The inverse of the cumulative standard normal probability function.

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Abstract

Some properties of the inverse of the function $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$ are studied. Its derivatives, integrals and asymptotic behavior are presented.

1 Introduction

It would be difficult to overestimate the importance of the standard normal (or Gauss) distribution. It finds widespread applications in almost every scientific discipline, e.g., probability theory, the theory of errors, heat conduction, biology, economics, physics, neural networks [10], etc. It plays a fundamental role in financial mathematics, being part of the Black-Scholes formula [2], and its inverse is used in computing the implied volatility of an option [9]. Yet, little is known about the properties of the inverse function, e.g., series expansions, asymptotic behavior, integral representations. The major work done has been in computing fast and accurate algorithms for numerical calculations [1].

Over the years a few articles have appeared with analytical studies of the closely related error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt$$

and its complement

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt \ .$$

Philip [11] introduced the notation "inverfc(x)" to denote the inverse of the complementary error function. He gave the first terms in the power series for inverfc(x), asymptotic formulas for small x in terms of continued logarithms,

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and some expressions for the derivatives and integrals. Carlitz [3], studied the arithmetic properties of the coefficients in the power series of inverfc(x). Strecok [12] computed the first 200 terms in the series of inverfc(x), and some expansions in series of Chebyshev polynomials. Finally, Fettis [6] studied inverfc(x) for small x, using an iterative sequence of logarithms.

The purpose of this paper is to present some new results on the derivatives, integrals, and asymptotics of the inverse of the cumulative standard normal probability function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

which we call $S(x)$. The function $N(x)$ is related to other important special functions like the error function erf(x)

$$N(x) = \frac{1}{2} \left[\operatorname{erf}\left(\frac{x}{\sqrt{2}} \right) + 1 \right],$$

the parabolic cylinder function $D_\nu(x)$

$$N(x) = \frac{1}{2} \left[2 - \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{4}} D_{-1}(x) \right],$$

and the incomplete Gamma function $\Gamma(\nu; x)$

$$N(x) = \frac{1}{2} \left[2 - \frac{1}{\sqrt{\pi}} \Gamma(\frac{1}{2}, \frac{x^2}{2}) \right].$$

In section 2 we derive an ODE satisfied by $S(x)$, and solve it using a power series. We introduce a family of polynomials P_n related to the calculation of higher derivatives of $S(x)$. In section 3 we study some properties of the P_n, such as relations between coefficients, recurrences, and generating functions. We also derive a general formula for P_n using the idea of "nested derivatives", and we compare the P_n with the Hermite polynomials H_n.

In section 4 we extend the definition of the P_n to $n < 0$ and use them to calculate the integrals of $S(x)$. We also compute the integrals of powers of $S(x)$ on the interval $[0, 1]$. Section 5 is dedicated to the asymptotic expansions of $S(x)$ for $x \to 0,\ x \to 1$, using the function Lambert W.

Finally, appendix A contains the first 20 non-zero coefficients in the series of $S(x)$, and the first 10 polynomials P_n.

2 Derivatives

Definition 1 *Let $S(x)$ denote the inverse of*

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

satisfying

$$S \circ N(x) = N \circ S(x) = x \tag{1}$$

Proposition 2 *$S(x)$ satisfies the IVP*

$$S'' = S(S')^2, \quad S\left(\tfrac{1}{2}\right) = 0,\ S'\left(\tfrac{1}{2}\right) = \sqrt{2\pi} \tag{2}$$

Proof. Since $N(0) = 1/2$, in follows that $S(1/2) = 0$. From (1) we get

$$S'[N(x)] = \frac{1}{N'(x)} = \sqrt{2\pi}e^{\frac{x^2}{2}} = \sqrt{2\pi}e^{\frac{S^2[N(x)]}{2}}$$

Substituting $N(x) = y$ we have

$$S'(y) = \sqrt{2\pi}e^{\frac{S^2(y)}{2}}, \quad S'(\frac{1}{2}) = \sqrt{2\pi}$$

Differentiating $\ln[S'(y)]$, we get (2). ■

Proposition 3 *The derivatives of $S(x)$ obey the recurrence formula*

$$S^{(n+2)}(x) = \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}S^{(n-i)}(x)S^{(i-j+1)}(x)S^{(j+1)}(x), \quad n \geq 0.$$

