$H(\text{div}; \mathbb{S})$-Conforming Finite Elements for Symmetric Tensors

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1 Introduction

In the mixed finite element method for linear elasticity, the stress tensor $\boldsymbol{\sigma}$ and the displacement $\boldsymbol{u}$ are approximated in the finite element spaces $H(\text{div}; \mathbb{S})$ and L^2, respectively, where $\mathbb{S}$ denotes the space of symmetric tensors. The corresponding discrete spaces Σ_h and

V_h must satisfy the inf–sup condition, whose validity requires that

$$\operatorname{div} \Sigma_h = V_h.$$

For any simplex T, define the trace-free polynomial space

$$\mathbb{B}_k(\operatorname{div}, T) := \{\sigma \in \mathbb{P}_k(T, \mathbb{S}) \mid \sigma|_{\partial T} \cdot \boldsymbol{n} = 0\}.$$

It holds that

$$\operatorname{div} \mathbb{B}_k(\operatorname{div}, T) = \mathbb{P}_{k-1}(T, \mathbb{S}) \cap RM^{\perp},$$

where $RM^{\perp}$ is the L^2-orthogonal complement of

$$RM := \mathbb{P}_0(T, \mathbb{R}^d) + \mathbb{P}_0(T, \mathbb{K})\boldsymbol{x},$$

and $\mathbb{K}$ denotes the space of d-dimensional skew-symmetric matrices.

Given a simplicial mesh $\mathcal{T}_h$, and an $H(\operatorname{div}, \mathbb{S})$-conforming finite element space Σ_h defined on $\mathcal{T}_h$, we make the following assumptions:

(A1) For every $T \in \mathcal{T}_h$, we have

$$\operatorname{div} \Sigma_T = \mathbb{P}_{k-1}(T, \mathbb{S}).$$

(A2) The following degrees of freedom are linearly independent on Σ_h:

$$\mathcal{N}_{F,\boldsymbol{q}}(\boldsymbol{\sigma}) := \int_F (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \boldsymbol{q} \, \mathrm{d}S, \quad \forall \boldsymbol{q} \in \mathbb{P}_1(T, \mathbb{R}^d), \ F \in \Delta_{d-1}\mathcal{T}_h.$$

Under these assumptions, we have

$$\operatorname{div} \Sigma_h = \mathbb{P}_{k-1}^{-1}(\mathcal{T}_h, \mathbb{S}),$$

where assumption (A2) ensures that

$$\prod_{T \in \mathcal{T}_h} RM(T) \subseteq \operatorname{div} \Sigma_h.$$

The Hu–Zhang element is an $H(\operatorname{div}, \mathbb{S})$-conforming finite element whose shape function space is $\mathbb{P}_k(T, \mathbb{S})$. Thus, it automatically satisfies assumption (A1). When $k \geq d+1$, it also satisfies assumption (A2). The Hu–Zhang element includes super-smooth degrees of freedom, which enforce normal continuity on all subsimplices of dimension less than or equal to $d-1$. These properties stem from the symmetry and smoothness of the shape function space.

To resolve this issue for $k \leq d$, we enrich the shape function space $\mathbb{P}_k(T,\mathbb{S})$ with additional $H(\mathrm{div};\mathbb{S})$-conforming functions. These functions are non-smooth and have nonzero traces $\boldsymbol{\sigma}\cdot\boldsymbol{n}$ on ∂T, thereby ensuring that the degrees of freedom in assumption (A2) remain linearly independent. Moreover, to satisfy assumption (A1), these functions are required to be divergence-free. We present the construction of these functions below.

2 Hybridizable $H(\mathrm{div};\mathbb{S})$-Conforming Finite Elements

According to the exactness of the elasticity complex, the functions in $H(\mathrm{div};\mathbb{S})$ with zero divergence form the image of the space U under the corresponding differential operator d, where U is the space preceding $H(\mathrm{div};\mathbb{S})$ in the elasticity complex:

$$U \xrightarrow{\mathrm{d}} H(\mathrm{div};\mathbb{S}) \xrightarrow{\mathrm{div}} L^2(\mathbb{R}^d) \tag{1}$$

Therefore, one can select functions from the image of d applied to U as additional basis functions. To construct functions that are not infinitely smooth, we introduce the following split simplex.

