

## Laplace Transform and Barrier Hitting Time Distributions

X. Sheldon Lin

Department of Statistics and Actuarial Science

University of Iowa

Iowa City, IA 52242

Email: shlin@stat.uiowa.edu

### 1 Introduction

Equity-indexed annuities (EIA's) are the fastest growing annuity products. An appealing feature of an EIA to its holder is that it contains path-dependent options which allow the policyholder to participate in favorable investment performance while maintaining a minimum guarantee on the benefit level. Studying the valuation problem of path-dependent options enables us to provide needed information to insurance companies on the costs of these options as well as investment strategy for their portfolios. For details, see Bacinello and Ortu (1993, 1994), and Bensman (1996).

Valuation of such options usually involves analysis of Brownian motion, which requires a good knowledge of stochastic processes theory and stochastic calculus. The martingale approach by Gerber and Shiu (1994, 1996) enables us to solve some valuation problems in option pricing without knowing deep results in stochastic calculus such as the Reflection Principle and Girsanov Theorem. What is needed is the Laplace transform of certain distributions known by actuaries.

In this paper, we show how to use Gerber and Shiu's approach to compute two defective density functions related to double barrier hitting probabilities of a geometric Brownian motion. The approach used here is simple and straightforward, and purely analytical. We then apply the formulas to value some exotic options whose payoffs are contingent on

barrier hitting time.

We begin with the basic properties of the Inverse Gaussian and the one-sided stable distribution. We then compute two double barrier hitting time distributions for a Brownian motion using the Gerber and Shiu method. The corresponding double barrier hitting time distributions are then derived for a geometric Brownian motion with drift. In the remaining sections we apply our results to various exotic options.

## 2 Inverse Gaussian distributions

Central to our discussion are two distributions: Inverse Gaussian distribution and one-sided stable distribution of index  $\frac{1}{2}$ . The first distribution has been used widely by actuaries to model claim distributions. The second distribution, although less familiar, is a limiting distribution of Inverse Gaussian distributions and well known as the first passage time of a standard Brownian motion. As we will see in later sections, certain barrier hitting time distributions can be decomposed into a series of Inverse Gaussian distributions and the price of certain path-dependent options can be expressed as a linear combination of one-sided stable distribution functions of index  $\frac{1}{2}$ .

The density function of an Inverse Gaussian distribution with shape parameter  $\alpha > 0$  and scale parameter  $\beta > 0$  is given by

$$f_{IG}(t) = \frac{\alpha}{\sqrt{2\pi\beta t^3}} e^{-\frac{1}{2\beta t}(\beta t - \alpha)^2}, \quad t > 0. \quad (1)$$

Let  $F_{IG}(t) = \int_0^t f_{IG}(y)dy$  be its distribution function (see Bower, et al. 1997, p. 39, and Panjer and Willmot, 1992, p. 114). Then

$$F_{IG}(t) = N\left(\frac{\beta t - \alpha}{\sqrt{\beta t}}\right) + e^{2\alpha} N\left(-\frac{\beta t + \alpha}{\sqrt{\beta t}}\right), \quad (2)$$

where  $N(x)$  is the standard normal distribution function. The Laplace transform of the

Inverse Gaussian distribution is

$$\int_0^\infty e^{-zt} f_{IG}(t) dt = e^{\alpha[1 - \sqrt{1 + 2z/\beta}]}, \quad z > 0. \quad (3)$$

The density function of a one-sided stable distribution of index  $\frac{1}{2}$  is

$$f(t; a) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}, \quad t > 0, \quad (4)$$

where parameter  $a > 0$  (see Feller, 1971, p. 52). A comparison with (1) indicates that this distribution is the limit of certain Inverse Gaussian distributions, if  $\alpha = a\sqrt{\beta}$  and  $\beta \rightarrow 0$ . Taking this limit in (3) we obtain the Laplace transform of the one-sided stable distribution of index  $\frac{1}{2}$ :

$$\int_0^\infty e^{-zt} f(t; a) dt = e^{-a\sqrt{2z}}, \quad z > 0. \quad (5)$$

For valuation purposes, we are also interested in

$$\int_0^T e^{-zt} f(t; a) dt, \quad \text{for some } T > 0.$$

To calculate this integral, we write the integrand as

$$e^{-a\sqrt{2z}} \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{(2zt - a\sqrt{2z})^2}{4zt}}.$$

