# Dimer Models in Statistical Mechanics
# Lecture Notes

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# Chapter 1

# Lecture 1

## 1.1 Plan

- Introduction
- Dimer models
- Thermodymanic limit "$D \to \infty$"
- Six-vertex model, Yang Baxter equation
- Limit shape, variational principle, correlation functions

## 1.2 Equilibrium statistical mechanics

**Conventions**:

- $X$: finite set of states of a physical system.
- $E : X \to \mathbb{R}$: for each $x \in X$ we have energy $E(x)$.

**Definition 1.2.1** (Bolzmann distribution)**.** Let $T$ be the temperature, K be Bolzmann constant;

$$\text{Prob}[x] \propto \exp(-\frac{E(x)}{kT})$$

gives a probability distribution of states, which is called Bolzmann distribution.

The normalizing factor of this distribution(also known as the **partition function**) is

$$Z = \sum_{x \in X} \exp(-\frac{E(x)}{kT})$$

so we can get the normalized probability:

$$\text{Prob}[x] = \frac{1}{Z}\exp(-\frac{E(x)}{kT})$$

**Observables** are functions on $X$. The expectation value of function$f : X \to \mathbb{R}$ is:

$$\langle f \rangle = \mathbb{E}[f] = \sum_x f(x)\text{Prob}(x) = \frac{\sum_x f(x)e^{-\frac{E(x)}{kT}}}{Z}$$

For finite $X$, finding $Z$ and $\langle f \rangle$ are combinatorial problems. (Some time very nice formulae)

## 1.3 Cell complexes

We will start the description of models in statistical mechanics on crystal lattices. Crystal can be irregular, more generally, can we quite irregular solid body. Proper mathematical name of such solid body is cell complex. Cell complexes and statistical mechanics on them also appear in a discretization of Euclidean QFT. They play the role of a discrete space-time.

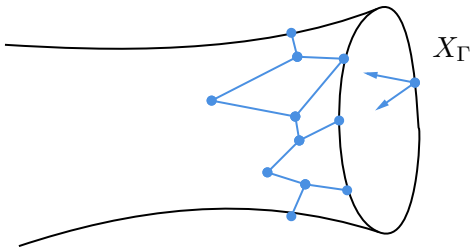
So, we first remind basic notions about cell complexes and then introduce local models in statistical mechanics on cell complexes.

For goals of statistical mechanics, language of cell complexes is not necessary. However, it is convenient and it is the right language for describing cutting and gluing properties of discrete space-time, and more general, for discrete geometry.

Cell complexes in question are cell decomposition of manifolds, possibly with boundary. Interesting "solvable" examples are known mostly in 2D, so we'll focus only on the 2D case.

**Definition 1.3.1.** A CW cell complex is the decomposition of a manifold into cells where each $k$-cell is homeomorphic to an open ball $B^k$ and cells are attached to each other in a natural way. A cell complex is called regular if homeomorphisms between open cells and open balls can be continued to the homeomorphism between corresponding closed cells and closed balls.

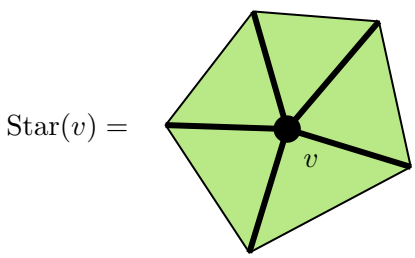
For two dimensional case, we will call 0-cell vertices, 1-cell edges and 2-cell faces. The collection of vertices and edges form a graph $\Gamma$ called 1-skeleton of this complex and it completely determined the complex. We can assume $X \cap \partial\Sigma$ is a cell complex for $\partial\Sigma$, i.e. on each connected component we have at least one vertex of $\Gamma$. This cell complex will be denoted as $X_\Gamma$.



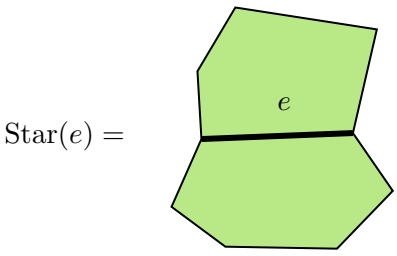

An important notion is the star of a subcomplex.

$$\mathrm{Star}(Y \subset X) = \bigcup_{\bar{c} \cap Y \neq \varnothing} c$$

**Example.** *for 2-dim. cell complex, stars of a vertex, an edge and a face(or their closure) are:*

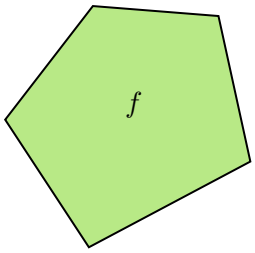



Star of a vertex consists of
all edges and faces adjacent to v



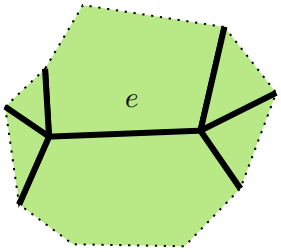

Star of an edge consists of
2 faces adjacent to e

Star$(f)$ =



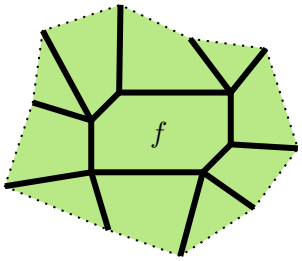

Star of a face is the face ifself

Star$(\overline{e})$ =




Star of a closed edge consists of edges adjacent to e and faces adjacent to its boundary

Star$(\overline{f})$ =

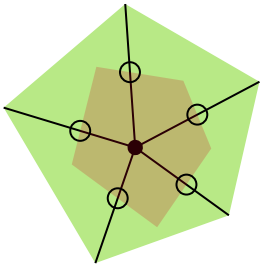



Star of a closed face consists of edges and faces adjacent to f and its boundary

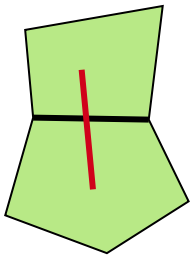
### 1.3.1 Dual cell complex

For a closed surface dual complex $X^\vee$ to $X$ is uniquely defined by a bijection $c \mapsto c^\vee \subset X^\vee$ where $c^\vee$ is the dual cell to $c$.

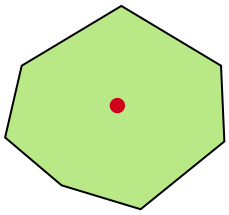
The dual cell is defined as a subset of Star$(c)$ such that $c^\vee \cap c$ is transeversive.



$c = v$



$c = e$

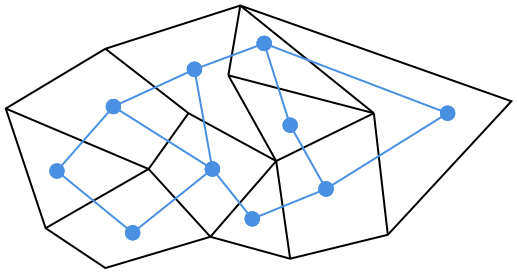


$c = f$

the dual of 0,1,2-cell (red part of the image)

We still need to clarify the dual structure near the boundary.

if $\partial\Sigma \neq \varnothing$ and $X$ is a regular CW complex we denote by $X^\vee$ the dual cell complex where the bijection $c \mapsto c^\vee$ only holds for $c$ out side the boundary. In other words, edges and vertices on the boundary do not have dual cells.

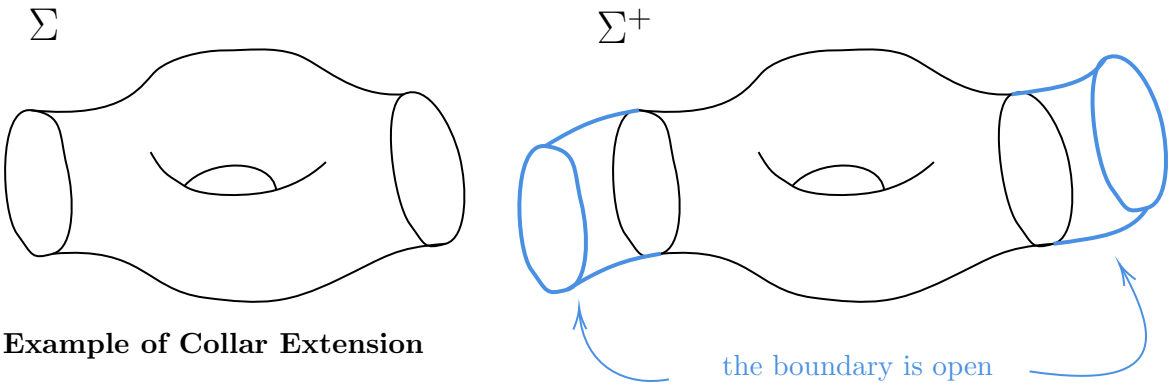


An example of dual of cell complex with boundary

### 1.3.2 Cell complexes, extensions and duals for surface with boundary

**Collared extensions**

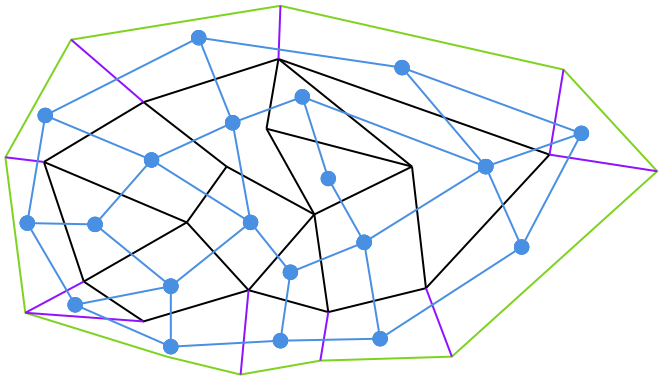
Let $\Sigma$ be a compact oriented surface. Denote by $\Sigma^+$ its collar extension. $\Sigma^+ \supset \Sigma$ and $\Sigma^+ \setminus \Sigma$ is homeomorphic to n open cylinders, one for each boundary component of $\Sigma$.




**Example of Collar Extension**

Assume we have a graph $\Gamma$ embedded into $\Sigma$ as a 1-skeleton of a regular CW-complex $X_\Gamma$ for $\Sigma$. We can denote $\Gamma$ an extension of $\Gamma$ to a skeleton of cell decomposition $X_\Gamma$ of $\Sigma^+$. We will define its dual as a cell complex with boundary.

In the following figure, $X_\Gamma$ is represented by black edges, with its collar extension in black purple and green. Green is not contained in $\Sigma^+$. The extended dual cell complex is in blue.



Cell decomposition (black) extended by purple and green edges

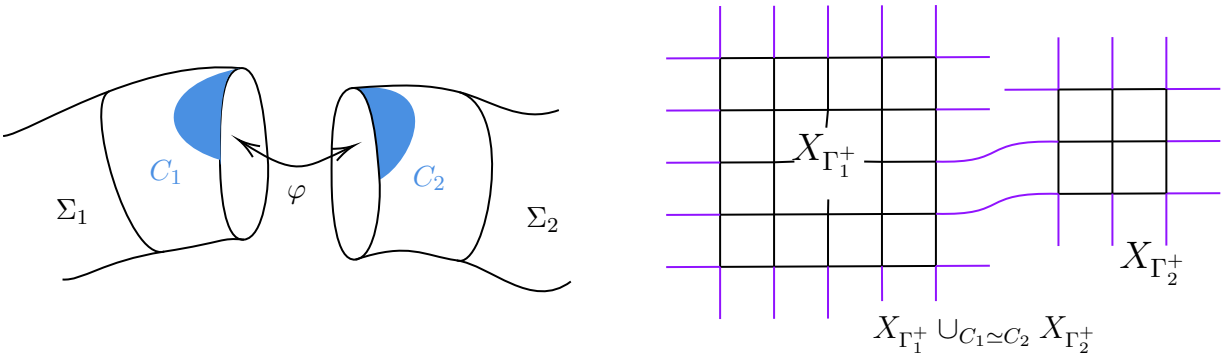
### 1.3.3 Cutting and gluing

Let $X_{\Gamma_1}$ and $X_{\Gamma_2}$ be cell decomposition of $\Sigma_{1,2}$ respectively and there collar extensions denoted as $X_{\Gamma_{1,2^+}}$.

Assume that

$$\varphi : \Sigma_1^+ \setminus \Sigma_1 \to \Sigma_2^+ \setminus \Sigma_2$$

is a mapping identifying part $C_1$ of the collar with part $C_2$ of the collar of $\Sigma_2^+$. Then we can glue them into $X_{\Gamma_1^+} \cup_{C_1 \simeq C_2} X_{\Gamma_2^+}$ as what follows.




Example of gluing two surfaces and gluing two complexes

### 1.3.4 chain complexes

- $\Gamma \subset \Sigma$ connected and simply connected lattice domain in oriented compact, possibly with boundary surface $\Sigma$.
- Chain complex: $C_\bullet(\Gamma) = C_0(\Gamma) \oplus C_1(\Gamma) \oplus C_2(\Gamma)$

$$C_0(\Gamma) = \{\sum_v c_v v, c_v \in \mathbb{R}\}, C_1(\Gamma) = \{\sum_e c_e e, c_e \in \mathbb{R}\}, C_2(\Gamma) = \{\sum_f c_f f, c_f \in \mathbb{R}\}$$

- Boundary operation $\partial$:
  - $\partial f = \sum_{e \in \partial f} (-1)^{\epsilon(e,f)} e$
  - $\partial e = e_+ - e_-$, $e$ point to $e_-$ from $e_+$
  - $\partial v = 0$

  where $\epsilon$ is $+1$ when $e$ is along $\partial f$, $-1$ when $e$ is opposite to $\partial f$.

Let $\Sigma^+$ be a collar extension of $\Sigma$ and $\Gamma^+$ extension of $\Gamma$. We note that border edges in $\Gamma^+$ are half-open, which contains only one boundary vertex. The cell complex $\partial\bar{\Sigma^+} \cap \bar{X_{\Gamma^+}}$ is uniquely determined by $\Gamma^+ \subset \Sigma^+$.

$$C_\bullet(X_{\Gamma^+}; \mathbb{Z}) \simeq C_\bullet(\bar{X_{\Gamma^+}}, \partial\bar{X_{\Gamma^+}}; \mathbb{Z})$$

then from Poincaré's duality, we have the non-degenerate intersection pairing for closed manifolds.

$$C_k(X_\Gamma; \mathbb{Z}) \otimes X_{2-k}(X_\Gamma^\vee; \mathbb{Z}) \to \mathbb{Z}$$

for manifold with boundary, this also holds as the map:

$$C_k(\bar{X_{\Gamma^+}}, \partial\bar{X_{\Gamma^+}}; \mathbb{Z}) = C_k(X_{\Gamma^+}; \mathbb{Z}) \otimes X_{2-k}(X_{\Gamma^+}^\vee; \mathbb{Z}) \to \mathbb{Z}$$

## 1.4 Local model of statistical mechanics on cell complexes

We can first define the space of states as following:

a) For each cell $c \subset X_\Gamma$ we define local space of states $S_c$ as a topological space with measure. In our case, this will be always a finite set.

b) the total space will be subset of product of all local space.

