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Pricing double barrier options using Laplace transforms

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Abstract. In this paper we address the pricing of double barrier options. To derive the density function of the first-hit times of the barriers, we analytically invert the Laplace transform by contour integration. With these barrier densities, we derive pricing formulæ for new types of barrier options: knock-out barrier options which pay a rebate when either one of the barriers is hit. Furthermore we discuss more complicated types of barrier options like double knock-in options.

Key words: Option pricing, Laplace transform, contour integration

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Editor's note:

Different solution methods for pricing double barrier options have already been proposed in the literature (see e.g. Kunitomo-Ikeda 1992, German-Yor 1996 and the following comment by Hui et al.). Nevertheless we decided to publish Pelsser's paper, since his method to use contour integration in the complex plane for Laplace transform inversion is different and may be of independent interest for related problems in finance. In a letter to the Editor Michael Schröder (University of Mannheim, Lehrstuhl Mathematik III) pointed out certain difficulties when applying this method uncritically in other contexts, in particular in choosing the integration path and checking the condition of Cauchy's Residue Theorem.

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1 Introduction

Barrier options have become very popular instruments in derivative markets. It is relatively straightforward to price and hedge ”single barrier” options. Valuation formulas have been available in the literature for quite a while, see Merton (1973) or Goldman et al. (1979). The valuation and hedging formulas have been incorporated in standard market software for options traders and clients. In fact, most derivatives firms view ”single barrier” options nowadays more like vanilla than exotic options.

One of the reasons why barrier options have become so popular, is the fact that they are cheaper than standard options, but offer a similar kind of protection. A natural extension to ”single barrier” options is to consider double barrier options. These are options which have a barrier above and below the price of the underlying, and the option gets knocked in or out as soon as one of the two barriers is hit.

Several papers have already analysed double knock-out call and put options using different methods. Kunitomo and Ikeda (1992) derive the probability density for staying between two (exponentially) curved boundaries. They express the density as an infinite sum of normal density functions, and the prices for double knock-out calls and puts are derived by integrating with respect to this density.

In a recent paper, Geman and Yor (1996) derive expressions for the Laplace transform of the double barrier option price. They invert the Laplace transform numerically to obtain option prices.

These papers deal, however, with only one type of double barrier option: double barrier knock-out calls and puts. In the markets a much wider variety of double barrier options is being traded. Especially, options which pay a fixed amount of money (a ”rebate”) as soon as one of the barriers is hit and double barrier knock-in options.

The contribution of this paper is twofold:

First, we analytically invert the Laplace transforms of the probability density functions using contour integration. Option prices are then calculated by integrating the option payoff with respect to the density functions. To our knowledge this is one of the first applications of contour integration to the area of option pricing theory. Although the same result can be derived by other methods, we believe that contour integration is a very powerful tool with many potential applications in finance.

Second, we derive analytical formulas for pricing a wider variety of double barrier options than has been treated so far in the literature.[1] We find formulas for options which give a constant payoff either ”at hit” or at maturity, we derive pricing formulas for double barrier options where the final payoff can be

[1] In a recent paper, Sidenius (1998) also derives valuation formulæ for knock-out options with rebates and knock-in options. His approach is based on ”path counting” which is basically the method of images. In another recent paper, Schröder (1998) derives the Kunitomo-Ikeda (1992) valuation formulas by inversion of Laplace Transforms derived in Geman-Yor (1996).

expressed as any power of the underlying value and we find valuation formulas for knock-in options.

The paper is organised as follows. In Sect. 2 we derive our analytical expressions for the probability density of the first passage time for the upper and lower barriers. In Sect. 3 we derive some pricing formulas for different kinds of double barrier options. Finally, we conclude in Sect. 4.

2 Barrier densities

If we make the assumption (which is standard) that the underlying asset of the option can be modeled as a geometric Brownian motion, we can model the log of the asset price (under the equivalent martingale measure) by the following stochastic differential equation

$$dz = \mu\, dt + \sigma\, dW, \tag{1}$$

where μ and σ are constants.

