Solving the puzzle in the interest rate market (Part 1 & 2)

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Abstract

Different anomalies have appeared in the interest rate market after the burst of the credit crunch. A wide wedge has opened between the market quotes of Forward Rate Agreements and their standard spot Libor replication, and large Basis Spreads have appeared for exchanging floating payments with different tenors. Here we tackle these issues under two aspects.

In Part 1 we focus on issues of direct interest to market practitioners. We show that the gap between FRA rates and their spot Libor replication can be explained by using the quoted Basis spreads. Then we explain the market patterns of the Basis spreads by modelling them as options on the credit worthiness of the counterparty. We also investigate analytically the FRA market payoff.

In Part 2 we study the mathematical representation of the interest rate market in the post-crisis reality. We introduce credit risk at market level, allowing for no-fault standard rule and collateralization. We use subfiltrations to model Libor rates, which now embed relevant credit risk although no default event is possible on Libor itself. We compute change of numeraire and convexity adjustments for collateralized derivatives tied to risky Libor.

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"Basis Virtutum Constantia"
Constancy is the basis of virtues. Latin saying.

1 Introduction

A number of anomalies have appeared in the interest rate market after the burst of the credit crunch in August 2007. Before this date Libor and OIS rates were tracking each other closely, the market quotes of Forward Rate Agreements (FRA) had a precise relationship with the spot Libor rates they are indexed to, flows of interest rate payments differing only for their frequency were considered equivalent, apart from a very little Basis spreads. After the beginning of the crisis large wedges have opened between quantities that were considered practically equivalent: there is a relevant gap between Libor and OIS, FRA rates cannot be anymore replicated using Libor spot rates, and floating legs differing only for the tenor are now separated by large Basis spreads.

These facts had a very strong impact on the financial community, since they questioned both our understanding of the working of the interest rate market during the credit crisis, and the techniques and relations used by all banks to construct the term structure of discount factors to be used for pricing all financial products. Various papers dealing with this new situation were recently put forward in the literature. Among the most relevant we recall Mercurio (2008), Ametrano and Bianchetti (2009), Bianchetti (2009), Henrard (2009). Most of these works focus on finding methodologies for building consistent interest rate models or curves also in the context of anomalous interest rate quotes, and focus on abstract frameworks. Mercurio (2008) takes FRA quotes as new separate inputs for a larger Libor market model, Bianchetti (2008) recognizes an analogy between FX pricing and the pricing of interest rate derivatives when the discounting is decoupled from the indexation of the rates in the payoff, Henrard (2008) follows an axiomatic approach to make the standard framework to term structure bootstrapping consistent with the multicurve situation generated by the presence of large Basis spreads.

In Part 1 we follow a different approach from the above literature, since we aim at explaining the new market patterns at a more fundamental and structural level. At the same time we aim at understanding the relationships among the different anomalies appeared in the market. We first show that the gap between FRA rates and their replication using spot Libor can be explained and replicated using the basis swap spreads, so that the two problems mentioned at the beginning can be reduced to only one. Then we develop a model that takes into account that the reference Libor rates embed options on the credit worthiness of the counterparty, and show that this model explains the basis swaps patterns during the crisis by taking as input the level of counterparty risk in the money market and credit volatility. This gives indications on how the post-crisis interest rate market can be modelled, showing in particular that flows of floating rates with different tenors should embed different levels of default risk and different level of default risk volatility.

Our analysis has some relation with the introductory part of Mercurio (2008), that defines FRA rates in terms of expectations of future survival probabilities. There such expectations are not modelled and can be higher or lower than the probabilities implied by spot quotes. In this work, instead, after showing that FRA rates can be fully explained via a tenor premium expressed by Basis swap spreads, we develop a credit model for Basis spreads as mentioned above. This model allows to understand why in the crisis there was always a negative difference between FRA rates and their standard Libor spot replication, and allows to replicate it approximately, based on credit data. Meanwhile we address the issue of reconciling the actual FRA market payoff and the payoff considered in their replication strategy, another issue arisen in Mercurio (2008).

The issues in Part 1 are of more interest to traders and market practitioners. In Part 2 we address issues of more interest to quantitative researchers involved in updating the standard tools of mathematical finance to the new post-crisis interest rate reality. The difference from the above literature is that we introduce a specific risk factor, credit risk, and we analyze how this alters the mathematics to link spot and forward quotes. We allow for realistic market features such as ISDA no-fault standard rule and collateralization. The tools used are mainly change of measure and subfiltrations. By allowing to separate default probabilities from default indicators, subfiltrations permit to model Libor, a rate which in these days embeds credit risk but with the peculiarity that no default event is possible on Libor itself.

Some results of Part 2 are related to the issues dealt with in Bianchetti (2008) and Henrard (2009). In our framework we work with different bonds (such as Libor bond and OIS bond) that embed different risks in spite of the fact that they all have the same non-defaultable unitary payoff at maturity. We show this does not represent a contradiction, since some of them are not tradable asset, and only a modification of them can be used as numeraires in pricing. Then we compute change of numeraire for collateralized products indexed to risky rates, and we see that the analogy between FX and pricing with discounting decoupled by indexing, first noticed in Bianchetti (2008), emerges here as an output. In our setting we also have an explicit equivalent of the spot rate of exchange, which is here a survival probability, a further element of analogy that is not made explicit in Bianchetti (2008). We also investigate the convexity adjustments involved in the decoupling of discounting from indexing, and estimate them numerically, finding very small numbers. A relevant consequence of Part 2 is the conclusion that credit risk alone would not explain the market patterns without the other elements introduced in Part 1.

The two parts are related by the fact that Part 2 provides a more formal foundation to some aspects that in Part 1 are only intuitive, while Part 1 releases some of the simplifications of Part 2. The seeming inversion in the order is due to the fact that the mathematics used in Part 1 is lighter, thus Part 2 can be skipped by the reader not interested in mathematical issues.

Part 1 is composed of 5 Sections. In Section 2 we present the standard "risk free" interest rate market model and we show on market data why it is not valid anymore. In Section 3 we summarize those results of Part 2 that can be of more practical interest. In Section 4 we analyze the FRA payoff and show the relationship between FRA's and basis spreads, finding an almost exact replication. In Section 5 we analyze qualitatively the possible financial motivations for large basis spreads. In Section 6 we implement this analysis in a credit volatility model that explains approximately the Basis spread patterns (or equivalently the FRA patterns) in the crisis.

Part 2 is composed of 3 Sections. In Section 7 we introduce counterparty risk and compute how the relations among rates are affected by this. In Section 8 we introduce subfiltrations to model products indexed to Libor. In Section 9 we compute the convexity adjustment due to the fact that collateralized derivatives linked to risky rates must be priced with two curves, and show a change of numeraire formally similar to FX change of measure.

Part I
Explaining Basis swaps and FRA in the credit crunch

2 The Rates Market before and after the crisis

Before August 2007, market operators usually thought in terms of one single term structure of *risk-free* or *riskless* interest rates. The concept of riskless does not refer to absence of interest rate risk, but to the absence of elements of credit or liquidity risk influencing in a non-negligible way the fair level of interest rates. In the next section we first review this classic setting, and then we show on recent market data why it is not valid anymore.

2.1 Before the Crisis

The setting reviewed in this section is known to most practitioners, however we find it important to recall it since what comes later puts just this setting under discussion.

We consider a set of contractual dates $T_0, \ldots, T_i, \ldots, T_N$. The *spot riskless interest rate* at time t with maturity T_i is the interest rate $R(t, T_i)$ applying to a deposit contract where a bank A lends a unit of money to a bank B from t (today) until T_i. In building a term structure of discount bonds to be used in the valuation of financial products, a relation is introduced between $R(t, T_i)$ and $P(t, T_i)$, the price of the *riskless zero-coupon bonds* maturity T_i. In standard no-arbitrage pricing, the latter is defined as

$$P(t, T_i) = \mathbb{E}_t \left[D(t, T_i) \right],$$

where $\mathbb{E}_t$ indicates expectation under the risk-adjusted or risk-neutral probability measure, given the information up to t. The flow of all market information is represented by a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. There exists a riskless money market account B_t and $D(t,T_i) = \frac{B_t}{B_{T_i}}$ is the riskless discount factor from T_i to t. In a market free of arbitrage opportunities, the relationship between $P(t,T_i)$ and $R(t,T_i)$ is clear. Buying a riskless bond with price $P(t,T_i)$ and maturity T_i, and lending an amount $P(t,T_i)$ of money until T_i to a riskless counterparty, are two strategies that expose an investor to the same cost at t and the same risk, so that also the return at T_i must be the same, leading to

$$
\begin{aligned}
(1) \qquad P(t,T_i)\left[1 + R(t,T_i)\alpha(t,T_i)\right] &= 1, \\
R(t,T_i) &= \frac{1}{\alpha(t,T_i)}\left[\frac{1}{P(t,T_i)} - 1\right],
\end{aligned}
$$

where we are using simple compounding and $\alpha(t,T_i)$ is the year fraction between t and T_i.

The real-world interbank market is not populated by completely riskless banks. Nonetheless the way market operators used to deal with the quotes of interest rate sensitive products to build curves of zero-coupon bonds corresponds to the assumption that the risk in the interbank lending market is negligible. This was justified by the actually low level of risk for the large majority of banks, and by the fact that interest rate derivatives products were usually indexed to Libor rates (or other similar rates such as Euribor in the Euro market). Libor is a trimmed average[1] of the unsecured inter-bank deposit rates at which funds can be borrowed by designated contributor banks. The banks belonging to the Libor world are selected to be the upper part of the banks world in terms of credit standing, a population that was considered virtually riskless before the crisis.

Thus the Libor rate $L_M(t,T_i)$ with maturity T_i was considered a good approximation to $R(t,T_i)$, in the sense that one could treat $L_M(t,T_i)$ as the riskless rate, and use it as a reference to define derivatives, and to build a curve of discount bonds,

$$
\begin{aligned}
(2)\ R(t,T_i) &= L_M(t,T_i), \\
P(t,T_i) &= \frac{1}{1 + R(t,T_i)\alpha(t,T_i)} = \frac{1}{1 + L_M(t,T_i)\alpha(t,T_i)} =: P_L(t,T_i).
\end{aligned}
$$

This leads to the possibility of very simple replication procedures to price fundamental interest rate derivatives such as swaps. The most basic swap is the Forward Rate Agreement (FRA). A T_i-maturity, T_{i-1}-fixing FRA has a payoff at T_i that, for the payer of the fixed rate, is given by[2]

$$
(3) \qquad \alpha(T_{i-1},T_i)(L_M(T_{i-1},T_i) - K)
$$

[1] Before averaging, the highest and the lowest quartiles of the distribution are eliminated.

[2] As pointed out in Mercurio (2008), formula (3) is the textbook representation of the FRA payoff but it does not exactly coincide with the market termsheet payoff. Mercurio (2008) presents the termsheet formula in the introductory section, although then he uses (3) to build his Market Model consistent with FRA quotes. Similarly we focus on (3) that is more tractable, but in Section 4.2 we present the termsheet formula and some analytic evidence that, for practical purposes, the two formulas are equivalent.

The FRA is quoted through its equilibrium rate $F_M(t;T_{i-1},T_i)$, corresponding to the level of K making such a deal fair at t. Under (2), in an arbitrage-free market this rate is not difficult to compute based on Libor spot quotes, even without observing quotes in the FRA market. In this setting Libor is both the rate at which the contract is indexed and the rate used to build a curve of discount bonds, so the FRA has a simple Libor-based replication.

Remark 1 ***(FRA Replication Strategy)*** *The unitary FRA payoff can be replicated at time t by the payer by borrowing $(1+K\alpha(T_{i-1},T_i))\,P_L(t,T_i)$ with maturity T_i , and lending $P(t,T_{i-1})$ with maturity T_{i-1}, reinvesting then the proceedings (an amount equal to 1) from T_{i-1} to T_i at Libor.*

In fact the payoff (3) is equivalent to

$$1+L_M(T_{i-1},T_i)\alpha(T_{i-1},T_i)-(1+K\alpha(T_{i-1},T_i)).$$

The term $1+K\alpha(T_{i-1},T_i)$, being deterministic, can be easily replicated by shorting at t a corresponding amount of T_i-maturity bonds. The first term is instead $1+L_M(T_{i-1},T_i)\alpha(T_{i-1},T_i)=1/P_L(T_{i-1},T_i)$, which can be replicated at T_{i-1} by buying an amount $1/P_L(T_{i-1},T_i)$ of T_i-maturity bonds with price $P_L(T_{i-1},T_i)$. This strategy has a unit cost at T_{i-1}, leading to a price at t for the replication strategy which is

$$FRA_{Std}(t;T_{i-1},T_i;K)=P_L(t,T_{i-1})-(1+K\alpha(T_{i-1},T_i))\,P_L(t,T_i)\,. \tag{4}$$

The level of the fixed rate K that gives a null price to the FRA at t is

$$F_{Std}(t;T_{i-1},T_i):=\left(\frac{P_L(t,T_{i-1})}{P_L(t,T_i)}-1\right)\frac{1}{\alpha(T_{i-1},T_i)}. \tag{5}$$

This is the textbooks' *Libor Standard Replication forward rate* for fixing at T_{i-1} and payment at T_i, set at t. A remark on the choices made about the notation is now in order.

Notation 2 *We indicate market quotes by the subscript M like in $L_M(t,T_i)$ or $F_M(t;T_{i-1},T_i)$. Variables which are defined as model-independent, unambiguous functions of market quotes have subscript that indicates the market quote they refer to, such as $P_L(t,T_i)$. Variables that represent the model replication of a market quote are identified by a subscript that indicates the model they are based upon, such as $F_{Std}(t;T_{i-1},T_i)$. Most other variables are theoretical quantities defined by a set of properties, such as the riskless rate $R(t,T_i)$.*

In the above setting it is also easy to price a *Money-Market Basis Swap*. This is a contract where: counterparty Y pays every α^Y units of time (tenor) the α^Y-Libor rate, while counterparty X pays every $\alpha^X<\alpha^Y$ units of time the α^X-Libor rate plus a spread Z. The spread is added to the leg with shorter tenor/higher frequency and set to the level that makes the deal fair at inception, see Tuckman and Porfirio (2003). According to Ametrano and Bianchetti

(2008), the current EUR market practice is slightly different since Basis Swaps are quoted as portfolios of two standard receiver fixed-for-floating swaps with the same $12m$-*tenor* fixed legs, and floating legs paying Libor with two different tenors.

Irrespective of the quotation system, when the two counterparties are riskless and the market is free of arbitrage opportunities, any floating leg fixing first time at T_0 and paying last time at T_N is worth $P(t,T_0) - P(t,T_N)$, no matter the frequency/tenor. In fact, if the fixing and payment dates are $[T_0, \ldots, T_N]$, the time-t discounted payoff is

$$A_t = \sum_{i=1}^{N} D(t,T_i)\,\alpha(T_{i-1},T_i)L_M(T_{i-1},T_i)$$

From the above FRA pricing one can derive the price

$$\Pi A_t = \sum_{i=1}^{N} P(t,T_i)\,\alpha(T_{i-1},T_i)F_{Std}(t;T_{i-1},T_i) = P(t,T_0) - P(t,T_N)$$

Thus the value of the spread Z setting the Basis swap price to zero is always $Z = 0$. This corresponds to intuition, in fact the deal could apparently be replicated without any basis spread.

Remark 3 ***(Basis swap replication strategy)****. If Y is a highly rated bank that can lend and borrow at libor, Y could lend* 1 *unit of currency at* $6m$ *frequency at the prevailing* $6m$*-Libor rate to another Libor counterparty X and borrow the same amount, for the same maturity, with the same Libor counterparty X, at* $12m$ *frequency. The cash flows for X and Y would be the same as in a basis swap and the deal would be fair since the two legs have the same value at inception.*

2.2 After the Crisis

If the above relationships are an acceptable, albeit approximate, representation of reality, then the equilibrium Basis Swap spreads in the market should be very low and the difference between the FRA market equilibrium rate $F_M(t;T_{i-1},T_i)$ and the level of the Standard Replication forward rate $F_{std}(t;T_{i-1},T_i)$ computed based on the prevailing Libor quotes $L_M(t,T_i)$ should also be in practice negligible. We see in Figure 1, reporting the difference between $F_{std}(t;t+6m,t+12m)$ and $F_M(t;t+6m,t+12m)$ with t covering a period of more than 6 years, that this corresponds to the market situation until July 2007.

