VARIANCE SWAPS

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Abstract. This article describes the theory, practical details, pricing, hedging, profit and loss calculation, and risks for variance swap contracts.

1. Theory

A theoretical variance swap contract specifies an underlying and observation times, $t_0 < t_1 < \cdots < t_n$. If the observation of the underlying at time t_j is X_j the floating leg of the variance swap pays

$$\frac{1}{t_n - t_0} \sum_{j=0}^{n-1} R_j^2,$$

where $R_j = \log X_{j+1}/X_j = \Delta \log X_j$, or $R_j = (X_{j+1} - X_j)/X_j = \Delta X_j/X_j$. The first is called the *log quotient* and is the most common, the second is called the *realized return*.

Variance swap prices are quoted in terms of volatility. If the quote is σ, the fixed leg pays σ^2 at t_n. Just like for interest rate swaps, the quote is chosen so the two legs have equal price at t_0.

To get a feel for why the sum is related to the square of the volatility, replace it by a continuous time version, $(1/t)\int_0^t (dX/X)^2$. Assuming X is geometric Brownian motion, $dX/X = \mu\, dt + \sigma\, dB$, so $(dX/X)^2 = \sigma^2\, dt$ and the integral reduces to σ^2.

Exercise 1.1. *Show* $(d \log X)^2 = (dX/X)^2$.

This shows that there is no difference between the log quotient and realized return contracts in continuous time assuming the underlying is an Ito process.

There is no need to make assumptions about the stochastic model of the underlying in order to value a variance swap. The cost of replicating the floating leg is determined by European options, assuming a wide range of strikes are available, and futures on the underlying, all having the same expiration date at variance swap maturity.

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Using a telescoping sum and Taylor's formula, we have

$$\begin{aligned} f(X_n) - f(X_0) &= \sum_{0\le j<n} f(X_{j+1}) - f(X_j) \\ &= \sum_{0\le j<n} f'(X_j)\,\Delta X_j + f''(X_j)\,\Delta X_j^2/2! + \cdots \end{aligned}$$

where cubic and higher order terms are not shown. If we take $f(x) = -2\log x$, then $f'(x) = -2/x$ and $f''(x) = 2/x^2$ so

$$\sum_{0\le j<n} (\Delta X_j/X_j)^2 = -2\log X_n/X_0 + \sum_{0\le j<n} (2/X_j)\,\Delta X_j + O((\Delta X_j/X_j)^3)$$

We see that the floating leg of a variance swap can be replicated using a static position in an European option with a logarithmic payoff, and a dynamic hedge in futures contracts

This can be generalized by taking $f(x) = -2\log x + 2x/z$, for any number z. We then have $f'(x) = -2/x + 2/z$ and $f''(x) = 2/x^2$, so

$$\sum_{0\le j<n} (\Delta X_j/X_j)^2 = -2\log X_n/X_0 + 2(X_n - X_0)/z + \sum_{0\le j<n} (2/X_j - 2/z)\,\Delta X_j$$

Taking the *expansion point* $z = X_0$ results in the initial futures hedge being zero.

If all strikes are available for trading, then any European option with positive, twice differentiable payoff can be replicated by a dollar position, a futures, and a portfolio of puts and calls.

$$f(x) = f(z) + f'(z)(x-z) + \int_0^z f''(k)(k-x)^+\,dk + \int_z^\infty f''(k)(x-k)^+\,dk$$

2. Practical Details

The first correction to be done is to the incorporate interest rates into to the dynamic hedge. The static hedge needs no correction since it pays out at t_n. The futures position over the period t_j to t_{j+1} pays at time t_{j+1} and we want the value at expiration to be $(2/X_j - 2/z)\,\Delta X_j$ so the number of contracts to purchase is $D(t_{j+1}, t_n)(2/X_j - 2/z)$, where $D(t,u)$ is the discount from time t to time u. (We are assuming deterministic interest rates and that the payment at time t_{j+1} is invested at the risk-free rate until time t_n.)

If X is a cash index then typically the dynamic hedge will be in futures. If S denotes spot and F denotes the forward price to time t_n, then $\Delta S_j = e^{-r(t_n - t_j)}\Delta F_j - \sum_{t_j<t<t_{j+1}} d_t$, where d_t is the dividend paid at time t. Typically the sum is empty and occasionally has one term, but it is possible more than one dividend payment occurs.