The case we want to consider is more complicated. We want to value double barrier options. This can be modeled by assuming that the process z is killed as soon as it hits one of the two barriers. Suppose we have two barriers, the lower barrier is located at 0, the upper barrier at the level l. This specification is general, since we can always shift the process z by a constant such that the lower barrier is placed at 0.

Let us consider the transition density function $p(t,x;s,y)$. It describes the probability density that the process z starts at time t at $z(t) = x$ and survives until time s and ends up at $z(s) = y$. Of course we have, $t \leq s$ and $0 \leq x, y \leq l$.

The transition density function $p(t,x;s,y)$ can be represented in terms of a Fourier series (see, e.g. Cox and Miller 1965):

$$\begin{aligned} p(t,x;s,y) &= e^{\frac{\mu}{\sigma^2}(y-x)} \frac{2}{l} \sum_{k=1}^{\infty} e^{-\lambda_k(s-t)} \sin(k\pi \frac{x}{l}) \sin(k\pi \frac{y}{l}) \\ \lambda_k &= \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{l^2}\right) \; . \end{aligned} \tag{2}$$

The choice for represention (2) has the additional advantage that analytical expressions (on a term-by-term basis) can be found for calculating options prices. Hence, there is no need to work with approximations, as in the case of the cumulative normal distribution function.

We have now characterised the density function of surviving until time s. This densitiy is used for pricing double knock-out options which get nullified as soon as one of the barriers is hit.

We are also interested in the density functions of hitting the upper and the lower barrier. These densities are used for pricing options which have a non-zero payoff as soon as one of the barriers is hit.

Let $g^+(t,x;s)$ denote the probability density function of first hitting the upper barrier at time s before the lower barrier is hit, given that the process started at (t,x). Let $g^-(t,x;s)$ denote the probability density of first hitting the lower barrier, before the upper barrier is hit.

The density $g^+(t,x;s)$ must satisfy the backward equation. Due to the fact that μ and σ are constants, we know that the function g^+ depends only on $s-t$. If we set $\tau = s - t$, with $\tau \geq 0$, we can write $g^+(t,x;s) = g^+(\tau,x)$. To obtain an expression for g^+ we consider the Laplace transform

$$\gamma^+(x;v) = \int_0^\infty e^{-v\tau} g^+(\tau,x)\, d\tau,$$

for any $v \geq 0$. It is well known (see, Geman and Yor 1996) that the transform can be expressed as

$$\begin{aligned} \theta(v) &= \frac{1}{\sigma^2}\sqrt{\mu^2 + 2\sigma^2 v} \\ \gamma^+(x;v) &= e^{\frac{\mu}{\sigma^2}(l-x)} \frac{\sinh\big(\theta(v)x\big)}{\sinh\big(\theta(v)l\big)}. \end{aligned} \tag{3}$$

We have written $\theta(v)$ to emphasize the dependence of θ on v.

To obtain the density for the upper barrier g^+, we now have to invert the Laplace transform γ^+. This can be done using Bromwich's Integral (see, Duffy 1994, Chapter 2.1.)

$$g^+(\tau,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\tau z}\gamma^+(x;z)\, dz\,, \tag{4}$$

where c lies to the right of any of the singularities of the function γ^+.

The integral (4) can be evaluated as follows. We can transform the line integral into a contour integral by choosing a path through the second and third quadrant. This has to be done with care since we have to take the topography of the poles into account and for the exact construction of the integration path we refer to Doetsch (1950), pp. 279 ff.[2]

The value of the contour integral we have constructed can now be determined by Cauchy's Residue Theorem (see, Duffy 1994, Chapter 1.4): Since the path we have added to transform the line-integral (4) into a contour integral makes no contribution to the integral, the line-integral must be equal, by the Residue Theorem, to the sum of the residues of the singularities enclosed in the contour.

