties:

Theorem 8 (*Minimum Dwell-Time*). *The following statements are equivalent:*

(a) *There exist matrices $P_i \in \mathbb{S}^n_{\succ 0}$, $i = 1,\dots,N$, such that the LMIs*

$$A_i^\intercal P_i + P_iA_i \prec 0 \tag{27}$$

and

$$e^{A_i^\intercal\bar{T}}P_ie^{A_i\bar{T}} - P_j \prec 0 \tag{28}$$

hold for all $i,j = 1,\dots,N$, $i \neq j$.

(b) *There exist matrix functions $R_i : [0,\bar{T}] \mapsto \mathbb{S}^n$, $i = 1,\dots,N$, $R_i(0) \in \mathbb{S}^n_{\succ 0}$, and a scalar $\varepsilon > 0$ such that the LMIs*

$$A_i^\intercal R_i(0) + R_i(0)A_i \prec 0 \tag{29}$$

$$A_i^\intercal R_i(\tau) + R_i(\tau)A_i - \dot{R}_i(\tau) \preceq 0 \tag{30}$$

and

$$-R_j(0) + R_i(\bar{T}) + \varepsilon I \preceq 0 \tag{31}$$

hold for all $\tau \in [0,\bar{T}]$ and all $i,j = 1,\dots,N$, $i \neq j$.

(c) *There exist matrix functions $S_i : [0,\bar{T}] \mapsto \mathbb{S}^n$, $i = 1,\dots,N$, $S_i(\bar{T}) \in \mathbb{S}^n_{\succ 0}$, and a scalar $\varepsilon > 0$ such that the LMIs*

$$A_i^\intercal S_i(\bar{T}) + S_i(\bar{T})A_i \prec 0 \tag{32}$$

$$A_i^\intercal S_i(\tau) + S_i(\tau)A_i + \dot{S}_i(\tau) \preceq 0 \tag{33}$$

and

$$S_i(0) - S_j(\bar{T}) + \varepsilon I \preceq 0 \tag{34}$$

hold for all $\tau \in [0,\bar{T}]$ and all $i,j = 1,\dots,N$, $i \neq j$.

Moreover, when one of the above statements holds, the switched system (1) is asymptotically stable for any sequence of switching instants in $\mathbb{I}_{\bar{T}}$. $\triangle$

Proof. The proof follows from Theorem 1 and Lemma 7. $\diamondsuit$

The following proposition states the sum of squares program associated with the conditions of Statement (b) of Theorem 8.

Proposition 9. *Let $\varepsilon, \nu, \mu\bar{T} > 0$ be given. Assume that the sum of squares program*

$$\begin{array}{ll} \textit{Find polynomials} & R_i, \Gamma_i : [0:\bar{T}] \to \mathbb{S}^n \\ \textit{such that} & R_i(\bar{T}) - \nu I_n \succeq 0 \\ & \Gamma_i(\tau) \text{ is SOS} \\ & -A_i^\intercal R_i(0) - R_i(0)A_i - \mu I \succeq 0 \\ & \dot{R}_i(\tau) - A_i^\intercal R_i(\tau) - R_i(\tau)A_i \\ & -\Gamma_i(\tau)\tau(\bar{T}-\tau) \text{ is SOS} \\ & -S_i(0) + S_j(\bar{T}) - \varepsilon I \succeq 0 \end{array} \tag{35}$$

where $i,j = 1,\dots,N$, $i \neq j$, is feasible. Then, the conditions of Statement (c) *of Theorem 8 hold.*

3.3. Robust stability under minimum dwell-time

Let us assume now that the system is uncertain and the (possibly time-varying) matrices A_i belong to the following polytopes

$$A_i \in \mathcal{A}_i := \mathrm{co}\left\{A_i^{[1]},\dots,A_i^{[M]}\right\} \tag{36}$$

for some $M > 0$ and all $i = 1,\dots,N$. In order to ensure the existence of solutions when the subsystems are time-varying, we assume, for instance, that the matrices $A_i(t)$ are absolutely continuous. We also define the set $\mathbf{\Phi}_i^\theta$ as

$$\mathbf{\Phi}_i^\theta := \{\Phi_i(\theta) : \ \Phi_i(s) \text{ solves (38)}, \lambda(s) \in \Lambda_M, \ s \in [0,\theta]\} \tag{37}$$

where Λ_M is the M-unit simplex and

$$\begin{aligned} \frac{d\Phi_i(s)}{ds} &= \left(\sum_{j=1}^M \lambda_j(s)A_i^{[j]}\right)\Phi_i(s), \\ \Phi_i(0) &= I, \ \lambda(s) \in \Lambda_M, \ s \geq 0. \end{aligned} \tag{38}$$

