

A note on a simple, accurate formula to compute implied standard deviations

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Abstract

We derive a simple, accurate formula to compute implied standard deviations for options priced in the classic framework developed by Black and Scholes (1973) and Merton (1973). When a stock price is equal to a discounted strike price, this formula reduces to a formula provided by Brenner and Subrahmanyam (1988). However, their formula's accuracy is sensitive to stock price deviations from a discounted strike price. The formula derived here extends the range of accuracy to a wide band of option moneyness.

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1. Introduction

In the classic option pricing framework developed by Black and Scholes (1973) and Merton (1973), the value of a European call option on a commodity is stated as

$$C = Me^{(b-r)T}\Phi(d) - Ke^{-rT}\Phi(d - \sigma\sqrt{T})$$

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$$d = \frac{\ln(M/K) + (b + \sigma^2/2)T}{\sigma\sqrt{T}} \quad (1)$$

where the commodity price, strike price, interest rate, and time until option expiration are denoted by M , K , r , and T , respectively. The instantaneous standard deviation of the commodity log-price is represented by σ , and $\Phi(\cdot)$ is the standard normal distribution function. The parameter b is the per unit-time cost of carrying the commodity. For a non-dividend paying stock, the cost of carry is equal to the interest rate, i.e., $b = r$, and Eq. (1) reduces to the Black–Scholes (1973) stock option pricing formula. For futures options, substitute the futures price (F) from the spot-futures parity condition, i.e., $F = Me^{bT}$, and Eq. (1) becomes Black's (Black, 1976) futures option pricing formula.

A useful property of the Black–Scholes–Merton option pricing model is that all model parameters except the log-price standard deviation are directly observable from market data. This allows a market-based estimate of a commodity's future price volatility to be obtained by inverting Eq. (1), thereby yielding an implied volatility. Originally suggested by Latane and Rendleman (1976), implied volatilities are extensively used in financial markets research.

Unfortunately, a closed-form solution for an implied standard deviation from Eq. (1) is unknown.¹ Typically, to obtain an implied standard deviation an iterative algorithm is implemented where successive iterations yield improved accuracy within limits of numerical precision set by the computer language used (Manaster and Koehler, 1982; Cox and Rubinstein, 1985; Chance, 1991; Stoll and Whaley, 1993). However, the nettlesome aspects of an iterative algorithm are that manual calculation is error-prone, spreadsheet implementation is cumbersome, and pedagogic presentation is tedious.

Brenner and Subrahmanyam (1988) provide an elegant formula to compute an implied stock return standard deviation that is accurate when a stock price is exactly equal to a discounted strike price. Letting S denote the price of a share of common stock, their formula is stated as follows².

$$\sigma\sqrt{T} = \sqrt{2\pi} \frac{C}{S} \quad (2)$$

By setting $S = Me^{(b-r)T}$, the Brenner–Subrahmanyam formula can be applied to European-style commodity options, and setting $S = Fe^{-rT}$ allows the use of European-style futures options. For simplicity, we initially restrict our discussion to stock options.

The accuracy of the well-known Brenner–Subrahmanyam formula depends on the assumption that a stock price is equal to a discounted strike price. While strike prices typically cluster around current stock prices, stock prices are rarely exactly

¹ Lai et al. (1992) provide a formula to calculate an implied standard deviation that requires numerical values for the derivatives $\partial C/\partial S$ and $\partial C/\partial K$. However, these partial derivatives are themselves functions of the underlying return volatility.

² Feinstein (1989) independently derives an essentially identical formula.

equal to a discounted strike price. In this paper, we derive a simple formula that yields accurate implied standard deviation values when stock prices deviate from discounted strike prices. When a stock price is equal to a discounted strike price, this formula reduces to the Brenner–Subrahmanyam formula exactly.