2.1 Split Simplex

Let T be a d-dimensional simplex with $d+1$ vertices denoted by $\boldsymbol{x}_0,\cdots,\boldsymbol{x}_d$. Define:

$$\boldsymbol{x}_c = \frac{1}{d+1}\sum_{i=0}^{d}\boldsymbol{x}_i, \quad \boldsymbol{t}_{ij} = \boldsymbol{x}_j - \boldsymbol{x}_i, \quad \boldsymbol{t}_{ic} = \boldsymbol{x}_c - \boldsymbol{x}_i.$$

For $f \in \Delta(T)$, define $\boldsymbol{t}_{fc} = \boldsymbol{t}_{f[0]c} = \boldsymbol{x}_c - \boldsymbol{x}_{f[0]}$.

By connecting $\boldsymbol{x}_c$ with each vertex $\boldsymbol{x}_i$, the simplex T is split into $d+1$ d-simplices, denoted collectively as T^R. Denote by T_i the sub-simplex in T^R that does not contain vertex $\boldsymbol{x}_i$. For each face $f \in \Delta_{d-1}T$, let T_{f^*} denote the sub-simplex that contains f. Let χ_{T_i} be the characteristic function on T_i.

Define λ_i as a continuous piecewise linear function on T^R such that

$$\lambda_i(\boldsymbol{x}_j) = \delta_{ij}, \quad \forall j = 0,\cdots,d, \quad \text{and} \quad \lambda_i(\boldsymbol{x}_c) = 0.$$

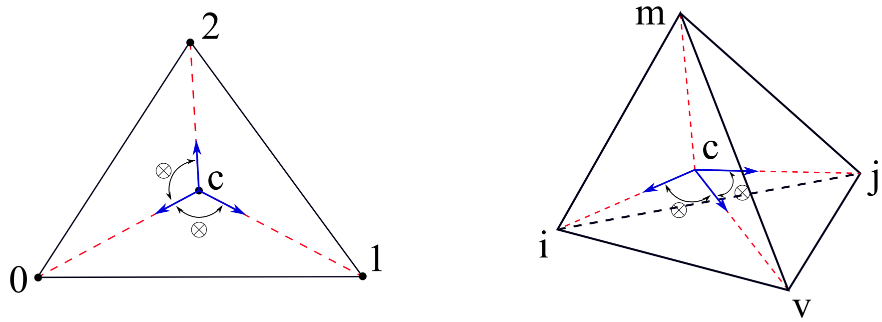

图 1: Split simplex

Lemma 1. For any tetrahedron T_i in T^R, with $0 \le i \le d$, the vectors $\{\boldsymbol{t}_{cm}\}_{m=0,\, m\neq i}^{d}$ form a basis of $\mathbb{R}^d$, and $\{\nabla\lambda_m|_{T_i}\}_{m=0,\, m\neq i}^{d}$ is the dual basis to it.

3 Mixed Finite-Element Method for Linear Elasticity

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a polygonal domain with boundary $\partial\Omega$. We study the boundary-value problem

$$\begin{cases} \mathcal{A}(\boldsymbol{\sigma}) = \varepsilon(\boldsymbol{u}) & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{u}$ is the displacement vector, and $\boldsymbol{f}$ is the body force. The strain tensor and the compliance tensor are given by

$$\varepsilon(\boldsymbol{u}) = \tfrac{1}{2}\big(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}}\big), \qquad \mathcal{A}(\boldsymbol{\sigma}) = \tfrac{1}{2\mu}\big(\boldsymbol{\sigma} - \lambda_0 \operatorname{tr}\boldsymbol{\sigma}\, I\big),$$

with Lamé parameters λ, $\mu > 0$. We partition Ω by a simplicial mesh $\mathcal{T}_h$ and define finite-element spaces Σ_h and V_h as before. The mixed finite-element scheme reads: find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \Sigma_h \times V_h$ such that