The concrete definition of $S_c$ depend on the model.
We will also define **local Boltzmann weight** as a function:

$$w_c : S_c \subset \prod_{c' \in \mathrm{Star}(\bar{c})} S'_c \to \mathbb{R}_{\geq 0}$$

$$s \mapsto \exp -\frac{E(s)}{kT}$$

Where $E(s)$ is the energy of local state $s \in S_c$. Then the Boltzmann measure is

$$\mathrm{Prob}(s) = \frac{w(s)}{Z_\Gamma}$$

where $w(s) = \prod w_c(s|_c)$ is weight of $s \in S_\Gamma$, and $Z_\Gamma$ is partition function $\sum w(s)$.

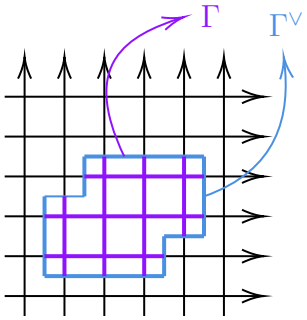
We will introduce specific models from now, do operations like cutting and gluing and study their partition functions.

### 1.4.1 Vertex models: 6-vertex models

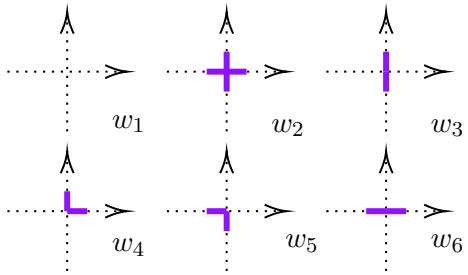
In these models Boltzmann weights are only assigned to vertices.(Edges have weight 1.)

**Data:**

- Lattice domain $D \subset \mathbb{Z}^2 \subset \mathbb{R}^2$ open, bounded by a subgraph domain $\Gamma^\vee$.
- States: path descending down that only turn right or down.




$\Gamma$ and $\Gamma^\vee$




six different vertices, with six nonzero weights $w_i$

- Partition function:

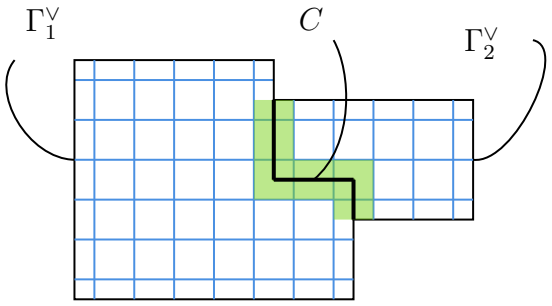
$$Z_\Gamma(b) = \sum_s \prod_{v \in \Gamma} w(s_v)$$

where $s$ are configurations of paths with entry/exit points $b$, $s_v$ is the path configuration around $v$ and $w(s_v)$ is one of the weights from above.

**Gluing two regions:**

If $\Gamma_1$ and $\Gamma_2$ are two regions with common subgraph $C$, we can derive the partition functions for $\Gamma_1$, $\Gamma_2$ and the result of gluing them to $\Gamma_1 \#_C \Gamma_2$ are related as follows:

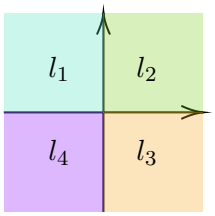
$$Z_{\Gamma_1 \#_c \Gamma_2}(b_1, b_2) = \sum_{b \subset C} Z_{\Gamma_1}(b_1, b) Z_{\Gamma_2}(b, b_2)$$




$b$ denotes the edges that intersect $C$ and occupied by pathes

### 1.4.2 Vertex model: The SOS(Solid-on-Solid) model

This is another example of vertex model on an oriented square grid where states are assigned to faces. The space of states for an ultra local neighbourhood of a vertex is defined as the set of non-negative integers,on per neighbouring face:




Local states of this model are given by the set
$S_v = \{(l_1, l_2, l_3, l_4) | l_i \geqslant 0, |l_i - l_{i+1}| = |l_4 - l_1| = 1\}$

Weights given by non-negative function $w(l_1, l_2, l_3, l_4)$.

For special choices of such function the model is integrable. Let $\Gamma$ be a closed domain in the a square lattice. It's extended dual $\Gamma^{\vee +}$ has open border faces.

**Partition function**

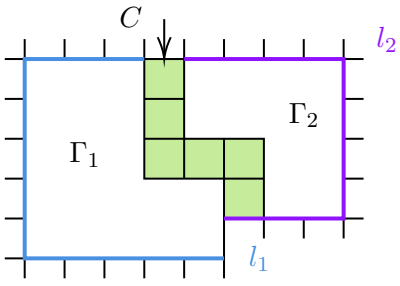
Fix states $l_1, l_2, ...$ on border faces, then

$$Z_\Gamma^{\mathrm{SOS}}(l) = \sum_{m \text{ on } F(\Gamma)} \prod_v w(m_v)$$

Here the sum is taken over configurations(states) on faces of $\Gamma$, $w(m_v)$ is the weight of this configuration $m_v$ on adjacent faces.

**Gluing two regions**

$\Gamma_1$ and $\Gamma_2$ are regions with common border faces $C \subset \partial\Gamma_{1,2}^\vee$, and $\Gamma_1 \# \Gamma_2$ is the result of the gluing along $C$, we have:




$Z_{\Gamma_1 \# \Gamma_2}(l_1, l_2) = \sum_l Z_{\Gamma_1}(l_1, l) \times Z_{\Gamma_2}(l, l_2)$

Gluing two regions in SOS model.

**The RSOS model**(Restricted SOS model)

In this case only an additional constraint on local states: $l_i <= r$

### 1.4.3 Edge models: The Ising model

**Data:**

- Lattice domain $D \subset \mathbb{Z}^2 \subset \mathbb{R}^2$
- Interaction constant: $J \in \mathbb{R}$
- States: $S = \{s : v(\Gamma) \to \{\pm\}, v \mapsto \sigma_v = \pm 1\}$ ($\sigma_v$ often referred as **spin** of vertex $v$.)
- Energy of state $s \in S$ defined as

$$E(s) = -J \sum_{\substack{\text{edges} \\ (v,v')}} \sigma_v \sigma_{v'}$$

  - ferromagnetic if $J > 0$
  - antiferromagnetic if $J < 0$

**Boltzmann distribution:**

$$Z = \sum_s \exp(-\frac{E(s)}{kT})$$

$$\text{Prob}(s) = \frac{1}{Z}\exp(-\frac{E(s)}{kT})$$

We can define **local correlation functions** for Ising model:

$$G_{v_1 \ldots v_n} = \sum_s \text{Prob}(s)\sigma_{v_1} \ldots \sigma_{v_n}$$

It characterizes how "spins" assigned to different vertices are correlated. Consider 2 limits:

1) $T \to 0$.
   If $J > 0$, $s_\pm := \{\sigma_v = \pm\}$ then (H.W.):

$$\text{Prob}(s_\pm) \to \frac{1}{2};$$

$$\text{Prob}(s) \to 0, s \neq s_\pm.$$

   Only strong correlated state exists.

2) $T \to \infty$.

$$\text{Prob}(s) \to \frac{1}{2^{|V|}}$$

   States on different vertices are distributed independently (totally decorrelated).

One of the main goals of statistical mechanics is to find local correlation functions.

**Thermodynamic limit:** $D \to \infty$

Typically, when talking about $D \to \infty$ we mean the following:

- Fix $\mathbb{D} \subset \mathbb{R}^2$ connected, simply connected.
- $\varphi_\epsilon : \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2$ for $\epsilon > 0$. (with width of each lattice $= \epsilon$.)
- $D_\epsilon := \varphi(\mathbb{Z}^2) \cap \mathbb{D}$.
- $D_\epsilon \to \infty$ means $\epsilon \to 0$, while microscopic distance is finite on $\mathbb{Z}$, macroscopic volume is finite in $\mathbb{D}$.

**Phase transition:** [this subsection is now left blank]

# Chapter 2

# Lecture 2

## 2.1 Dimer models on graphs

**Definition 2.1.1** (Dimer configuration)**.** Dimer configuration on $\Gamma$ is a perfect matching on vertices connected by edges:

- occupied edges cannot overlap on vertices
- all vertices should be covered.

**Energy function:**

$$E : E(\Gamma) \to \mathbb{R}, e \mapsto E(e).$$

$E(e)$ is energy of occupying edge $e$. Therefore, the energy of a dimer configuration $\mathcal{D} \subset E(\Gamma)$ is:

$$E(\mathcal{D}) = \sum_{e \in \mathcal{D}} E(e)$$

Here we do not have "dynamical" interaction.

**Bolzmann distribution:**

$$\text{Prob}(\mathcal{D}) \propto \exp(-\frac{1}{kT} E(\mathcal{D})) = \prod_{e \in \mathcal{D}} w(e)$$

where $w(e)$ defined as $\exp(-\frac{1}{kT} E(e))$. After normalization we will get

$$Z = \sum_{\mathcal{D}} \prod_{e \in \mathcal{D}} w(e)$$

$$\text{Prob}(\mathcal{D}) = \frac{1}{Z} \prod_{e \in D} w(e)$$

*Remark.*

1) For a physicist:

   One can think of a dimer as a pair of fermions, one at each end. This agrees with **Pauli principle**: 2 identival fermions cannot be in the same state.

   Dimer covering can be regarded as a dense covering of a graph by such pairs connected by edges.

2) If we relax denseness condition and the exclution condition, but do not allow any edge to be occupied twice:

$$Z = \sum_{\text{occupation}} \prod_{e\in\mathcal{D}} w(e) = \prod_{e\in E(\Gamma)} (1 + w(e))$$

3) If we relax dimer conditions, and allow multiple occupation:

$$Z = \prod_{e\in E(\Gamma)} (1 - w(e))^{-1}$$

**Local correlation functions:**

Local observables, conditional probabilities $\langle\sigma_{e_1}...\sigma_{e_n}\rangle = \text{Prob}(e_i \subset \mathcal{D})$ can also be regarded as expectation values of characteristic functions on dimer configurations:

$$\sigma_e(\mathcal{D}) = \begin{cases} 1, e \in \mathcal{D} \\ 0, e \notin \mathcal{D} \end{cases}$$

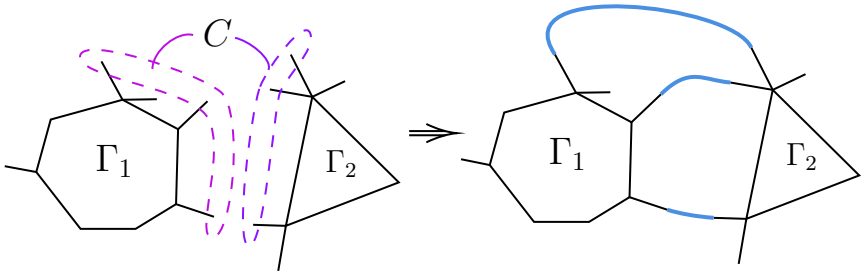
Claim (H.W.):

1) $\langle\sigma_{e_1}...\sigma_{e_n}\rangle = \sum_{\mathcal{D}\subset\Gamma} \sigma_{e_1}(\mathcal{D})...\sigma_{e_n}(\mathcal{D})\text{Prob}(\mathcal{D})$

2) $\langle\sigma_{e_1}...\sigma_{e_n}\rangle = \prod_{i=1}^{n} w(e_i) \frac{\partial^n}{\partial w(e_1)...\partial w(e_n)} \ln Z$

### 2.1.1 Gluing graphs and dimer partition functions

Define the border of a graph $b(\Gamma)$ as the set of 1-valent vertices, and adjacent edges, and edge open at one end. If a subgraph $C \subset b(\Gamma_1)$ is identified with a subgraph in $b(\Gamma_2)$, we can make new graph $\Gamma_1 \# \Gamma_2$, the result of the gluing $\Gamma_1$ to $\Gamma_2$ via C.




A dimer partition function on a graph $\Gamma$ with fixed boundary conditions is

$$Z_\Gamma(\mathcal{D}_b) = \sum_{\substack{\mathcal{D}\subset\Gamma \\ \mathcal{D}|_b(\Gamma)=\mathcal{D}_b}} \prod_{e\in\Gamma_{\text{int}}\cap\mathcal{D}} w(e)$$

We take the sum over all possible dimer configurations on $\Gamma$ with fixed dimer configuration $D_b$ on the border of $\Gamma$.

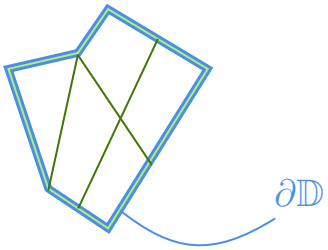
If we glue a triple $(\Gamma_1, \Gamma_2, C)$ where $C$ identified as subgroup in each $\Gamma_i$, then we can derive the total partition function as:

$$Z_{\Gamma_1\#_c\Gamma_2}(\mathcal{D}_{b_1}, \mathcal{D}_{b_2}) = \sum_{\mathcal{D}_b\subset C} \prod_{e\in\mathcal{D}_b} w(e) Z_{\Gamma_1}(\mathcal{D}_{b_1}, \mathcal{D}_b) Z_{\Gamma_2}(\mathcal{D}_b, \mathcal{D}_{b_2})$$

where $\mathcal{D}_b$ are dimer configuration on C, $\mathcal{D}_{b_i}$ are those on rest of $b(\Gamma_i)$.

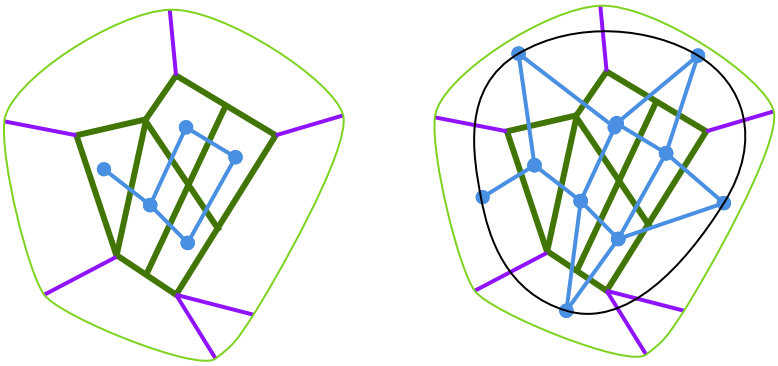
### 2.1.2 Dimers and tilings

let $\Gamma \subset \mathbb{D} \subset \mathbb{R}^2$ be a plane graph (finite, connected) defining a cell complex $X_\Gamma = \Gamma$ with:




- •vertices 0-cells
- •edges 1-cells
- •faces(regions) 2-cells

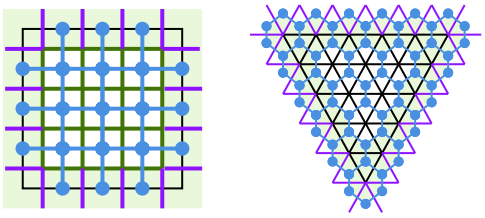
We assume that $\Gamma|_{\partial\mathbb{D}}$is a subcomplex of $X_\Gamma$, i.e., vertices and edges of $\Gamma$ on $\partial\mathbb{D}$(in blue). Remind that $X_\Gamma$ defined earlier does not have boundaries. In order to give a functorial description of dimer models and equivalent tilings we need cutting and gluing operations. For this we need an extension of $\Gamma$ and of the dual cell complex with we outlined earlier. Recall collar extension (lime and purple):



Unextended dual cell complex $X_\Gamma^+$ (left)
and extended dual cell complex $X_{\Gamma^+}^\vee$(right)

The extended dual complex is constructed as the following way: First, we choose a collar extension $\mathbb{D}^+ \supset \mathbb{D}$ and an extension $\Gamma^+$ of $\Gamma$ by edges ending with 1-valent vertices laying on $\partial\mathbb{D}^+$.(Remark: This is not contained in $\Gamma$)

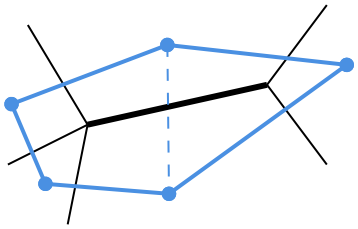
Such extension defines a cell complex $X_{\Gamma^+}$ and we can use it's dual as extended dual complex. When working on lattice domain in $\mathbb{R}^2$, the extension become natural as extension by "one layer".



