Cornell University

ORIE 5961: Applied Financial Engineering

Price-Based Hedging: Hedge When You Can, Not When You Have to

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1 – Introduction

The field of financial mathematics is, among many different areas of knowledge, one that has experiencing a tremendous growth in terms of pure research and applications in the past decades. Although having its own limitations as a modeling framework, the original work of Fisher Black, Myron Scholes and Robert Merton is widely regarded as a turning point where financial derivatives started being priced in accordance to a (reasonable) theoretical approach instead of treated by practitioners by "thumb rules" and rough approximations. As a direct consequence of their work the word has observed an enormous growth in the derivatives markets, which reached the peak notional value of more than eleven times the World´s total GDP before the Subprime Crisis of 2007-2008.

Under the theoretical perspective, the work of Black, Scholes and Merton had two main complimentary findings that relates respectively to the problem of pricing and hedging financial derivatives. By defining (under certain hypotheses) the geometric Brownian motion as the governing stochastic dynamics of the spot price for the underlying instrument and constructing a hedged portfolio with self-financing trading strategies involving the underlying asset and the money market account the authors essentially obtained a closed form equation to specifically price vanilla options – as the solution of a partial differential equation – and, additionally, what should be the corresponding positions on the spot and money market accounts to perfectly hedge the position on that option.

The limitations of the Black and Scholes framework are largely known by scholars and practitioners and alterative modeling approaches to deal with these limitations – as the development of more complex models as stochastic volatility and local volatility models were proposed to deal with the problem of assuming a flat volatility for the dynamics of the spot price. By coherently calibrating the model with market prices of some traded instruments a financial institution could possibly price a derivative instrument under assumptions closer to real world observations, providing it competitive advantage in terms of pricing.

However the hedging problem also falls into another problem of the Black and Scholes hedging limitation – the fact that in a real world situation one could not continuously rebalance (or re-hedge) his or her position in a financial derivative by using the spot and the money market account as the primary instruments. Firstly, in a mathematical sense the idea of a risky underlying following the Brownian motion dynamics (that evolves by continuous paths almost surely) is definitely an approximation of the real markets. In practice, prices moves as new trades (buy and sell orders) are executed and performed by human traders. Moreover, even taking the limit of the time interval to be granularly smaller to measure the typical "cents" variations, a self financing trading strategy that constantly rebalances the hedged portfolio with such high frequency would be hard (or even impossible) to be implemented – for instance, given its transaction costs.

Therefore, the motivation of the present project is to develop and evaluate real word trading strategies that deals with the constraint that one could not re-hedge a position in a financial derivative by continuously trading. In other words, these strategies should use pre-determined rules to re-hedge the derivative exposure in a discrete-time setting, but still considering the simplification hypotheses of the Black and Scholes framework – for example, the constant (or flat) volatility structure. Efficiency criterions should be defined to measure the effectiveness of these strategies, comparing each of them for different combinations of parameters and across their peers (other strategies).

2 – Technical Approach and Product Development

Given the technical (theoretical) motivation of the present work, the main goal is how to develop a structured methodology to compute (or simulate) those strategies and, as a consequence, provide as a final output an user-friendly environment where these computations could be timely performed and used in a trading environment. In order to better illustrate the final deliverable of the project one could refer to the following Figure 1.

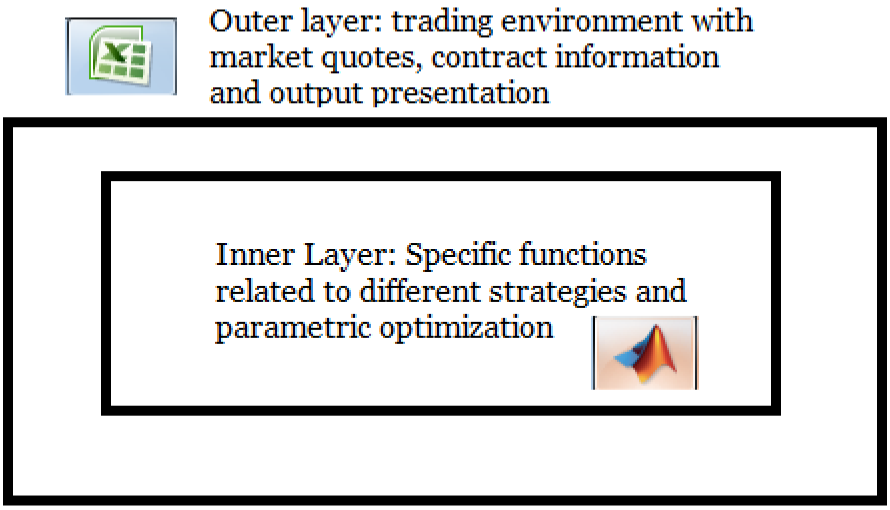

Figure 1: Product Structure

The individual function of each of these "layers" and the relationship between them is better described by the following paragraphs.

Inner Layer: That structure can be seen as the core calculator (or back engine) that comprises the basic functions that should be executed (individually and collectively) to obtain reasonable and meaningful results. For example, this layer should contain individual coded functions corresponding to each different trading strategy that must be considered in the present work. Additionally, functions that take into consideration parametric optimizations and trading (or performance) comparisons across different strategies, although can be classified in an upper "sub-layer", can also be considered part of the core calculator.

One point that should be remarkable in this aspect is that all functions of the inner layer should be developed with a coherent structure of inputs and outputs as generic mathematical entities. To exemplify this statement, one can consider the case of the hedging strategy of a short position in a call option that takes into consideration the placement of symmetric upper and lower limit orders (whose difference is given by $2\Delta S_{min}$) in the underlying order book for each finite time variation Δt - this strategy will be clearly described by Section 3.2. This function will probably will have as its main inputs the initial price of the underlying S_0, the strike price of the option K, the option´s time to maturity T, so and on.

Hence, rather than coding a function that specifically computes the hedging strategy for an "option on the S&P500 index maturing on December 2012", this function can be called limit_order_hedging and has the corresponding inputs and outputs (for example, the expected terminal P&L function) accordingly defined as variables.

It is also worthy to mention that the core calculator should take advantage on the use of efficient programming languages (C++) or even mathematical computing environments (as MATLAB, given its suitability of dealing with matrix problems) for the main development of such functions.

Outer Layer: Although the inner layer corresponds to the computational essence of the problem, as a simple (and separate) tool it does not provide the user the main objectives of its development. In order to simulate and obtain valuable results for a real word situation a trader in a sell side firm should be able, for instance, to take the basic function of the limit order hedging strategy and apply it for a specific option in the market (say, an option on the USD/GBP FX parity) and find how the variance of the terminal P&L would change by simply changing parameter ΔS_{min} *coeteris paribus*.

Alternatively, he or she could be focused on finding the maximum value of parameter Δt that makes the terminal P&L not to exceed the range of 10 cents (given some confidence level). These examples illustrate the importance of the outer layer, which connects the abstract structures and functions of the inner layer with the market quotes and financial instruments in order to provide the trader with meaningful results.

This example is also important to illustrate the essential difference between different input variables of this function. For example, it was previously mentioned that the initial price of the underlying S_0, the strike price of the option K, the option´s time to maturity T, the risk free interest rate r, the upper-lower limit order difference ΔS_{min} and the finite rebalancing interval Δt are some of the input parameters of the core function limit_order_hedging. However there are some conceptual differences on the intrinsic nature of these variables. While T and K are specific characteristics of the option contract that we are simulating and S_0 and r correspond to market prices (or quotes), ΔS_{min} and Δt are parameters of the trading strategy that, differently from the other variables, are totally under the control of the user.

A final example that shows the essence of that difference is the case where the trader is concerned of finding what is the combination (or ordered pair $(\Delta S_{min}, \Delta t)$) that optimizes (maximizes) the expected terminal P&L adjusted to the P&L histogram range – a risk-adjusted return measure.

In order to provide a user-friendly final solution that could be executed by traders or risk managers with different levels of proficiency on computational methods, it is important that the outer layer should be developed in a classically established (but efficient) environment like Excel.

3- Hedging Strategies

3.1 - Traditional hedging strategy

This section describes the mathematical aspects and intuition of each individual trading strategy to be evaluated. We define $(\Omega, \mathcal{F}, \mathrm{F}, \mathrm{P})$ as being a complete probability space, where Ω is a sample space, $\mathcal{F}$ is the σ-algebra of that space, $\mathrm{F} = (\mathcal{F}_t)_{t\geq 0}$ is a filtration and P is the associated probability measure. The underlying instrument (risky asset) is modeled by a stochastic process adapted to the filtration $\mathrm{F} = (\mathcal{F}_t)_{t\geq 0}$ and represented by $(S_t)_{t\geq 0}$. Unless it is mentioned the opposite, we assume that S_t follows the stochastic differential equation for the Black and Scholes model:

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W_t}$$

Where $\widetilde{W_t}$ is a Brownian motion under the risk neutral probability measure $\tilde{P}$.To simplify our notation we may refer as $S(t)$ for the spot process, $C(t)$ for the price of the call option issued (computed by the Black and Scholes formula), and $B(t) = e^{rt}$ as the price of the money market account – where r represents the risk-free interest rate, considered to be a constant deterministic parameter for the Black and Scholes framework.

As it is well known by the literature, the Black and Scholes price of the call option $C(t)$ with underlying price $S(t)$, strike price K, maturity T and volatility σ is given by:

$$C(t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_{1,2} = \frac{ln\left(\frac{S(t)}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T-t}}$$

And $N(.)$ represents the standard normal cumulative distribution function. The corresponding positions on the spot instrument and the money market are respectively denominated $Q_S(t)$ and $Q_B(t)$, being also adapted processes. Finally, one could define the profit-and-loss (P&L) function for the short position in a call option hedged by self-financing trading strategies in both the underlying and the bank account by:

$$P\&L(t) = Q_S(t)S(t) + Q_B(t)B(t) - C(t)$$

One could argue, by construction, that this position will be hedged by positive (long) positions of $Q_S(t)$ and negative (short) positions of $Q_B(t)$, but we will keep that sign notation in the P&L function for simplification purposes, in addition to the fact that $P\&L(0) = 0$, since the position is hedged at $t = 0$.

For (Monte Carlo) simulation purposes the Stochastic Differential Equation that governs the dynamics of the risky asset is represented by finite differences ΔS_t that occur in a finite time set Δt. Because the simulation consists of generating a determined number of paths for the time evolution of S(t) we will refer by using indexes $j = 1, 2, 3, \dots, N_{sim}$ to represent different paths.

$$\Delta S_{ij} = rS_{ij}\Delta t + \sigma S_{ij}\Delta\widetilde{W_{ij}}$$

$$S_{i+1,j} = \Delta S_{ij} + S_{ij}$$

For each path $j = 1, 2, 3, \dots, N_{sim}$ and each discrete time point $i = 0, 1, 2, 3, \dots$.. Hence, for the remaining of this paper we will use the index notation as it was defined to represent the time simulation dimension and the path dimension properly associated with each variable. For example, one hedging strategy $\left(Q_S(t), Q_B(t)\right)$ can take different values for different paths and different time steps, which will lead to the notation of $Q_{S_{ij}}$ and $Q_{B_{ij}}$. On the other hand, the deterministic price of the money market depends only on the time evolution (not on the path that has been simulated for the spot price). Hence it can be simply stated by $B_i = e^{rt(i)}$.

Finally, but not less important, there must be a defined relationship between the time indexes $i = 0, 1, 2, 3, \dots$ and the "real" time measurement t. Usually one can write $t(i) = i\Delta t$, for $i = 0, 1, 2, 3, \dots$.. The final index would be normally defined to match the option maturity T.

Given that preliminary background, we can start the description of the hedging strategies themselves:

3.2 - Innovation: Upper-Lower Limit Order Hedge – First Approach

Although the idea of this strategy was already mentioned in the previous paragraphs, this section describes the theoretical background of the limit order hedging and the assumptions that were made for this initially simplified approach. Given the fact that in a real world setting one could never trade continuously in both the money market account and the underlying in order to perfectly hedge the position, a possible idea to approximate the Black and Scholes hedging is the use of pairs of limit orders with pre-computed quantities that are intended to re-hedge the position for the following evolution of the market prices.

Limit orders are trading orders (buy or sell orders) where the trader must inform the quantity that he wants to buy or sell and the price that he is willing to pay (or receive) for the security. As a trade-off aspect, when place limit orders the trader actually does not know the time when the orders will be executed – since the execution is conditioned to the market price reaching the order level. Mathematically speaking, the execution time of limit orders are usually modeled as stopping times.

A complimentary concept of limit orders is the so called *market order*. In this case, the trader wants to buy or sell a determined number (or position) of a security instantaneously, meaning that he or she will know the quantity to be purchased or sold but the execution price will be determined by the market (supply and demand structure of the order book).

Conceptually, as time evolves and the market price of the option and the underlying instrument change, the trader will be placing pairs of upper and lower limit orders – symmetrically with respect to the current observable level of the underlying price $S(t)$ in order to ensure that as the underlying price changes and the original P&L started deviating from zero the limit orders could be executed and the position will be theoretically re-hedged. Figure 2 provides a better understanding of this strategy.

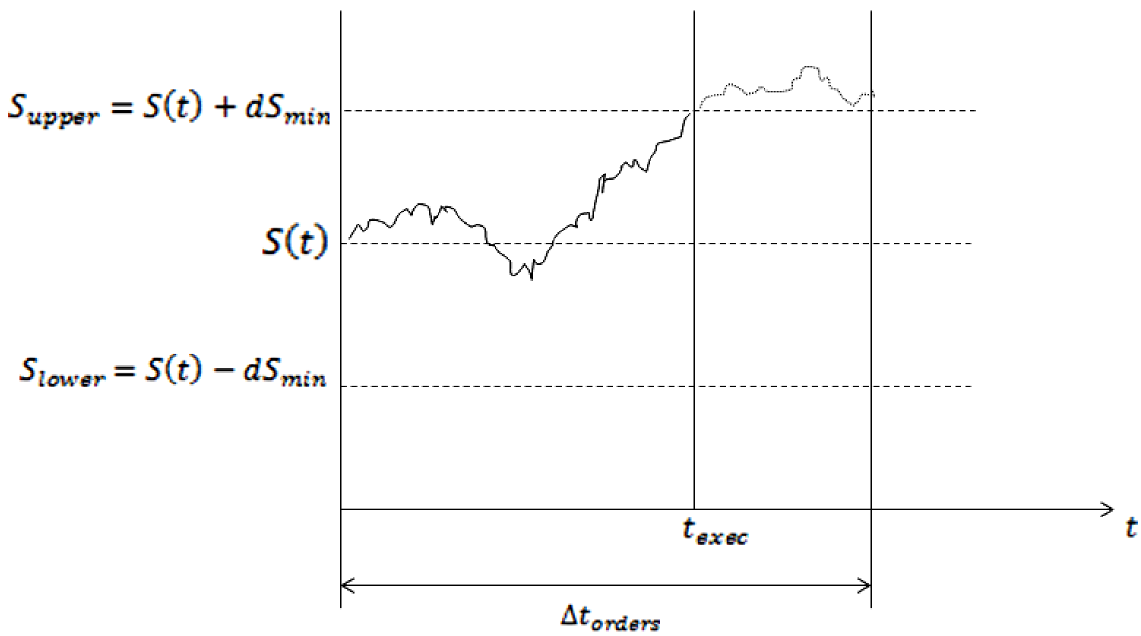

Figure 2: Conceptual representation of the limit order re-hedging

As one could see, starting from time t (where the underlying value is known to be $\mathrm{S}(t)$), the trader will place two limit orders at price levels $S_{upper} = \mathrm{S}(t) + dS_{min}$ and $S_{lower} = \mathrm{S}(t) - dS_{min}$. Because he (or she) does not know the execution time for either the upper and the lower orders (represented by the stopping time t_{exec} at Figure 2), he initially defines the inter-placement time Δt_{orders} where he will always place a new pair of symmetric limit orders by taking into consideration the current price level of the underlying. As we can see in the illustration of Figure 2, there could be the case where one (and exactly one) of the orders will be executed before the placement of a new pair. On the other hand, if the underlying does not deviate from its current level within the following time Δt_{orders} the old limit orders must be cancelled by the trader and new orders will be placed – now taking into consideration the current (novel) level of $\mathrm{S}(t)$.