$$\begin{aligned} \big(\mathcal{A}\boldsymbol{\sigma}_h, \boldsymbol{\tau}\big) - \big(\operatorname{div}\boldsymbol{\tau}, \boldsymbol{u}_h\big) &= 0 && \forall \boldsymbol{\tau} \in \Sigma_h, \\ \big(\operatorname{div}\boldsymbol{\sigma}_h, \boldsymbol{v}\big) &= \big(\boldsymbol{f}, \boldsymbol{v}\big) && \forall \boldsymbol{v} \in V_h. \end{aligned}$$

3.1 Two-Dimensional Case

In two dimensions, the space U in the elasticity complex is H^2, and the differential operator $\mathrm{d} = J$ is defined by:

$$J(u) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla^2 u \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2}$$

We now define H^2-conforming functions on the split triangle T^R. Let the three vertices be $\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2$. For vertex v, let $[i,j] = v^*$ (the opposite edge). Define

$$\psi_v = \lambda_v^3 \lambda_i \chi_{T_j} - \lambda_v^3 \lambda_j \chi_{T_i}.$$

Lemma 2. $\psi_v \in H^2(T^R)$.

证明. It suffices to prove that $\nabla\psi_v$ is continuous across T^R, i.e., the jump across each edge $[v,c]$, $[i,c]$, and $[j,c]$ is zero.

$$\nabla\psi_v = (3\lambda_v^2\lambda_i \nabla\lambda_v + \lambda_v^3 \nabla\lambda_i)\chi_{T_j} - (3\lambda_v^2\lambda_j\nabla\lambda_v + \lambda_v^3\nabla\lambda_j)\chi_{T_i}.$$

Since λ_v vanishes on $[i,c]$ and $[j,c]$, the jump across these two edges is zero. On $[v,c]$, both λ_i and λ_j vanish, so

$$[\![\nabla\psi_v]\!]|_{[v,c]} = \lambda_v^3(\nabla\lambda_i + \nabla\lambda_j).$$

Because T_i and T_j have equal area, $\nabla\lambda_i$ and $\nabla\lambda_j$ are equal in magnitude but opposite in direction. Hence the jump is zero on $[v,c]$, and the lemma follows. □

Clearly, ψ_v is linearly independent of $\mathbb{P}_k(T,\mathbb{S})$. We further have the following:

Lemma 3. There does not exist $\boldsymbol{q} \in \mathbb{P}_k(T,\mathbb{S})$ such that $(\boldsymbol{q} - J(\psi_v))|_{\partial T} \cdot \boldsymbol{n} = 0$.

证明. $J(\psi_v)$ is discontinuous across T^R:

$$J(\psi_v) : (\boldsymbol{t}_{cv} \otimes \boldsymbol{t}_{cv}) = \nabla^2\psi_v : (\boldsymbol{n}_{cv} \otimes \boldsymbol{n}_{cv}) = \frac{\partial^2\psi_v}{\partial n_{cv}^2}.$$

Since ψ_v has a discontinuous second normal derivative across $[v,c]$, we conclude that $J(\psi_v)|_{T_i} : (\boldsymbol{t}_{ci} \otimes \boldsymbol{t}_{ci})$ and $J(\psi_v)|_{T_j} : (\boldsymbol{t}_{cj} \otimes \boldsymbol{t}_{cj})$ differ at $\boldsymbol{x}_v$. Thus $J(\psi_v) \cdot \boldsymbol{n}$ is discontinuous on ∂T.

