A class of term structures for SVI Implied Volatility

Sébastien Gurrieri *

Risk Management Department, Mizuho Securities
Tokyo

Abstract

We derive a set of sufficient conditions on the parametric forms of the Stochastic Volatility Inspired (SVI) implied volatility model parameters in order to satisfy the no-calendar spread arbitrage constraint while preserving the (necessary) condition of no-strike arbitrage. We propose a strategy to find solutions to these constraints and give one such example. We fit it to the market data of USD/JPY and show that the good fitting quality of SVI model is essentially preserved. We exhibit the Dupire local volatilities based on the original and term structure models and illustrate how the term structure of the implied volatility leads to a much smoother behavior in the time direction and more reasonable call calendar spread prices. We also show that the local volatility calibration remains robust even in the presence of arbitrage in the market data.

Keywords: SVI, term structure, no-arbitrage, calendar spread, local volatility, implied volatility.

*Dr. Sébastien Gurrieri
Otemachi First Square
1-5-1, Otemachi, Chiyoda-ku, Tokyo 100-0004, Japan
tel: +81-3-5208-3736
fax: +81-3-3516-9744
email: sebastien.gurrieri@mizuho-sc.com

Contents

1 Introduction

Many different methods have been proposed to parametrize the implied volatility surface, including modeling of the vanilla prices, of the implied volatility, or even of the probability density (see (Rebonato, 2004) for a good review). (Alexander, 2001) and (Brigo & Mercurio, 2000) suggest to model the probability density by a mixture of lognormal distributions at each maturity, preventing arbitrage in the strike direction and ensuring smoothness of the probability density. (Bloch, 2010) gives an example of term–structure for the lognormal mixture parameters and shows how to achieve no-arbitrage in the time direction as well. In this manner, the whole implied volatility surface is arbitrage-free and smooth enough to be used safely to derive local volatilities. However, the implied volatility function is available only indirectly, through the calculation of Black prices and subsequent numerical inversion. Furthermore, the relation between the model parameters and the implied volatility is not always intuitive.

As traders often prefer manipulating the implied volatility rather than the density, approaches modeling the implied volatility function $\theta(K, T)$ are very popular. Among them, the Stochastic Volatility Inspired (SVI) model is particularly appealing (Gatheral, 2004). It is smooth in the strike direction, and its 5 parameters at each maturity have intuitive interpretations in terms of implied volatility changes. Furthermore, it has proper high/low strike limiting behaviors, and has close relations to Heston model (Gatheral & Jacquier, 2010).

On the other hand, the no-arbitrage conditions are much more difficult to enforce than in a density-modeling approach. (Rogers & Tehranchi, 2008) give a necessary condition for no-strike arbitrage. This was translated into a condition on SVI parameters by (Gatheral, 2004), who further observes that provided the model is fitted to the market and the constraint is enforced, arbitrage almost never occurs in practice.

However, as far as we know, the question of no-arbitrage in SVI surface in the time direction does not seem to be addressed yet. It is our purpose to fill in this gap in the present document.

Adapting the strategy of (Bloch, 2010) to the case of SVI, we allow the model parameters to be functions of time and derive a set of sufficient conditions on these functional forms such that no calendar spread arbitrage can exist. We further prove the existence of solutions, and propose a strategy to find explicit examples.

We then fit one such example of surface to the market data of USD/JPY and show that the good fitting quality of the original SVI model is essentially preserved. We display the behavior of Dupire local volatilities calculated from the original SVI and the term structure surfaces. We show that the roughness of the market data, translating into discontinuities in the local volatility, is also responsible for odd prices for call calendar spread produces, and show that these difficulties do not exist in the term structure model. We further show that, in the presence of arbitrage opportunities in the market, although the original model may lead to ill-defined local volatilities, the term structure model does not. This makes the term structure SVI surface particularly suitable for pricing exotics under a Dupire local volatility framework.

In section 2 we review the original formulation of SVI model and show a typical example of fit to the market data, in order to illustrate the qualities of the model as well as its shortcomings.

In section 3 we derive a set of sufficient constraints on the parametric forms $a(T)$, $b(T)$, $\rho(T)$, $m(T)$, $\sigma(T)$ in order to satisfy the no-arbitrage condition in the time direction. These constraints also preserve the no-arbitrage necessary condition in the strike direction, as in the original formulation of the model. However, they do not fully fix the parametric forms such that we are left with some flexibility to choose the remaining degrees of freedom.

