UNIVERSITY OF BRITISH COLUMBIA

Math 103 Syllabus

Contents

Version

Version 1.0 This file was created by Carmen Bruni in 2013, a then graduate student at the University of British Columbia.

Version 1.1 Updated by Carmen Bruni to correct numerous typos in the document. Updated the license and mission statement.

Mission Statement

The goal of this document is to provide a clear vision of Math 103 that can be utilized by both instructors and students to gain a perspective on what will be covered during a term. Throughout, learning goals are presented to help make it clear what instructors should teach and what knowledge students should expect to obtain upon successful completion of the course.

The Math 102 and 103 sequence is very different from its two sister first year calculus sequences offered at UBC. This sequence focuses heavily on biological applications and emphasizes understanding how these ideas play a role in real world situations. As such, many proofs are deemphasized if not altogether omitted (the main exception being the content on series).

The document also makes extensive references to the Math Educational Resources wiki. This project began in 2012 out of a desire to create a free working resource for undergraduates. Graduate students volunteer their time towards the development of this project. The site can be found at

http://wiki.ubc.ca/Science:Math_Educational_Resources

Here, students can find relevant exam questions **by exam** or **by topic**. Videos are also embedded on topic links to help reinforce concepts and aid in reminding students of the core ideas. The original author of this document, Carmen Bruni, included pencasts on the wiki which were created specifically for this course. Combining this with many of the examples present in this document makes this not only a syllabus, but an extremely valuable resource for students to ensure that they are learning the correct material in order to excel in this course.

Solutions when provided are presented only in a minimalist form. Students working through the problems are encouraged to write out complete comprehensive solutions. These solutions are only present as a time saving measure for students. Students are more than welcomed and even encouraged to discuss questions found in this document with their instructors or fellow peers. Concerns or confusions found on the wiki should be discussed on the wiki itself.

Any comments about this current version are welcomed and can be sent to Carmen Bruni at cbruni@alumni.ubc.ca. The TeX file is also available upon request.

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- Carmen Bruni, December 5th, 2013 (modified May 4th, 2014)

Prerequisites

The following is a base set of skills one needs in order to do well in this course. Any problems in these areas should seek immediate rectification.

- Adding, subtracting, multiplying and dividing rational functions (that is, functions where the numerator and denominator are both polynomials and the denominator is nonzero).

- Solving for roots of quadratic polynomials using all of the following methods: factoring, completing the square, and the quadratic formula. This also includes finding common factors in expressions

- Solving systems of equations

- Manipulate exponents including but not limited to $x^a x^b = x^{a+b}$, $(x^a)^b = x^{ab}$, $\frac{x^a}{x^b} = x^{a-b}$, $y = e^x \Leftrightarrow \ln(y) = x$.

- Derivatives and graphs of elementary functions including x^a where a is a real number, trigonometric, inverse trigonometric, logarithmic and exponential functions.

- All rules of differentiation (power, product, quotient, chain rules).

- Understanding and being able to compute limits of functions.

- Basic familiarity with differential equations (what are they, how they are used).

- A working understanding of trigonometry, including but not limited to drawing the function of $\sin(x)$,$\cos(x)$,$\tan(x)$ and values of these functions at $0,\pi/6,\pi/4,\pi/3,\pi/2$ and these values added with multiples of $\pi/2$.

Solutions are included here to help with solving problems. These are in general *not intended to be full solutions* but are included to help you verify that your work is correct.

Learning goal sections roughly correspond to weeks of the course which also roughly correspond to chapters in the course notes.

Throughout this article, an implicit emphasis will be to apply many of these learning goals in a variety of biological applications not explicitly given in the exercises. These will be a mix of WebWork problems as well as examples covered in class. The best way to practice is to attempt problems and make sure you're familiar with examples covered in the notes (even if your instructor does not cover them in class).

1 Area, Volume and Sigma Notation

Learning Goal 1.1. Know formulas for the areas and perimeters of basic shapes, including triangles, squares, regular n-gons, parallelograms and circles. [Knowledge]

Learning Goal 1.2. Explain Archimedes method for computing the exact value of π. [Knowledge, Evaluation]

Learning Goal 1.3. Know formulas for the surface areas and volumes of basic shapes, include cubes, rectangular boxes, cylinders and spheres. [Knowledge]

Learning Goal 1.4. Convert between sigma notation and standard notation. [Comprehension]

Exercise 1.1 Write out the summation $\displaystyle\sum_{i=0}^{5} e^i$ in standard notation

Exercise 1.2 Write out the sum $2 + 5 + 10 + 17$ using sigma notation

Exercise 1.3 Write out the summation $\displaystyle\sum_{i=1}^{3} f(i+2)$ in standard notation where $f(x) = \ln(2x)$

Learning Goal 1.5. Calculate sums using the following identities: [Analysis]

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Exercise 1.4 Compute $\displaystyle\sum_{i=1}^{10} i^2$

Learning Goal 1.6. Using the above identities and the linearity of summations, manipulate sums into a form that can be evaluated using those formulas and then compute their sums. [Application and Analysis]

Exercise 1.5 Compute $\displaystyle\sum_{i=5}^{15}(i-1)^2$

Exercise 1.6 Compute $\displaystyle\sum_{i=2}^{10}(i^3+2i+1)$

Exercise 1.7 Compute $\displaystyle\sum_{i=0}^{n}n$ where n is a positive integer.

Learning Goal 1.7. Evaluate why a given answer to a summation question cannot be correct. [Evaluation]

Exercise 1.8 A student computes $\displaystyle\sum_{i=4}^{20}i^2$ to be $\frac{19(20)(41)}{6}$. Why is this not correct?

Learning Goal 1.8. Convert a summation using a change of variables. [Comprehension]

Exercise 1.9 Which of the following sums is equivalent to $\displaystyle\sum_{i=7}^{29}(i-1)^3+i$?

(i) $\displaystyle\sum_{j=1}^{23}(j+5)^3+j+5$

(ii) $\displaystyle\sum_{j=1}^{22}(j+6)^3+j+6$

(iii) $\displaystyle\sum_{j=3}^{25}(j+4)^3+j+5$

(iv) $\displaystyle\sum_{j=10}^{32}(j+2)^3+j+3$

(v) $\displaystyle\sum_{j=5}^{27}(j+1)^3+j+2$

Learning Goal 1.9. Convert expressions to closed forms (that is, a form that is in an easy to put into a calculator form).

Exercise 1.10 Convert $-1 + 2 - 3 + 4 - 5 + 6 - \ldots - (2n - 1) + 2n$ into a closed form.

