UNIFIED DERIVATIVES

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1 Background

Before learning about derivative securities you need to know something about who the market participants are, the primary securities they are based on, and the markets available for trading.

The name *derivative securities* is a bit misleading. A derivative is quite different from a security like a stock, bond, or currency. If you stuff a deriviative under your mattress it will eventually be worthless. Derivatives don't provide equity in a company that allows you to influence decisions on how it is run. If a company goes bankrupt there is no preferred treatment on the assets of the company that bond holders receive. Currencies are a very different creature that big banks control access to.

> *A derivative is a legal contract between two counterparties for future exchanges of cash flows.*

Owning a security involves cash flows. Stocks pay dividends. Bonds pay coupons on a regular schedule and principal at maturity. In the spot foreign exchange market you get paid the difference between the rates of interest in the currency pairs you hold each day.

Derivatives use market traded securities to manufacture arbitrary cash flow streams. There is a mathematical theory that can be used to determine the trades required to do so. As with any theory, it has assumptions that hold to greater or lesser degree.

It is important to distinguish the *price* of an instrument from the cash flows associated with owning it. Market traded instruments have prices. There is a mathematial theory that can provide a *value* for a derivative instrument given the cash flows specified in a contract.

The value a model provides is not a price. It is used for pricing derivatives, and people do transactions based on that, but one of the serious shortcomings of the theory is that it does not provide a good measure of how accuate the value is. Risk management of that uncertainty is still in it's early stages.

1.1 Participants

Every transaction has a *buyer* and a *seller*. Don't confuse these nouns with the verbs buy and sell. The buyer always decides whether or not to buy or sell based on the prices offered by a seller. A buyer can also decide to go *long* or *short*. If a buyer goes long it means they own something that they believe increases in value if the market goes up. If a buyer goes short, they believe they make money if the market goes down. Prices are determined by the market. The value of a transaction is a subjective judgement made by a buyer, but models can be used to make this more objective.

You can already figure out what *buy side/sell side* means. These are also referred to as price *taker/maker*.

Sellers make a market for buyers. They provide prices for buyers who decide to buy at the ask, or sell at the bid. The ask price is also called the offer, but the seller is not really offering anything other than to allow the buyer to transact. The price also depends on the amount being transacted and who the buyer and seller are. Markets are never perfectly liquid, something the classical theory of mathematical finance likes to assume. Sellers make their living off of this asymmetry, among other things.

There are also *exchanges* and *broker-dealers* that facilitate trading. They act as both buyers and sellers to provide *liquidity*. The exchange aggregates quotes and shows their customers the best bid/ask spread currently available. They take a fraction of that spread and make their money based on the *volume* of trades. Broker-dealers provide a similar service but often hold an inventory of trades when they can't match a buyer and a seller or want to trade their own account. Broker-dealers can take advantage of market movements instead of making money only on the spread. Or take a hit on market movements. Unlike broker-dealers, no exchange has every gone bankrupt.

1.2 Securities

There are many more securities than just currencies, bonds, and stocks. Commodities are also securities and futures are derivatives that allow speculators to reduce the risk producers faced. Convertible bonds start out as bonds but contain a provision allowing the holder to convert them into shares of stock. Mortgage backed securities have cash flows determined by the interest and principal paid on a collection of mortgages. Credit default swaps provide insurance in the event a company defaults.

There are even securities that are based on the weather or future earnings of celebrities like David Bowie. The mathematical theory behind valuing these products is not up to the inventivness of those trying to make a buck off of people who have money but don't really understand what they are buying.

1.3 Markets

Exchanges provide nearly instantaneous access to transactions via *market orders*. The price cannot be guaranteed by the exchange. You get *filled* depending on the *limit orders* in the current *order book*.

A limit order is a way for buyers to act like sellers. You can offer to buy or sell some quantity of an instrument at a price. If the market moves to the level you offer, then you can get filled at the price you set. But that might never happen.

Opening an account on an exchange involve giving them *margin*. If the exchange sells you a *futures* the price and cash flow associated with entering the contract is zero. In fact, the price of a futures is always zero. Your margin account is adjusted each day with the change in the price of the underlying times then number of contracts you hold. If your margin goes to zero you get a *margin call*. If you cannot supply that, the exchange takes you out of your contracts and keeps your margin.