The quantities of underlying related to the upper and lower limit orders are also a variable that must be coherently defined for this strategy. Since the current number of stocks held (at current time t) is known and assuming that the position should be re-hedged if reached either the upper and the lower order levels, the quantities $\mathrm{Q}_{\mathrm{upper}}(t)$ and $Q_{lower}(t)$ can be estimated as $Q_{upper}(t) = \Delta_{BS}\left(S_{upper}, K, r, \sigma, t_{exec},, T\right) - Q_S(t)$ and $Q_{lower}(t) = \Delta_{BS}(S_{lower}, K, r, \sigma, t_{exec},, T) - Q_S(t)$, where $\Delta_{BS}(.)$ represents the classical Black and Scholes delta function. All the parameters related to these expressions are known at time t, except the execution time t_{exec}, that is a random variable in the interval $[t.t + \Delta t_{orders}]$ or even can be inexistent (or infinity piecewise) if the orders were not executed for that time period. Because the quantities must be informed at time t (placement of the order pair) one could approximate t_{exec} by any point $t_{exec} \in [t.t + \Delta t_{orders}]$.

Given the conceptual description of how the use of limit order pairs can be implemented to a real world re-hedging strategy we can now specify the first approach that was considered to simulate it. As it was mentioned before, the dynamics of the spot price is simulated according to the following equation:

$$\Delta S_{ij} = rS_{ij}\Delta t + \sigma S_{ij}\Delta \widetilde{W_{ij}}$$

$$S_{i+1,j} = \Delta S_{ij} + S_{ij}$$

Where $S_{0,j} = S_0$ and the $\Delta \widetilde{W_{ij}}$ terms are obtained by generating samples of the normal distribution whose standard deviation is properly scaled to match the time increment Δt. In this first approach, however, we considered a limitation for the time dimension: both the Brownian motion time increment and the time interval where new limit orders must be placed are assumed to be the same. As a reference of Figure 2, the discrete time where the finite variations of the underlying are simulated is consistent with Δt_{orders}, which means that $\Delta t = \Delta t_{orders}$ and all the discrete points S_{ij} will be simulated only for $t(i) = i\Delta t_{orders}$ where $i = 0, 1, 2, \ldots$ so and on.

The main problem of that initial approach is that the spot could actually reach the upper or the lower order prices for time points before the next point of $t(i+1) = t(i) + \Delta t$ - while in this case the limit order check is accomplished just at the same frequency of the re-hedging problem. This limitation is clearly seen by the illustrative example of Figure 2. However, that model is worthy of simulation, as our first attempt.

The idea of the algorithm (trading strategy) is the following one. Starting from $S_{0,j} = S_0$ we construct the corresponding delta hedged positions at time $t = 0$ (called $Q_{S_{0j}}$ and $Q_{B_{0j}}$) in order to ensure the position is originally hedged and the initial P&L is equal to zero. The value of $Q_{S_{0j}}$ is simply $Q_{S_{0j}} = \Delta_{BS}(S, K, r, \sigma, t = 0, T)$, the Black and Scholes delta and $Q_{B_{0j}}$ is given by $Q_{B_{0j}} = C(t = 0) - Q_{S_{0j}} S_0$.

For each $i = 0, 1, 2, 3, \ldots$ and $j = 1, 2, 3, \ldots$ the finite increments $\Delta S_{ij} = r S_{ij} \Delta t + \sigma S_{ij} \Delta \widetilde{W_{\iota J}}$ are computed and one must verify whether $\Delta S_{ij} > \Delta S_{min}$ (hit upper order), $\Delta S_{ij} < -\Delta S_{min}$ (hit lower order), and otherwise.

Because the order execution check is accomplished just after the next time increment Δt one can define the corresponding quantities (sizes) of the upper and lower orders to be placed for each time point i to ensure that, if we reached either the upper or the lower limits (in the next time point), the terminal quantity of underlying held should be equal to the Black and Scholes delta at the next time point $i + 1$, respectively with prices $S_{ij} + \Delta S_{min}$ and $S_{ij} - \Delta S_{min}$. Conceptually it means that the approximation of t_{exec} for the right end extreme point of each subinterval is considered and the upper and lower quantities are given by

$$Q_{upper_{ij}} = \Delta_{BS}\big(S_{ij} + \Delta S_{min}, K, r, \sigma, t(i + 1), T\big) - Q_{S_{ij}}$$

$$Q_{lower_{ij}} = \Delta_{BS}\big(S_{ij} - \Delta S_{min}, K, r, \sigma, t(i + 1), T\big) - Q_{S_{ij}}$$

Observe that the time dimension t is already adjusted for the next point of $i + 1$. However it is not because the upper and lower orders were placed that one can guarantee they will be executed. Hence, indicator functions for the upper and lower hits are also created, being called $I_{upper_{ij}}$ and $I_{lower_{ij}}$. They can be expressed as:

$$I_{upper_{ij}} = \begin{cases} 1, & if\ \Delta S_{ij} > \Delta S_{min} \\ 0, & otherwise \end{cases}$$

$$I_{lower_{ij}} = \begin{cases} 1, & if\ \Delta S_{ij} < -\Delta S_{min} \\ 0, & otherwise \end{cases}$$

The next position on the stock (underlying) instrument is simply computed as a function of the indicator variables:

$$\begin{aligned} Q_{S_{i+1,j}} = Q_{S_{i,j}} \left(1 - I_{upper_{ij}} - I_{lower_{ij}}\right) + \Delta_{BS}\big(S_{ij} + \mathit{\Delta} S_{min}, K, r, \sigma, t(i + 1), T\big) I_{upper_{ij}} \\ + \Delta_{BS}\big(S_{ij} - \mathit{\Delta} S_{min}, K, r, \sigma, t(i + 1), T\big) I_{lower_{ij}} \end{aligned}$$

It means that, if both $I_{upper_{ij}} = I_{lower_{ij}} = 0$ then the number of underlying held remains unaltered, $Q_{S_{i+1,j}} = Q_{S_{i,j}}$. Otherwise $\mathrm{Q_{S_{i+1,j}}} = \Delta_{BS}\big(S_{ij} + \mathit{\Delta} S_{min}, K, r, \sigma, t(i + 1), T\big)$ if we reached the upper limit and $\Delta_{BS}\big(S_{ij} - \mathit{\Delta} S_{min}, K, r, \sigma, t(i + 1), T\big)$ if we reached the lower limit.

The corresponding position in the money market account is given by the self financing trading condition, meaning that:

$$Q_{B_{i+1,j}} = \frac{S_{i+1,j}\left(Q_{S_{i,j}} - Q_{S_{i+1,j}}\right) + B_{i+1} Q_{B_{i,j}}}{B_{i+1}}, \quad i = 0, 1, \ldots. \quad j = 1, 2, \ldots N_{sim}$$

3.3 - Upper-Lower Limit Order Hedge – Second Approach

As it was mentioned before, one of the limitations of the first attempt is the fact that the time granularity considered to simulate the Brownian motion (and consequently the dynamics of the underlying and the corresponding prices of the money account and the call option) is exactly the same as the determined order placement time interval, having the direct (and undesired) consequence of making the order execution only to be checked at the following time step – where new limit orders will be placed. This problem is illustrated at Figure 3 below:

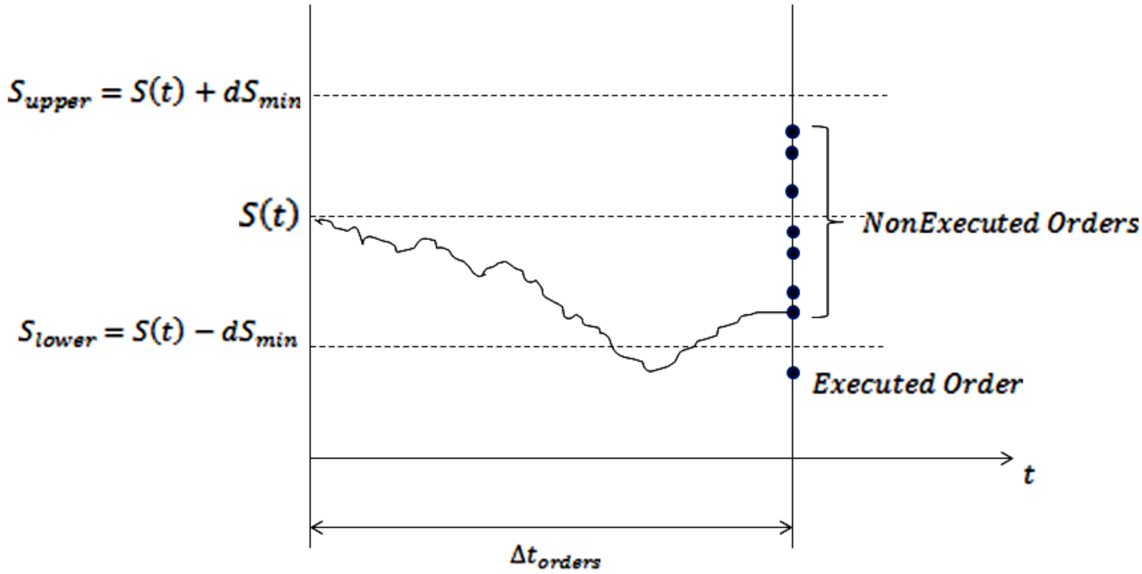

Figure 3: Limitations of the First Simulation Approach

As one will notice, having the simulated spot (market) prices only at the times where new orders will be placed labels of 7 out of 8 paths as their orders having not being executed for that time frame. However there might be some paths where the orders were indeed executed for intermediate times.

One possible way to increase the time granularity of the evolution of market prices is demonstrated by the following Figure 4. The market dynamics is simulated in a Monte Carlo framework exactly as it was for the first attempt:

$$\Delta S_{ij} = rS_{ij}\Delta t + \sigma S_{ij}\Delta\widetilde{W_{ij}}$$

$$S_{i+1,j} = \Delta S_{ij} + S_{ij}$$

The intrinsic change is the periodic order placement, which does not follow the structure of $\Delta \mathrm{t} = \Delta t_{orders}$ anymore. Instead, one can define a multiplicative factor for the time dimension as being $t_{factor} = \frac{\Delta t_{orders}}{\Delta_t}$ – for the example of Figure 4 we have $t_{factor} = 5$, for instance. It means that market prices will be simulated for $t(i) = i\Delta t$ where $i = 0, 1, 2, \ldots$ but the pairs of limit orders will be placed only for $t_{orders}(i^*) = i^*\Delta t_{orders}$ where $i^* = (t_{factor})i = 0, t_{factor}, 2t_{factor}, \ldots$, so and on.

For simulation purposes it is worthy to define an indicator vector with the same dimension of the number of times $t(i)$ and whose entities will be equal to one if $t(i)$ represents a time where limit orders will be placed and zero otherwise. It means that:

$$I_{orders}(i) = \begin{cases} 1, & if\ mod(i, t_{factor}) = 0 \\ 0, & otherwise \end{cases}$$

Where $mod(i, t_{factor})$ represents the remainder of the division of i by t_{factor}.

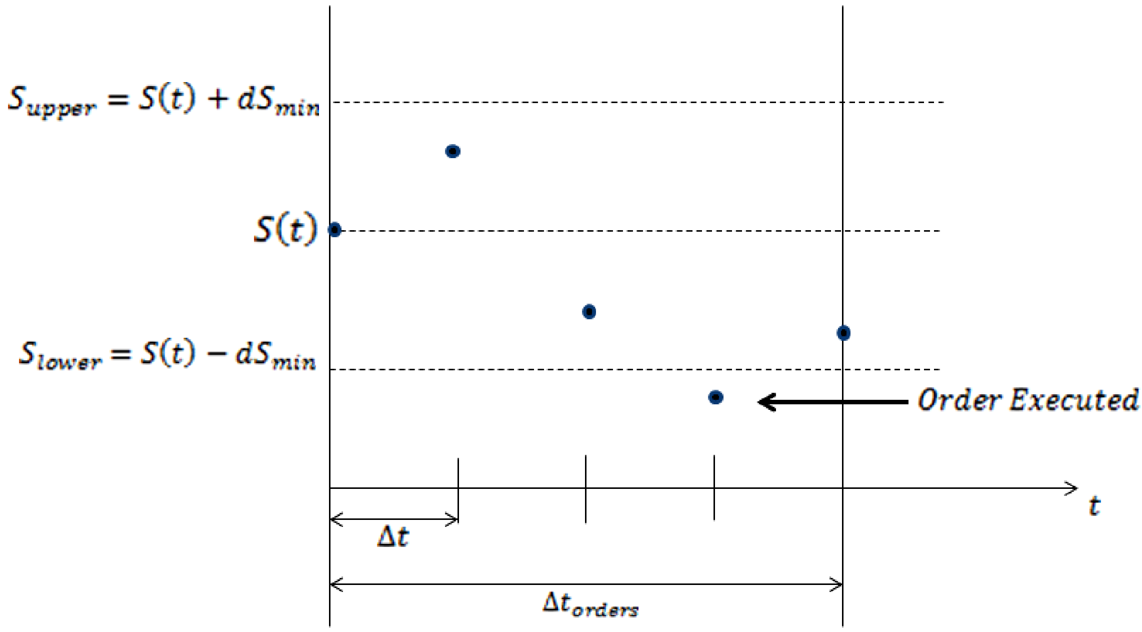

Figure 4: Simulation Framework for the Second Approach

As a direct consequence, the current levels of the upper and lower limit orders as a function of time must be properly defined.

$$S_{upper_{ij}} = (S_{i,j} + dS_{min})\ I_{orders}(i) + S_{upper_{i-1.j}}(1 - I_{orders}(i))$$

$$S_{lower_{ij}} = (S_{i,j} - dS_{min})\ I_{orders}(i) + S_{lower_{i-1.j}}(1 - I_{orders}(i))$$

These expressions mean that both the upper and lower levels for limit orders will be updated respectively for $S_{i,j} + dS_{min}$ and $S_{i,j} - dS_{min}$ only at the placement order times. For times where market prices are simulated but new limit orders are not placed the levels of $S_{upper_{ij}}$ and $S_{lower_{ij}}$ remain exactly the same as they were in the previous time.