Learning Goal 1.10. State the definition of a geometric series. [Knowledge]

Learning Goal 1.11. Compute sums using $\displaystyle\sum_{i=0}^{N} ar^i = \frac{a(1 - r^{N+1})}{1 - r}$ [Application]

Exercise 1.11 Compute $\displaystyle\sum_{i=0}^{9} 3 \cdot 2^i$

Exercise 1.12 Compute $\displaystyle\sum_{i=1}^{9} 3 \cdot 2^i$

Exercise 1.13 Compute $\displaystyle\sum_{i=8}^{13} \frac{3 \cdot 5^{i+1}}{2^{2i}}$

Learning Goal 1.12. Compute sums by expanding into standard notation and noticing patterns [Analysis]

Exercise 1.14 Simplify $\displaystyle\sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right)$ as a fraction

Exercise 1.15 Compute $\displaystyle\sum_{n=1}^{50} (-1)^n n$

Learning Goal 1.13. Define a partial sum [Knowledge]

Learning Goal 1.14. State what it means for an infinite series to converge [Knowledge]

Learning Goal 1.15. Compute sums using $\displaystyle\sum_{i=0}^{\infty} ar^i = \frac{a}{1 - r}$ [Application]

Exercise 1.16 Find the sum of the geometric series $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \frac{80}{81} - ...$

Exercise 1.17 Write 2.445353535353... as a fraction

Exercise 1.18 Determine if $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$ converges. If so, compute its value.

Exercise 1.19 Compute or state it diverges $\sum_{i=0}^{\infty} 5 \left(\frac{1}{3} \right)^i$

Exercise 1.20 Compute or state it diverges $\sum_{i=0}^{\infty} 5 \cdot 2^i$

Exercise 1.21 Compute or state it diverges $\sum_{i=2}^{\infty} 3 \cdot 2^{-i}$

Learning Goal 1.16. Use geometric series in biological applications to determine limiting behaviours. [Application]

Exercise 1.22 Compute the maximum and minimum amount of a drug remaining in a patient in the limit (that is, as time tends to infinity) given that the patient takes an 80 milligram dose once a day at the same time each day and the drug has a half life of 22 hours.

External Example. Check out section 1.7 in the notes (page 18) on bifurcating and trifurcating trees.

Relevant exam questions from previous years can be found at the following links

- wiki.ubc.ca/Category:MER Tag Geometric series
- wiki.ubc.ca/Category:MER Tag Summations

2   Riemann Sums and Integration

Learning Goal 2.1. Define a Riemann Sum. [Knowledge]

Learning Goal 2.2. Approximate the area under a function using left, right and midpoint rules. Also, identify when an estimate is an overestimate or underestimate.[Analysis]

Exercise 2.1   Approximate the area under the curve $f(x) = \ln(x)$ between $x = 1$ and $x = 5$ using the left, right and midpoint rules with $n = 4$ intervals (rectangles)

Learning Goal 2.3. Define a definite integral. Also, be able to sketch a region as given in a definite integral. [Knowledge]

Learning Goal 2.4. Using Riemann sums, compute the area under a function. Reworded, evaluate a definite integral using the definition of a definite integral. [Application]

Exercise 2.2   Compute the area under the curve $y = x^2$ between $x = 1$ and $x = 3$ using Riemann sums (that is, using the definition of a definite integral). *No marks will be given to solutions not using the definition directly.*

Exercise 2.3   Compute the area under the curve $y = 2x^2 + x$ between $x = 0$ and $x = 3$ using Riemann sums (that is, using the definition of a definite integral). *No marks will be given to solutions not using the definition directly.*

Learning Goal 2.5. Convert between a Riemann sum and a definite integral and vice versa. [Comprehension]

Exercise 2.4   Write $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\tfrac{2i}{n}\right)^2 + 1\right)\left(\tfrac{2}{n}\right)$ as a definite integral

Exercise 2.5   Write $\displaystyle\int_{2}^{5} (e^{x^2} + x)\, dx$ as a Riemann sum.

Exercise 2.6   Write $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \left(\tfrac{\pi}{8n}\right) \tan\left(\tfrac{i\pi}{40n}\right)$ as a definite integral

Learning Goal 2.6. Compute integrals using high school geometry. That is, recognize an integral as a signed area and compute it. [Application]

Exercise 2.7 Evaluate $\int_{-2}^{2} f(x)\,dx$ where $f(x) = \begin{cases} 3 & \text{if } x \leq -1 \\ x & \text{if } x > -1 \end{cases}$

Exercise 2.8 Evaluate $\int_{-2}^{2} \sqrt{4 - x^2} + 1\,dx$

Learning Goal 2.7. Compute the area of objects given a defining equation using Riemann sums. [Application]

Exercise 2.9 Compute the area of a leaf given by the equation $y^2 = (x^2 - 1)^2$ for $-1 \leq x \leq 1$ using Riemann sums.

Learning Goal 2.8. State properties of a definite integral and give justification of their truth, including the following; in what follows, let $f(x)$ and $g(x)$ be integrable on $[a, b]$ and let c be a real number:

- Trivial integral $\int_{a}^{a} f(x)\,dx = 0$

- Linearity $\int_{a}^{b} (f(x) \pm g(x))\,dx = \int_{a}^{b} f(x)\,dx \pm \int_{a}^{b} g(x)\,dx$ and
  $\int_{a}^{b} cf(x)\,dx = c \int_{a}^{b} f(x)\,dx$

- Area of a rectangle $\int_{a}^{b} c\,dx = c(b - a)$

- Switching directions $\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx$

- Adding parts: If $a \leq c \leq b$, then $\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$

- Positivity: If $f(x) \geq 0$ for every value of x between a and b, then $\int_{a}^{b} f(z)\,dz \geq 0$

- Boundedness: If $h(x) \leq f(x) \leq g(x)$ for every value of x between a and b, then
  $\int_{a}^{b} h(z)\,dz \leq \int_{a}^{b} f(z)\,dz \leq \int_{a}^{b} g(z)\,dz$

Relevant exam questions from previous years can be found at the following links

- wiki.ubc.ca/Category:MER Tag Riemann sum

- wiki.ubc.ca/Category:MER Tag Integral properties

3   Fundamental Theorem of Calculus

Learning Goal 3.1. Define an integrand, dummy variable and a definite integral. [Knowledge]

Learning Goal 3.2. Define an antiderivative. [Knowledge]

Learning Goal 3.3. State the Fundamental Theorem of Calculus (parts 1 and 2) [Knowledge]

Learning Goal 3.4. Calculate integrals using the Fundamental Theorem of Calculus [Analysis]

Exercise 3.1 Compute $\displaystyle\int_1^2 3x^2 \, dx$

Exercise 3.2 Compute $\displaystyle\int_{-2}^2 e^x + x^5 \, dx$

Exercise 3.3 Compute $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Learning Goal 3.5. Compute derivatives of functions using the Fundamental Theorem of Calculus (and possibly using the chain rule) [Application]

Exercise 3.4 Compute $\displaystyle\frac{d}{dx}\int_0^x e^{t^2} \, dt$

Exercise 3.5 Compute $\displaystyle\frac{d}{dx}\int_{\sin(x)}^{x^2} t^t \, dt$

Learning Goal 3.6. Identify when the Fundamental Theorem of Calculus cannot be used. Examples include when a function is undefined on a region, when an integral is over a singularity (point of discontinuity), or when an endpoint has a singularity. Note that later in the course we will discuss a technique that can sometime be used to evaluate such integrals. [Comprehension]

Exercise 3.6 Compute $\int_{-1}^{1} \frac{1}{x}\, dx$ using the Fundamental Theorem of Calculus or state why this is impossible.