Proof. Taking the n$^{\text{th}}$ derivative of (2), and using Leibnitz's theorem, we have

$$\begin{aligned} S^{(n+2)} &= \sum_{i=0}^{n}\binom{n}{i}S^{(n-i)}(S'S')^{(i)} \\ &= \sum_{i=0}^{n}\binom{n}{i}S^{(n-i)}\sum_{j=0}^{i}\binom{i}{j}S^{(i-j+1)}S^{(j+1)} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}S^{(n-i)}S^{(i-j+1)}S^{(j+1)}. \end{aligned}$$

■

Corollary 4 *If $D_n = \frac{d^n S}{dx^n}(\frac{1}{2})$, then*

$$D_{2n} = 0, \quad n \geq 0.$$

Putting

$$D_n = (2\pi)^{\frac{n}{2}}C_n, \tag{3}$$

we can write

$$S(x) = \sum_{n\geq 0}(2\pi)^{\frac{2n+1}{2}}\frac{C_{2n+1}}{(2n+1)!}(x - \frac{1}{2})^{2n+1}$$

where

$$C_1 = 1,\ C_3 = 1,\ C_5 = 7,\ C_7 = 127,\ \ldots \quad .$$

Proposition 5

$$S^{(n)} = P_{n-1}(S)(S')^n \quad n \geq 1 \tag{4}$$

where $P_n(x)$ is a polynomial of degree n satisfying the forward recurrence

$$P_0(x) = 1, \quad P_n(x) = P'_{n-1}(x) + nxP_{n-1}(x), \quad n \geq 1 \tag{5}$$

so that

$$P_1(x) = x, \quad P_2(x) = 1 + 2x^2, \quad P_3(x) = 7x + 6x^3, \ \ldots \ .$$

Proof. We use induction on n. For $n = 2$ the result follows from (2). If we assume the result is true for n then

$$\begin{aligned} S^{(n+1)} &= [P_{n-1}(S)(S')^n]' \\ &= P'_{n-1}(S)S'(S')^n + P_{n-1}(S)n(S')^{n-1}S'' \\ &= P'_{n-1}(S)(S')^{n+1} + P_{n-1}(S)n(S')^{n-1}S(S')^2 \\ &= [P'_{n-1}(S) + nSP_{n-1}(S)](S')^{n+1} \\ &= P_n(S)(S')^{n+1} \end{aligned}$$

Since $P_{n-1}(x)$ is a polynomial of degree $n-1$ by hypothesis, is clear that

$$P_n(x) = P'_{n-1}(x) + nxP_{n-1}(x)$$

is a polynomial of degree n. ■

Corollary 6 *Setting $x = 0$ in (4), we obtain*

$$C_n = P_{n-1}(0)$$

3 The polynomials $P_n(x)$

Lemma 7 *If we write*

$$P_n(x) = \sum_{k=0}^{n} Q_k^n x^k$$

we have

$$\begin{aligned} Q_0^n &= Q_1^{n-1} \\ Q_k^n &= nQ_{k-1}^{n-1} + (k+1)Q_{k+1}^{n-1}, \quad k = 1, \ldots, n-2 \\ Q_k^n &= nQ_k^{n-1} \quad k = n-1, \ n \end{aligned} \tag{6}$$

In particular,

$$Q_n^n = n!$$

Proof.

$$\begin{aligned}
&\sum_{k=0}^{n} Q_k^n x^k \\
&= P_n = \frac{d}{dx} P_{n-1} + nxP_{n-1} \\
&= \sum_{k=0}^{n-1} Q_k^{n-1} k x^{k-1} + \sum_{k=0}^{n-1} n Q_k^{n-1} x^{k+1} \\
&= \sum_{k=0}^{n-2} Q_{k+1}^{n-1} (k+1) x^k + \sum_{k=1}^{n} n Q_{k-1}^{n-1} x^k
\end{aligned}$$

■

Corollary 8 *In matrix form (6) reads $A^{(n)}Q^{n-1} = Q^n$, where $A^{(n)} \in \Re^{(n+1)\ \times\ n}$ is a tridiagonal matrix given by*

$$A_{i,j}^{(n)} = \begin{cases} i, \quad j = i+1, \quad i = 1, \ldots, n-1 \\ n, \quad j = i-1, \quad i = 2, \ldots, n+1 \\ \qquad 0, \qquad ow \end{cases}$$

and

$$Q^n = \begin{bmatrix} Q_0^n \\ Q_1^n \\ Q_2^n \\ \vdots \\ Q_n^n \end{bmatrix}$$