OTC transaction are done between two counterparties and involve machinery more complicated than setting up a margin account with an exchange. A single trade can take weeks or months of negotiation and involve setting *collateral agreements* in place of margin accounts. Exchanges provide a fixed menu of trades, the value added by what used to be called investment banks are custom tailored trades.

New ways of trading are continually invented. Dark pools are markets with limited access to enable large trades to occur without the price movements that would otherwise happen if traded on an exchange. Swap Execution Facilities (SEF) are similar to exchanges for interest rate swaps. They help standardize OTC trades and provide centralized *clearing* to mitigate counterparty risk. There are many more examples.

The mathematical models and software implementations are always far behind the inventiveness of the financial world. Even the simple models currently used do not have a well developed theory for managing risk and the implementations do not permit correlations between different markets. These are difficult problems that this note will make a first stab at addressing.

2 Finance

In order to apply mathematics to finance we need to map the complicated reality of the financial world to concepts amenable to mathematical analysis. This involves making assumptions, but it allows rigorous methods to be applied that can extend a trader's intuition or mitigate a risk manager's uncertainty.

2.1 Assumptions

2.1.1 Trading Times

It is customary in the literature to model time as a real number, usually time in years since a given epoch. Unfortunately, this leads to unrealistic models. E.g., the Doubling Paradox. We assume trading times are discrete and correspond to dates. The time increments can be arbitrarily small and there is no upper bound on time.

2.1.2 Single Currency

To simplify exposition we will assume a single currency is used. There is a standard procedure for incorporating multiple currencies.

2.1.3 Perfect Liquidity

This means every instrument can be bought or sold in any quantity at a single price. The previous two assumptions are innocuous, but this assumption is a drastic deviation from how markets work. Buying or selling even the smallest quantity of an instrument involves a bid-ask spread and buying large quantities typically increases the spread. There is a finite amount of any instrument available and some instruments cannot be sold short.

2.1.4 No Arbitrage

This is the most ridiculous assumption, but also the most crucial. The potential existence of arbitrage is a major driving force. Market participants get paid to identify and eliminate arbitrage. This is also a reason that makes this assumption more plausible.

It has been empirically verified that giving traders models that are not arbitrage free results in them selling undervalued and buying overvalued instruments. Eventually the real world catches up and the company takes a P&L hit.

2.2 Definitions

(t_j) – **Trading Times** The set of all times at which trading can occur.

(X_j) – **Prices** Each X_j is a vector of market instrument prices at time t_j.

(C_j) – **Cash Flows** Each X_j is a vector of cash flows instrument holders receive at time t_j if the instrument is held at time t_{j-1}. Most components of C_j are zero.

(Γ_j) – **Trades** Each Γ_j is the amount traded in each instrument at time t_j. Most components of Γ_j are zero.

(Δ_j) – **Position** Define $\Delta_j = \sum_{i<j} \Gamma_j$. This is the total amount of each instrument held at time t_j.

(A_j) – **Account** Define $A_j = \Delta_{j-1} \cdot C_j - \Gamma_j \cdot X_j$. The amount of money reflected in the trade blotter at time t_j is the cash flows from the existing position less the cost of current securities traded. Note that no cash flows accrue to current trades.

3 Unified Derivatives

The last formula is the skeleton key to derivatives. It can be used to turn market traded instruments into any sequence of cash flows, assuming you can find the appropriate trades (Γ_j). The value of the cash flows is the cost of setting up the initial trade: $\Gamma_0 \cdot X_0$.

Unfortunately, this never happens in the real world. No hedge is perfect. Tragically, the Nobel prize winning Black-Scholes/Merton theory has had a pernicious influence on the mathematical finance world: continuous time trading and perfect replication are a mathematical fiction.

The hard problem that is yet to be solved is how to manage risk under uncertainty. We are still in early stages. What follows is an attempt to map the complicated financial world more faithfully to mathematics with an eye to efficient software implementation.

The starting point is a clear understanding of how arbitrage can only be defined in terms of a model.

3.1 Arbitrage

Arbitrage exists if there are trades (Γ_j) such that $\sum_j \Gamma_j = 0$, $A_0 > 0$, and $A_j \geq 0$, $j > 0$.