Exercise 3.7 Compute $\int_{0}^{\pi} \tan(x)\, dx$ or state why this is impossible.

Learning Goal 3.7. Verify antiderivatives using differentiation. [Evaluation]

Exercise 3.8 A student computes an antiderivative of $\sec(x) - \tan(x)\sin(x)$ to be $\tan(x)\cos(x)$. Is the student correct?

Learning Goal 3.8. Identify use of absolute values in the logarithmic integral. [Knowledge, Evaluation]

Exercise 3.9 A student computes $\displaystyle\int_{-3}^{-1} \frac{1}{x}\, dx$ to be $\ln(-1) - \ln(-3)$. Explain why the answer is incorrect and how to fix it.

Learning Goal 3.9. Define even and odd functions and use this property to compute integrals. [Knowledge and Application]

Exercise 3.10 Compute $\displaystyle\int_{-10}^{10} \sin^{103}(x)\cos^{102}(x^{11})\, dx$

Learning Goal 3.10. Give an expression for the area function $A(x)$ of a given function $f(t)$ from a fixed endpoint and be able to draw this function. Reworded, be able to sketch the area under a function given a picture of the function. [Knowledge, Analysis]

Learning Goal 3.11. Compute the area of objects given a defining equation. [Application]

Exercise 3.11 Compute the area of a leaf given by the equation $y^2 = (x^2 - 1)^2$ for $-1 \leq x \leq 1$.

Learning Goal 3.12. Calculate the area between two curves [Analysis]

Exercise 3.12 Compute the area between $y = x^2$ and $y = x^3$.

Exercise 3.13 Compute the area of the region bounded by $y = 2x^2 + 10$ and $y = 4x + 16$

Exercise 3.14 Compute the area of the region bounded by $y = x - 2$ and $y^2 = x$.

Relevant exam questions from previous years can be found at the following links

- wiki.ubc.ca/Category:MER Tag Fundamental theorem of calculus
- wiki.ubc.ca/Category:MER Tag Integration using symmetry
- wiki.ubc.ca/Category:MER Tag Antiderivative sketching
- wiki.ubc.ca/Category:MER Tag Area between two curves

4 Applications of Integrals (primarily to velocities and rates)

Learning Goal 4.1. Define distance, displacement, velocity and acceleration and explain the relationship between these values using derivatives and integrals. [Knowledge]

Learning Goal 4.2. Compute the solution to the differential equation $\frac{dv}{dt} = g - kv$ where $g = 9.8m/s^2$ is the acceleration due to gravity and k is the drag coefficient and describe its terminal velocity (including the case when $k = 0$ and g is an arbitrary acceleration). [Knowledge, Application]

Learning Goal 4.3. Describe the difference between distance and displacement and know how to compute both given a velocity function (or an acceleration function). [Comprehension, Application]

Exercise 4.1 Compute the velocity as a function of time given the acceleration function $a(t) = t + 4$ and $v(0) = 5$

Exercise 4.2 Compute the displacement and the total distance traveled of a particle given that its velocity at time t is given by $v(t) = 2t^2 - 10t + 8$ from $t = 1$ to $t = 4$

Exercise 4.3 Compute the displacement and the total distance traveled of a particle given that its velocity at time t is given by $v(t) = t^2 - t - 6$ from $t = 1$ to $t = 4$

Learning Goal 4.4. Understand how rates of changes of functions relate to the original function. This is referred to as the net change theorem. [Comprehension]

Exercise 4.4 A tree grows at a rate of $g(t) = 10 + 15t^2$ cm/year. How tall is the tree after 10 years?

Exercise 4.5 The rate of change of a cup of tea is given by $f(t) = 50e^{-2t}$ degrees Celsius per minute. Compute the total change in temperature of the coffee between $t = 1$ and $t = 5$.

External Example. For more examples, see 4.3, 4.4 and 4.6 in your notes (pages 72-85).

Learning Goal 4.5. State the formula for average value. [Knowledge]

Learning Goal 4.6. Compute average values of functions on bounded domains. [Application]

Exercise 4.6 Find the average value of $f(x) = \sin(x)$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Relevant exam questions from previous years can be found at the following links

- wiki.ubc.ca/Category:MER Tag Average value
- wiki.ubc.ca/Category:MER Tag Net change theorem

5   Applications of integrals to volume, mass and length

Learning Goal 5.1. State formulas for the mass of an object given a density function, the centre of mass of a one dimensional object. [Knowledge]

Learning Goal 5.2. Compute densities and centre of mass of a one dimensional object [Application]

Exercise 5.1   Compute the mass, centre of mass and average mass of a bar of length 10 cm and whose density is linear with distance given by $\rho(x) = 3x$.

Learning Goal 5.3. Understand the difference between average mass and the point where a one dimensional object can be split to get two parts of equal mass. [Comprehension, Evaluation]

Exercise 5.2   For a bar of length 10cm, given the density function $\rho(x) = 3x$, compute the point where the bar is split into two equal masses and compute the average mass.

Learning Goal 5.4. State the formulas for volumes of revolution of the region between two curves rotated about the x-axis and the y-axis. [Knowledge]

Learning Goal 5.5. Compute the volume of revolution of a region around lines of the form $x = a$ and $y = b$ where a and b are given real numbers. [Application]

Exercise 5.3   Compute the volume of revolution of the solid described by revolving the region between the curves $y = 2x^{@}$ and $y = x^3$ about the x-axis.

Exercise 5.4   Compute the volume of revolution of the solid described by revolving the region between the y-axis, the line $y = a$ for a given constant a and $y = \sqrt{x}$ about the x-axis.

Exercise 5.5   Compute the volume of revolution of the solid described by revolving the region between the curves $y = x^2$ and $y = x$ about the line $x = -2$.

Exercise 5.6   Compute the volume of revolution of the solid described by revolving the region between the curves $y = x^2$, $x = 1$ and $y = 0$ about the line $y = 1$.

Exercise 5.7   Compute the volume of revolution of the solid described by revolving the region between the curves $y = x^2$, $x = 1$ and $y = 0$ about the line $x = 1$.

Exercise 5.8 Compute the volume of revolution of the solid described by revolving the region between the x-axis and $y = x^2 - 1$ about the x-axis.

Exercise 5.9 Compute the volume of revolution of the solid described by revolving the region between the x-axis and $y = x^2 - 1$ about the y-axis.

Exercise 5.10 Compute the volume of revolution of the solid described by revolving the region between the x-axis and $y = x^2 - 1$ about the line $y = -1$.

Exercise 5.11 Compute the volume of revolution of the solid described by revolving the region between the x-axis and $y = x^2 - 1$ about the line $x = -1$.

Learning Goal 5.6. State the formula for computing the arc length of a function. [Knowledge]

Learning Goal 5.7. Compute the arc length of an object. [Application]

Exercise 5.12 Find the arc length of $y = \int_1^x \sqrt{t - 1}\, dt$ between $x = 1$ and $x = 4$.

Exercise 5.13 Find the arc length of $y = \frac{x^3}{3} + \frac{1}{4x}$ between $x = 1$ and $x = 2$.

