Generalized Jarrow-Rudd Approximate Option Valuation

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1 Outline

Instead of working with the difference of the cumulants from a lognormal distribution as in Jarrow-Rudd [**?**] we work with the difference of the cumulants of a normal distribution. This is more natural since the cumulants of a normal distribution are easier to interpret.

In section 2 we recall some basic fact about cumulants. Section 3 uses an Edgeworth expansion to give an explicit formula for the cumulative distribution function of the perturbed distribution. Option valuation is considered in section 4. We show how the Esscher transform can be calculated using the same method as in section 3. The K-model is described in section 5 as a parameterization that seems to fit market data well. The last section collects some general remarks.

2 Cumulants

The *cumulants*, (κ_n), of a random variable X are defined by

$$\log Ee^{uX} = \sum_{n=0}^{\infty} \kappa_n \frac{u^n}{n!}$$

Taking $u = 0$ yields $\kappa_0 = 0$. Since $(d/du)^n \log Ee^{uX}|_{u=0} = \kappa_n$ it is easy to work out that $\kappa_1 = EX$ and $\kappa_2 = \mathrm{Var}(X)$. Higher order cumulants are less intuitive.

Cumulants have some handy properties. The cumulants of a random variable plus a constant are the same except the first cumulant is increased by the constant. More generally, the cumulants of the sum of two independent random variables are the sums of the cumulants. The n-th cumulant is homogeneous of degree n, $\kappa_n(cX) = c^n\kappa_n(X)$.

Another nice property of cumulants is that they are more likely to exists than moments, $m_n = EX^n$. The relationship between cumulants and moments involves Bell polynomials[**?**].

$$Ee^{uX} = \sum_{n=0}^{\infty} m_n \frac{u^n}{n!} = \exp(\sum_{n=1}^{\infty} \kappa_n \frac{u^n}{n!}) = \sum_{n=0}^{\infty} B_n(\kappa_1, \dots, \kappa_n) \frac{u^n}{n!}$$

where $B_n(\kappa_1, \dots, \kappa_n)$ is the n-th complete Bell polynomial. This is just a special case of the Faà di Bruno formula first proved by Louis François Antoine Arborgast in 1800[**?**]. Bell polynomials satisfy the recurrence [**?**] $B_0 = 1$ and

$$B_{n+1}(x_1, \dots, x_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(x_1, \dots, x_{n-k}) x_{k+1}.$$

2.1 Examples

If X is normal then $Ee^X = e^{EX + \operatorname{Var}(X)/2}$ so $Ee^{uX} = e^{uEX + u^2 \operatorname{Var}(X)/2}$ showing the third and higher order cumulants vanish, if X is Poisson with mean μ then $\kappa_n = \mu$ for all n, and if X is exponential with mean μ then $\kappa_n = (n-1)!\mu^n$.

3 Edgeworth Expansion

Given random variables X and Y, we have

$$\log Ee^{iuY} - \log Ee^{iuX} = \sum_{n=1}^{\infty} \Delta\kappa_n (iu)^n / n!$$

where $(\Delta\kappa_n)$ are the differences of the cumulants of X and Y, so

$$Ee^{iuY} = Ee^{iuX} \sum_{n=0}^{\infty} \frac{B_n}{n!} (iu)^n.$$

Let F and G be the cumulative distribution functions of X and Y respectively. The simple fact that the Fourier transform of F' is $-iu\hat{F}(u)$ implies the Fourier transform of the n-th derivative $F^{(n)}$ is $(-iu)^n \hat{F}(u)$. Taking the inverse Fourier transform shows

$$G(x) = \sum_{n=0}^{\infty} (-1)^n \frac{B_n}{n!} F^{(n)}(x)$$

3.1 Hermite Polynomials

The derivatives the standard normal cumulative distribution can be computed using Hermite polynomials[?]. The standard result[?] is

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$

and they satisfy the recurence $H_0(x) = 1$, $H_1(x) = x$ and

$$H_{n+1}(x) = xH_n(x) - nH_{n-1}(x).$$

Note some authors use $He_n(x)$ instead of $H_n(x)$.

4 Option Valuation

The forward value of a put on a stock S with strike k is

$$E\max\{k - S, 0\} = kP(S \leq k) - ES\, P^*(S \leq k)$$

where $dP^*/dP = S/ES$ is the *Esscher transform*. As we will see, this is the analog of $E\max\{k - S, 0\} = kN(-d_2) - sN(-d_1)$ when S is lognormal.

4.1 Black Model

The Black model is $S = S_t = se^{-\sigma^2 t/2 + \sigma B_t}$ where (B_t) is standard Brownian motion and t is time in years. Note $ES = s$ and $S \leq k$ if and only if $B_t/\sqrt{t} \leq z$ where $z = (\log k/s + \sigma^2 t/2)/\sigma\sqrt{t}$. In the usual notation $z = -d_2$.