The trading strategy must be closed out at some point, make a positive amount on the first trade, and never lose money thereafter.

You can double down on losses to keep A_j non-negative but eventually you run out of capital. Closing out Leeson's position caused the demise of the UK's oldest investment bank.

In the literature you will see arbitrage defined as $A_0 = 0$, $A_j \geq 0$, and $A_j > 0$ with non-zero probability. This is sufficient to prove a mathematical theorem, but not what a trader would consider to be arbitrage without more information on how positive the payoff is and how likely it is to occur.

Even this stronger definition of arbitrage [@Gar1981] is not sufficient. In addition to knowing how much they make up front, traders also want to know how much capital they will tie up to make that amount. A crude measure is to slap absolute values around every number and only consider trading if $-\Gamma_0 \cdot X_0/|\Gamma_0| \cdot |X_0|$ is sufficiently large.

Furthermore, the value of $|\Delta_{j-1}| \cdot |C_j| - |\Gamma_j| \cdot |X_j|$ over the life of the trade relative to A_j will be something risk managers look at every day.

But wait, that's not all. If A_j becomes significantly positive risk managers will pressure traders to modify the trading strategy to capture the value as quickly as possible.

In what follows we will ignore these considerations.

4 Mathematics

Contrary to popular belief, stochastic processes are not needed for financial modeling. All you need to know about are algebras and measures. Positive measures with mass one show up, but they are not the probability of anything.

4.1 Outcomes

The set of everything that can happen, all possible *outcomes*, is denoted by Ω. It is usually a collection of possible instrument price trajectories but it could also include current and future news, social media data, etc. No, really. It can be the set of EVERYTHING that can happen. It may not be practical to implement, but math allows us to think big.

4.2 Algebras

A subset of outcomes is an *event*. The outcome of rolling a die is represented by a set: $\{1, 2, 3, 4, 5, 6\}$. The event of rolling an even number is the subset $\{2, 4, 6\}$. An *algebra* is a collection of sets closed under complement and union. By De Morgan's laws, algebras are also closed under intersection. These are natural axioms when talking about an event not happening, or either of two events occurring.

A *partition* is a collection of events, $(A_j)_j$, such that $\Omega = \cup_j A_j$ where $A_i \cap A_j = \emptyset$ if $i \neq j$. For an algebra, $\mathcal{F}$, $A \in \mathcal{F}$ is an *atom* if $B \subseteq A$, $B \in \mathcal{F}$ imply $B = A$ or $B = \emptyset$. If an algebra $\mathcal{F}$ is finite the atoms form a partition. (Exercise: prove this.)

The algebra $\{\emptyset, \Omega\}$ represents no information. The *power set* $\mathcal{P}(\Omega) = \{A : A \subseteq \Omega\}$ represents complete information. Knowing which atom $\omega \in \Omega$ belongs to represents *partial information.* E.g., the partition $\{\{1, 3, 5\}, \{2, 4, 6\}\}$ represents knowing whether a die roll is even or odd.

4.3 Measures

A *(finitely additive) measure* is a function $\Pi\colon \mathcal{F} \to \mathbb{R}$ such that $\Pi(A \cup B) = \Pi(A) + \Pi(B)$ if $A \cap B = \emptyset$. The set of finitely additive measures is denoted $ba(\Omega, \mathcal{F})$.

A function $X\colon \Omega \to \mathbb{R}$ is *measurable* if $\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$ for all $x \in \mathbb{R}$. If $\mathcal{F}$ is finite, measurable is the same as being constant on atoms. The set of bounded measurable functions is denoted $B(\Omega, \mathcal{F})$.

Finitely additive measures are the vector space dual of bounded measurable functions. The dual pairing is $\langle X, \Pi\rangle = \int_\Omega X\, d\Pi$. If $\mathcal{F}$ is finite the integral is just the sum over atoms.

For $X \in B(\Omega, \mathcal{F})$ and $\Pi \in ba(\Omega, \mathcal{F})$ define $X\Pi \in ba(\Omega, \mathcal{F})$ by $\langle Y, X\Pi\rangle = \langle YX, \Pi\rangle$ for all $Y \in B(\Omega, \mathcal{F})$.