The complexity of the set $\mathbf{\Phi}_i^\theta$ emphasizes the difficulty of characterizing uncertain sets in the discrete-time framework. The proposed framework, however, allows us to circumvent this problem as shown by the following result:

Theorem 10 (*Robust Minimum Dwell-Time*). *The following statements are equivalent:*

(a) *There exist matrix functions $R_i : [0,\bar{T}] \mapsto \mathbb{S}^n$, $R_i(0) \in \mathbb{S}^n_{\succ 0}$, $i = 1,\dots,N$, and a scalar $\varepsilon > 0$ such that the LMIs*

$$\left(A_i^{[k]}\right)^\intercal R_i(0) + R_i(0)A_i^{[k]} \prec 0 \tag{39}$$

$$\left(A_i^{[k]}\right)^\intercal R_i(\tau) + R_i(\tau)A_i^{[k]} - \dot{R}_i(\tau) \preceq 0 \tag{40}$$

and

$$-R_j(0) + R_i(\bar{T}) + \varepsilon I \preceq 0 \tag{41}$$

hold for all $\tau \in [0,\bar{T}]$, all $i,j = 1,\dots,N$, $i \neq j$ and all $k = 1,\dots,M$.

(b) *There exist matrix functions $S_i : [0,\bar{T}] \mapsto \mathbb{S}^n$, $S_i(\bar{T}) \in \mathbb{S}^n_{\succ 0}$, $i = 1,\dots,N$, and a scalar $\varepsilon > 0$ such that the LMIs*

$$\left(A_i^{[k]}\right)^\intercal S_i(\bar{T}) + S_i(\bar{T})A_i^{[k]} \prec 0 \tag{42}$$

$$\left(A_i^{[k]}\right)^\intercal S_i(\tau) + S_i(\tau)A_i^{[k]} + \dot{S}_i(\tau) \preceq 0 \tag{43}$$

and

$$S_i(0) - S_j(\bar{T}) + \varepsilon I \preceq 0 \tag{44}$$

hold for all $\tau \in [0,\bar{T}]$, all $i,j = 1,\dots,N$, $i \neq j$ and all $k = 1,\dots,M$.

Moreover, when one of the above statements holds, then there exist matrices $P_i \in \mathbb{S}^n_{\succ 0}$, $i = 1, \dots, N$, such that the LMIs

$$A_i^{\intercal} P_i + P_i A_i \prec 0 \tag{45}$$

and

$$\Phi_i(\bar{T})^{\intercal} P_i \Phi_i(\bar{T}) - P_j \prec 0 \tag{46}$$

hold for all $\Phi_i(\bar{T}) \in \boldsymbol{\Phi}_i^{\bar{T}}$, all $A_i \in \mathscr{A}_i$ and all $i, j = 1, \dots, N$, $i \neq j$ proving therefore that the uncertain switched system (1)–(36) is asymptotically stable for any sequence of switching instants in $\mathbb{I}_{\bar{T}}$. △

Proof. The proof follows the same lines as the one of Theorem 8 and exploit very standard arguments on the convexity of the stability conditions and the convexity of the polytopes $\mathscr{A}_i$. It is thus omitted. ◊

Remark 11. The above result has been obtained using a common Lyapunov function. It is possible to consider more general Lyapunov functions in order to reduce the conservatism at the expense of an increase of the computational complexity. For instance, when λ is time-invariant, the Lyapunov function $V(x, \lambda, \tau) = \sum_{j=1}^{N} \lambda_j P_i^{[j]}(\tau)$ can be used. However, the resulting condition would be quadratic in λ and therefore nonconvex in general. To recover convex conditions, the approaches in [38,39] or in [40] can be used, the latter one being asymptotically exact. Basically, any robust analysis method can be applied as the conditions are convex in the matrices of the system.