With the help of these matrices, we have an expression for the coefficients of $P_n(x)$

$$Q^n = \prod_{k=1}^{n} A^{(n-k+1)} = A^{(n)} A^{(n-1)} \cdots A^{(1)}$$

Proposition 9 *The polynomials $P_n(x)$ satisfy the recurrence relation*

$$P_{n+1}(x) = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} P_{n-i-1}(x) P_{i-j}(x) P_j(x)$$

Proof.

$$
\begin{aligned}
&P_{n+1}(S)(S')^{n+2}\\
=&S^{(n+2)} = \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}S^{(n-i)}S^{(i-j+1)}S^{(j+1)}\\
&= \sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}P_{n-i-1}(S)(S')^{n-i}P_{i-j}(S)(S')^{i-j+1}P_j(S)(S')^{j+1}\\
&= (S')^{n+2}\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}P_{n-i-1}(S)P_{i-j}(S)P_j(S).
\end{aligned}
$$

■

Proposition 10 *The zeros of the polynomials $P_n(x)$ are purely imaginary for $n \geq 1$*

Lemma 11 ***Proof.*** *For $n = 1$ the result is obviously true. Assuming that it is true for n, and that $P_n(x)$ is written like*

$$P_n(x) = n!\prod_{k=1}^{n}(z - z_k), \quad \mathrm{Re}(z_k) = 0,\ 1 \leq k \leq n \tag{7}$$

we have two possibilities for $z^*,\ P_{n+1}(z^*) = 0$:

1. *$z^* = z_k$, for some $1 \leq k \leq n$.*

   In this case, $\mathrm{Re}(z^*) = 0$ and the lemma is proved.

2. *$z^* \neq z_k$, for all $1 \leq k \leq n$.*

   From (5) we get

$$
\begin{aligned}
\frac{P_{n+1}(x)}{P_n(x)} &= \frac{d}{dx}\ln\left[P_n(x)\right] + (n+1)x\\
&= \sum_{k=1}^{n}\frac{1}{z - z_k} + (n+1)x
\end{aligned}
$$

   using (7). After evaluating at $z = z^*$ we obtain

$$0 = \sum_{k=1}^{n}\frac{1}{z^* - z_k} + (n+1)z^*$$

and taking $\mathrm{Re}(\bullet)$

$$
\begin{aligned}
0 &= \mathrm{Re}\left[\sum_{k=1}^{n} \frac{1}{z^* - z_k} + (n+1)z^*\right] \\
&= \sum_{k=1}^{n} \frac{\mathrm{Re}\,(z^* - z_k)}{\left|z^* - z_k\right|^2} + (n+1)\,\mathrm{Re}(z^*) \\
&= \mathrm{Re}(z^*)\left[\sum_{k=1}^{n} \frac{1}{\left|z^* - z_k\right|^2} + (n+1)\right]
\end{aligned}
$$

which implies that $\mathrm{Re}(z^*) = 0.$

■

Proposition 12 *The exponential generating function of the polynomials* $P_n(x)$ *is*

$$
\sum_{k \geq 0} P_k(x) \frac{t^k}{k!} = e^{\frac{1}{2} S^2 [N(x) + tN'(x)] - \frac{x^2}{2}}
$$

Proof. Since

$$
\begin{aligned}
F(x,t) &= \sum_{k \geq 0} P_k(x) \frac{t^k}{k!} \\
&= 1 + \sum_{k \geq 1} \frac{d}{dx} P_{k-1}(x) \frac{t^k}{k!} + \sum_{k \geq 1} kx P_{k-1}(x) \frac{t^k}{k!} \\
&= 1 + \sum_{k \geq 0} \frac{d}{dx} P_k(x) \frac{t^{k+1}}{(k+1)k!} + \sum_{k \geq 0} x P_k(x) \frac{t^{k+1}}{k!} \\
&= 1 + \int_0^t \frac{\partial}{\partial x} F(x,s) ds + xtF(x,t)
\end{aligned}
$$

it follows that $F(x,t)$ satisfies the differential-integral equation

$$
1 + (xt - 1)F(x,t) + \frac{\partial}{\partial x} \int_0^t F(x,s) ds = 0 \tag{8}
$$

Differentiating (8) with respect to t we get

$$
xF(x,t) + (xt - 1)\frac{\partial}{\partial t} F(x,t) + \frac{\partial}{\partial x} F(x,t) = 0
$$

whose general solution is of the form

$$
F(x,t) = e^{-\frac{x^2}{2}} G\left(te^{-\frac{x^2}{2}} + \sqrt{2\pi}\left[N(x) - \frac{1}{2}\right]\right)
$$

for some function $G(z)$.