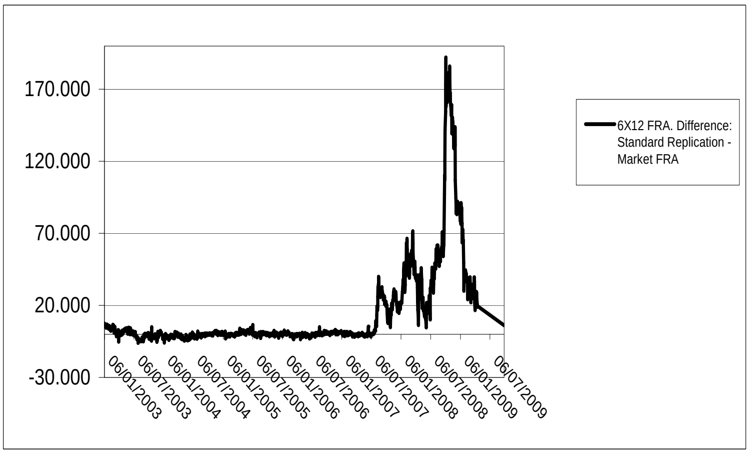

Figure 1. Standard Replication - Market FRA ($6m$ fixing, $12m$ payment)

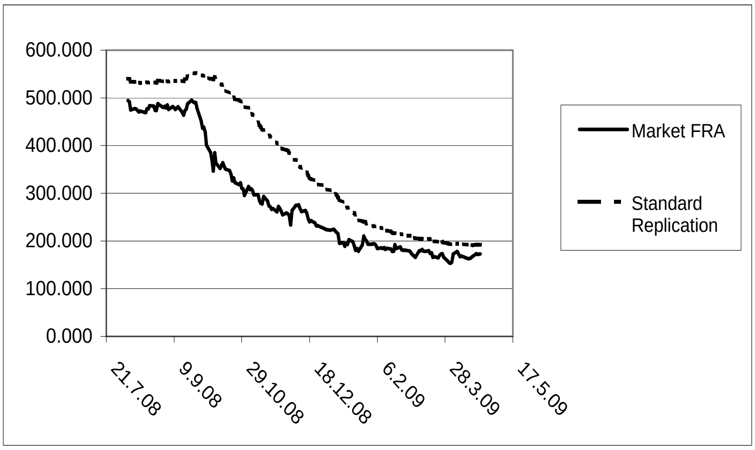

Figure 2. Market FRA and Standard Replication ($6m$ fixing, $12m$ payment)

Although the market FRA rate F_M and the Standard Replication F_{std} never exactly coincide, the difference averages 0.88bp (0.000088) in the three years preceding July 2007. After July 2007, a gap $F_{std} - F_M$ explodes, and remains clearly positive, averaging to 50bp from August 2007 to May 2009. In Figure 2 we can see in more detail both F_{std} and F_M in the second half of 2008 and first half of 2009.

Analogously, the Basis swap spreads widened from very few basis points to much larger values after the crisis. From August 2008 to April 2009, the Basis swap spread to exchange 6 Month Libor with 12 Month Libor over 1 year was strongly positive and averaged 40bps, as we see in Figure 3.

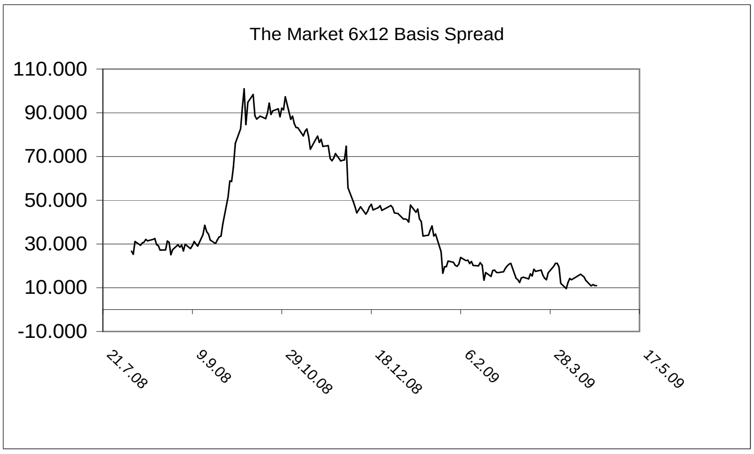

Figure 3. Basis swap spread 6X12, maturity 1y.

These events questioned the setting we reviewed in Section 2, that for the majority of banks was the foundation of the construction of the term structure of discount bonds, a fundamental object since it underlies the valuation of all financial derivatives. There are several assumptions underlying the setting of Section 2, and it is not immediate to understand which ones can still be kept as a reasonable approximation to the reality and which ones should instead be discarded and replaced with new ones. The task of detecting these new assumptions is particularly complex since they should be able to explain not only the existence of the discrepancies, but also their size and in particular their sign, analogously to how the previous assumptions could justify a negligible discrepancy.

The current large discrepancy could be simply explained by assuming that the market has become arbitrage-prone and thus even objects that should in principle be very close have diverged. This, however, could not explain why discrepancies showed clear patterns and even an unambiguous sign.

We recalled in the Introduction that Morini (2008) and Mercurio (2008) focus on the discrepancies between FRA's and the Standard Replication forward, and invoke credit and liquidity issues to justify such discrepancies. This corresponds to discarding the assumption of riskless counterparties in the language of Section 2, and appears to be a view which is shared by most market operators and financial researchers. The evidence is that actually the above discrepancies erupted when another major discrepancy arose in the market: the discrepancy between Libor and OIS (Overnight Indexed Swaps) rates, as we can see in Figure 4.

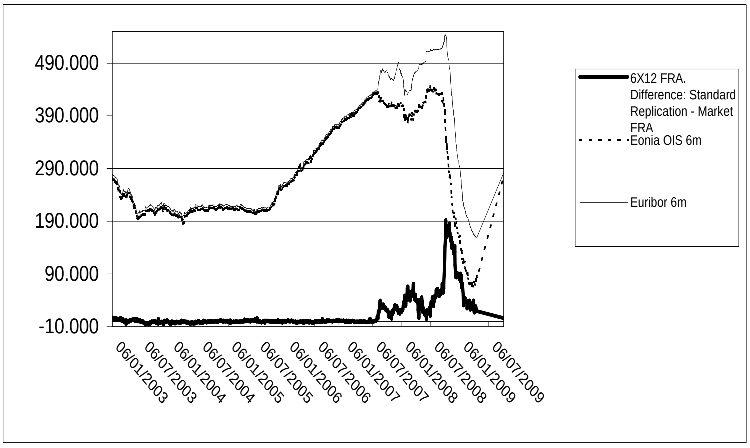

Figure 4. OIS (6m), Libor (6m), and FRA anomalies.

An OIS is a fixed/floating interest rate swap with the floating leg tied to a published index of a daily overnight reference rate, for example the Eonia rate in the Euro market. Since an overnight rate refers to lending for an extremely short period of time, it is assumed to incorporate negligible credit or liquidity risk.[3] The OIS rate is usually intended as good indications of market expectations about future overnight lending transactions over the swap term. Thus the relevant difference between OIS and Libor is considered as an indication of credit or liquidity problems that may affect the counterparties over lending for periods longer than one day.

Before the crisis the spread between Libor and OIS was so little that it was acceptable consider both quotes risk free, and it was reasonable to use (2). When the spread grows OIS is definitely a better approximation for a riskless rate, as confirmed by OIS rates being lower than Libor (by 90 bps on average for a $6m$ maturity from August 2007 to April 2009). It is therefore becoming frequent among financial players to use the OIS swap curve to build a riskless term structure, see Wood (2009). In our simple context, we can follow this by replacing (2) with

$$\begin{aligned} R\left(t, T_i\right) &= E_M\left(t, T_i\right), \\ P(t, T_i) &= \frac{1}{1 + R\left(t, T_i\right)\alpha(t, T_i)} = \frac{1}{1 + E_M(t, T_i)\alpha(t, T_i)} =: P_E(t, T_i). \end{aligned}$$

where E_M indicates a OIS rate and P_E will be called the OIS bond.

Now we need a different definition for Libor, and a different way to deal with FRA quotes, that take into account Libor default and liquidity risk. Morini (2008) introduces bilateral counterparty credit risk, without explicitly considering liquidity. We follow this approach for the reasons recalled below.

[3]We will go back later to this market wisdom, that the reader may find not a foregone conclusion.

2.3 Liquidity or credit risk?

Liquidity problems for Libor banks are among of the main reasons advocated to explain the gap opened between Libor and OIS in the crisis, besides credit (see for example the Bank of International Settlements research by Michaud and Upper (2008)). Since the FRA-forward gap and the Basis swap spread widened when the Libor-OIS gap did, it appears a crucial element to consider. However we need some precision in defining what we mean by liquidity problems. As shown in Acerbi and Scandolo (2008), by Liquidity risk one may mean:

1. Funding Liquidity risk: the risk of running short of available funds.
2. Market Liquidity risk: the risk of having large exposures to markets where it is difficult to sell a security.
3. Systemic liquidity risk: the risk of a global crisis where it is difficult to borrow.

We add two elements to the analysis in Acerbi and Scandolo (2008). First we point out that these three aspects do not really appear a problem for a bank unless we have them together. In fact, if a banks has problem 1), but not 2), it will be able to liquidate its assets to get funding liquidity. Even if 1) and 2) are present, when 3) is lacking the bank should be able to borrow funds to overcome 1) and 2), at least in the short term.

Secondly, we notice that it is difficult to disentangle these elements from credit risk, in particular when one is analyzing not the default risk of one counterparty in a single derivative deal but a money market with bilateral credit risk. In fact, funding liquidity risk for a bank X is normally strongly correlated to the risk of default of X, since funding liquidity is measured by the cost of financing of a bank and an increase of this cost is usually both a cause and consequence of an increase in risk of default. As for market liquidity risk, since we are analyzing a deposit market, it refers to the difficulties of transferring a specific deposit for a specific counterparty Y, and as such it is always strongly correlated to the risk of default of Y. As for the systemic risk, we now know even too well that this is strongly correlated to the risk of default of the generic Libor counterparty. Thus for the problem at hands one has to be careful to draw too precise a line between credit and liquidity risk, since it may lead to an unnecessary multiplication of the actual risk factors. The same view is expressed in Duffie and Singleton (1997), where the credit spread modelled must be intended as including both credit risk and liquidity differentials, and in Collin-Dufresne and Solnik (2001) where the authors claim that the two effects cannot be disentangled, absent a theory for liquidity. In spite of this, we hint at some model possibilities to separate credit and liquidity at the end of Section 6.

3 The Rates Market when banks can default

This section is a summary of those findings of Part 2 that can be relevant to Part 1, since they introduce some important concepts or quantities. To simplify the notation, in the following we concentrate on FRA contracts that fix in 6 months and pay in $12m$, and on $6m/12m$ Basis swaps. We set $\alpha = 6m$, and we take payments happening exactly at multiples of α, so that α is to be intended both as a calendar time ($6m$ after the beginning of our time line) and as a year fraction (a period of $6m$).

Following Morini (2008), in Part 2 we analyze in detail what happens to the above setting if we introduce risk of default for Libor banks. In Section 7 we first compute the price of the FRA replicating strategy of Remark 1 as if it was put in place by two defaultable counterparties A (receiver of the fixed rate) and B (payer) whose defaultable bonds are $P^A(t,\alpha)$ and $P^B(t,\alpha)$, and there are no clauses that can mitigate default risk. The resulting equilibrium rate of this deal is

$$F_{Def}^{A,B}(t;\alpha,2\alpha) = \frac{1}{\alpha}\left(\frac{P^A(t,\alpha)}{P^B(t,2\alpha)} - 1\right).$$

This rate is different for any different couples of counterparties. However we work with two counterparties A and B that we consider typical players in the Libor world, namely two potential Libor contributors, so that Libor quotes should give an indication for their default or liquidity risk. We define $\mathbb{L}_t$ to be the set of Libor counterparties at t and we make the following homogeneity assumption:

Assumption 4 ***(Homogeneity)*** *For any counterparty $X^t \in \mathbb{L}_t$ the interest rate applying at t to a deposit until T_i is $L^{X^t}(t,T_i)$, and we assume*

$$\begin{aligned} L^{X^t}(t,T_i) &= L_M(t,T_i), \\ P^{X^t}(t,T_i) &= \frac{1}{1+L_M(t,T_i)\alpha(t,T_i)} = P_L(t,T_i) \end{aligned}$$

With this assumption the equilibrium rate for $A, B \in \mathbb{L}_t$ is unique for any couple of counterparties and given by

$$F_{Def}(t;\alpha,2\alpha) = \frac{1}{\alpha}\left(\frac{P_L(t,\alpha)}{P_L(t,2\alpha)} - 1\right).$$

This rate $F_{Def}(t;\alpha,2\alpha)$ coincides with the trivial replication $F_{std}(t;\alpha,2\alpha)$. This result shows that when counterparties are defaultable the classic forward $F_{std}(t;\alpha,2\alpha)$ keeps a precise financial meaning as the equilibrium rate of a tradable defaultable lending-borrowing strategy with different maturities, put in place by two risky counterparties that are typical Libor counterparties. Thus $F_{std}(t;\alpha,2\alpha)$ can still be considered the "forward rate" for Libor counterparties.

In Section 8 we point out that the real market FRA does not coincide with this replication strategy since there are counterparty risk mitigation clauses. The first one is the "no-fault" or "two-way payment rule": in case of default of one of the two counterparties, the other counterparty loses (assuming null recovery) the positive part of the net present value of the residual deal. With this provision, that must be introduced bilaterally, namely taking into account that both counterparties can default, it becomes not anymore possible to price simply through replication, but we have to introduce a framework for credit modelling.

We work in a modelling framework whose special case is the classic reduced-form or intensity model of Duffie and Singleton (1997, 1999) and Lando (1998), the market standard for the pricing of simple credit derivatives such as credit default swaps. In this framework we show that it is possible to model the fact that Libor is tied to risky counterparties but it never defaults, because, thanks to the use of *subfiltrations*, one can separate the counterparty-specific default indicator $1_{\{\tau^X > s\}}$ from default probabilities. Second, this setting allows to deal consistently with different bonds which embed different risks in spite of the fact that they give the same non-defaultable payoff of 1 at T, such as the Libor bond $P_L(t,T)$ and the OIS bond $P_E(t,T)$. We show that the latter is considered a tradable asset, while the Libor bond $P_L(t,T)$ is not a tradable asset, but it can be used to define a numeraire to perform a change of measure that provides the FRA equilibrium rate in closed form.

In order to reach this closed-form, we make an additional assumption:

Assumption 5 *(**Persistency**, or **Libor today remains Libor**) We assume for counterparty A that*

$$A \in \mathbb{L}_t, \tau^A > \alpha \Longrightarrow A \in \mathbb{L}_\alpha,$$

where τ^A is the default time of A. This means that a counterparty which is today a good representative of the Libor world will remain a good representative of the Libor world until the fixing of the deal if it does not default. To put it differently, we model a market where future Libor contributors will be so similar to how current Libor contributors will be in the future that we can identify Libor in the future with any survived counterparty that is Libor today.

Even with this more realistic payoff the equilibrium rate of the FRA is

$$F_{Net}(t;\alpha,2\alpha) = \frac{1}{\alpha}\left(\frac{P_L(t,\alpha)}{P_L(t,2\alpha)} - 1\right),$$

so that it coincides with the standard replication forward rate $F_{std}(t;\alpha,2\alpha)$[4], confirming that this rate has a precise financial meaning also in the presence of credit risk but leaving market anomalies unexplained.