That is, the integrand is  $e^{-a\sqrt{2z}}$  times the Inverse Gaussian density with parameters  $\alpha = a\sqrt{2z}$ ,  $\beta = 2z$ . From this and (2) it follows that

$$\int_0^T e^{-zt} f(t; a) dt = e^{-a\sqrt{2z}} N\left(\frac{\sqrt{2z}T - a}{\sqrt{T}}\right) + e^{a\sqrt{2z}} N\left(-\frac{\sqrt{2z}T + a}{\sqrt{T}}\right). \quad (6)$$

Finally, we extend the function (4) to negative  $a$ . For  $a < 0$ , we define  $f_a(t) = -f_{|a|}(t)$ . Thus, for  $a < 0$ ,

$$\int_0^T e^{-zt} f(t; a) dt = -e^{a\sqrt{2z}} N\left(\frac{\sqrt{2z}T + a}{\sqrt{T}}\right) - e^{-a\sqrt{2z}} N\left(-\frac{\sqrt{2z}T - a}{\sqrt{T}}\right). \quad (7)$$

### 3 Double Barrier Hitting Time Distributions

In this section we begin with three barrier hitting times of a Brownian motion. We assume that there is an upper barrier and a lower barrier. The first hitting time is defined as the first time the Brownian motion hits one of the two barriers, the second one is the first time the Brownian motion hits the lower barrier without hitting the upper barrier earlier, and the third one is the first time the Brownian motion hits the upper barrier without hitting the lower barrier earlier.

Let  $\{W(t), t > 0\}$  be a standard Brownian motion and  $X(t)$  be a Brownian motion with drift parameter  $\mu$  and diffusion parameter  $\sigma$ , i.e.,  $X(t) = \mu t + \sigma W(t)$ . the parameters  $\mu$  and  $\sigma$  are assumed to be arbitrary nonnegative constants. Let  $a < 0 < b$ . Define

$$T_{a,b} = \begin{cases} \inf\{t; X(t) = a, \text{ or } X(t) = b\}, & \text{if such a } t \text{ exists} \\ \infty, & \text{if } X(t) \text{ never hits the barriers.} \end{cases} \quad (8)$$

Here  $T_{a,b}$  is the first time the Brownian motion  $\{X(t)\}$  hits one of two barriers  $x = a$  and  $x = b$ . Further, define

$$T_a = T_{a,b}, \text{ if } X(T_{a,b}) \neq b; \quad T_b = T_{a,b}, \text{ if } X(T_{a,b}) \neq a. \quad (9)$$

Then,  $T_a$  is the first time  $\{X(t)\}$  hits the lower barrier  $x = a$  without hitting  $x = b$  earlier, while  $T_b$  is the first time  $\{X(t)\}$  hits the upper barrier  $x = b$  without hitting  $x = a$  earlier.

In the following, we will identify the defective densities of  $T_a$  and  $T_b$ . The density of  $T_{a,b}$  then is the sum of these two.

We first compute the Laplace transforms of  $T_a$  and  $T_b$  using the Gerber-Shiu Technique. We then express them as a series of the Laplace transforms of stable distributions.

It is well know that for any real  $\lambda$ , the exponential

$$Z_\lambda(t) = e^{\lambda X(t) - \lambda \mu t - \frac{1}{2} \lambda^2 \sigma^2 t}$$

is a martingale. For an arbitrary  $z > 0$ , let  $\lambda_1$  and  $\lambda_2$  be the negative root and the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2\lambda^2 + \mu\lambda - z = 0,$$

respectively. Then,

$$\lambda_1 = \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 z}}{\sigma^2}, \quad \lambda_2 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 z}}{\sigma^2}. \quad (10)$$