From (8) we know that $F(x,0) = 1$, and hence

$$G\left(\sqrt{2\pi}\left[N(x) - \frac{1}{2}\right]\right) = e^{\frac{x^2}{2}},$$

which implies that

$$G(z) = e^{\frac{1}{2}S^2\left(\frac{z}{\sqrt{2\pi}} + \frac{1}{2}\right)}.$$

Therefore,

$$F(x,t) = e^{\frac{1}{2}S^2[N(x)+tN'(x)] - \frac{x^2}{2}}$$

■

Definition 13 *We define the "nested derivative" $\mathfrak{D}^{(n)}$ by*

$$\mathfrak{D}^{(0)}[f](x) \equiv 1$$

$$\mathfrak{D}^{(n)}[f]\,(x) = \frac{d}{dx}\left\{f(x) \times \mathfrak{D}^{(n-1)}[f]\,(x)\right\}, \quad n \geq 1$$

Example 14 *The following examples serve to illustrate the calculation of the "nested derivatives" for some elementary functions.*

1. $$\mathfrak{D}^{(n)}\left[e^{ax}\right] = n!a^n e^{nax}$$

2. $$\mathfrak{D}^{(n)}\left[x\right] = 1$$

3. $$\mathfrak{D}^{(n)}\left[x^2\right] = (n+1)!x^n$$

Proposition 15 *With the help of the "nested derivatives" we can represent the polynomials $P_n(x)$ by*

$$P_n(x) = e^{-\frac{n}{2}x^2}\mathfrak{D}^{(n)}\left[e^{\frac{x^2}{2}}\right]$$

Proof. We use induction on n. For $n = 0$ the result follows from the definition of $\mathfrak{D}^{(n)}$. Assuming the result is true for $n-1$

$$\begin{aligned}
P_n(x) &= P'_{n-1}(x) + nxP_{n-1}(x) \\
&= \frac{d}{dx}[e^{-\frac{(n-1)}{2}x^2}\mathfrak{D}^{(n-1)}(e^{\frac{x^2}{2}})] + nxe^{-\frac{(n-1)}{2}x^2}\mathfrak{D}^{(n-1)}(e^{\frac{x^2}{2}}) \\
&= -(n-1)xe^{-\frac{(n-1)}{2}x^2}\mathfrak{D}^{(n-1)}(e^{\frac{x^2}{2}}) + e^{-\frac{(n-1)}{2}x^2}\frac{d}{dx}[\mathfrak{D}^{(n)}(e^{\frac{x^2}{2}})] + \\
&+ nxe^{-\frac{(n-1)}{2}x^2}\mathfrak{D}^{(n-1)}(e^{\frac{x^2}{2}}) \\
&= e^{-\frac{(n-1)}{2}x^2}\left[x\mathfrak{D}^{(n-1)}(e^{\frac{x^2}{2}}) + \frac{d}{dx}\mathfrak{D}^{(n-1)}(e^{\frac{x^2}{2}})\right] \\
&= e^{-\frac{(n-1)}{2}x^2}e^{-\frac{1}{2}x^2}\frac{d}{dx}\left[e^{\frac{1}{2}x^2}\mathfrak{D}^{(n-1)}(e^{\frac{x^2}{2}})\right] \\
&= e^{-\frac{n}{2}x^2}\mathfrak{D}^{(n)}\left[e^{\frac{x^2}{2}}\right]
\end{aligned}$$

■

Summary 16 *We conclude this section by comparing the properties of $P_n(x)$ with the well known formulas for the Hermite polynomials $H_n(x)$ [8]. Since the H_n are deeply related to the function $N(x)$, we would expect to see some similarities between the H_n and the P_n.*

$P_n(x)$	$H_n(x)$
$P_n(x) = e^{-\frac{n}{2}x^2}\mathfrak{D}^{(n)}(e^{\frac{x^2}{2}})$	$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$
$\sum\limits_{k\geq 0} P_k(x)\frac{t^k}{k!} = e^{\frac{1}{2}S^2[N(x)+tN'(x)]-\frac{x^2}{2}}$	$\sum\limits_{k\geq 0} H_k(x)\frac{t^k}{k!} = e^{2xt-t^2}$
$P_n(x) = P'_{n-1}(x) + nxP_{n-1}(x)$	$H_n(x) = -H'_{n-1}(x) + 2xH_{n-1}(x)$
$S^{(n)} = P_{n-1}(S)(S')^n$	$N^{(n)} = \left(\frac{-1}{\sqrt{2}}\right)^{n-1} H_{n-1}\left(\frac{x}{\sqrt{2}}\right)N'$