If there exists $\boldsymbol{q} \in \mathbb{P}_k(T,\mathbb{S})$ such that $(\boldsymbol{q} - J(\psi_v)) \cdot \boldsymbol{n} = 0$ on ∂T, then $\boldsymbol{q}$ would also be discontinuous on ∂T, which is impossible. The lemma is proved. □

Theorem 1. Define

$$\Sigma_T = \mathbb{P}_k(T, \mathbb{S}) + \mathrm{span}\{J(\psi_i)\}_{i=0}^{2}.$$

Then $\Sigma_T \subset H(\mathrm{div}; \mathbb{S})$, and the following degrees of freedom are unisolvent:

$$(\boldsymbol{\sigma} \cdot \boldsymbol{n}, \boldsymbol{q}) \quad \forall e \in \Delta_1 T,\ \boldsymbol{q} \in \mathbb{P}_k(T, \mathbb{R}^2), \tag{3}$$

$$(\boldsymbol{\sigma} : \boldsymbol{q}) \quad \forall \boldsymbol{q} \in \mathbb{B}_k(\mathrm{div}; T). \tag{4}$$

证明. The $H(\mathrm{div}; \mathbb{S})$-conformity is clear. We now prove unisolvency of the degrees of freedom.

First, count the degrees of freedom. The first kind contributes $6(k+1)$ DOFs. Using geometric decomposition, the second kind contributes

$$\dim\left(\mathbb{B}_k(\mathrm{div}; T)\right) = \frac{3(k-1)(k-2)}{2} + 3(k-1).$$

The total dimension of the space Σ_T is

$$\dim(\mathbb{P}_k(T, \mathbb{S})) + 3 = \frac{3(k+1)(k+2)}{2} + 3.$$

A direct computation shows this equals the total number of DOFs.

Now let $\boldsymbol{\sigma} \in \Sigma_T$ with all DOFs zero. Since $\boldsymbol{\sigma} \cdot \boldsymbol{n} \in \mathbb{P}_k(T, \mathbb{R}^2)$, the vanishing of the DOFs in (3) implies $\boldsymbol{\sigma} \cdot \boldsymbol{n}|_{\partial T} = 0$, hence $\boldsymbol{\sigma} \in \mathbb{B}_k(\mathrm{div}; T)$ by Lemma 6. Then (4) implies $\boldsymbol{\sigma} = 0$, proving unisolvency. □

Define the global finite element space:

$$\Sigma_h := \left\{ \boldsymbol{\sigma} \in L^2(\Omega) \;\middle|\; \boldsymbol{\sigma}|_T \in \Sigma_T \text{ for all } T \in \mathcal{T}_h, \text{ and DOFs (3) are single-valued} \right\}.$$

Then Σ_h is an $H(\mathrm{div}; \mathbb{S})$-conforming finite element space.

3.2 Arbitrary Dimensional Case

While the complex-based approach allows extension to arbitrary dimensions, the elasticity complex in dimensions higher than 2 is technically involved. In this section, we present an alternative constructive approach.

We begin by examining the structure of the function $J(\psi_v)$ introduced in the 2D case. Define the rotated gradient

$$\nabla^\perp f = \begin{pmatrix} -\frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \nabla f.$$

Then by the definition of J in (2), we have

$$\begin{aligned} J(\psi_v) = 6\big(&\lambda_v^2 \operatorname{sym}(\nabla^\perp \lambda_i \otimes \nabla^\perp \lambda_v) + \lambda_v \lambda_i \nabla^\perp \lambda_v \otimes \nabla^\perp \lambda_v\big)\chi_{T_j} \\ &- 6\big(\lambda_v^2 \operatorname{sym}(\nabla^\perp \lambda_j \otimes \nabla^\perp \lambda_v) + \lambda_v \lambda_j \nabla^\perp \lambda_v \otimes \nabla^\perp \lambda_v\big)\chi_{T_i}. \end{aligned}$$

Note that there exist constants c_1, c_2 such that

$$\nabla^\perp \lambda_v = c_1 \boldsymbol{t}_{ci} \chi_{T_j} + c_2 \boldsymbol{t}_{cj} \chi_{T_i}.$$

Let us define:

$$\begin{aligned} &\psi_v^0 = \lambda_v \lambda_i (\boldsymbol{t}_{ci} \otimes \boldsymbol{t}_{ci}) \chi_{T_j}, \quad \psi_v^1 = \lambda_v \lambda_j (\boldsymbol{t}_{cj} \otimes \boldsymbol{t}_{cj}) \chi_{T_i}, \\ &\psi_v^2 = \lambda_v^2 \big(\operatorname{sym}(\boldsymbol{t}_{cv} \otimes \boldsymbol{t}_{ci}) \chi_{T_j} - \operatorname{sym}(\boldsymbol{t}_{cv} \otimes \boldsymbol{t}_{cj}) \chi_{T_i}\big). \end{aligned}$$

Then $J(\psi_v)$ is a linear combination of ψ_v^0, ψ_v^1, and ψ_v^2.