[4]The fact that, under a standard symmetric credit risk setting, the definition of the equilibrium rate of swaps in terms of Libor rates is independent from the riskiness of the counterparties has confirmations in the literature, although derived in frameworks different from the change of measure used here. Under assumptions similar to those made above, but in an econometric setting, Duffie and Huang (1996) and Sorensen and Bollier (1994) find indeed that "a swap between two parties of similar credit quality should entail no default risk premium in either direction because of the symmetric nature of the contract", to put it as in Collin-Dufresne and Solnik (2001), who add that, in spite of this "the swap term structure will be different from (and above) the risk-free term structure, because the swap rate payments are indexed on six-month Libor, which is a default risky rate".

Finally in Section 9 of Part 2 we point out that in the FRA market there is an additional provision we have not yet taken into account: collateralization. We introduce it while keeping the above assumptions 4 and 5. Collateralization is seen by market operators as eliminating risk of default, thus the FRA payoff needs to be discounted with a default-free discount factor. The price is

$$FRA_{Col}\left(t;\alpha,2\alpha;K\right)=\mathbb{E}_t\left[D\left(t,2\alpha\right)\alpha\left(L_M(\alpha,2\alpha)-K\right)\right]. \tag{6}$$

Thus there is an inconsistency between the indexing of the payoff rate, which is Libor, and the indexing of the discount factor, which is the riskless discount factor, namely OIS. This gives rise to a convexity adjustment similar to the one that enters the pricing of futures, in-arrears swaps and CMS swaps. We compute that the equilibrium rate of the FRA is

$$F_{Col}\left(t;\alpha,2\alpha\right)=F_{Std}\left(t;\alpha,2\alpha\right)+CA\left(t;\alpha,2\alpha\right),$$

where $CA\left(t;\alpha,2\alpha\right)$ is the convexity adjustment. On estimated parameters we compute that $CA\left(t;\alpha,2\alpha\right)<1bp$, even changing the period selected for the historical estimation. This shows that the convexity adjustment can account only for a very small fraction of the discrepancy between $F_M\left(t;\alpha,2\alpha\right)$ and $F_{Std}\left(t;\alpha,2\alpha\right)$, that in the crisis has been on average $50bp$. Thus in the following we neglect $CA\left(t;\alpha,2\alpha\right)$.

The convexity adjustment is computed in Part 2 via change of numeraire from a measure associated to a default-dependent numeraire to a measure associated to a default-free numeraire. In this context change of numeraire depends on the dynamics of a quantity that recalls the *forward rate of exchange* of FX modelling, with the Libor bond from the indexing curve that plays the role of the foreign bond, the OIS bond from the discounting playing the role of the domestic bond, and a conditional survival probability that replaces the *spot rate of exchange*. This is similar to the FX analogy detected by Bianchetti (2008) in a more abstract setting.

Part 2 introduces a number of techniques that can be of some use in updating the standard mathematical representation of the interest rate market. But for the purposes of Part 1 the most relevant conclusion of Part 2 is that credit risk associated to Assumptions 4 and 5 does not explain the market patterns. There are in market reality important elements which are different from the representation given in this section. The issue is tackled in the following sections.

4 The Link between Forward rate Agreements and Basis swaps

We first show in this section that the large discrepancy between the market FRA and the Standard Replication forward rate during the credit crunch is just one aspect of the large quotes for Basis swaps observed in the market in the same period. This appears a fact overlooked in the literature, although not unknown to experienced practitioners (Schiavi (2009)).

4.1 A Basis-consistent replication of the FRA rate

We analyze Basis swaps in order to understand if there is a relationship between the growth of Basis swap spreads in the crisis and the anomalies in the FRA market. It is clear that in the crisis Basis Swaps cannot be dealt with in the riskless setting of Section 2. As we did with FRA's, now we have to take into account that Basis swaps are collateralized contracts that suffer no risk of default, but they are indexed to Libor rates that are now perceived as risky.

We recalled in Section 2 that in the current EUR market practice Basis Swaps are quoted as portfolios of two standard receiver fixed-for-floating swaps with the same $12m$-*tenor* fixed legs and different floating legs. The Basis spread Z is given as the difference between the fixed leg of the swap whose floating leg has longer tenor, and the fixed leg of the other swap.

We consider a simple Basis swap: the $6m/12m$ $(\alpha/2\alpha)$ basis swap with maturity $12m$, a quoted contract. In $6m/12m$ Basis swap the spread Z is actually an addition to the fixed leg of the $12m$-tenor swap. In this special case where for the $12m$ swap the floating and the fixed leg have the same frequency, we can neglect the fixed legs and say that the two counterparties pay two floating legs, one with $6m$ frequency and one with $12m$ frequency, and the spread Z is subtracted to the $12m$-tenor leg. This is the convention we follow in the rest of the paper. We omit the frequencies of the two legs since they remain $\alpha/2\alpha$. Once the relations are clear for this example, the generalization to more tenors or longer maturities should not be difficult.

The price of the Basis swap is computed as the expectation of the Libor-dependent payoff discounted with riskless rate, as we did with FRA in (6):

$$\begin{aligned}&Basis\,(0;2\alpha;Z) \;=\\ &\mathbb{E}_0\left[D\,(0,\alpha)\,\alpha L_M(0,\alpha)+D\,(0,2\alpha)\,\alpha L_M(\alpha,2\alpha)-D\,(0,2\alpha)\,2\alpha\,(L_M(0,2\alpha)-Z)\right],\end{aligned}$$

that is

$$\begin{aligned}Basis\,(0;2\alpha;Z) \;&=\; \mathbb{E}_0\left[D\,(0,2\alpha)\,\alpha\,(L_M(\alpha,2\alpha)\right]+\\ &-P_E\,(0,2\alpha)\left(\frac{1}{P_L(0,2\alpha)}-1-2Z\alpha-\frac{P_E\,(0,\alpha)}{P_E\,(0,2\alpha)}\left(\frac{1}{P_L(0,\alpha)}-1\right)\right).\end{aligned}$$

If we define

$$\tilde{K}\,(Z)=\left(\frac{1}{P_L(0,2\alpha)}-1-2Z\alpha-\frac{P_E\,(0,\alpha)}{P_E\,(0,2\alpha)}\left(\frac{1}{P_L(0,\alpha)}-1\right)\right)/\alpha$$

we have

$$Basis\,(0;2\alpha;Z)=\mathbb{E}_0\left[D\,(0,2\alpha)\,\alpha\,(L_M(\alpha,2\alpha)\right]-P_E\,(0,2\alpha)\,\tilde{K}\,(Z)\,\alpha. \tag{7}$$

We now analyze further $\tilde{K}(Z)$:

$$\begin{aligned}
\tilde{K}(Z) &= \left(\frac{1}{P_L(0,2\alpha)} - \frac{P_E(0,\alpha)}{P_E(0,2\alpha)}\frac{1}{P_L(0,\alpha)} + \frac{P_E(0,\alpha)}{P_E(0,2\alpha)} - 1 - 2Z\alpha\right)/\alpha \\
&= \frac{1}{P_L(0,\alpha)}\left(\frac{P_L(0,\alpha)}{P_L(0,2\alpha)} - \frac{P_E(0,\alpha)}{P_E(0,2\alpha)}\right) + \left(\frac{P_E(0,\alpha)}{P_E(0,2\alpha)} - 1 - 2Z\alpha\right)/\alpha \\
&= E_{Std}(0;\alpha,2\alpha) + \frac{1}{P_L(0,\alpha)}\left(F_{Std}(0;\alpha,2\alpha) - E_{Std}(0;\alpha,2\alpha)\right) - 2Z
\end{aligned}$$

or alternatively

$$\tilde{K}(Z) = F_{Std}(0;\alpha,2\alpha) + \left(\frac{1}{P_L(0,\alpha)} - 1\right)\left(F_{Std}(0;\alpha,2\alpha) - E_{Std}(0;\alpha,2\alpha)\right) - 2Z.$$

Now compare the Basis swap price (7) with the FRA price (6). We see that if one sets $K = \tilde{K}(Z)$ the FRA price is equal to the price of a Basis swap where the spread is set to Z.

In fact both FRA and Basis swap involve the exchange of two legs. One leg is deterministic and fixed today: for the FRA it is the payment of the fixed leg at 2α, for the basis swap it is given by the payment of the 2α leg at 2α minus the first payment of the α leg. The other leg is the only one stochastic and for both contracts it corresponds to the payment of $L_M(\alpha, 2\alpha)$ at α. Both contracts are collateralized and therefore discounting must be done at the riskless rate.

Another way to understand the equivalence between FRA and Basis swap is analyzing the FRA replication strategy of Remark 3. The fixed leg is replicated by a strategy with maturity 2α, while the floating leg is replicated by a strategy with maturity α, followed by another lending from α to 2α. One leg has α tenor, the other leg has 2α tenor. It is clear that a FRA replication is affected by the presence of non-negligible Basis swap spreads in the market.

We see from the above relations that a FRA is fair when we set $K = \tilde{K}(B(0;2\alpha))$, where $B(0;2\alpha)$ is the equilibrium value for the Basis spread Z. Thus we have found a replication

$$\begin{aligned}
&F_B(0;\alpha,2\alpha) = \\
&F_{Std}(0;\alpha,2\alpha) + \left(\frac{1}{P_L(0,\alpha)} - 1\right)\left(F_{Std}(0;\alpha,2\alpha) - E_{Std}(0;\alpha,2\alpha)\right) - 2B(0;2\alpha).
\end{aligned}$$

of the equilibrium value of the FRA rate. For the crisis period the relevant term in the difference $F_B - F_{Std}$ is the term $-2B(0;2\alpha)$, while the other term is much smaller. Now we check if replacing the trivial replication F_{Std} of the equilibrium FRA with our new replication F_B that takes into account the basis we are able to reduce the large discrepancy we observed in Figure 2.

This is done in Figure 6, where we see that the basis-consistent replication F_B of the FRA rate is indistinguishable from the market FRA rate, even though this replication uses Libor spot data, Basis swap data and no information from market FRA. We are now able to replicate the FRA with other market quotes as well as the Standard Replication forward was able to replicate the FRA before the crisis. This evidence confirms that the reason for the gap between market FRA's and their standard replication is the presence of a large Basis spread that must introduced in the replication, as we did above. This leads to the remark made in the next section.

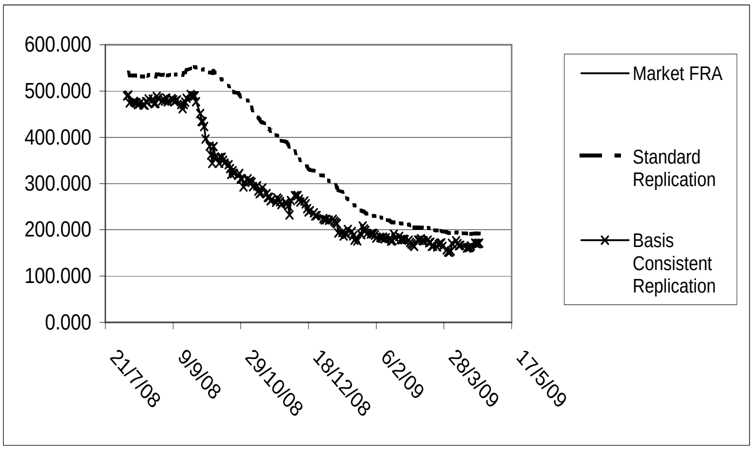

Figure 6. 6X12 Market FRA, Standard Replication, and Basis-consistent Replication

4.2 Reconciling Textbooks' FRA with Real Market's FRA

In the definition of the Forward Rate Agreement used so far, the payoff is *paid at* 2α and it is given by

$$(L_M(\alpha, 2\alpha) - K)\,\alpha. \tag{8}$$

so that the price is

$$FRA\,(0;\alpha,2\alpha,K) = \mathbb{E}_0\left[D\,(0,2\alpha)\,\alpha\,(L_M(\alpha,2\alpha) - K)\right].$$

We recall that according to the Change of Numeraire technique, see Brigo and Mercurio (2006), given two numeraires $N1$, $N2$, the following holds for any tradable X

$$X_0 = \mathbb{E}^{N1}\left[\frac{N1_0}{N1_T}X_T\right] = \mathbb{E}^{N2}\left[\frac{N2_0}{N2_T}X_T\right]$$

where $\mathbb{E}^N$ indicates the probability measure associated to numeraire N. We can apply change of numeraire to FRA pricing, finding

$$\begin{aligned}
FRA\,(0;\alpha,2\alpha,K) &= \mathbb{E}_0\left[D\,(0,2\alpha)\,\alpha\,(L_M(\alpha,2\alpha) - K)\right] \\
&= \mathbb{E}_0\left[\frac{B\,(0)}{B\,(2\alpha)}\alpha\,(L_M(\alpha,2\alpha) - K)\right] \\
&= P_E\,(0,2\alpha)\,\alpha\mathbb{E}_0^{P_E(\cdot,2\alpha)}\left[L_M(\alpha,2\alpha) - K\right].
\end{aligned}$$

where $\mathbb{E}_0^{P_E(\cdot,2\alpha)}$ indicates expectation under the measure associated to the bond numeraire $P_E\,(\cdot,2\alpha)$.

The payoff (8) is the FRA payoff reported in most textbooks, with few exceptions (for example Myron and Swannell (1991)). The termsheet payoff of FRA used in the market is different, since it provides for the *payment at* α of a payoff given by

$$\frac{L_M(\alpha, 2\alpha) - K}{1 + L_M(\alpha, 2\alpha)\alpha}\alpha, \tag{9}$$

namely, compared to (8), it pays earlier a payoff which is Libor-discounted. Thus the price is

$$FRA^{MKT}(0; \alpha, 2\alpha, K) = \mathbb{E}_0\left[D(0,\alpha)\,\alpha\left(\frac{L_M(\alpha, 2\alpha) - K}{1 + L_M(\alpha, 2\alpha)\alpha}\right)\right].$$

This is pointed out in also in Mercurio (2008), that presents (9) in the introductory section, although then he uses (8) to build his Market Model consistent with FRA quotes. In order to understand the relation between (8) and (9) we perform a few transformations to (9):

$$\begin{aligned} FRA^{MKT}(0; \alpha, 2\alpha, K) &= \mathbb{E}_0\left[D(0,\alpha)\,\alpha\left(\frac{L_M(\alpha, 2\alpha) - K}{1 + L_M(\alpha, 2\alpha)\alpha}\right)\right] \\ &= \alpha\mathbb{E}_0\left[\frac{B(0)}{B(\alpha)}P_L(\alpha, 2\alpha)\,(L_M(\alpha, 2\alpha) - K)\right]. \end{aligned}$$

If we assume that $P_L(\cdot, 2\alpha)$ is a valid numeraire, we have

$$\begin{aligned} FRA^{MKT}(0; \alpha, 2\alpha, K) &= P_L(0, 2\alpha)\,\alpha\mathbb{E}_0^{P_L(\cdot,2\alpha)}\left[\frac{P_L(\alpha, 2\alpha)}{P_L(\alpha, 2\alpha)}(L_M(\alpha, 2\alpha) - K)\right] \\ &= P_L(0, 2\alpha)\,\alpha\mathbb{E}_0^{P_L(\cdot,2\alpha)}\left[(L_M(\alpha, 2\alpha) - K)\right]. \end{aligned}$$

Thus under this precise market payoff (9) the equilibrium rate is

$$\mathbb{E}_0^{P_L(\cdot,2\alpha)}\left[L_M(\alpha, 2\alpha)\right]$$

to be compared with the equilibrium rate

$$\mathbb{E}_0^{P_E(\cdot,2\alpha)}\left[L_M(\alpha, 2\alpha)\right]$$

that we obtain under the text-book payoff (8) used so far.