In section 4 we propose an example of fully specified term structure for SVI model which satisfies the no-calendar spread arbitrage and the (necessary) no-strike arbitrage conditions. We then test it numerically in terms of quality of the fit and of the corresponding local volatility, of the prices of call calendar spreads, and show its robustness in the presence of arbitrage opportunities in the market. We conclude in section 5.

2 Review of SVI Model

2.1 Original formulation

In its original formulation (Gatheral, 2004), SVI model is defined at each maturity T in terms of the 5 parameters a, b, ρ, m, σ such that the square of the implied volatility $\theta(K,T)$ is

$$\begin{aligned} \theta^2(K,T) &= v(x,T) = a + b\Big(\rho(x-m) + \sqrt{(x-m)^2+\sigma^2}\Big) \\ x &= \ln(K/F(T)) \end{aligned} \tag{1}$$

where $F(T)$ is the forward and the parameters lie in the following definition domain

$$b > 0 \tag{2}$$

$$\sigma \geq 0 \tag{3}$$

$$\rho \in [-1,1] \tag{4}$$

$$a \geq -b\sigma\sqrt{1-\rho^2}. \tag{5}$$

(Rogers & Tehranchi, 2008) derived the necessary condition for no-strike arbitrage

$$\forall x,\, \forall T, \quad |\frac{\partial v(x,T)}{\partial x}| \leq 4, \tag{6}$$

which (Gatheral, 2004) translated into the additional constraint on the parameters

$$b \leq \frac{4}{(1+|\rho|)T}. \tag{7}$$

The function $v(x,T)$ (at fixed T) exhibits a global minimum at the point (x^*, v^*) with

$$x^* = m - \frac{\rho\sigma}{\sqrt{1-\rho^2}} \tag{8}$$

$$v^* = a + b\sigma\sqrt{1-\rho^2}. \tag{9}$$

Changes in the parameters a, b, ρ, m, σ produce easily understandable changes in $\theta(x,T)$, which was nicely illustrated in (Gatheral, 2004).

2.2 Fit to the market data

As an illustration, we take the FX index USD/JPY on the date 2010/07/02 and fit SVI model at 11 maturities ranging from 1W to 5Y. In table 1 we display the market volatilities together with the model error (absolute difference) and in table 2 we give the corresponding SVI parameters.

Table 1: Market Implied Volatilities and SVI Model Error

	Market Implied Volatility					Model Error				
T	10Δ L	25Δ L	ATM	25Δ H	10Δ H	10Δ L	25Δ L	ATM	25Δ H	10Δ H
1W	16.13%	14.68%	13.53%	12.88%	12.73%	0.00%	0.01%	-0.01%	0.01%	0.00%
1M	16.30%	14.44%	13.10%	12.26%	12.10%	-0.01%	0.02%	-0.03%	0.02%	-0.01%
2M	17.15%	15.00%	13.55%	12.60%	12.55%	-0.02%	0.05%	-0.06%	0.04%	-0.02%
3M	17.45%	15.05%	13.55%	12.55%	12.65%	-0.03%	0.09%	-0.09%	0.07%	-0.03%
6M	18.30%	15.60%	14.00%	12.90%	13.10%	-0.04%	0.11%	-0.11%	0.08%	-0.04%
9M	18.75%	15.85%	14.10%	12.85%	13.05%	-0.05%	0.12%	-0.11%	0.07%	-0.03%
1Y	19.60%	16.40%	14.60%	13.20%	13.40%	-0.06%	0.15%	-0.14%	0.09%	-0.04%
2Y	20.74%	17.12%	15.00%	13.62%	13.32%	-0.09%	0.10%	0.00%	-0.08%	0.05%
3Y	21.06%	17.37%	15.15%	13.57%	13.27%	-0.06%	0.06%	0.01%	-0.05%	0.03%
4Y	21.59%	17.71%	15.40%	13.61%	13.33%	-0.05%	0.04%	0.01%	-0.04%	0.02%
5Y	22.30%	18.17%	15.70%	13.67%	13.31%	-0.06%	0.04%	0.03%	-0.05%	0.03%