Lemma 4. The functions $\psi_v^0, \psi_v^1, \psi_v^2$ defined above belong to $H(\operatorname{div}; \mathbb{S})$, and there exists a linear combination of them whose divergence is zero.

证明. We only need to verify normal continuity across the internal edges $[v, c]$, $[i, c]$, and $[j, c]$. The functions ψ_v^0 and ψ_v^1 vanish on these edges. ψ_v^2 vanishes on $[i, c]$ and $[j, c]$, and on $[v, c]$ we compute

$$[\![\psi_v^2 \cdot \boldsymbol{n}_{[v,c]}]\!] = \frac{1}{2} \lambda_v^2 ((\boldsymbol{t}_{ci} + \boldsymbol{t}_{cj}) \cdot \boldsymbol{n}_{[v,c]}) \boldsymbol{t}_{cv}.$$

Since

$$\boldsymbol{t}_{ci} + \boldsymbol{t}_{cj} = \boldsymbol{x}_c - \boldsymbol{x}_i + \boldsymbol{x}_c - \boldsymbol{x}_j = \boldsymbol{x}_v - \boldsymbol{x}_c = -\boldsymbol{t}_{cv},$$

the jump vanishes, establishing $H(\operatorname{div}; \mathbb{S})$-conformity.

Next, compute the divergence:

$$\operatorname{div}(\psi_v^0) = \lambda_v \boldsymbol{t}_{ci} \chi_{T_j}, \quad \operatorname{div}(\psi_v^1) = \lambda_v \boldsymbol{t}_{cj} \chi_{T_i}, \quad \operatorname{div}(\psi_v^2) = \lambda_v (\boldsymbol{t}_{ci} \chi_{T_j} - \boldsymbol{t}_{cj} \chi_{T_i}).$$

Thus,

$$\operatorname{div}(\psi_v^2 - \psi_v^0 + \psi_v^1) = 0.$$

$\square$

The lemma shows that we can construct the desired functions in a constructive manner. The key is to use tensor products of vectors from $\{\boldsymbol{t}_{c0}, \boldsymbol{t}_{c1}, \ldots, \boldsymbol{t}_{cd}\}$, which are dual to $\{\nabla\lambda_0, \nabla\lambda_1, \ldots, \nabla\lambda_d\}$ as shown in Lemma 1, simplifying divergence calculations. We now generalize this to arbitrary dimensions.

Let T be a d-simplex, and let e be an l-face of T with $l < d-1$. For $\alpha \in \mathbb{T}_k^l$, define the monomial

$$\lambda_e^\alpha = \lambda_{e[0]}^{\alpha_0}\lambda_{e[1]}^{\alpha_1}\cdots\lambda_{e[l]}^{\alpha_l}.$$

We define three types of $H(\operatorname{div};\mathbb{S})$-conforming functions as in the 2D case. For $\alpha \in \mathring{\mathbb{T}}_k^l := \{\alpha \in \mathbb{T}_k^l \mid \alpha_i \neq 0,\ \forall i\}$ and $i < j \in e^*$, define