Exercise 5.14 Find the arc length of $y = 1 + 6x^{3/2}$ between $x = 0$ and $x = 1$.

Relevant exam questions from previous years can be found at the following links

- wiki.ubc.ca/Category:MER Tag Center of mass

- wiki.ubc.ca/Category:MER Tag Solid of revolution

- wiki.ubc.ca/Category:MER Tag Arc length

- wiki.ubc.ca/Category:MER Tag Mass

6 Integration Techniques

Learning Goal 6.1. Understand the interpretation of $\frac{dy}{dx}$ via the perspective of differentials. [Comprehension]

Learning Goal 6.2. Explain the difference between definite and indefinite integrals (for example, explain why the latter has an arbitrary constant in their solution). [Evaluation]

Learning Goal 6.3. State the substitution rule for indefinite integrals. [Knowledge]

Learning Goal 6.4. State the substitution rule for definite integrals. [Knowledge]

Learning Goal 6.5. Using the substitution rule, compute definite and indefinite integrals. [Application]

Exercise 6.1 Compute $\int (x+1)^{102}\, dx$.

Exercise 6.2 Compute $\int_0^\pi \sin(x)(\cos(x))^2\, dx$.

Exercise 6.3 Compute $\int x\sqrt{x+7}\, dx$

Learning Goal 6.6. Using the substitution rule, compute definite and indefinite integrals using a multiplication by one technique. [Application]

Exercise 6.4 Compute $\int \sec(x)\, dx$

Exercise 6.5 Compute $\int \csc(x)\, dx$ (Hint: multiply top and bottom by $-(\csc(x)\cot(x))$ and recall that $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$ and $\frac{d}{dx}\cot(x) = -\csc(x)^2$)

Learning Goal 6.7. Using the substitution rule, compute definite and indefinite integrals by first factoring a perfect square in the denominator. [Application]

Exercise 6.6 Compute $\displaystyle\int \frac{dx}{4x^2 - 12x + 9}$ and $\displaystyle\int_0^1 \frac{dx}{4x^2 - 12x + 9}$

Learning Goal 6.8. Using the substitution rule, compute definite and indefinite integrals by first completing the square in the denominator. [Application]

Exercise 6.7 Compute $\displaystyle\int \frac{dx}{x^2 + 10x + 50}$

Learning Goal 6.9. State formulas for the centroid (centre of mass) of an object of uniform density of a two dimensional object. [Knowledge]

Learning Goal 6.10. Using the substitution rule, compute definite and indefinite integrals of powers of $\sin(x)$ and $\cos(x)$ in the case when the powers are either odd or even and positive given certain formulas. See the introduction for given formulas. [Application]

Exercise 6.8 Compute $\displaystyle\int \sin^2(x)\,dx$

Exercise 6.9 Compute $\displaystyle\int \sin(x) \cos^3(x)\,dx$

Exercise 6.10 Compute $\displaystyle\int \cos^3(x)\,dx$

Exercise 6.11 Compute $\displaystyle\int \frac{\cos^3(x)\,dx}{\sin(x)}$

Learning Goal 6.11. Using the substitution rule, compute definite and indefinite integrals of powers of $\tan(x)$ and $\sec(x)$ in the case when the power of $\tan(x)$ is odd or when the power of $\sec(x)$ is even and both of the degrees are nonzero (an exception is when the integral is just $\sec(x)$). See the introduction for given formulas. [Application]

Exercise 6.12 Compute $\displaystyle\int \sec(x)\,dx$

Exercise 6.13 Compute $\displaystyle\int \tan(x)\,dx$

Exercise 6.14 Compute $\int \sec(x)\tan^3(x)\,dx$

Exercise 6.15 Compute $\int \sec^2(x)\tan^2(x)\,dx$

Learning Goal 6.12. Using the method of trigonometric substitution, solve integrals, both definite and indefinite, of rational functions and radicals (polynomials divided by polynomials and possibly multiplied by square roots) and solve. Techniques of completing the square might also be required. [Comprehension, Application]

Exercise 6.16 Compute $\int_{1/2}^{1} \frac{dx}{\sqrt{2x-x^2}}$

Exercise 6.17 Compute $\int \frac{\sqrt{9-x^2}}{x^2}\,dx$

Exercise 6.18 Compute $\int \frac{dx}{x^2\sqrt{x^2+4}}$

Exercise 6.19 Compute $\int \frac{\sqrt{x^2-9}}{x^3}\,dx$

Learning Goal 6.13. Using the method of partial fractions, compute integrals of rational functions (polynomials divided by polynomials) where the degree of the numerator is less than the degree of the denominator and the degree of the denominator is at most two. [Comprehension, Application]

Exercise 6.20 Compute $\int_{0}^{1} \frac{x-1}{x^2+3x+2}\,dx$

Exercise 6.21 Compute $\int_{0}^{1} \frac{2x+3}{x^2+2x+1}\,dx$

Exercise 6.22 Compute $\int \frac{dx}{x\sqrt{x+1}}$

Learning Goal 6.14. Using integration by parts (possibly multiple times or in conjunction with other methods), compute definite and indefinite integrals. [Application]

Exercise 6.23 Compute $\int_{1}^{\ln(2)} xe^x\,dx$

Exercise 6.24 Compute $\int x\ln(x)\,dx$

Exercise 6.25 Compute $\int \ln(x)\,dx$

Exercise 6.26 Compute $\int \ln(x)/x \, dx$

Exercise 6.27 Compute $\int x^2 \cos(x) \, dx$

Exercise 6.28 Compute $\int x \sin(2x + 3) \, dx$

Learning Goal 6.15. Compute integrals using integration by parts where the integral cycles. [Application]

Exercise 6.29 Compute $\int e^x \sin(x) \, dx$

Learning Goal 6.16. Identify which of the above techniques is appropriate and compute integrals using a method of choice. [Comprehension, Application]

Relevant exam questions from previous years can be found at the following links

- wiki.ubc.ca/Category:MER Tag Substitution

- wiki.ubc.ca/Category:MER Tag Integration by parts

- wiki.ubc.ca/Category:MER Tag Trigonometric integral

- wiki.ubc.ca/Category:MER Tag Trigonometric substitution

- wiki.ubc.ca/Category:MER Tag Partial fractions

- wiki.ubc.ca/Category:MER Tag Integrals that cycle

7   Improper Integrals

Learning Goal 7.1. Describe the two types of improper integrals discussed in this course. [Knowledge]

Learning Goal 7.2. Show divergence of integrals for improper integrals even if not prompted that the integral is improper. [Application, Evaluation, Proof]

Exercise 7.1   Compute $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\, dx$.