Letting $X = B_t/\sqrt{t}$ and so $F = N$ is the standard normal cumulative distributing function we have $P(X \leq z) = F(z)$ and $P^*(X \leq z) = P(X + \sigma\sqrt{t} \leq z) = F(z - \sigma\sqrt{t})$ where we use the fact

$$Ee^U f(V) = Ee^U Ef(V + \mathrm{Cov}(U, V))$$

if U and V are jointly normal. In the standard notation $z - \sigma\sqrt{t} = -d_1$.

4.2 Generalized Black Model

The generalized Black model is $S = se^{-\kappa(t,\sigma) + \sigma C_t}$ where (C_t) is any stochastic process and $\kappa(t, \sigma) = \log E\exp(\sigma C_t)$. Note $ES = s$ and $S \leq k$ if and only if $C_t/\sqrt{t} \leq (\log k/s + \kappa(t, \sigma))/\sigma\sqrt{t}$.

Letting $Y = C_t/\sqrt{t}$ and G be the corresponding cumulative distribution function, $G(z) = P(Y \le z)$, we need to compute

$$G^*(z) = P^*(Y \le z) = Ee^{-\kappa(t,\sigma)+\sigma\sqrt{t}Y} 1(Y \le z).$$

Letting $\gamma = \sigma\sqrt{t}$, the cumulants of G^* can be found from

$$\begin{aligned}
\log Ee^{uY^*} &= \log Ee^{-\kappa(t,\gamma)+\gamma Y} e^{uY} \\
&= -\kappa(t,\gamma) + \log Ee^{(\gamma+u)Y} \\
&= -\sum_{n=1}^\infty \kappa_n \frac{\gamma^n}{n!} + \kappa_n \frac{(u+\gamma)^n}{n!} \\
&= \sum_{n=1}^\infty \sum_{k=0}^{n-1} \kappa_n \binom{n}{k} \frac{\gamma^{n-k}u^k}{n!} \\
&= \sum_{k=1}^\infty \sum_{n=k+1}^\infty \kappa_n \binom{n}{k} \frac{\gamma^{n-k}u^k}{n!} \\
&= \sum_{k=1}^\infty \sum_{n=k+1}^\infty \kappa_n \frac{\gamma^{n-k}}{(n-k)!} \frac{u^k}{k!} \\
&= \sum_{k=1}^\infty (\sum_{n=1}^\infty \kappa_{n+k} \frac{\gamma^n}{n!}) \frac{u^k}{k!}
\end{aligned}$$

This shows the cumulants of Y^* are $\kappa^*_k = \sum_{n=1}^\infty \kappa_{n+k}\gamma^n/n!$ so we can also compute $P(Y^* \le z) = G^*(z)$ using the same technique as we did for G.

If G is standard normal then all cumulants vanish except $\kappa_2 = 1$. The only nonzero cumulants of G^* are $\kappa_1 = \gamma$ and $\kappa_2 = 1$. The corresponding reduced Bell polynomials are $B_n = \gamma^n/n!$ and we see the above formula is just the power series expansion of $G(z-\gamma)$.

5 K-model

Kolmogorov's version of the Lévy-Khintchine theorem[**?**] states that if a random variable X is infinitely divisible there exists a number γ and a non-decreasing function G defined on the real line such that

$$\log Ee^{iuX} = iu\gamma + \int_{-\infty}^\infty K_u(x)\, dG(x),$$

where $K_u(x) = (e^{iux} - 1 - iux)/x^2$. Note the first cumulant of X is γ and for $n \geq 2$, $\kappa_n = \int_{-\infty}^{\infty} x^{n-2}\,dG(x)$. In particular the variance of X is $\int_{-\infty}^{\infty} dG(x) = G(\infty) - G(-\infty)$.

The K-model takes $\gamma = 0$ and G of the form

$$G(x) = \begin{cases} ae^{x/\alpha} & x < 0 \\ 1 - be^{-x/\beta} & x > 0 \end{cases}$$

Note G jumps by $1 - a - b$ at the origin. If $a = b = 0$ this reduces to a standard normal distribution.

The cumulants are simple to compute: $\kappa_1 = 0$, $\kappa_2 = 1$, and $\kappa_{n+2} = (a(-\alpha)^n + b\beta^n)n!$ for $n > 1$.

6 Remarks

The Gram-Charlier A series expands the quotients of cumulative distribution functions G/F using Hermite polynomials, but does not have asymptotic convergence, whereas the Edgeworth expansion involves the quotient of characteristic functions $\hat{G}/\hat{F}$ in terms of cumulants and does have asymptotic convergence, ignoring some dainty facts [?].