Table 2: SVI Model Parameters

T	a	b	ρ	m	σ
1W	1.09%	0.192	-50%	1.03%	3.16%
1M	0.82%	0.133	-50%	1.64%	5.57%
2M	0.97%	0.105	-50%	2.17%	6.60%
3M	1.01%	0.092	-50%	2.22%	7.03%
6M	1.06%	0.073	-50%	2.84%	9.60%
9M	1.09%	0.062	-50%	4.05%	10.45%
1Y	1.04%	0.060	-50%	4.53%	13.52%
2Y	0.52%	0.053	-50%	5.77%	27.41%
3Y	0.35%	0.047	-50%	6.94%	35.08%
4Y	0.26%	0.043	-50%	8.02%	40.47%
5Y	0.21%	0.041	-50%	10.23%	44.20%

As noticed in (Zeliade, 2009), the fit of this model to the market is not trivial, as there are often local minima. Moreover, the handling of the parameter a may prove difficult, as it may vary over several orders of magnitude, with an unclear sign. We find for example that the Nelder-Mead method or even the Simulated Annealing tend to perform quite poorly.

(Zeliade, 2009) uses a decomposition of the minimization problem into an analytical and a numerical one, which improves the situation by reducing the number of dimensions of the numerical optimization. Alternatively, we found that these difficulties can be circumvented by using a different kind of optimization algorithm, namely the Differential Evolution (Storn & Price, 1995), together with extra constraints. The results in table 1 show that this strategy performs well for this optimization problem.

Indeed, a direct optimization on the 5 SVI parameters without further constraints tends to produce rather unstable parameters across maturities. With the view to calculating local volatilities, this is quite an undesirable feature. To improve the situation, we fix the correlation (to $\rho = -50\%$) and impose additional constraints relating maturities to each other:

1. aT is non-decreasing in time
2. v^*T is non-decreasing in time
3. $\sigma^2 T$ is non-decreasing in time.

Thanks to these constraints, the stability across maturities is improved, without significant loss of fitting quality. This is however an illustration of the fact that with such a large number of parameters (including all maturities), many different parameter sets can lead to an excellent fit. This in turns means that the optimization procedure may give ambiguous results, largely dependent on the algorithm and its control parameters.

3 Imposing No-Arbitrage

3.1 Change of variables

Although the original formulation of SVI model is already elegant in the sense that the behavior of $\theta(x,T)$ in terms of changes of a, b, ρ, m, σ is visually clear, we prefer a different formulation, more suitable for the optimization and for the analysis of the no-arbitrage condition in the time direction.

We would like to use the knowledge of the minimum (x^*, v^*) to rewrite the function $v(x)$ so as to make this minimum more apparent. To do so, we simply replace the couple (m, a) in eq. (1) by their expressions (8) and (9) and we find

$$v(x,T) = v^* + b\Big(\rho(x - x^*) + n_T(x - x^*) - \lambda\Big) \tag{10}$$

$$n_T(X) = \sqrt{X^2 - 2\rho X\lambda + \lambda^2} \tag{11}$$

in terms of the 5 parameters $v^*, b, \rho, x^*, \lambda$, related to the original a, b, ρ, m, λ by

$$
\begin{aligned}
v^* &= a + b\sigma\sqrt{1-\rho^2} && (12)\\
b &= b && (13)\\
\rho &= \rho && (14)\\
x^* &= m - \frac{\rho\sigma}{\sqrt{1-\rho^2}} && (15)\\
\lambda &= \frac{\sigma}{\sqrt{1-\rho^2}}. && (16)
\end{aligned}
$$

In these modified definitions, the parameter v^* truly represents the level of the variance, which is not the case for the original a.