Proposition 12. *Let $\varepsilon, \nu, \mu, \bar{T} > 0$ be given. Assume that the sum of squares program*

$$\begin{array}{rl} \textit{Find polynomials} & R_i, \Gamma_i^{[k]} : [0 : \bar{T}] \to \mathbb{S}^n \\ \textit{such that} & R_i(\bar{T}) - \nu I_n \succeq 0 \\ & \Gamma_i^{[k]}(\tau) \text{ is SOS} \\ & -\left(A_i^{[k]}\right)^{\intercal} R_i(0) - R_i(0) A_i^{[k]} - \mu I \succeq 0 \\ & \dot{R}_i(\tau) - \left(A_i^{[k]}\right)^{\intercal} R_i(\tau) - R_i(\tau) A_i^{[k]} \\ & -\Gamma_i^{[k]}(\tau)\tau(\bar{T} - \tau) \text{ is SOS} \\ & -S_i(0) + S(\bar{T}) - \varepsilon I \succeq 0 \end{array} \tag{47}$$

where $i, j = 1 \dots, N$, $i \neq j$, and $k = 1, \dots, M$ is feasible. Then, the conditions of Statement (b) *of Theorem 10 hold.*

3.4. Examples

Illustrative examples are given here. The stability conditions are verified using the sum-of-squares programs via the package SOSTOOLS [30] and the SDP solver SeDuMi [34]. Thus, in the examples below, the matrix functions R_i's will be considered as polynomials of chosen degree. From Theorem 3, it is expected to reach reasonably close suboptimal values for some sufficiently large polynomial orders. The results are compared to those obtained using looped-functionals [23] and piecewise-linear matrix-valued functions [26]. We consider the following systems

(a) Example 1 [5]

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.5 \end{bmatrix}. \tag{48}$$

(b) Example 2 [41]

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 \\ -9 & -1 \end{bmatrix}. \tag{49}$$

(c) Example 3 [41]

$$A_1 = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 0 \\ -2 & 1 & -1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 0 & 6 \\ -2 & -1 & -5 \\ 0 & 3 & -1 \end{bmatrix}. \tag{50}$$

(d) Example 4 [23]

$$\begin{aligned} \mathscr{A}_1 &= \mathrm{co}\left\{\begin{bmatrix} 0 & 1 \\ -2-\kappa & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -2+\kappa & -1 \end{bmatrix}\right\}, \\ \mathscr{A}_2 &= \mathrm{co}\left\{\begin{bmatrix} 0 & 1 \\ -9-\kappa & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -9+\kappa & -1 \end{bmatrix}\right\} \end{aligned} \tag{51}$$

for some $\kappa > 0$ representing the amplitude of the perturbation.

We then get the following remarks and conclusions:

(a) Regarding the first example, using the initial result on minimum-dwell-time of [17], the upper-bound 6.66 on the minimum dwell-time is found. Using the minimum dwell-time result of [5], i.e. statement (b) of Theorem 8, the upper bound 2.7508 is obtained. On the other hand, the use of the statement (c) or the statement (d) of Theorem 8 yields the minimum dwell-time estimates in Table 1. We can see that the proposed method allows to approach the upper-bound on the minimum dwell-time as the degree of the R_i's increases. The results are compared with those of [26,23]. Even if the results obtained with looped-functionals [23] outperform, in accuracy, those of the paper and those obtained using piecewise-linear functions [26], this is in spite of a very high computational cost. By comparing accuracy versus computational complexity, it is clear that the proposed approach is, for this example, the method of choice. This is also supported by the results of Table 2 where the proposed approach also yields optimization problems that are faster to build and solve.

(b) Regarding the second and the third example, the same remarks hold with the difference that we obtain the exact bound using the SOS approach in the second example while it is very close in the second one.

(c) Regarding the fourth example, similar conclusions can be drawn. The looped-functional approach leads to the most accurate results but along with the highest computational complexity (see Table 3). The proposed approach is comparable with the one based on piecewise linear functions in terms of accuracy. However, the latter one is much more complex numerically. The proposed approach is again the method of choice for this example.