**Bijection between dimers and tilings**

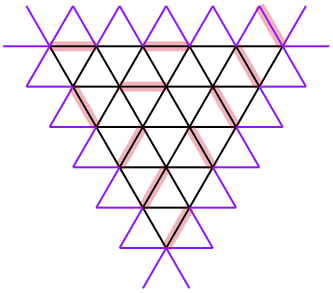
As before a dimer configuration on $\Gamma^+$ means that some 1-valent vertices may not be matched to adjacent ones.

The bijection between dimers on $\Gamma^+$ and double tiles inside $X_{\Gamma^+}$js natural:

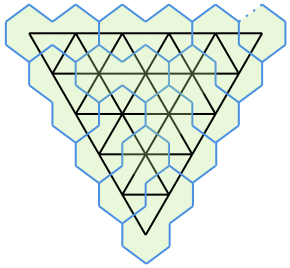


The natural bijection:
Dimers in $\Gamma$ (black);
and 2-cell pair it corresponds to (blue).

If a dimer is on an extension edge of $\Gamma^+$, it corresponds to half of double tile, i.e. a single cell of $X^\vee_{\Gamma^+}$:



Dimers on $\Gamma_v$

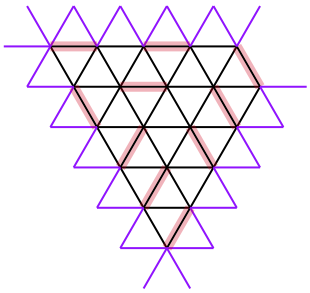


Corresponding tiling of $X^\vee_{\Gamma^+_v}$, by double tiles and one single tile.

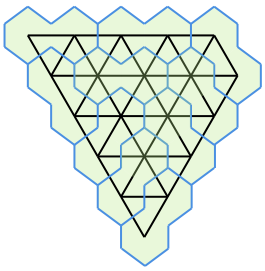
We can also consider about tilings with no single tiles. Given a dimer configuration $\mathcal{D} \subset \Gamma^+$ with dimers occupying edges outside $\Gamma$. Remove vertices $v_1, v_2, v_3, ...$ adjacent to these edges, Denote the resulting graph $\Gamma_{v_1,v_2,v_3,...}$.

Clearly $\mathcal{D} \cap \Gamma_{v_1,v_2,v_3,...}$ is a dimer covering of $\Gamma_{v_1,v_2,v_3,...}$. Let $\Gamma^+_{v_1,v_2,v_3,...}$ be the graph $\Gamma^+$ with vertices $v_1, v_2, v_3, ...$ and all edges adjacent to these vertices being removed.

Then, we can derive a new bijection between dimer configurations on $\Gamma_{v_1,v_2,v_3,...}$ and tilings of $X^\vee_{\Gamma^+_{v_1,v_2,v_3,...}}$ by double tiles as below:



Dimers on $\Gamma_{v_1,v_2,...}$



Corresponding tiling of $X^\vee_{\Gamma^+_{v_1,v_2,...}}$, by double tiles only

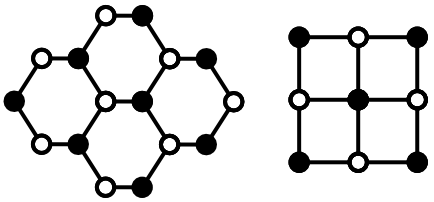
## 2.2 Biparitite graphs

**Definition 2.2.1.** $\Gamma$ is said to have a **bipartite structure** if vertices of $\Gamma$ are devided into 2 types(say black and white):

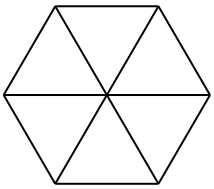
$$V(\Gamma) = B \sqcup W$$

such that there are no edges between black and black or white and white vertices.

**Example.** *Here are some examples and non-examples:*

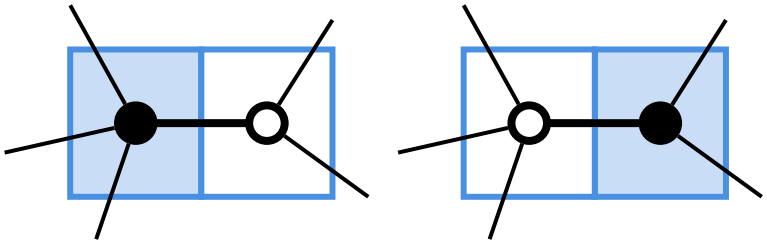


*Example of bipartite graphs*



*An non-example: Remind that there are no bipartite structure on a triangular lattice*

On a dual cell complex to a bipartite graph we have cell of two types, and bipartite structure means no two 2-cells with same colour in the dual complex share an edge.



Colors of faces on dual cell corresponding to colors on vertices in $\Gamma$

# Chapter 3

# Lecture 3

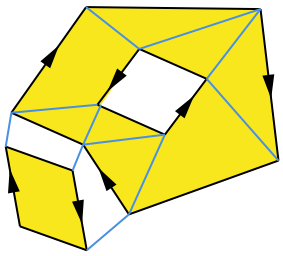
## 3.1 Height functions on plane graphs

Here we will describe an equivalence between dimer configurations on a plane bipartite graph and random surfaces.

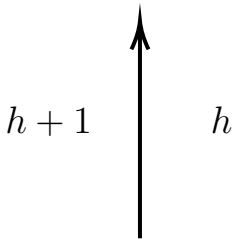
Let $\mathcal{D}_1, \mathcal{D}_2 \subset \Gamma$ be a pair of dimer coverings of $\Gamma$. They form a collection of non-intersecting non-selfintersecting closed path covering all vertices of $\Gamma$.

Paths of length $> 2$ are connected components of symmetric difference $D_1 \Delta D_2$, we will call them composition cycles. Now assume $\Gamma$ bipartite and orient edges occupied by $\mathcal{D}$ from black points to white points. This gives an orientation to composition cycle $\mathcal{D}_1 - \mathcal{D}_2$.

Now, assume the graph is planar, without 1-valent vertices, defining a cell complex for a connected simply connected domain $\Gamma \subset \mathbb{D} \subset \mathbb{R}^2$. consider them as level curves of a function $h_{\mathcal{D}_1,\mathcal{D}_2}$ on faces of the cell complex $X_\Gamma$. The function (called **height function**) is unique up to a constant.



Yellow: Height -1
White: Height 0




In order to fix the height function, we can set the outer part of $\Gamma$ is a cell called "outer cell", and it has feight 0.

It's easy to check that

$$h_{\mathcal{D}_1\mathcal{D}_2} + h_{\mathcal{D}_2\mathcal{D}_3} = h_{\mathcal{D}_1\mathcal{D}_3}$$

define the space $\mathcal{H}_{\Gamma,\mathcal{D}_0}$ as space of relative height functions parallel to $\mathcal{D}_0$. This space is in bijection with dimer configurations.

### 3.1.1 A more general definition of height function

**Notation:**

- $\Gamma \subset \mathbb{D} \subset \mathbb{R}^2$ connected and simply connected lattice domain
- Chain complex: $C_\bullet(\Gamma) = C_0(\Gamma) \oplus C_1(\Gamma) \oplus C_2(\Gamma)$

$$C_0(\Gamma) = \{\sum_v c_v v, c_v \in \mathbb{R}\}, C_1(\Gamma) = \{\sum_e c_e e, c_e \in \mathbb{R}\}, C_2(\Gamma) = \{\sum_f c_f f, c_f \in \mathbb{R}\}$$

- Boundary operation $\partial$:
  - $\partial f = \sum_{e\in\partial f}(-1)^{\epsilon(e,f)}e$
  - $\partial e = e_+ - e_-$, $e$ point to $e_-$ from $e_+$
  - $\partial v = 0$

  where $\epsilon$ is $+1$ when $e$ is counter-clockwise, $-1$ when $e$ is clockwise.

**Theorem 3.1.1** ((H.W.)). $\partial^2 = 0$

The story keeps the same when dealing with collar extension of $\Gamma \subset \mathbb{D}$ into $\Gamma^+ \subset \mathbb{D}^+$, providing we dismiss the cell at boundary and define

$$C_\bullet(X_{\Gamma^+;\mathbb{Z}}) \simeq C_\bullet(\bar{X}_{\Gamma^+}, \partial\bar{X}_{\Gamma^+};\mathbb{Z})$$

Assume that a dimer configuration on $\Gamma^+$ is a matching of vertices of $\Gamma^+$ connected by edges, perfect everywhere, except, may be some of the 1-valent vertices. In other words, some vertices are not covered by dimers.

We can define 1st homology of $C_\bullet(\Gamma^+)$ as

$$H_1 = \frac{Z_1}{B_1}$$

Where $Z_1 = \{c \in C_1, \partial c = 0\} \subset C_1$ set of closed cycles; $B_1 = c \in C_1, c = \partial a \subset Z_1$ set of boundaries. Since $\Gamma$ is asked to be simply connected, we can prove that $H_1 = 0$. This means, $c \in C_1, \partial c = 0$ implies $c = \partial h$ for some $h \in C_2$.

Now assume $\Gamma$ is a biparite graph. We can orient each edge from black point to white point. Let $\mathcal{D}$ be a dimer configuration, define

$$\hat{\mathcal{D}} = \sum_{\mathcal{D}} e \in C_1(\Gamma)$$

$$\partial\hat{\mathcal{D}} = \sum_{b\in B} b - \sum_{w\in W} w = B - W$$

Consider $\omega \in C_1(\Gamma)$ s.t. $\partial\omega = B - W$, we can get $\partial(\mathcal{D} - \omega) = 0$ and therefore we can define $h_{\mathcal{D},\omega}$ s.t. $\partial h_{\mathcal{D},\omega} = \mathcal{D} - \omega$. This 2-chain can also be explicitly constructed, and it is a function of faces, say **height function**.

**Explicit construction of $h_{\mathcal{D},\omega}$:**

Fix $f_0$, a face on $\Gamma$, a path $\gamma$ for each face $f$ to connected to $f_0$ and the value $h_{\mathcal{D},\omega}(f_0)$, Then:

$$h_{\mathcal{D},\omega}(f) := \sum_{e\cap\gamma\neq\varnothing} \epsilon(e,\gamma)(\sigma_{\mathcal{D}}(e) - \omega(e)) + h_{\mathcal{D},\omega}(f_0)$$

where $\epsilon(e,\gamma)$ is defined to be $+1$ if $\gamma$ walk through e from right side, and $-1$ from left side. In fact, the function we constructed is independent to $\gamma$, and has boundary $\hat{\mathcal{D}} - \omega$.

This equation holds for $\mathcal{D} - \omega = \partial h$, so $\partial h(e) = h(f+) - h(f-)$, where $f_\pm$ is the face at the left/right side of the edge(with orientation). So the height function varies along the path from $f_0$. Since we can globally define the function uniquely after $h(f_0)$ fixed, this construction is independent to the path we chose.

**Remark:** This definition extends height functions on $\Gamma$ to height functions on $\Gamma^+$, with some of them has non trivial value "outside" the boundary.(As a comparison, we can always fix height of the outer cell as 0 when defining height function on $\Gamma$.)

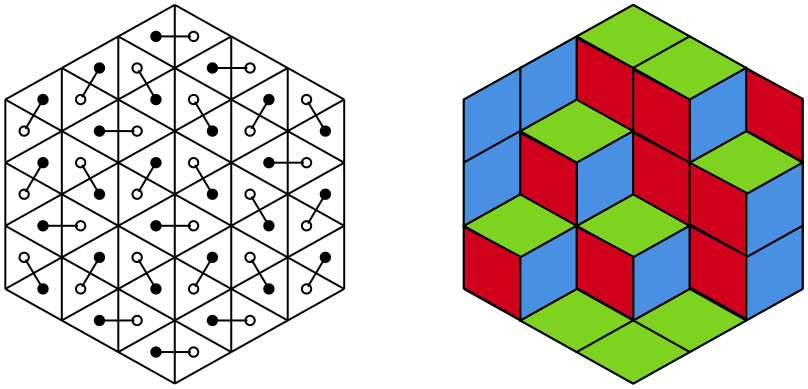
Denote the space of such height functions on $X_{\Gamma^+}$ as $\mathcal{H}_{\Gamma^+,\omega}$. It has a subspace of functions corresponding to dimer configurations on $\Gamma$ without dimers on border edges of $\Gamma^+$, where all faces near the border has same value.

$$\mathcal{H}^{(0)}_{\Gamma^+,\omega} = \{h : F(\Gamma^+) \to \mathbb{R} | \partial h = \hat{\mathcal{D}} - \omega \text{ for dimer config.} \mathcal{D}\}$$

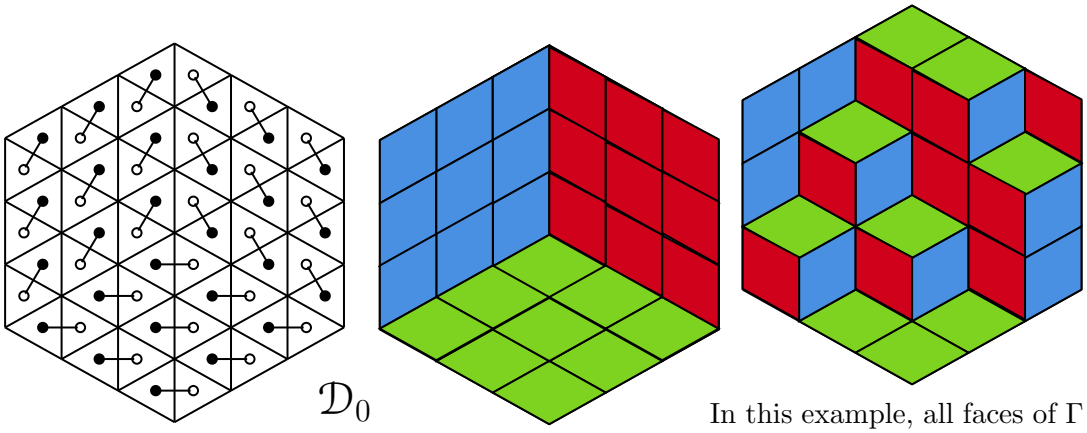
## 3.2 Examples of height functions

Now let us consider some examples and connect the above construction to relative height functions constracted earlier.