Indicator functions for whether the upper and lower levels were hit are now defined in terms of the absolute value of $S_{i,j}$ (and not by the incremental value $dS_{i,j}$ as it was accomplished in our first attempt).

$$I_{upper\ hit_{ij}} = \begin{cases} 1, & if\ S_{ij} > S_{upper_{ij}} \\ 0, & otherwise \end{cases}$$

$$I_{lower\ hit_{ij}} = \begin{cases} 1, & if\ S_{ij} < S_{lower_{ij}} \\ 0, & otherwise \end{cases}$$

One should notice, however, that the simple definition of $I_{upper\ hit_{ij}}$ and $I_{lower\ hit_{ij}}$ themselves are necessary but not sufficient conditions to determine the points where the re-hedging will be performed (if it is the case) and whether it will occur for the upper or the lower levels. In fact the only characteristic of $I_{upper\ hit_{ij}}$ and $I_{lower\ hit_{ij}}$can isolatedly provide is about the current level of the spot price comparatively to the limit order levels – and that both of them could not be equal to one for the same time index i for the same path j. To determine the point where the re-hedging will be performed one must keep track of the evolution of $I_{upper\ hit_{ij}}$ and $I_{lower\ hit_{ij}}$ between two consecutive placement indexes i^* and record the first (and only the first) point where either $I_{upper\ hit_{ij}}$ or $I_{lower\ hit_{ij}}$ was equal to 1. This point represents the discrete approximation of the moment where one of the limit orders was executed and, after this point the limit orders are now disregarded – until the placement of other orders.

Hence, we define $I_{upper\ rehedge_{ij}}$ and $I_{lower\ rehedge_{ij}}$ as:

$$I_{upper\ rehedge_{ij}} = I_{upper\ hit_{ij}}\left(1 - I_{orders}(i)\right)\left(1 - \prod_{k=i^*}^{i-1} I_{upper\ hit_{kj}}\right)\left(1 - \prod_{k=i^*}^{i-1} I_{lower\ hit_{kj}}\right)$$

$$I_{lower\ rehedge_{ij}} = I_{lower\ hit_{ij}}\left(1 - I_{orders}(i)\right)\left(1 - \prod_{k=i^*}^{i-1} I_{upper\ hit_{kj}}\right)\left(1 - \prod_{k=i^*}^{i-1} I_{lower\ hit_{kj}}\right)$$

For $i^* \leq i < (i+1)^*$. The quantities associated with the upper and lower limit orders are given by the following expressions:

$$Q_{upper_{ij}} = \left\{\Delta_{BS}\left(S_{upper_{ij}}, K, r, \sigma, t(i), T\right) - Q_{S_{ij}}\right\} I_{orders}(i) + Q_{upper_{i-1j}}\left(1 - I_{orders}(i)\right)$$

$$Q_{lower_{ij}} = \left\{\Delta_{BS}\left(S_{lower_{ij}}, K, r, \sigma, t(i), T\right) - Q_{S_{ij}}\right\} I_{orders}(i) + Q_{lower_{i-1j}}\left(1 - I_{orders}(i)\right)$$

Observe that, in this case the "deltas" for the upper and lower limit orders are approximated by assuming that the stopping time is actually the moment where the order pair is originally placed (not the right end of the inter-placement time). In analogy, one can also define the corresponding Black and Scholes deltas associated with the limit orders as:

$$\Delta_{upper_{ij}} = \Delta_{BS}\left(S_{upper_{ij}}, K, r, \sigma, t(i), T\right) I_{orders}(i) + \Delta_{upper_{i-1j}}\left(1 - I_{orders}(i)\right)$$

$$\Delta_{lower_{ij}} = \Delta_{BS}\left(S_{lower_{ij}}, K, r, \sigma, t(i), T\right) I_{orders}(i) + \Delta_{lower_{i-1j}}\left(1 - I_{orders}(i)\right)$$

Although limit order pairs are placed only for multiple indexes of t_{factor} the check of the re-hedging is performed for each time index i. It means that the position on the underlying for the following step is given by:

$$Q_{S_{ij}} = Q_{S_{i-1,j}}\left(1 - I_{upper\ rehedge_{ij}} - I_{lower\ rehedge_{ij}}\right) + \varDelta_{upper_{ij}} I_{upper\ rehedge_{ij}} + \varDelta_{lower_{ij}} I_{lower\ rehedge_{ij}}$$

Finally, the self financing trading condition is implemented for each simulated index i since the information if the upper or lower orders were executed (and the portfolio rebalanced) is already comprised for the time series of $Q_{S_{i,j}}$. Hence we can write that:

$$Q_{B_{i,j}} = \frac{S_{i,j}\left(Q_{S_{i-1,j}} - Q_{S_{i,j}}\right) + B_i Q_{B_{i-1,j}}}{B_i}, \quad i = 1, \ldots . \quad j = 1, 2, \ldots N_{sim}$$

4- Analysis of Results

This section comprises the simulation results for each strategy considered, as well as parametric optimizations of the hedging effectiveness of such strategies. Starting from the limit order hedging strategy (simplified modeling approach) with $S_0 = 100, K = 110,$ $\sigma = 30\%, r = 5\%, T = \frac{1}{12}$ year, $\Delta t = \frac{1}{24.21.12}$ year (or simply 1 hour) and $\Delta S_{min} = 0.1$ one will find the following results for the terminal P&L.

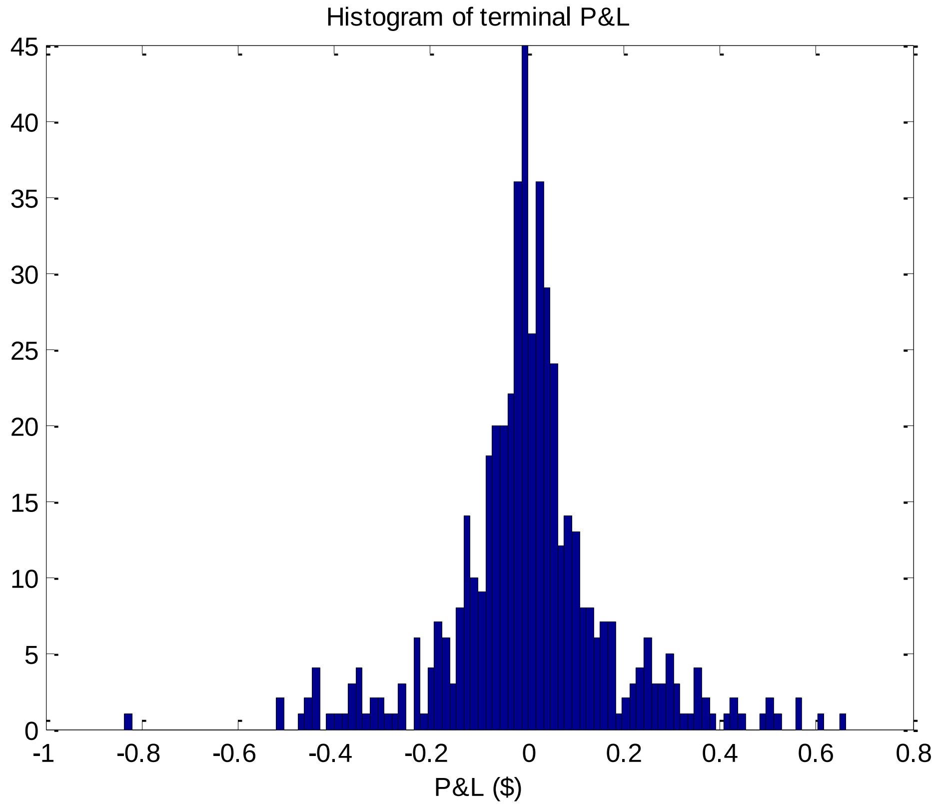

Figure 5: Histogram of Terminal P&L for $S_0 = 100, K = 110, \sigma = 30\%, r = 5\%,$ $T = \frac{1}{12}$ year, $\Delta t = \frac{1}{24.21.12}$ year and $\Delta S_{min} = 0.1$

In order to ensure that simulation was correctly performed one could see the information from Figure A1 to Figure A4 which are presented in the Appendix section. For the indicator variables only the first 10 paths are plotted (Figure A1) while the evolution of the limit orders placed (Figure A2), the simulation of market prices (Figure A3) and corresponding market exposures (Figure A4) represent the first 50 paths.

The intuition says that if one increased the inter-placement time from one hour to four hours (or $\Delta t = \frac{4}{24.21.12}$ year) it must decrease the effectiveness of the hedging strategy – even though the number of paths actually reaching the upper or lower limits must increase, as the finite differences of the underlying prices must increase as we are simulating them for a larger time horizon. The terminal histogram for $S_0 = 100, K = 110,$ $\sigma = 30\%, r = 5\%, T = \frac{1}{12}$ year, $\Delta t = \frac{4}{24.21.12}$ year (4 hours) and $\Delta S_{min} = 0.1$ is given by Figure 6:

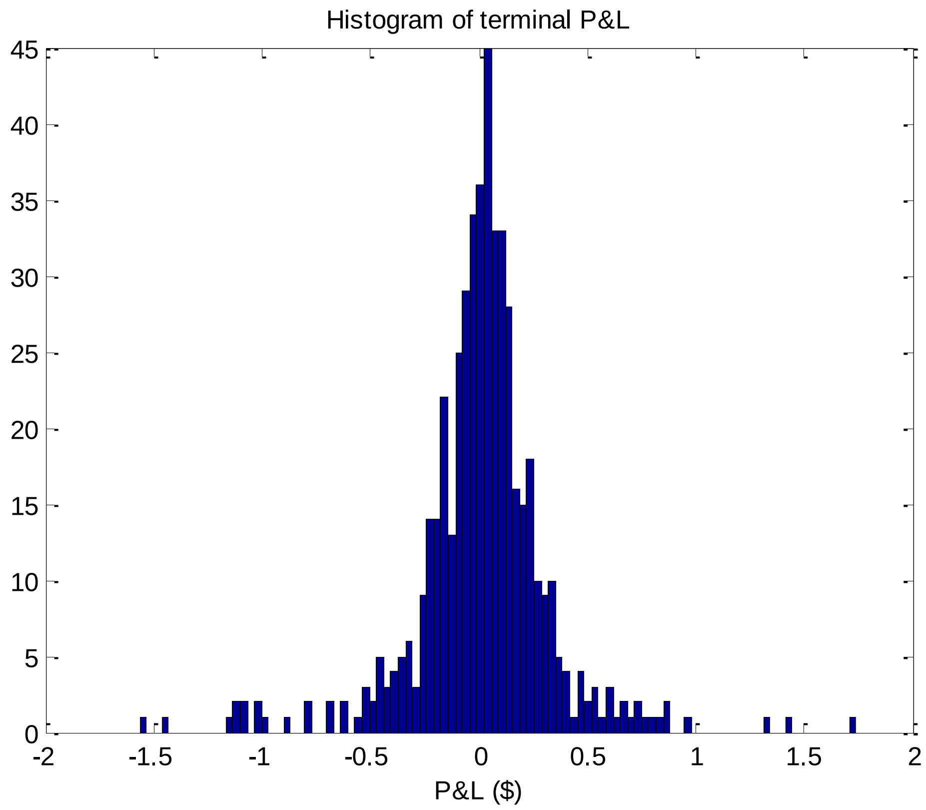

Figure 6: Histogram of Terminal P&L for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = \frac{4}{24.21.12}$ year and $\Delta S_{min} = 0.1$

As one could notice, the histogram presented for Figure 6 shows a higher variance when compared with the first simulation (Figure 5). The evolution of the first 10 paths for the dummy variables (indicating whether upper and lower re-hedging were performed) can be seen by Figure A5 (Appendix). Because the inter-placement time was increased by a multiplicative factor of 4 the rebalancing occurs in a clearer way.

We must also expect that when the upper-lower order spread (represented by $2\Delta S_{\min}$) increases then the terminal P&L will also show a higher variance, as the frequency of re-hedging (hitting frequency of limit orders) will definitely decrease.

By taking $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = \frac{4}{24.21.12}$ year but increasing ΔS_{min} to one dollar, which means $\Delta S_{min} = 1$ one will find the histogram of Figure 7:

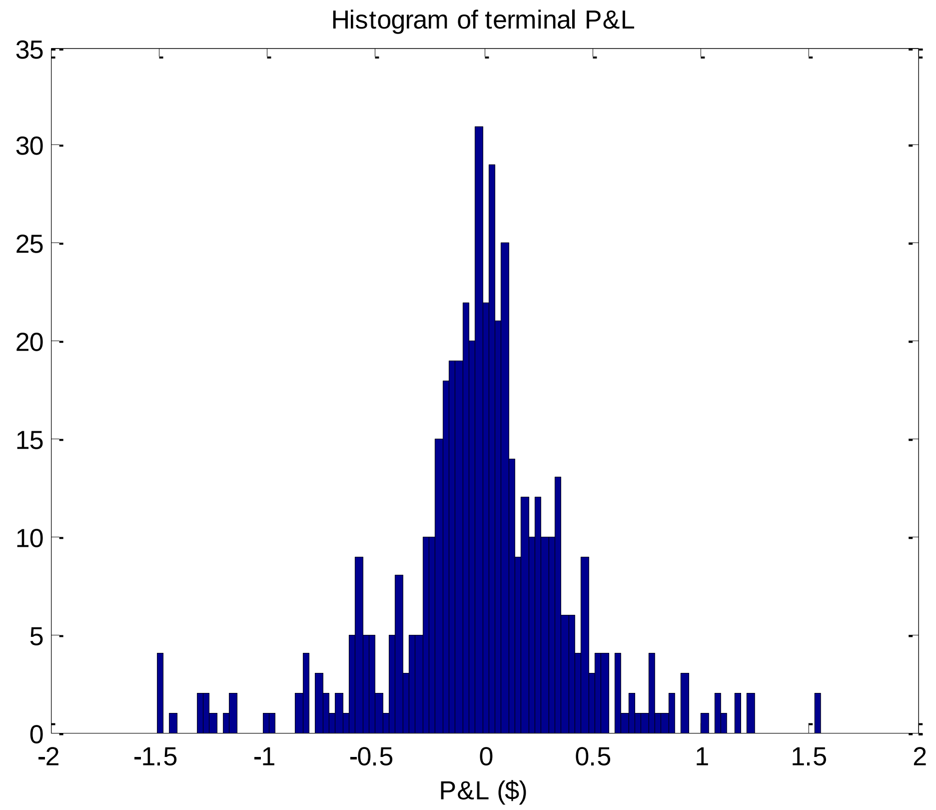

Figure 7: Histogram of Terminal P&L for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = \frac{4}{24.21.12}$ year and $\Delta S_{min} = 1$

Although the extreme points of the histogram did not vary considerably when compared with the previous simulation (Figure 6) it is remarkable how the tail probabilities increased as the spread increased. Figure A6 and Figure A7 also provide a comparative information of the hedging effectiveness of the limit orders as it is clear how Figure A6 shows a lower re-hedging frequency (when compared with Figure A5) and Figure A7 indicates paths for financial exposures and P&L spreader than previous Figure A4 (first simulation).