Learning Goal 7.3. Compute improper integrals when they exist (using proper notation, that is, breaking into one sided limits as necessary). [Application]

Exercise 7.2   Compute $\displaystyle\int_{0}^{\infty} e^{-4x}\, dx$

Exercise 7.3   Compute $\displaystyle\int_{1}^{\infty} \frac{1}{x^e}\, dx$

Exercise 7.4   Compute $\displaystyle\int_{-1}^{0} \frac{e^{1/x}}{x^3}\, dx$

Learning Goal 7.4. State the integral comparison test. [Knowledge]

Learning Goal 7.5. Use the integral comparison test to show that an integral converges. [Proof]

Exercise 7.5   Show that the following integral is convergent $\int_{0}^{\infty} \frac{x}{x^3+1}\, dx$

Learning Goal 7.6. State l'Hopital's rule. [Knowledge]

Learning Goal 7.7. Use l'Hopital's rule to compute limits of functions. [Application]

Exercise 7.6   Compute $\displaystyle\lim_{x\to\infty}\frac{\ln(x)}{x}$

Exercise 7.7   Compute $\displaystyle\lim_{x\to\infty}\frac{e^x}{x^2}$

Exercise 7.8   Compute $\displaystyle\lim_{x\to\infty}\frac{\ln\ln x}{\ln x}$

Relevant exam questions from previous years can be found at the following links

- wiki.ubc.ca/Category:MER Tag Improper integral

- wiki.ubc.ca/Category:MER Tag Integral comparison test

- wiki.ubc.ca/Category:MER Tag L'Hopital's rule

8 Continuous Probability Distributions

Learning Goal 8.1. Define a probability density function. [Knowledge]

Learning Goal 8.2. Define a cumulative distribution function. [Knowledge]

Learning Goal 8.3. Normalize functions so that they become probability density functions. [Comprehension, Application]

Exercise 8.1 Normalize $f(x) = \cos(\pi x/6) + 2$ on $0 \le x \le 9$ so that it is a probability density function.

Learning Goal 8.4. Given a probability density function, compute a cumulative distribution function and vice versa. [Application]

Exercise 8.2 In the previous example, compute the cumulative distribution function of the associated probability density function.

Learning Goal 8.5. Given a probability density function or a cumulative density function, compute the probability that a random variable takes on a value between a and b. [Analysis]

Exercise 8.3 Let $p(x) = x/2$ be a probability density function defined on $[0, 2]$. Compute the probability that $0.5 \le x \le 1.5$.

Learning Goal 8.6. Define the median and the mean, also known as the average or expected value, of a probability density function. [Knowledge]

Learning Goal 8.7. Compute the median x value and the mean of a probability density function. [Application]

Exercise 8.4 Compute the mean and median of $f(t) = 5e^{-5t}$ given that this function defines a probability density function on $[0, \infty)$.

External Example. See question 1.(b) on the Math 103 exam from 2011

External Example. See also Section 8.3.2 on page 162 of the course notes.

External Example. Raindrops in section 8.4 of your notes

External Example. See question 6 parts a through c on the Math 103 exam from 2013

Exercise 8.5 Suppose there is a mysterious fish species in the sea, where for each fish in the species its weight w in grams follows the probability density function

$$p(w) = \frac{2}{\sqrt{\pi}} e^{-w^2}$$

where $0 \leq w < \infty$ (there is no limit for the maximum weight). For some mysterious reason , each fish has a concentration of radioactive material. Suppose that the amount of radioactive material m (in micro milligrams) in each fish in this species depends on the weight w of the fish as $m = \ln(1 + w)$. Find the probability density function of m.

Learning Goal 8.13. Compute the n-th moment (for $n = 0, 1, 2$), variance and standard deviation of a given probability density function. [Application]

Exercise 8.6 Compute the n-th moment for $n = 0, 1, 2$, variance and standard deviation of $f(t) = e^{-t}$ given that this function defines a probability density function on $[0, \infty)$.

Relevant exam questions from previous years can be found at the following links

- wiki.ubc.ca/Category:MER Tag Probability density function

- wiki.ubc.ca/Category:MER Tag Standard deviation (continuous)

- wiki.ubc.ca/Category:MER Tag Mean (continuous)

- wiki.ubc.ca/Category:MER Tag Median (continuous)

- wiki.ubc.ca/Category:MER Tag Cumulative distribution function

9 Differential Equations

Learning Goal 9.1. Define what a differential equation is. [Knowledge]

Learning Goal 9.2. Solve a (first order) differential equation using separation of variables. [Application]

Exercise 9.1 Solve $\frac{dy}{dt} = ky$

Exercise 9.2 Solve $\frac{dy}{dt} = \frac{y}{t}$

Learning Goal 9.3. Solve an initial value problem using separation of variables. [Application]

Exercise 9.3 Solve $\frac{dy}{dt} = ky^2$ subject to $y(0) = 7$ and $y(1) = -14/5$

Exercise 9.4 Solve $\frac{dy}{dt} = 3t^2 e^{-y}$ subject to $y(0) = 1$

Learning Goal 9.4. Define a steady state. [Knowledge]

Learning Goal 9.5. Compute steady states (equilibria) of differential equations. [Application]

Exercise 9.5 Find the steady states of $\frac{dy}{dx} = y^2 + 3y + 2$

Learning Goal 9.6. Given a graph of a derivative, be able to interpret the information and make inferences about the original function. [Comprehension]

External Example. Check out question 5 (b) from the Math 103 exam from 2013.

Learning Goal 9.7. Construct differential equations from a given scenario and solve for their solution (ie understanding that rates of changed are measured by rate in minus rate out). [Application]

External Example. Blood alcohol and chemical kinetics found on page 187-190 in section 9.4 of your notes

External Example. Torricelli's law as stated on page 190 and 191 of the course notes (section 9.5)

Exercise 9.6 Mixing problem. A tank contains 20kg of salt dissolved in 5000L of water. Brine that contains 0.03kg or salt per litre enters the tank at a rate of 25L/min. The solution is kept thoroughly mixed and drains out at the same rate. How much salt remains in the tank after half an hour?

External Example. Here is an example from the Math 102 differential calculus exam from December 2012 Question C4a.

Learning Goal 9.8. Describe the logistic equation and explain why it is a better model than the naive population growth model $\frac{dy}{dx} = ky$. [Knowledge, Evaluation]

Relevant exam questions from previous years can be found at the following links

- wiki.ubc.ca/Category:MER Tag Differential equation

- wiki.ubc.ca/Category:MER Tag Separation of variables

- wiki.ubc.ca/Category:MER Tag Initial value problem

- wiki.ubc.ca/Category:MER Tag Steady states

10 Sequences

Learning Goal 10.1. Define a sequence (in the context of this course). Also, define the head and tail of a sequence. [Knowledge]

External Example. A good exercise is to take the idea of Newton's method from the previous term and put it in the context of sequences.

Learning Goal 10.2. Compute the first terms of a sequence given a formula (including recursive formulas) or given the first few terms. [Comprehension]

Learning Goal 10.3. Be able to manipulate sequences and demonstrate an understanding of the sequence notation. Give examples of sequences, including the harmonic sequence and the Fibonacci sequence. [Knowledge]

Learning Goal 10.4. Understand and define what it means for a sequence to converge (formal definition not required but a working understanding is required). [Knowledge, Comprehension]

Learning Goal 10.5. Define what it means for a sequence to be bounded above, bounded below, bounded, monotonic, increasing, decreasing, non-increasing and non-decreasing. [Knowledge]

Learning Goal 10.6. Understand the sentence "the head of a sequence does not determine convergence". [Comprehension]

Learning Goal 10.7. Compute the convergence of sequences both formally and using known facts about growth rates of functions. [Application]

Exercise 10.1 Determine the convergence of $\left(\frac{k!}{k^k}\right)_{k \geq 0}$. If it converges, compute the limit.