We obtain two martingales:

$$M_1(t) = Z_{\lambda_1}(t) = e^{\lambda_1 X(t) - zt} \quad (11)$$

and

$$M_2(t) = Z_{\lambda_2}(t) = e^{\lambda_2 X(t) - zt}. \quad (12)$$

By the optional sampling theorem, we have

$$E[M_i(T_{a,b})] = 1, \quad i = 1, 2.$$

It follows from the law of total probability that

$$\begin{aligned} E(e^{-zT_a})e^{a\lambda_1} + E(e^{-zT_b})e^{b\lambda_1} &= 1, \\ E(e^{-zT_a})e^{a\lambda_2} + E(e^{-zT_b})e^{b\lambda_2} &= 1. \end{aligned}$$

Solving the linear equations above yields the Laplace transforms

$$E(e^{-zT_a}) = \frac{e^{b\lambda_2} - e^{b\lambda_1}}{e^{a\lambda_1 + b\lambda_2} - e^{b\lambda_1 + a\lambda_2}} \quad (13)$$

and

$$E(e^{-zT_b}) = \frac{e^{a\lambda_1} - e^{a\lambda_2}}{e^{a\lambda_1 + b\lambda_2} - e^{b\lambda_1 + a\lambda_2}}. \quad (14)$$

Since

$$\frac{1}{e^{a\lambda_1 + b\lambda_2} - e^{b\lambda_1 + a\lambda_2}} = e^{-a\lambda_1 - b\lambda_2} \sum_{n=0}^{\infty} e^{-n(b-a)(\lambda_2 - \lambda_1)},$$

we have

$$E(e^{-zT_a}) = \sum_{n=0}^{\infty} e^{-n(b-a)(\lambda_2-\lambda_1)-a\lambda_1} = \sum_{n=0}^{\infty} e^{-(n+1)(b-a)(\lambda_2-\lambda_1)-a\lambda_2}, \quad (15)$$

$$E(e^{-zT_b}) = \sum_{n=0}^{\infty} e^{-n(b-a)(\lambda_2-\lambda_1)-b\lambda_2} = \sum_{n=0}^{\infty} e^{-(n+1)(b-a)(\lambda_2-\lambda_1)-b\lambda_1}. \quad (16)$$

We now identify the defective density functions of  $T_a$  and  $T_b$  through the Laplace transforms (15) and (16). To do so, we first identify functions that correspond to the terms of the series in (15) and (16). Let  $a_n = \frac{1}{\sigma}[2n(b-a) - a]$  and  $b_n = \frac{1}{\sigma}[2n(b-a) + b]$ , for  $n = \dots, -2, -1, 0, 1, 2, \dots$ , and let  $g_a(t)$  and  $g_b(t)$  represent the defective density function of  $T_a$  and  $T_b$ , respectively. From (10), the terms of the first series in (15) can be rewritten as

$$\begin{aligned} e^{-n(b-a)(\lambda_2-\lambda_1)-a\lambda_1} &= e^{\frac{a\mu}{\sigma^2} - \frac{a n \mu}{\sigma}} \sqrt{1 + \frac{2\sigma^2 z}{\mu^2}} \\ &= e^{\frac{a\mu}{\sigma^2} - \frac{a n \mu}{\sigma}} e^{\frac{a n \mu}{\sigma}} \left[1 - \sqrt{1 + \frac{2\sigma^2 z}{\mu^2}}\right]. \end{aligned}$$

The second factor above is the Laplace transform of the Inverse Gaussian distribution with parameters  $\alpha = \frac{a n \mu}{\sigma}$ ,  $\beta = \frac{\mu^2}{\sigma^2}$ . Thus, it follows from (1) and (3) that the corresponding function to this expression is

$$e^{\frac{a\mu}{\sigma^2} - \frac{a n \mu}{\sigma}} \frac{a_n}{\sqrt{2\pi t^3}} e^{-\frac{(\mu t - a n \sigma)^2}{2\sigma^2 t}} = e^{\frac{a\mu}{\sigma^2} - \frac{1}{2}(\frac{\mu}{\sigma})^2 t} f(t; a_n), \quad n = 0, 1, \dots. \quad (17)$$