It is important to notice that all histograms were roughly centered at a terminal P&L of zero, which is expected for this hedging strategy. As a general conclusion one can affirm that this first attempt is definitely a good approximation of a real world limit order hedging strategy – although having the limitation of the intermediate times not being evaluated for the market prices.

Essentially the trader should balance the trade-off between parameters ΔS_{min} and Δt in order to ensure that the variance of his (or her) terminal P&L will satisfy some requirements. Having ΔS_{min} and Δt as control parameters (to be decided by the trader) he (or she) must understand that as $\Delta S_{min} \to 0$ and $\Delta t \to 0$ we are certainly approaching the limit situation of a continuously trading Black and Scholes framework. Unfortunately, taking $\Delta t \to 0$ and $\Delta S_{min} \to 0$ could make the strategy unfeasible in terms of transaction costs. Hence, running the simulation for different combinations of parameters ΔS_{min} and Δt and, for each simulation analyzing the statistical properties of the terminal P&L is certainly an analysis that is extremely valuable.

Hence, for the parametric study, one could vary these inputs in certain space. By taking the same option of $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year and varying

$\Delta t = \{0.5\text{h},\ 1.0\text{h}, 1.5\text{h}, 2.0\text{h}, 3.0\text{h}\ 4.0\text{h}, 6.0\text{h}\}$ and $dS_{min} = \{0.05, 0.10, \dots, 0.1\}$ with the corresponding combinations of the Cartesian product of both sets one will find the following surfaces, respectively measuring the Expected Terminal P&L, the P&L range and the conditional loss (expected terminal P&L given that P&L is negative). Each simulation was executed now for $\mathrm{N_{sim}} = 5000$.

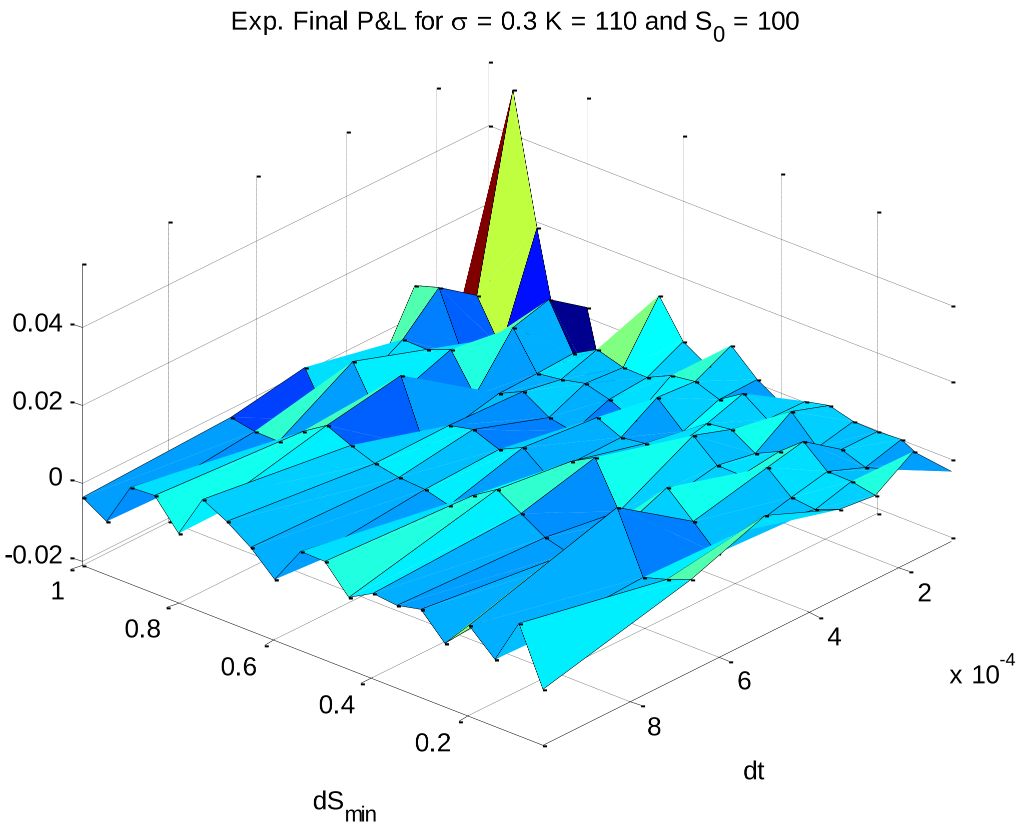

Figure 8: Expected value of terminal P&L

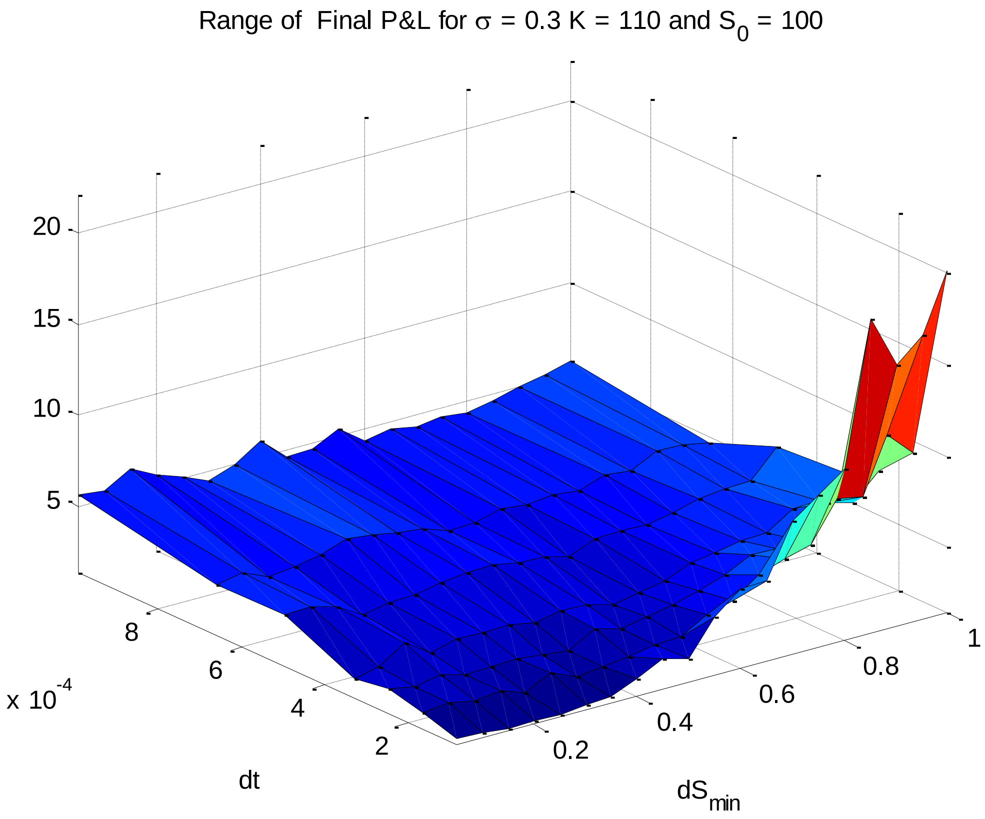

Figure 9: Range of terminal P&*L*

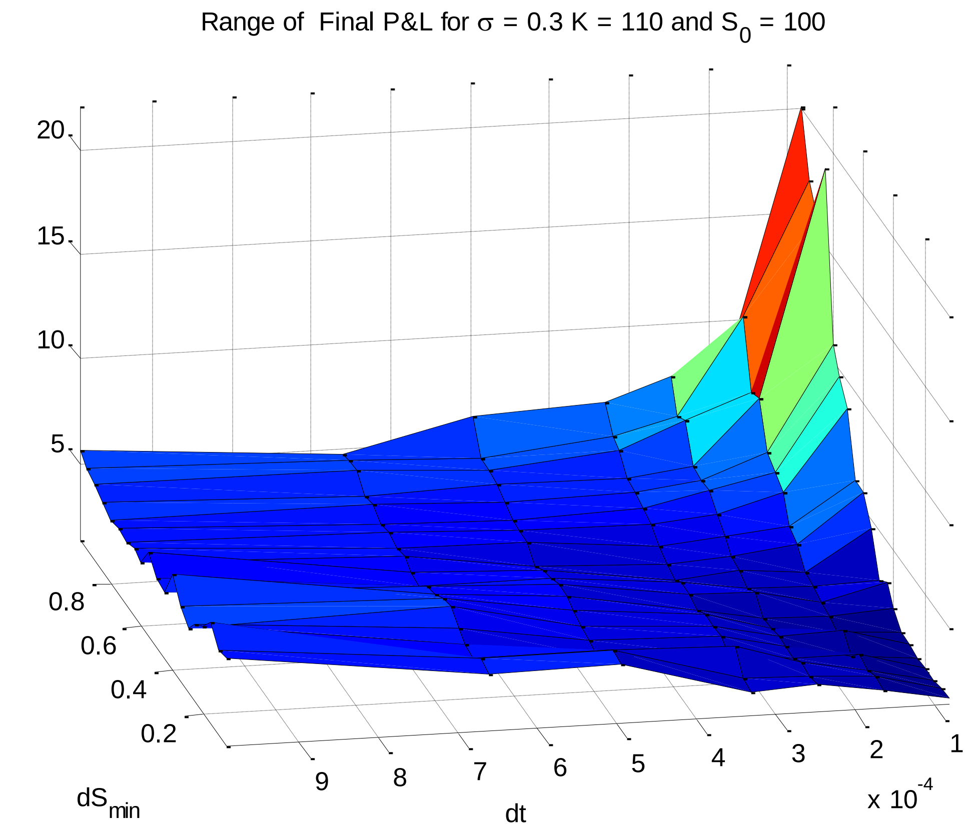

Figure 10: Range of terminal P&*L* - Different Perspective

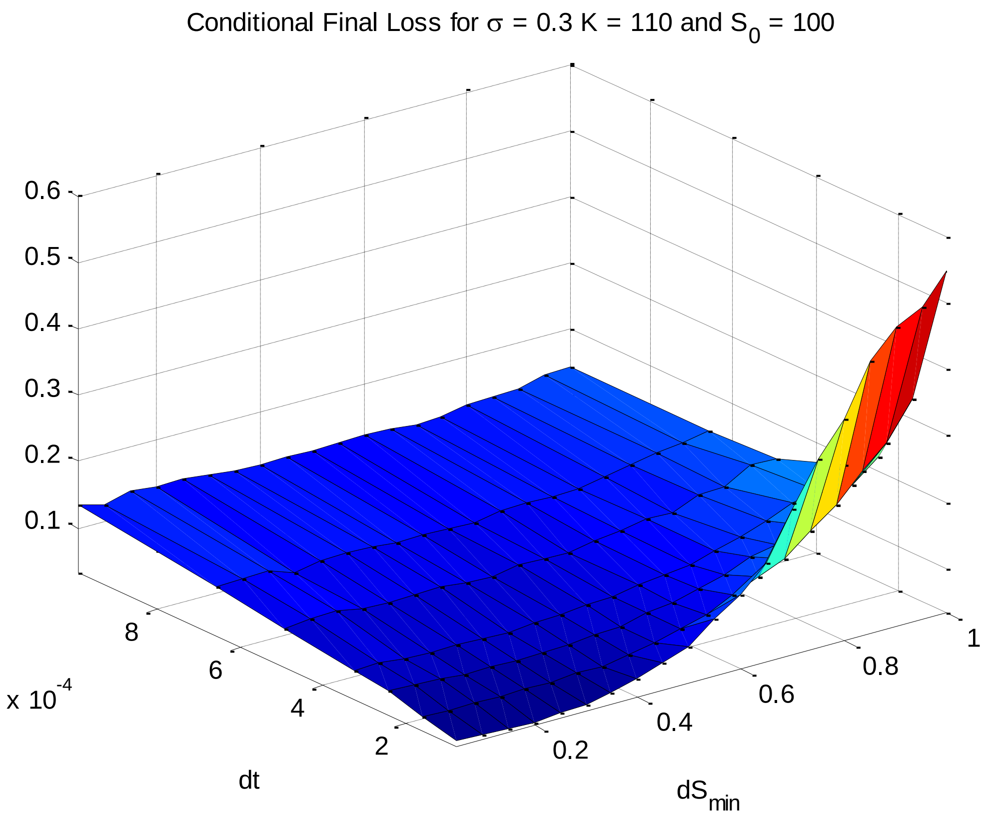

Figure 11: Conditional Terminal Loss

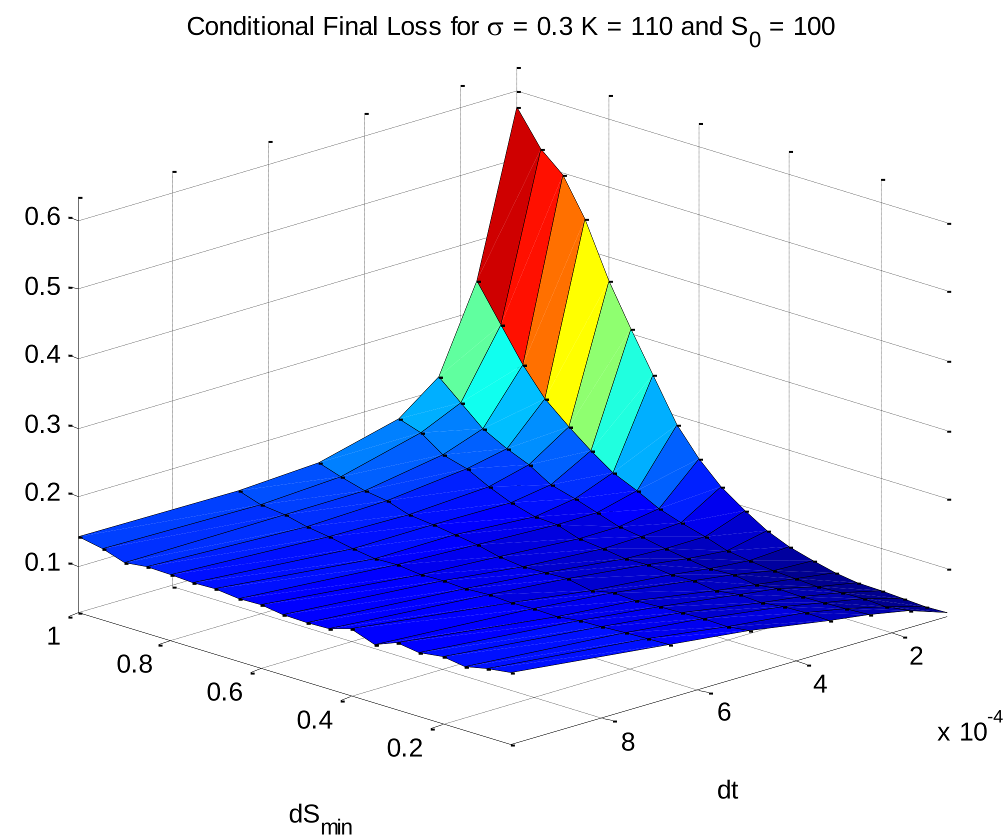

Figure 12: Conditional Terminal Loss – Different Perspective

Our next research direction is to work on how to figure out the optimized limit order level that can guarantee the delta-hedging as perfect as possible.

