Normal Jarrow-Rudd

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Abstract

The Black-Scholes/Merton model has been Nobel prize winning successful, but parameterizing models that fit volatility smiles at even a single maturity for European options has been a remarkably intractable problem.

Instead of working with the difference of the cumulants from a log-normal distribution as in Jarrow-Rudd [6] we work with the difference of the cumulants from a normal distribution. This is more natural since all but the first two cumulants of a normal distribution are zero. For other distributions third and fourth cumulant are closely related to the skew and kurtosis.

The mathematics involved is fairly elementary and a software implementation is available for you to try out your ideas on how this can be used.

1 Outline

The forward value of an European put option having strike k and forward F at expiration is

$$E\max\{k - F, 0\} = E(k - F)1(F \leq k) = kP(F \leq k) - fP^*(F \leq k)$$

where $f = EF$ and $dP^*/dP = F/f$ is the *Esscher transform*[5]. We derive formulas for $P(F \leq k)$ and $P^*(F \leq k)$ that can be computed efficiently when $\log F$ is approximately normal.

This generalizes the Black formula $E\max\{k-F, 0\} = k\Phi(-d_2)-f\Phi(-d_1)$ where $F = fe^{-\sigma^2 t/2+\sigma B_t}$, (B_t) is standard Brownian motion, Φ is the standard normal cumulative distribution function, $d_1 = (\sigma^2 t/2 + \log f/k)/\sigma\sqrt{t}$ and $d_2 = d_1 - \sigma\sqrt{t}$.

In section 2 we recall some basic facts about cumulants. Section 3 uses an Edgeworth expansion and Bell polynomials to show their relationship to moments. The next section describes how Hermite polynomials give an explicit formula for the cumulative distribution function of the perturbed distribution. Option valuation is considered in section 4 where we show how the Esscher transform can be calculated using the method in section 3. The last section collects some general remarks.

2 Cumulants

The *cumulants*, (κ_n), of a random variable X are defined by

$$\kappa(s) = \log Ee^{sX} = \sum_{n=1}^{\infty} \kappa_n \frac{s^n}{n!}$$

Since $(d/ds)^n \kappa(s)|_{s=0} = \kappa_n$ it is easy to work out that $\kappa_1 = EX$ and $\kappa_2 = \mathrm{Var}(X)$. Higher order cumulants are less intuitive but the third and fourth are related to skew and kurtosis.

Cumulants have some handy properties. The cumulants of a random variable plus a constant are the same except the first cumulant is increased by the constant. More generally, the cumulants of the sum of two independent random variables are the sums of their cumulants. They scale homogeneously, the n-th cumulant of a constant times a random variable is $\kappa_n(cX) = c^n \kappa_n(X)$.

The relationship between cumulants and moments, $m_n = EX^n$, involves Bell polynomials[3].

$$Ee^{sX} = \sum_{n=0}^{\infty} m_n \frac{s^n}{n!} = \exp\Bigl(\sum_{n=1}^{\infty} \kappa_n \frac{s^n}{n!}\Bigr) = \sum_{n=0}^{\infty} B_n(\kappa_1, \dots, \kappa_n) \frac{s^n}{n!}$$

where $B_n(\kappa_1, \dots, \kappa_n)$ is the n-th complete Bell polynomial. This is just a special case of the Faà di Bruno formula first proved by Louis François Antoine Arborgast in 1800[2]. Bell polynomials satisfy the recurrence [4] $B_0 = 1$ and

$$B_{n+1}(x_1, \dots, x_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x_1, \dots, x_{n-k}) x_{k+1}.$$

3 Edgeworth Expansion

Given random variables X and Y, we have

$$\log Ee^{iuY} - \log Ee^{iuX} = \sum_{n=1}^{\infty} \Delta\kappa_n (iu)^n / n!$$

where $(\Delta\kappa_n)$ are the differences of the cumulants of X and Y, so

$$Ee^{iuY} = Ee^{iuX}\sum_{n=0}^{\infty} B_n(\Delta\kappa_1, ..., \Delta\kappa_n)(iu)^n/n!.$$

Let F and G be the cumulative distribution functions of X and Y respectively. Since the Fourier transform, $\hat{F}(u) = Ee^{iuX}$, of F' is $-iu\hat{F}(u)$, the Fourier transform of the n-th derivative $F^{(n)}$ is $(-iu)^n\hat{F}(u)$. Taking the inverse Fourier transform shows

$$G(x) = \sum_{n=0}^{\infty}(-1)^n\frac{B_n}{n!}F^{(n)}(x)$$

3.1 Hermite Polynomials

The derivatives the standard normal cumulative distribution can be computed using Hermite polynomials[1] pp. 793–801. Recall the Hermite polynomials are the result of the Gram-Schmidt process applied to the basis $(x^n)_{n\geq 0}$ on the Hilbert space having inner product $(f,g) = \int_{-\infty}^{\infty} f(x)g(x)\, e^{-x^2/2}dx$. The standard results are

$$H_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}$$

and they satisfy the recurrence $H_0(x) = 1$, $H_1(x) = x$ and

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x).$$

Note some authors use $He_n(x)$ instead of $H_n(x)$.