4 Integrals of S(x)

Definition 17 *We define the n^{th} repeated integral of $S(x)$ by*

$$S^{(-n)}(x) = \int_0^x \int_0^{x_1} \cdots \int_0^{x_{n-1}} S(x_n)\ dx_n dx_{n-1} \ldots dx_1, \qquad n \geq 1.$$

Lemma 18 *The backward recurrence formula for the $P_n(x)$ is*

$$P_{n-1}(x) = e^{-\frac{n}{2}x^2}\left[P_{n-1}(0) + \int_0^x e^{\frac{n}{2}t^2}P_n(t)dt\right] \tag{9}$$

Proof. It follows immediately from solving the ODE for P_{n-1} in terms of P_n. ■

Proposition 19 *Using (9) to define $P_n(x)$ for $n < 0$ yields*

$$\begin{aligned} P_{-1}(x) &= x \\ P_{-2}(x) &= -1 \\ P_{-3}(x) &= -\sqrt{\pi}e^{x^2}N\left(\sqrt{2}x\right) \end{aligned} \tag{10}$$

and the relation

$$S^{(n)} = P_{n-1}(S)(S')^n$$

still holds.

Proof. For $n = 0$ we have

$$P_{-1}(x) = P_{-1}(0) + x$$
$$S = S^{(0)} = P_{-1}(S)$$

so $P_{-1}(0) = 0$. For $n = -1$

$$P_{-2}(x) = e^{\frac{x^2}{2}} \left[P_{-2}(0) + 1 \right] - 1$$

We can calculate $S^{(-1)}$ explicitly by

$$\begin{aligned} S^{(-1)}(x) &= \int_0^x S(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{S(x)} S[N(z)]e^{-\frac{z^2}{2}} dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{S(x)} z e^{-\frac{z^2}{2}} dz \\ &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{S(x)^2}{2}} = -(S')^{-1} \end{aligned}$$

Hence, $P_{-2}(0) = -1$.
Finally, for $n = -2$

$$P_{-3}(x) = e^{x^2} \left[P_{-3}(0) - \sqrt{\pi} N(\sqrt{2}x) + \frac{\sqrt{\pi}}{2} \right]$$

A similar calculation as the one above, making a change of variables $t = N(z)$ in the integral of $S^{(-1)}(x)$ yields

$$S^{(-2)}(x) = -\frac{1}{2\sqrt{\pi}} N[\sqrt{2}S(x)]$$

and we conclude that $P_{-3}(0) = -\frac{\sqrt{\pi}}{2}$. ∎

Corollary 20 *The function $S(x)$ satisfies the functional relations*

$$S'(x)S^{(-1)}(x) = -1 \tag{11}$$

$$S\left[-2\sqrt{\pi}S^{(-2)}(x)\right] = \sqrt{2}S(x)$$

Corollary 21 *The coefficients C_n satisfy the recurrence relation*

$$C_{n+1} = \sum_{j=0}^{n-1} \binom{n}{j+1} C_j C_{n-j}, \quad n \geq 1$$

Proof. *Taking a derivative in (11), we have*

$$
\begin{aligned}
0 &= \sum_{k=0}^{n} \binom{n}{k} S^{(k-1)} S^{(n-k+1)} \\
&= S^{(-1)} S^{(n+1)} + \sum_{j=0}^{n-1} \binom{n}{j+1} S^{(j)} S^{(n-j)}
\end{aligned}
$$

Evaluating at $x = \frac{1}{2}$*, we obtain*

$$\sum_{j=0}^{n-1} \binom{n}{j+1} S^{(j)} \left(\frac{1}{2}\right) S^{(n-j)} \left(\frac{1}{2}\right) = \frac{1}{\sqrt{2\pi}} S^{(n+1)} \left(\frac{1}{2}\right)$$

and the result follows from (3) after dividing both sides by $(2\pi)^n$. ■

Proposition 22 *The integral of the* n^{th} *power of* $S(x)$ *is given by the formula*

$$\int_0^1 S^n(x)dx = \begin{cases} \prod_{i=1}^{k} (2i+1), & n = 2k, \quad k \geq 1 \\ 0, & n = 2k+1, \quad k \geq 0 \end{cases}$$

Proof.