4. Stability of switched systems under mode-dependent dwell-time

Let us now consider stability under mode-dependent dwell-time. It is important to stress that the notion of mode-dependent dwell-time is understood in the sense of [23] and is vastly different from the ones from [42]. In the considered framework, some subsystems can indeed be unstable. A necessary condition for this method to apply is that at least one subsystem be asymptotically stable.

Let us now define the set of switching signals we shall consider in this section. The set of upper and lower bounds for the dwell-time for each mode is given by

$$\mathcal{T} := [T_1^{\min}, T_1^{\max}] \times \cdots \times [T_N^{\min}, T_N^{\max}] \tag{52}$$

where $0 < T_i^{\min} < T_i^{\max} \leq \infty$, $i = 1, \dots, N$. With this in mind, we can define the set of switching signals

$$\mathbb{I}_{\mathcal{T}} := \left\{\{t_1, t_2, \dots\} : T_k := t_{k+1} - t_k \in [T_{\sigma(t_k)}^{\min}, T_{\sigma(t_k)}^{\max}),\ k \in \mathbb{N}\right\}. \tag{53}$$

Since $T_i^{\min}$ is uniformly bounded away from 0 for all $i = 1, \dots, N$, then Zeno behavior is avoided.

Table 3
Upper bounds on the minimal dwell-time of system (49)–(51) determined using statement (b) of Theorem 10 (constant uncertainties) and (45)–(46) of Theorem 10 (time-varying uncertainties) for different orders for R_i.

		$\kappa = 0.1$	$\kappa = 0.3$	$\kappa = 0.5$	$\kappa = 0.7$	$\kappa = 0.9$	$\kappa = 1.1$	$\kappa = 1.3$	No. vars.
Theorem 10, (a)	$\deg(R_i) = 2$	0.6833	0.8035	0.9697	1.1888	1.4929	1.9698	2.8789	258/84
	$\deg(R_i) = 4$	0.6792	0.7485	0.8115	0.9122	1.1277	1.5062	2.4590	462/108
	$\deg(R_i) = 6$	0.6784	0.7411	0.7972	0.8769	1.0037	1.1977	1.9374	730/132
Th. 13, [23]	$\deg(Z_{ij}) = 5$	0.6759	0.7413	0.7976	0.8786	1.0004	1.1835	1.7174	4310/536
Cor. 1, [26]	$K = 80$	0.6805	0.7471	0.8141	0.9088	1.0610	1.3938	1.9603	3232/486
Theorem 10, (45)–(46)	101 pts. grid	0.6759	0.7299	0.7689	0.8129	0.8673	0.9512	1.1476	832/6

4.1. A preliminary result

Let us now recall the following result taken back from [23]:

Theorem 13. *Assume there exist matrices $P_i \in \mathbb{S}^n_+$, $i = 1, \ldots, N$, such that the LMIs*

$$e^{A_i^{\intercal} T_i} P_i e^{A_i T_i} - P_j \prec 0 \tag{54}$$

hold for all $T_i \in [T_i^{\min}, T_i^{\max}]$ and all $i, j = 1, \ldots, N$, $i \neq j$.

Then, the switched system (1) is asymptotically stable for any sequence of switching instants in $\mathbb{I}_{\mathcal{T}}$. △

Proof. See [23]. ◊

In the above result, the dwell-times lie within a certain range of values. It is however possible to have the upper-bound $T_i^{\max} = \infty$ for certain modes by slightly modifying the conditions. This is explained in the remark below; see also [23].

Remark 14. When $T_i^{\max} = \infty$ for some indices i, it is then necessary that subsystem i be stable, in a similar way as in minimal dwell-time results. In this case, inequality (54) considered at $T_i = \infty$ can be substituted by $A_i^{\intercal} P_i + A_i P_i \prec 0$. The stability conditions for subsystem i then become

$$e^{A_i^{\intercal} T_i^{\min}} P_i e^{A_i T_i^{\min}} - P_j \prec 0, \quad j = 1, \ldots, N, \ j \neq i \tag{55}$$

and

$$A_i^{\intercal} P_i + A_i P_i \prec 0. \tag{56}$$

It is important to mention that the provided approach for mode-dependent dwell-time is radically different from the standard approaches, such as the one in [42]. In the current approach, a discrete-time condition is used to explicitly characterize a range of values for the dwell-times, while in most of the approaches continuous-time conditions are considered. Additionally, the proposed approach does not require stability of all the subsystems, and is thus applicable to a wider class of switched systems. The price to pay is the difficulty for extending the results to uncertain systems, a problem which is resolved by the proposed approach relying on looped-functionals.