3.2 No-Arbitrage constraints

The implied volatility surface is free of calendar spread arbitrage when

$$\forall x,\, \forall T, \quad \frac{\partial (T v(x,T))}{\partial T}|_x \geq 0. \tag{17}$$

We allow the parameters $v^*, b, \rho, x^*, \lambda$ to be functions of the time T, and of it only, and try to find conditions on them such that (17) is satisfied. Our statement, proved in appendix A, is that there is no calendar spread arbitrage if the functions $v^*(T), b(T), \rho(T), x^*(T), \lambda(T)$ satisfy the following conditions:

$$
\begin{aligned}
b(T) &\geq 0 && (18)\\
\lambda(T) &\geq 0 && (19)\\
\rho(T) &\in [-1, 1] && (20)\\
v^*(T) &\geq 0 && (21)\\
Tb(T) &\leq \frac{4}{1+|\rho|} && (22)\\
\frac{\partial Tb(T)}{\partial T} &\geq 0 && (23)\\
\frac{\partial \rho}{\partial T} &= 0 && (24)\\
\frac{\partial x^*}{\partial T} &= -\rho\frac{\partial \lambda}{\partial T} && (25)\\
\frac{\partial T v^*(T)}{\partial T} &\geq bT(1-\rho^2)\frac{\partial \lambda}{\partial T} \quad \text{when} \quad \frac{\partial \lambda}{\partial T} \geq 0, && (26)\\
\frac{\partial T v^*(T)}{\partial T} &\geq -bT(1-\rho^2)\left(\frac{1}{\sqrt{1-\rho^2}} - 1\right)\frac{\partial \lambda}{\partial T} \quad \text{when} \quad \frac{\partial \lambda}{\partial T} \leq 0. && (27)
\end{aligned}
$$

3.3 Existence of solutions

It is actually easy to find solutions to the system of constraints above. Indeed, one can start by specifying a constant value for the correlation ρ following (20). Then we choose an increasing function of time $Tb(T)$ bounded by above by the number $\frac{4}{1+|\rho|}$. Then we choose a monotonic function of time $\lambda(T)$, say increasing, and we obtain the differential of $x^*(T)$ by (25). For $x^*(T)$, we just need to choose the initial value $x^*(0)$. Finally, we can choose the differential of $Tv^*(T)$ as its bound in (26) plus a positive function of time, and finally choose $v^*(0)$.

The difficulty here is not to find a solution, but to find a solution that has good fitting properties and is financially reasonable. We address this issue in the next section.

4 An Example of no-arbitrage SVI term structure

Our goal is to find solutions that satisfy the sufficient constraints obtained in the previous section, have good fitting properties, and are financially reasonable. To this end, we develop a strategy that may be applied to find many different types of solutions, illustrate it with one explicit example, and test the resulting surface numerically. We find the parametric forms in the following order.

Parameter function $\rho(T)$

We choose a constant correlation parameter $\rho(T) = \rho$.

Parameter function $x^*(T)$

Its differential is fixed by (25) such that $x^*(T)$ can be expressed in terms of the parameters of $\lambda(T)$ and its initial value x_0^* as

$$x^*(T) = x_0^* - \rho(\lambda(T) - \lambda(0)). \tag{28}$$

Parameter function $b(T)$

We want $Tb(T)$ to be increasing to satisfy the no-calendar spread arbitrage condition, but we also want $b(T)$ to be decreasing, since we know the skew/smile strength should decrease with time. There are many ways to achieve this, and we find that the following gives a good fitting ability:

$$Tb(T) = \alpha T^{\beta} \tag{29}$$

with the 2 parameters $\alpha \geq 0$ and $0 < \beta < 1$. Note that the no-strike arbitrage necessary condition (7) cannot be satisfied for all T, since $Tb(T)$ is not bounded from above in the direction of large T. However, in practical situations, T is always limited as trades do not to exceed a certain limiting maturity. We can use this fact to satisfy the no-strike arbitrage necessary condition in all practical applications, by imposing

$$\alpha T_{Max}^{\beta} \leq \frac{4}{1 + |\rho|}. \tag{30}$$

where T_{Max} is the latest maturity involved in our portfolio.

Note that $b(T)$ goes to infinity at $T = 0$, but this is not necessarily a problem as we never fit nor use the implied volatility function at $T = 0$.

Parameter function $\lambda(T)$

We choose the monotonically increasing function of time

$$\lambda(T) = \lambda_0 + \frac{\gamma}{\delta + 1} T^{\delta+1} \tag{31}$$

with the 3 positive parameters $\lambda_0 \geq 0$, $\gamma \geq 0$, and $\delta > -1$.