In Section 8 and 9 of Part 2, under the hypothesis made there, we do not consider $P_L(t, 2\alpha)$ as a numeraire, but rather we use a slight modification $\tilde{P}(t, 2\alpha)$ that appears a more natural and correct numeraire. Then we compute, on market estimated parameters, that the expectation of L_M under the measure associated to $\tilde{P}(t, 2\alpha)$ differs from the expectation of L_M under the measure associated to $P_E(\cdot, 2\alpha)$ by less than one basis point. Thus, if we neglect the technicality associated to the slight mathematical difference between $\tilde{P}(t, 2\alpha)$ and $P_L(t, 2\alpha)$, we can conclude that the difference between the market payoff (9) and the text-book payoff (8) is of little practical relevance for the computation of the equilibrium FRA rate. This conclusion is strongly confirmed by the fact that the Basis-consistent replication of the FRA equilibrium rate we outlined in the previous section captures the market rate even though we have used the text-book payoff (8) and not the market payoff (9).

5 Explaining FRA and Basis Swaps in the crisis

The previous section has shown that the gap between FRA market quotes and the Standard Replication can be bridged using the Basis swap spreads, so that the two problems we are analyzing can be reduced to a single one. This moves our focus from explaining the FRA gap to explaining why we have such large Basis swaps in the crisis. By "explaining" we mean the possibility to replicate, at least approximately, the behaviour of Basis swap spread during the crisis using more fundamental market quantities, and clarify analytically why in the market, during the crisis, the spread $B^{\alpha/2\alpha}$ paid by the leg with shorter tenor to the leg with longer tenor has been large and positive. Which is equivalent to explaining why the market FRA rate has been, in the crisis, remarkably smaller than the Standard Replication forward.

We have already seen that we have to abandon the riskless setting of Section 2 and that Basis swaps in a credit crisis are different from the replication of Remark 3, since

1. In a basis swap no-one really borrows or lends to a Libor counterparty.
2. A basis swap is a collateralized derivative, so there is no counterparty default risk.
3. Payments are indexed to risky unsecured Libor.

We have also seen that default risk for Libor banks is not sufficient, in itself, to explain anomalies like those we have seen in the crisis, not even considering collateralization. This can be based on the analysis of Part 2 reported in Section 3, on a FRA which is equivalent to a Basis swap $\alpha/2\alpha$ with maturity α. We have to move to considering additional issues.

Market operators appear to have an intuitive justification of the fact that in the crisis a large positive spread needs to be paid by the payer of the leg with shorter tenor to the payer of the leg with longer tenor. The market explanation usually starts from a conjecture that one can find, given as an unexplained axiom, for example in Tuckman and Porfirio (2003).

Conjecture 6 ***(An axiom for the market)*** *Lending at* $12m$ *Libor involves more counterparty/liquidity risk than rolling lending at* $6m$ *Libor.*

As a consequence, in the replication strategy of Remark 3 applied in a risky world, the receiver of the leg with longer tenor ($12m$) suffers a higher counterparty/liquidity risk, that will be compensated by a higher market level of the $12m$-Libor compared to the one implied by the $6m$-Libor. When in a Basis swap one exchanges the same flows as in the replicating strategy but eliminating any counterparty/liquidity risk by collateralization and indexing to Libor rather than actual lending, this higher level of the $12m$-Libor will is not justified anymore by a higher risk. Thus in the Basis swap the receiver of the leg with longer tenor ($12m$) will have to compensate this advantage by adding a spread to her payments.

If the reader agrees with Conjecture 6, the above reasoning explains why the non-negative basis $B^{x/y}$ needs always to be added to the shorter-tenor leg of a basis swap and thus why the FRA rate needs always to be lower than (or equal to) the corresponding replicated forward rate. However now we want to analyze qualitatively the foundations of Conjecture 6, and then we want find a quantitative framework to express Basis swap spreads.

The first step is to consider the various fundamental explanations usually given to justify Conjecture 6.

5.1 Lower loss due to default: only one coupon

There is one obvious advantage for the $6m$ roller in the risky Libor world of Conjecture 6. When default happens in the period $6m - 12m$ (between $6m$ from now and $12m$ from now) the $12m$ lender loses all the interest, while the $6m$ lender loses only the interest for the period $6m - 12m$, having already cashed-in the interest for $0 - 6m$. We will take this into account, but we will see later that quantitatively the advantage is largely insufficient to justify the basis observed in 2007, 2008 and 2009.

On the other hand, lending lasts $12m$ for both the $12m$ lender and the $6m$ roller. If the counterparty default happens in the $12m$ period, both lenders lose the notional.

5.2 Exiting at par when credit conditions worsen

One may see another credit advantage in $6m$ rolling under risky Libor. The advantage is that after $6m$, if the counterparty credit conditions have strongly worsened, one can stop lending with no cost, and move to lending to a better counterparty. The $12m$ lender, instead, for doing the same will have first to unwind or transfer its deposit (if possible) at a cost that incorporates the increased risk of default, to be compared to the $6m$ lender that exits at par.

However, by itself this does not imply that the $6m$ roller has a monetary advantage. In fact the $6m$ roller does not have an option to exit at par after $6m$. He will always exit at par after $6m$, including the opposite situations: when the counterparty credit conditions have improved rather than worsened. In these cases the $6m$ lender will exit at par while the $12m$ lender will have a gain. Thus the expected gain of the $6m$ lender compared to the $12m$ lender when the counterparty worsens is compensated by its expected loss when the counterparty gets better.

This symmetry can be broken down by some considerations on how the market works, but that are difficult to model. Unwinding a financing contract is not so common in the market. A first reason is that, due to bid-ask spread, this cannot be done at a theoretical fair value. Secondly, and more importantly, there are commercial reasons that restrain banks from unwinding a funding contract. However a worsening in the credit quality of a borrower can have non-linear negative effects for the lender, for example if the rating worsens there can be negative consequences also on the regulatory capital point of view, let alone raising concerns on the solvability of the lending bank itself (the 2002 ISDA Master agreement even provides for a clause of automatic unwinding of a deal if the rating of the counterparty worsens (ATE)).

These considerations can be more important of commercial considerations. As a consequence, unwinding is likely not to happen when it would be convenient for the $12m$ borrower, but only when it involves a loss in fair value, and when the bid-ask spread is likely to be large. This breaks the above symmetry, but this effect is difficult to quantify.

5.3 A liquidity advantage in 6m lending?

Michaud and Upper (2008) claim explicitly that in the analysis of the money market in the crisis "it is difficult to disentangle credit and liquidity factors", as we already pointed out in Section 2.3. The liquidity risk considered by Michaud and Upper (2008) appears to be mainly funding liquidity. With reference to funding liquidity, it can be an advantage to exit at par after $6m$ since the lender may be in need of funding liquidity (its credit risk has grown, or in any case it funding costs have increased). Here the considerations of 5.2 apply. One may exit also in the $12m$ contract, and exiting at fair value can be more ore less convenient than exiting at par, so that a longer tenor by itself on average does not represent either an advantage of a disadvantage. We have seen in 5.2 that there are elements in market reality that can break the symmetry. One is bid-ask spread. The other element is the bias towards unwinding when it involves a fair value loss, but this is true only when the unwinding is performed for credit reasons (the credit risk of the borrower) rather than funding liquidity reasons (the credit risk of the lender). In case of liquidity reasons, under this aspect the symmetry is broken only if we also assume correlation between credit risk of the borrower and credit risk of the lender.

5.4 Libor anomalies

Practitioners from large Libor contributors hint that Libor was not a reliable indication for inter-bank borrowing during the crisis, see Peng et al. (2008). The argument is that Libor was understating actual interbank lending, as confirmed by a number of market observations, since "any bank posting a high Libor level runs the risk of being perceived as needing funding". Thus we can infer that banks in higher need of funding were not posting their actual funding cost but a lower rate. As Cho and Rosemberg (2008) put it, for the purposes of the fixing the bank has an incentive to quote a lower interest rate publicly than it would be prepared to pay in a private transaction. Some confirmation of this hypothesis come also from Michaud and Upper (2008), that mention that banks with increasing credit risk, as measured by the CDS market, do not appear to have quoted significantly higher Libor rates than banks with lower credit risk. This can be interpreted in two ways: either credit risk was not relevant to the cost of term funding of Libor banks, or, as hinted by Peng et al. (2008), Libor did not reflect the actual term funding particularly for distressed banks. Another anomaly observed during the crisis is the so-called *turn-of-the-year effect*: the Libor-OIS spread showed a positive jump when the maturity of the contracts reached the end of the year.

5.5 Borrower (and lender) will not be Libor for ever

Only in the above paragraph we have explicitly taken into account the first characteristic of a Basis swap that makes it different from a standard replication strategy: here no-one really borrows or lends to a Libor counterparty, but simply payments are indexed to the prevailing Libor fixing rate. The Instructions to BBA Libor contributor Banks state that a Libor contributor bank will contribute the rate at which it could borrow funds for unsecured deposits. Additionally, according to British Bank Association (2009a),

"*Decisions on individual banks were taken on the basis of scale of activity in the London market, perceived expertise in the currency concerned, reputation, and due consideration of credit standing*

"*The BBA is committed to reviewing the Panels at least twice annually.*

and according to British Bank Association (2009b)

"*The banks represented on the panels are the most active in the cash markets and have the highest credit ratings*

This shows that when the credit standing of a bank in the Libor panel worsens compared to the rest of the banks in the panel, its borrowing (and lending) rate cannot anymore be a representative of Libor. The bank will go out of the panel, which is periodically revised (or it will adjust its contribution to make it acceptable, but this is discussed below). Thus there is a floor to how much a current Libor bank's credit standing can worsen for it to influence the future Libor fixing. This floor turns into a cap on the borrowing rate of a Libor counterparty, that needs to be respected for the bank to remain a Libor counterparty. This cap regards only the spread of a counterparty vs OIS, since the component in each lending rate which is riskless OIS is the same for all counterparties. As a consequence the expected spread over OIS of a future Libor counterparty is constantly lower than the expected spread over OIS in the future of a bank which is now a Libor counterparty. Indeed when the latter increases too much the bank will not be Libor anymore at that future date.

Thus there is an additional reason why the replication strategy of the basis swap in Remark 3, that appears to justify a zero Basis swap spread, is naive. If an investor wants to replicate the actual flows of the shorter-tenor leg of a Basis swap, she should lend 1 for $6m$ at the prevailing $6m$-Libor rate to another Libor counterparty X, then after $6m$ she should check if the credit standing of X is still sufficiently high for the bank to belong to the uppermost group of banks in terms of credit standing. If the answer is yes, she can actually lend the money again to the same counterparty X. Otherwise she will have to choose another counterparty Y with a higher credit standing. Thus the $6m$ leg implicitly embeds this bias towards the uppermost group of banks.

The counterparty risk in the $6m$ leg is obviously lower than in the $12m$ leg because the replication strategy of the $6m$ leg includes the possible requirement of moving after $6m$ to a counterparty that is better than the previous one in terms of credit standing, by definition of Libor . Thus the expectation of the survival probability of the borrower of the $6m$ leg in the $6m$-$12m$ period is higher than the survival probability of the borrower of the $12m$ in the same period, explaining why, on average, the $6m$ leg embeds less counterparty risk than the $12m$ leg. This justifies the existence of a positive basis to be added to the $6m$ leg when the flows are replicated in the collateralized Basis swap free of counterparty risk.

Differently from some of the considerations above, this fact is unambiguous. It is also mentioned in the previous literature, although in contexts different from this one. To the best of our knowledge, the first to mention this issue in Libor modeling is Grinblatt (1995). He points out that longer term Libor is the rate of a loan to an AA or AAA-rated borrower, and that an investor which is AA or AAA at an initial date may not be an AA or AAA investor some time later, while a sequence of shorter term Libor investments has the AA or AAA rating of the borrower "refreshed" periodically. Subsequently Collin-Dufresne and Solnik (2001) use this fact to build an econometric explanation of the spread between swaps and corporate bonds. Instead in the following we will implement this fact in a simpler model where Libor turns out to include an option on a forward spread.

Also the considerations of Section 5.3 enter the picture, telling us that not only a deal can exit from the Libor world because of a worsening of the credit conditions of the borrowers, but also because of a worsening of the lender's funding cost (credit conditions of the lender). The analysis of Section 5.4 tells that not necessarily the worsening of the conditions of a bank lead it to go out of the Libor panel. More simply, the bank may not update its contributions to its higher perceived credit risk, and give instead a more optimistic contribution. The effect on the Libor fixing is the same as exiting from the panel and being replaced by a healthier bank. This can be considered particularly important in the latter crisis when the interbank market had become strongly illiquid, so that Libor was less linked to actually traded quotes.

As regards the "turn-of-the year" effect observed in the Libor-OIS spread, it seems that when a date of balance-sheet report is between inception and maturity of a deal it is more likely that one of the two counterparties will have its credit standing revised during the life of the deal, increasing pressure on longer tenor rates versus shorter tenor rates.

Finally, Michaud and Upper (2008) and Cho and Rosemberg (2008) make a precise analysis of the Libor contributions from the different panel banks and find that in the Euro money market, that we consider in our empirical application, the dispersion was particularly high during the crisis (from 2 bps pre-crisis to around 20bps when the crisis erupted), which should increase the relevance of the "counterparty replacement" factor under analysis.

Although this is surely a partial representation of the problem it seems the single most relevant aspect to introduce in our analysis.

6 A model for Basis and FRA. Libor as an option

In this section we implement the above ideas in a simple model that we test on market data. We start from the expression of the Basis swap price given by (7), that implies a Basis equilibrium spread $B(0,2\alpha)$ which is

$$
\begin{aligned}
(10) \qquad B(0,2\alpha) &= \frac{1}{2}F_{Std}(0;\alpha,2\alpha)+\frac{1}{2}\left(\frac{1}{P_L(0,\alpha)}-1\right)\times \\
&\times\left(F_{Std}(0;\alpha,2\alpha)-E_{Std}(0;\alpha,2\alpha)\right)-\frac{1}{2}\frac{\mathbb{E}\left[D(0,2\alpha)\,\alpha\left(L_M(\alpha,2\alpha)\right]\right.}{P_E(0,2\alpha)\,\alpha}
\end{aligned}
$$

The only quantity that is now model-dependent is $\mathbb{E}\left[D(0,2\alpha)\,\alpha\left(L_M(\alpha,2\alpha)\right]\right.$. Before introducing the model that according to Section 5.5 is most realistic, we see what would happen to the Basis spread in simpler models.

First we consider the case of a riskless market where $L_M(0,T)=E_M(0,T)$, as it was, approximately, before the crisis. The Basis price would be:

$$
Basis\,(0,2\alpha;B) = P_E(0,2\alpha)\,\alpha E_{Std}(0;\alpha,2\alpha)-P_E(0,2\alpha)\,\alpha E_{Std}(0;\alpha,2\alpha)+P_E(0,2\alpha)\,\alpha 2B,
$$

leading to an equilibrium spread $B(0,2\alpha) = 0$.

In case the market is not riskless, as it happens in a credit crisis, we have $L_M(t,T)\neq E_M(t,T)$. There are now different possible hypothesis on the meaning of Libor, that lead to different results. We keep the homogeneity of Assumption 4, that implies

$$
\begin{aligned}
L^{X^0}(0,\alpha) &= L_M(0,\alpha)=F_{Std}(0;0,\alpha), \\
L^{X^\alpha}(\alpha,2\alpha) &= L_M(\alpha,2\alpha)=F_{Std}(\alpha;\alpha,2\alpha),
\end{aligned}
$$

recalling that L^{X^t} is the rate applying to a counterparty which is Libor at t. We need to make additional assumptions to reach results about FRA or Basis swaps. In the setting of Section 3, we added Assumption 5, namely that the future Libor $L_M(\alpha,2\alpha)$ will coincide, at fixing, with the rate $L^{X^0}(\alpha,2\alpha)$ applying in the future to the current Libor counterparty X^0, that is

$$
L_M(\alpha,2\alpha)=L^{X^0}(\alpha,2\alpha),\tau^{X^0}>\alpha.
$$

This is true only when a generic counterparty which is Libor today will remain Libor if it does not default (or equivalently when we consider a theoretical synthetic forward rate agreement tied to the future spot rate $L^{X^0}(\alpha,2\alpha)$, rather than being tied to $L^{X^\alpha}(\alpha,2\alpha)$). In this case we have, neglecting the small convexity adjustment

$$
\mathbb{E}\left[D(0,2\alpha)\,\alpha\left(L_M(\alpha,2\alpha)\right]\right.=P_E(0,2\alpha)\,\alpha F_{Std}(0;\alpha,2\alpha)
$$

so that

$$
B(0,2\alpha) = \frac{1}{2}\left[\left(\frac{1}{P_L(0,\alpha)}-1\right)\left(F_{Std}(0;\alpha,2\alpha)-E_{Std}(0;\alpha,2\alpha)\right)\right].
$$

This assumes that there is no upper bias in Libor due to the refreshment of Libor counterparty described in Section 5.5. However it includes consideration mentioned in Section 5.1: in $6m$ rolling the interest payment is broken into two payments of which the one at $6m$ is subject to lower default risk. In Figure 7 we show the Basis spread during the crisis computed according to these hypotheses, compared to the actual behaviour of the market Basis spread. We are far away from explaining the market patterns.