Exercise 10.2 Determine the convergence of $(1, 1, 1, 1, ...)$. If it converges, compute the limit.

Exercise 10.3 Determine the convergence of $(1, 1/2, 1/3, 1/4, ...)$. If it converges, compute the limit.

Exercise 10.4 Determine the convergence of $\left(\frac{k}{\ln(k)^{102}}\right)_{k \geq 0}$. If it converges, compute the limit.

Learning Goal 10.8. Use $a_n = e^{\ln a_n}$ to compute limits. [Application]

Exercise 10.5 Determine if the sequence defined by $a_n = \left(1 + \frac{1}{n}\right)^n$ converges and if so compute the limit.

Learning Goal 10.9. State l'Hopital's rule (for sequences). [Knowledge]

Learning Goal 10.10. Use l'Hopital's rule to compute limits of sequences. [Application]

External Example. See the section on improper integrals in this file for some examples.

Learning Goal 10.11. State the squeeze theorem. [Knowledge]

Learning Goal 10.12. Use the squeeze theorem to compute limits. [Application]

Exercise 10.6 Compute if the sequence $\{\frac{\sin(n)}{n}\}_{n=1}^{\infty}$ converges.

Learning Goal 10.13. Use the fact that if $f(x)$ is a continuous real valued function and $\lim_{k \to \infty} a_k$ exists, then $\lim_{k \to \infty} f(a_k) = f(\lim_{k \to \infty} a_k)$. Recognize examples where this fails but we still have a limit. [Application]

Exercise 10.7 Compute if the sequence $\{\frac{\sin(1/n)}{(1/n)}\}_{n=1}^{\infty}$ converges.

Exercise 10.8 Compute if the sequence $\{\sin(n\pi)\}_{n=1}^{\infty}$ converges.

Learning Goal 10.14. Explain the process of cobwebbing and that the process computes. [Evaluation]

Learning Goal 10.15. Compute examples of a cobwebbing both numerically and graphically given a function, an initial point and a number of iterations. [Application]

External Example. See the 2013 exam, question 5 part (a) on cobwebbing.

Learning Goal 10.16. Discuss and compute the fixed points (or equilibria or steady states) of a [recursive] sequence and be able to distinguish this from the steady states of a differential equation. [Comprehension, Application, Evaluation]

External Example. See the 2013 exam question 5 problems (a) and (b) on steady states.

Learning Goal 10.17. Define what it means for fixed points of a recurrence sequence to be stable or unstable both analytically and graphically. [Knowledge]

Learning Goal 10.18. Classify fixed points as stable or unstable. [Comprehension]

External Example. See the examples on the difference equation and logistic maps in section 10.7-10.8 (pages 221-227) in the notes.

Learning Goal 10.19. Compare the logistic map to the logistic differential equation and evaluate their similarities and differences. [Evaluation]

Relevant exam questions from previous years can be found at the following links

- wiki.ubc.ca/Category:MER Tag Sequences

- wiki.ubc.ca/Category:MER Tag Steady states (sequences)

- wiki.ubc.ca/Category:MER Tag Cobwebbing

11   Series

Learning Goal 11.1. Define a series (in the context of this course). [Knowledge]

Learning Goal 11.2. Define a partial sum. [Knowledge]

Learning Goal 11.3. State what it means for an infinite series to converge. [Knowledge]

Learning Goal 11.4. State the harmonic series and justify why it diverges. [Knowledge, Evaluation]

Learning Goal 11.5. Give conditions for an infinite geometric series to converge and evaluate such series. [Knowledge]

External Example. Refer to chapter 1 in this document for practice exercises.

Learning Goal 11.6. Use telescoping series to evaluate infinite series (you will likely need to apply a partial fractions technique as well). [Application]

Exercise 11.1 Compute $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

Exercise 11.2 Compute $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

Learning Goal 11.7. State the divergence test. [Knowledge]

Learning Goal 11.8. Know examples of sequences $\{a_n\}_{n=1}^{\infty}$ where a_n converges to 0 but $\sum_{n=1}^{\infty} a_n$ still diverges. [Knowledge, Comprehension]

> **Learning Goal 11.9.** Apply the divergence test to show a series diverges. [Application, Proof]

Exercise 11.3 Determine if $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ converges.

Exercise 11.4 Determine if $\sum_{n=1}^{\infty} \frac{n(n+2)}{n^2+3}$ converges.

> **Learning Goal 11.10.** State the integral test. [Knowledge]

> **Learning Goal 11.11.** Apply the integral test to show a series converges or diverges. [Application, Proof]

Exercise 11.5 Determine if $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converges.

Exercise 11.6 Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 9}$ converges.

Exercise 11.7 Determine for which values of p does the sum $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converge.

> **Learning Goal 11.12.** State the p-series test. [Knowledge]

> **Learning Goal 11.13.** Apply the p-series test to show a series converges or diverges. [Application, Proof]

Exercise 11.8 Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.000000001}}$ converges.

> **Learning Goal 11.14.** State the comparison test. [Knowledge]

> **Learning Goal 11.15.** Apply the comparison test to show a series converges or diverges. [Application, Proof]

Exercise 11.9 Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n^7 + 9}$ converges.

Learning Goal 11.16. State the absolute comparison test. [Knowledge]

Learning Goal 11.17. Apply the absolute comparison test to show a series converges or diverges. [Application, Proof]

Exercise 11.10 Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n^7 + 9}$ converges.

Learning Goal 11.18. Describe what the notation $n!$ means for a nonnegative integer n. [Knowledge]

Learning Goal 11.19. State the ratio test. [Knowledge]

Learning Goal 11.20. Apply the ratio test to show a series converges or diverges. [Application, Proof]

Exercise 11.11 Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 2^n}{n!}$ converges.

Learning Goal 11.21. Determine the convergence of a series by using one of the previous tests (without prompt). [Application, Proof]

Exercise 11.12 Determine which of the following series converge or diverge.

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \arctan(n)$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 1}{3n^4 + 1}$

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

(v) $\displaystyle\sum_{n=3}^{\infty} \frac{n!}{2^n}$

(vi) $\displaystyle\sum_{n=3}^{\infty} \frac{n^3-1}{n^3+1}$

(vii) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{\sqrt{n^2+1}}$

Learning Goal 11.22. Apply the ratio test to show the interval of convergence for a power series. [Application, Proof]

Exercise 11.13 Find the interval of convergence for $\displaystyle\sum_{n=3}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$.