Results from Figure 8 to Figure 12 depict the relationship between limit order spread and the order-placement frequency to the characteristics of the terminal P&L. As one must observe, Figure 8 shows that the expected value of the terminal P&L is roughly zero regarding of the simulated set of parameters. The peak of expected terminal P&L is definitely due to an outlier result for that simulation, as the strategy is definitely expected to have a zero expected P&L.

More interesting results are observed regarding the range of the terminal P&L and the conditional loss for the portfolio hedging strategy. Figures 9 to 12 clearly show that there is a relationship between ΔS_{min} and Δt where these "risk measurements" seem to be optimal – specifically as we increase Δt the corresponding value of ΔS_{min} within this optimal subset also increases. It can be possibly related to the ratio of limit orders which are executed – a parameter that can also be plotted in 3D shapes as for Figure 9 to Figure 12.

However, in order to investigate the inter-execution time for the limit orders and the expected ratio of re-hedging one could use the second model (described by section 3.2) instead of the first approach. As a first result, in order to validate the outputs of the second model comparatively with the first one, it worths to take the first simulation of $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year , and $\Delta S_{min} = 0.1$ but assuming $\Delta t = \frac{1}{24.21.12.20}$ year (or 3 minutes) for the simulation of market prices and $t_{factor} = 20$, in order to match the inter-placement time to 1 hour.

This simulation leads to the following histogram for the terminal P&L. At the first glance one could see that the histogram´s range seems to be even lower than the one obtained for Figure 5, although they represent the same inter-placement times. However this result totally makes sense, as the present simulation considers the re-hedging possibility for intermediate times (between the placements of the limit orders) as we have discussed before.

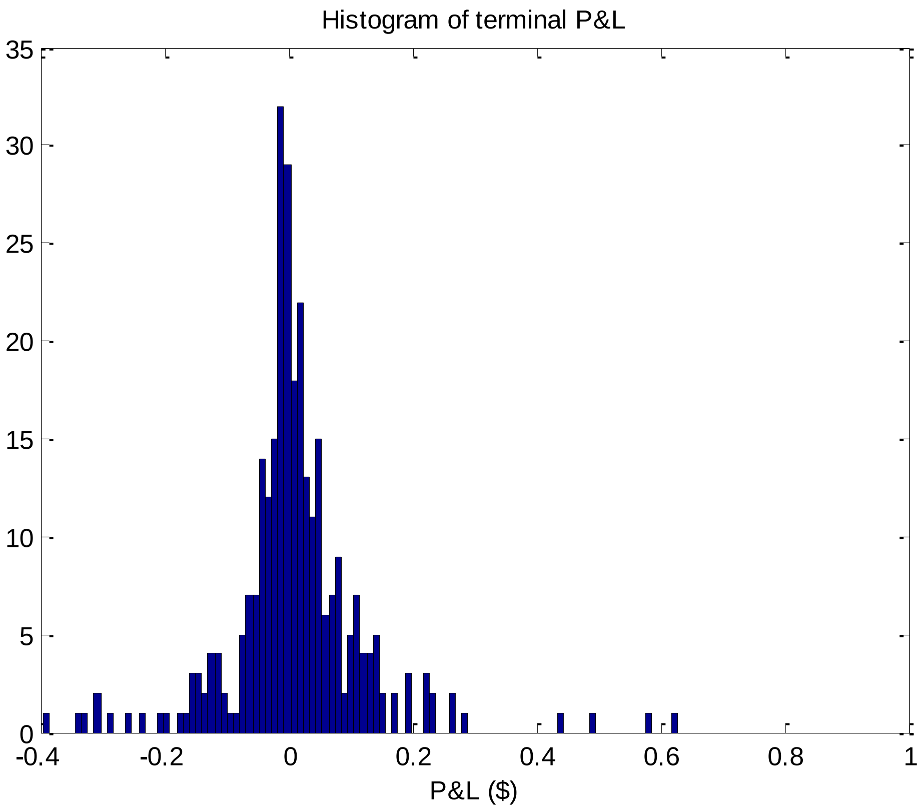

Figure 13: Histogram of Terminal P&L for $S_0 = 100, K = 110, \sigma = 30\%, r = 5\%,$ $T = \frac{1}{12}\ year,\ \Delta t = \frac{1}{24.21.12;20}$ year, $t_{factor} = 20$ and $\Delta S_{min} = 0.1$

Perhaps more important than the histogram for the terminal P&L is the information of the inter-hedging times (times between the execution, not placement, of two limit orders), whose histogram can be seen by Figure 14.

The effectiveness of the limit orders (defined as the ratio between the total number of orders placed and the total number of orders executed) is easily computed from that simulation. Specifically in this case we achieved a execution ratio of 99.93%.

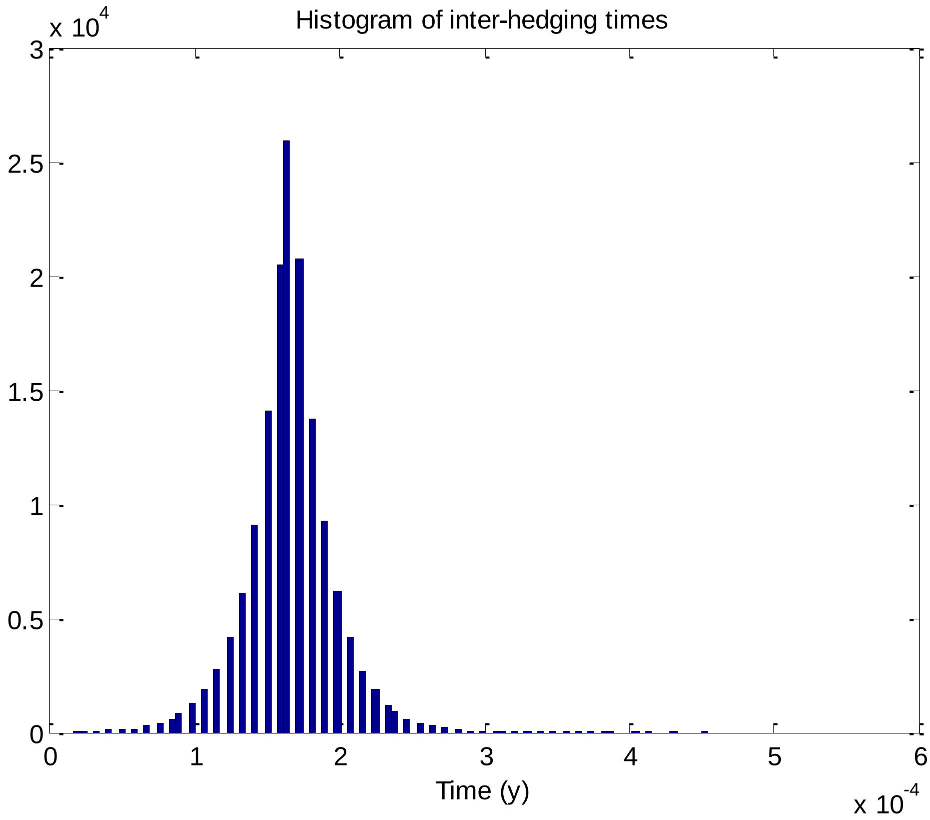

Figure 13: Histogram of Inter-hedging Times for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = \frac{1}{24.21.12;20}$ year, $t_{factor} = 20$ and $\Delta S_{min} = 0.1$

Keeping the same simulation parameters but increasing ΔS_{min} to $\Delta S_{min} = 1$ one will find the following histograms of Figure 14 and Figure 15. As one can see, we experienced huge losses for the left tail of the terminal P&L in this simulation, indicating that parameter ΔS_{min} was chosen to be much higher than ideally it must be to ensure an effective hedging strategy.

For comparison purposes, the execution ratio for the limit orders was equal to 1.383% in this case, indicating that the strategy hardly can be called a hedging strategy.

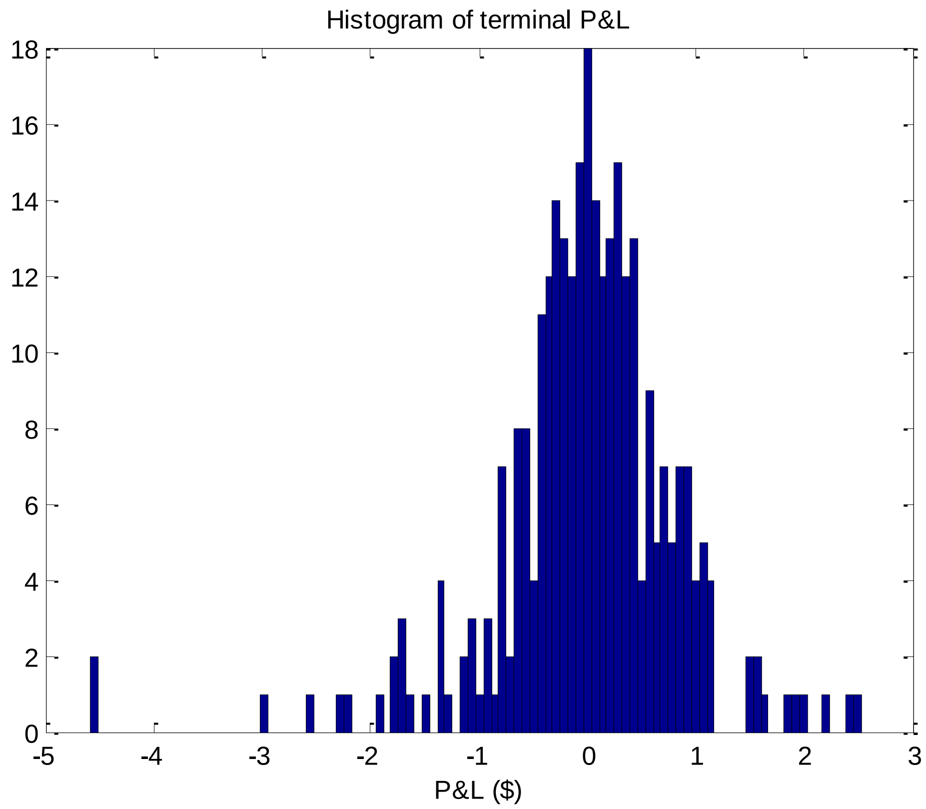

Figure 14: Histogram of Terminal P&L for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = \frac{1}{24.21.12;20}$ year, $t_{factor} = 20$ and $\Delta S_{min} = 1$

The histogram of the inter-execution times also provides an indicative measure of how the hedging is not effective in this case. As one can easily see, in some cases the inter-hedging times have reached values of 0.06 years, which is slightly less than one month – the maturity of this option. In other words, for some paths the upper and lower limit orders, even though placed for every hour, only were executed once or twice during the whole life of the option.

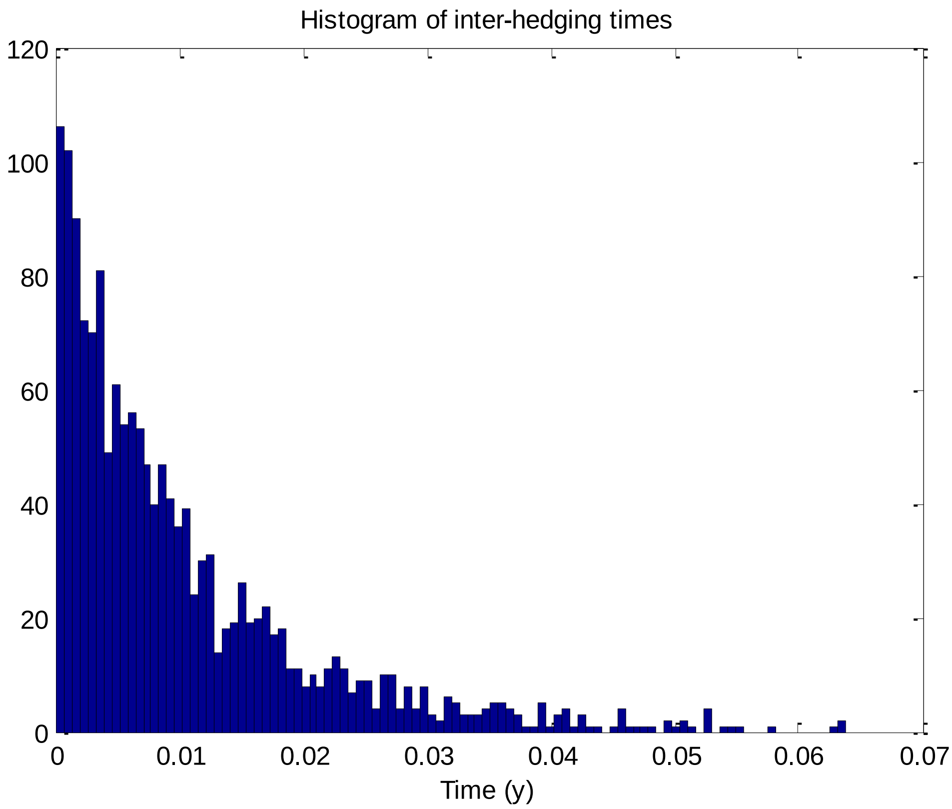

Figure 15: Histogram of Inter-hedging Times for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = \frac{1}{24.21.12;20}$ year, $t_{factor} = 20$ and $\Delta S_{min} = 1$

As an intermediate simulation, taking $\Delta S_{min} = 0.5$ leads to the results of Figure 16 and Figure 17. It is possible to see a better effectiveness (when compared with the last simulation), with a execution ratio of 28.90%