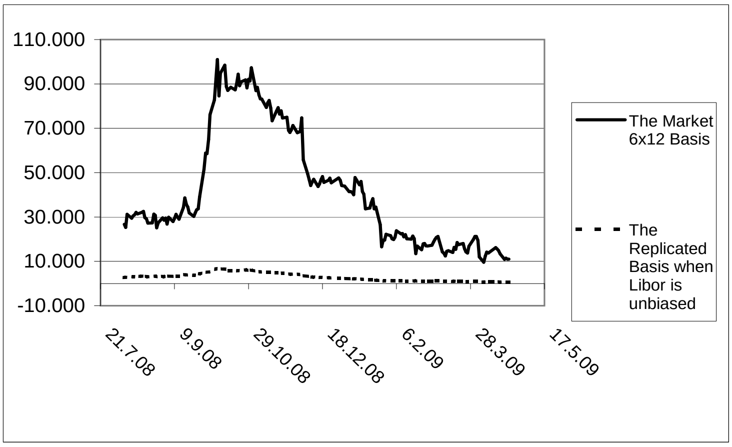

Figure 7

Now we can drop Assumption 5. We do not assume anymore that the future Libor $L_M(\alpha, 2\alpha)$ will coincide, at fixing, with the future rate $L^{X^0}(\alpha, 2\alpha)$ applying to the generic current Libor counterparty X^0. We start again from (10), but now

$$L_M(\alpha, 2\alpha) = L^{X^\alpha}(\alpha, 2\alpha) \neq L^{X^0}(\alpha, 2\alpha).$$

What is $L_M(\alpha, 2\alpha)$ in this new context? How it relates to $L^{X^0}(\alpha, 2\alpha)$?

We saw in Section 5.5 that the difference between $L_M(\alpha, 2\alpha)$ and $L^{X^0}(\alpha, 2\alpha)$ can only regard the spread of such rates over the riskless OIS rate, which is a component that does not depend on the specific counterparty considered. Thus we define the spread over OIS implicit in Libor,

$$S_M(\alpha, 2\alpha) := L_M(\alpha, 2\alpha) - E_M(\alpha, 2\alpha),$$

and the spread over OIS for the generic X^0 Libor counterparty,

$$S^{X^0}(\alpha, 2\alpha) := L^{X^0}(\alpha, 2\alpha) - E_M(\alpha, 2\alpha).$$

For the generic X^0 Libor counterparty it makes sense also to define a forward spread

$$S^{X^0}(t, \alpha, 2\alpha) := F_{Std}(t; \alpha, 2\alpha) - E_{Std}(t; \alpha, 2\alpha).$$

To implement the ideas of Section 5.5, we assume that Libor starts evolving like the rate of the generic libor counterparty X^0, but it is capped by the fact

X^0 will be excluded from the Libor panel at α if its spread over OIS is higher than the forward spread implicit in spot quotations at 0,

$$S^{X^0}(\alpha, 2\alpha) > S^{X^0}(0, \alpha, 2\alpha).$$

We could have chosen a different level for a counterparty to be excluded from the Libor world, but this choice has the advantage to be the simplest representation of an underperformer. This changes the strategy we have to implement to replicate the value of a Libor-indexed Basis swap:

Proposition 7 ***(An option in Libor quotes)*** *In the replication of a Libor floating leg the initial counterparty X^0 will be changed at α, to move to a Libor counterparty Y^α with a different spread $S^{Y^\alpha}(\alpha, 2\alpha)$, whenever the spread over OIS of the original counterparty, $S^{X^0}(\alpha, 2\alpha)$, is higher than the forward spread computed at* 0*. This means*

$$\begin{aligned} S_M(\alpha, 2\alpha) &= S^{X^0}(\alpha, 2\alpha)\, 1_{\left\{S^{X^0}(\alpha,2\alpha) < S^{X^0}(0,\alpha,2\alpha)\right\}} + S^{Y^\alpha}(\alpha, 2\alpha)\, 1_{\left\{S^{X^0}(\alpha,2\alpha) > S^{X^0}(0,\alpha,2\alpha)\right\}} \\ &= S^{X^0}(\alpha, 2\alpha) - 1_{\left\{S^{X^0}(\alpha,2\alpha) > S^{X^0}(0,\alpha,2\alpha)\right\}} \left(S^{X^0}(\alpha, 2\alpha) - S^{Y^\alpha}(\alpha, 2\alpha)\right). \end{aligned}$$

We need to compute, with $\mathbb{E} = \mathbb{E}^{P_E(\cdot,2\alpha)}$,

$$\begin{aligned} &\frac{\mathbb{E}\left[D(0, 2\alpha)\, \alpha L_M(\alpha, 2\alpha)\right]}{P_E(0, 2\alpha)\, \alpha} \\ &\quad = \mathbb{E}\left[L_M(\alpha, 2\alpha)\right] = \mathbb{E}\left[E_M(\alpha, 2\alpha) + S_M(\alpha, 2\alpha)\right] \end{aligned} \tag{11}$$

$$\begin{aligned} &= \mathbb{E}\left[E_M(\alpha, 2\alpha) + S^{X^0}(\alpha, 2\alpha) - 1_{\left\{S^{X^0}(\alpha,2\alpha) > S^{X^0}(0,\alpha,2\alpha)\right\}} \left(S^{X^0}(\alpha, 2\alpha) - S^{Y^\alpha}(\alpha, 2\alpha)\right)\right] \\ &= \mathbb{E}\left[L^{X^0}(\alpha, 2\alpha) - 1_{\left\{S^{X^0}(\alpha,2\alpha) > S^{X^0}(0,\alpha,2\alpha)\right\}} \left(S^{X^0}(\alpha, 2\alpha) - S^{Y^\alpha}(\alpha, 2\alpha)\right)\right] \\ &= F_{Std}(0; \alpha, 2\alpha) - \mathbb{E}\left[1_{\left\{S^{X^0}(\alpha,2\alpha) > S^{X^0}(0,\alpha,2\alpha)\right\}} \left(S^{X^0}(\alpha, 2\alpha) - S^{Y^\alpha}(\alpha, 2\alpha)\right)\right]. \end{aligned}$$

When, because of the above condition, the counterparty is changed, we move to a counterparty Y^α that, differently from the original one, has outperformed. It must be $S^{Y^\alpha}(\alpha, 2\alpha) \leq S^{X^0}(0, \alpha, 2\alpha)$, but what precisely will be $S^{Y^\alpha}(\alpha, 2\alpha)$?

We make again the simplest assumption: if the spread of our counterparty $S^{X^0}(\alpha, 2\alpha)$ is higher than $S^{X^0}(0, \alpha, 2\alpha)$, we move to a counterparty Y^α that took a symmetric path with respect to $S^{X^0}(0, \alpha, 2\alpha)$ (we may say that when the "barrier" $S^{X^0}(0, \alpha, 2\alpha)$ is touched, the path of the Libor spread is bounced back/reflected). This means that when the spread of our counterparty is *higher* than the forward spread by an amount $S^{X^0}(\alpha, 2\alpha) - S^{X^0}(0, \alpha, 2\alpha)$, we move to a counterparty whose spread is *lower* than the forward spread by the same amount, so that $S^{Y^\alpha}(\alpha, 2\alpha) = S^{X^0}(0, \alpha, 2\alpha) - \left(S^{X^0}(\alpha, 2\alpha) - S^{X^0}(0, \alpha, 2\alpha)\right) =$

$2S^{X^0}(0,\alpha,2\alpha) - S^{X^0}(\alpha,2\alpha)$. This leads to

$$
\begin{aligned}
& \mathbb{E}\left[1_{\left\{S^{X^0}(\alpha,2\alpha)>S^{X^0}(0,\alpha,2\alpha)\right\}}\left(S^{X^0}(\alpha,2\alpha)-S^{Y^\alpha}(\alpha,2\alpha)\right)\right] \\
= \; & \mathbb{E}\left[1_{\left\{S^{X^0}(\alpha,2\alpha)>S^{X^0}(0,\alpha,2\alpha)\right\}}\left(S^{X^0}(\alpha,2\alpha)-2S^{X^0}(0,\alpha,2\alpha)+S^{X^0}(\alpha,2\alpha)\right)\right] \\
= \; & 2\times\mathbb{E}\left[1_{\left\{S^{X^0}(\alpha,2\alpha)>S^{X^0}(0,\alpha,2\alpha)\right\}}\left(S^{X^0}(\alpha,2\alpha)-S^{X^0}(0,\alpha,2\alpha)\right)\right]
\end{aligned}
$$

If we go on neglecting no-arbitrage drifts and we assume that $S^{X^0}(t,\alpha,2\alpha)$ evolves as a geometric brownian motion

$$
dS^{X^0}(t,\alpha,2\alpha) = S^{X^0}(t,\alpha,2\alpha)\,\sigma_\alpha dW_\alpha(t), \tag{12}
$$

we can price the above option with the standard Black and Scholes formula,

$$
\begin{aligned}
BlackCall\left(X,K,\sigma\sqrt{T}\right) &= XN(d_1) - KN(d_2) \\
d_1 &= \frac{\ln(X/K)+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \qquad d_2 = \frac{\ln(X/K)-\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}.
\end{aligned}
$$

We obtain that implies

$$
\begin{aligned}
B(0,2\alpha) &= \frac{1}{2}\left(\frac{1}{P_L(0,\alpha)}-1\right)\left(F_{Std}(0;\alpha,2\alpha)-E_{Std}(0;\alpha,2\alpha)\right)+ \\
&+BlackCall\left(S^{X^0}(0,\alpha,2\alpha), S^{X^0}(0,\alpha,2\alpha), \sigma_\alpha\sqrt{\alpha}\right).
\end{aligned} \tag{13}
$$

This formula for the Basis is extremely simple. The only input we do not have available is the volatility. Since we are working on Euro data, as a proxy we use the volatility of the i-Traxx Index spread, the average credit spread of the 125 most liquid entities in the Euro market. We extract the volatility information from the ATM options on the $5y$ i-Traxx spread with expiry $\alpha = 6m$, that are quoted daily. Options with $6m$ expiry for a $6m$ spread would be more appropriate, but such options are not traded in the market.

The simple formula (13), based uniquely on Euribor and Eonia OIS data, with a credit volatility input, yields a good replication of the historical behaviour of the traded $6m/12m$ basis, as shown in Figure 8.

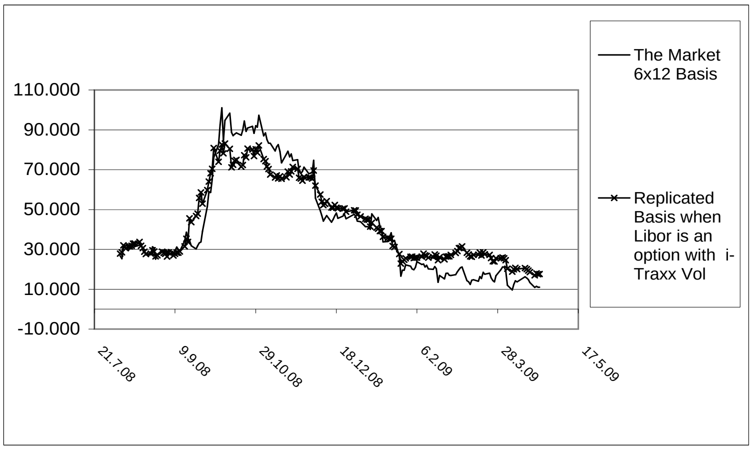

Figure 8

Some of the discrepancies between the historical pattern and our replication can be explained also by the lack of a more appropriate volatility input. For example, in the aftermath of Lehman default, the option replication tends to underestimate the historical pattern. This can be related to the fact that we are using as volatility proxy a value that refers to the 125 most liquid companies in Europe, an index composed by more than a half by corporate, nonfinancial firms, while the Libor spread should refer to financials only. The volatility of financials was perceived higher than the average volatility after Lehman's default, so that our volatility input may underestimate the correct one. This issue may partially explain also why, in the final part of the sample, referring to a period where banks reported very positive results and were perceived as less risky, the replicated basis is higher than the quoted one.

In spite of the details, the relevance of this result is to show that an extremely simple explanatory model based on counterparty risk in Libor, and on the optionality implicit in Libor, allows an approximate replication of the Basis swap (and FRA) market patterns, even though in the model there are no parameters used to fit to the Basis swap or FRA market. This representation of the Basis as an option has also the advantage of giving a general explanation of why the Basis gets higher the higher the difference in tenor between the two legs of the swap. A $6m$-tenor Libor leg is lower than a $1y$-tenor Libor leg since $6m$ leg involves a stream of counterparties whose credit risk is checked and refreshed every $6m$, down to reaching an overnight leg (OIS) whose risk of default is checked and refreshed overnight. This risk is sufficiently low to consider such a leg approximately riskless.

This result could be a starting point for a more general model where flows of Libor rates with different tenors embed different levels of default risk and different levels of default risk volatility, to be consistent with the financial evidence that there is a periodic updating of credit risk in Libor, and with the empirical patterns observed in the crisis.

Then some critical observations are in order. The model is obviously too simple to represent a detailed representation of the Basis swap or FRA dynamics. The spread behaviour could be represented by more realistic stochastic dynamics. Various elements of realism could be added even without changing the assumptions on the stochastic dynamics. For example, we modelled Libor as a kind of cap contract, but one may notice that in normal market conditions Libor is rather a collar, since there is not only a cap on the level of the forward rate of a counterparty for it to be considered Libor, but also a floor. In fact there are counterparties which are considered less risky than Libor, and in any case Libor is a trimmed average where both the lowest and highest quartiles are excluded. This may justify, in market conditions different from the recent crisis, even negative Basis Swap spreads.

We also point out that the choice made about $S^{Y^{\alpha}}(\alpha, 2\alpha)$ has the implication that this spread over OIS can turn negative in some case. More generally the choice appears optimistic about the Libor counterparty that substitutes the original one when the latter underperforms. A less optimistic choice, however, would underestimate the market basis, if the remaining assumptions are unchanged. With this regard, we point out that there are certainly few reasonable alternative assumptions that would lead to a result similar to (13). For example we could be less optimistic about the substitute counterparty $S^{Y^{\alpha}}$, but we could bring into the picture some of the elements mentioned in Section 5 and not introduced explicitly in the final simple model. In particular we could allow for the fact that the conditions of a lending contract depend not only on the borrower's credit conditions, but also on the lender's credit/liquidity conditions. And a lending contract can exit the Libor world non only because of credit problems of the borrower, as we assumed here, but also for liquidity problems of the lender, as we only hinted at in Section 5.3. Research is being carried on in this direction, as remarked in the Conclusions, where we present some possible directions for application, extension and improvement of the framework outlined here.

Part II

Modelling the Rates Market when banks can default

This Part starts from the standard results for a riskless market presented in Section 2, and explores how the mathematical no-arbitrage relations are modified when we introduce market-wide counterparty risk, with increasing level of realism. Our focus will be on Forward Rate Agreements, and techniques used are either replication, when possible, or change of measure/numeraire/filtration. The latter techniques are somewhat an extension of replication when simple replication is not possible, and in fact include replication as a special, lucky case.