4.2. Nominal stability under mode-dependent dwell-time

The following result provides affine conditions for checking stability under mode-dependent dwell-time:

Theorem 15 (*Mode-Dependent Dwell-Time*)**.** *The following statements are equivalent:*

(a) *There exist matrices $P_i \in \mathbb{S}^n_{\succ 0}$, $i = 1, \ldots, N$, such that the LMIs*

$$e^{A_i^{\intercal} T_i} P_i e^{A_i T_i} - P_j \prec 0 \tag{57}$$

hold for all $T_i \in [T_i^{\min}, T_i^{\max}]$ and for all $i, j = 1, \ldots, N$, $i \neq j$.

(b) *There exist matrix functions $R_i : [0, \bar{T}] \mapsto \mathbb{S}^n$, $i = 1, \ldots, N$, $R_i(0) \in \mathbb{S}^n_{\succ 0}$, and a scalar $\varepsilon > 0$ such that the LMIs*

$$A_i^{\intercal} R_i(\tau) + R_i(\tau) A_i - \dot{R}_i(\tau) \preceq 0 \tag{58}$$

and

$$-R_j(0) + R_i(T_i) + \varepsilon I \preceq 0 \tag{59}$$

hold for all $\tau \in [0, T_i^{\max}]$, $T_i \in [T_i^{\min}, T_i^{\max}]$ and all $i, j = 1, \ldots, N$, $i \neq j$.

Moreover, when one of the above statements holds, the switched system (1) is asymptotically stable for any sequence of switching instants in $\mathbb{I}_{\mathcal{T}}$. △

Proof. The proof is similar to the one of Theorem 1 and is only sketched. The proof that the statement (a). implies the statement (b). follows from the fact that $\bar{R}_i(\tau) = e^{A_i^{\intercal}\tau} R_i(0) e^{A_i \tau}$ solves the conditions of the second statement. Conversely, solving for (58) yields $R_i(\tau) \succeq e^{A_i^{\intercal}\tau} R_i(0) e^{A_i \tau}$ for all $\tau \in [0, T_i^{\max}]$. Using now (59), we get that

$$e^{A_i^{\intercal} T_i} R_i(0) e^{A_i T_i} - R_j(0) + \varepsilon I \preceq 0$$

for all $T_i \in [T_i^{\min}, T_i^{\max}]$. This inequality is equivalent to (57). The proof is complete. ◊

Example 16. Let us consider the switched system (1) with 2 modes and matrices [23]

$$A_1 = \begin{bmatrix} -2 & 1 \\ 5 & -3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.2 \end{bmatrix}. \tag{60}$$

This system cannot be stable under minimum dwell-time since the second subsystem is unstable. Note also that other mode-dependent results such the ones in [1,42] cannot be applied since a necessary condition is the stability of all the subsystems.

Now let $T_1 \in [T_1^{\min}, \infty)$. The goal is then to find the range of $T_2 \in [T_2^{\min}, T_2^{\max}]$ for which the switched system is asymptotically stable. As the first subsystem satisfies a minimum dwell-time condition, Remark 14 applies and the conditions (55) and (56) are considered for mode 1. Note that, in this case, we have the equality $P_i = R_i(0)$. We now use the affine conditions (58)–(59) for the characterization of the mode-dependent dwell-time for the second subsystem. Choosing $T_2^{\min} = 0.001$, we get the results gathered in Table 4. Note that in the first case, the computed maximal $T_2^{\max}$ is equal to the one obtained in the periodic switching case (necessary condition). We can hence conclude on the nonconservatism of the approach for this particular case. As in [23], the results are valid in both the cases of constant and uncertain dwell-times, and time-varying dwell-times.

In Table 5, we can see that the approach based on looped-functionals [23] yields better results but has a much higher computational complexity. The proposed approach is again preferable here due to low conservatism and low computational complexity.