$$\begin{aligned}
\psi_{ij,0}^{e,\alpha} &= \lambda_e^{\alpha-\epsilon_0}\lambda_i\, \boldsymbol{t}_{ci} \otimes \boldsymbol{t}_{ci}\, \chi_{T_j},\\
\psi_{ij,1}^{e,\alpha} &= \lambda_e^{\alpha-\epsilon_0}\lambda_j\, \boldsymbol{t}_{cj} \otimes \boldsymbol{t}_{cj}\, \chi_{T_i},\\
\psi_{ij,2}^{e,\alpha} &= \lambda_e^{\alpha}\big(\operatorname{sym}(\boldsymbol{t}_{ce} \otimes \boldsymbol{t}_{ci})\chi_{T_j} - \operatorname{sym}(\boldsymbol{t}_{ce} \otimes \boldsymbol{t}_{cj})\chi_{T_i}\big),
\end{aligned}$$

where $\epsilon_0 = (1, 0, \ldots, 0)$ is a vector of length $l+1$.

Lemma 5. For all $\alpha \in \mathring{\mathbb{T}}_k^l$ and $i < j \in e^*$, the functions $\psi_{ij,0}^{e,\alpha}, \psi_{ij,1}^{e,\alpha}, \psi_{ij,2}^{e,\alpha}$ belong to $H(\operatorname{div};\mathbb{S})$, and there exists a linear combination whose divergence vanishes.

证明. Let $\bar{f} = f \cup \{\boldsymbol{x}_c\} \in \Delta\mathring{T}^R$ denote an interior face of T^R. We verify normal continuity across each $\bar{f}$ for $f \in \Delta_{d-2}T$.

For $\psi_{ij,0}^{e,\alpha}$, if $\bar{f} \notin \Delta_{d-1}T_j$, the function vanishes on $\bar{f}$. If $q = T_j \setminus \bar{f} = i$, then λ_i vanishes on $\bar{f}$. If $q \neq i$, then $\nabla\lambda_q$ is normal to $\bar{f}$ and orthogonal to $\boldsymbol{t}_{ci}$ by Lemma 1, so the normal component of $\psi_{ij,0}^{e,\alpha}$ vanishes. Similar arguments apply to $\psi_{ij,1}^{e,\alpha}$.

For $\psi_{ij,2}^{e,\alpha}$, the only nontrivial case is when $f = [i,j]^*$. In that case, choosing $\boldsymbol{n}_{\bar{f}} = \nabla\lambda_i|_{T_j}$ and noting

$$\boldsymbol{t}_{ci} + \boldsymbol{t}_{cj} = -\boldsymbol{t}_{ce}, \quad \nabla\lambda_i|_{T_j} \cdot \boldsymbol{t}_{ci} = 1, \quad \nabla\lambda_j|_{T_i} \cdot \boldsymbol{t}_{cj} = 1,$$

we compute

$$[\![\psi_{ij,2}^{e,\alpha} \cdot \boldsymbol{n}_{\bar{f}}]\!] = \frac{1}{2}\lambda_e^{\alpha}\,(\boldsymbol{t}_{ci} + \boldsymbol{t}_{cj}) \cdot \nabla\lambda_i\,\boldsymbol{t}_{ce} = 0.$$

The divergence is:

$$\begin{aligned}
\operatorname{div}(\psi_{ij,0}^{e,\alpha}) &= \lambda_e^{\alpha-\epsilon_0}\boldsymbol{t}_{ci}\chi_{T_j}, \\
\operatorname{div}(\psi_{ij,1}^{e,\alpha}) &= \lambda_e^{\alpha-\epsilon_0}\boldsymbol{t}_{cj}\chi_{T_i}, \\
\operatorname{div}(\psi_{ij,2}^{e,\alpha}) &= \frac{\alpha_0}{2}\lambda_e^{\alpha-\epsilon_0}(\boldsymbol{t}_{ci}\chi_{T_j} - \boldsymbol{t}_{cj}\chi_{T_i}).
\end{aligned}$$

Let

$$\psi_{ij}^{e,\alpha} = -\frac{\alpha_0}{2}\psi_{ij,0}^{e,\alpha} + \frac{\alpha_0}{2}\psi_{ij,1}^{e,\alpha} + \psi_{ij,2}^{e,\alpha}.$$