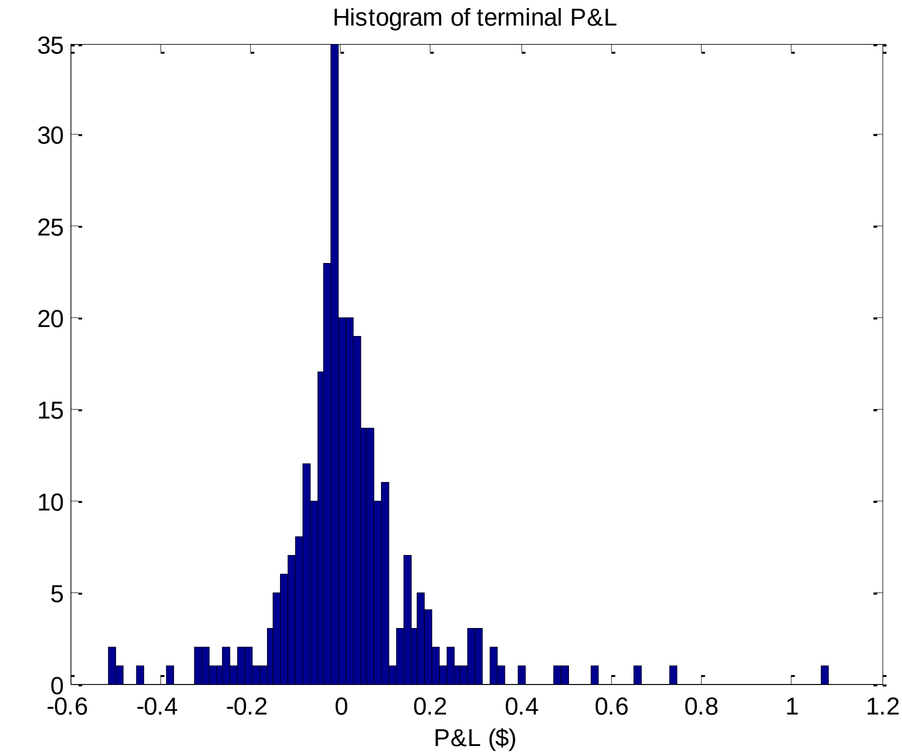

Figure 16: Histogram of Terminal P&L for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = \frac{1}{24.21.12;20}$ year, $t_{factor} = 20$ and $\Delta S_{min} = 0.5$

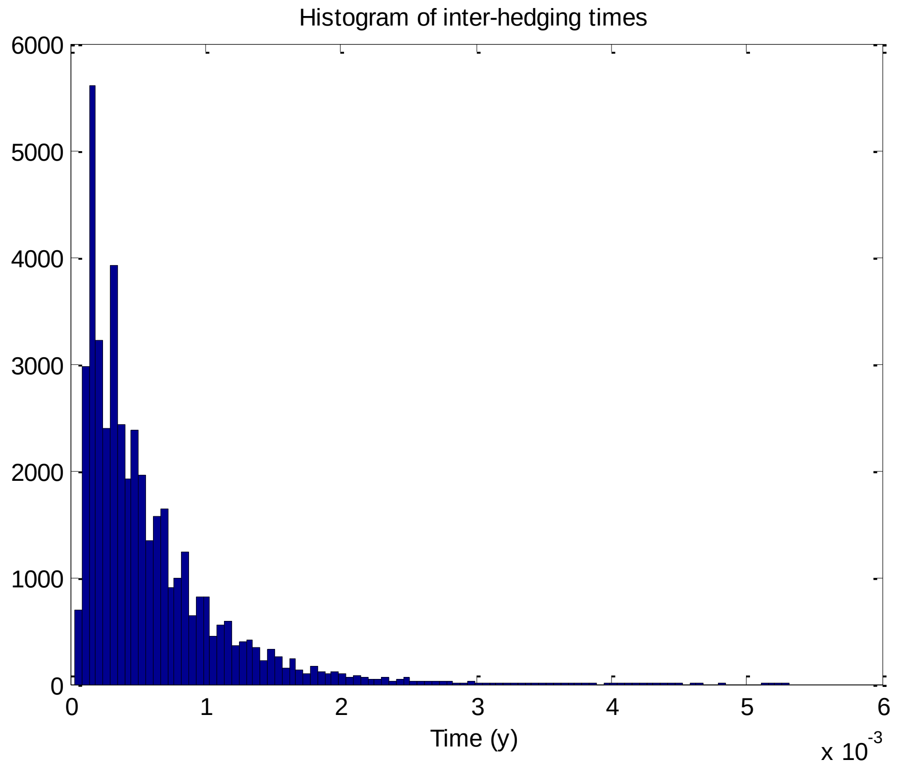

Figure 17: Histogram of Inter-hedging Times for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = \frac{1}{24.21.12;20}$ year, $t_{factor} = 20$ and $\Delta S_{min} = 0.5$

We can know focus at the parametric study of the hedging efficiency (similar parameter space that we considered for the first modeling approach). For the combination (or Cartesian product) of parameters $\Delta t_{orders} = \{0.5\text{h}, 1.0\text{h}, 1.5\text{h}, 2.0\text{h}, 3.0\text{h}\ 4.0\text{h}, 6.0\text{h}\}$ and $dS_{min} = \{0.05, 0.10, \dots, 0.1\}$ we run 140 (7 times 20) different simulations and plot the corresponding values of Expected Terminal P&L, Standard deviation of Terminal P&L, Conditional loss for Terminal P&L and, more important, Expected Inter-hedging times and Execution Ratio. These two last parameters are only possible to be studies because of the intermediate time simulations for the dynamics of the market prices. For each simulation, 500 paths were considered.

The following Figure 18 shows the expected terminal P&L for each point on the space $(\Delta t_{orders}, dS_{min})$. Observe that the terminal P&L is roughly zero for all simulations, as we already expected for this limit order strategy.

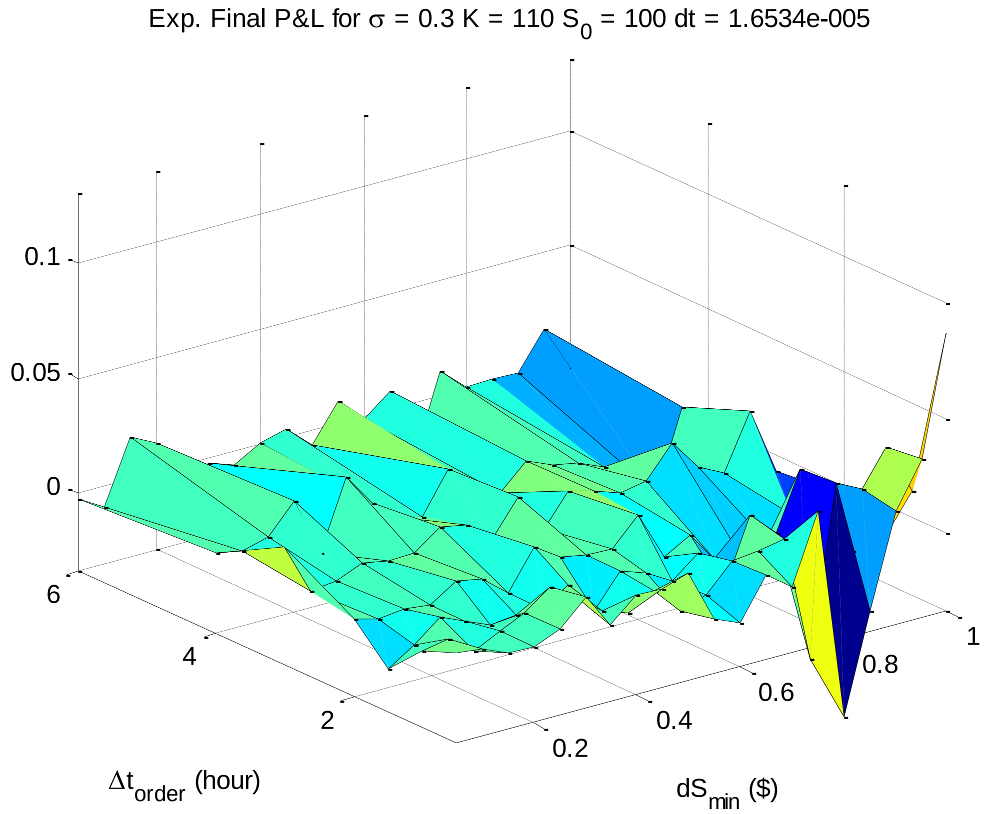

Figure 18: Expected value of terminal P&L - Second model

The standard deviation and the conditional loss of the terminal P&L, variables that measures the efficacy of the hedging strategy on its main purpose, are shown at Figure 19 and Figure 20. As we already expected, as we increase dS_{min} and decrease Δt_{orders} the number of orders actually executed decreases considerably and most of the limit orders are cancelled after Δt_{orders}, which makes the strategy to be not so different than a situation where the initial hedge is constructed at $t = 0$ but, after that, no re-hedging is performed.

Results for the expected inter-hedging times are represented by Figure 21 and Figure 22.