For expressions  $e^{-(n+1)(b-a)(\lambda_2-\lambda_1)-a\lambda_2}$ ,  $n = 0, 1, \dots$ , let  $m = -n-1$ . Then  $m = -1, -2, \dots$  and we have

$$e^{-(n+1)(b-a)(\lambda_2-\lambda_1)-a\lambda_2} = e^{\frac{a\mu}{\sigma^2} + \frac{a m \mu}{\sigma}} e^{-\frac{a m \mu}{\sigma}} \left[1 - \sqrt{1 + \frac{2\sigma^2 z}{\mu^2}}\right].$$

Thus, the functions corresponding to the terms in the second series in (15) are

$$e^{\frac{a\mu}{\sigma^2} + \frac{a m \mu}{\sigma}} \frac{a_m}{\sqrt{2\pi t^3}} e^{-\frac{(\mu t + a m \sigma)^2}{2\sigma^2 t}} = e^{\frac{a\mu}{\sigma^2} - \frac{1}{2}(\frac{\mu}{\sigma})^2 t} f(t; a_m), \quad m = -1, -2, \dots. \quad (18)$$

Together with (17) and (18), we obtain the density of  $T_a$ :

$$g_a(t) = e^{\frac{a\mu}{\sigma^2} - \frac{1}{2}(\frac{\mu}{\sigma})^2 t} \sum_{n=-\infty}^{\infty} f(t; a_n), \quad (19)$$

where  $f(t; u_n)$  is defined in (4).

A similar argument yields

$$g_b(t) = e^{\frac{2\mu}{\sigma^2} - \frac{1}{2}(\frac{\mu}{\sigma})^2 t} \sum_{n=-\infty}^{\infty} f(t; b_n). \quad (20)$$

To apply the above results to option pricing, we consider a geometric Brownian motion as the price of a stock under the risk-neutral probability measure. Under a Black-Scholes economy with one stock with price process  $\{S(t)\}$  and one riskfree bond with constant force of interest  $r$ , it is well know that, if the stock pays no dividends, the price process  $\{S(t)\}$  under the risk-neutral probability measure follows a geometric Brownian motion

$$S(t) = S e^{(r - \frac{1}{2}\sigma^2)t - \sigma W(t)}, \quad (21)$$

where  $S$  is the initial stock price at time 0 and  $\{W(t)\}$  is a standard Brownian motion. In this case,  $S(t) = S e^{X(t)}$ ,  $X(t)$  is a Brownian motion with drift parameter

$$\mu = r - \frac{1}{2}\sigma^2 \quad (22)$$

and diffusion parameter  $\sigma$ .

Let  $L$  and  $U$  be a lower barrier and an upper barrier for  $\{S(t)\}$  with  $0 < L < S < U$ . Denote  $\tau_L$  and  $\tau_U$  as the first hitting time to the lower barrier without hitting the upper barrier earlier and the first hitting time to the upper barrier without hitting the lower barrier earlier, respectively, i.e.,

$$\tau_L = \inf\{t; S(t) = L, L < S(s) < U \text{ for all } s \in [0, t)\},$$

$$\tau_U = \inf\{t; S(t) = U, L < S(s) < U \text{ for all } s \in [0, t)\}.$$

To find the density for  $\tau_L$  and  $\tau_U$ , let

$$a = \ln \left[ \frac{L}{S} \right], \quad b = \ln \left[ \frac{U}{S} \right]. \quad (23)$$

Thus, the density  $g_L(t; S)$  of  $\tau_L$  is  $g_a(t)$ . It follows from (19) and (22) that

$$g_L(t; S) = \left[ \frac{L}{S} \right]^{(r\sigma^{-2} + \frac{1}{2})} e^{-\frac{1}{2}(r\sigma^{-1} - \frac{1}{2}\sigma)^2 t} \sum_{n=-\infty}^{\infty} f(t; a_n). \quad (24)$$

Similarly, it follow from (20) and (22) that the density  $g_U(t; S)$  of  $\tau_U$  is

$$g_U(t; S) = \left[ \frac{U}{S} \right]^{(r\sigma^{-2} - \frac{1}{2})} e^{-\frac{1}{2}(r\sigma^{-1} + \frac{1}{2}\sigma)^2 t} \sum_{n=-\infty}^{\infty} f(t; b_n). \quad (25)$$

Here  $a_n = \frac{1}{\sigma} \ln \frac{L(2n+1)}{L^{2n+1}}$ , and  $b_n = \frac{1}{\sigma} \ln \frac{U(2n+1)}{L^{2n+1}}$ .