Putting this all together we get

$$G(x) = F(x) - \sum_{n=1}^{\infty} B_n(\Delta\kappa_1, \dots, \Delta\kappa_n)e^{-x^2/2}H_{n-1}(x)/\sqrt{2\pi}$$

4 The Esscher Transform

If X is standard normal then $P^*(X \leq z) = Ee^{-s^2/2+sX}1(X \leq z)) = P(X + s \leq z) = \Phi(z - s)$ where we use the fact

$$Ee^M f(N) = Ee^M Ef(N + \mathrm{Cov}(M, N))$$

if M and N are jointly normal. Here $M = -s^2/2 + sX$, $N = X$, and $f(x) = 1(x \leq z)$.

If Y is a pertubation of X then for $G(z) = P(Y \leq z)$, we need to compute

$$G^*(z) = P^*(Y \leq z) = Ee^{-\kappa(s)+sY}1(Y \leq z).$$

The cumulants of Y^* can be found from

$$\begin{aligned}
\log Ee^{uY^*} &= \log Ee^{-\kappa(s)+sY}e^{uY} \\
&= -\kappa(s) + \kappa(s+u) \\
&= \sum_{n=1}^{\infty} -\kappa_n \frac{s^n}{n!} + \kappa_n \frac{(u+s)^n}{n!} \\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} \kappa_n \binom{n}{k} \frac{s^{n-k}u^k}{n!} \\
&= \sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty} \kappa_n \binom{n}{k} \frac{s^{n-k}u^k}{n!} \\
&= \sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty} \kappa_n \frac{s^{n-k}}{(n-k)!}\frac{u^k}{k!} \\
&= \sum_{k=1}^{\infty}(\sum_{n=1}^{\infty} \kappa_{n+k} \frac{s^n}{n!}) \frac{u^k}{k!}
\end{aligned}$$

This shows the cumulants of Y^* are $\kappa_n^* = \sum_{k=1}^{\infty} \kappa_{n+k} s^k/k!$ so we can also compute $P(Y^* \leq z) = G^*(z)$ using the same technique as for G.

If G is standard normal then all cumulants vanish except $\kappa_2 = 1$. The only nonzero cumulants of G^* are $\kappa_1 = s$ and $\kappa_2 = 1$. The corresponding complete Bell polynomials are $B_n = s^n/n!$ and we see the above formula is just the Taylor series expansion of $G(z - s)$.

4.1 Examples

Recall $\kappa_n^* = \sum_{k=0}^{\infty} \kappa_{n+k} s^k/k!$.

(Normal)**.** If X is normal with mean μ and variance σ^2 then $\kappa(s) = \mu s + \sigma^2 s^2/$ so $\kappa_1 = \mu$ and $\kappa_2 = \sigma^2$ are the only non-zero cumulants. We also have $\kappa_1^* = \mu + \sigma^2 s$. These formulas follow from $Ee^X = e^{\mu+\sigma^2/2}$.

If the cumulants of a random variable vanish after some some point, then it must be normal[8] (Theorem 7.3.5). Something to keep in mind with computer implementations.

(Poisson). If X is Poisson with mean μ then $\kappa(s) = \mu(e^s - 1)$ so $\kappa_n = \mu$ for all n and $\kappa_n^* = \mu e^s$.

(Exponential). If X is exponential with mean μ then $\kappa(s) = -\log(1 - \mu s)$ and $\kappa_n = (n-1)!\mu^n$ so $\kappa_n^* = (n-1)!\mu^n/(1+\mu s)^n$. since $\sum_{k=0}^\infty (n+k-1)!x^k/k! = (n-1)!/(1-x)^n$.

(Gamma). If X is gamma with mean α/β and variance α/β^2 then $\kappa(s) = -\alpha\log(1 - s/\beta)$ and $\kappa_n = (n-1)!\alpha/\beta^n$ so $\kappa_n^* = (n-1)!\alpha/\beta^n/(1-s/\beta)^n$.

Note gamma is exponential when $\alpha = 1$ and $\beta = 1/\mu$.

4.2 Lévy Processes

Kolmogorov's precursor to the Lévy-Khintchine theorem[7] states that if a random variable X is infinitely divisible and has finite variance there exists a number γ and a non-decreasing function G defined on the real line such that

$$\kappa(s) = \log Ee^{sX} = s\gamma + \int_{-\infty}^{\infty} K_s(x)\,dG(x),$$

where $K_s(x) = (e^{sx} - 1 - sx)/x^2 = \sum_{n=2}^\infty s^n x^{n-2}/n!$. Note the first cumulant of X is γ and for $n \geq 2$, $\kappa_n = \int_{-\infty}^\infty x^{n-2}\,dG(x)$.

Since $K_s(0) = s^2/2$ is the cumulant of the standard normal distribution and $a^2 K_s(a) + as$ is the cumulant of a Poisson distrbution having mean a, infinitely divisible random variables can be approximated by a normal plus a linear combination of independent Poisson distributions.

5 Remarks

The Gram-Charlier A series expands the quotients of cumulative distribution functions G/F using Hermite polynomials, but does not have asymptotic convergence, whereas the Edgeworth expansion involves the quotient of characteristic functions $\hat{G}/\hat{F}$ in terms of cumulants and does have asymptotic convergence, ignoring some dainty facts [9].

If (X_t) is a Lévy process then X_1 is infinitely divisible and $\log Ee^{sX_t} = t\kappa(s)$. A consequence is that the volatility smile at a single maturity determines the entire volatility surface, a fact that may indicate Lévy processes are not appropriate for modeling stock prices.

A software implementation in C++ is available at `https://fmsgjr.codeplex.com` and Excel add-ins at `https://xllgjr.codeplex.com`.

References

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