Let $\Pi|_\mathcal{G}$ be the measure Π restricted to the algebra $\mathcal{G}$. If Π is a probability measure *conditional expectation*, $E[X|\mathcal{G}]$, is defined by $\langle E[X|\mathcal{G}], \Pi\rangle = \langle X, \Pi|_\mathcal{G}\rangle$, i.e, the adjoint of conditional expectation is restriction.

4.4 Filtrations

A *filtration* is an increasing collection of algebras $(\mathcal{F}_t)_{t\in T}$, i.e., $\mathcal{F}_t \subseteq \mathcal{F}_u$ if $t < u$. A filtration represents information available over time.

Consider flipping a coin: T,H,H,... can be modeled by the base 2 representation of $\omega \in [0, 1)$, $\omega = .011\ldots_2 = \sum_{j>0} \omega_j 2^{-j}$, $\omega_j \in \{0, 1\}$. The algebra $\mathcal{F}_j = \{[\frac{i}{2^j}, \frac{i+1}{2^j}) : 0 \leq i < 2^j\}$ represents knowing the first j digits

4.5 The Fundamental Theorem of Asset Pricing

There is no arbitrage iff there exist positive scalar measures (Π_j) with

$$X_j\Pi_j = (C_{j+1} + X_{j+1})\Pi_{j+1}|_{\mathcal{F}_j}$$

By induction

$$X_j\Pi_j = (\sum_{j<i\leq k} C_i\Pi_i + X_k\Pi_k)|_{\mathcal{F}_j}$$

We can multiply the (Π_j) by any positive $\mathcal{F}_0$-measurable function so replacing Π_j by $\Pi_j\Pi_0^{-1}$ allows us to assume $\Pi_0 = 1$.

Under this assumption the deflators are unique if the market is complete (the so called Second fundamental theorem) but this is never true for realistic models. We will see later that there is a canonical way of choosing deflators for most models. In the case of non-stochastic rates $\Pi_j = D(j)P$ where $D(j) is the discount to timet_j and P is a probability measure.

4.5.1 FTAP (easy direction)

Suppose there exist price deflators with $X_j\Pi_j = (C_{j+1} + X_{j+1})\Pi_{j+1}|_{\mathcal{F}_j}$. Using $A_{j+1} = \Delta_j \cdot C_{j+1} - \Gamma_{j+1} \cdot X_{j+1}$

$$
\begin{aligned}
\Delta_j \cdot X_j\Pi_j &= \underline{\Delta_j} \cdot (\underline{C_{j+1}} + X_{j+1})\Pi_{j+1}|_{\mathcal{F}_j} \\
&= (\mathbf{A_{j+1}} + \underline{\mathbf{\Gamma_{j+1}}} \cdot \mathbf{X_{j+1}} + \underline{\Delta_j} \cdot X_{j+1})\Pi_{j+1}|_{\mathcal{F}_j} \\
&= (A_{j+1} + \boldsymbol{\Delta}_{\mathbf{j+1}} \cdot X_{j+1})\Pi_{j+1}|_{\mathcal{F}_j}
\end{aligned}
$$

By induction $\Delta_j \cdot X_j\Pi_j = (\sum_{j<i\leq k} A_i\Pi_i + \Delta_k \cdot X_k\Pi_k)|_{\mathcal{F}_j}$. Note if $\sum_j \Gamma_j = 0$ and $A_i \geq 0$, $i > 0$ then $A_0 \leq 0$ since $-A_0 = \Delta_0 \cdot X_0 = \sum_{j>0} A_j\Pi_j|_{\mathcal{F}_0} \geq 0$.

4.6 The Money Shot of Derivatives

Note the similarity between

$$\underline{X_j}\Pi_j = (\mathbf{C_{j+1}} + \underline{X_{j+1}})\Pi_{j+1}|_{\mathcal{F}_j}$$

and

$$\underline{\Delta_j \cdot X_j}\Pi_j = (\mathbf{A_{j+1}} + \underline{\Delta_{j+1} \cdot X_{j+1}})\Pi_{j+1}|_{\mathcal{F}_j}$$

This is the foundation of a unified approach to derivative securities. A derivative is an exchange of cash flows. *If* we can find trades (Γ_j) such that the account (A_j) replicates those cash flows then the value $\Delta_j \cdot X_j$ plays the analog of the price of a market instrument. Dynamically trading market instruments allows us to synthetically create (approximations to) new market-like instruments.