Parameter function $v^*(T)$

This is the most difficult step as $v^*(T)$ gives the level of the implied volatility, such that it must have sufficient fitting ability. However, we also want it to satisfy the condition (26), so we take

$$\begin{aligned} Tv^*(T) &= (1-\rho^2)\int_0^T ub(u)\frac{\partial\lambda}{du}du + \int_0^T F(u)du \quad (32) \\ &= \frac{\alpha\gamma(1-\rho^2)}{\beta+\delta+1}T^{\beta+\delta+1} + \int_0^T F(u)du \quad (33) \end{aligned}$$

where $F(u)$ is the positive function

$$F(u) = s_\infty^2 + (Bu + s_0^2 - s_\infty^2)e^{-\frac{u}{\tau}} \quad (34)$$

with the 4 parameters s_0, s_∞, B, τ and the constraints $\tau > 0$ and

$$B \geq 0 \quad (35)$$

or

$$\begin{aligned} B &< 0 \quad (36) \\ s_\infty^2 + B\tau e^{-1+\frac{s_0^2 - s_\infty^2}{B\tau}} &\geq 0 \quad (37) \end{aligned}$$

for positivity. The integration of $F(u)$ is straightforward and we finally obtain

$$v^*(T) = \frac{\alpha\gamma(1-\rho^2)}{\beta+\delta+1}T^{\beta+\delta} - B\tau + s_\infty^2 - \frac{\tau}{T}(BT + B\tau + s_0^2 - s_\infty^2)(e^{-\frac{T}{\tau}} - 1). \quad (38)$$

4.1 Fit to the market data

We start from the same market data as in table 1, i.e. options on USD/JPY on 2010/07/02, and optimize on the 11 parameters $s_0, s_\infty, B, \tau, \alpha, \beta, \rho, x_0^*, \lambda_0, \gamma, \delta$, with the constraints described above. We find the following errors between the model and market implied volatilities

Table 3: Term Structure SVI Model Error

	Model Error				
T	10Δ L	25Δ L	ATM	25Δ H	10Δ H
1W	-0.46%	-0.27%	-0.12%	-0.14%	-0.41%
1M	0.00%	0.10%	0.15%	0.23%	0.07%
2M	-0.12%	-0.11%	-0.20%	-0.06%	-0.26%
3M	0.05%	0.09%	-0.06%	0.11%	-0.20%
6M	0.21%	0.18%	-0.10%	0.10%	-0.26%
9M	0.36%	0.34%	0.11%	0.40%	0.04%
1Y	0.09%	0.14%	-0.15%	0.21%	-0.15%
2Y	-0.02%	0.08%	-0.05%	0.01%	0.07%
3Y	0.04%	0.10%	0.04%	0.09%	0.04%
4Y	-0.13%	0.00%	0.03%	0.08%	-0.07%
5Y	-0.45%	-0.19%	0.00%	0.13%	-0.04%

Although the fitting errors have increased compared to the original SVI model in table 1, they remain consistent with typical bid-ask spreads. The parameters that produce this surface are

Table 4: Term Structure SVI parameters

s_0	s_∞	B	τ	α	β	ρ	x_0	λ_0	γ	δ
11.72%	9.79%	1.57%	0.946	5.33%	68.99%	-80.00%	7.33%	8.63%	0.122	0.011

4.2 Calendar Spread Smoothness and Local Volatility

Calculating a local volatility with Dupire's formula (Dupire, 1994) assumes that the implied volatility surface is smooth in particular in the time direction. For the original SVI model, the parameters are known only at a discrete set of maturities, such that an interpolation scheme must be chosen to provide parameters at any chosen time. The local volatility will then depend on this choice, resulting in an ambiguity and possible unsmooth patterns.

In fig. 1, we display the local volatilities obtained from the original SVI surface with linear interpolation, with cubic spline interpolation, and finally the one obtained from the term structure SVI. The strike is fixed at ATM and the maturities run up to 5Y on the horizontal axis.

