Normal Short Rate Model

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The normal short rate model is $f_t = \phi(t) + \sigma(t)B_t$, where B_t is standard Brownian motion. The stochastic discount is $D_t = \exp(-\int_0^t f_s ds)$.

The discount to time t is $D(t) = ED_t$. %Using $Ee^N = \exp(EN + 1/2\,Var\,N)$ we need to compute $E\log D_t = -\int_0^t \phi(s)\,ds$ and $Var\log D_t$. Since $Cov(B_u, B_v) = \min\{u, v\}$,

$$\begin{aligned}
Var(\int_0^t B_s ds) &= Cov(\int_0^t B_s ds, \int_0^t B_s ds) \\
&= \int_0^t \int_0^t \min\{u, v\}\,du\,dv \\
&= \int_0^t (\int_0^v u du + v \int_v^t du)\,dv \\
&= \int_0^t v^2/2 + v(t - v)\,dv \\
&= \int_0^t vt - v^2/2\,dv \\
&= v^3/2 - v^3/6 \\
&= v^3/3
\end{aligned}$$

It follows $Var\log D(t) = \sigma^2 t^3/3$ if $\sigma(t) = \sigma$ is constant and $D(t) = \exp(-\int_0^t \phi(s)\,ds + \sigma^2 t^3/6)$. Note $D(t) = \exp(-\int_0^t (\phi(s) - \sigma^2 s^2/2)\,ds)$ so the forward is $f(t) = \phi(t) - \sigma^2 t^2/2$.

The difference between the future and forward, $\sigma^2 t/2$, is called the *convexity*.

The price of a zero coupon bond at time t paying one unit at time u is $D_t(u) = E[D_u/D_t|t] = \exp(-\int_t^u f_s\,ds)|_t$ and has a closed form solution. Since $d(tB_t) = t\,dB_t + B_t\,dt$,

$$
\begin{aligned}
\int_t^u B_s ds &= \int_t^u d(sB_s) - sdB_s \\
&= uB_u - tB_t - \int_t^u sdB_s \\
&= uB_u(-uB_t + uB_t) - tB_t - \int_t^u sdB_s \\
&= (uB_u - uB_t) + uB_t - tB_t - \int_t^u sdB_s \\
&= (u-t)B_t + \int_t^u (u-s)dB_s.
\end{aligned}
$$

Now we use the fact that $M_t = \exp(-\int_0^t a(s)^2\,ds/2 + \int_0^t a(s)\,dB_s)$ is a martigale for any function $a(s)$, so $EM_u/M_t|_t = 1$ and taking $a(s) = \sigma(u-s)$ we have

$$
\begin{aligned}
E\exp(int_t^u \sigma(u-s)\,dB_s)|_t &= \exp(\int_t^u \sigma^2(u-s)^2\,ds/2) \\
&= \exp(-\sigma^2(u-s)^3/6|_t^u) \\
&= \exp(\sigma^2(u-t)^3/6).
\end{aligned}
$$

hence

$$
E\exp(\int_t^u B_s\,ds) = \exp((u-t)B_t + (u-t)^3/6).
$$

Note

$$
E\exp(-\int_t^u B_s\,ds) = \exp(-(u-t)B_t + (u-t)^3/6)
$$

since we can replace (B_t) by $(-B_t)$. Putting these facts together yields

$$
\begin{aligned}
D_t(u) &= ED_u/D_t|_t \\
&= E\exp(-\int_u^t f_s\,ds)|_t \\
&= E\exp(-\int_u^t (\phi(s)+\sigma B_s)\,ds)|_t \\
&= E\exp(-\int_u^t \phi(s)\,ds - \sigma(u-t)B_t + \sigma^2(u-t)^3/6).
\end{aligned}
$$

Note $D_t(u)$ is lognormal and

$$
E\log D_t(u) = -\int_u^t \phi(s)ds + \sigma^2(u-t)^3/6
$$

$$
Var\log D_t(u) = \sigma^2(u-t)^2 t
$$

Define $\Phi(t) = \exp(-\int_0^t \phi(s)\,ds)$. Since $\log D(t) = \log\Phi(t) + \sigma^2 t^3/6$ we have

$$
\begin{aligned}
E\log D_t(u) &= \log D(u)/D(t) - \sigma^2(u^3-t^3)/6 + \sigma^2(u-t)^3/6 \\
&= \log D(u)/D(t) + \sigma^2(-3u^2t + 3ut^2)/6 \\
&= \log D(u)/D(t) - \sigma^2 ut(u-t)/2
\end{aligned}
$$

The *forward rate* at time t over the interval $[u,v]$ is $F_t(u,v) = (D_t(u)/D_t(v) - 1)/(v-u)$.

A caplet pays $(v-u)\max\{F_u(u,v)-k,0\}$ at time v. It has value

$$
\begin{aligned}
c &= E(v-u)\max\{F_u(u,v)-k,0\}D_v \\
&= E\max\{1/D_u(v) - 1 - (v-u)k, 0\}D_u(v)D_u \\
&= E\max\{1-(1+(v-u)k)D_u(v), 0\}D_u \\
&= E\max\{1-(1+(v-u)k)D_u(v)e^{\gamma}, 0\}ED_u
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma &= Cov(\log D_u(v), \log D_u) \\
&= Cov(-\sigma(v-u)B_u, -\int_0^u \sigma B_s \, ds \\
&= \sigma^2(v-u)\int_0^u Cov(B_u, B_s)\, ds \\
&= \sigma^2(v-u)\int_0^u s\, ds \\
&= \sigma^2(v-u)u^2/2
\end{aligned}
$$