**Example 1:** Let $\Gamma$ be a region in a hexagonal lattice. Extend it by on layer of the lattice. Let $X^{\vee}_{\Gamma^+}$ be the dual cell complex. Here the bijection is shown as following:

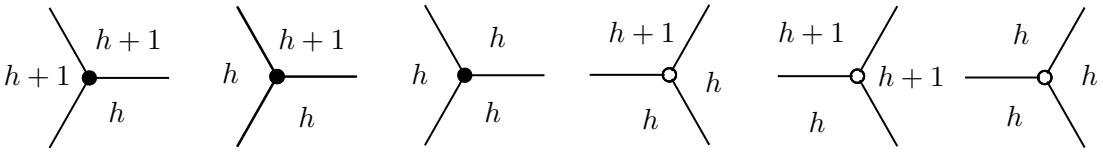


Naturally, this tilling can be naturally recognized as piles of boxes in a cubic room. The height function of $\mathcal{D}$ to $\mathcal{D}_0$ has a natural meaning: the distance of the closest corner to the floor/wall along the $(1,1,1)$ direction.




In this example, all faces of $\Gamma$ has height 0 or 1

We still have other choices. If we choose $w(e) = 1/3$ for all edges(we always can do this if all vertices has the same valence), then the height function will be the height along the (1,1,1) direction to the lowest point. If we let $w(e)$ has non-zero value only on "horizonal" edges, then the height function will be the vertical height from the floor. These height function satisfy one of the following condition that follows.



## 3.3 Partition function

Here we will use the same notations as earlier in the discussion of the bijection between tilings and dimers.

We want to rewrite the dimer partition function in terms of height functions. $\Gamma$, $\Gamma^+$ are set as earlier and 1- valent vertices are removed. Then the dimer pertition function on $\Gamma^+$ with a fixed boundary dimer covering $\mathcal{D}_b \subset b(\Gamma^+) = \Gamma^+ \setminus \Gamma$ is

$$Z_{\Gamma^+}(\mathcal{D}_0) = \sum_{\mathcal{D}|_b = \mathcal{D}_b} \prod_{e \subset \Gamma \cap \mathcal{D}} w(e)$$

with Boltzmann distribution:

$$\mathrm{Prob}_{\Gamma^+}(\mathcal{D}|\mathcal{D}) = \frac{\prod_{e \subset \Gamma \cap \mathcal{D}} w(e)}{Z^+_\Gamma(\mathcal{D}_b)}$$

where $w(e) = \exp(-E(e))$, $E$ is the energy function. We have:

$$\begin{aligned}\sum_{e \in \mathcal{D}} E(e) &= \sum_{e \in \Gamma^+} E(e)\sigma_{\mathcal{D}}(e) = \sum_{\Gamma^+} \omega(e) + \sum_{\Gamma^+} E(e)(\sigma_{\mathcal{D}}(e) - \omega(e)) \\ &= \sum_{e \in \Gamma^+} E(e)\omega(e) + \sum_{f \in F(\Gamma)} h_{\mathcal{D},\omega}(f) \sum_{e \in \partial f} \epsilon(e,f)E(e) + \sum_{f_b} h_{\mathcal{D},\omega}(f) \sum_{e \in \partial f_b \cap \Gamma} \epsilon(e,f)E(e)\end{aligned}$$

where $\epsilon$ defined as $+1$ when $e$ is counter-clockwise, $-1$ when clockwise. Define:

$$q_f = \prod_{e \in \partial f} w(e)^{\epsilon(e,f)}$$

$$q_{f_b} = \prod_{e \in \partial f \cap \Gamma} w(e)^{\epsilon(e,f)}$$

then we can conclude that

$$\prod_{\mathcal{D}} w(e) = \prod_{\Gamma} w(e)^{\omega(e)} \prod_{f} q_f^{h_{\mathcal{D},\omega}(f)} \prod_{f_b} q_{f_b}^{h_{\mathcal{D},\omega}(f_b)}$$

Now we can define the space of height function with boundary condition fixed $\mathcal{H}_{\Gamma^+,\omega}(\mathcal{D}_b)$.

**Defining space of height function without dimer pictures**

First, we need the space of boundary height functions

$$\mathcal{H}^b_{\Gamma^+,\omega} = \{h : F(\Gamma^+ \setminus \Gamma) \to \mathbb{R} | h(f_+) - h(f_-) = \epsilon(e)(\kappa - \omega(e)), \kappa = 0, 1\}$$

here $\epsilon(e)$ is 1 if $e$ comes into $\Gamma$, otherwise $-1$. We can think $\kappa = 1$ implies the appearance of an 1-valent vertex on boundary of $\Gamma$.

Given $h_b$ a boundary height function, we can define the space of

$$\mathcal{H}_{\Gamma^+,\omega}(h_b) = \{h : F(\Gamma) \to \mathbb{R} | h(f_+) - h(f_-) = \kappa - \omega(e), \kappa = 0, 1 \text{ and } \kappa = 1 \text{ only once in each Star}(v)\}$$

where $f_\pm$ represents the left/right face of $e$.We have those theorems:

**Theorem 3.3.1.** *1)*

$$Z_{\Gamma^+}(\mathcal{D}_b) = \prod_{e \in \Gamma} w(e)^{\omega(e)} Z_{\Gamma^+}(h_b)$$

*where $h_b$ is the height function $h_{\mathcal{D},\omega}$ for boundary(only depends on $(\Gamma^+, \omega, \mathcal{D}_b)$), and:*

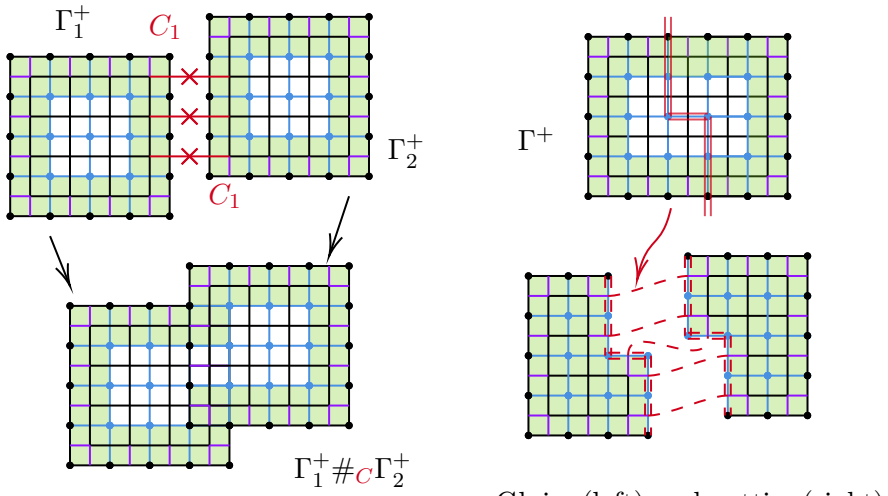
$$Z_{\Gamma^+}(h_b) = \prod_{f_b} q_{f_b}^{h_b(f_b)} \sum_{h \in \mathcal{H}_{\Gamma^+,\omega}(\mathcal{D}_b)} \prod_{f \in F(\Gamma)} q_f^{h(f)}$$

*2) we can define*

$$\mathrm{Prob}_{\Gamma^+,\omega}(h_b) = \frac{\prod_{f\in F(\Gamma)} q_f^{h(f)}}{\sum_{h\in\mathcal{H}_{\Gamma^+,\omega}(\mathcal{D}_b)} \prod_{f\in F(\Gamma)} q_f^{h(f)}} = \mathrm{Prob}_{\Gamma^+}(\mathcal{D}|\mathcal{D}_b)$$

## 3.4 Gluing properties

Now Let us reformulate the gluing property of dimer partition functions in terms of height function partition functions. We set $(\Gamma_1, \Gamma_1^+, \Gamma_2, \Gamma_2^+, C_1 \simeq C_2)$where $\Gamma^+$ collar extension of $\Gamma_i$ with subgraph of boundary $C_i$.

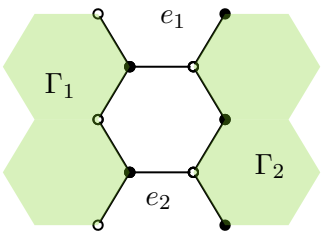



Gluing(left) and cutting(right)

We translate the gluing property of partition function into height partition function to get the form:

$$Z_{\Gamma_1^+\#_C\Gamma_2^+}(h_{b_1}, h_{b_2}) = \prod_{b_1(\Gamma_1^+)} q_f^{h(f)} \prod_{b_2(\Gamma_2^+)} q_f^{h(f)} \sum_{\mathcal{H}_{C,\omega}} Z_{\Gamma_1^+}(h_{b_1}, h_C) Z_{\Gamma_2^+}(h_C, h_{b_2}) \prod_C \tilde{q}_f^{h_c(f)}$$

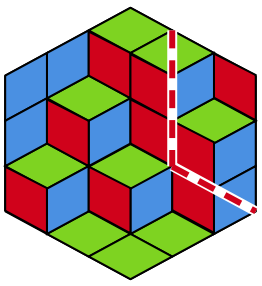
where $\mathcal{H}_{C,\omega}$ is the space of height functions on the subcomplex $C$, $\tilde{q}_f$ defined as:




$$\tilde{q}_f = q_f^L q_f^R \omega(e_1)^{\epsilon(e_1,f)} \omega(e_2)^{\epsilon(e_2,f)}$$

$$q_f^L = \prod_{\Gamma_1\cap\partial f} \omega(e)^{\epsilon(e,f)};\ q_f^R = \prod_{\Gamma_2\cap\partial f} \omega(e)^{\epsilon(e,f)}$$

An example of cutting in Dimer model which often gives 1-valent vertices(1-cell tile in dual cell complex) is shown below.



**Example:**
A typical result of cutting a tiled dual cell complex:
Pair of tilings including 1-cell tile at boundary

# Chapter 4

# Lecture 4

## 4.1 Kosteleyn solution

This chapter we will introduce computation of the partition function and of correlation functions in terms of Pfaffians.

**Definition 4.1.1** (Kasteleyn orientation)**.** A Kasteleyn orientation of a plane graph is a orientation of edges such that $\prod_{\partial f} \epsilon(e, f) = -1$. If an oriented edge $e$ has a same orientation as in $K$, say $\epsilon_e^K = 1$, otherwise $\epsilon_e^K = -1$.

Kasteleyn orientation always exists. We can construct it following the procedure below:

- choose a spanning tree of $\Gamma^\vee$ $T$
- orient edges that do not intersect $T$ arbitrarily.
- orient edges on the boundary of faces corresponding to leaves of $T$ satisfying the definition.
- remove those leaves, and countinue this procedure.

We call 2 K.-orientations $K$, $K'$ are equivalent if one an be obtained from another by a sequence of orientation-reserving operation in $\mathrm{Star}(v)$ for some vertex $v$. Luckily, For plane graphs all K.-orientations are in fact equivalent. (H.W.)

Let $K_1, K_2$ be 2 Kasteleyn orientations define:

$$\epsilon_{K_1,K_2}(e) = \begin{cases} 1, K_1, K_2 \text{ have same orientation;} \\ -1, \text{otherwise.} \end{cases}$$

it is a representation $\rho_{K_1,K_2} : \pi_1(\epsilon) \to \mathbb{Z}_2$.

**Lemma 4.1.2.** *Each representation can be obtained in this way. (H.W.)*

## 4.2 Kosteleyn matrix

Let $w(e) > 0$ be weights of edges and $K$ be a Kasteleyn orientation, define skew-symmetric matrix:

$$A_{ij}^K = \begin{cases} w(ij), i \longrightarrow j \\ -w(ij), i \longleftarrow j \\ 0, \text{not connected.} \end{cases}$$

**Theorem 4.2.1.**

$$\mathrm{Pf}(A^k) = \epsilon^K \sum_{\mathcal{D}\subset\Gamma} \prod_{e\in\mathcal{D}} w(e)$$

*where for $n \times n$ matrix $A$, $n$ even,*

$$\mathrm{Pf}(A) = \frac{1}{2^{\frac{n}{2}}(\frac{n}{2}!)} \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^{n/2} A_{\sigma_{2i-1}\sigma_{2i}}$$

$$\epsilon^K = \epsilon^K_{\sigma_1\sigma_2}...\epsilon^K_{\sigma_{n-1}\sigma_n}$$

*Proof.* Dimers are perfect matchings on edges connected by edges, we can get a natural bijection between perfect matchings on $(1, 2, ..., n)$ and $S_n/(S_2^{\frac{n}{2}} \times S_{\frac{n}{2}})$ and therefore we can derive the equation.

We still need to clarify why $\epsilon^K$ is well defined. For two dimer configurations $\mathcal{D}, \mathcal{D}'$,

$$\epsilon^K(D)\epsilon^K(D) = (-1)^\sigma(-1)^{\sigma'}\epsilon^K_{\sigma_1\sigma_2}...\epsilon^K_{\sigma_{n-1}\sigma_n}$$

we only need to clarify, contribution from each connected components of $\mathcal{D}\Delta\mathcal{D}'$ is trivial.(for same edge, they always give trivial contribution) Let $\tau$ be the permutation shifting vertices by one edge along cycle $C \subset \Gamma$, with length >= 4. Since $\tau$ is an odd permutation, it contributes $-1$ to each $(-1)^\sigma(-1)^{\sigma'}$.

For what occurs in $\prod \epsilon$, we need to consider how many edges have orientation "opposed to" $C$, which we will denote as $n_C^K$ from now on. Assume $C$ be clockwise(it doesn't matter because $C$ has even edges), we will claim:

**Lemma 4.2.2.**

$$(-1)^{n_C^K} = -(-1)^{\#C}$$

*where $\#C$ is number of vertices inside $C$.*

This claim is proved by induction. When gluing two adjacent cycles, the number of new interior vertices is decided by number of edges in their intersection, which each give a $(-1)$ as in $(-1)^{\#C}$, with an extra $-1$ from the definition of K.-orientation.

Since each cycle can be glued by finite boundaries of faces, this equation holds for every cycle. Meanwhile, from the definition of Dimer configuration, number of vertices strictly inside $C$ must be even, so each cycle will contribute an extra $-1$ in $\prod \epsilon$ term. We have proved the main theorem. □

# Chapter 5

# Lecture 5

## 5.1 Grassmann Integral

Before computing correlation functions, a small digression into Grassman integrals. Consider the exterior algebra of $\mathbb{C}^n$

$$\Lambda^\bullet \mathbb{C}^n = \langle a_i | a_i a_j = -a_j a_i, i, j = 1, ..., n \rangle$$

Choose a basis in $\Lambda^n \mathbb{C}^n$, say $e_1 \wedge ... \wedge e_n$, with would correspond to monomial $a_1 ... a_n$. Any $f \in \Lambda^\bullet \mathbb{C}^n$can be written as:

$$f = f_n a_1 ... a_n + \text{lower terms}$$

We will define

$$f_n = \int f \mathrm{d}a$$

*Remark.* Usually d$a$ is written as d$a_n$...d$a_1$ and the integral is computed using the rules:

$$\int a_i \mathrm{d}a_i = 1, \quad \int \mathrm{d}a_i = 0, \quad a_i \mathrm{d}a_j = -\mathrm{d}a_j a_i$$

**Lemma 5.1.1.** *Let $A = (a_{ij})$ be a skew-symmetric $n \times n$ matrix, $n$ is even, then:*

$$\mathrm{Pf}(A) = \int \exp(\frac{1}{2}(a, Aa)) \mathrm{d}a$$

*where* $\exp(x) = \sum_{>=0} \frac{1}{n!} x^n$.