## 4 Pricing Single Barrier Options

In this section we consider exotic options whose payoffs depend on the hitting time distribution of their underlying asset price to a single barrier. Although the single barrier hitting probabilities can be derived by the reflection principle, they are simply a special case of our results in the last section.

Two cases are considered: the barrier level is less than the initial price of the underlying asset, i.e.,  $L < S$ , and the barrier level is greater than the initial price of the underlying,  $S < U$ . We denote  $g_L(t)$  for the density of the former and  $g_U(t)$  for the density of the latter. It is obvious that

$$g_L(t; S) = \lim_{U \rightarrow \infty} g_L(t; S), \text{ and } g_U(t; S) = \lim_{L \rightarrow 0} g_U(t; S),$$

where  $g_U(t; S)$  and  $g_L(t; S)$  are given in (24) and (25). Note that for  $n > 0$ ,

$$\lim_{U \rightarrow \infty} a_n = \infty, \quad \lim_{L \rightarrow 0} b_n = \infty;$$

for  $n < 0$ ,

$$\lim_{U \rightarrow \infty} a_n = -\infty, \quad \lim_{L \rightarrow 0} b_n = -\infty;$$

and

$$a_0 = \frac{1}{\sigma} \ln \left[ \frac{S}{L} \right], \quad b_0 = \frac{1}{\sigma} \ln \left[ \frac{U}{S} \right].$$



Thus, we have

$$g_l(t; S) = \left[ \frac{L}{S} \right]^{(r\sigma^{-2} - \frac{1}{2})} e^{-\frac{1}{2}(r\sigma^{-1} - \frac{1}{2}\sigma)^2 t} f(t; a_0), \quad (26)$$

and

$$g_u(t; S) = \left[ \frac{U}{S} \right]^{(r\sigma^{-2} - \frac{1}{2})} e^{-\frac{1}{2}(r\sigma^{-1} - \frac{1}{2}\sigma)^2 t} f(t; b_0). \quad (27)$$

In the following we use the density functions obtained above to evaluate continuous lookback options. Lookback options have been used in several equity-indexed annuities. For example, an EIA which indexes on highest daily point of S&P 500 over 6 years contains a lookback option with fixed strike price over 6 years.

In this section, we always assume the Black-Scholes economy with one stock and one riskfree bond as discussed in section 3. The stock pays no dividends. Consider a European lookback call which pays at maturity  $T$

$$\max \left[ \max_{0 \leq t \leq T} S(t) - K, 0 \right]. \quad (28)$$

The price of this lookback option is

$$e^{-rT} E \left\{ \max \left[ \max_{0 \leq S(t) \leq T} S(t) - K, 0 \right] \right\}. \quad (29)$$

The expectation is taken under the risk-neutral probability measure. To evaluate, we need to identify the distribution of  $\max_{0 \leq S(t) \leq T} S(t)$ . Define

$$\overline{G}_1(U) = \begin{cases} \int_0^T g_u(t; S) dt, & \text{if } U \geq S \\ 1, & \text{if } U < S \end{cases} \quad (30)$$

Then  $\overline{G}_1(U) = \Pr\{\max_{0 \leq S(t) \leq T} S(t) \geq U\}$  is the survival function of  $\max_{0 \leq S(t) \leq T} S(t)$ . Thus,

$$E \left\{ \max \left[ \max_{0 \leq S(t) \leq T} S(t) - K, 0 \right] \right\} = \int_K^\infty \overline{G}_1(x) dx, \quad (31)$$

which is analogous to the formula for the stop-loss premium in insurance.