This can be seen also in the context of a riskless market. If, instead of replication, we had used change of numeraire for pricing a FRA, we would have reached the same result. Setting $\alpha_i = \alpha(T_{i-1}, T_i)$, the FRA price is

$$FRA_{Std}(t; T_{i-1}, T_i; K) = \mathbb{E}_t [D(t, T_i)\, \alpha_i (L_M(T_{i-1}, T_i) - K)]$$

We can move from the risk neutral measure to the forward measure $Q^{P_L(\cdot,T_i)}$ associated to the Libor bond $P_L(t, T_i)$ (that in a riskless market is simply a riskless tradable bond), finding

$$FRA_{Std}(t; T_{i-1}, T_i; K) = P_L(t, T_i)\, \alpha_i \mathbb{E}_t^{P_L(\cdot,T_i)} [(L_M(T_{i-1}, T_i) - K)] .$$

Finding K that sets this to zero, we have

$$F_{Std}(t; T_{i-1}, T_i) = \mathbb{E}_t^{P_L(\cdot,T_i)} [L_M(T_{i-1}, T_i)] .$$

Thanks to (2),

$$\begin{aligned} \mathbb{E}_t^{P_L(\cdot,T_i)} [L_M(T_{i-1}, T_i)] &= \mathbb{E}_t^{P_L(\cdot,T_i)} \left[\frac{1}{\alpha_i} \left(\frac{1}{P_L(T_{i-1}, T_i)} - 1 \right) \right] \\ &= \mathbb{E}_t^{P_L(\cdot,T_i)} [F_{std}(T_{i-1}; T_{i-1}, T_i)] \end{aligned}$$

But $F_{std}(t; T_{i-1}, T_i)$ is a martingale under $Q^{P_L(\cdot,T_i)}$ since it is a tradable asset $P_L(t, T_{i-1})$ divided by the numeraire $P_L(t, T_i)$, so that we have

$$\mathbb{E}_t^{P_L(\cdot,T_i)} [F_{std}(T_{i-1}; T_{i-1}, T_i)] = F_{std}(t; T_{i-1}, T_i) = \left(\frac{P_L(t, T_{i-1})}{P_L(t, T_i)} - 1 \right) \frac{1}{\alpha_i}.$$

7 Forward rate Agreements when counterparties are defaultable

The first consequence of the credit crunch is that we can no longer assume that bank counterparties are riskless. In a defaultable bond we replace the payoff 1 at T_i with the payoff at T_i

$$\mathbb{R} + 1_{\{\tau^B > T_i\}} (1 - \mathbb{R}) ,$$

where $\mathbb{R}$ is the deterministic recovery rate, τ^B is default-time of the bond issuer B and we have assumed that the recovery payments always happen at maturity T_i, an acceptable approximation particularly for short-term deals like those we consider in the practical application of our analysis. The price of this defaultable bond at time t is

$$\begin{aligned} (14) \qquad P^B(t, T_i) \quad &: = \mathbb{E}_t \left[D(t, T_i) \left(\mathbb{R} + 1_{\{\tau^B > T_i\}} (1 - \mathbb{R}) \right) \right] \\ &= P(t, T_i)\, \mathbb{R} + \mathbb{E}_t \left[D(t, T_i) \left(1_{\{\tau^B > T_i\}} \right) \right] (1 - \mathbb{R}) . \end{aligned}$$

We look for a no-arbitrage relationship replacing (1) for defaultable counterparties. Suppose two counterparties A and B agree on the following deal with inception t: at time $T_{i-1} \geq t$ counterparty A pays 1 (if A has not defaulted earlier, otherwise only a fraction $\mathbb{R}$ is paid), while the other counterparty B pays at $T_i \geq T_{i-1}$ the amount $1 + K\alpha\left(T_{i-1}, T_i\right)$ (if B has not defaulted earlier, otherwise only a fraction $\mathbb{R}$ is paid). For the deal to be fair at t we need

$$
\begin{aligned}
(15) \qquad & \mathbb{E}_t\left[D\left(t, T_{i-1}\right)\left(\mathbb{R} + 1_{\{\tau^A > T_{i-1}\}}\left(1 - \mathbb{R}\right)\right)\right] \\
= \; & \mathbb{E}_t\left[D\left(t, T_i\right)\left(\mathbb{R} + 1_{\{\tau^B > T_i\}}\left(1 - \mathbb{R}\right)\right)\left(1 + K\alpha\left(T_{i-1}, T_i\right)\right)\right]
\end{aligned}
$$

If we take $T_{i-1} = t$, we have

$$
(16) \qquad \left(\mathbb{R} + 1_{\{\tau^A > t\}}\left(1 - \mathbb{R}\right)\right) = P^B\left(t, T_i\right)\left(1 + K\alpha\left(t, T_i\right)\right)
$$

The equilibrium rate is

$$
L^{A,B}(t, T_i) = \frac{1}{\alpha\left(t, T_i\right)}\left(\frac{\mathbb{R} + 1_{\{\tau^A > t\}}\left(1 - \mathbb{R}\right)}{P^B\left(t, T_i\right)} - 1\right) = \frac{1}{\alpha\left(t, T_i\right)}\left(\frac{P^A\left(t, t\right)}{P^B\left(t, T_i\right)} - 1\right).
$$

If additionally we are sure that both A and B are alive at t ($\tau^A, \tau^B > t$), we are describing a B-issued bond bought by A at t (or a deposit where A lends to B), and (16) expresses the relation of a risky spot rate with a risky bond, leading to the equilibrium rate

$$
L^B(t, T_i) = \frac{1}{\alpha\left(t, T_i\right)}\left(\frac{1}{P^B\left(t, T_i\right)} - 1\right).
$$

If we take $T_{i-1} > t$, we are describing the equivalent of the FRA replicating strategy of Remark 1 in a defaultable world. The equilibrium relation for this payoff is derived from (15),

$$
P^A\left(t, T_{i-1}\right) = P^B\left(t, T_i\right)\left(1 + K\alpha\left(T_{i-1}, T_i\right)\right)
$$

leading to the following equilibrium rate for this replicating strategy of the defaultable FRA

$$
F^{A,B}_{Def}\left(t; T_{i-1}, T_i\right) = \frac{1}{\alpha_i}\left(\frac{P^A\left(t, T_{i-1}\right)}{P^B\left(t, T_i\right)} - 1\right).
$$

The last part of the strategy should include a floating payment by A for the period from T_{i-1} to T_i. This makes sense only if we assume A and B to be alive at T_{i-1}. In this case the money paid by A at T_{i-1} can be reinvested by lending again to A. This floating payment can actually be introduced without changing the value of the strategy, since

$$
1 = P^A\left(T_{i-1}, T_i\right)\left(1 + L^A(T_{i-1}, T_i)\alpha_i\right),
$$

namely the payment of 1 made by A to B at T_{i-1} is financially equivalent to the payment $\left(1 + L^A(T_{i-1}, T_i)\alpha_i\right)$ at T_i.

Remark 8 ***(Defaultable FRA?)*** *We will see in the next sections that a FRA in the market is not a defaultable contract. A theoretical defaultable FRA, however, should certainly pay a floating leg indexed to Libor. Thus the above strategy replicates a defaultable FRA contract only when the rate $L^A(T_{i-1}, T_i)$ paid by the survived counterparty A is very similar to Libor. This introduces an important issue considered again in the following of Part 2, and analyzed under an economic point of view in Part 1.*

The rate $F^{A,B}_{Def}(t; T_{i-1}, T_i)$ cannot correspond to the FRA market quote $F_M(t; T_{i-1}, T_i)$ since the former depends on two specific counterparties while $F_M(t; T_{i-1}, T_i)$ is a rate unique for all the market. When default risk becomes non-negligible, counterparties become different from one another and therefore it is not trivial to model how the market comes to a single FRA equilibrium rate. We can reach a unique FRA equilibrium rate in a simple way if we treat all counterparties that are potential Libor contributors as having a homogeneous default probability expressed by Libor quotes.[5] Defining $\mathbb{L}_t$ as the set of counterparties that are potential Libor contributors a t, we have

$$L^B(t, T_i) = L^A(t, T_i) = L^{X^t}(t, T_i) = L_M(t, T_i) \quad A, B \in \mathbb{L}_t \tag{17}$$

where X^t is the generic Libor counterparty at t ($X^t \in \mathbb{L}_t$). Consequently, taking into account that a counterparty must first be alive at t for being a potential Libor contributor at t,

$$P^B(t, T_i) = \frac{1}{1 + L_M(t, T_i)\alpha(t, T_i)} = P_L(t, T_i) = P^{X^t}(t, T_i),$$

If the two counterparties of a FRA belong to the Libor world $\mathbb{L}_t$,

$$\begin{aligned} F^{A,B}_{Def}(t; T_{i-1}, T_i) &= F^{X^t}_{Def}(t; T_{i-1}, T_i) \\ &= \frac{1}{\alpha_i}\left(\frac{P_L(t, T_{i-1})}{P_L(t, T_i)} - 1\right) = F_{Std}(t; T_{i-1}, T_i) \end{aligned} \tag{18}$$

Thus under this credit setting we are back to (5) and we cannot explain why $F_M(t; T_{i-1}, T_i)$ has been different from $F_{Std}(t; T_{i-1}, T_i)$.

8 Forward rate Agreements with netting "no-fault" rule

In this section we introduce more realism in the treatment of FRA's and at the same time we try and keep simple formulas. The current standard for bilateral contracts such as swaps, supported by ISDA standard documentation, provides for the "no-fault" or "two-way payment rule", where if A defaults at τ^A the payoff for B will be

$$\left[Rec \cdot \left(NPV^B_{\tau^A}\right)^+ - \left(-NPV^B_{\tau^A}\right)^+\right],$$

[5] This is called Homogeneity Assumption in Section 3 of Part 1.

where NPV_t^X is the riskless net present value of the residual deal for counterparty X at time t. Furthermore here we have bilateral counterparty risk. In this context we cannot perform a replication of cashflows like in Remark 1, but we can still compute the price at inception t of a Forward Rate Agreement between two Libor counterparties $A, B \in \mathbb{L}_t$. To simplify notation and algebra we consider now the case when recovery is $R = 0$.

The price of the FRA to the payer B is the risk neutral expectation of its payoff:

$$
\begin{aligned}
FRA_{Net}^{A,B}(t; T_{i-1}, T_i; K) &= \alpha_i \mathbb{E}_t \left[D(t, T_i) \left(L_M(T_{i-1}, T_i) - K \right) \right] + \\
-\alpha_i \mathbb{E}_t &\left[D(t, T_i) 1_{\{\tau^A \le T_i\}} \left(L_M(T_{i-1}, T_i) - K \right)^+ \right] + \\
+\alpha_i \mathbb{E}_t &\left[D(t, T_i) 1_{\{\tau^B \le T_i\}} \left(K - L_M(T_{i-1}, T_i) \right)^+ \right].
\end{aligned}
$$

where the last two terms take into account bilateral counterparty risk along the lines of Brigo and Masetti (2005).

Now we need a model for credit risk. We work in a modelling framework based mainly on Bielecki and Rutkowski (2001) and Jamshidian (2004). The total market information, expressed by the filtration $(\mathcal{F}_s)_{s \ge 0}$, is divided into subfiltrations defined by

$$
\begin{aligned}
\mathcal{F}_t &= \mathcal{H}_t \vee_{J=1}^n \mathcal{J}_t^J \\
\mathcal{J}_t^J &= \sigma \left(\left\{ \tau^J > u \right\}, u \le t \right),
\end{aligned}
\tag{19}
$$

where $\left(\mathcal{J}_s^J \right)_{s \ge 0}$ is the natural filtration of the default time τ^J of the J-th market player, while $\mathcal{H}_t$ is the *no-default information*, information up to t on economic quantities which affect default probability, such as the default free interest rates, but excluding specific information on happening of default. The notation $\mathcal{A} \vee \mathcal{B}$ indicates the joint σ-algebra generated by the σ-algebras $\mathcal{A}$ and $\mathcal{B}$. A more detailed analysis of the range of financial hypotheses that can specify the representation of (19) is beyond the scope of this Part 2, so here we limit ourselves to the additional explicit and implicit hypothesis made below.

A special case of this setting, in the single name world, is the intensity model of Duffie and Singleton (1997, 1999) and Lando (1998), a standard in the Credit Default Swap (CDS) market. There the default time is modelled as $\tau^J = \inf \left\{ t : \int_0^t \lambda_s^J ds \ge \varepsilon^J \right\}$, and the positive process λ_s^J, $s \ge 0$, called *default intensity* of the name J, is adapted to $\mathcal{H}_t$, while ε^J is a random variable independent of no-default quantities. The process driving the default event in this model is called a Cox Process. We remain in a more general setting, but we make two assumptions that are typical of the standard market Cox Process setting. First we assume what Jamshidian (2004) calls *conditional independence for subfiltrations*, a property which is called *martingale invariance* by Bielecki and Rutkowski (2001), secondly we assume *positivity* of conditional survival probabilities.

Assumption 9 *(**Martingale Invariance**) Every (square-integrable) $\mathcal{H}$-martingale is also a $\mathcal{F}$-martingale, so that for $X_T \in \mathcal{H}_T$*

$$\mathbb{E}\left[X_T|\mathcal{H}_t\right] = \mathbb{E}\left[X_T|\mathcal{F}_t\right], \quad t \le T.$$

Assumption 10 *(**Positivity**) The survival probability conditional on no-default information is strictly positive*

$$\mathbb{Q}\left(\tau^J > t|\mathcal{H}_t\right) > 0, \quad t \ge 0.$$

In this framework a defaultable payoff $\mathbf{Y} = 1_{\{\tau>t\}}\mathbf{Y}$ depending on no-default information and on the default time τ of a given player can be priced using only a default indicator and no-default information. Dellacherie (1972) and Bielecki and Rutkowski (2001) show that

$$\mathbb{E}\left[\mathbf{Y}|\mathcal{F}_t\right] = \frac{\mathbf{1}_{\{\tau>t\}}}{\mathbb{Q}\left(\tau > t|\mathcal{H}_t\right)}\mathbb{E}\left[\mathbf{Y}|\mathcal{H}_t\right]. \tag{20}$$

We continue with the homogeneity assumption (17). Since in (17) Libor is defined on non-defaulted counterparties which are all the same, we assume $L(t, T_i)$ adapted to no-default information $\mathcal{H}_t$. We can rewrite the FRA price as

$$\begin{aligned} FRA^{A,B}_{Net}\left(t; T_{i-1}, T_i; K\right) &= \alpha_i \mathbb{E}_t\left[D\left(t, T_i\right) 1_{\{\tau^A > T_i\}}\left(L_M(T_{i-1}, T_i) - K\right)^+\right] \\ -\alpha_i \mathbb{E}_t\left[D\left(t, T_i\right) 1_{\{\tau^B > T_i\}}\left(K - L_M(T_{i-1}, T_i)\right)^+\right] \end{aligned}$$

so that we can now apply (20),

$$\begin{aligned} FRA^{A,B}_{Net}\left(t; T_{i-1}, T_i; K\right) &= \frac{1_{\{\tau^A>t\}}}{H\Pr^A_t}\alpha_i \mathbb{E}_{Ht}\left[D\left(t, T_i\right) 1_{\{\tau^A > T_i\}}\left(L_M(T_{i-1}, T_i) - K\right)^+\right] \\ -\frac{1_{\{\tau^B>t\}}}{H\Pr^B_t}\alpha_i \mathbb{E}_{Ht}\left[D\left(t, T_i\right) 1_{\{\tau^B > T_i\}}\left(K - L_M(T_{i-1}, T_i)\right)^+\right], \end{aligned}$$

where $\mathbb{E}_{Ht}\left[\cdot\right] := \mathbb{E}\left[\cdot|\mathcal{H}_t\right]$ and $H\Pr^B_t := \mathbb{Q}\left(\tau^B > t|\mathcal{H}_t\right)$, that is the $\mathcal{H}_t$-conditional survival probability of B up to t. Using law of iterated expectations,

$$\begin{aligned} FRA^{A,B}_{Net}\left(t; T_{i-1}, T_i; K\right) &= \frac{1_{\{\tau^A>t\}}}{H\Pr^A_t}\alpha_i \mathbb{E}_{Ht}\left[\mathbb{E}_{HT_i}\left[D\left(t, T_i\right) 1_{\{\tau^A > T_i\}}\left(L_M(T_{i-1}, T_i) - K\right)^+\right]\right] \\ -\frac{1_{\{\tau^B>t\}}}{H\Pr^B_t}\alpha_i \mathbb{E}_{Ht}\left[\mathbb{E}_{HT_i}\left[D\left(t, T_i\right) 1_{\{\tau^B > T_i\}}\left(K - L_M(T_{i-1}, T_i)\right)^+\right]\right] \end{aligned}$$