*Proof.* Since $\int (a, Aa)^k \mathrm{d}a = 0$ for all $k \neq \frac{n}{2}$,

$$\begin{aligned} \int (a, Aa)^{\frac{n}{2}} \mathrm{d}a &= \sum_{\{i\}\{j\}} A_{i_1 j_1} ... A_{i_{\frac{n}{2}} j_{\frac{n}{2}}} \int a_{i_1} a_{j_1} ... a_{j_{\frac{n}{2}}} \mathrm{d}a \\ &= \sum_\sigma A_{\sigma_1 \sigma_2} ... A_{\sigma_{n-1} \sigma_n} \int a_{\sigma_1} a_{\sigma_2} ... a_{sigma_n} \mathrm{d}a \\ &= \sum_\sigma (-1)^\sigma A_{\sigma_1 \sigma_2} ... A_{\sigma_{n-1} \sigma_n} \end{aligned}$$

Thus

$$\int e^{\frac{1}{2}(a, Aa)} \mathrm{d}a = \frac{1}{2^{\frac{n}{2}} (\frac{n}{2})!} \sum_\sigma (-1)^\sigma A_{\sigma_1 \sigma_2} ... A_{\sigma_{n-1} \sigma_n} = \mathrm{Pf}(A)$$

□

**Proposition 5.1.2.** *1)* $\mathrm{Pf}(A)^2 = \det(A)$

*2) If* $\mathbb{C}^n = B \oplus W, \dim(B) = \dim(W) = \frac{n}{2}$ *and*

$$A = \begin{pmatrix} 0 & K^{BW} \\ K^{WB} & 0 \end{pmatrix}$$

*Then* $\mathrm{Pf}(A) = \det(K^{BW}K^{WB}) = \det(K^{WB}K^{BW})$.

## 5.2 Correlation functions

Now let us compute local correlation functions. Recall:

$$\langle \sigma_{i_1 j_1} ... \sigma_{i_k j_k} \rangle := \mathbb{E}(\sigma_{i_1 j_1} ... \sigma_{i_k j_k}) = \frac{1}{Z} \sum_{\mathcal{D}} w(\mathcal{D}) \sigma_{i_1 j_1}(\mathcal{D}) ... \sigma_{i_k j_k}(\mathcal{D})$$

It can be also regarded as conditional probability of $(i_1, j_1), ..., (i_k, j_k) \in \mathcal{D}$. The following theorem expresses local correlation functions in terms of the Kasteleyn matrix:

**Theorem 5.2.1.** *when* $(i_a, j_a) \neq (i_b, j_b)$

$$\langle \sigma_{i_1 j_1} ... \sigma_{i_k j_k} \rangle = (-1)^k a^K_{i_1 j_1} ... a^K_{i_k j_k} \mathrm{Pf}(A^K_{ab})^{-1}|_{a,b=i_1,j_1,...,i_k,j_k}.$$

*Proof.* Differenting $Z^{\mathrm{dimer}}_\Gamma$ in $w(e)$ we have

$$\langle \sigma_{e_1} ... \sigma_{e_k} \rangle = \frac{1}{Z} w(e_1) ... w(e_k) \frac{\partial^k Z}{\partial w(e_1) ... \partial w(e_k)} = \frac{1}{\mathrm{Pf}(A)} A^K_{i_1 j_1} ... A^K_{i_k j_k} \frac{\partial^k Z}{\partial A^K_{i_1 j_1} ... \partial A^K_{i_k j_k}}$$

we can assume $i_1 < j_1 < ... < i_k < j_k$
We know

$$\mathrm{Pf}(A^k) = \int e^{\frac{1}{2}(a, Aa)} \mathrm{d}a$$

$i < j$, then $\frac{\partial}{\partial a^k_i j} \frac{1}{2}(a, A^K a) = \frac{\partial}{\partial a^k_i j} \sum_{k<l} a_k (A^K_{kl}) a_l = a_i a_j$
From here

$$\frac{\partial^k Z}{\partial A^K_{i_1 j_1} ... \partial A^K_{i_k j_k}} = \int e^{\frac{1}{2}(a, A^K a)} a_{i_1} a_{j_1} ... a_{j_k} \mathrm{d}a$$

Thus, we need to compute such integrals. Consider

$$\int e^{\frac{1}{2}(a, Aa) + (\eta, a)} \mathrm{d}a$$

where $\eta$ is an odd variable.Since

$$\frac{1}{2}(a, Aa) + (\eta, a) = \frac{1}{2}(a - A^{-1}\eta, A(a - A^{-1}\eta)) - \frac{1}{2}(A^{-1}\eta, \eta)$$

the integral will be $\mathrm{Pf}(A) e^{\frac{1}{2}(\eta, A^{-1}\eta)}$.
Now expand the exponents in powers of $\eta$. We have

$$(\eta, A^{-1}\eta) = \sum_{\{i\}\{j\}} A^{-1}_{i_1 j_1} ... A^{-1}_{i_k j_k} \eta_{i_1} \eta_{j_1} ... \eta_{j_k} = \sum_{l_1 < ... < l_{2k}} \eta_1 ... \eta_{2k} \sum_\sigma (-1)^\sigma A^{-1}_{\sigma_1 \sigma_2} ... A^{-1}_{\sigma_{2k-1} \sigma_k}$$

Then, from here:

$$e^{\frac{1}{2}(\eta,A^{-1}\eta)}\sum_{k>=0}\sum_{l_1<...<l_{2k}}\eta_1...\eta_{2k}\mathrm{Pf}(A^K_{ab})^{-1}|_{a,b=i_1,j_1,...,i_k,j_k}.$$

Compare coefficients in the result with in

$$\begin{aligned}\int e^{\frac{1}{2}(a,Aa)+(\eta,a)}\mathrm{d}a &= \frac{1}{m!}\sum_{m>=0}\int e^{\frac{1}{2}(a,Aa)}(\eta,a)^m\mathrm{d}a\\ &= \sum_{m>=0}\int e^{\frac{1}{2}(a,Aa)}\sum_{l_1<...<l_m}\eta_{l_1}...\eta_{l_m}(-1)^{m(m-1)/2}a_{l_1}...a_{l_m}\mathrm{d}a\\ &= \sum_{k>=0}\sum_{l_1<...<l_{2k}}\eta_{l_1}...\eta_{l_{2k}}\int e^{\frac{1}{2}(a,Aa)}(-1)^k a_{l_1}...a_{l_{2k}}\mathrm{d}a\end{aligned}$$

Then we complete the proof.

□

**Theorem 5.2.2.** *We have those identities(Left for H.W.):*

*1)*

$$\frac{\partial^k\mathrm{Pf}(A)}{\partial a_{i_1j_1}\partial a_{i_kj_k}} = (-1)^{\sigma(I)}\mathrm{Pf}(A_I)$$

*where $A_I$ denotes the matrix with columns and rows $i_s, j_s$ being removed, with $\sigma : (1, 2, ..., 2n) \mapsto (i_1, j_1, i_2, ..., j_k, 1, ..., \hat{i}_i, ..., \hat{j}_k, ..., 2n)$.*

*2)*

$$(-1)^{\sigma(I)}\mathrm{Pf}(A_I) = (-1)^k\mathrm{Pf}(A)\mathrm{Pf}((A^{-1}_{ab})_{a,b\in I})$$

In particular:

$$\langle\sigma_{ij}\rangle = -a_{ij}(A^k)^{-1}_{ij}$$

$$\langle\sigma_{i_1j_1}\sigma_{i_2j_2}\rangle = a_{i_1j_1}a_{i_2j_2}((A)^{-1}_{i_1j_1}(A)^{-1}_{i_2j_2} + (A)^{-1}_{i_1j_2}(A)^{-1}_{i_2j_1} - (A)^{-1}_{i_1i_2}(A)^{-1}_{j_1j_2})$$

# Chapter 6

# Lecture 6

In this chapter we are going to introduce Ising models and their relationship to dimers.

## 6.1 Ising model on a graph $\Gamma$

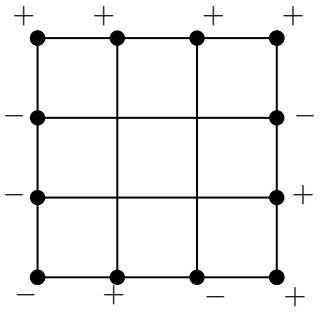
As an Example, we will assume $\Gamma \subset \mathbb{Z}^2$, connected and simply connected. $X = \{+, -\}^{V(\Gamma)}$ is the space of state.

**Energy function:**

$$E(\sigma) = -\sum_{e \in E(\Gamma)} J_e \sigma_{e_1} \sigma_{e_2}$$

is the usual form of Energy function in Ising model, with $J_e > 0$ exchange energy, and $e_{1,2}$ be boundary vertices of edge $e$. Sometime we need a modified version of Energy function, with an additional term $\sum_{v \in V(\Gamma)} H_v \sigma_v$, called Ising model with magnetic field. (Imagine $H_v$ be the strength of magnetic field at $v$)

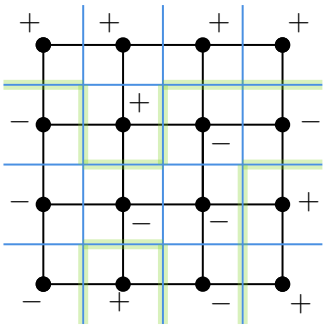
**Boundary Conditions:**




For the Ising model the matural choice of boundary conditions is fixing boundary spins. For a planar domain $\mathcal{D}$ and $\Gamma \subset \mathcal{D}$ defining a cell complex $X_\Gamma$ for $\mathcal{D}$ , fix spins on vertices from $\partial X_\Gamma$.

Consider the Ising model on a graph $\Gamma$ which is a connected, simply connected domain of a square lattice in $\mathbb{R}^2$. Fix configurations of spins in $\partial X_\Gamma$.

Let $(X_\Gamma^\vee)^+$ be the extension of the dual cell complex by semiopen edges.



The graph $\Gamma$ is in black, $\Gamma_+^\vee$ is in blue,
semi open edges are border edges.

Green walls seperating $+$ and $-$ are border edges.

We can get a bijection:

$$\{\text{Ising configurationson } \Gamma\} \simeq \{\text{Domain walls seperating } + \text{ and } -\}$$

Let us assume $J_e$ only depends on the edge is vertical or horizontal. The Energy on Ising configuration is

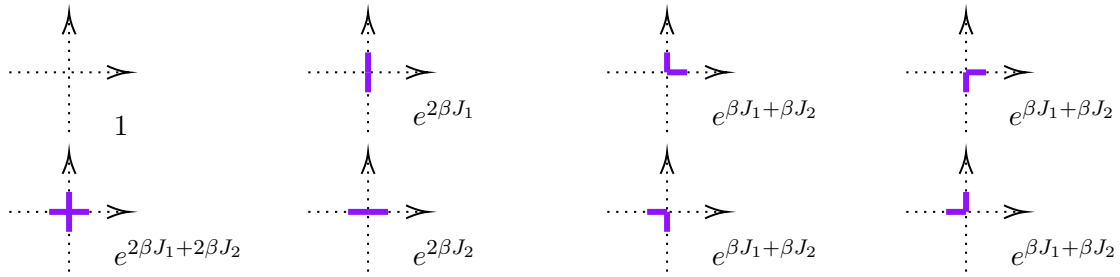
$$E(\sigma) = J_1 \sum_{\text{vertical}} \sigma_i\sigma_j + J_2 \sum_{\text{horizontal}} \sigma_i\sigma_j$$

The Boltzmann weight of a spin configuration we can write as

$$e^{-\beta E(\sigma)} = e^{-\beta(J_1|E_v|+J_2|E_h|)} \prod_{ver.wall} e^{2\beta J_1} \prod_{hor.wall} e^{2\beta J_2}$$

where $E_v$ is the number of vertical edges, $E_h$ is the number of horizontal edges.

so, use the language of domain walls. each vertex in the dual graph give weight as follows:

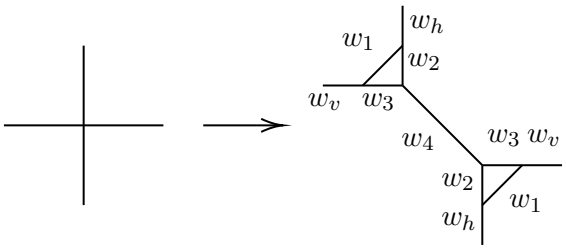



## 6.2 A mapping of Ising model to dimers

Now let us describe the mapping of Ising model to dimer model. We'll do it in few steps.

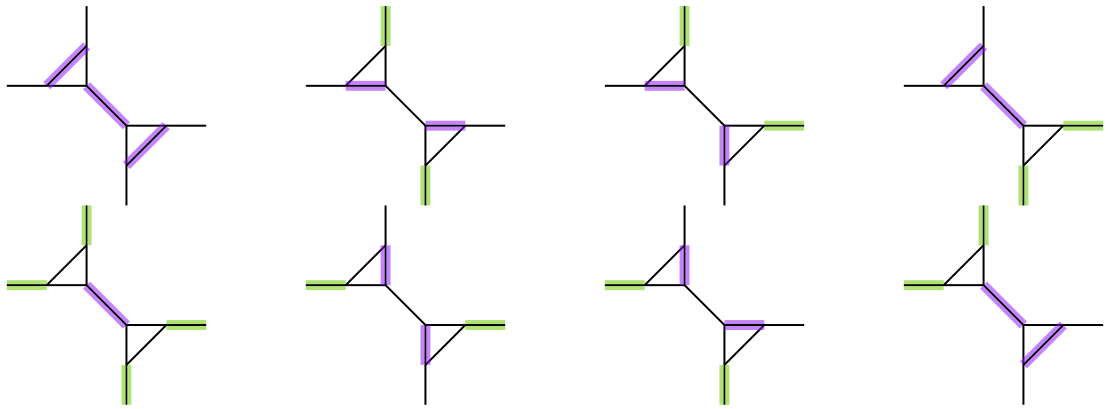
**Step1:**

Stretch the lattice $\Gamma_+^\vee$ as it is shown here:




**Step2:**

Extend Ising domain walls to dimer configuration as it is shown below:



**Step3:**

we need the new model give the same weight. Naturally we need those identities:

$$w_h w_n w_4 = e^{2\beta J_1 + 2\beta J_2}, \quad w_1^2 w_4 = 1$$

$$w_v w_3^2 = e^{2\beta J_1}, \quad w_h w_2^2 = e^{2\beta J_2}$$

$$(w_h w_v)^{1/2} w_2 w_3 = (w_h w_v)^{1/2} w_1 w_4 = e^{\beta J_1 + \beta J_2}$$

then, by choosing an appropriate gauge transformation $w(e) \mapsto s(e_+)w(e)s(e_-)$, we can set $w_h = w_v = 1$, then:

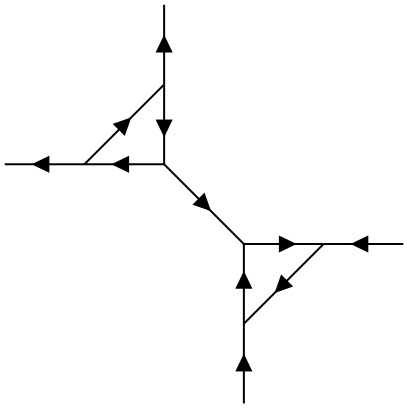
$$w_4 = e^{2\beta J_1 + 2\beta J_2}, w_3 = e^{\beta J_1}, w_2 = e^{\beta J_2}, w_1 = e^{-\beta J_1 - \beta J_2}$$

We have completed the construction.