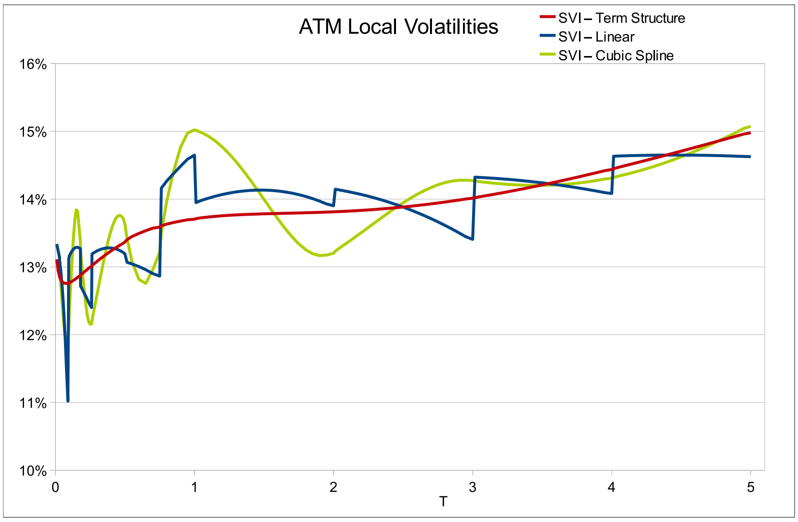

Figure 1: Local Volatilities based on different Implied Volatility models.

We can observe a rough behavior of the local volatility based on SVI with linear interpolation, with sudden jumps from one time to the other. These jumps are smoothed out when using the cubic spline interpolation, but the resulting local volatility still suffers from oscillations that are difficult to justify. Note on the other hand how the local volatility based on the term structure SVI surface is perfectly smooth and has a more reasonable behavior.

Although the roughness of the local volatilities based on interpolating the original SVI parameters is clear, it is less clear how this impacts the prices of derivative products. Indeed, the paths of the underlying at a time t depend on the integral of the local volatility up to t, which is continuous, even if the local volatility itself is not. Therefore, we expect to see suspicious behaviors mostly in products that depend on 2 maturities, and for which these 2 maturities are quite close to each other. For example, calendar spread options starting and one date before the discontinuity and ending a few months later, may suffer from the discontinuity.

In fig. 2, we illustrate this by calculating calendar spreads of calls at ATM starting at a maturity T and ending 3 months later at $T + 3M$. We move T around the maturity $9M$, which sees a discontinuity in the local volatility, as can be observed in fig. 1 above.

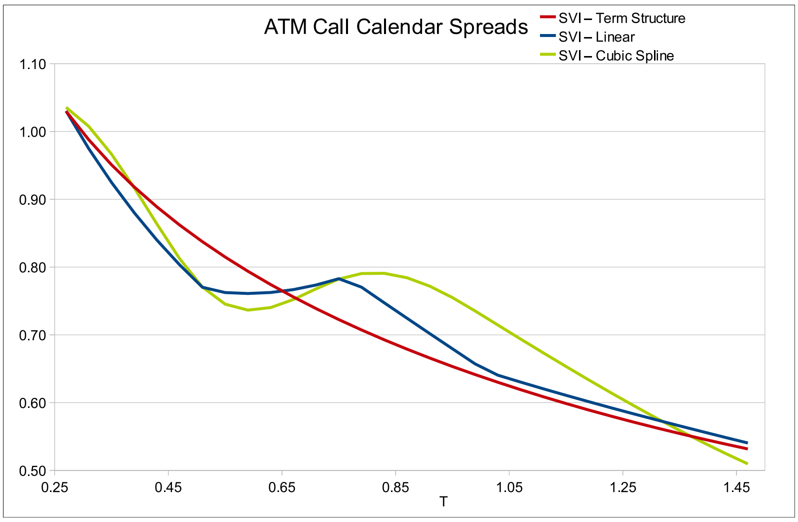

Figure 2: ATM call 3M calendar spreads based on different Implied Volatility models.

The slight roughness in the market data at 9M, which led to a sudden jump in the local volatility around $T = 0.75$ in fig. 1, also translates into oscillating prices for the calendar spreads around this maturity.

4.3 No-Arbitrage and Local Volatility

The market data that we have used until now, given in table 1, is of good quality as there was no calendar spread arbitrage. We would like to observe what happens to the local volatility in the presence of calendar spread arbitrage, so let us introduce a small arbitrage opportunity artificially. We bump the market implied volatilities at maturity 6M by +1%, and at 9M by -1%. We obtain the result in fig. 3.

