**Cornell University**

**ORIE 5961: Applied Financial Engineering**



**Price-Based Hedging: Hedge When You Can, Not When You Have to**

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**1 – Introduction**

The field of financial mathematics is, among many different areas of knowledge, one that has experiencing a tremendous growth in terms of pure research and applications in the past decades. Although having its own limitations as a modeling framework, the original work of Fisher Black, Myron Scholes and Robert Merton is widely regarded as a turning point where financial derivatives started being priced in accordance to a (reasonable) theoretical approach instead of treated by practitioners by “thumb rules” and rough approximations. As a direct consequence of their work the word has observed an enormous growth in the derivatives markets, which reached the peak notional value of more than eleven times the World´s total GDP before the Subprime Crisis of 2007-2008.

Under the theoretical perspective, the work of Black, Scholes and Merton had two main complimentary findings that relates respectively to the problem of pricing and hedging financial derivatives. By defining (under certain hypotheses) the geometric Brownian motion as the governing stochastic dynamics of the spot price for the underlying instrument and constructing a hedged portfolio with self-financing trading strategies involving the underlying asset and the money market account the authors essentially obtained a closed form equation to specifically price vanilla options – as the solution of a partial differential equation – and, additionally, what should be the corresponding positions on the spot and money market accounts to perfectly hedge the position on that option.

The limitations of the Black and Scholes framework are largely known by scholars and practitioners and alterative modeling approaches to deal with these limitations – as the development of more complex models as stochastic volatility and local volatility models were proposed to deal with the problem of assuming a flat volatility for the dynamics of the spot price. By coherently calibrating the model with market prices of some traded instruments a financial institution could possibly price a derivative instrument under assumptions closer to real world observations, providing it competitive advantage in terms of pricing.

However the hedging problem also falls into another problem of the Black and Scholes hedging limitation – the fact that in a real world situation one could not continuously rebalance (or re-hedge) his or her position in a financial derivative by using the spot and the money market account as the primary instruments. Firstly, in a mathematical sense the idea of a risky underlying following the Brownian motion dynamics (that evolves by continuous paths almost surely) is definitely an approximation of the real markets. In practice, prices moves as new trades (buy and sell orders) are executed and performed by human traders. Moreover, even taking the limit of the time interval to be granularly smaller to measure the typical “cents” variations, a self financing trading strategy that constantly rebalances the hedged portfolio with such high frequency would be hard (or even impossible) to be implemented – for instance, given its transaction costs.

Therefore, the motivation of the present project is to develop and evaluate real word trading strategies that deals with the constraint that one could not re-hedge a position in a financial derivative by continuously trading. In other words, these strategies should use pre-determined rules to re-hedge the derivative exposure in a discrete-time setting, but still considering the simplification hypotheses of the Black and Scholes framework – for example, the constant (or flat) volatility structure. Efficiency criterions should be defined to measure the effectiveness of these strategies, comparing each of them for different combinations of parameters and across their peers (other strategies).

**2 – Technical Approach and Product Development**

Given the technical (theoretical) motivation of the present work, the main goal is how to develop a structured methodology to compute (or simulate) those strategies and, as a consequence, provide as a final output an user-friendly environment where these computations could be timely performed and used in a trading environment. In order to better illustrate the final deliverable of the project one could refer to the following Figure 1.

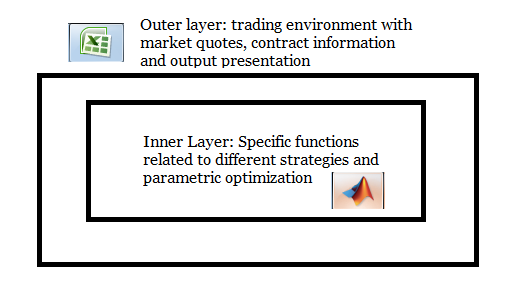
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Figure 1: Product Structure

The individual function of each of these “layers” and the relationship between them is better described by the following paragraphs.