Let us therefore find the singularities of the function $e^{\tau z}\gamma^+(x;z)$. Singularities can only be caused if the term in the denominator of γ^+ goes to zero. Using the identity $\sinh(z) = -i\sin(iz)$, we find that $\sinh\big(\theta(z)l\big)$ is zero if $i\theta(z)l = k\pi$, for k integer. Solving for z yields $z_k = -\frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{l^2}\right)$. So, for $k = 0, 1, 2, \ldots$ we have identified all the singularities z_k of the function $e^{\tau z}\gamma^+(x;z)$. The residue for each singularity z_k is given by

[2] The author would like to thank Michael Schröder for pointing this out.

$$\mathrm{Res}(z_k) = e^{\tau z_k} e^{\frac{\mu}{\sigma^2}(l-x)} \frac{\sigma^2}{l^2} k\pi \sin(k\pi \frac{l-x}{l}). \quad (5)$$

Thus, summing up all of the residues gives the following expression for the density function of hitting the upper barrier $g^+(t, x; s)$ (remember that we set $\tau = s - t$):

$$g^+(t, x; s) = e^{\frac{\mu}{\sigma^2}(l-x)} \frac{\sigma^2}{l^2} \sum_{k=1}^{\infty} e^{z_k(s-t)} k\pi \sin(k\pi \frac{l-x}{l}). \quad (6)$$

This is a well known expression in the literature on Brownian Motion. See, for example, Knight (1981).

An expression for the density g^- can be derived in a similar fashion.

3 Valuation formulas

With the analytical expressions we have derived for the transition density p and the barrier densities g^+ and g^- we can calculate prices for various types of double barrier options.

In the cases we will analyse, we take the underlying to be an F/X-rate.[3] Let $S(t)$ be the spot exchange rate today. Let r_d be the domestic interest rate, r_f the foreign interest rate and σ the volatility of the exchange rate. Let U be the upper barrier and L be the lower barrier with $L < S(t) < U$.

If we divide by L and take logarithms we obtain for $s > t$ that $z(s) = \log(S(s)/L)$ where z is the process defined in (1) with $x = z(t) = \log(S(t)/L)$ and $l = \log(U/L)$. The drift-term μ of the process z (under the equivalent martingale measure) is equal to $\mu = r_d - r_f - \frac{1}{2}\sigma^2$. For all options we denote the maturity date by T.

3.1 Constant payoff at maturity

The simplest kind of double barrier is an option which pays a constant amount at the maturity of the option. Suppose we receive an amount K_U if the upper barrier is hit first, an amount K_L if the lower barrier is hit first and an amount K is neither barrier is hit during the life. All amounts are payed at maturity T. The value $V_{\mathrm{CPM}}(t)$ of this option is equal to

$$V_{\mathrm{CPM}}(t) = e^{-r_d(T-t)} \Big(K_U P^+(T) + K_L P^-(T) + K \left(1 - P^+(T) - P^-(T)\right) \Big), \quad (7)$$

where $P^+(T)$ and $P^-(T)$ denote the probability of hitting first the upper and the lower barrier respectively before time T. The probability of surviving until time T is equal to $1 - P^+ - P^-$. To find P^+ and P^- we have to integrate over the barrier densities. To find an expression for these integrals we rewrite them as

[3] Instead of an F/X-rate, the formulas can also be applied to equity or commodities with a continuous dividend-yield δ, by setting $r_f = \delta$.

$$P^{\pm}(T) = \int_t^T g^{\pm}(t,x;s)\,ds \quad = \int_t^{\infty} g^{\pm}(t,x;s)\,ds - \int_T^{\infty} g^{\pm}(t,x;s)\,ds \\ = \gamma^{\pm}(x;0) - \int_T^{\infty} g^{\pm}(t,x;s)\,ds. \tag{8}$$