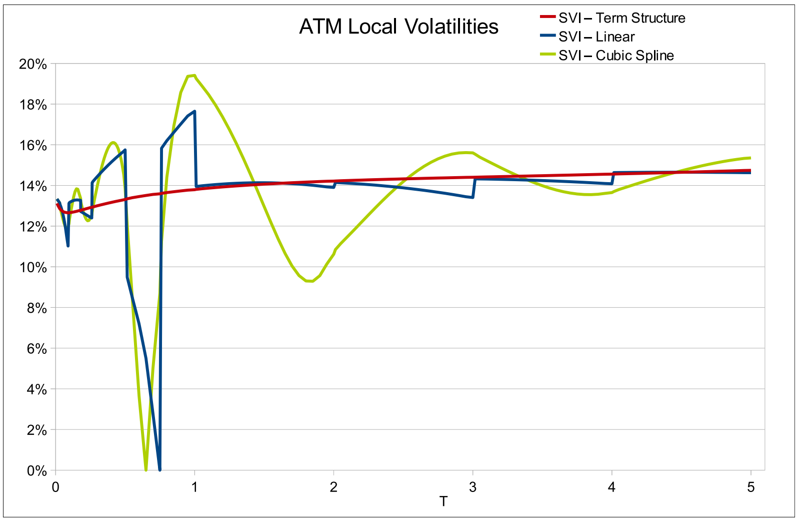

Figure 3: ATM call 3M calendar spreads based on different Implied Volatility models.

Both interpolated SVI models not only have much wilder discontinuities now, but also produced ill-defined local volatilities at one point in time. We have set the local volatility to 0% at this point in order to be able to draw a graphic, but the reader should keep in mind that in fact the local volatility does not exist at this point. On the other hand, the local volatility based on the term structure SVI does not suffer from this bump, since calendar spread cannot exist due to the constraints in the time direction.

5 Conclusion

In this work we have derived a series of sufficient conditions on the parametric forms of SVI parameters $a(T), b(T), \rho(T), m(T), \sigma(T)$ such that the surface is free of calendar spread arbitrage. We have shown that the necessary condition of no-strike arbitrage of (Rogers & Tehranchi, 2008) and (Gatheral, 2004) can be preserved, and we proposed a strategy to find explicit examples of solutions to these constraints.

We further gave one example of term structure SVI surface and illustrated its good fitting properties. We derived local volatilities based on the original and the term structure surface, subject of this work, and showed how the term structure leads to an improvement in the shape of the local volatilities in the time direction. This improvement is reflected in the prices of call calendar spreads, which have a more reasonable behavior under the term structure model. We have also shown how the Dupire local volatilities are properly defined even in the presence of arbitrage in the market, which makes this implied volatility model particularly suitable for pricing of exotics under a Dupire local volatility framework.

As we have found a whole class of parameter functions, we believe that other explicit solutions may be found that would have even better properties in terms of fitting or would appear more intuitive than, for example eq. (38). Note however, as we have shown in this work, that close fit to the market data may not always be desirable, especially when the data is rough or has arbitrage. We hope to investigate and report on these issues in a close future.

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A Proofs

Proposition: With the constraints

$$\frac{\partial T v^*(T)}{\partial T} \geq 0 \tag{39}$$
$$\frac{\partial T b(T)}{\partial T} \geq 0, \tag{40}$$

and (46), (48), (52), (53) below, there can be no calendar spread arbitrage.

Proof:

Our goal is to find constraints on the functions $v^*(T), b(T), \rho(T), x^*(T), \lambda(T)$ such that the no-calendar spread arbitrage condition (17) is satisfied.

First of all, let us define the function $N(x,T)$ as

$$N(x,T) = \rho(T)(x - x^*(T)) + n_T(x - x^*(T)) - \lambda(T), \tag{41}$$

where $n_T(.)$ was defined in (11). We can rewrite $v(x,T)$ as

$$v(x,T) = v^*(T) + b(T)N(x,T). \tag{42}$$

Now we want to impose the positivity of the T-differential of $Tv(x,T)$

$$\frac{\partial T v(x,T)}{\partial T} = \frac{\partial T v^*(T)}{\partial T} + N(x,T)\frac{\partial T b(T)}{\partial T} + Tb(T)\frac{\partial N(x,T)}{\partial T} \tag{43}$$

