Option Valuation

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Abstract

The Black-Sholes/Merton model has been Nobel prize winning successful, but parameterizing models that fit market data has been a remarkably recalcitrant problem. This note explores using cumulants of the log of the the risk-neutral distribution of the underlying to achieve this.

Instead of working with the difference of the cumulants from a log-normal distribution as in Jarrow-Rudd [?] we work with the difference of the cumulants of a normal distribution. This is more natural since the cumulants of a normal distribution are easier to interpret.

Just as with moments, not every cumulant sequence corresponds to a random variable. In the case of infinitely divisible random variables having finite variance, there is a condition similar to the Hamburger moment problem that ensures this.

1 Outline

In section 2 we recall some basic fact about cumulants. Section 3 uses an Edgeworth expansion to give an explicit formula for the cumulative distribution function of the perturbed distribution. Option valuation is considered in section 4. We show how the Esscher transform can be calculated using the same method as in section 3. The K-model is described in section 5 as a parameterization that seems to fit market data well. The last section collects some general remarks.

2 Cumulants

The *cumulants*, (κ_n), of a random variable X are defined by

$$\kappa(u) = \log Ee^{uX} = \sum_{n=0}^{\infty} \kappa_n \frac{u^n}{n!}$$

Taking $u = 0$ yields $\kappa_0 = 0$. Since $(d/du)^n \kappa(u)|_{u=0} = \kappa_n$ it is easy to work out that $\kappa_1 = EX$ and $\kappa_2 = \mathrm{Var}(X)$. Higher order cumulants are less intuitive.

Cumulants have some handy properties. The cumulants of a random variable plus a constant are the same except the first cumulant is increased by the constant. More generally, the cumulants of the sum of two independent random variables are the sums of their cumulants. They scale homogeneously, the n-th cumulant of a constant times a random variable is $\kappa_n(cX) = c^n \kappa_n(X)$.

Another nice property of cumulants is that they are more likely to exists than moments, $m_n = EX^n$. (Why?) The relationship between cumulants and moments involves Bell polynomials[**?**].

$$Ee^{uX} = \sum_{n=0}^{\infty} m_n \frac{u^n}{n!} = \exp(\sum_{n=1}^{\infty} \kappa_n \frac{u^n}{n!}) = \sum_{n=0}^{\infty} B_n(\kappa_1, \dots, \kappa_n) \frac{u^n}{n!}$$

where $B_n(\kappa_1, \dots, \kappa_n)$ is the n-th complete Bell polynomial. This is just a special case of the Faà di Bruno formula first proved by Louis François Antoine Arborgast in 1800[**?**]. Bell polynomials satisfy the recurrence [**?**] $B_0 = 1$ and

$$B_{n+1}(x_1, \dots, x_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x_1, \dots, x_{n-k}) x_{k+1}.$$

2.1 Examples

(Normal). If X is normal then $Ee^X = e^{EX + \mathrm{Var}(X)/2}$ so $Ee^{uX} = e^{uEX + u^2 \mathrm{Var}(X)/2}$ showing the third and higher order cumulants vanish. If the cumulants of a random variable vanish after some some point, then it must be normal[**?**]. Something to keep in mind with computer implementations.

(Poisson). If X is Poisson with mean μ then $\kappa_n = \mu$ for all n.

(Compound Poisson). If Y is Poisson with mean μ and Z_j are independent and identically distributed, define $X = \sum_{j=1}^{Y} Z_j$. The cumulants of X are $\kappa_n =?$.

$$
\begin{aligned}
Ee^{uX} &= \sum_{k=0}^{\infty} Ee^{u(Z_1+\cdots Z_k)}\mu^k/k!\, e^{-\mu} \\
&= \sum_{k=0}^{\infty} (Ee^{uZ_1})^k \mu^k/k!\, e^{-\mu} \\
&= \sum_{k=0}^{\infty} (Ee^{uZ_1}\mu)^k/k!\, e^{-\mu} \\
&= e^{\mu Ee^{uZ_1}}\, e^{-\mu} \\
&= e^{\mu(Ee^{uZ_1}-1)}
\end{aligned}
$$

Define $\lambda(u) = \log Ee^{uZ_1}$. Then $\log Ee^{uX} = \mu(e^{\lambda(u)} - 1)$.

(Exponential). If X is exponential with mean μ then $\kappa_n = (n-1)!\mu^n$.