Table 4
Maximal $T_2^{\max}$ for system (60) computed for different values of $T_1^{\min}$.

	degree of R_i	$T_1^{\min}=1$	$T_1^{\min}=2$	$T_1^{\min}=5$	$T_1^{\min}=7$	No. vars.
Theorem 15, (b)	2.	1.2636	2.3864	4.3570	4.7630	252/132
	4.	1.2847	2.5473	6.2140	8.3711	314/114
Theorem 11 in [23]deg(Z_{ij}) = 2	–	1.2847	2.5471	6.2149	8.5753	2086/351
Periodic switching case	–	1.2847	2.5471	6.2158	8.5804	–

Table 5
Comparison of the computation times (in seconds) of the different methods for system (60). The format is "solving time/preprocessing time ".

	$T_1^{\min}=1$	$T_1^{\min}=2$	$T_1^{\min}=5$	$T_1^{\min}=7$
Theorem 15, (b), deg(R) = 4	0.38/0.63	0.59/0.64	1.41/0.64	1.04/0.66
Theorem 11 in [23], deg(Z_{ij}) = 2	0.72/2.63	0.68/2.62	0.89/2.62	0.9/2.62

5. Stabilization of switched systems with guaranteed minimum and mode-dependent dwell-time

Stabilization using state-feedback is considered in this section. Robust stabilization is omitted since it straightforwardly follows from nominal stabilization and robust stability analysis. To derive our nominal stabilization result, let us redefine system (1) as

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u_{\sigma(t)}(t) \tag{61}$$

where $u_i \in \mathbb{R}^{m_i}$, $i = 1, \dots, N$, are the control inputs. We further assume that the control law is given by

$$u_{\sigma(t)}(t) = K_{\sigma(t)}(t)x(t) \tag{62}$$

where

$$K_{\sigma(t_k+\tau)}(t_k+\tau) = \begin{cases} \tilde{K}_{\sigma(t_k)}(\tau) & \text{if } \tau \in [0, \bar{T}) \\ \tilde{K}_{\sigma(t_k)}(\bar{T}) & \text{if } \tau \in [\bar{T}, T_k) \end{cases} \tag{63}$$

where the functions $\tilde{K}_i : [0, \bar{T}] \to \mathbb{R}^{m_i \times n}$, $i = 1, \dots, N$, have to be determined such that the closed-loop system (61)–(62)–(63) is asymptotically stable with prescribed minimum dwell-time $\bar{T}$.

5.1. Minimum dwell-time stabilization

Theorem 17 (*Stabilization with Minimum Dwell-Time*)**.** *The following statements are equivalent:*

(a) *There exist matrices $P_i \in \mathbb{S}^n_{\succ 0}$, $i = 1, \dots, N$ such that the LMIs*

$$(A_i + B_i\tilde{K}_i(\bar{T}))P_i + P_i(A_i + B_i\tilde{K}_i(\bar{T}))^{\intercal} \prec 0 \tag{64}$$

and

$$\Psi_i(\bar{T})P_i\Psi_i(\bar{T})^{\intercal} - P_j \prec 0 \tag{65}$$

hold for all $i, j = 1, \dots, N$, $i \neq j$ where

$$\frac{d\Psi_i(s)}{ds} = \left(A_i + B_i\tilde{K}_i(s)\right)\Psi_i(s), \quad \Psi_i(0) = I, \ s \geq 0. \tag{66}$$

(b) *There exist matrix functions $S_i : [0, \bar{T}] \mapsto \mathbb{S}^n$, $S_i(\bar{T}) \in \mathbb{S}^n_{\succ 0}$, $U_i : [0, \bar{T}] \mapsto \mathbb{R}^{m_i \times n}$, $i = 1, \dots, N$ and a scalar $\varepsilon > 0$ such that the LMIs*

$$\mathrm{Sym}[A_iS_i(\bar{T}) + B_iU_i(\bar{T})] \prec 0 \tag{67}$$

$$\mathrm{Sym}[A_iS_i(\tau) + B_iU_i(\tau)] + \dot{S}_i(\tau) \preceq 0 \tag{68}$$

and

$$S_i(0) - S_j(\bar{T}) + \varepsilon I \preceq 0 \tag{69}$$

hold for all $\tau \in [0, \bar{T}]$ and all $i, j = 1, \dots, N$, $i \neq j$.

