Risk-Neutral Measure and HJM

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1 Introduction

The purpose of this short document is to provide non-rigorous intuition for some well known results in the theory of pricing interest rate derivatives.

2 Bond Price Process From $\mathbb{P}$ to $\mathbb{Q}$

We suppose there exists a money market account $\beta_t \equiv e^{\int_0^t r_s ds}$, where r is a real-valued process called the short rate. We assume that r never goes to negative infinity, so β is always strictly positive. We have:

$$\frac{d\beta_t}{\beta_t} = r_t dt, \quad t \in [0, \bar{T}]. \tag{1}$$

We say the money market account β is a positive process of bounded variation. The process starts at one and grows randomly over time. If the short rate r is positive at some time t, then the money market account increases at that time by a random amount. Although values of the money market account after time 0 have positive variance, we say the instantaneous volatility of the money market account is always zero. The uncertainty in the value of the money market account at some time t arises through the cumulative effect of the random drift $r_s \beta_s ds$ at all prior times $s \in [0, t)$. While all increments in β are random, all squared increments of β vanish. Hence, at any time $t \geq 0$, a one dollar investment in the money market account will grow at the short rate r_t with certainty.

Let $\{P_t(T); t \in [0, T]\}$ be the time t price of a pure discount (i.e. zero coupon) bond paying one dollar at time $T \in [0, \bar{T})$. We assume that the bond price process is continuous over time (no jumps in price). Let $m_t(T) \equiv E_t^{\mathbb{P}} dP_t(T)$ be the stochastic process giving the expected change in the bond price at time $t \in [0, T]$. Let $s_t(T)$ be the stochastic process giving the "square root of the variance rate" of changes in the bond price at time $t \in [0, T]$, i.e. for each $T \in [0, \bar{T}]$, the bond price solves the following SDE:

$$dP_t(T) = m_t(T)dt + s_t(T)dW_t, \quad t \in [0, T]. \tag{2}$$

Here, W is a standard Brownian motion under the (real world or statistical) probability measure $\mathbb{P}$. Importantly, (2) holds for all $T \in [0, \bar{T}]$ and W does not depend on T. We say we have a one factor model for the term structure of bond prices. This is a very strong assumption (not supported empirically).

You can think of the stochastic processes $m_t(T)$ and $s_t(T)$ as standing for the mean and standard deviation of changes in the price of the T- maturity bond at time t. These two stochastic processes can depend on the bond price level $P_t(T)$ and on any other quantity measurable at time t. If we divide $m_t(T)$ and $s_t(T)$ by $P_t(T)$, then we get the stochastic processes describing expected return and volatility respectively. The expressions will be simpler if we work with $m_t(T)$ and $s_t(T)$, (i.e. the mean and standard deviation of price changes) rather than the expected return and volatility (i.e. the mean and standard deviation of returns). Although the following derivation assumes that bond prices cannot jump, it allows jumps in the stochastic processes $m_t(T)$ and $s_t(T)$.

For some fixed maturity T_1, consider holding $\frac{1}{s_t(T_1)}$ units of the bond over the time interval $[t, t+dt]$. The cost of this investment at time t is $\frac{P_t(T_1)}{s_t(T_1)}$ dollars. Suppose we fully finance the cost of this investment by shorting $\frac{P_t(T_1)}{s_t(T_1)}$ dollars worth of the money market account at time t. Hence, we ourselves put up nothing. We say the portfolio has zero cost. At time $t+dt$, we liquidate this zero cost portfolio of bonds and the money market account. The gain (possibly negative) which is realized at time $t+dt$ is:

$$\frac{1}{s_t(T_1)}dP_t(T_1) - \frac{P_t(T_1)}{s_t(T_1)}\frac{d\beta_t}{\beta_t} = \frac{1}{s_t(T_1)}[dP_t(T_1) - r_t P_t(T_1)], \tag{3}$$

from (1). We observe that the gains on a dynamic position in an asset differs from the total derivative of the product of number of shares with price per share.