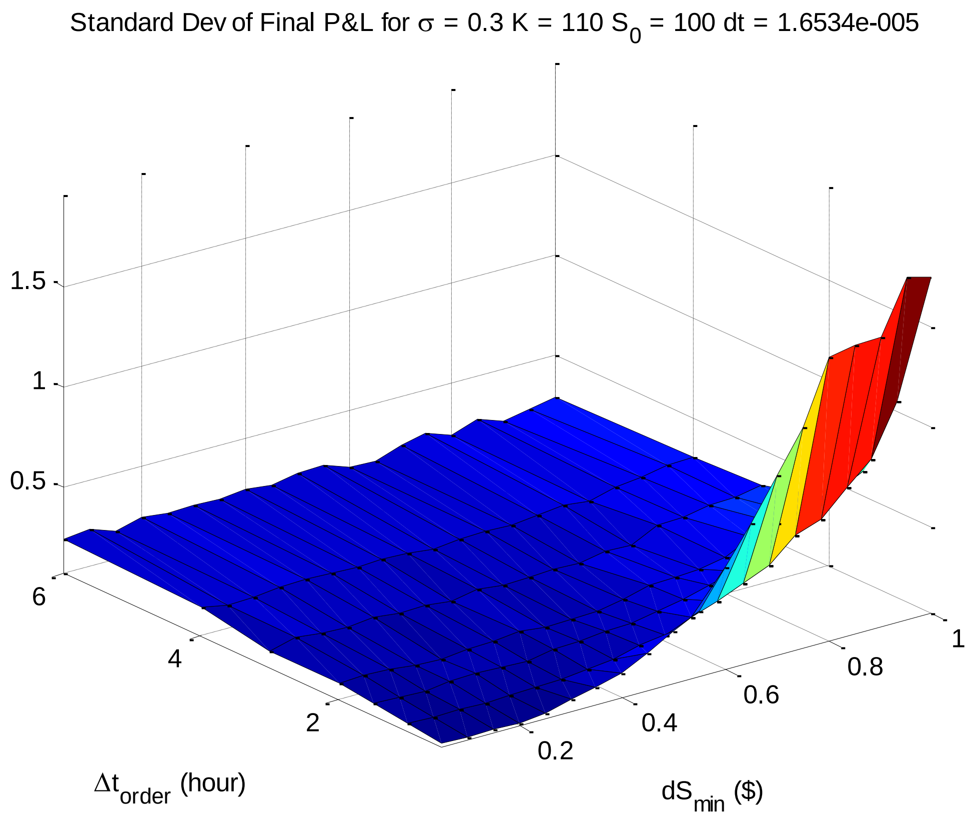

Figure 19: Standard Deviation of terminal P&L - Second model

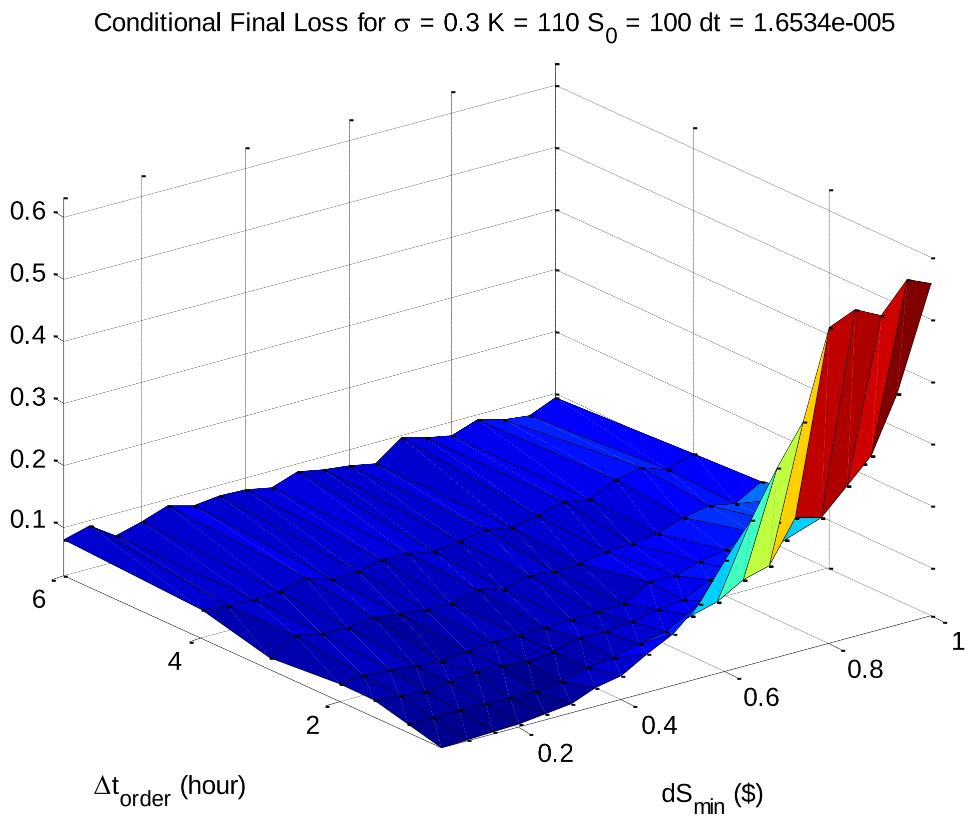

Figure 20: Conditional Loss of terminal P&L - Second model

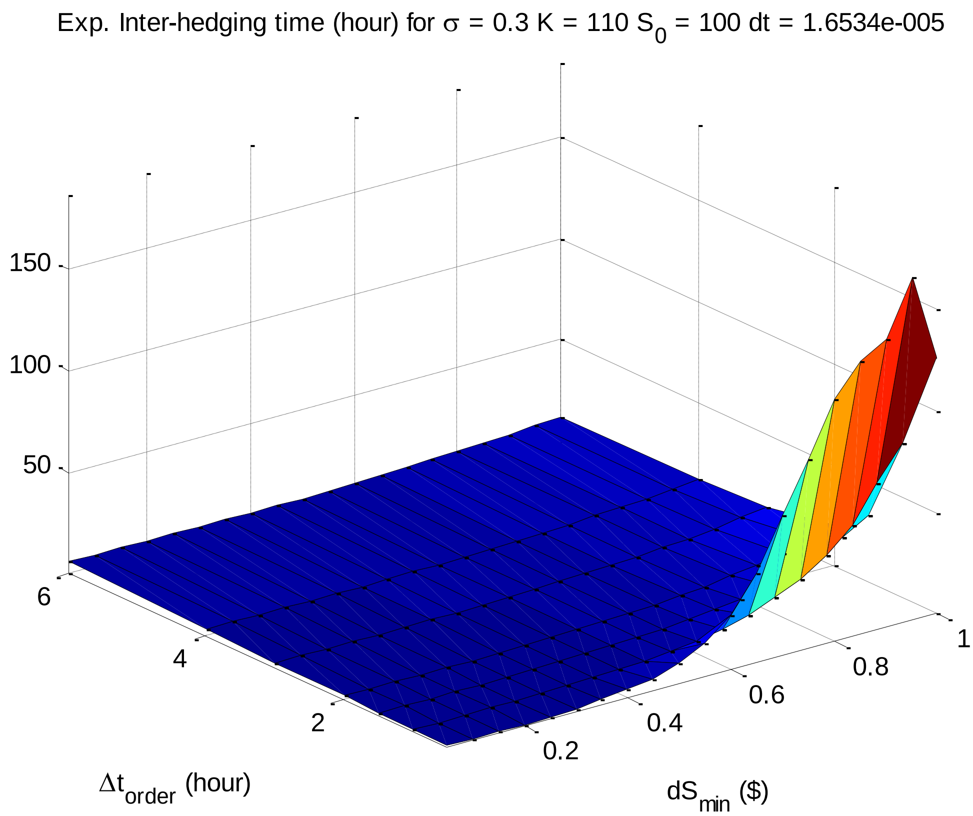

Figure 21: Expected Inter-hedging Times - Second model

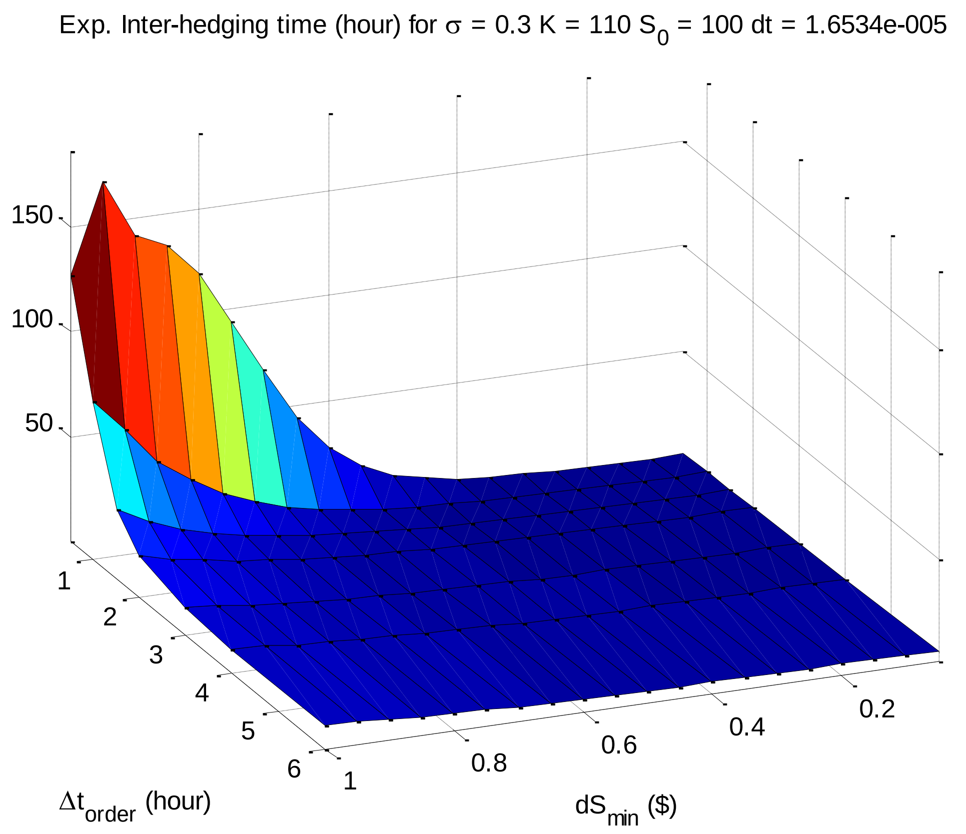

Figure 22: Expected Inter-hedging Times - Second model (Different Perspective)

As we observed from Figure 19 and Figure 20, the situation tends to the theoretical Black and Scholes hedging (continuously rebalancing) when we make $\Delta t_{orders} \to 0$ and $dS_{min} \to 0$. For lower values of dS_{min}, increasing values of Δt_{orders} actually worsen the hedging strategy (terminal standard deviation and conditional loss), since it does not change the execution ratio of the limit orders considerably (since it should be high even for low values of Δt_{orders}) but it makes the rebalancing to be not so frequent. On the other hand, for higher values of dS_{min} decreasing Δt_{orders} can definitely reduce the execution ratio – and therefore the efficiency of the strategy.

Figure 21 and Figure 22 depict that for lower values of dS_{min} the increase on Δt_{orders} hardly affects the inter-hedging times, since the limit orders are so close that most of the orders are executed in a very short time period after their placements (and before the placement of the consecutive limit orders).

The investigation of how the execution ratio is affected by the change on dS_{min} and Δt_{orders} and how it connects with the hedging efficiency is plotted by the following Figure 23 and Figure 24 (different perspective)

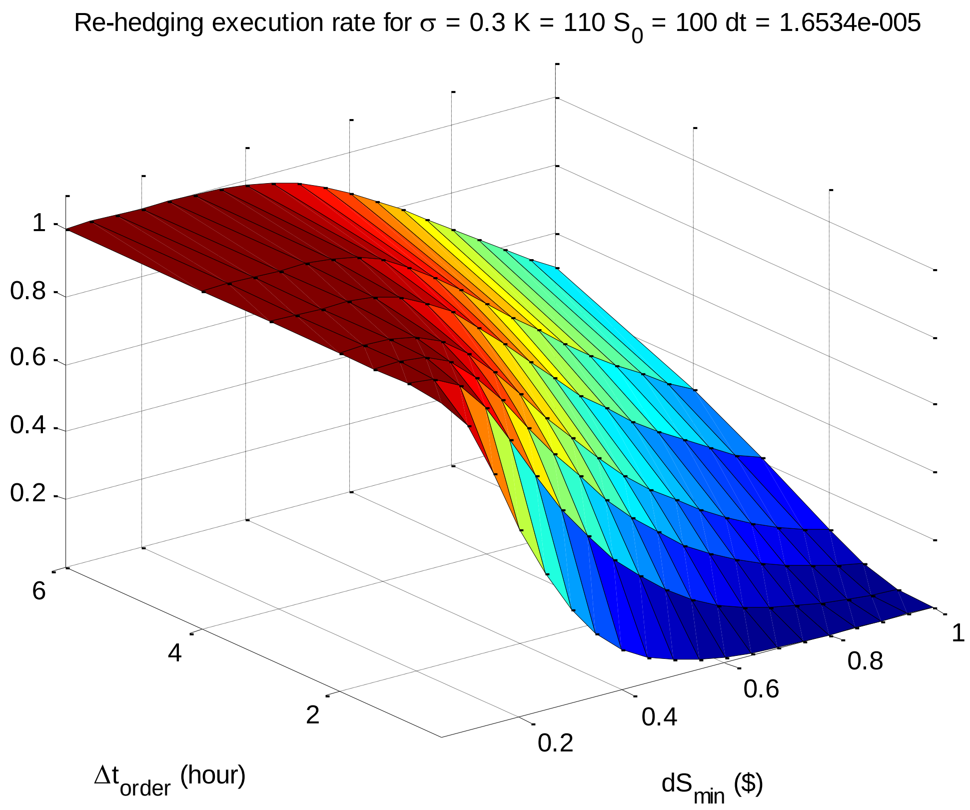

Figure 23: Expected Execution Ratio - Second model

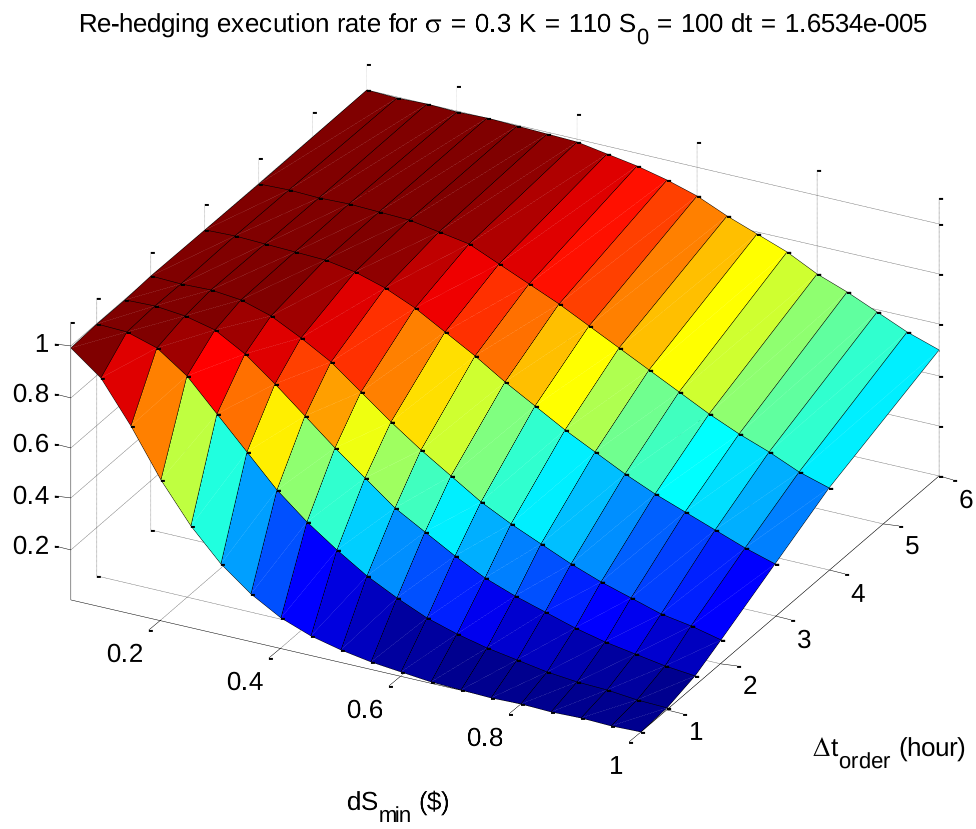

Figure 24: Expected Execution Ratio - Second model (Different Perspective)

As the intuition would suggest, the execution rate for the orders placed (either the lower or the upper) increases as we decrease dS_{min} and increase Δt_{orders}. By comparing the two plotted variables of Figure 24 (Execution Ratio) and Figure 19 (standard deviation of terminal P&L) one can see the following trade-off.

Ideally the trader wants to ensure lower values for the standard deviation of the terminal P&L, which means that he or she needs to avoid regions of high dS_{min} and low Δt_{orders}. In fact, the best case scenario for the terminal standard deviation will be simply to reduce dS_{min} and Δt_{orders} as much as possible. On the other hand, we can clearly see at Figure 24 that, for a fixed value of dS_{min} decreasing Δt_{orders} usually leads to lower execution ratios.

Finally, even though we have not included transaction costs into the modeling approach, it is known that in the real world an extremely high execution ration of the limit orders could lead to higher transaction costs. Hence, roughly speaking the trader must find a region whose terminal standard deviation does not exceed a limit that he or she considers acceptable and also whose execution ratio is not considerably high – due to transaction costs.

An illustrative way to study this trade off is to take the portfolio´s standard deviations and execution ratio results of all 140 simulations (related to pairs of dS_{min} and Δt_{orders}) and observe these points by the scatted plots of Figure 25.

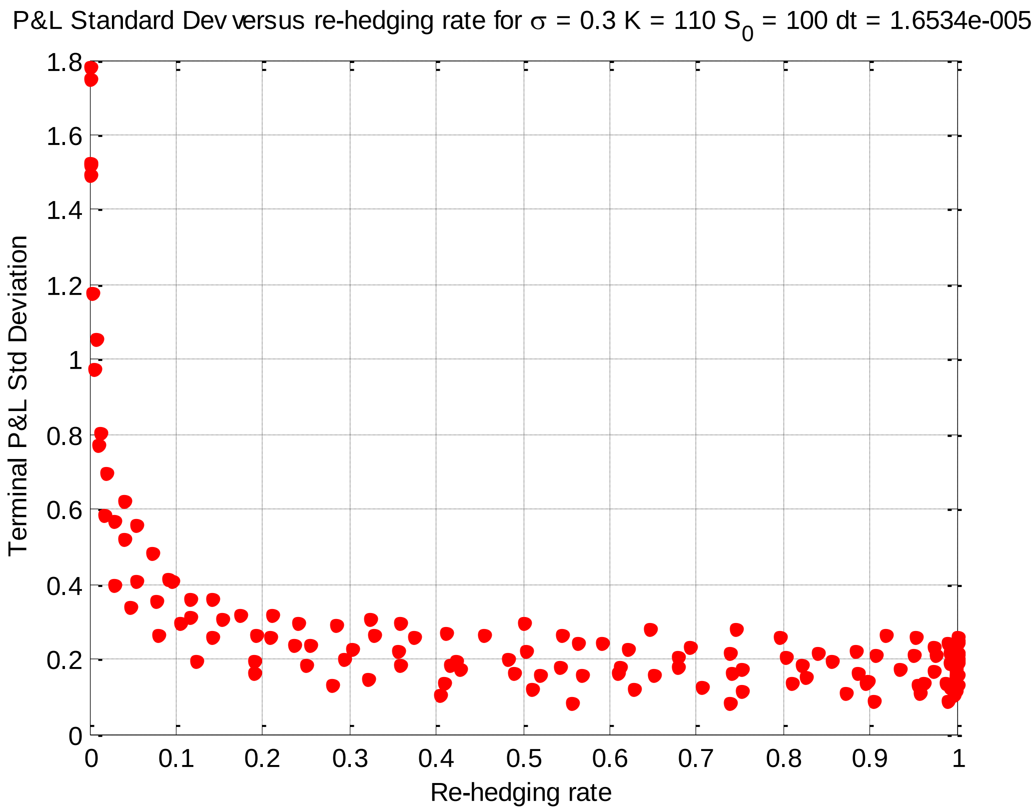

Figure 25: Trade-off Between Expected Execution Ratio and Portfolio´s Standard Deviation

Having this plot the trader could, for example, draw a horizontal line on the maximum standard deviation that is acceptable for this hedging strategy. By doing this he or she can simply select the point within this line that ensures the lower execution ratio.

5 - Work in progress

From practical views, taken transaction cost into consideration, it is not profitable to place limit order in a very high frequency. For next step, we are going to derive another trading strategy which is more similar to realistic trading activities. Instead of placing limit orders for each step of stock movement, we are going to place limit orders on fixed intervals (dTao). For example, the stock movement is based on minute level where the limit orders are only placed on hour levels. This time we are going to draw relationship between limit order intervals (dTao) and limit order levels (dSmin) and volatility of P&L. For each fixed dTao, we can calculated the dSmin that minimized the volatility of final P&L. Our goal for next step is to figure out the function between dSmin and dTao, so as to provide practical strategies for traders to choose the level of limit order based on their preference of limit order placement frequency.