Then $\operatorname{div}(\psi_{ij}^{e,\alpha}) = 0$. □

Lemma 6. There does not exist $\boldsymbol{q} \in \mathbb{P}_k(T, \mathbb{S})$ such that

$$(\boldsymbol{q} - \psi_{ij}^{e,\alpha}) \cdot \boldsymbol{n} = 0 \quad \text{on } \partial T.$$

证明. The proof is similar to Lemma 6 in the 2D case. □

Theorem 2. Let

$$V_T := \mathbb{P}_k(T, \mathbb{S}) \oplus \bigoplus_{l=0}^{d-2} \bigoplus_{e \in \Delta_l T} \operatorname{span}(\Phi_e^k),$$

where

$$\Phi_e^k = \left\{ \psi_{ij}^{e,\alpha} \mid i < j \in e^*,\ \alpha \in \mathring{\mathbb{T}}_k^l \right\}.$$

Then $V_T \subset H(\operatorname{div}; \mathbb{S})$, and the following degrees of freedom are unisolvent:

$$(\boldsymbol{\sigma} \cdot \boldsymbol{n}, \boldsymbol{q})_F \quad \forall F \in \Delta_{d-1}T,\ \boldsymbol{q} \in \mathbb{P}_k(F, \mathbb{R}^d), \tag{5}$$

$$(\boldsymbol{\sigma} : \boldsymbol{q})_T \quad \forall \boldsymbol{q} \in \mathbb{B}_k(\operatorname{div}; T). \tag{6}$$

证明. The proof is analogous to that of Theorem 1. □

Define the global finite element space

$$\Sigma_h := \left\{ \boldsymbol{\sigma} \in L^2(\Omega) \;\middle|\; \boldsymbol{\sigma}|_T \in V_T,\ \forall T \in \mathcal{T}_h,\ \text{and DOFs (5) are single-valued} \right\}.$$

Theorem 3. We have $\operatorname{div}\Sigma_h = \mathbb{P}_{k-1}^{-1}(\mathcal{T}_h, \mathbb{S})$, and there exists a constant $C > 0$ such that

$$\sup_{\boldsymbol{\tau}\in\Sigma_h} \frac{(\operatorname{div}\boldsymbol{\tau}, \boldsymbol{v})}{\|\boldsymbol{\tau}\|_{H(\operatorname{div})}} \geq C\|\boldsymbol{v}\|_{L^2}. \tag{7}$$

Proof omitted. $\square$

4 Numerical Experiments

4.1 Mixed Finite Element Method for Linear Elasticity

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a polygonal domain with boundary $\partial\Omega$. We consider the linear elasticity problem:

$$\begin{cases} \mathcal{A}(\boldsymbol{\sigma}) - \varepsilon(\boldsymbol{u}) = 0, & \text{in } \Omega, \\ -\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}, & \text{in } \Omega, \\ \boldsymbol{u} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{u}$ is the displacement vector, and $\boldsymbol{f}$ is the body force. The strain tensor $\varepsilon(\boldsymbol{u})$ and the compliance operator $\mathcal{A}(\boldsymbol{\sigma})$ are defined as:

$$\varepsilon(\boldsymbol{u}) = \tfrac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{T}}), \quad \mathcal{A}(\boldsymbol{\sigma}) = \lambda_0\boldsymbol{\sigma} - \lambda_1 \operatorname{tr}(\boldsymbol{\sigma})I.$$

Here, λ and μ are the Lamé constants, and the coefficients are given by

$$\lambda_0 = \frac{1}{2\mu}, \quad \lambda_1 = \frac{\lambda}{2\mu(2\mu + d\lambda)},$$

where I denotes the identity tensor.