Since $L(t, T_i)$ is adapted to no-default information $\mathcal{H}_t$,

$$\begin{aligned} FRA^{A,B}_{Net}\left(t; T_{i-1}, T_i; K\right) &= \frac{1_{\{\tau^A>t\}}}{H\Pr^A_t}\alpha_i \mathbb{E}_{Ht}\left[D\left(t, T_i\right) H\Pr^A_{T_i}\left(L_M(T_{i-1}, T_i) - K\right)^+\right] \\ -\frac{1_{\{\tau^B>t\}}}{H\Pr^B_t}\alpha_i \mathbb{E}_{Ht}\left[D\left(t, T_i\right) H\Pr^B_{T_i}\left(K - L_M(T_{i-1}, T_i)\right)^+\right]. \end{aligned}$$

Thanks to the fact that $A, B \in \mathbb{L}_t$, we can set

$$1_{\{\tau^B > t\}} = 1_{\{\tau^A > t\}} = 1, \tag{21}$$

while the homogeneity assumption (17) implies

$$P^A(t, T_i) = P^B(t, T_i) = P_L(t, T_i) = P^{X^t}(t, T_i).$$

We can price with (20) also the defaultable bond of the generic counterparty $X^t \in \mathbb{L}_t$. From the bond definition (14), with $R = 0$,

$$\begin{aligned} P^{X^t}(t, T_i) &= \mathbb{E}_t\left[D(t, T_i)\, 1_{\{\tau^{X^t} > T_i\}}\right] = \frac{1_{\{\tau^{X^t} > t\}}}{H\,\mathrm{Pr}_t^{X^t}} \mathbb{E}_{Ht}\left[D(t, T_i)\, H\,\mathrm{Pr}_{T_i}^{X^t}\right] \\ &= \mathbb{E}_{Ht}\left[D(t, T_i) \frac{H\,\mathrm{Pr}_{T_i}^{X^t}}{H\,\mathrm{Pr}_t^{X^t}}\right], \end{aligned}$$

where the last passage comes from (21). At a generic time $s > t$ we have

$$P^{X^t}(s, T_i) = \frac{1_{\{\tau^{X^t} > s\}}}{H\,\mathrm{Pr}_s^{X^t}} \mathbb{E}_{Hs}\left[D(s, T_i)\, H\,\mathrm{Pr}_{T_i}^{X^t}\right].$$

Now we understand that the homogeneity assumption (17) leads also to

$$H\,\mathrm{Pr}_{T_i}^{B} = H\,\mathrm{Pr}_{T_i}^{A} = H\,\mathrm{Pr}_{T_i}^{X^t}, \tag{22}$$

so that the FRA price is

$$\begin{aligned} FRA_{Net}^{A,B}(t; T_{i-1}, T_i; K) &= FRA_{Net}^{X^t}(t; T_{i-1}, T_i; K) = \\ \alpha_i \mathbb{E}_{Ht}&\left[D(t, T_i) \frac{H\,\mathrm{Pr}_{T_i}^{X^t}}{H\,\mathrm{Pr}_t^{X^t}} \left(L_M(T_{i-1}, T_i) - K\right)\right]. \end{aligned} \tag{23}$$

This shows that a swap with symmetric counterparty risk can be priced as a simple defaultable payoff, where the survival probability to use is the one of the generic counterparty $X^t \in \mathbb{L}_t$ (replacing $\frac{H\,\mathrm{Pr}_{T_i}^{X^t}}{H\,\mathrm{Pr}_t^{X^t}}$ with $R + \frac{H\,\mathrm{Pr}_{T_i}^{X^t}}{H\,\mathrm{Pr}_t^{X^t}}(1 - R)$ this holds also for deterministic recovery).

Now we can compute the expectation in (23) by change of numeraire. In line with Jamshidian (2004), Brigo (2005) and Brigo and Morini (2005), for computing an $\mathcal{H}_t$-conditional expectation we take as a numeraire the $\mathcal{H}_t$-conditional expectation of a payoff. We choose the so called "no-default value" of $P^{X^t}(s, T_i)$, namely the quantity

$$\tilde{P}^{X^t}(s, T_i) := \mathbb{E}_{Hs}\left[D(s, T_i)\, 1_{\{\tau^{X^t} > T_i\}}\right] = \mathbb{E}_{Hs}\left[D(s, T_i)\, H\,\mathrm{Pr}_{T_i}^{X^t}\right].$$

In the following we set $\tau = \tau^{X^t}$. As expected for a numeraire, we have

$$\frac{\tilde{P}^{X^t}(s, T_i)}{B(s)} = \mathbb{E}_{Hs}\left[\frac{\tilde{P}^{X^t}(T_{i-1}, T_i)}{B(T_{i-1})}\right],$$

since
(24)
$$\mathbb{E}_{Hs}\left[\frac{\tilde{P}^{X^t}(T_{i-1},T_i)}{B(T_{i-1})}\right]=\mathbb{E}_{Hs}\left[\frac{\mathbb{E}_{HT_{i-1}}\left[D(T_{i-1},T_i)\,1_{\{\tau>T_i\}}\right]}{B(T_{i-1})}\right]=\mathbb{E}_{Hs}\left[\frac{1_{\{\tau>T_i\}}}{B(T_i)}\right].$$

Taking as numeraire $\tilde{P}^{X^t}(s,T_i)$, we define on $\mathcal{H}_{T_{i-1}}$ the pricing measure $\mathbb{Q}^{\tilde{P}^{X^t}(\cdot,T_i)}$ through the Radon-Nikodym derivative

$$\tilde{Z}_{T_{i-1}}=\left.\frac{d\mathbb{Q}^{\tilde{P}^{X^t}(\cdot,T_i)}}{d\mathbb{Q}}\right|_{\mathcal{H}_{T_{i-1}}}=\frac{B_0\;\tilde{P}^{X^t}(T_{i-1},T_i)}{\tilde{P}^{X^t}(0,T_i)\;\;B_{T_{i-1}}},$$

which is well defined since one can prove that its restriction to $\mathcal{H}_s$ is

$$\tilde{Z}_s=\mathbb{E}\left[\left.\left.\frac{d\mathbb{Q}^{\tilde{P}^{X^t}(\cdot,T_i)}}{d\mathbb{Q}}\right|_{\mathcal{H}_{T_{i-1}}}\right|\mathcal{H}_s\right]=\frac{B_0\;\tilde{P}^{X^t}(s,T_i)}{\tilde{P}^{X^t}(0,T_i)\;\;B_s}.$$

We indicate with $\mathbb{E}^{\tilde{P}^{X^t}(\cdot,T_i)}[\cdot]$ the expectation under this pricing measure, and we find

$$\begin{aligned}FRA^{X^t}_{Net}(t;T_{i-1},T_i;K) &= \tilde{P}^{X^t}(t,T_i)\,\alpha_i\mathbb{E}^{\tilde{P}^{X^t}(\cdot,T_i)}_{Ht}\left[\frac{\frac{H\,\mathrm{Pr}^{X^t}_{T_i}}{H\,\mathrm{Pr}^{X^t}_{t}}\left(L_M(T_{i-1},T_i)-K\right)}{\tilde{P}^{X^t}(T_i,T_i)}\right]\\ &= \frac{\tilde{P}^{X^t}(t,T_i)}{H\,\mathrm{Pr}^{X^t}_t}\alpha_i\mathbb{E}^{\tilde{P}^{X^t}(\cdot,T_i)}_{Ht}\left[\frac{H\,\mathrm{Pr}^{X^t}_{T_i}\left(L_M(T_{i-1},T_i)-K\right)}{H\,\mathrm{Pr}^{X^t}_{T_i}}\right]\\ &= P_L(t,T_i)\,\alpha_i\mathbb{E}^{\tilde{P}^{X^t}(\cdot,T_i)}_{Ht}\left[\left(L_M(T_{i-1},T_i)-K\right)\right].\end{aligned}$$

Now we can impose this to be 0, finding

$$F^{X^t}_{Net}(t;T_{i-1},T_i)\;=\;\mathbb{E}^{\tilde{P}^{X^t}(\cdot,T_i)}_{Ht}\left[L_M(T_{i-1},T_i)\right]=\mathbb{E}^{\tilde{P}^{X^t}(\cdot,T_i)}_{Ht}\left[\frac{1}{\alpha_i}\left(\frac{1}{P_L(T_{i-1},T_i)}-1\right)\right].$$

Notice that $P_L(s,T_i)$, $t<s\le T_i$, is a variable whose relationship with some other quantities in the model is not clear yet. In fact in (17) we have assumed $P_L(s,T_i)=P^{X_s}(s,T_i)\,,\forall s$, namely for any s the Libor bond at s corresponds to the bond of the generic Libor counterparty at s, X^s. However the numeraire that defines this pricing measure is $\tilde{P}^{X^t}(\cdot,T_i)$, which is the no-default value of $P^{X^t}(s,T_i)$, namely the no-default price at s of the bond of a counterparty which is Libor at t, and not necessary at s, $t\le s\le T_i$.

We can solve the expectation easily if we assume further that not only the credit conditions of my counterparty today are expressed by Libor,

$$P_L(s,T_i)=:P^{X_s}(s,T_i)\,,$$

as implied by homogeneity, but also that, in case counterparty X^t is alive at s, it will have credit conditions analogous to those expressed by future Libor at s,[6]

$$P_L(s,T_i) = P^{X_t}(s,T_i), \text{ with } s>t, \tau^{X^t}>s. \tag{25}$$

The assumption (25) corresponds to modelling a market where *future* Libor contributors will be so similar to how *current* Libor contributors will be *in the future* that we can identify Libor in the future with any current Libor counterparty, until the latter is alive. To put it in different terms, we are modelling a market where a counterparty which is Libor today will remain Libor until it defaults.

Under (25) we have that $P_L(s,T_i)$ is equal to the bond of any counterparty X^t alive at s, namely to the value of the bond of X^t without the default indicator, $P_L(s,T_i) = \frac{\tilde{P}^{X^t}(s,T_i)}{H\Pr_s^{X^t}}$. We implicitly assume that there will always be at least one counterparty X^t alive at s to be taken as a reference. We have

$$\begin{aligned}
& \mathbb{E}_{Ht}^{\tilde{P}^{X^t}(\cdot,T_i)}\left[\frac{1}{\alpha_i}\left(\frac{1}{P_L(T_{i-1},T_i)}-1\right)\right] \\
= {} & \mathbb{E}_{Ht}^{\tilde{P}^{X^t}(\cdot,T_i)}\left[\frac{1}{\alpha_i}\left(\frac{H\Pr_{T_{i-1}}^{X^t}}{\tilde{P}^{X^t}(T_{i-1},T_i)}-1\right)\right] \\
= {} & \mathbb{E}_{Ht}^{\tilde{P}^{X^t}(\cdot,T_i)}\left[\frac{1}{\alpha_i}\left(\frac{\tilde{P}^{X^t}(T_{i-1},T_{i-1})}{\tilde{P}^{X^t}(T_{i-1},T_i)}-1\right)\right]
\end{aligned} \tag{26}$$

This is the expectation of a quantity which is the no-default price of the bond of a Libor counterparty divided by the numeraire at T_{i-1}, and we can prove that $\frac{1}{\alpha_i}\left(\frac{\tilde{P}^{X^t}(s,T_{i-1})}{\tilde{P}^{X^t}(s,T_i)}-1\right)$ is a martingale under the associated pricing measure, conditional to H_t. According to the Bayes rule for conditional change of numeraire,

$$\begin{aligned}
& \mathbb{E}_{Hs}^{\tilde{P}^{X^t}(\cdot,T_i)}\left[\frac{1}{\alpha_i}\left(\frac{\tilde{P}^{X^t}(T_{i-1},T_{i-1})}{\tilde{P}^{X^t}(T_{i-1},T_i)}-1\right)\right] \\
& = \mathbb{E}_{Hs}\left[\frac{\tilde{Z}_{T_{i-1}}}{\tilde{Z}_s}\frac{1}{\alpha_i}\left(\frac{\tilde{P}^{X^t}(T_{i-1},T_{i-1})}{\tilde{P}^{X^t}(T_{i-1},T_i)}-1\right)\right] \\
& = \frac{1}{\alpha_i}\mathbb{E}_{Hs}\left[\left(\frac{\mathbb{E}_{HT_{i-1}}\left[D(s,T_{i-1})1_{\{\tau>T_{i-1}\}}-D(s,T_i)1_{\{\tau>T_i\}}\right]}{\mathbb{E}_{Hs}\left[D(s,T_i)1_{\{\tau>T_i\}}\right]}\right)\right] \\
& = \frac{\mathbb{E}_{Hs}\left[\mathbb{E}_{HT_{i-1}}\left[D(s,T_{i-1})1_{\{\tau>T_{i-1}\}}-D(s,T_i)1_{\{\tau>T_i\}}\right]\right]}{\alpha_i\mathbb{E}_{Hs}\left[D(s,T_i)1_{\{\tau>T_i\}}\right]} \\
& = \frac{1}{\alpha_i}\left(\frac{\tilde{P}^{X^t}(s,T_{i-1})}{\tilde{P}^{X^t}(s,T_i)}-1\right).
\end{aligned} \tag{27}$$

[6]This is called Persistency Assumption in Section 3 of Part 1.

Additionally, notice that

$$(28) \qquad \frac{1}{\alpha_i}\left(\frac{\tilde{P}^{X^t}(s,T_{i-1})}{\tilde{P}^{X^t}(s,T_i)}-1\right) = \frac{1}{\alpha_i}\left(\frac{\tilde{P}^{X^t}(s,T_{i-1})/H\,\mathrm{Pr}_s^{X^t}}{\tilde{P}^{X^t}(s,T_i)/H\,\mathrm{Pr}_s^{X^t}}-1\right)$$

$$= \frac{1}{\alpha_i}\left(\frac{P_L(s,T_{i-1})}{P_L(s,T_i)}-1\right) = F_{Std}(s;T_{i-1},T_i),$$

leading to

$$F_{Net}(t;T_{i-1},T_i) = F_{Std}(t;T_{i-1},T_i).$$

The equilibrium rate is still the standard forward. This confirms that this rate keeps a financial meaning, under certain hypothesis, even in a defaultable market, but also implies that this setting based on (25) does not explain the market rate $F_M(t;T_{i-1},T_i)$ during the crisis.

The contribution of this section is to show that the formalism of Jamshidian (2004) and Brigo (2005), where one can separate the default indicator from the conditional default probability, allows to build a no-arbitrage model where Libor is the rate tied to the value of the defaultable bond of the generic Libor counterparty, and at the same time Libor itself never defaults. Additionally we have the following remark.

Remark 11 ***(Bonds as numeraires: same payoff but different risk)*** *With reference to the issues dealt with in Bianchetti (2008) and Henrard (2009), one can notice that this setting allows to deal consistently with different bonds, such as the Libor bond $P_L(t,T)$ and the OIS bond $P_E(t,T)$, which embed different risk in spite of the fact that they give the same non-defaultable payoff of 1 at T. In our setting this does not represent an arbitrage or an inconsistency since we treat only the OIS bond as a tradable asset, and in fact we will use it as a numeraire in the next section, while $P_L(t,T)$ is not a tradable asset and only a modification of it can be used as a numeraire.*
As a consequence, and differently from what one would expect in standard change of numeraire, in (26) we do not treat $P_L(T_{i-1},T_i)$ at the denominator as the numeraire (in fact $P_L(t,T_i)$ is not the $\mathcal{H}$-expectation of a payoff) and we treat 1 at numerator as the value at maturity of a Libor bond $P_L(t,T_{i-1})$ but this bond is not intended as a standard tradable asset (otherwise $P_L(t,T_{i-1})$ would be indistinguishable from $P_E(t,T_{i-1})$). The solution to the conundrum is given by the fact that, when we use the definition of $P_L(T_{i-1},T_i)$, we find a new ratio where the denominator is actually the numeraire and the numerator is a default-dependent quantity such that we can prove the martingale property.