Moreover, when one of the above statements holds, the closed-loop system (61)–(62)–(63) is asymptotically stable for all switching instants in $\mathbb{I}_{\bar{T}}$ and suitable matrix functions $\tilde{K}_i$ are given by

$$\tilde{K}_i(\tau) = U_i(\tau)S_i(\tau)^{-1}. \tag{70}$$

Proof. The goal is then to show that statement (a) is a necessary and sufficient condition for the existence of a stabilizing state-feedback of the form (62)–(63) for system (61), in the sense of Theorem 8. The closed-loop system is given by

$$\dot{x}(t) = (A_i + B_i\tilde{K}_i(t - t_k))x(t), \quad t \in [t_k, t_{k+1}). \tag{71}$$

The key idea is to use Lemma 7 and Theorem 8 to prove stability of the closed-loop system. To simplify the derivation of convex synthesis conditions, the adjoint system with reverse-time of (71) given by

$$\dot{y}(t) = (A_i + B_i\tilde{K}_i(t - t_k))^{\intercal}y(t), \quad t \in (t_k, t_{k+1}] \tag{72}$$

is considered. The crucial point here is that proving stability of (72) is equivalent to proving stability of (71). Noting first that for all $\theta \geq \bar{T}$, $\tilde{K}_i(\theta) = \tilde{K}_i(\bar{T})$ and following the same arguments as in the proof of Lemma 7, we find that condition (21) exactly becomes condition (64). In the same way, condition (22) equivalently becomes condition (76), which proves exactness of statement (a).

Equivalence with statement (b) follows from Theorem 8 and from the change of variables $U_i(\tau) = \tilde{K}_i(\tau)P_i(\tau)$. The proof is complete. ◊

Example 18. Let us consider the system (61) with matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -5 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{73}$$

Setting $\bar{T} = 0.1$ and choosing the polynomials S and Y as first-order polynomials, we obtain the control gains given in Eq. (74) in Box I, where $d_1(\tau) = 0.6915 + 4.2476\tau + 1.9162\tau^2$ and $d_2(\tau) = 2.1420 + 3.7553\tau + 0.9021\tau^2$. The state and control-gain trajectories are depicted in Figs. 1 and 2 where we can see that the obtained controller stabilizes the switched system correctly. Note that it is also possible to identify phases where the controller maintains its value to $\tilde{K}_i(\bar{T})$.

5.2. Mode-dependent dwell-time stabilization

For mode-dependent dwell-time stabilization, the control law becomes

$$u_{\sigma(t_k+\tau)}(t_k+\tau) = K_{\sigma(t_k+\tau)}(t_k+\tau)x(t_k+\tau) \tag{75}$$

where $\tau \in (T^{\min}_{\sigma(t_k)}, T^{\max}_{\sigma(t_k)}]$ and $K_i : [0, T_i^{\max}] \to \mathbb{R}^{m_i \times n}$, $i = 1, \dots, N$.

$$
\begin{aligned}
K_1(\tau) &= \frac{1}{d_1(\tau)}\begin{bmatrix}0.9251 + 15.4274\tau + 1.5713\tau^2 & -3.7623 + 7.1348\tau + 1.2093\tau^2\end{bmatrix} \\
K_2(\tau) &= \frac{1}{d_2(\tau)}\begin{bmatrix}-2.8369 + 1.6128\tau + 0.4961\tau^2 & -15.4803 - 21.3893\tau - 4.2782\tau^2\end{bmatrix}
\end{aligned}
\tag{74}
$$

Box I.