for all strikes x. A simple analysis of $N(x,T)$ shows that

$$\forall x, \forall T, \quad N(x,T) \geq 0, \tag{44}$$

or in other words, the minimum at $v^*(T)$ is global. Furthermore, $Tv^*(T)$ and $Tb(T)$ are functions of T only, so imposing that they increase with time is not a problem. Note also that the no-strike arbitrage necessary condition (7) is consistent with this, simply meaning that the function $Tb(T)$ is increasing and bounded by $\frac{4}{1+|\rho(T)|}$. The main difficulty lies instead in the T-differential of $N(x,T)$, which we decompose as

$$\frac{\partial N(x,T)}{\partial T} = \frac{\partial \rho}{\partial T}(x - x^*) - \Big(\rho\frac{\partial x^*}{\partial T} + \frac{\partial \lambda}{\partial T}\Big) + \frac{\partial n_T}{\partial T}. \tag{45}$$

Our first constraint is to assume that the correlation is constant, i.e.

$$\frac{\partial \rho}{\partial T} = 0. \tag{46}$$

As indeed the parameter ρ represents a correlation in the limiting Heston model, and as estimating a correlation, all the more time-dependent, is rather difficult, this assumption seems financially reasonable. The excellent quality of fit of the original SVI model, with constant correlation, also backs up this assumption.

Next we turn to the differential of n_T. We write it as

$$\frac{\partial n_T}{\partial T} = \frac{1}{n_T}\Big[-(x-x^*)(\frac{\partial x^*}{\partial T} + \rho\frac{\partial \lambda}{\partial T}) + \lambda(\rho\frac{\partial x^*}{\partial T} + \frac{\partial \lambda}{\partial T})\Big] \tag{47}$$

and impose the additional constraint

$$\frac{\partial x^*}{\partial T} + \rho\frac{\partial \lambda}{\partial T} = 0 \tag{48}$$

such that we obtain

$$\frac{\partial N(x,T)}{\partial T} \quad = \quad (1-\rho^2)(\frac{\lambda}{n_T} - 1)\frac{\partial \lambda}{\partial T}. \tag{49}$$

Note that $n_T = n_T(x - x^*(T))$ is a function of x so the sign of (49) will depend on the value of the strike. We study its variations on the whole range of $x \in \mathbb{R}$ and find

$$0 \leq \quad \frac{\lambda}{n_T} - 1 \quad \leq \frac{1}{\sqrt{1-\rho^2}} - 1, \quad \rho(x-x^*) \in [0, 2\rho^2\lambda] \tag{50}$$

$$-1 \leq \quad \frac{\lambda}{n_T} - 1 \quad \leq 0, \quad \rho(x-x^*) \in \mathbb{R} - \{[0, 2\rho^2\lambda]\}. \tag{51}$$

This entails that $\dfrac{\partial N(x,T)}{\partial T}$ is not positive for all x. However, we show below that with suitable constraints, the combination $\dfrac{\partial Tv^*(T)}{\partial T} + Tb(T)\dfrac{\partial N(x,T)}{\partial T}$ can be made positive for all x.

We need to distinguish 2 cases, depending on the sign of $\dfrac{\partial \lambda}{\partial T}$.

Case 1: $\dfrac{\partial \lambda}{\partial T} \geq 0$

In the range (50), the positivity of (49) is trivially satisfied but in the range (51), it is violated. We make use of the term $\dfrac{\partial T v^*(T)}{\partial T}$ to compensate for this violation. As we want the positivity to hold for all x, we want to compensate for the worst case scenario in (51), which is $\dfrac{\lambda}{n_T} - 1 \approx -1$. One way to achieve this is to enforce

$$\frac{\partial (T v^*(T))}{\partial T} \geq bT(1-\rho^2)\frac{\partial \lambda}{\partial T}. \tag{52}$$

Case 2: $\dfrac{\partial \lambda}{\partial T} \leq 0$

In the range (51), the positivity of (49) is trivially satisfied but in the range (50), it is violated. The worst case scenario in (50), which is $\dfrac{\lambda}{n_T} - 1 = \dfrac{1}{\sqrt{1-\rho^2}} - 1$ is compensated by

$$\frac{\partial (T v^*(T))}{\partial T} \geq -bT(1-\rho^2)(\frac{1}{\sqrt{1-\rho^2}} - 1)\frac{\partial \lambda}{\partial T}. \tag{53}$$