(Gamma). If X is gamma with mean α/β and variance α/β^2 then $\kappa_n = (n-1)!\alpha/\beta^n$. Note gamma is exponential when $\alpha = 1$ and $\beta = 1/\mu$.

$$
\begin{aligned}
Ee^{uX} &= \int_0^\infty e^{ux}x^{\alpha-1}e^{-\beta x}\frac{\beta^\alpha}{\Gamma(\alpha)}\,dx \\
&= \int_0^\infty x^{\alpha-1}e^{-(\beta-u)x}\frac{\beta^\alpha}{\Gamma(\alpha)}\,dx \\
&= \frac{\Gamma(\alpha)}{(\beta-u)^\alpha}\frac{\beta^\alpha}{\Gamma(\alpha)} \\
&= (1-u/\beta)^{-\alpha}
\end{aligned}
$$

so $\log Ee^{uX} = \alpha\sum_{n=1}^\infty (u/\beta)^n/n$.

3 Edgeworth Expansion

Given random variables X and Y, we have

$$
\log Ee^{iuY} - \log Ee^{iuX} = \sum_{n=1}^\infty \Delta\kappa_n (iu)^n/n!
$$

where $(\Delta\kappa_n)$ are the differences of the cumulants of X and Y, so

$$
Ee^{iuY} = Ee^{iuX}\sum_{n=0}^\infty B_n(\Delta\kappa_1, ..., \Delta\kappa_n)(iu)^n/n!.
$$

Let F and G be the cumulative distribution functions of X and Y respectively. Since the Fourier transform of F' is $-iu\hat{F}(u)$, the Fourier transform of the n-th derivative $F^{(n)}$ is $(-iu)^n\hat{F}(u)$. Taking the inverse Fourier transform shows

$$G(x) = \sum_{n=0}^{\infty}(-1)^n \frac{B_n}{n!} F^{(n)}(x)$$

3.1 Hermite Polynomials

The derivatives the standard normal cumulative distribution can be computed using Hermite polynomials[**?**]. The standard result[**?**] is

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

and they satisfy the recurrence $H_0(x) = 1$, $H_1(x) = x$ and

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x).$$

Note some authors use $He_n(x)$ instead of $H_n(x)$.

4 Option Valuation

The forward value of a put on a stock S with strike k is

$$E\max\{k - S, 0\} = kP(S \leq k) - ES\,P^*(S \leq k)$$

where $dP^*/dP = S/ES$ is the *Esscher transform*[**?**]. This is the analog of $E\max\{k - S, 0\} = kN(-d_2) - sN(-d_1)$ when S is lognormal.

4.1 Black Model

The Black model is $S = S_t = se^{-\sigma^2 t/2 + \sigma B_t}$ where (B_t) is standard Brownian motion and t is time in years. Note $ES = s$ and $S \leq k$ if and only if $B_t/\sqrt{t} \leq z$ where $z = (\log k/s + \sigma^2 t/2)/\sigma\sqrt{t}$. In the usual notation $z = -d_2$.

Letting $X = B_t/\sqrt{t}$ and so $F = N$ is the standard normal cumulative distributing function we have $P(X \leq z) = F(z)$ and $P^*(X \leq z) = P(X + \sigma\sqrt{t} \leq z) = F(z - \sigma\sqrt{t})$ where we use the fact

$$Ee^U f(V) = Ee^U Ef(V + \mathrm{Cov}(U, V))$$

if U and V are jointly normal. In the standard notation $z - \sigma\sqrt{t} = -d_1$.

4.2 Generalized Black Model

The generalized Black model is $S = se^{-\kappa(t,\sigma)+\sigma X_t}$ where (X_t) is any stochastic process and $\kappa(t,\sigma) = \log E\exp(\sigma X_t)$. Note $ES = s$ and $S \leq k$ if and only if $X_t/\sqrt{t} \leq (\log k/s + \kappa(t,\sigma))/\sigma\sqrt{t}$.

Letting $Y = X_t/\sqrt{t}$ and G be the corresponding cumulative distribution function, $G(z) = P(Y \leq z)$, we need to compute

$$G^*(z) = P^*(Y \leq z) = Ee^{-\kappa(t,\sigma)+\sigma\sqrt{t}Y} 1(Y \leq z).$$

Letting $\gamma = \sigma\sqrt{t}$, the cumulants of G^* can be found from

$$\begin{aligned}
\log Ee^{uY^*} &= \log Ee^{-\kappa(t,\gamma)+\gamma Y} e^{uY} \\
&= -\kappa(t,\gamma) + \log Ee^{(\gamma+u)Y} \\
&= -\sum_{n=1}^{\infty} \kappa_n \frac{\gamma^n}{n!} + \kappa_n \frac{(u+\gamma)^n}{n!} \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \kappa_n \binom{n}{k} \frac{\gamma^{n-k} u^k}{n!} \\
&= \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \kappa_n \binom{n}{k} \frac{\gamma^{n-k} u^k}{n!} \\
&= \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \kappa_n \frac{\gamma^{n-k}}{(n-k)!} \frac{u^k}{k!} \\
&= \sum_{k=1}^{\infty} (\sum_{n=1}^{\infty} \kappa_{n+k} \frac{\gamma^n}{n!}) \frac{u^k}{k!}
\end{aligned}$$

This shows the cumulants of Y^* are $\kappa^*_k = \sum_{n=1}^{\infty} \kappa_{n+k}\gamma^n/n!$ so we can also compute $P(Y^* \leq z) = G^*(z)$ using the same technique as we did for G.