Inner Layer: That structure can be seen as the core calculator (or back engine) that comprises the basic functions that should be executed (individually and collectively) to obtain reasonable and meaningful results. For example, this layer should contain individual coded functions corresponding to each different trading strategy that must be considered in the present work. Additionally, functions that take into consideration parametric optimizations and trading (or performance) comparisons across different strategies, although can be classified in an upper “sub-layer”, can also be considered part of the core calculator.

One point that should be remarkable in this aspect is that all functions of the inner layer should be developed with a coherent structure of inputs and outputs as generic mathematical entities. To exemplify this statement, one can consider the case of the hedging strategy of a short position in a call option that places upper and lower limit orders (whose difference is given by ) in the underlying order book for each finite time variation . This function will probably will have as its main inputs the initial price of the underlying , the strike price of the option , the option´s time to maturity , so and on.

Hence, rather than coding a function that specifically computes the hedging strategy for an “option on the S&P500 index maturing on December 2012”, this function can be called and has the corresponding inputs and outputs (for example, the expected terminal P&L function) accordingly defined as variables.

It is also worthy to mention that the core calculator should take advantage on the use of efficient programming languages (C++) or even mathematical computing environments (as MATLAB, given its suitability of dealing with matrix problems) for the main development of such functions.

Outer Layer: Although the inner layer corresponds to the computational essence of the problem, as a simple (and separate) tool it does not provide the user the main objectives of its development. In order to simulate and obtain valuable results for a real word situation a trader in a sell side firm should be able, for instance, to take the basic function of the limit order hedging strategy and apply it for a specific option in the market (say, an option on the USD/GBP FX parity) and find how the variance of the terminal P&L would change by simply changing parameter *coeteris paribus*.

Alternatively, he or she could be focused on finding the maximum value of parameter that makes the terminal P&L not to exceed the range of 10 cents (given some confidence level). These examples illustrate the importance of the outer layer, which connects the abstract structures and functions of the inner layer with the market quotes and financial instruments in order to provide the trader with meaningful results.

This example is also important to illustrate the essential difference between different input variables of this function. For example, it was previously mentioned that the initial price of the underlying , the strike price of the option , the option´s time to maturity , the risk free interest rate , the upper-lower limit order difference and the finite rebalancing interval are some of the input parameters of the core function . However there are some conceptual differences on the intrinsic nature of these variables. While and are specific characteristics of the option contract that we are simulating and and correspond to market prices (or quotes), and are parameters of the trading strategy that, differently from the other variables, are totally under the control of the user.

A final example that shows the essence of that difference is the case where the trader is concerned of finding what is the combination (or ordered pair ) that optimizes (maximizes) the expected terminal P&L adjusted to the P&L histogram range – a risk-adjusted return measure.

In order to provide a user-friendly final solution that could be executed by traders or risk managers with different levels of proficiency on computational methods, it is important that the outer layer should be developed in a classically established (but efficient) environment like Excel.

**3- Hedging Strategies**

This section describes the mathematical aspects and intuition of each individual trading strategy to be evaluated. We define as being a complete probability space, where is a sample space, is the -algebra of that space, is a filtration and is the associated probability measure. The underlying instrument (risky asset) is modeled by a stochastic process adapted to the filtration and represented by . Unless it is mentioned the opposite, we assume that follows the stochastic differential equation for the Black and Scholes model:

Where is a Brownian motion under the risk neutral probability measure .To simplify our notation we may refer as for the spot process, for the price of the call option issued (computed by the Black and Scholes formula), and as the price of the money market account – where represents the risk-free interest rate.

As it is well known by the literature, the Black and Scholes price of the call option with underlying price , strike price , maturity and volatility is given by:

And represents the standard normal cumulative distribution function. The corresponding positions on the spot instrument and the money market are respectively denominated and , being also adapted processes. Finally, one could define the profit-and-loss (P&L) function for the short position in a call option hedged by self-financing trading strategies in both the underlying and the bank account by:

One could argue, by construction, that this position will be hedged by positive (long) positions of and negative (short) positions of , but we will keep that sign notation in the P&L function for simplification purposes, in addition to the fact that , since the position is hedged at .