This framework could be of some help in adapting to the new post-crisis situation the techniques typical in fixed income modelling, that also underlie curve bootstrapping. On the other hand we point out that in Section 5 and 6 of Part 1 we debate the realism of (25) and then we release it, so the techniques used here should go along the considerations made there.

9 Forward rate Agreements with Collateral Agreement

In the above derivations we assumed that the credit risk in a FRA contract is the same as the credit risk of the counterparties in unsecured lending, so that the discounting can be associated to the Libor bond $P_L(t, T_i)$. However FRA contracts are usually stipulated within collateral agreements. Thus the credit risk is commonly considered negligible, justifying the choice of a curve as riskless as possible for discounting:

$$
\begin{aligned}
FRA_{Col}^{X_t}(t; T_{i-1}, T_i; K) &= \alpha_i \mathbb{E}_t \left[D(t, T_i) \left(L_M(T_{i-1}, T_i) - K \right) \right] \\
&= P_E(t, T_i) \, \alpha_i \mathbb{E}_t^{P_E(\cdot, T_i)} \left[\left(L_M(T_{i-1}, T_i) - K \right) \right],
\end{aligned}
$$

(29)
$$
\begin{aligned}
F_{Col}^{X_t}(t; T_{i-1}, T_i) &= \mathbb{E}_t^{P_E(\cdot, T_i)} \left[L_M(T_{i-1}, T_i) \right] \\
&= \mathbb{E}_t^{P_E(\cdot, T_i)} \left[\frac{1}{\alpha_i} \left(\frac{1}{P_L(T_{i-1}, T_i)} - 1 \right) \right] \\
&= \mathbb{E}_t^{P_E(\cdot, T_i)} \left[F_{Std}(T_{i-1}; T_{i-1}, T_i) \right].
\end{aligned}
$$

Since $F_{Std}(t; T_{i-1}, T_i) := \frac{1}{\alpha(T_{i-1}, T_i)} \left(\frac{P_L(t, T_{i-1})}{P_L(t, T_i)} - 1 \right)$, now this is not a martingale under the pricing measure, thus

(30)
$$
F_{Col}^{X_t}(t; T_{i-1}, T_i) \neq F_{Std}(t; T_{i-1}, T_i).
$$

Remark 12 ***(Convexity Adjustment)****. The above "measure mismatch" means that the collateralized FRA in a world of non-negligible default risk stands to the theoretical FRA of a riskless world similarly to how a Futures contract of a riskless world stands to a FRA of a riskless world. Futures contracts eliminate the correlation between discount and payoff that is typical of standard FRA. Similarly, in the crisis the growth of counterparty risk led, for market collateralized FRA's, to a reduction in the correlation between discount and payoff that is typical of standard FRA. In Futures pricing this is dealt with by the use of a convexity adjustment, which is a simplified way to express a "measure mismatch".*

We want to verify whether (30) can explain a discrepancy with the sign and the size we observe in the market. In this setting the FRA rate $F_{Col}^{X_t}(t; T_{i-1}, T_i)$ will depend on the dynamics of $F_{Std}(t; T_{i-1}, T_i)$ under the $P_E(\cdot, T_i)$-measure. We would like to use the results (27) of the previous section where we have shown that $F_{Std}(t; T_{i-1}, T_i)$ is a martingale under the pricing measure $\mathbb{Q}^{\tilde{P}^{X^t}(\cdot, T_i)}$ defined on $\mathcal{H}$, and apply change of numeraire and Girsanov theorem in the setting of the Libor Market Model of Brace, Gatarek and Musiela (1997).

Thanks to the Martingale Property of Assumption 9, we have

$$
\mathbb{E}[X_T | \mathcal{H}_t] = \mathbb{E}[X_T | \mathcal{F}_t]
$$

for $\mathcal{H}_T$-measurable X_T, like in the reduced-form or intensity model which is the standard for credit derivatives. This implies that the above expectation (29) can be replaced by an $\mathcal{H}_t$-expectation

$$\mathbb{E}_{Ht}^{P_E(\cdot,T_i)}\left[F_{Std}\left(T_{i-1};T_{i-1},T_i\right)\right],$$

so that we are looking for the $\mathcal{H}_t$-expectation under measure $\mathbb{Q}^{P_E(\cdot,T_i)}$ of a quantity that is an $\mathcal{H}_t$-martingale under $\mathbb{Q}^{\tilde{P}^{X^t}(\cdot,T_i)}$ and we can apply standard change of numeraire.

For computing the change of measure giving the dynamics of $F_{Std}\left(t;T_{i-1},T_i\right)$ under the measure associated to the OIS bond $P_E\left(t,T_i\right)$ we need the dynamics of the "survival probabilities"

$$\frac{\tilde{P}^{X^0}\left(t,T_i\right)}{P_E\left(t,T_i\right)}=\frac{\mathbb{Q}\left(\tau>t|\mathcal{H}_t\right)P_L\left(t,T_i\right)}{P_E\left(t,T_i\right)}=:Q\left(t,T_i\right)$$

which is analogous to the *default risk factor* of Schonbucher (2004), with the difference that in Schonbucher (2004) $\mathbb{Q}\left(\tau>t|\mathcal{H}_t\right)$ does not appear. This follows from the fact that, differently from us, Schonbucher (2004) works with probability measures which are not equivalent to the real world or risk-neutral probability measures.

Remark 13 ***(Cross-currency analogy)*** *An analogy emerges between FX cross-currency change of numeraire and the above change of numeraire for collateralized interest rate derivatives tied to risky rates. In fact $Q\left(t,T_i\right)$ has the same form as the forward exchange rate $X\left(t,T_i\right)$ that regulates cross-currency change of numeraire,*

$$X\left(t,T_i\right)=\frac{x\left(t\right)P_F\left(t,T_i\right)}{P_D\left(t,T_i\right)}$$

where $P_F\left(t,T_i\right)$, $P_D\left(t,T_i\right)$ are respectively the foreign and domestic bond prices, and $x\left(t\right)$ is the spot exchange rate that converts the foreign bond in domestic currency. Such analogy has been first noticed by Bianchetti (2008) in a more abstract setting. In this setting it is one consequence of the introduction of credit risk, and additionally we have a further element of analogy. In cross-currency application, when changing to the domestic measure the foreign bond cannot be used as a numeraire unless it is converted in domestic currency by the spot rate of exchange $x\left(t\right)$. In our credit setting the Libor bond cannot be used as a numeraire unless it is converted through the conditional survival probability $\mathbb{Q}\left(\tau>t|\mathcal{H}_t\right)$, that plays the role of the spot rate of exchange $x\left(t\right)$. This term is not made explicit in Bianchetti (2008), where it is set to 1 at $t=0$.

We will work with the following vector process, where for notational simplicity we write $F_{Std}\left(t;T_{i-1},T_i\right)=F\left(t;T_{i-1},T_i\right)$,

$$\left[F\left(t;T_0,T_1\right),\ldots,F\left(t;T_{N-1},T_N\right),Q\left(t,T_1\right),\ldots,Q\left(t,T_N\right)\right]'$$

associated to the set of fixing-payment dates $[T_0, \ldots, T_N]$, called tenor structure. We start from the martingale dynamics of $F(t; T_{i-1}, T_i)$ under $\tilde{P}^{X^0}(t, T_i)$ measure, with brownian motion $W^{L_i}(t)$, and of $Q(t, T_i)$ under $P_E(t, T_i)$ measure, with brownian motion $W^{E_i}(t)$. We assume for simplicity lognormal dynamics,

$$\begin{aligned} dF(t; T_{i-1}, T_i) &= \sigma^F_{T_i}(t) F(t; T_{i-1}, T_i)\ dW^{L_i}_{F_i}(t), \quad i = 1, \ldots, N, \\ dQ(t, T_i) &= \sigma^Q_{T_i}(t) Q(t, T_i)\ dW^{E_i}_{Q_i}(t), \quad i = 1, \ldots, N. \end{aligned}$$

The instantaneous correlation of the vector brownian motion is Σ. The dynamics of $F(t; T_{i-1}, T_i)$ under the $P_E(t, T_i)$ measure is regulated by

$$dW^{L_i}(t) = dW^{E_i}(t) - \Sigma \mathrm{DC}\left(\ln\left(Q(t, T_i)\right)\right)' dt, \tag{31}$$

where $\mathrm{DC}(\cdot)$ stands for Diffusion Coefficient, see for example Brigo and Mercurio (2006).

For simplicity, and for consistency with the rest of the paper, we focus on the FRA with fixing at α and payment at 2α with underlying rate $L_M(\alpha, 2\alpha)$. We work on a simplified tenor structure $[T_0 = \alpha, T_1 = 2\alpha]$ with only $F(t; \alpha, 2\alpha) = F(t; T_0, T_1)$ and $Q(t, 2\alpha) = Q(t, T_1)$. In this simplified context, we have

$$\mathrm{DC}\left(\ln\left(Q(t, T_i)\right)\right) = \left[0, \sigma^Q_{2\alpha}(t)\right]$$

Now we take into account the instantaneous correlation matrix Σ, that in this case is fully expressed by the correlation ρ between $dW_{F_1}(t)$ and $dW_{Q_1}(t)$,

$$\begin{aligned} dW^{L_1}(t) &= dW^{E_1} - \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \sigma^Q_{2\alpha}(t) \end{bmatrix} dt, \\ dW^{L_1}_{F_1}(t) &= dW^{E_1}_{F_1} - \rho \sigma^Q_{2\alpha}(t) dt, \end{aligned}$$

$$\begin{aligned} dF(t; \alpha, 2\alpha) &= \sigma^F_{2\alpha}(t) F(t; \alpha, 2\alpha)\ dW^{L_1}_{F_1}(t) \\ dF(t; \alpha, 2\alpha) &= -\rho \sigma^Q_{2\alpha}(t) \sigma^F_{2\alpha}(t) F(t; \alpha, 2\alpha)\, dt + \sigma^F_{2\alpha}(t) F(t; \alpha, 2\alpha)\, dW^{E_1}_{F_1} \end{aligned}$$

Under flat parameters

$$\begin{aligned} F^{X_0}_{Col}(0; \alpha, 2\alpha) &= \mathbb{E}^{P_E(\cdot, 2\alpha)}_0 \left[F(\alpha; \alpha, 2\alpha)\right] = \mathbb{E}^{P_E(\cdot, 2\alpha)}_0 \left[F_{Std}(\alpha; \alpha, 2\alpha)\right], \\ F^{X_0}_{Col}(0; \alpha, 2\alpha) &= F_{Std}(0; \alpha, 2\alpha) \exp\left(-\alpha \rho \sigma^Q_{2\alpha} \sigma^F_{2\alpha}\right), \end{aligned}$$

a result that, as expected, reminds of Bianchetti (2008) *quanto-adjusted rate*, with the differences pointed out above.

If we could estimate the value of the variables and parameters in the above relation, we would have a model estimation of the gap between the market FRA F_M, as replicated by $F^{X_t}_{Col}(t; \alpha, 2\alpha)$, and the standard replication F_{Std}. The gap in the market was around $-50bp$. It is immediate to see that if there was negligible credit volatility ($\sigma^Q_{2\alpha} \approx 0$), like in the pre-crisis time, F_M-$F_{Std} \approx 0$.

Otherwise the sign of the gap depends on the correlation between conditional default probability and rates. If the correlation was positive, the gap would be negative, as it was in the market.

We now use market guesses on the values of the parameters to assess the model gap. Based on a simple historical estimation on Libor and OIS data, considering the crisis period from July 2007 to May 2009,

$$\sigma_{2\alpha}^{F} = 12\%, \sigma_{2\alpha}^{Q} = 0.7\%, \rho = 0.6\%$$

while on a longer period, from January 2003 to May 2009,

$$\sigma_{2\alpha}^{F} = 19\%, \sigma_{2\alpha}^{Q} = 0.4\%, \rho = 0.8\%$$

The average value of $F_{Std}(0;\alpha,2\alpha)$ was 4.3% in the crisis period and 3.2% in the longer one. This would lead to

$$F_{Col}^{X_t}(t;\alpha,2\alpha) - F_{Std}(0;\alpha,2\alpha) \approx \begin{cases} \text{-0.02}bp & \text{(based on crisis period)} \\ \text{-0.01}bp & \text{(based on longer period)} \end{cases}$$

The estimations of the parameters are very rough. In spite of this, the resulting numbers are so low that they strongly suggest that the gap due to this measure-mismatch/convexity-adjustment is much smaller than the average $50bp$ observed during the crisis. These results are confirmed by the analysis of Section 4.2, where we show that even neglecting the above no-arbitrage drift we replicate very well market quotes, so that the no-arbitrage drift appears really negligible. This leads to the relevant conclusion that credit risk alone does not explain the market patterns, and that there must be some relevant point in market reality which is different from the representation given in Part 2.

This issue is tackled in Part 1. Indeed in Section 6 we abandon (25), assuming instead, consistently with market reality, that in terms of credit quality the future Libor counterparties will be on average better than current Libor counterparties in the future. Within the focus of Part 1, this point is crucial since it allows to explain market patterns. In this Part 2, instead, the focus is on proposing a formal framework to model the interest rate market in the presence of counterparty risk. This framework should be joined with the results of Part 1. In Section 6 we show how, in a less formal context, one can take into account the properties of Libor by representing this rate as an option.

10 Conclusions

In this work we studied the changes that happened to the interest rate market after the burst of the credit crisis. We focused on two issues: the gap that opened between forward rate agreement (FRA) quotes and their Libor based replication, and the appearance of large Basis spreads required to set fair swaps exchanging floating payments with different tenors.

In Part 1 we have shown that the two problems can be reduced to only one, since by using explicitly Basis spreads one can still replicate FRA exactly. We analyze different financial issues that may explain the patterns that FRA's and Basis swaps had in the crisis. Based on this we build a model, where Libor is an option on the credit spread of the counterparty, that approximately fits the empirical evidence. We also investigate how the market payoff of FRA's can be written as a simpler payoff priced under a particular measure.

In Part 2 we have analyzed how an interest rate market with default risk can be modelled. We use subfiltrations to model Libor as a market rate tied to risky contracts. We show a change of measure for a market with collateralized counterparty risk that turns out similar to cross-currency change of measure where survival probability replaces the rate of exchange. We analyze the convexity adjustments that can enter the pricing of collateralized derivatives tied to risky rates.

The results shown here give a number of indications on how to model an interest rates market with counterparty risk , supported by analytic results, empirical testing and financial considerations. Flows of floating rates with different frequency should embed different levels of default risk and different level of default risk volatility, to represent the fact that the credit quality of the Libor counterparty is periodically updated. Otherwise credit risk alone does not explain the market patterns. Products like FRAs are affected by this and must be priced taking this into account. Convexity adjustments can arise from the decoupling of the interest rate curves, but they may be negligible. We also propose a framework based on change of measure and subfiltrations to price collateralized derivatives depending on non-defaultable Libor rates tied to risky products, and make some financial considerations for the introduction in the model of other factors beside counterparty risk, in particular liquidity risk.

Many of the issues dealt with here are open to further research in the direction of making interest rate modelling more consistent with the real situation of the market after the credit crunch. In particular the simple structural model we develop in Part 1 to explain Basis spreads could be elaborated and extended to include explicitly liquidity risk, and to become a general model for all tenors down to the overnight, almost riskless, tenor. As for the more formal framework of Part 2, it proposes some solutions to open issues that still require analysis and investigation.

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