2. Quadratic approximations

A simple, accurate formula to compute implied standard deviations can be obtained from an appropriate quadratic approximation. Following the method employed by Brenner and Subrahmanyam, we make use of this expansion of the normal distribution function.³

$$\Phi(z) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \left(z - \frac{z^3}{6} + \frac{z^5}{40} + \dots \right). \quad (3)$$

Substituting expansions of the normal probabilities $\Phi(d)$ and $\Phi(d - \sigma\sqrt{T})$ according to Eq. (3) into the Black–Scholes call price formula yields this call price approximation when cubic and higher order terms are ignored.

$$C = S \left(\frac{1}{2} + \frac{d}{\sqrt{2\pi}} \right) - X \left(\frac{1}{2} + \frac{d - \sigma\sqrt{T}}{\sqrt{2\pi}} \right). \quad (4)$$

For simplicity, in Eq. (4) and hereafter we write a discounted strike price as $X = Ke^{-rT}$. Since the argument d defined in Eq. (1) is a function of the return standard deviation, Eq. (4) can be manipulated to yield the following quadratic equation in the quantity $\sigma\sqrt{T}$.

$$\sigma^2 T (S + X) - \sigma\sqrt{T} \sqrt{8\pi} \left(C - \frac{S - X}{2} \right) + 2(S - X) \ln(S/X) = 0. \quad (5)$$

All real roots of Eq. (5) are non-negative. The largest root is

$$\begin{aligned} \sigma\sqrt{T} = & \sqrt{2\pi} \left(\frac{C - \frac{S - X}{2}}{S + X} \right) \\ & + \sqrt{2\pi \left(\frac{C - \frac{S - X}{2}}{S + X} \right)^2 - \frac{2(S - X) \ln(S/X)}{S + X}}. \end{aligned} \quad (6)$$

³ Stuart and Ord (1987, p. 184).

Only the largest root reduces to the original Brenner–Subrahmanyam formula when a stock price is equal to a discounted strike price. Therefore, the root given in Eq. (6) is the economically correct solution.⁴

The accuracy of the quadratic formula in Eq. (6) can be significantly improved by minimizing its concavity as follows. First, for simplicity only, we substitute the logarithmic approximation $\ln(S/X) \approx 2(S-X)/(S+X)$ into (6)⁵, and then replace the value '4' with the parameter α to obtain this restatement of the quadratic formula.

$$\sigma\sqrt{T} = \sqrt{2\pi} \left(\frac{C - \frac{S-X}{2}}{S+X} \right) + \sqrt{2\pi \left(\frac{C - \frac{S-X}{2}}{S+X} \right)^2 - \alpha \left(\frac{S-X}{S+X} \right)^2}. \quad (7)$$

Notice that when the stock price is equal to the discounted strike price, Eq. (7) reduces to the original Brenner–Subrahmanyam formula independently of the parameter α . Thus, we use the parameter α to minimize the concavity of Eq. (7) without affecting at-the-money accuracy. We choose a value for α such that Eq. (7) is approximately linear in the stock price with a slope of zero in a neighborhood of where the stock price is equal to the discounted strike price. Noting that the call price in (7) is a function of the stock price, we evaluate the second derivative of the right-hand side of (7) with respect to the stock price where it is equal to the discounted strike price. We then set this second derivative equal to zero and solve for the resulting value of the parameter α to obtain the following expression.

$$\alpha = \frac{4\pi C}{\sigma\sqrt{T}S} \varphi(\sigma\sqrt{T}/2) + \pi \left(\Phi(\sigma\sqrt{T}/2) - \frac{1}{2} \right)^2. \quad (8)$$

In the expression above, $\varphi(\cdot)$ denotes the standard normal density function. Substituting the Brenner–Subrahmanyam formula from (2) into (8), we obtain this simplification.

$$\alpha = \sqrt{8\pi} \varphi(\sigma\sqrt{T}/2) + \pi \left(\Phi(\sigma\sqrt{T}/2) - \frac{1}{2} \right)^2. \quad (9)$$

Evaluating Eq. (9) using realistic parameter values, i.e., $\sigma \leq 1$ and $T \leq 1$, yields values of α close to 2. For example, $\sigma = 1$ and $T = 1$ yields $\alpha = 1.88$. When $\sigma = 0$, we have $\alpha = 2$. For simplicity, we henceforth set a parameter value of $\alpha = 2$. Substituting $\alpha = 2$ into Eq. (9) and rearranging terms yields the following

⁴ Following a different approach, Chance (1993) develops a quadratic formula that requires knowledge of an at-the-money implied volatility and utilizes that information to obtain a value for the implied volatility of an option that is away-from-the-money.