If G is standard normal then all cumulants vanish except $\kappa_2 = 1$. The only nonzero cumulants of G^* are $\kappa_1 = \gamma$ and $\kappa_2 = 1$. The corresponding reduced Bell polynomials are $B_n = \gamma^n/n!$ and we see the above formula is just the Taylor series expansion of $G(z - \gamma)$.

4.3 Computations

Explicit formulas the first few Bell polynomials:

$$
\begin{aligned}
B_0 &= 1 \\
B_1 &= x_1 \\
B_2 &= B_1 x_1 + B_0 x_2 \\
&= x_1^2 + x_2 \\
B_3 &= B_2 x_1 + 2B_1 x_2 + B_0 x_3 \\
&= x_1^3 + x_1 x_2 + 2x_1 x_2 + x_3 \\
&= x_1^3 + 3x_1 x_2 + x_3 \\
B_4 &= B_3 x_1 + 3B_2 x_2 + 3B_1 x_3 + B_0 x_4 \\
&= (x_1^3 + 3x_1 x_2 + x_3)x_1 + 3(x_1^2 + x_2)x_2 + 3x_1 x_3 + x_4 \\
&= x_1^4 + 6x_1^2 x_2 + 4x_1 x_3 + 3x_2^2 + x_4
\end{aligned}
$$

Explicit formulas for the first few Hermite polynomials:

$$
\begin{aligned}
H_0 &= 1 \\
H_1 &= x \\
H_2 &= x^2 - 1 \\
H_3 &= x^3 - x \\
H_4 &= x^4 - 6x^2 + 3 \\
H_5 &= x^5 - 10x^3 + 15x \\
H_6 &= x^6 - 15x^4 + 45x^2 - 15
\end{aligned}
$$

Assuming $\kappa_1 = \kappa_2 = 0$ the first few terms of the Edgeworth expansion are:

$$
\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} (-1)^n B_n F^{(n)}(x)/n! \\
&= F(x) - B_1 F'(x) + B_2 F''(x)/2 - B_3 F^{(3)}(x)/6 + B_4 F^{(4)}/24 \\
&= F(x) + (-\kappa_3(x^2-1)/6 - \kappa_4(x^3-x)/24)e^{-x^2/2}/\sqrt{2\pi}
\end{aligned}
$$

The distribution is unimodal if and only if the second derivative has exactly one root.

$$\begin{aligned} G''(x) &= \sum_{n=0}^{\infty}(-1)^n B_n F^{(n+1)}(x)/n! \\ &= F^{(2)}(x) - B_1 F^{(3)}(x) + B_2 F^{(4)}(x)/2 - B_3 F^{(5)}(x)/6 + B_4 F^{(6)}/24 \\ &= (-x - \kappa_3(x^4 - 6x^2 + 3)/6 - \kappa_4(x^5 - 10x^3 + 15x)/24)e^{-x^2/2}/\sqrt{2\pi} \end{aligned}$$

4.4 Lévy Processes

Let $X = aN + bP + c$ where N is standard normal and P is Poisson with mean μ.

$EX = b\mu + c$ and $\operatorname{Var} X = a^2 + b^2$. Taking $b = \sqrt{1 - a^2}$

Kolmogorov's version of the Lévy-Khintchine theorem[**?**] states that if a random variable X is infinitely divisible there exists a number γ and a non-decreasing function G defined on the real line such that

$$\kappa(u) = \log E e^{uX} = u\gamma + \int_{-\infty}^{\infty} K_u(x)\, dG(x),$$

where $K_u(x) = (e^{ux} - 1 - ux)/x^2$. Note the first cumulant of X is γ and for $n \geq 2$, $\kappa_n = \int_{-\infty}^{\infty} x^{n-2}\, dG(x)$. In particular the variance of X is $\int_{-\infty}^{\infty} dG(x) = G(\infty) - G(-\infty)$.

If (X_t) is a Lévy process then X_1 is infinitely divisible and $\log E e^{uX_t} = t\kappa(u)$.

5 Remarks

The Gram-Charlier A series expands the quotients of cumulative distribution functions G/F using Hermite polynomials, but does not have asymptotic convergence, whereas the Edgeworth expansion involves the quotient of characteristic functions $\hat{G}/\hat{F}$ in terms of cumulants and does have asymptotic convergence, ignoring some dainty facts [**?**].