Let $\mathcal{T}_h$ be a simplicial mesh of Ω, and define the discrete stress space Σ_h and displacement space $V_h := \mathbb{P}_{k-1}^{-1}(\mathcal{T}_h)$. The mixed finite element method for the linear elasticity system seeks $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \Sigma_h \times V_h$ such that:

$$\begin{aligned} a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, \boldsymbol{u}_h) &= 0, \quad \forall\, \boldsymbol{\tau}_h \in \Sigma_h, \\ b(\boldsymbol{\sigma}_h, \boldsymbol{v}_h) &= (-\boldsymbol{f}, \boldsymbol{v}_h), \quad \forall\, \boldsymbol{v}_h \in V_h, \end{aligned} \tag{8}$$

where the bilinear forms are defined by

$$a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) = (\mathcal{A}(\boldsymbol{\sigma}_h), \boldsymbol{\tau}_h), \quad b(\boldsymbol{\tau}_h, \boldsymbol{u}_h) = (\operatorname{div}\boldsymbol{\tau}_h, \boldsymbol{u}_h).$$

4.2 Two-Dimensional Example

Let the exact solution be given by

$$\boldsymbol{u} = (\sin(5x)\sin(7y), \cos(5x)\cos(4y)),$$

on the domain $\Omega = (0,1)^2$, with parameters $\lambda_0 = 4$ and $\lambda_1 = 1$.

Figure 2 shows the numerical results for polynomial degrees $k = 2$, 3, and 5. The observed convergence rates confirm the theoretical estimates:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2} = O(h^{k+1}), \quad \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2} = O(h^k).$$

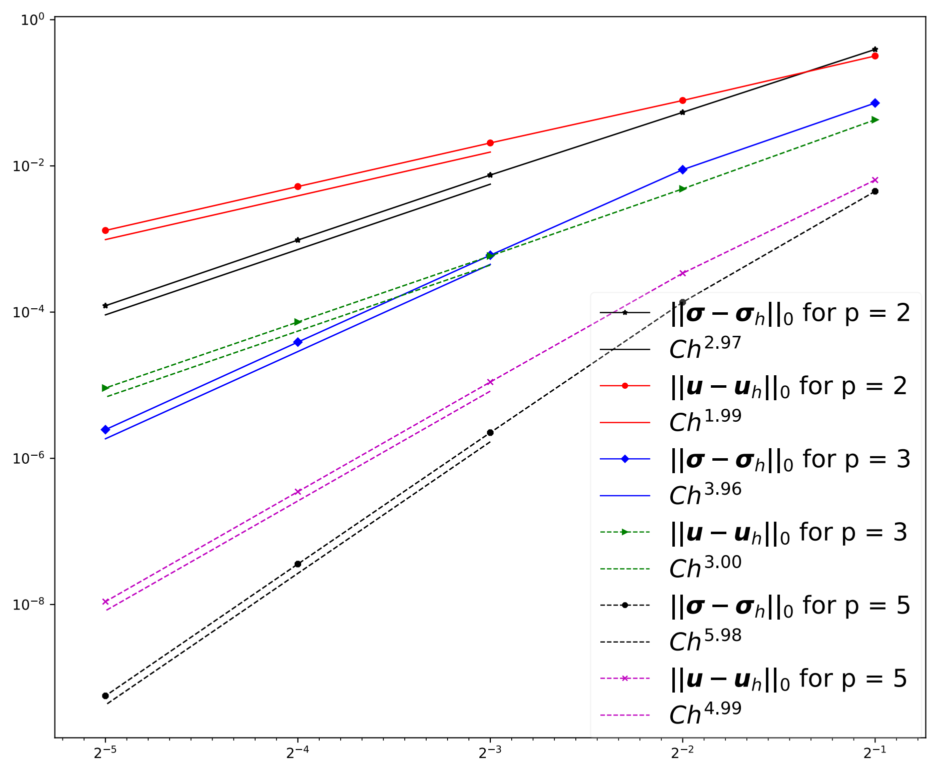

图 2: Convergence results for $k = 2, 3, 5$.

5 Conclusion

We have constructed a family of $H(\mathrm{div}, \mathbb{S})$-conforming finite elements that extend the Hu–Zhang framework to any polynomial degree $k \geq 2$. Under mild assumptions, the discrete divergence operator is onto, guaranteeing the inf–sup stability of the mixed formulation. Numerical experiments confirm the optimal convergence orders predicted by the theory.