⁵ Gradshteyn and Ryzhik (1990), Eq. 1.542.2.

formula to compute an implied standard deviation.

$$\sigma\sqrt{T} = \frac{\sqrt{2\pi}}{S+X} \left(C - \frac{S-X}{2} + \sqrt{\left(C - \frac{S-X}{2} \right)^2 - \frac{(S-X)^2}{\pi}} \right). \quad (10)$$

Hereafter, we refer to Eq. (10) above as the improved quadratic formula. Recall that $X = Ke^{-rT}$ represents a discounted strike price and the substitutions $S = Me^{(b-r)T}$ and $S = Fe^{-rT}$ extends Eq. (10) to the cases of commodity options and futures options, respectively.

In graphical analyses not reported here, we found that with option maturities of 3 months or more, the improved quadratic formula in Eq. (10) provides near perfect accuracy for stock prices within ± 10 percent of a discounted strike price. With shorter maturities, say, 1 month, the range of precision is reduced to about ± 5 percent of a discounted strike price. In practice, option contracts are regularly opened for trading with strike prices set to continually bracket current stock prices. In addition, option trading is concentrated in near-the-money contracts. For example, Barone-Adesi and Whaley (1986) report that in a sample of 697,733 stock option transactions, about 41 percent are within ± 5 percent of being exactly at-the-money, and more than 67 percent are within ± 10 percent. Stephan and Whaley (1990) report that in a sample of 950,346 stock option transactions, about 51 percent are within ± 5 percent and about 78 percent are within ± 10 percent of being at-the-money.

To compare the accuracy of the original Brenner–Subrahmanyam formula in Eq. (2) with the improved quadratic formula in Eq. (10) using actual price data, Table 1 presents implied volatility calculations using stock option prices reported in the financial press. For each stock, two strike prices are selected that bracket the current stock price. All of these American-style options have maturities of 29 days, but no ex-dividend dates occurred during their remaining lives and therefore the Black–Scholes model is appropriate. In Table 1, we see that the original Brenner–Subrahmanyam formula (column 5) is sensitive to the moneyness of the option used to compute implied standard deviation values. By contrast, the improved quadratic formula (column 6) yields implied standard deviation values that are nearly identical to actual Black–Scholes implied standard deviations (column 7).

For stock index options, trading is also concentrated in near-the-money contracts. For example, Day and Lewis (1988) report that for several index options, most contracts trade within 5 index points of being exactly at the money. Index options are available in strike price increments of 5 index points, which means that a cash index value is never more than $2\frac{1}{2}$ index points away from an available strike price. Since current index levels are around 600 for the heavily traded S&P 100 (OEX) and S&P 500 (SPX) contracts, $\pm 2\frac{1}{2}$ index points is less than a ± 1 percent deviation between cash and strike prices.

Table 1
Comparison of computed and actual implied standard deviations (ISDs)

Stock	Price	Strike	Call	Implied standard deviations (%)		
				Brenner–Subrahmanyam, Eq. (2)	Improved quadratic, Eq. (10)	Actual ISD
Borland	22 1/4	20	3 3/8	134.90	85.05	85.50
		22 1/2	1 1/2	59.95	63.52	63.60
Ford	52 1/8	50	2 3/4	46.92	23.11	23.36
		55	1/2	8.53	24.31	24.71
Gen Elec	88 1/2	85	4 1/8	41.45	16.28	16.99
		90	1 1/16	10.68	16.20	16.21
IBM	54	50	4 3/4	78.22	34.64	35.48
		55	1 7/16	23.67	30.05	30.06
Microsoft	84 1/4	80	5 3/4	60.69	32.61	32.82
		85	2 5/8	27.71	30.41	30.42
Tel Mex	52 3/4	50	3 5/8	61.11	31.77	32.03
		55	13/16	13.70	27.37	27.66
Unisys	13 1/2	12 1/2	1 5/16	102.93	65.72	65.95
		15	7/16	28.82	62.41	62.90

Comparison of implied standard deviations computed from the original Brenner–Subrahmanyam formula in Eq. (2), and the improved quadratic formula in Eq. (10), and actual Black–Scholes implied standard deviations. Examples are based on closing prices reported in the financial press for options expiring in 29 days. The interest rate used is 3 percent. No ex-dividend dates occurred during this period.