Integrating on a term-by-term basis, we find for

$$\begin{aligned} P^{+}(T) &= e^{\frac{\mu}{\sigma^2}(l-x)} \left(\frac{\sinh(\frac{\mu}{\sigma^2}x)}{\sinh(\frac{\mu}{\sigma^2}l)} - \frac{\sigma^2}{l^2} \sum_{k=1}^{\infty} \frac{e^{-\lambda_k(T-t)}}{\lambda_k} k\pi \sin(k\pi \frac{l-x}{l}) \right), \\ P^{-}(T) &= e^{-\frac{\mu}{\sigma^2}x} \left(\frac{\sinh(\frac{\mu}{\sigma^2}(l-x))}{\sinh(\frac{\mu}{\sigma^2}l)} - \frac{\sigma^2}{l^2} \sum_{k=1}^{\infty} \frac{e^{-\lambda_k(T-t)}}{\lambda_k} k\pi \sin(k\pi \frac{x}{l}) \right). \end{aligned} \tag{9}$$

3.2 Rebate at hit

A more realistic payoff scheme, which is used often in knock-out options, is to offer a rebate as soon as the option hits one of the barriers. Suppose we receive an amount K_U at the moment the upper barrier is hit first. The value $V_{\mathrm{RAHU}}(t)$ is given by

$$V_{\mathrm{RAHU}}(t) = K_U \int_t^T e^{-r_d(s-t)} g^{+}(t,x;s)\,ds. \tag{10}$$

Solving this integral involves finding a primitive for terms of the form $e^{-r_d(s-t)}\ e^{-\lambda_k(s-t)}$. We obtain a value for the integral in a simpler way, if we bring r_d inside the λ_k as follows

$$r_d + \lambda_k = \frac{1}{2}\left(\frac{2\sigma^2 r_d + \mu^2}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{l^2} \right) = \frac{1}{2}\left(\frac{\mu'^2}{\sigma^2} + \frac{k^2\pi^2\sigma^2}{l^2} \right) = \lambda'_k \tag{11}$$

with

$$\mu' = \sqrt{\mu^2 + 2\sigma^2 r_d} \tag{12}$$

If we denote g'^{+} as the barrier density with drift μ', then we obtain

$$\begin{aligned} V_{\mathrm{RAHU}}(t) &= K_U\, e^{\frac{\mu-\mu'}{\sigma^2}(l-x)} \int_t^T g'^{+}(t,x;s)\,ds \\ &= K_U\, e^{\frac{\mu}{\sigma^2}(l-x)} \left(\frac{\sinh(\frac{\mu'}{\sigma^2}x)}{\sinh(\frac{\mu'}{\sigma^2}l)} - \frac{\sigma^2}{l^2} \sum_{k=1}^{\infty} \frac{e^{-\lambda'_k(T-t)}}{\lambda'_k} k\pi \sin(k\pi \frac{l-x}{l}) \right). \end{aligned} \tag{13}$$

Similarly, we find that the value of an amount K_L received as soon as the lower barrier is hit first, can be expressed as

$$\begin{aligned} V_{\mathrm{RAHL}}(t) &= K_L e^{-\frac{\mu-\mu'}{\sigma^2}x} \int_t^T g'^{-}(t,x;s)\,ds \\ &= K_L e^{-\frac{\mu}{\sigma^2}x} \left(\frac{\sinh(\frac{\mu'}{\sigma^2}(l-x))}{\sinh(\frac{\mu'}{\sigma^2}l)} - \frac{\sigma^2}{l^2} \sum_{k=1}^{\infty} \frac{e^{-\lambda'_k(T-t)}}{\lambda'_k} k\pi \sin(k\pi \frac{x}{l}) \right). \end{aligned} \tag{14}$$

3.3 Double knock-out

Another payoff we want to consider, are double knock-out options.[4] Suppose we have a double knock-out call, with a payoff $\max\{S(T)-K,0\}$, if the price of S hits neither barrier during the life $[t,T]$ of the option. The value at t is given by

$$V_{\mathrm{DKOC}}(t) = e^{-r_d(T-t)} \int_0^l \max\{Le^y - K, 0\} p(t,x;T,y)\, dy. \tag{15}$$