**Corollary 6.2.1.**

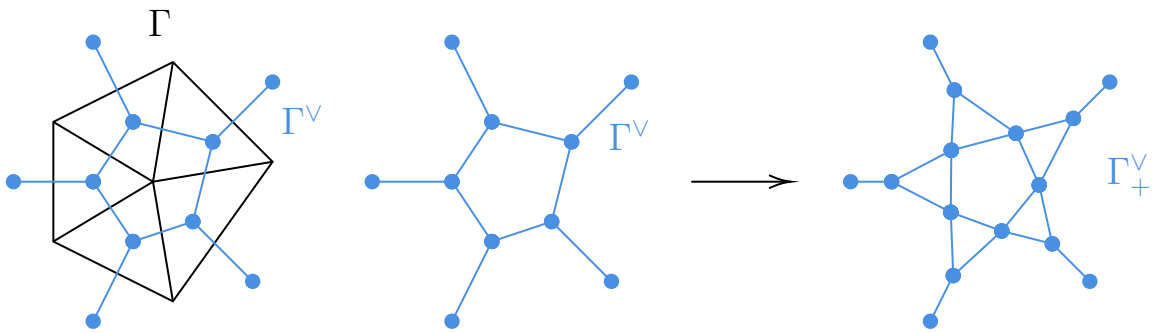
$$Z_{\mathrm{Ising}_\Gamma} = e^{-\beta(J_1|E_v| + J_2|E_h|)} Z_{\mathrm{Dimer}_{\Gamma^\vee_+}}$$

*We can choose a K.orientation of the stretched square lattice as*



*This gives a Pfaffian formula for the Ising partition function on a planar square grid.*

This map can be extended to other graphs without 1-valent vertices, for example:




Homework: Prove that this stretching gives a mapping of the Ising model on $\Gamma$ with weights $\exp(-\beta J_e)$ to a dimer model on $\hat{\Gamma^\vee}$ with appropriate weights. Compute weights.

## 6.3 Correlation functions

$\Gamma$ be a square lattice, 2 dimensional.

Consider 2 adjacent vertices, and $e$ be the cell dual to their connecting edge, then we will claim:

$$\sigma_1(S)\sigma_2(S) = 1 - 2\sigma_e(D)$$

, where $S$ is an Ising configuration and $D$ be the corresponding dimer configuration on the stretched dual graph $\hat{\Gamma^\vee}$.

**Corollary 6.3.1.** *Let $u, v \in V(\Gamma)$ and $\gamma$ be a simple path connecting $u, v$, then:*

$$\sigma_u(S)\sigma_v(S) = \prod_{e \in \gamma \subset \hat{\Gamma^\vee}} (1 - 2\sigma_e(D))$$

We proved this theorem:

**Theorem 6.3.2.**

$$\langle \sigma_u \sigma_v \rangle = \langle \prod_{e \in \gamma \subset \hat{\Gamma^\vee}} (1 - 2\sigma_e) \rangle$$

Homework: Compute this expectation value in terms of a Kasteleyn matrix on $\hat{\Gamma^\vee}$ assuming $+$ boundary conditions.

# Chapter 7

# Lecture 7

Let $\Sigma_g$ be oriented surface of genus $g$. Here we assume $\Sigma_G$ is closed without boundary.

Assume $\Gamma \subset \Sigma_g$ defines a CW-complex $X_\Gamma$. The homology group $H_1(X_\Gamma, \mathbb{Z}_2)$ has $2g$ generators and the intersection pairing defines a non-degenerate symmetric form on this $\mathbb{Z}_2$ vector field.

## 7.1 Kasteleyn orientations

**Definition 7.1.1.** An orientation of edges of $\Gamma \subset \Sigma_g$ is Kasteleyn if for each face $f$ of $X_\Gamma$:

$$\prod_{e \in \partial f} \epsilon(e, f) = -1$$

where $\epsilon(e, f)$ is the relative orientation number.

**Theorem 7.1.2.** *Kasteleyn orientations exist iff the number of vertices of* $\Gamma$ *is even.*

*Proof.* Assume that $K$ is a Kasteleyn orientation of $\Gamma \subset \Sigma_g$, closed, compact, oriented. Then,

$$\prod_{F(X_\Gamma)} (-1) = \prod_{F(X_\Gamma)} \prod_{\partial f} \epsilon(e, f) = \prod_{E(X_\Gamma)} (-1)$$

thus,

$$(-1)^{|E|} = (-1)^{|F|}, \quad |E| + |F| = 0 (\text{mod } 2)$$

so, by Euler characteristic of $X_\Gamma$:

$$|V| + |E| + |F| = 0 (\text{mod } 2)$$

so $|V|$ is even.

On the other side, if $|V|$ is even, then $|E| + |F| = 0 (\text{mod } 2)$.

**Lemma 7.1.3.** *Assume* $|V|$ *is even and* $K$ *is an orientation of* $\Gamma$ *such that all faces of* $X_\Gamma$*, except* $f_0$ *are Kasteleyn oriented. Then* $f_0$ *is also Kasteleyn oriented.*

*Proof.*

$$\prod_{\partial f_0} \epsilon(e, f_0) \prod_{F(X_\Gamma) - \{f_0\}} (-1) = \prod_{F(X_\Gamma)} \prod_{\partial f} \epsilon(e, f) = \prod_{E(X_\Gamma)} (-1) = \prod_{F(X_\Gamma)} (-1)$$

so $\prod_{\partial f_0} \epsilon(e, f_0)$ must be $-1$. □

Now we can construct a K.orientation for $\Gamma \subset \Sigma_g$.

First, we choose a face $f_0$ on $X_\Gamma$ and paths $\alpha_1, ..., \alpha_g, \beta_g, ..., \beta_g$ on $\Gamma^\vee$ such that they form a basis in $H_1(\Sigma_g, \mathbb{Z}_2)$, and cutting along them gives a disk $\Sigma' \subset \Sigma/$. Choose a spanning tree for $\Gamma'$, and use the same arguments as planar graph and finally use the lemma, we will prove the result. □

Now let us describe equivalence classes. Recall that an orientation reserving operation on a vertex means reserving direction of all edges adjacent to it, and two K.orientations are equivalent iff one is result of another after finite orientation reserving operations.

**Theorem 7.1.4.** *There are exactly $2^{2g}$ equivalence classes of K.orientations of edges of cell complex $X_\Gamma$ on $\Sigma_g$.*

*Proof.* Assume that $K'$ and $K''$ are two K.orientations. Consider:

$$\epsilon^{K',K''}(e) = \begin{cases} 1, \text{if } K' \text{ agrees with K" on e} \\ -1, \text{otherwise} \end{cases}$$

The K. condition implies for each face $f$:

$$\prod_{e \in \partial f} \epsilon^{K',K''}(e) = 1$$

**Digression: computing equivalent classes**

Let $X_\Gamma^\vee$ be the cell complex dual to $X_\Gamma$ and $\Gamma^\vee$ be it's 1-spine(dual graph). Recall that $C_k(X, \mathfrak{k})$ is the space is $\mathfrak{k}$-linear combinations of $k$-cells, $C^k(X, \mathfrak{k})$ be its dual, the space of $\mathfrak{k}$-valued functions on $X_k$ with the differential:

$$\mathrm{d} : C^k \to C^{k+1}, \quad \mathrm{d}f_0(e) = f_0(e_+) - f_0(e-)$$
$$\mathrm{d}f_1(f) = \sum_{e \in \partial f} (-1)^{\epsilon(e, \partial f)} f_1(e), \quad \mathrm{d}f_2 = 0$$

we have the linear isomorphism

$$C_k(X^\vee) \simeq C^{n-k}(X), \quad c^\vee \mapsto \delta_c$$

In our case, $\mathfrak{k} = \mathbb{Z}_2$. Write $\epsilon^{K',K''}(e) = (-1)^{c^{K',K''}}$, then The K. condition of

$$\prod_{e \in \partial f} \epsilon^{K',K''}(e) = 1$$

is equivalent to $c^{K',K''} \in C^1(X_\Gamma, \mathbb{Z}_2)$ exact. Invert the orientation $K''$ of all edges adjacent to $v \in V(\Gamma)$ result in $\epsilon^{K',K''}(e)$ becomes $\epsilon^{K',K''}(e)(-1)^{\varphi_v(e)}$, where $\varphi_v$ is the characteristic function of $\mathrm{Star}(v)$.

So, for $v \in \partial e$, this changes cocycle $c^{K',K''}$ by adding a coboundary $\mathrm{d}\delta_v$.

Taking into account the isomorphism $C_1(X_\Gamma) \simeq C^1(X_\Gamma^\vee)$ we can write any closed cochain as $\mathbb{Z}_2$-combination of a basis in $H^1(X_\Gamma, \mathbb{Z}_2)$ up to a coboundary, and $K$ and $K'$ are equivalent iff $c^{K,K'}$ is a coboundary. Then we have proved the bijection between equivalent classes and $H^1(X_\Gamma, \mathbb{Z}_2)$. □

There is a convenient way to choose representatives of equivalence classes of K.orientations of surface graphs. Choose closed simple paths $A_i$ on $\Gamma^\vee$ representing a basis on $H^1(X_\Gamma, \mathbb{Z}_2)$.

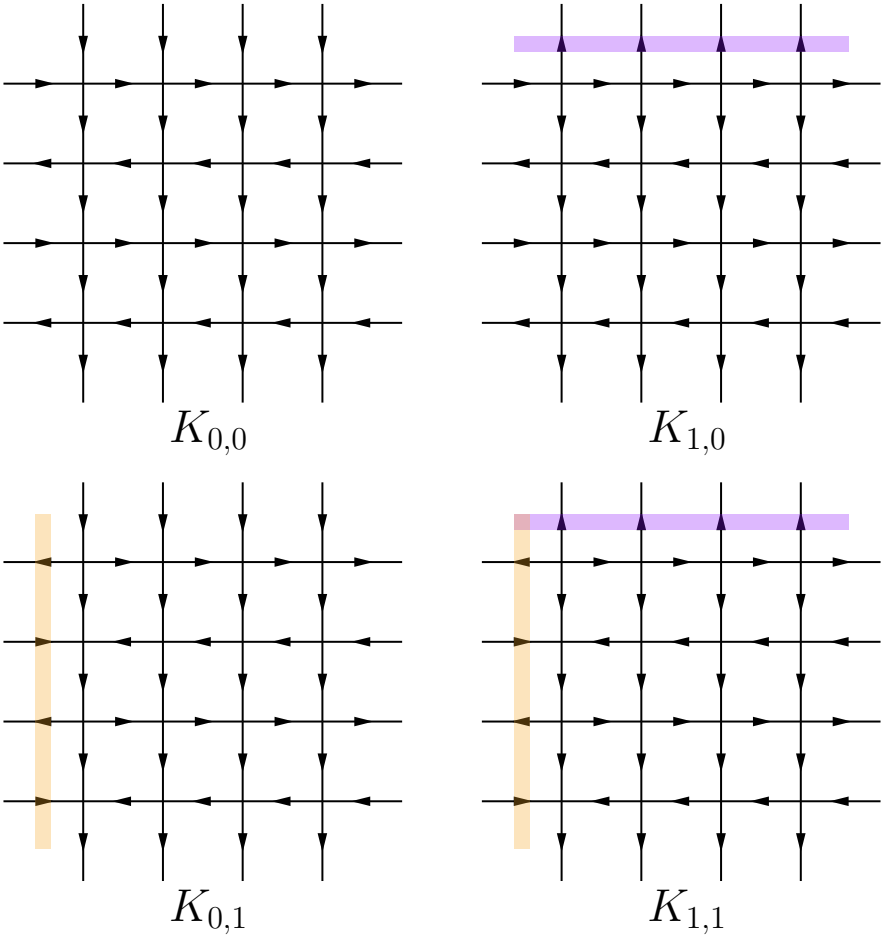
**Theorem 7.1.5.** *Reversing orientations of edges of $\Gamma$ intersecting any of $A_i$ brings any K.-orientation to a nonequivalent one.*

*Proof.* Let $K$ be a K.orientation of $\Gamma$. Reversing he orientation of edges intersecting $A_i$ results in a K.orientation.

Indeed, if a face $f$ does not intersect $A_i$, the K. condition does not change, If it intersects $A_i$, two edges from $\partial f$ do this, and therefore reversing their orientation does not violate the K.condition.

So, after reversing all those edges, we get a new K.orientation, say $K'$. We have $c^{K,K'} = A_i$, means they are not equivalent. □

**Corollary 7.1.6.** *Let $K$ be a K. orientation, $(\epsilon_1, ..., \epsilon_{2g})$ be a sequence $\epsilon_i = 0, 1$, and $K_{\epsilon_1,...,\epsilon_{2g}}$ be the orientation obtained from $K$ reversing orientation off all edges intersecting $A_i$ if $\epsilon_i = 1$ and returning the same orientation if $\epsilon_i = 0$. Then we find a set of representive of equivalence classes of K-orientations.*




## 7.2 The Pffafian formula

Let K be a K.orientation of $\Gamma \subset \Sigma_g$. As before, we assume that $\Gamma \subset \Sigma_g$ defines a cell complex $X_\Gamma$.

Fix a reference dimer configuration $\mathcal{D}_0 \subset \Gamma$ and let us compute $\epsilon^K(\mathcal{D}_0)\mathrm{Pf}(A^\Gamma)$ as we did in the plane case.

$$\epsilon^K(\mathcal{D}_0)\mathrm{Pf}(A^\Gamma) = \sum_{\mathcal{D}} \epsilon^K(\mathcal{D}_0)\epsilon^K(\mathcal{D}) \prod_{e\in\mathcal{D}} w(e)$$

Just as in the plane case

$$\epsilon^K(\mathcal{D}_0)\epsilon^K(\mathcal{D}) = \prod_{C \subset \mathcal{D}\Delta\mathcal{D}_0} (-1)^{n^K(C)+1}$$

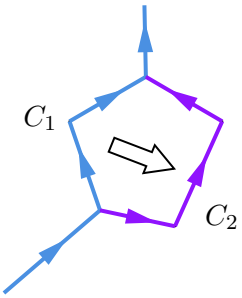
where we choose some orientation of each closed curve $C \subset \mathcal{D}\Delta\mathcal{D}_0 \subset \Gamma$ and $n_C^K$ is the number of edges in $C$ with the orientation opposite to $C$. In the planar case we proved that $n_C^K$ is always odd, however it is not true in a surface case: it (mod 2) depends on the curve $C$, not only on its homology class in $H_1$!

We have a lemma shows the fact:

**Lemma 7.2.1.** *Move $C$ to the right (according to the orientation of $C$ and $\Sigma$) by a face $f$, Then*

$$n_C^K - n_{C'}^K = (|C \cap \partial f| + 1) \mod 2$$

*Proof.* We have:




$$n_C^K - n_{C'}^K = n_{C_1}^K - n_{C_2}^K$$
$$n_{\partial f}^K = |C_1| + n_{C_1}^K - n_{C_2}^K \equiv 1 \mod 2$$

Therefore we proved this lemma.