Implied volatilities from stock index options are often used to assess total risk for the stock market. Although these options are frequently American style, researchers typically rely on implied standard deviations based on the Black–Scholes formula. For example, Day and Lewis (1988), Day and Lewis (1992), Schwert (1990), and Resnick et al. (1993) obtain Black–Scholes implied standard deviations from American option prices. Since the price of an American option is not less than the price of a European option, implied standard deviations so obtained are biased upwards. However, the early exercise premium for these options is on average quite small. For example, Harvey and Whaley (1992) show that the early exercise premium for at-the-money American calls on the S & P 100 index is about two to three cents on average.

Table 2 illustrates the accuracy of the improved quadratic formula when used with European-style and American-style stock index options. This illustration assumes a true return standard deviation of $\sigma = 15$ percent and a cash index level of 400. Strike prices of 390, 400, and 410 bracket the cash index. Option maturities of 1 month and 3 months are used. The interest rate is $r = 4$ percent and the cost of carry is $b = 0$. For options on a cash stock index, the cost of carry is

Table 2
Accuracy of implied standard deviations (ISDs)

Strike price	European option price	ISD (%)	American option price	ISD (%)
<i>1-month call options</i>				
390	12.91	14.91	12.92	14.93
400	6.89	15.00	6.89	15.01
410	3.09	14.92	3.09	14.93
<i>3-month call options</i>				
390	17.31	14.99	17.34	15.04
400	11.85	15.00	11.87	15.03
410	7.69	14.99	7.70	15.01
<i>1-month put options</i>				
390	2.95	14.91	2.98	15.00
400	6.89	15.00	6.97	15.18
410	13.06	14.92	13.26	15.42
<i>3-month put options</i>				
390	7.41	14.99	7.59	15.24
400	11.85	15.00	12.16	15.40
410	17.59	14.99	18.12	15.69

Implied standard deviations computed using the improved quadratic formula in Eq. (10) where the true standard deviation is 15 percent, the cash price is 400, the interest rate and dividend yield are 4 percent. Examples are based on American and European option prices calculated using the Barone-Adesi and Whaley (1987) algorithm and the Black–Scholes–Merton formula in Eq. (1), respectively.

the interest rate less the dividend yield, i.e., $b = r - d$, implying a dividend yield of $d = 4$ percent. For each strike price and maturity combination, European call and put option prices are calculated using the Black–Scholes–Merton formula in Eq. (1) and American call and put option prices are calculated using algorithms developed by Barone-Adesi and Whaley (1987). From these European and American option prices, implied standard deviations are calculated using Eq. (10).

In Table 2, column 1 lists strike prices. Column 3 lists implied standard deviations (ISDs) calculated from European option prices listed in column 2. Column 5 lists implied standard deviations calculated from American option prices listed in column 4. Table 2 reveals that for near-the-money options, American option prices are usually close to corresponding European option prices. This is especially so for call options, suggesting that implied standard deviations from near-the-money European calls and American calls are similar. But counterexamples are possible. For example, Harvey and Whaley (1992) show that early exercise premia for S&P 100 index options and a lumpy dividend stream are small, on average, but can be significant for in-the-money options. In these cases, an implied standard deviation that assumes a European option might be useful only as a first approximation. However, for European option prices the implied standard deviations reported in Table 2 calculated using the improved quadratic formula are within 10 basis points of the correct standard deviation value.

3. Summary

This paper provides a simple, accurate formula to compute Black–Scholes implied standard deviations. This formula is an extension of the original Brenner–Subrahmanyam (1988) formula. With option maturities of 3 months or more, the formula derived here provides near perfect accuracy for stock prices within ± 10 percent of a discounted strike price. With shorter maturities, say, 1 month, the range of precision is reduced to about ± 5 percent of a discounted strike price. Since option trading volume is concentrated in near-the-money contracts, implied volatilities computed from near-the-money contracts are more reliable estimates (Harvey and Whaley, 1991). As a result, financial researchers typically rely on volatility estimates from near-the-money contracts and restricting the formula derived here to near-the-money options is consistent with current practice.

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