It follows from (6) that

$$\begin{aligned} & \int_0^T g_u(t; S) dt \\ &= N\left(\frac{\ln \frac{S}{U} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) + \left[\frac{S}{U}\right]^{1-2r\sigma^{-2}} N\left(\frac{\ln \frac{S}{U} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right). \end{aligned} \quad (32)$$

Denote

$$d(x, y) = \frac{\ln \frac{x}{y} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}. \quad (33)$$

Then, if  $K \geq S$ , the integration of the first term on the right hand side of (32) yields

$$\begin{aligned} \int_K^\infty N(d(x, S)) dx &= \int_K^\infty \int_{-\infty}^{d(x, S)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy dx \\ &= \int_{-\infty}^{d(K, S)} \int_K^{Se^{-\sigma\sqrt{T}y + (r - \frac{1}{2}\sigma^2)T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dx dy \\ &= \int_{-\infty}^{d(K, S)} \left[ Se^{-\sigma\sqrt{T}y + (r - \frac{1}{2}\sigma^2)T} - K \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\ &= e^{rT} S N\left(\frac{\ln \frac{S}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - K N\left(\frac{\ln \frac{S}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right), \end{aligned} \quad (34)$$

and the integration of the second term of (32) yields

$$\begin{aligned} & \int_K^\infty \left[\frac{S}{x}\right]^{1-2r\sigma^{-2}} N\left(d(x, S) - 2(r\sigma^{-1} - \frac{1}{2}\sigma)\sqrt{T}\right) dx \\ &= S^{1-2r\sigma^{-2}} \int_{-\infty}^{d(K, S) - 2(r\sigma^{-1} - \frac{1}{2}\sigma)\sqrt{T}} \int_K^{Se^{-\sigma\sqrt{T}y - (r - \frac{1}{2}\sigma^2)T}} x^{2r\sigma^{-2}-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dx dy \\ &= \frac{\sigma^2}{2r} S^{1-2r\sigma^{-2}} \int_{-\infty}^{d(K, S) - 2(r\sigma^{-1} - \frac{1}{2}\sigma)\sqrt{T}} \left[ S^{2r\sigma^{-2}} e^{-2r\sigma^{-1}\sqrt{T}y - 2r\sigma^{-2}(r - \frac{1}{2}\sigma^2)T} - K^{2r\sigma^{-2}} \right] \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy \\ &= \frac{\sigma^2}{2r} e^{rT} S N\left(\frac{\ln \frac{S}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - \frac{\sigma^2}{2r} S^{1-2r\sigma^{-2}} K^{2r\sigma^{-2}} N\left(\frac{\ln \frac{S}{K} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right). \end{aligned} \quad (35)$$

Together with (34) and (35), we have from (29) that the price of this lookback is

$$\begin{aligned} & \left(1 + \frac{\sigma^2}{2r}\right) S N\left(\frac{\ln \frac{S}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\ & - e^{-rT} K N\left(\frac{\ln \frac{S}{K} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) - \frac{\sigma^2}{2r} e^{-rT} S^{1-2r\sigma^{-2}} K^{2r\sigma^{-2}} N\left(\frac{\ln \frac{S}{K} - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \end{aligned} \quad (36)$$

If  $K < S$ , the price is

$$e^{-rT} \int_K^\infty \bar{G}_1(x) dx = e^{-rT} (S - K) + e^{-rT} \int_S^\infty \bar{G}_1(x) dx. \quad (37)$$

The second term on the right hand side of (37) is given by (36) with  $K = S$ .

## 5 Pricing Double Barrier Options

In this section, we consider a double lookback option whose payoff is contingent on the spread between the maximum and the minimum of a stock. It pays

$$\max \left[ \max_{0 \leq t \leq T} S(t) - \min_{0 \leq t \leq T} S(t) - K, 0 \right]$$

at maturity  $T$ . The value of this option is

$$e^{-rT} E \left\{ \max \left[ \max_{0 \leq t \leq T} S(t) - \min_{0 \leq t \leq T} S(t) - K, 0 \right] \right\}, \quad (38)$$

where  $K$  is the strike price.