Relevant exam questions from previous years can be found at the following links

- wiki.ubc.ca/Category:MER Tag Series

- wiki.ubc.ca/Category:MER Tag Power series

12 Taylor Series

Learning Goal 12.1. Define the nth degree Taylor polynomial and the Taylor series of a function $f(x)$ centred at a point $x = a$ (typically a, our centre, will be 0). [Knowledge]

Learning Goal 12.2. Explain what Taylor series are used for. [Evaluation]

Learning Goal 12.3. Know the power series expansions for $1/(1-x)$, $\ln(1-x)$, e^x, $\sin(x)$, $\cos(x)$, $\arctan(x)$. [Knowledge, Application]

Learning Goal 12.4. Compute the Taylor series expansion (either the first few terms or the entire series in sigma notation) of a function either directly using the definition or using the above known Taylor series and combinations of differentiating or integrating. [Application]

Exercise 12.1 Compute the Taylor series for $1/(1-x)^3$ centred at 0.

External Example. Check out the final problem on the MATH 103 exam from 2013.

Learning Goal 12.5. Compute the Taylor series expansion (either the first few terms or the entire series in sigma notation) of a function using the above known Taylor series and a combination of multiplying, composing and or adding Taylor series. [Application]

Exercise 12.2 Compute the first 3 nonzero terms for the Taylor series for $x^2 \sin(x^3)$ centred at 0.

Exercise 12.3 Compute the Taylor series for $x^3 \cos(x^2)$ using summation notation centred at 0.

Learning Goal 12.6. Use Taylor series to evaluate limits. [Comprehension, Application]

Learning Goal 12.7. Use Taylor series to evaluate derivatives of functions. [Comprehension, Application]

Exercise 12.4 Compute the 19th and 20th derivative of $x \arctan(x^2)$ at the point $x = 0$.

Learning Goal 12.8. Use Taylor series to evaluate integrals of functions. [Comprehension, Application]

External Example. See question 7 part (b) from the 2011 exam in Math 103.

Exercise 12.5 Evaluate $\displaystyle\int_0^{1/2} \sum_{n=0}^{\infty} nx^{n-1}\, dx$

Exercise 12.6 Evaluate $\displaystyle\int_0^{1/2} \sin(t^2)\, dt$ using Taylor series

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Learning Goal 12.9. Use Taylor series to determine solutions to differential equations. [Comprehension, Application]

External Example. See question 2 part (c) from the 2011 exam in Math 103.

Exercise 12.7 Using a Taylor series centred at 0, solve $\frac{dy}{dx} = 1 + xy$ subject to $y(0) = 3$. What are the first four non-zero terms?

Relevant exam questions from previous years can be found at the following links

- wiki.ubc.ca/Category:MER Tag Taylor series
- wiki.ubc.ca/Category:MER Tag Power series

13 Summary

Knowledge (Total : 54). These types of question refer to rote memorization of information and the ability to repeat the information exactly.

Comprehension (Total : 23). This involves the ability to demonstrate an understanding of key facts of gained knowledge.

Application (Total : 61). This involves taking the knowledge points and actually using the information to solve concrete problems.

Analysis (Total : 8). This involves breaking down complex parts into simple components and applying skills learnt to each part.

Evaluation (Total : 13). This involves making judgements or offering explanations on certain content.

Proof (Total : 11). This could have been grouped with evaluation above but warrants a bit of extra emphasis. In problems with this flag, the process is far more valuable than the direct answer to a problem. With these problem, there is more of a focus on understand how parts fits together and being able to explain your answer with proper justification to another person.

14 Solutions

Solution to 1.1 $1 + e + e^2 + e^3 + e^4 + e^5$

Solution to 1.2 $\displaystyle\sum_{i=1}^{4}(i^2 + 1)$ (there are many answers)

Solution to 1.3 $\ln(6) + \ln(8) + \ln(10)$

Solution to 1.4 385

Solution to 1.5 1001

Solution to 1.6 3141

Solution to 1.7 $n(n+1)$ (this is not a typo!)

Solution to 1.8 The answer can be reduced in lowest terms to $\frac{19(10)(41)}{3}$ which is not an integer. Since we are adding integers, we need to get an integer answer. Thus the answer cannot be correct.

Solution to 1.9 $\displaystyle\sum_{j=5}^{27}(j+1)^3 + j + 2$

Solution to 1.10 n

Solution to 1.11 3069

Solution to 1.12 3066

Solution to 1.13 $\dfrac{15\left(\frac{5}{4}\right)^8\left(1-\left(\frac{5}{4}\right)^6\right)}{1-\frac{5}{4}}$

Solution to 1.14 $1 - \frac{1}{N+1}$ (Telescoping sum)

Solution to 1.15 25

Solution to 1.16 3

Solution to 1.17 $24209/9900$.

Solution to 1.18 $1/7$

Solution to 1.19 $\frac{15}{2}$

Solution to 1.20 This sum diverges.

Solution to 1.21 $\frac{3}{2}$

Solution to 1.22 Max: $80/(1 - 2^{-12/11})$, Min: $80 \cdot 2^{-12/11}/(1 - 2^{-12/11})$

Solution to 2.1 Left: $\ln(2) + \ln(3) + \ln(4)$

Right: $\ln(2) + \ln(3) + \ln(4) + \ln(5)$

Midpoint: $\ln(1.5) + \ln(2.5) + \ln(3.5) + \ln(4.5)$

Solution to 2.2 $\frac{26}{3}$

Solution to 2.3 $\frac{45}{2}$

Solution to 2.4 $\int_0^2 (x^2 + 1)\, dx$ (there are other answers)

Solution to 2.5 $\lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{3}{n}\right)\left(e^{(2+3i/n)^2} + (2 + 3i/n)\right)$

Solution to 2.6 $\int_0^{\pi/8} \tan(x/5)\, dx$ (there are other answers)

Solution to 2.7 $9/2$

Solution to 2.8 $2\pi + 4$

Solution to 2.9 $8/3$

Solution to 3.1 7

Solution to 3.2 $e^2 - e^{-2}$

Solution to 3.3 $\frac{\pi}{2}$

Solution to 3.4 e^{x^2}

Solution to 3.5 $2x(x^2)^{x^2} - \cos(x)(\sin(x))^{\sin(x)}$

Solution to 3.6 The function has a discontinuity at $x = 0$ and so the Fundamental Theorem of Calculus cannot be used.

Solution to 3.7 This function has a singularity (point of discontinuity) at $\pi/2$ and so cannot be integrated using techniques up to this point.

Solution to 3.8 Differentiating $\tan(x)\cos(x)$ gives

$$\sec^2(x)\cos(x) - \tan(x)\sin(x)$$

which is equivalent to the original function.

Solution to 3.9 Domains of logarithms are positive real numbers. Correct by recalling that
$$\int_{-3}^{-1} \frac{1}{x} \, dx = \ln|-1| - \ln|-3|.$$

Solution to 3.10 0

Solution to 3.11 8/3

Solution to 3.12 1/12

Solution to 3.13 64/3

Solution to 3.14 9/2

Solution to 4.1 $v(t) = t^2/2 + 4t + 5$

Solution to 4.2 Displacement -9 and the distance traveled 9 (integrate $|v(t)|$).

Solution to 4.3 Displacement $-9/2$ and the distance traveled 61/6 (integrate $|v(t)|$).

Solution to 4.4 5100

Solution to 4.5 $25/e^2 - 25/e^{10}$

Solution to 4.6 0

Solution to 5.1 Mass is 150, average mass is 15, centre of mass is 20/3.