The option is in-the-money for $Le^y > K \iff y > \log(K/L) = d$. If we assume $0 \le d \le l$ (the other cases are trivial) then we get

$$\begin{aligned} V_{\mathrm{DKOC}}(t) &= e^{-r_d(T-t)} \int_d^l (Le^y - K) p(t,x;T,y)\, dy \\ &= e^{-r_d(T-t)} \Big(L \int_d^l e^y p(t,x;T,y)\, dy - K \int_d^l p(t,x;T,y)\, dy \Big). \end{aligned} \tag{16}$$

Both integrals involve finding the primitive for terms of the form $e^{ay}\sin(by)$. The primitive for these terms is given by

$$\int e^{ay}\sin(by)\, dy = e^{ay} \frac{a\sin(by) - b\cos(by)}{a^2 + b^2}.$$

Hence, if we set $Q(\alpha, y) = \int e^{\alpha y} p(t,x;T,y)\, dy$, we obtain for Q

$$Q(\alpha,y) = \frac{2}{l} e^{\frac{\mu}{\sigma^2}(y-x)} e^{\alpha y} \sum_{k=1}^{\infty} e^{-\lambda_k(T-t)} \sin(k\pi\frac{x}{l}) \left(\frac{(\frac{\mu}{\sigma^2} + \alpha)\sin(k\pi\frac{y}{l}) - \frac{k\pi}{l}\cos(k\pi\frac{y}{l})}{(\frac{\mu}{\sigma^2} + \alpha)^2 + \frac{k^2\pi^2}{l^2}} \right). \tag{17}$$

The value of the double knock-out call can now be expressed as

$$V_{\mathrm{DKOC}}(t) = e^{-r_d(T-t)} \Big(L\big(Q(1,l) - Q(1,d)\big) - K\big(Q(0,l) - Q(0,d)\big)\Big). \tag{18}$$

The value of a double knock-out put is given by

$$\begin{aligned} V_{\mathrm{DKOP}}(t) &= e^{-r_d(T-t)} \int_0^d (K - Le^y) p(t,x;T,y)\, dy \\ &= e^{-r_d(T-t)} \Big(K \int_0^d p(t,x;T,y)\, dy - L \int_0^d e^y p(t,x;T,y)\, dy \Big), \end{aligned} \tag{19}$$

which can be expressed as

$$V_{\mathrm{DKOP}}(t) = e^{-r_d(T-t)} \Big(K\big(Q(0,d) - Q(0,0)\big) - L\big(Q(1,d) - Q(1,0)\big)\Big). \tag{20}$$

The derivation given above, also holds for options which have a payoff which depends on $S(T)^\alpha$. Normal call and put payoffs have $\alpha = 1$. However, a so-called ”bull/bear” contract has a payoff of $\max\{0, \frac{S(T)-K}{S(T)}\} = \max\{0, 1 - KS(T)^{-1}\}$, which can be valued in our framework with $\alpha = -1$.

[4] An alternative method of deriving the expressions given in this section, is to invert the Laplace transform of the option price given in (2.11) of Geman and Yor (1996) using a contour integration. This would involve a tedious calculation, which is sometimes left as an exercise (see, Wilmott et al. 1995).

3.4 Knock-in options

Thus far, we have only considered knock-out options. However, we can also consider double-barrier knock-in options. A knock-in option can be viewed as a "rebate-at-hit" option. However, at the time one of the barriers is hit, not a constant amount is payed, but a payoff equal to a standard Black-Scholes (1973) formula. For example, a double barrier knock-in option with knocks in a call if the upper barrier is hit first, or knocks in a put of the lower barrier is hit first has a value given by

$$V_{\mathrm{DKI}}(t) = \int_t^T e^{-r_d(s-t)} \mathbf{C}(s;U) g^{+}(t,x;s)\, ds + \int_t^T e^{-r_d(s-t)} \mathbf{P}(s;L) g^{-}(t,x;s)\, ds, \tag{21}$$

where $\mathbf{C}(s;U)$ denotes the (Black-Scholes) value of a call-option at time s, with spot-price $S(s) = U$, and $\mathbf{P}(s;L)$ denotes the put option. The integrals above cannot be solved analytically, but using a numerical integration routine it is straightforward to obtain an accurate value for the one-dimensional integrals.