$$
\begin{aligned}
\int_0^1 S^n(x)dx &= \int_{-\infty}^{\infty} z^n N'(z)dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^n e^{-\frac{1}{2}z^2} dz
\end{aligned}
$$

The last integral can be computed exactly [7], to yield the result. ■

5 Asymptotics

Definition 23 *We'll denote by* $\mathrm{LW}(x)$ *the function Lambert W [4],*

$$\mathrm{LW}(x) e^{\mathrm{LW}(x)} = x \tag{12}$$

This function has the series representation [5]

$$\mathrm{LW}(x) = \sum_{n \geq 1} \frac{(-n)^{n-1}}{n!} x^n,$$

the derivative

$$\frac{d}{dx} \mathrm{LW} = \frac{\mathrm{LW}(x)}{x[1 + \mathrm{LW}(x)]} \quad \text{if } x \neq 0,$$

and it has the asymptotic behavior

$$\mathrm{LW}(x) \sim \ln(x) - \ln[\ln(x)] \quad x \to \infty.$$

Proposition 24

$$S(x) \sim g_0(x) = -\sqrt{\mathrm{LW}\left(\frac{1}{2\pi x^2}\right)}, \quad x \to 0$$
$$S(x) \sim g_1(x) = \sqrt{\mathrm{LW}\left(\frac{1}{2\pi (x-1)^2}\right)}, \quad x \to 1$$

Both functions $g_0(x)$ and $g_1(x)$ satisfy the ODE

$$g'' = g(g')^2\left[1 + \frac{2}{g^2(1+g^2)}\right] \sim g(g')^2, \quad \textit{for} \quad |g| \to \infty$$

Proof.

$$N(x) \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{x}, \quad x \to -\infty$$
$$t \sim \frac{1}{\sqrt{2\pi}} e^{-\frac{S(t)^2}{2}} \frac{1}{S(t)}, \quad t \to 0$$
$$S(t) e^{\frac{S(t)^2}{2}} \sim \frac{1}{\sqrt{2\pi} t}, \quad t \to 0$$
$$S^2(t) e^{S^2(t)} \sim \frac{1}{2\pi t^2}, \quad t \to 0$$

Using the definition of $\mathrm{LW}(x)$ we have

$$S^2(t) \sim \mathrm{LW}\left(\frac{1}{2\pi t^2}\right), \quad t \to 0$$

or

$$S(t) \sim -\sqrt{\mathrm{LW}\left(\frac{1}{2\pi t^2}\right)}, \quad t \to 0$$

The case $x \to 1$ is completely analogous. ■

Corollary 25 *Combining the above expressions, we get*

$$S(x) \sim (2x-1)\sqrt{\mathrm{LW}\left(\frac{1}{2\pi x^2 (x-1)^2}\right)}, \quad x \to 0, \ x \to 1 \tag{13}$$

Conclusion 26 *We have presented some results on the operations of the calculus, intrarelationships, and asymptotic analysis of the function $S(x)$ and a family of polynomials associated with it. The objective of this work is to provide a reference for researchers from different disciplines, who may encounter the Normal distribution in one of its many applications.*

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6 Appendix

The first 10 $P_n(x)$ are

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= 1 + 2x^2 \\
P_3(x) &= 7x + 6x^3 \\
P_4(x) &= 7 + 46x^2 + 24x^4 \\
P_5(x) &= 127x + 326x^3 + 120x^5 \\
P_6(x) &= 127 + 1740x^2 + 2556x^4 + 720x^6 \\
P_7(x) &= 4369x + 22404x^3 + 22212x^5 + 5040x^7 \\
P_8(x) &= 4369 + 102164x^2 + 290292x^4 + 212976x^6 + 40320x^8 \\
P_9(x) &= 243649x + 2080644x^3 + 3890484x^5 + 2239344x^7 + 362880x^9 \\
P_{10}(x) &= 243649 + 8678422x^2 + 40258860x^4 + 54580248x^6 + \\
&+ 25659360x^8 + 3628800x^{10}
\end{aligned}
$$

The first few odd C_n are

n	C_n
1	1
3	1
5	7
7	127
9	4369
11	243649
13	20036983
15	2280356863
17	343141433761
19	65967241200001
21	15773461423793767
23	4591227123230945407
25	1598351733247609852849
27	655782249799531714375489
29	313160404864973852338669783
31	172201668512657346455126457343
33	108026349476762041127839800617281
35	76683701969726780307420968904733441
37	61154674195324330125295778531172438727
39	54441029530574028687402753586278549396607
41	53789884101606550209324949796685518122943569

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