Solution to 5.2 Average mass is 15. Point to cut the bar in half to get two pieces of equal mass is $x = 5\sqrt{2}$.

Solution to 5.3 $\int_0^2 \pi(4x^4 - x^6) \, dx = \frac{256\pi}{35}$

Solution to 5.4 $\int_0^{a^2} \pi(a^2 - x) \, dx = a^4\pi/2$

Solution to 5.5 $5\pi/6$

Solution to 5.6 $7\pi/15$

Solution to 5.7 $\pi/6$

Solution to 5.8 $\int_{-1}^{1} \pi(x^2 - 1)^2 \, dx = 16\pi/15$

Solution to 5.9 $\int_{-1}^{0} \pi(\sqrt{y+1})^2 \, dy = \pi/2$

Solution to 5.10 $\int_{-1}^{1} \pi(1^2 - (x^2 - 1 + 1)^2) \, dx = 8\pi/5$

Solution to 5.11 $\int_{-1}^{0} \pi((\sqrt{y+1}+1)^2 - (1-\sqrt{y+1})^2)\,dy = 8\pi/3$

Solution to 5.12 $14/3$

Solution to 5.13 $59/24$

Solution to 5.14 $-2/243 + 164/(243\sqrt{82})$

Solution to 6.1 $(x+1)^{103}/103 + C$

Solution to 6.2 $2/3$

Solution to 6.3 $2(x+7)^{3/2}(3x-14)/15$

Solution to 6.4 $\ln|\sec(x) + \tan(x)| + C$

Solution to 6.5 $-\ln|\csc(x) + \cot(x)| + C$

Solution to 6.6 $-1/(2(2x-3)) + C$ and $1/3$

Solution to 6.7 $\arctan((x+5)/5)/5 + C$

Solution to 6.8 $x/2 - \sin(2x)/4 + C$

Solution to 6.9 $-\cos^4(x)/4 + C$

Solution to 6.10 $\sin(x) - \sin^3(x)/3 + C$

Solution to 6.11 $\ln|\sin(x)| - \frac{\sin^2(x)}{2} + C$

Solution to 6.12 $\ln|\sec(x) + \tan(x)| + C$

Solution to 6.13 $-\ln|\cos(x)| + C$ or $\ln|\sec(x)| + C$

Solution to 6.14 $\sec^3(x)/3 - \sec(x) + C$

Solution to 6.15 $\tan^3(x)/3 + C$

Solution to 6.16 $\pi/6$

Solution to 6.17 $-\frac{\sqrt{9-x^2}}{x} - \arcsin(x/3) + C$

Solution to 6.18 $-\frac{\sqrt{x^2+4}}{4x} + C$

Solution to 6.19 $\operatorname{arcsec}(x/3)/6 - \sqrt{x^2-9}/(2x^2) + C$

Solution to 6.20 $3\ln 3 - 5\ln 2$

Solution to 6.21 $2\ln 2 - 1/2$

Solution to 6.22 $\ln|\sqrt{x+1}-1| - \ln|\sqrt{x+1}+1| + C$

Solution to 6.23 $\ln(4) - 2$

Solution to 6.24 $x^2 \ln(x)^2/2 - x^2/4 + C$

Solution to 6.25 $x\ln(x) - x + C$

Solution to 6.26 $\ln(x)^2/2 + C$ (you probably want to use a substitution here!)

Solution to 6.27 $x^2 \sin(x) - 2\sin(x) + 2x\cos(x) + C$

Solution to 6.28 $\sin(2x+3)/4 - (2x+3)\cos(2x+3)/4 + 3\cos(2x+3)/4 + C$

Solution to 6.29 $-e^x \cos(x)/2 + e^x \sin(x)/2 + C$.

Solution to 7.1 This integral is improper (there is a discontinuity at 0) and also the integral diverges (don't forget to show your work!).

Solution to 7.2 $1/4$

Solution to 7.3 $1/(e-1)$.

Solution to 7.4 $-2/e$.

Solution to 7.5 Briefly, $\int_0^\infty \frac{x}{x^3+1}\, dx < \int_0^\infty \frac{1}{x^2}\, dx$ which converges (you will need to justify these claims on an exam for full marks).

Solution to 7.6 0

Solution to 7.7 ∞

Solution to 7.8 0

Solution to 8.1 $\pi(\cos(\pi x/6) + 2)/(-6 + 18\pi)$

Solution to 8.2 $(3\sin(\pi x/6) + \pi x)/(-3 + 9\pi)$

Solution to 8.3 $1/2$

Solution to 8.4 Mean: $1/5$. Median: $(\ln 2)/5$.

Solution to 8.5 $q(m)\frac{2}{\sqrt{\pi}}e^{m-(e^m-1)2}$

Solution to 8.6 The n-th moments are $1, 2, 4$ for $n = 0, 1, 2$ (for a bonus problem, show that the n-th moment is $n!$ for any n. Variance is 1 and so is the standard deviation.

Solution to 9.1 $y = Ce^{kt}$

Solution to 9.2 $y = kt$

Solution to 9.3 $y = \frac{14}{2-7t^2}$

Solution to 9.4 $y = \ln(t^3 + e)$

Solution to 9.5 $y = -1, -2$

Solution to 9.6 $150 - 130e^{-3/20}$

Solution to 10.1 0 since $k! \ll k^k$ (this is sufficient justification in this case).

Solution to 10.2 Converges to 1

Solution to 10.3 Converges to 0

Solution to 10.4 Diverges since $k \gg \ln(k)^{102}$.

Solution to 10.5 Converges to e.

Solution to 10.6 Yes converges to 0.

Solution to 10.7 Yes converges to 1 (think of this as a function and recognize this as the derivative of $\sin(x)$ after substituting $x = 1/n$ or use l'Hopital's rule.

Solution to 10.8 Yes converges to 0 (this is a constant sequence!)

Solution to 11.1 $3/2$

Solution to 11.2 $11/6$

Solution to 11.3 Diverges by the divergence test.

Solution to 11.4 Diverges by the divergence test.

Solution to 11.5 Diverges

Solution to 11.6 Converges

Solution to 11.7 $p > 1$

Solution to 11.8 Converges

Solution to 11.9 Converges

Solution to 11.10 Converges

Solution to 11.11 Converges

Solution to 11.12 Diverges (integral test), diverges (divergence test), converges (comparison + p-series test), converges (absolute + p-series test), diverges (ratio test), diverges (divergence test), diverges (comparison + p-series test).

Solution to 11.13 $-5 < x < 1$.

Solution to 12.1 $\displaystyle\sum_{n=2}^{\infty} \frac{n(n-1)}{2} x^{n-2}$.

Solution to 12.2 $x^5 - x^{11}/6 + x^{17}/120$

Solution to 12.3 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n)!}$

Solution to 12.4 $19!/9$, 0

Solution to 12.5 2

Solution to 12.6 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (1/2)^{4n+3}}{(2n+1)!(4n+3)}$

Solution to 12.7 $y = 3 + x + 3x^2/2 + x^3/2$