□

Thus, if wee will find a function $l(C)$ on curves on $\Gamma$ such that it changes as $n_C^K$ when we deform $C$, and $l + n$ only depend the homology class of $C$.

Let $\mathcal{D} \subset \Gamma$ be a dimer configuration. Define $l_{\mathcal{D}}(C)$ for oriented closed curve $C$ be the number of vertices on $C$ adjacent to dimers pointing to the left of $C$.

**Lemma 7.2.2.** *Let $C'$ be the result of moving $C$ to the right by a face $f$ as above, then:*

$$l_{\mathcal{D}}(C) - l_{\mathcal{D}}(C') == (|C_1| + 1) \mod 2$$

*Proof.* Consider first the case when dimer point to the left at each $v \in C_1^{\text{int}}$, It is clear that in this case the equality holds.

If one edge of $C_1$ is occupied by a dimer, then:

- if the dimer edge does not have vertices in the boundary, then the difference of $l$ is even, thus 0 mod 2,
- if this dimer edge has one boundary vertex of $C \cap \partial f$ as boundary, then $l_{\mathcal{D}}(C_1)$ changes by $-1$, $l_{\mathcal{D}}(C_2)$ changes by 1.
- if this dimer configuration contain $C_1$, then $l_{\mathcal{D}}(C_1)$ changes by 0, $l_{\mathcal{D}}(C_2)$ changes by 2.

Then we have considered all the cases. □

**Corollary 7.2.3.** *The sum $1 + n_C^K + l_{\mathcal{D}}(C)$ depends only on the homology class of $C$, $[C] \in H_1(\Sigma_g, \mathbb{Z}_2)$. We denote it as $q_{\mathcal{D}}^K$.*

**Lemma 7.2.4.** *If $\alpha = [\mathcal{D}\Delta\mathcal{D}_0]$, then*

$$q_{\mathcal{D}_0}^K(\alpha) = \sum_{i=1}^{m} (n^K(C_i) + 1)$$

This is because $l_{\mathcal{D}}(C_i) = 0$ for cycles in $\mathcal{D}\Delta\mathcal{D}_0$.
Let's continue the computation of the Pfaffians.We have

$$\epsilon^K(\mathcal{D}_0)\mathrm{Pf}(A^K) = \sum_{\mathcal{D}} \epsilon^K(\mathcal{D})\epsilon^K(\mathcal{D}_0)w(\mathcal{D}) = \sum_{\alpha\in H^1(\Sigma_g,\mathbb{Z}_2)} (-1)^{q^K_{\mathcal{D}_0}(\alpha)} Z_\alpha \tag{7.2.1}$$

The other hand, the partition function is

$$Z(\Gamma) = \sum_{\alpha} Z_\alpha$$

We still need to deal with the equation 7.2.1.

## 7.3 Quadratic forms on a $\mathbb{Z}_2$-vector field

Let $H$ be a $\mathbb{Z}_2$ vector space with a nondegenerate symmetric bilinear form $(\bullet, \bullet)$.

**Definition 7.3.1.** A quadratic form on $H$, is a mapping $q : H \to \mathbb{Z}_2$, s.t.

$$q(\alpha + \beta) = q(\alpha) + q(\beta) + (\alpha.\beta)$$

We have some basic facts of quadratic forms:

- The number of quadratic forms on $H$ is equal to the number of elements in $H$.
- The set $Q(H)$ of such forms is affine space over $H^* = \mathrm{Hom}(H, \mathbb{Z}_2)$. Indeed, difference of two quadratic form is a linear form.
- the Arf invariant of a quadratic form is defined as

$$\mathrm{Arf}(q) = \{\text{the element in } \mathbb{Z}_2 \text{ that occur most often as value of } q\}$$

More explicitly:

$$\mathrm{Arf}(q) = \frac{1}{\sqrt{q}} \sum_{\alpha\in H} (-1)^{q(\alpha)}$$

*Proof.* 1) Take a basis$\{\alpha_i\}$, and consider the bijection $Q(H) \to 2^{\dim(H)}$, $q \mapsto (q(\alpha_i))$.

2) Consider $\Delta$ be difference of 2 quadratic forms. We can prove $\Delta(\alpha+\beta) = \Delta(\alpha)+\Delta(\beta)$, so it is linear. Conversely, add any nontrivial linear map to $q$ will give a different quadratic form from $q$.

3) This property is equivalent to that $q$ take value $\mathrm{Arf}(q)$ for exactly $2^{2g-1} + 2^{g-1}$ times. We only need to prove $A(q) = \frac{1}{\sqrt{q}} \sum_{\alpha\in H}(-1)^{q(\alpha)}$ is always $\pm 1$, i.e.

$$\frac{1}{q} \sum_{\alpha,\beta\in H} (-1)^{q(\alpha)+q(\beta)} = 1$$

Fix $\alpha + \beta = \gamma \neq 0$, we consider $(x, \gamma) \neq 0$ since intersection form is non-degenerate, then $(\alpha, \beta) \mapsto (\alpha + x, \beta + x)$ induce a bijection between $\{(\alpha, \beta) = 0\}$ and $\{(\alpha, \beta) = 1\}$, so contribution of $\{\alpha + \beta \neq 0\}$ to the sum is 0. Then:

$$\frac{1}{q} \sum_{\alpha,\beta\in H} (-1)^{q(\alpha)+q(\beta)} = \frac{1}{q} \sum_{\alpha\in H} (-1)^{2q(\alpha)} = 1$$

□

It satisfies the following identities:

1)

$$\frac{1}{\sqrt{q}} \sum_{q \in Q(H)} (-1)^{\mathrm{Arf}(q)+q(\alpha)}$$

2)

$$\mathrm{Arf}(q_1) - \mathrm{Arf}(q_2) = q_1(\Delta_{12})$$

where $\Delta_{12}$ representing linear form $q_1 - q_2 = (\Delta_{12}, \alpha)$

We will need quadratic forms over $H_1(\Sigma, \mathbb{Z}_2)$. We will also use natural isomorphisms $H_1(\Sigma, \mathbb{Z}_2) \simeq H_1(X_\Gamma, \mathbb{Z}_2) \simeq H_1(X_\Gamma^\vee, \mathbb{Z}_2)$.

We claim the theorem:

**Theorem 7.3.2** (Cimasoni, R.)**.** *There is a unique quadratic form $q_{\mathcal{D}}^K$ on $H_1(\Sigma, \mathbb{Z}_2)$ which is given by our definition on simple curves as above, with bilinear form being the intersection form.*

## 7.4 Spin structure and quadratic form: A sketch of last theorem

Recall the general definition of spin structure: denote the frame bundle of $\Sigma$, which is an $SO(n)$ bundle $P_{SO} \to \Sigma$. We call a spin structure a principal $\mathrm{Spin}(n)$-bundle $P$ and a 2-1 map $P \to P_{SO}$ have 2-1 covering map $\mathrm{Spin}(n) \to SO(n)$ on each fibre.

This is equivalent to give a cohomology class $\xi \in H^1(P_{SO}; \mathbb{Z}_2)$, whose value on $[c]$ determined by if the point go back to itself or its antipode after walking through the lift of $c$ continuously by one round. In order it give the canonical 2-1 map locally, we require the restriction of $\xi$ gives generator of $H^1(SO(n), \mathbb{Z}_2)$ on each fibre.

Luckily, in dimension 2, the structure is more easy to understand. For 2-dimensional surfaces, the second Stiefel-Whitney class always vanish, so it always admits a spin structure. Moreover, the map $\mathrm{Spin}(n) \to SO(n)$ is just the canonical degree 2 map $S_1 \to S_1$, and we may recognize the frame bundle as the unit tangent bundle.

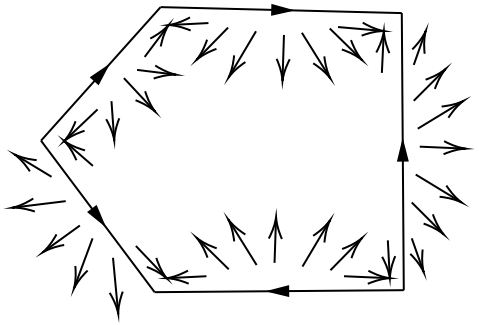
We can construct a correspondence between $H^1(P_{SO}, \mathbb{Z}_2)$ and set of sections on unit tangent bundle on $\Sigma$ with finite even-degree singularities $\mathcal{V}_{\mathrm{ev}}(\Sigma)$:

$$\mathcal{V}_{\mathrm{ev}}(\Sigma) \to H^1(P_{SO}, \mathbb{Z}_2)$$

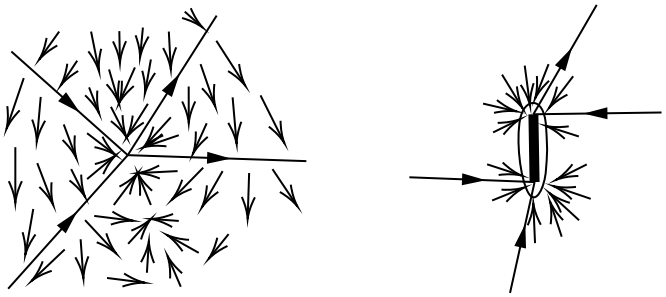
$$f \mapsto \xi_f : ([c] \mapsto \text{winding number of } f \text{ in } c^* P_{SO} \bmod 2$$

The number of spin structure is also $2^{2g}$ on $\Sigma_g$, which is equal to the number of different K.orientations. This is not an coincidence. Precisely, K.orientation can be understand as a combinatorial rewriting of spin structure. We will construct a vector field section in $\mathcal{V}_{\mathrm{ev}}(\Sigma)$ to give the correspondence.

First, we give the section on the boundary of each face by rotate counter-clockwise/clockwise by $\pi$ when walk along/against an oriented edge, with the direction near the boundary point to the boundary vertex. Here is an example:



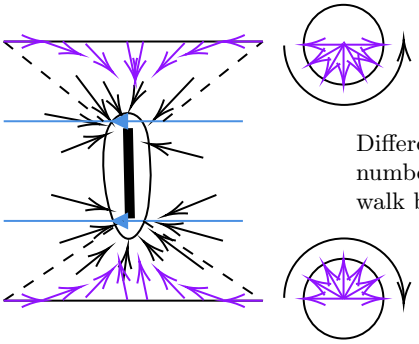
After that, we extend the section inside the face, ensuring only even-degree singularities appears. Now only odd-degree singularities are exactly those vertex of $\Gamma$. By choosing a dimer configuration, we can pair them into even-degree singularities.



This will give a vector field $f_{\mathcal{D}}^K$ in $\mathcal{V}_{\text{ev}}(\Sigma)$, which is correspond to a spin structure, denoted as an element $\xi$ in $H^1(\Sigma, \mathbb{Z}_2)$.

Now we need to give the quadratic form from our spin structure $\xi$. If a 1-chain $\alpha \in H_1(\Sigma, \mathbb{Z}_2)$ is represented by sum of $m$ disjoint simple closed curves $\gamma_1, ..., \gamma_m$, and $q_\xi(\alpha) := \sum(\xi(\gamma_i) + 1)$. In order to prove this definition gives $q_{\mathcal{D}}^K$ when the spin structure is given by K.orientation $K$ and dimer configuration $\mathcal{D}$, only prove when $\alpha$ is given by a simple closed oriented curve $c$. Assume $c$ hs $n$ vertices.

Let us compute the winding number of $f_{\mathcal{D}}^K$ on $c$. Each edge with same orientation as $K$ contributes $+1/2$, otherwise $-1/2$. Points adjacent to dimer at left/right contributes $\pm 1/2$. Since number of points on $c$ paired to point also on $c$ is even, we can get $\xi(c) = 1/2 * (n - 2 * n_c^K) + 1/2 * (2l_{\mathcal{D}}(c) - n) + \text{even terms} = n_c^K + l_{\mathcal{D}}(c) \mod 2 = q_{\mathcal{D}}^K + 1$. That is what we want, so we proved the theorem claimed in last section.



Different contribution of winding number when a closed curve walk by a dimer pairing at different side

## 7.5 The dimer partition function as the sum of Pfaffians

Now Let us return to the computation of the partition function of a dimer model on a surface graph. First we need a result:

**Lemma 7.5.1.** *i) Quadratic form $q_{\mathcal{D}}^K$ constructed earlier depends only on the equivalence class $[K]$ of $K$.*

*ii) The mapping $[K] \mapsto q_{\mathcal{D}}^K$ is a bijection between equivalence classes of K.orientations and the space of quadratic forms on $H_1(\Sigma, \mathbb{Z}_2)$.*

*Proof.* ii) Compute the difference between two quadratic forms $q_{\mathcal{D}}^K$ for different orientations $K$ and $K'$. Let $C$ be collection of closed curves on $\Gamma$.

$$q_{\mathcal{D}}^K - q_{\mathcal{D}}^{K'} = n^K(C) - n^{K'}(C) = c^{K,K'}(C)$$

Here $c^{K,K'}$ defined as the cocycle $[1_{\{e|K \neq K'\}}] \in H^1(\Sigma_g, \mathbb{Z}_2)$. Since it take all possible value in $H^1 \simeq H_1^\vee$, we can prove the lemma. □

**Theorem 7.5.2** (Cimasoni,Reshetikhin.)**.**

$$(\star)Z_{\Gamma\subset\Sigma_g}=\frac{1}{2^g}\sum_{[K]}(-1)^{\mathrm{Arf}(q^K_{\mathcal{D}_0})}\epsilon^K(\mathcal{D}_0)\mathrm{Pf}(A^K)$$

*Proof.* We proved earlier that

$$\epsilon^K(\mathcal{D}_0)\mathrm{Pf}(A^K)=\sum_{\alpha}(-1)^{q^K_{\mathcal{D}_0}(\alpha)}Z_\alpha(\Gamma)$$

so since $[K]\mapsto q^K_{\mathcal{D}}$ is a bijection, by properties we have proved:

$$\sum_{[K]}(-1)^{\mathrm{Arf}(q^K_{\mathcal{D}_0})}\epsilon^K(\mathcal{D}_0)\mathrm{Pf}(A^K)=\sum_{\alpha}\sum_{q}(-1)^{\mathrm{Arf}(q)+q(\alpha)}Z_{\alpha,\mathcal{D}_0}=2^gZ(\Gamma)$$

□

*Remark.* We can always select representive from each $K_0\in[K]$ s.t. $\epsilon^{K_0}=0$: If the representive satisfy the condition, do nothing; otherwise do orientation reversing operation for one vertex.

so we proved $\frac{1}{2^g}\sum_{[K]}(-1)^{\mathrm{Arf}(q^K_{\mathcal{D}_0})}\epsilon^K(\mathcal{D}_0)\mathrm{Pf}(A^K)$ do not depend on $\mathcal{D}$. Actually, this is true for each summand too.