Denote

$$\bar{G}(U, L) = \int_0^T g_U(t; S) dt + \int_0^T g_L(t; S) dt, \quad L \leq S \leq U.$$

Then

$$\bar{G}(U, L) = \Pr \left\{ \max_{0 \leq t \leq T} S(t) \geq U \text{ or } \min_{0 \leq t \leq T} S(t) \leq L \right\}.$$

Thus,

$$\begin{aligned} & E \left\{ \max \left[ \max_{0 \leq t \leq T} S(t) - \min_{0 \leq t \leq T} S(t) - K, 0 \right] \right\} \\ &= \int_0^S \int_S^\infty \max[x - y - K, 0] \frac{\partial^2 \bar{G}(x, y)}{\partial x \partial y} dx dy \\ &= \int_0^S \int_{\max(S, y+K)}^\infty (x - y - K) \frac{\partial^2 \bar{G}(x, y)}{\partial x \partial y} dx dy. \end{aligned}$$

If  $S \leq K$ , integration by parts and the fact that  $\bar{G}_2(y) = \lim_{x \rightarrow \infty} \bar{G}(x, y)$  yield

$$\begin{aligned}
& E \left\{ \max \left[ \max_{0 \leq t \leq T} S(t) - \min_{0 \leq t \leq T} S(t) - K, 0 \right] \right\} \\
&= \int_0^S \int_{y+K}^{\infty} \frac{\partial [\bar{G}_2(y) - \bar{G}(x, y)]}{\partial y} dx dy \\
&= \int_K^{S+K} \int_0^{x-K} \frac{\partial [\bar{G}_2(y) - \bar{G}(x, y)]}{\partial y} dy dx + \int_{S+K}^{\infty} \int_0^S \frac{\partial [\bar{G}_2(y) - \bar{G}(x, y)]}{\partial y} dy dx \\
&= \int_K^{S+K} [\bar{G}_1(x) + \bar{G}_2(x-K) - \bar{G}(x, x-K)] dx + \int_{S+K}^{\infty} \bar{G}_1(x) dx \\
&= \int_K^{\infty} \bar{G}_1(x) dx + \int_0^S \bar{G}_2(y) dy - \int_0^S \bar{G}(x+K, x) dx. \tag{39}
\end{aligned}$$

If  $S > K$ ,

$$\begin{aligned}
& E \left\{ \max \left[ \max_{0 \leq t \leq T} S(t) - \min_{0 \leq t \leq T} S(t) - K, 0 \right] \right\} \\
&= \int_0^{S-K} \int_S^{\infty} (x-y-K) \frac{\partial^2 \bar{G}(x, y)}{\partial x \partial y} dx dy + \int_{S-K}^S \int_{y+K}^{\infty} (x-y-K) \frac{\partial^2 \bar{G}(x, y)}{\partial x \partial y} dx dy \\
&= \int_0^{S-K} (S-y-K) \frac{\partial \bar{G}_2(y)}{\partial y} dy + \int_0^{S-K} \int_S^{\infty} \frac{\partial [\bar{G}_2(y) - \bar{G}(x, y)]}{\partial y} dx dy \\
&+ \int_{S-K}^S \int_{y+K}^{\infty} \frac{\partial [\bar{G}_2(y) - \bar{G}(x, y)]}{\partial y} dx dy \\
&= \int_0^{S-K} \bar{G}_2(y) dy + \int_S^{\infty} \int_0^{S-K} \frac{\partial [\bar{G}_2(y) - \bar{G}(x, y)]}{\partial y} dy dx \\
&+ \int_S^{S+K} \int_{S-K}^{x-K} \frac{\partial [\bar{G}_2(y) - \bar{G}(x, y)]}{\partial y} dy dx + \int_{S+K}^{\infty} \int_{S-K}^S \frac{\partial [\bar{G}_2(y) - \bar{G}(x, y)]}{\partial y} dy dx \\
&= \int_S^{\infty} \bar{G}_1(x) dx + \int_0^S \bar{G}_2(y) dy - \int_{S-K}^S \bar{G}(x+K, x) dx. \tag{40}
\end{aligned}$$

All the integrals above can be evaluated explicitly.

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