Substituting (2) in (3) implies that the gain on the zero cost portfolio has the form:

$$\frac{m_t(T_1) - r_t P_t(T_1)}{s_t(T_1)}dt + dW_t.$$

The coefficient of dt is the fraction $\frac{m_t(T_1)-r_tP_t(T_1)}{s_t(T_1)}$. If we divide the numerator and the denominator by the bond price $P_t(T)$, then we get the Sharpe ratio, i.e. the ratio of expected excess return to volatility. This ratio is the reward per unit time that investors expect for bearing one unit of exposure to the standard Brownian motion. Assuming $s_t(T)$ is positive, then bond prices and the standard Brownian motion move in the same direction. If this common move is positively correlated with the market, then the standard thinking is that this reward must be positive to compensate investors for the risk borne. However, the math allows the reward to have either sign. If bond prices and the SBM move opposite to the market, then the the standard thinking is that this reward would be negative. In this case, the bond acts as insurance against market declines, so the heavy demand for bonds pushes their prices up and their expected returns down below the benchmark short rate.

The main takeaway of this result ia that the reward for bearing unit exposure to the zero mean increments of a standard Brownian motion need not vanish. Starting with Bachelier, many have argued that the price of every zero mean payoff should vanish. The evidence for this assertion is highly non-supportive.

If we go through the same thought process using a bond with maturity date $T_2 \neq T_1$, then we get that the gain on this zero cost portfolio has the form:

$$\frac{m_t(T_2) - r_t P_t(T_2)}{s_t(T_2)}dt + dW_t.$$

Notice that both zero cost portfolios offer a reward for unit exposure to increments in the same SBM. If the rewards differ, then an investor could arbitrage by being long the portfolio with the higher reward and short the other. No arbitrage implies that the rewards cannot differ i.e..

$$\frac{m_t(T_1) - r_t P_t(T_1)}{s_t(T_1)} = \frac{m_t(T_2) - r_t P_t(T_2)}{s_t(T_2)}$$

Since T_1 and T_2 are arbitrarily chosen maturity dates, no arbitrage implies that there exists a stochastic process λ_t independent of T such that for all T:

$$\frac{m_t(T) - r_t P_t(T)}{s_t(T)} = \lambda_t. \tag{4}$$

We call λ_t the market price of standard Brownian risk.

The existence of the λ process, independent of T implies that there exists an associated positive local martingale M, also independent of T, defined by:

$$M_t = e^{-\int_0^t \lambda_s dW_s - \frac{1}{2}\int_0^t \lambda_s^2 ds}.$$

The local martingale M starts at one and fluctuates though positive values. We have that:

$$\frac{dM_t}{M_t} = -\lambda_t dW_t. \tag{5}$$

Fixing t at the maximum maturity date $\bar{T}$, the random variable $M_{\bar{T}}$ has mean one under $\mathbb{P}$ and is positive. As such, it can be used to change probability measure. Let $\mathbb{Q}$ be the probability measure defined via the Radon Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = M_{\bar{T}}.$$

By Girsanov's theorem, there exists a standard Brownian motion Z under $\mathbb{Q}$, which is independent of T and such that the bond price $P_t(T)$ solves the following SDE:

$$dP_t(T) = m_t(T)dt + \frac{dM_t}{M_t} dP_t(T) + s_t(T)dZ_t, \quad t \in [0, T]. \tag{6}$$

A reference for this form of Girsanov's Theorem is Revuz and Yor or Protter. Multiplying the right hand sides of (2) and (5), we see that the drift correction simplifies to:

$$\frac{dM_t}{M_t} dP_t(T) = -\lambda_t s_t(T)dt. \tag{7}$$

Substituting (4) in (7) implies that the drift correction further simplifies to:

$$\frac{dM_t}{M_t} dP_t(T) = -[m_t - r_t P_t(T)]dt. \tag{8}$$

Finally, substituting (8) in (9) implies that the bond price $P_t(T)$ solves the following SDE:

$$dP_t(T) = r_t P_t(T)dt + s_t(T)dZ_t, \quad t \in [0, T]. \tag{9}$$

3 From Bond Prices to Forward Interest Rates under $\mathbb{Q}$

In this section, we determine the implications of no arbitrage for the dynamics of forward interest rates under the recently introduced pricing measure $\mathbb{Q}$. It will now be convenient to switch from using normal vol's to lognormal vol's, so let $\Omega_t(T) \equiv \frac{s_t(T)}{P_t(T)}$ be the bond's return volatility. Then no arbitrage implies the existence of the pricing measure $\mathbb{Q}$ such that:

$$\frac{dP_t(T)}{P_t(T)} = r_t dt + \Omega_t(T) dZ_t, \tag{10}$$

where recall Z is a standard Brownian motion under the pricing measure $\mathbb{Q}$. Under this measure, all assets have an expected return at time t equal to the spot rate r_t. Since the bonds have no coupons, their prices all drift up at the same rate r. In contrast, bonds of different maturities T differ in terms of their instantaneous volatilities $\Omega_t(T)$. In general, these volatilities can themselves be stochastic processes, but they cannot be arbitrarily specified. We need bond prices to hit one at maturity from below and we furthermore want bond prices to decrease in T at all future times. This latter condition is equivalent to instantaneous forward rates always being non-negative.

HJM decided to work with these instantaneous forward rates, rather than bond prices. One can imagine a deterministic economy with constant forward rates over time, but the only economy with constant bond prices has zero interest rates always. The HJM framework will lead to bond prices hitting one at maturity, but the framework will in general allow forward rates to go negative, so it does not necessarily capture the other desired properties. Let $f_t(T)$ denote the forward rate quoted at time $t \geq 0$ for a loan which starts at a later time $T \geq t$ and ends a moment later. This instantaneous forward rate can be extracted at time t from a given differentiable discount curve $P_t(T)$ via the relation:

$$f_t(T) = -\frac{\partial}{\partial T} \ln P_t(T). \tag{11}$$

In words, up to sign, the instantaneous forward rate is just the logarithmic derivative of a bond price with respect to its term. Suppose that we wish to specify the dynamics of the forward rate curve $\{f(T) : T \in [t, \bar{T}]\}$ in a manner that is consistent with the progress achieved in (10). We show how to do this using a relatively simple mathematical argument first given in Flesaker[1] and Hughston[2]. Using Itô's lemma on (10) implies that the log price dynamics are given by:

$$d \ln P_t(T) = \left(r_t - \frac{\Omega_t^2(T)}{2} \right) dt + \Omega_t(T) dZ_t. \tag{12}$$

The logarithmic return $d \ln P_t(T)$ has a lower mean than the arithmetic return $\frac{dP_t(T)}{P_t(T)}$ due to the concavity of the log function. The larger the bond variance $\Omega_t^2(T)$, the greater is this difference. If we now differentiate (12) with respect to T and negate, (11) implies that:

$$df_t(T) = \Omega_t(T)\Omega_t'(T)dt - \Omega_t'(T)dZ_t. \tag{13}$$

[1] I am grateful to my former Bloomberg colleague Bjorn Flesaker for pointing out this reference.

Let:

$$\sigma_t(T) \equiv -\Omega_t'(T) \tag{14}$$

denote the instantaneous (normal) volatility of the forward rate $f_t(T)$. Integrating (14) in T implies:

$$\Omega_t(T) = -\int_t^T \sigma_t(u)du. \tag{15}$$

Substituting (14) and (15) in (13) yields the famous HJM equation:

$$df_t(T) = \sigma_t(T)\int_t^T \sigma_t(u)dudt + \sigma_t(T)dZ_t. \tag{16}$$

With the right choice of the volatility processes $\sigma_t(T)$, forward rates can be kept positive. Surprisingly, choosing each $\sigma_t(T)$ to be proportional to $f_t(T)$ failed for a technical reason, but HJM demonstrated reasonable specifications of $\sigma_t(T)$ which did ensure positivity of forward interest rates.

HJM actually presented a generalization of (16) which involved multiple Brownian motions. So long as the number of factors was finite, this generalization was trivial from a mathematical perspective. However, from a financial perspective, the ability to accomodate more than one factor was important, because empirical research has shown that three factors drive the yield curve. As a result, one cannot hedge a thirty year bond using overnight lending in practice. However, one could use the multi-factor HJM framework to find a dynamic hedge using multiple securities.

References

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[2] Flesaker B., and L. Hughston, 1996, "Dynamic Models for Yield Curve Evolution", published in Mathematics of Derivative Securities edited by M. Dempster and S. Pliska, Cambridge University Press.