Reread the above paragraph.

The '*If*' is a big if. This can never be achieve in practice, but the classical theory of mathematical finance tends to ignore this reality. It is fun to ignore because then the theory tells us how to find the trades to 'replicate' the cash flows.

Since $-A_0 = \Gamma_0 \cdot X_0 = \sum_{j>0} A_j\Pi_j|_{\mathcal{F}_0}$ we can find the initial trade by taking a derivative with respect to price at time t_0: $\Gamma_0 = (d/dX_0)\sum_{j>0} A_j\Pi_j(\Omega)$. Note $\Gamma_0 = \Delta_0$ and this is our initial delta hedge in the classical theory.

In fact, $\Delta_j = (d/dX_j)\sum_{k>j} A_k\Pi_k/\Pi_j|_{\mathcal{F}_j}$ can be given rigorous mathematical meaning to allow us to calculate all the delta's over the life of the trade.

4.6.1 FTAP (hard direction)

Stephen Ross [@Ros1978] gave the first proof of the FTAP using the Hahn-Banach theorem. His primary contribution was to extend the Black-Scholes/Merton result from a bond, stock, and option to an arbitrary collection of instruments and show it was essentially a geometric fact.

Fischer Black and Myron Scholes gave a mathematically incorrect derivation of their eponymous partial differential equation that was immediately corrected by Robert Merton who understood the Ito calculus better than they did. Ross's proof neglected to establish an essential condition required for the application of the Hahn-Banach theorem and this led to a sequence of papers addressing the matter. The current state of the art is [@DelSch1994] that contains a 61 page proof of the current formulation.

But who cares? No arbitrage implies deflators exist, but we can find plenty.

Using the fact that $e^{-\sigma^2 t/2 + \sigma B_t}$ is a martingale, where (B_t) is Brownian motion, it follows $X_t = (e^{rt}, se^{(r-\sigma^2/2)t + \sigma B_t})$, $\Pi_t = e^{-rt}P$, where P is Wiener measure, is a arbitrage free model. This is the Black-Scholes/Merton model for a bond and non-dividend paying stock, only without the unnecessary detour through assuming a 'real-world' drift, self-financing conditions, and the complicated machinery of the Ito calculus.

Finding arbitrage free models is not difficult. Finding models that reflect market dynamics with parameters that can be fit to market data is the holy grail.

4.7 Examples

Let's define some models and start putting them to use.

4.7.1 Put-Call Parity

Let $T = \{0, 1\}$, $\Omega = [0, \infty)$, $\mathcal{F}_0 = \{\Omega\}$, $\mathcal{F}_1 = \mathcal{P}(\Omega)$, $X_0 = (1, s, c, p)$, $X_1(\omega) = (R, \omega, 0, 0)$, and $C_1(\omega) = (0, 0, \max\{\omega - k, 0\}, \max\{k - \omega, 0\})$.

This is a one period model of a bond with realized return R, a stock that can have any non-negative value, and a call and put both struck at k. Note the call and put have zero price at expiration. They are derivatives hence specified by their cash flows.

No arbitrage implies there is a deflator, Π, such that

$$(1, s, c, p) = \langle (R, \omega, \max\{\omega - k, 0\}, \max\{k - \omega, 0\}), \Pi \rangle.$$

Taking $\Gamma_0 = (k/R, -1, 1, -1)$ and computing $\Gamma_0 \cdot X_0$ gives $k/R - s + c - p = \langle k - \omega + \max\{\omega - k, 0\} - \max\{k - \omega, 0\}), \Pi \rangle = 0$. *Put-call parity* is $c - p = s - k/R$. We don't really need a model for this, but it illustrates how to use the mathematical framework.

Note that Π is not unique but no matter which Π we use we come to the same conclusion. The first thing a trader will check with any new model is put-call parity. If that does not hold, the model must be wrong.