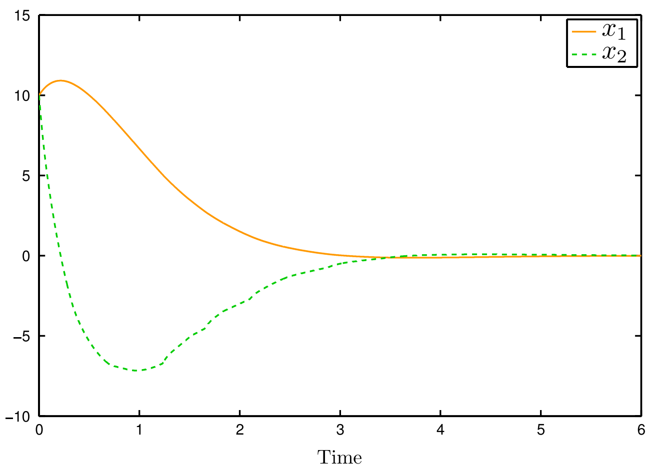

Fig. 1. State trajectories of the closed-loop system (61)–(73).

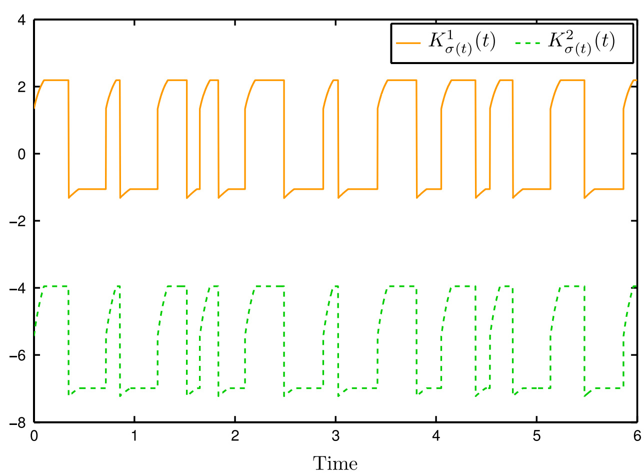

Fig. 2. Control gain trajectories of the closed-loop system (61)–(73).

Theorem 19 (*Stabilization with Mode-Dependent Dwell-Time*)**.** *The following statements are equivalent:*

(a) *There exist matrices* $P_i \in \mathbb{S}^n_{\succ 0}$, $i = 1, \dots, N$ *such that the LMIs*

$$\Psi_i(T_i)P_i\Psi_i(T_i)^\intercal - P_j \prec 0 \tag{76}$$

hold for all $T_i \in [T_i^{\min}, T_i^{\max}]$, $i, j = 1, \dots, N$, $i \neq j$ *where*

$$\frac{d\Psi_i(s)}{ds} = (A_i + B_iK_i(s))\,\Psi_i(s), \quad \Psi_i(0) = I,\ s \geq 0. \tag{77}$$

(b) *There exist matrix functions* $S_i : [0, \bar{T}] \mapsto \mathbb{S}^n$, $S_i(\bar{T}) \in \mathbb{S}^n_{\succ 0}$, $U_i : [0, T_i^{\max}] \mapsto \mathbb{R}^{m_i \times n}$, $i = 1, \dots, N$ *and a scalar* $\varepsilon > 0$ *such that the LMIs*

$$\mathrm{Sym}[A_iS_i(\tau) + B_iU_i(\tau)] + \dot{S}_i(\tau) \preceq 0 \tag{78}$$

and

$$S_i(0) - S_j(T_i) + \varepsilon I \preceq 0 \tag{79}$$

hold for all $\tau \in [0, T_i^{\max}]$, $T_i \in [T_i^{\min}, T_i^{\max}]$ *and all* $i, j = 1, \dots, N$, $i \neq j$.

Moreover, when one of the above statements holds, the closed-loop system (61)–(75) is asymptotically stable for all switching instants in $\mathbb{I}_{\mathcal{T}}$ *and suitable matrix functions* K_i *are given by*

$$K_i(\tau) = U_i(\tau)S_i(\tau)^{-1}. \tag{80}$$

Proof. The proof follows the same lines as the proof of Theorem 17. ◊

6. Conclusion

New conditions for characterizing minimum and mode-dependent dwell-times for uncertain linear switched systems with time-varying uncertainties have been provided. Thanks to the structural properties of the stability criterion, convex state-feedback design conditions have been derived. It has been shown that the proposed approach is numerically less complex than previous ones while still maintaining a reasonable accuracy. Possible extensions include the use of homogeneous Lyapunov functions as in [19] and the consideration of nonlinear switched systems with polynomial vector field.

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