For (Monte Carlo) simulation purposes the Stochastic Differential Equation that governs the dynamics of the risky asset is represented by finite differences that occur in a finite time set . Because the simulation consists of generating a determined number of paths for the time evolution of we will refer by using indexes to represent different paths.

For each path and each discrete time point . Hence, for the remaining of this paper we will use the index notation as it was defined to represent the time simulation dimension and the path dimension properly associated with each variable. For example, one hedging strategy can take different values for different paths and different time steps, which will lead to the notation of and . On the other hand, the deterministic price of the money market depends only on the time evolution (not on the path that has been simulated for the spot price). Hence it can be simply stated by .

Finally, but not less important, there must be a defined relationship between the time indexes and the “real” time measurement . Usually one can write , for . The final index would be normally defined to match the option maturity .

Given that preliminary background, we can start the description of the hedging strategies themselves:

*3.1 Upper-Lower Limit Order Hedge – First Approach*

The idea of this strategy was already mentioned in the previous paragraphs. The dynamics of the spot price is simulated according to the following equation:

Where and the terms are obtained by generating samples of the normal distribution whose standard deviation is properly scaled to match the time increment . In this first approach, however, we considered a limitation for the time dimension: both the Brownian motion time increment and the time interval where new limit orders must be placed are assumed to be the same. The main problem of that initial approach is that the spot could actually reach the upper or the lower order prices for time points before the next point of - while in this case the limit order check is accomplished just at the same frequency of the re-hedging problem. However, that model is worthy of simulation, as our first attempt.

The idea of the algorithm (trading strategy) is the following one. Starting from we construct the corresponding delta hedged positions at time (called and ) in order to ensure the position is originally hedged and the initial P&L is equal to zero. The value of is simply , the Black and Scholes delta and is given by .

For each and is computed and one should verify whether (hit upper order), (hit lower order), and otherwise.

The corresponding quantities (sizes) of the upper and lower orders to be placed for each time point are computed in order to ensure that, if we reached either the upper or the lower limits, the terminal quantity of underlying held should be equal to the Black and Scholes delta at the next time point , respectively with prices and . It means that the upper and lower quantities are given by

Observe that the time dimension is already adjusted for the next point of . However it is not because the upper and lower orders were placed that one can guarantee they will be executed. Hence, indicator functions for the upper and lower hits are also created, being called and . They can be expressed as:

The next position on the stock (underlying) instrument is simply computed as a function of the indicator variables:

It means that, if both then the number of underlying held remains unaltered, . Otherwise if we reached the upper limit and if we reached the lower limit.

The corresponding position in the money market account is given by the self financing trading condition, meaning that:

**4- Analysis of Results**

This section comprises the simulation results for each strategy considered, as well as parametric optimizations of the hedging effectiveness of such strategies. Starting from , , , , , and one will find the following Figures 1 to Figure 5.



Figure 1: Monte Carlo Simulation of the Indicator Variables

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Figure 2: Monte Carlo Simulation of the Size of Upper and Lower Orders Placed

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Figure 3: Monte Carlo Simulation of the Assets and and their Positions

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Figure 4: Monte Carlo Simulation of Exposures , and

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Figure 5: Histogram of the terminal

Regarding the parametric study, one could vary parameters in certain space. By taking the same option of , , , , and varying and with the corresponding combinations of the Cartesian product of both sets one will find the following surfaces, respectively measuring the Expected Terminal P&L, the P&L range and the conditional loss (expected terminal P&L given that P&L is negative).

Each simulation was executed now for .



Figure 6: Expected value of terminal



Figure 7: Range of terminal



Figure 8: Range of terminal - Different Perspective



Figure 9: Conditional Terminal Loss



Figure 10: Conditional Terminal Loss – Different Perspective