4.7.2 Cost of Carry

Let $T = \{0,1\}$, $\Omega = [0,\infty)$, $\mathcal{F}_0 = \{\Omega\}$, $\mathcal{F}_1 = \mathcal{P}(\Omega)$, $X_0 = (1, s, 0)$, $X_1(\omega) = (R, \omega, 0)$, and $C_1(\omega) = (0, 0, \omega - f)$

This models a bond with realized return R, a stock that can have any non-negative value, and an at-the-money forward. Forwards have price zero at expiration No arbitrage implies $(1, s, 0) = \langle (R, \omega, \omega - f), \Pi \rangle$. Taking $\Gamma_0 = (f/R, -1, 1)$ and computing $\Gamma_0 \cdot X_0$ gives $f/R - s = \langle f - \omega + \omega - f, \Pi \rangle = 0$. The *cost-of-carry* is $f = Rs$. We also don't really need a model for this.

4.7.3 One Period Binomial Model

Let $T = \{0,1\}$, $\Omega = \{S^-, S^+\}$, $X_0 = (1, s)$, $X_1(\omega) = (R, \omega)$, and $C_j = 0$. This models a bond with realized return R and a stock that goes from s to either S^- or S^+. We can find π^- and π^+ such that $(1, s) = (R, S^-)\pi^- + (R, S^+)\pi^+$: $\pi^- = (S^+/R - s)/(S^+ - S^-)$, $\pi^+ = (s - S^-/R)/(S^+ - S^-)$. Define the measure Π on Ω by $\Pi(S^-) = \pi^-$ and $\Pi(S^+) = \pi^+$.

Suppose a derivative pays $V(\omega)$ at time 1. There exists $\Gamma_0 = (m, n)$ such that $\Gamma_0 \cdot X_1 = V$ on Ω. Solving $mR + nS^- = V(S_-)$, $mR + nS^+ = V(S_+)$ for m, n yields $n = (V(S_+) - V(S_-))/(S_+ - S_-)$, and

$$v = \Gamma_0 \cdot X_0 = m + ns = \frac{1}{R}\left(\frac{S_+ - Rs}{S_+ - S_-}V(S_-) + \frac{Rs - S_-}{S_+ - S_-}V(S_+)\right)$$

The deflator is unique in the one period binomial model, and this makes it a poor model. Since there are only two possible outcomes every payoff is (equivalent to) a linear payoff. And that is just wrong.

Note that the stock hedge $n = \partial v/\partial s$ is the derivative of option value with respect to underlying.

4.7.3.1 Alternate Parameterization

The above model is not the usual way of paramterizing the OPB model as taught in MBA programs.

Let $\Omega = \{d, u\}$, $X_0 = (1, s)$, and $X_1(\omega) = (R, s\omega)$. This models a bond with realized return R and a stock that goes from s to either sd or su. There exists $\Gamma_0 = (m, n)$ such that $\Gamma_0 \cdot X_1 = V$. Solving $mR + nsd = V(sd)$, $mR + nsu = V(su)$ for m, n gives $n = (V(su) - V(sd))/(su - sd)$, and

$$v = \Gamma_0 \cdot X_0 = m + ns = \frac{1}{R}\left(\frac{u - R}{u - d}V(sd) + \frac{R - d}{u - d}V(su)\right)$$

Note that the stock hedge $n \neq dv/ds$ in general. E.g., for a butterfly spread with support inside the interval (sd, su) we have $dv/ds = 0$ since $V'(sd) = V'(su) = 0$.

Parameterization is important.

4.8 Delta Hedging

Suppose an option pays $V(\omega)$ in a general one period model. For a one period model, $X_1 = 0$. If we can find Γ_0 such that $\Gamma_0 \cdot C_1 = V$ then using the existence of a deflator, $\Pi_1 = \Pi$, we have $v = \Gamma_0 \cdot X_0 = \langle \Gamma_0 \cdot C_1), \Pi \rangle$ is the value of the option. We can find the *delta hedge* by $\Gamma_0 = dv/dX_0$.

If the market is not complete we aren't guaranteed that we can find a hedge that replicates the option payoff. If we fix a deflator then we can minimize the mean square error:

$$\min_{\Gamma_0} \langle (\Gamma_0 \cdot C_1 - V)^2, \Pi \rangle.$$

Of course this depends on the deflator but, as we will see, there are usually canonical deflators available for a model. The solution to this problem is $\Gamma_0 = \langle C_1 C_1^T, \Pi \rangle^{-1} \langle V C_1, \Pi \rangle$.

We can use this minimum as a measure of the quality of the hedge.