$S_j e^{(r-q)(t_n-t_j)} = F_j$, where r is the continuously compounded risk-free interest rate and q is the continuously compounded dividend yield. We assume forwards and futures prices are the same. (This is the case if interest rates are deterministic.) We have

$$\begin{aligned}
\Delta F_j &= F_{j+1} - F_j \\
&= S_{j+1}e^{(r-q)(t_n-t_{j+1})} - S_j e^{(r-q)(t_n-t_j)} \\
&= e^{(r-q)(t_n-t_{j+1})}(S_{j+1} - e^{(r-q)(t_{j+1}-t_j)}S_j) \\
&\approx e^{(r-q)(t_n-t_{j+1})}\Delta S_j.
\end{aligned}$$

Therefore $e^{-r(t_n-t_{j+1})}(2/S_j - 2/z)\Delta S_j = e^{(-2r+q)(t_n-t_{j+1})}(2/S_j - 2/z)\Delta F_j$ so the dynamic hedge in futures is $e^{(-2r+q)(t_n-t_{j+1})}(2/F_j e^{-(r-q)(t_n-t_j)} - 2/z)$.

Real variance swap contracts specify more than just the underlying and observation dates. There is a notional, N, a multiplier, M, a nominal number of observations per year, f, and a bias, b. The payoff if short the floating leg is

$$MN\left(\sigma^2 - \frac{f}{n-b}\sum_{0\le j<n} R_j^2\right)$$

at expiration, t_n. Almost always, M is chosen to be 10,000. For daily observations, f is an integer close to 252, while for weekly observations it is 52. The value of b is usually 1, 2, or 3 and is subject to change after the client agrees on the par volatility for the trade. Note that $(n-b)/f$ is approximately equal to $t_n - t_0$.

2.1. **Static Hedge.** We assume that only options expiring at strikes $k_1 < k_2 < \cdots < k_m$ can be traded and find a piecewise linear approximation to the payoff $\phi(x) = \phi(x; f, z) = -2\log x/f + 2(x-f)/z$. Note $\phi'(x) = -2/x + 2/z$ and $\phi''(x) = 2/x^2$. It is clear that $\phi(x)$ is concave up, and has a single minimum at $x = z$ at which it is negative unless $z = f$ since $\phi(f) = 0$.

In addition to the option strikes, a low strike $l < k_1$ and a high strike $h > k_m$ are chosen. The static hedge to use is the piecewise linear payoff having knot points $(l, k_1, \ldots, k_m, h)$ with linear extrapolation below l and above h. Typically puts are used for strikes less than the forward and calls are used for strikes greater than the forward.

In the event the underlying is on a cash index, there is no adjustment necessary since futures converge to cash at expiration.

2.2. **Dynamic Hedge.** The dynamic hedge is

$$e^{-r(t_n-t_{j+1})}MN(2/X_j - 2/z)f/c(n-b),$$

where c is the futures contract size.

3. Profit and Loss

The most important explanatory value for daily P&L is the cubic term in the Taylor expansion. For realized return variance swaps, this value is $(2/3)(\Delta X/X)^3$ and for log quotient swaps it is $(-1/3)(\Delta X/X)^3$ when long the floating leg. These should be multiplied by $MNf/(n-b)$ to get the dollar amount of the P&L mismatch.

Exercise 3.1. *Show* $(\Delta X/X)^2 = (\Delta \log X)^2 + (\Delta X/X)^3 + O((\Delta X/X)^4)$.

Since it costs nothing to trade futures, the value of a variance swap is the cost of setting up the static hedge.

4. Risks

Since ϕ is concave up, the suggested hedge subreplicates if the underlying expires below l or above h, and superreplicates otherwise. If you are short the floating leg, then it is conservative to take $l = k_1$ and $h = k_m$. If you are long the floating leg, there will be a shortfall if the underlying expires below l or above h. The most important consideration is when the underlying expires below l since there is unlimited loss potential.

The main consideration for the dynamic hedge in futures is that they trade in discrete contract sizes so the notional on the variance swap should be sufficiently large, or the book should contain a sufficient number of trades on the same underlying to make this effect negligible.

Since the formula for valuing variance swaps neglects cubic and higher order terms, there is also risk due to jumps in the underlying. This risk is captured by the cubic term described in the Profit and Loss section.

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