4 Numerical implementation

If we want to use the formulas given above to calculate prices of double barrier options, we have to truncate the infinite sums to a finite number of terms. Fortunately, the $\exp\{-\lambda_k(s-t)\}$ terms decline very fast to zero.

To determine the number of terms needed, we propose the following. Set ϵ to a small number, say $\epsilon = 10^{-10}$. We can then find the number k, for which $\exp\{-\lambda_k(s-t)\} < \epsilon$. This number is given by

$$k > \sqrt{\frac{-2\frac{\log \epsilon}{s-t} - \frac{\mu^2}{\sigma^2}}{\frac{\pi^2 \sigma^2}{l^2}}}. \tag{22}$$

If we set k^* to the smallest integer that satisfies the inequality, we can truncate our sums at k^*. The error we now make is of order ϵ.

To assess the performance of our implementation, we have compiled Table 1. In this table, we compare the calculation times of a double knock-out call options with the calculation times of the formulas derived by Kunitomo and Ikeda (1992). As they consider only knock-out options with no rebate, we used these options for our comparison. Kunitomo-Ikeda recommend using 11 terms in their summation ($n = -5, \ldots, 5$), which is what we used to time the calculation of their prices.

Alternative valuation schemes, like the numerical inversion of the Laplace Transform as proposed by Geman-Yor (1996), or finite difference methods to solve the partial differential equation numerically have not been taken into consideration since it is clear that these methods are not efficient from a computation-time point of view.

Table 1. Calculation time of double knock-out call options ($S = 1000, r_d = 0.05, r_f = 0.00, K = 1000$)

	U	L	Price	KI	P
$T - t = \frac{1}{12}$					
$\sigma = 0.2$	1500	500	25.12	5.8	29.9
	1200	800	24.76	5.9	11.6
	1050	950	2.15	7.6	3.2
$\sigma = 0.3$	1500	500	36.58	5.8	19.9
	1200	800	29.45	6.3	8.2
	1050	950	0.27	8.6	2.5
$\sigma = 0.4$	1500	500	47.85	5.8	15.5
	1200	800	25.84	6.4	6.2
	1050	950	0.02	9.4	1.8
$T - t = \frac{1}{2}$					
$\sigma = 0.2$	1500	500	66.13	6.1	13.8
	1200	800	22.08	6.7	5.3
	1050	950	0.00	9.4	1.7
$\sigma = 0.3$	1500	500	67.88	6.1	8.7
	1200	800	9.26	7.4	3.8
	1050	950	0.00	9.5	1.6
$\sigma = 0.4$	1500	500	53.35	6.5	6.8
	1200	800	3.14	7.6	3.7
	1050	950	0.00	9.4	1.0

"U" denotes level of upper barrier.
"L" denotes level of lower barrier.
"Price" is price of option.
"KI" and "P" denote times (in seconds) to compute 10000 times a price with formulas from Kunitomo-Ikeda and this paper respectively.

From the table follows that for options with short maturities and barriers relatively far away from spot, the Kunitomo-Ikeda formula is faster to compute than our valuation formula. However, for long maturities and barriers relatively close, we find that our valuation formulas are much faster to evaluate, up to a factor of 10. It seems that for implementation in a trading system, significant improvements in calculation times for double-barrier options can be obtained by using a combination of the Kunitomo-Ikeda for short-dated and our valuation formulas for long-dated options. The optimal choice to switch from one set of formulas to the other can be determined by assessing how many terms are needed in the summation, using the method outlined above.

5 Conclusions

In this paper we have provided valuation formulas for a wide range of double-barrier knock-out and knock-in options. We derived Laplace transforms which we inverted analytically using contour integration. With the analytical expressions obtained, we can efficiently calculate values for double-barrier options, without having to resort to numerical inversion methods. Furthermore, we show that our valuation formulas can lead to substantial improvements in computation time.

To our knowledge this has been one of the first applications of contour integration to an option pricing problem. Given the power of this approach, we think that many more applications will follow.

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