Appendix – Additional Simulation Plots

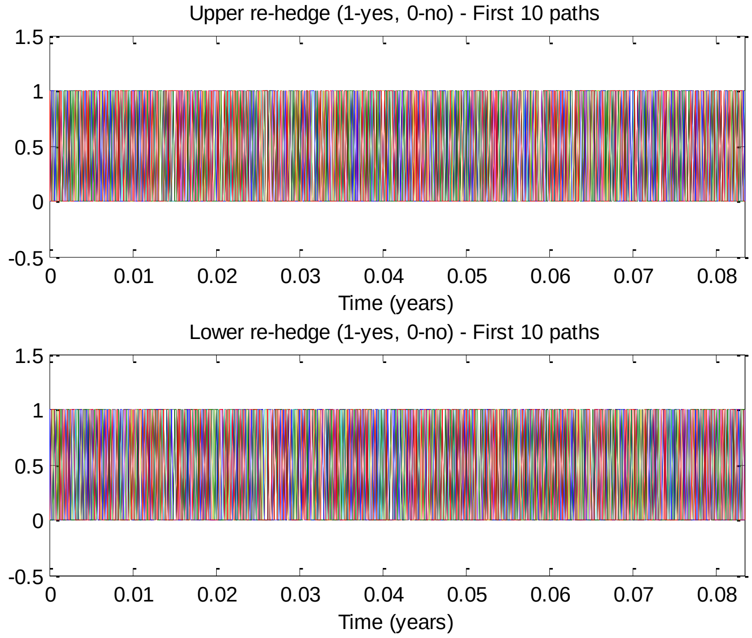

Figure A1: Indicator Variables for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = \frac{1}{24.21.12}$ year and $\Delta S_{min} = 0.1$

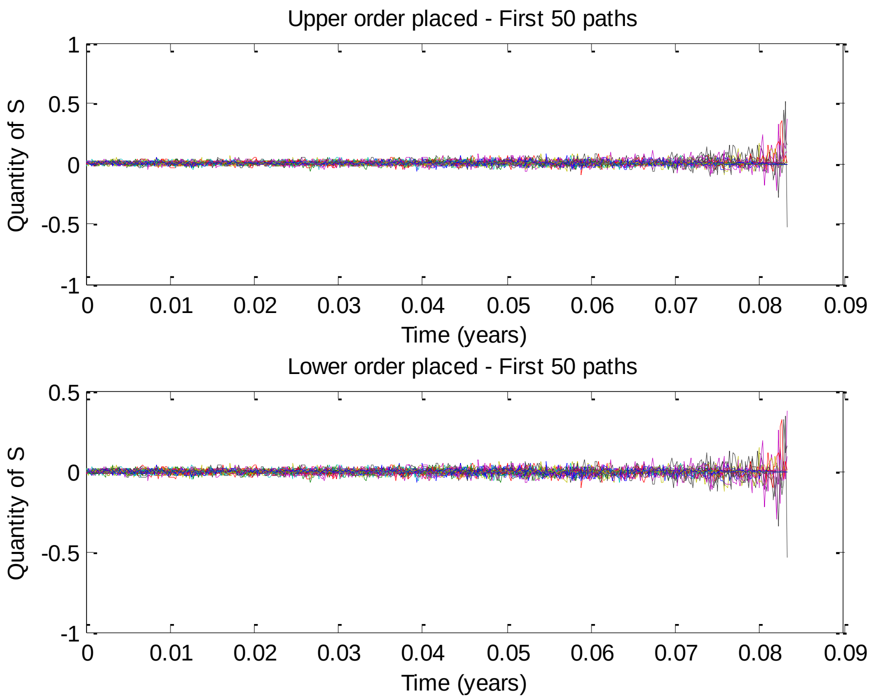

Figure A2: Limit Orders for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = 1/(24.21.12)$ year and $\Delta S_{min} = 0.1$

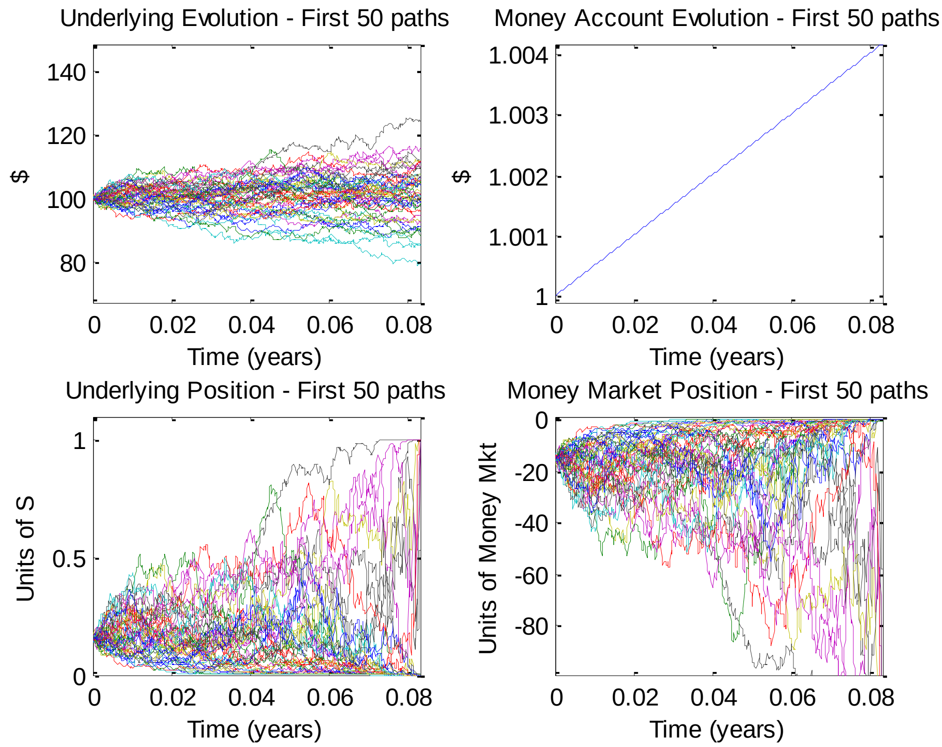

Figure A3: Market Prices for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = 1/(24.21.12)$ year and $\Delta S_{min} = 0.1$

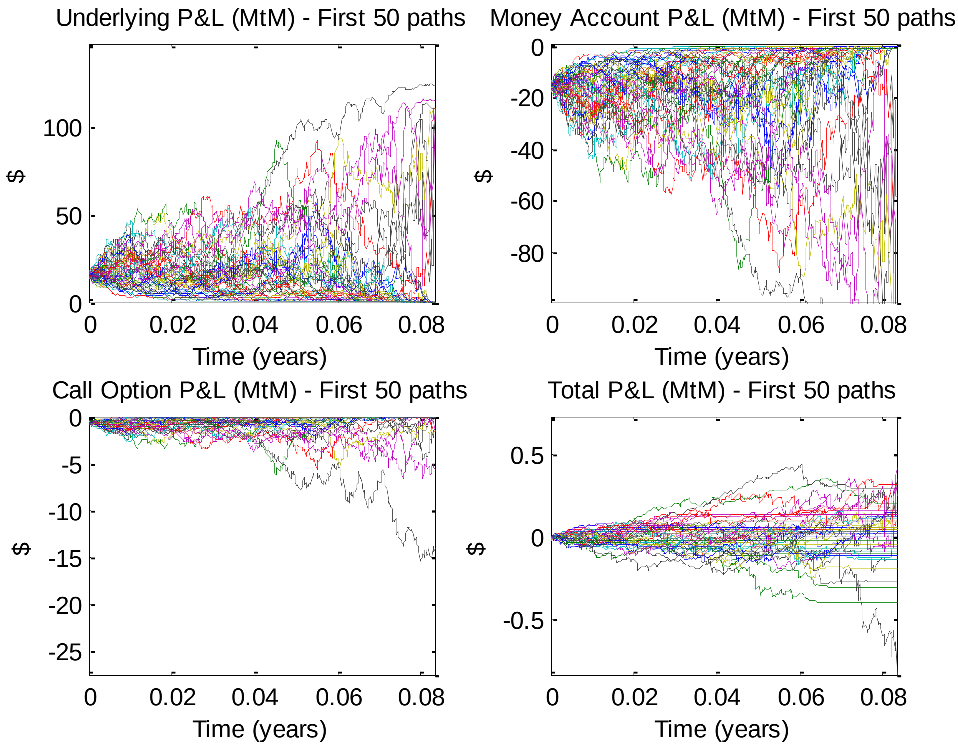

Figure A4: Market Exposures for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = 1/(24.21.12)$ year and $\Delta S_{min} = 0.1$

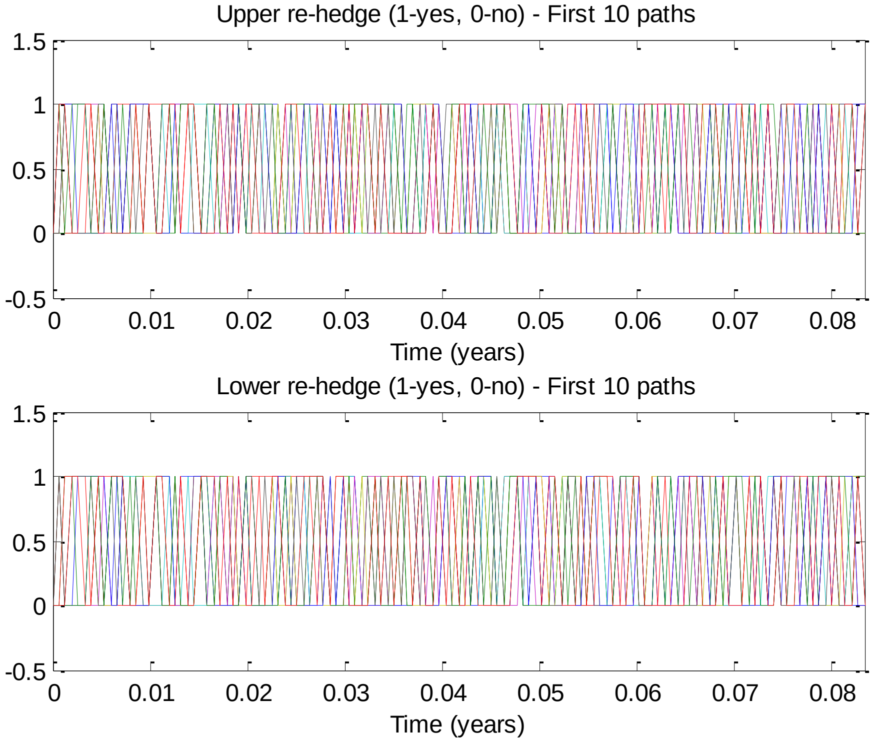

Figure A5: Indicator Variables for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = 4/(24.21.12)$ year and $\Delta S_{min} = 0.1$

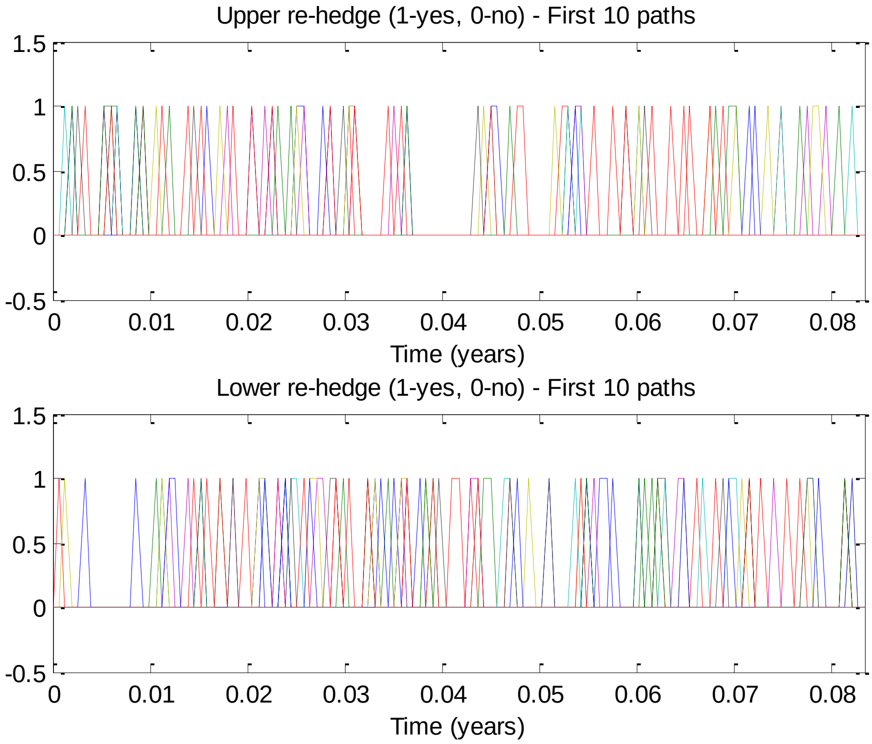

Figure A6: Indicator Variables for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = 4/(24.21.12)$ year and $\Delta S_{min} = 1$

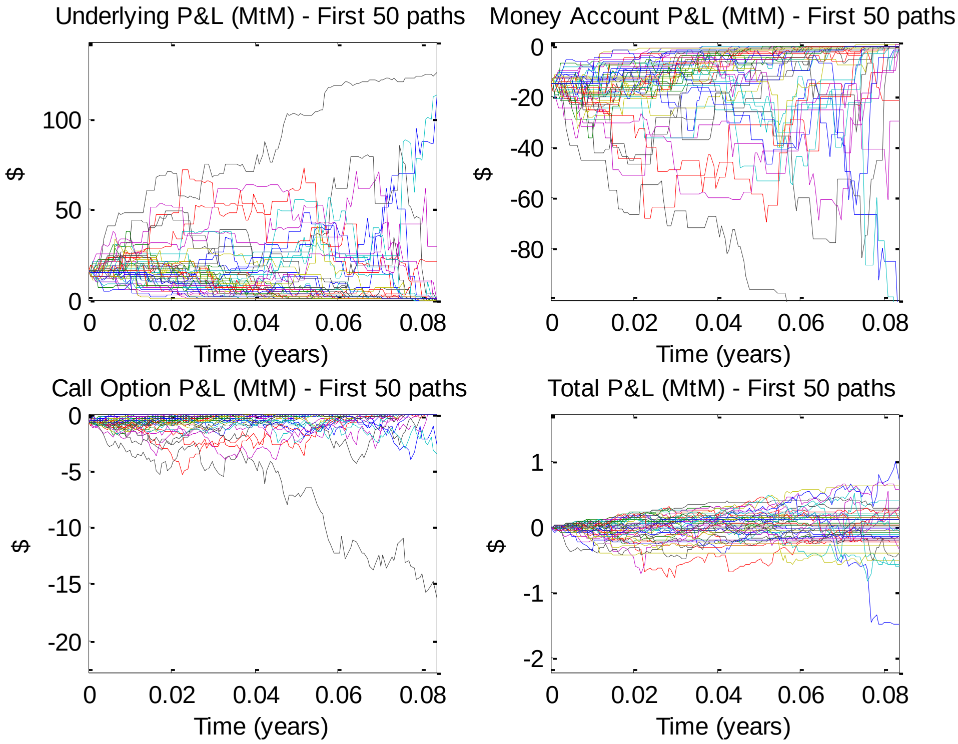

Figure A7: Market Exposures for $S_0 = 100$, $K = 110$, $\sigma = 30\%$, $r = 5\%$, $T = \frac{1}{12}$ year, $\Delta t = 4/(24.21.12)$ year and $\Delta S_{min} = 1$