**Lemma 7.5.3.** $(-1)^{\mathrm{Arf}(q^K_{\mathcal{D}})}\epsilon^K(\mathcal{D})$ *does not depend on* $\mathcal{D}$*.*

*Proof.* From the properties of Arf invariant we have

$$\mathrm{Arf}(q^K_{\mathcal{D}_1})+\mathrm{Arf}(q^K_{\mathcal{D}_2})=q^K_{\mathcal{D}_1}(\Delta_{\mathcal{D}_1\mathcal{D}_2})\mod 2$$

By definition of $q^K_{\mathcal{D}}$, if $\alpha$ is represented by a closed path $C\subset\Gamma$, $q^K_{\mathcal{D}}(\alpha)=n^K_X+l_{\mathcal{D}}(C)+1$, and

$$q^K_{\mathcal{D}_1}(\alpha)+q^K_{\mathcal{D}_2}(\alpha)=l_{\mathcal{D}_1}(C)+l_{\mathcal{D}_2}(C)=([D_1\Delta D_2],\alpha)\mod 2$$

Therefore

$$\Delta_{\mathcal{D}_1\mathcal{D}_2}=[D_1\Delta D_2]$$

,so

$$\mathrm{Arf}(q^K_{\mathcal{D}_1})+\mathrm{Arf}(q^K_{\mathcal{D}_2})=\sum_{i=1}^{m}(n^K_{C_i}+1)\mod 2$$

, where $\{c_i\}$ are connected components of $D_1\Delta D_2$. Thus:

$$(-1)^{\mathrm{Arf}(q^K_{\mathcal{D}_1})+\mathrm{Arf}(q^K_{\mathcal{D}_2})}\epsilon^K(\mathcal{D}_1)\epsilon^K(\mathcal{D}_2)=(-1)^{\sum_{i=1}^{m}(n^K_{C_i}+1)}\epsilon^K(\mathcal{D}_1)\epsilon^K(\mathcal{D}_2)=1$$

Then we proved this lemma. □

Now we can express partial partition functions $Z_{\alpha,\mathcal{D}_0}$ in terms of Pfaffians.

**Theorem 7.5.4.**

$$Z_{\alpha,\mathcal{D}_0}=\frac{1}{2^{2g}}\sum_{[K]}(-1)^{q^K_{\mathcal{D}_0}(\alpha)}\epsilon^K(\mathcal{D}_0)\mathrm{Pf}(A^K)$$

*Proof.* We only need the identity

$$\sum_{\alpha}(-1)^{q(\alpha)+q'(\alpha)}=\begin{cases}2^{2g}, & q=q'\\ 0. & q\neq q'\end{cases}$$

To prove it just choose a basis $\{\alpha_i\}$ in $H_1(\Sigma, \mathbb{Z}_2)$, and let

$$\alpha = \epsilon_1\alpha_1 + \epsilon_2\alpha_2 + ... + \epsilon_{2g}\alpha_{2g}$$

so

$$\sum_\alpha (-1)^{q(\alpha)+q'(\alpha)} = \prod_i (1 + (-1)^{q(\alpha_i)+q'(\alpha_i)})$$

and we can prove the identity. (Remark: $q + q'$ is linear.)

□

**Another convenient version of formula ($\star$)**

Choose closed paths $A_i$ on $\Gamma^\vee$ such that $[A_i]$ form a basis in $H_1(\Sigma, \mathbb{Z}_2)$. Denote $K_{\epsilon_1...\epsilon_{2g}}$ the K.orientation obtained from $K$ by reversing the orientation of edges of $\Gamma$ intersecting $A_i$ when $\epsilon_i = 1$ and keeping it the same as in $K$ when $\epsilon_i = 0$. These orientations represent all equivalence classes of K.orientations. Denote $T_{ij} = ([A_i], [A_j])$

**Theorem 7.5.5.** *Choose a K.orientation of* $\Gamma$ *such that* $(-1)^{\mathrm{Arf}(q^K_{\mathcal{D}_0})}\epsilon^K(\mathcal{D}_0) = 1$ *for some reference dimer configuration, then:*

$$Z_{\Gamma\subset\Sigma_g} = \frac{1}{2^g}\sum_{\epsilon_i=0,1} (-1)^{\sum_i \epsilon_i(n^K_{C_i}+1)+\sum_{i<j}\epsilon_i\epsilon_j T_{ij}}\mathrm{Pf}(A_{K\epsilon})$$

*Here $C_i$ is a path on $\Gamma$ which is parallel to the left of $A_i$.*

*Proof.* First let us prove that we can choose $K$ such that $(-1)^{\mathrm{Arf}(a^K_{\mathcal{D}_0})}\epsilon^K(\mathcal{D}_0) = 1$. If we choose a $K$ with $(-1)^{\mathrm{Arf}(a^K_{\mathcal{D}_0})}\epsilon^K(\mathcal{D}_0) = -1$, we can simply change $K$ by an orientation reserving operation. Recall the partition function:

$$Z_{\Gamma\subset\Sigma_g} = \frac{1}{2^g}\sum_\epsilon (-1)^{\mathrm{Arf}(q^{K_\epsilon}_{\mathcal{D}_0})}\epsilon^{K_\epsilon}(\mathcal{D}_0)\mathrm{Pf}(A^K)$$

By our assumption, this is equal to

$$Z_{\Gamma\subset\Sigma_g} = \frac{1}{2^g}\sum_\epsilon (-1)^{\mathrm{Arf}(q^{K_\epsilon}_{\mathcal{D}_0})+\mathrm{Arf}(q^K_{\mathcal{D}_0})}\epsilon^{K_\epsilon}(\mathcal{D}_0)\epsilon^K(\mathcal{D}_0)\mathrm{Pf}(A^K)$$

Define $\Delta_\epsilon$ to be $\sum A_i\epsilon_i$, since for closed oriented curve $c$,

$$q^{K_\epsilon}_{\mathcal{D}_0}(c) + q^K_{\mathcal{D}_0}(c) = n^{K_\epsilon}_c - n^K_c = (\Delta_\epsilon, c)$$

we have

$$\mathrm{Arf}(q^{K_\epsilon}_{\mathcal{D}_0}) + \mathrm{Arf}(q^K_{\mathcal{D}_0}) = q^K_{\mathcal{D}_0}([\Delta_\epsilon])$$

Again, we have

$$\epsilon^{K_\epsilon}(\mathcal{D}_0)\epsilon^K(\mathcal{D}_0) = (-1)^{\langle\Delta_\epsilon,\mathcal{D}_0\rangle}$$

so

$$(-1)^{\mathrm{Arf}(q^{K_\epsilon}_{\mathcal{D}_0})+\mathrm{Arf}(q^K_{\mathcal{D}_0})}\epsilon^{K_\epsilon}(\mathcal{D}_0)\epsilon^K(\mathcal{D}_0) = (-1)^{\sum_i \epsilon_i(q^K_{\mathcal{D}_0}([A_i])+(A_i,\mathcal{D}_0))+\sum_{i<j}\epsilon_i\epsilon_j T_i j}$$

By $l_{\mathcal{D}_0}(c_i) = \langle A_i, \mathcal{D}_0\rangle$ from the definition of $c_i$, we will get

$$(-1)^{\mathrm{Arf}(q^{K_\epsilon}_{\mathcal{D}_0})+\mathrm{Arf}(q^K_{\mathcal{D}_0})}\epsilon^{K_\epsilon}(\mathcal{D}_0)\epsilon^K(\mathcal{D}_0) = (-1)^{\sum_i \epsilon_i(n^K_{c_i})+\sum_{i<j}\epsilon_i\epsilon_j T_i j}$$

which is the form we want. □

## 7.6 Pfaffian solution for bipartite graphs

**Plane graphs:**

Let $\Gamma$ be a bipartite plane graph. Choose a bipartite orientation of $\Gamma$, say from black to white vertices. Then any other orientation of $\Gamma$ can be encoded by assigning $\pm$ to an edge, $+$ if the orientation agrees with it in the bipartite orientation, and $-$ if not. Let $s_K$ be signs of an orientation $K$.

The Kasteleyn condition becomes

$$\prod_{\partial f} s_K(e) = -(-1)^{\frac{\partial f}{2}}$$

for any face $f$.

In terms of the decomposition of vertices into black and white, the Kasteleyn metric is:

$$A^K = \begin{pmatrix} 0 & K^{BW} \\ -K^{WB} & 0 \end{pmatrix}$$

where $K^{BW} : B \to W$, $K^{WB} : w \to B$ and $K^{BW} = (K^{WB})^t$, and the matrix defined as

$$K^{BW} b = \sum_{w \text{ adjacent to } b} s_K(w,b) w(w,b) w$$

where $w(w,b)$ is the weight of the edge connecting vertices $w$ and $b$.

*Remark* (Kasteleyn-Perkus orientations). Note we can take complex entries in K.matrix, by saying $s_K \in S^1 \subset \mathbb{C}$ and $K^{BW} = (K^{WB})^*$. These complex orientations are called Kasteleyn-Perkus orientations.

Here the Kasteleyn conditions can be regarded as conditions for complex numbers $s(e)$:

$$\prod_{i=1}^{k} \frac{s(b_i, w_i)}{s(w_i, b_{i+1})} = (-1)^{k+1}$$

for edges from $\partial f$, with $|\partial f| = 2k$, hold for every face $f$.

(it seems should not end, but is does end in the lecture notes.)

# Chapter 8

# Lecture 8

## 8.1 Height functions for surface bipartite graphs

Here we define the height function for surfaces with no boundary. Let $\Sigma_g$be compact oriented closed surface, $\Gamma \subset \Sigma_g$ be a surface bipartite graph, i.e. we fix an embedding $\Gamma \subset \Sigma_g$, such that all faces are contractible. Denote the resulting cell complex $X_\Gamma$. We assume that 2-cell are oriented as $\Sigma_g$ and edge are oriented from black vertices to white.

A dimer configuration defines a chain

$$\hat{\mathcal{D}} = \sum_{\mathcal{D}} e \in C_1(X_\Gamma, \mathbb{Z})$$

It is clear that

$$\partial\hat{\mathcal{D}} = \sum b - \sum w$$

Let $\omega_0$ be any chain with the same boundary:

$$\partial\omega = \sum b - \sum w$$

then the difference $\hat{\mathcal{D}} - \omega$ is a cycle, i.e. $\partial(\hat{\mathcal{D}} - \omega) = 0$

*Remark.* One can think of $C^i$ as a discrete analog of the space of $i$-th differential form on $\Sigma$. The differential d as a discrete analog of de Rham differential. Because the space $C^i$ comes with a basis $\delta_c$ of functions supported by a cell, we have a natural scalar product

$$(\delta_c, \delta_{c'}) = \delta_{c,c'}$$

and naturally we have operation $\mathrm{d}^*$: the dual map of $d$. Fixing a scalar product is analogous to fixing a metric which means we can also think of elements of $C^i$ as polyvector fields.

In particular elements $C^1$ can be regarded as vector fields. If $\beta \in C^1(X_\Gamma, \mathbb{R})$, $\beta(e)$ can be regarded as the flow along $e$ in the direction of the edge. With this interpretation $d * \beta$ is just the divergence of this vector field, i.e. $d^*\beta(v)$ is the amount of the flow that sink into v. If $d^* = 0$, it means the vector field has no divergence,i.e. the amount of the flow coming to $v$ through adjacent edges is equal to the amount of flow leaving $v$ through the adjecent edges.

Now, let $c_1, ..., c_{2g}$ be a system of simple curves, making a basis of $H_1(\Sigma, \mathbb{Z})$. Then we can write

$$\hat{\mathcal{D}} - \omega = \sum_{i=1}^{2g} m_i c_i + \partial h$$

where

$$h = \sum_f h(f) f \in C_2(X_\Gamma, \mathbb{Z})$$

is a 2-cycle describing the height function on any open contractible domain and $m_i$ are coordinated of the class $[\hat{\mathcal{D}} - \omega]$ in the basis $[c_i]$ in $H_1(\Sigma, \mathbb{Z})$.

Locally, when $e \notin C_i$,

$$\sigma(e) - \omega(e) = h(f_+) - h(f_-)$$

where $f_\pm$ lies on the left/right side of $e$. This defines a height function on any open neighbourhood $U$ on $X_\Gamma$ (i.e. a contractible subcomplex, which is a decomposition of a disc) as follows

what is decomp of a disc mean?

- choose a reference face $f_0 \subset U$ and a path on $\Gamma^\vee$ connecting $f_0$ and $f$,
- define $h(f_0) = h_0$ and we have formula $(\star)$

$$h_{\mathcal{D},\omega}(f) = \sum_{e \cap \gamma \neq \varnothing} \epsilon(e, \gamma)(\sigma_{\mathcal{D}}(e) - \omega(e)) + h_0$$

where $\epsilon(e, \gamma) = \pm 1$ if $\gamma$ walks through $e$ from right/left.

It is convenient to describe height functions on $X_\Gamma$ by cutting $\Sigma$ along a system of fundamental cycles.

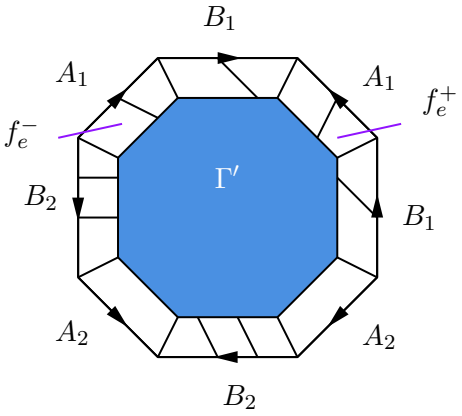
Let us assume that $c_1, ..., c_{2g}$ are oriented simple curves $A_1, ..., A_g, B_1, ..., B_g \subset \Gamma$, such that $A_i \cap A_j = B_i \cap B_j = \varnothing$, and $A_i \cap B_j = \delta_{i,j}$.Their homology classes form a basis in $H_1(\Sigma, \mathbb{Z})$.

They cut $\Sigma$ into a connected simply connected surface homeomorphic to a disc. Denote by $\Gamma'$ the plane graph which is the result of cutting $\Gamma$ along these curves.

*Remark.* When cutting $\Gamma$ along $\{a_i\} \cup \{b_i\}$ we split edges in $a_i$ and $b_i$. The resulting plane graph has boundary consisting of $\{a_i\} \cup \{b_i\}$ and $\{-a_i\} \cup \{-b_i\}$. It is a cell complex for the fundamental domain $\Sigma' = \Sigma - \{a_i\} \cup \{b_i\}$.

Let $h_{\mathcal{D},\omega} := h_{\mathcal{D},\omega}|_{\Sigma'}$ be a height function on $\Sigma'$ normalized $h(f_0) = h_0$ at some reference face $f_0$. It is determined by our formula $(\star)$ and defines a function on faces of $X_\Gamma \setminus \{-a_i\} \cup \{-b_i\}$

Denote:




$$\Delta_{A_i}(e) = h_{\mathcal{D},\omega}(f_e^+) - h_{\mathcal{D},\omega}(f_e^-)$$

$$\Delta_{B_i}(e) = h_{\mathcal{D},\omega}(f_e^+) - h_{\mathcal{D},\omega}(f_e^-)$$

Now let us compute the weight of a dimer configuration in terms of the corresponding height function $h_{\mathcal{D},\omega}$

Assume

$$w(e) = \exp(-E(e))$$

then

$$\begin{aligned}-\ln w(\mathcal{D}) &= \sum \sigma_{\mathcal{D}}(e)E(e)\\ &= \sum E(e)\omega(e) + (h(f_e^+) - h(f_e^-))E(e)\\ &= \sum E(e)\omega(e) + \sum(\sum_{\partial f}\tilde{\epsilon}(e,\partial f)E(e))h(f)\\ &+ \sum_{c\in\{A_i\}\sum_{e\in c}\cup\{B_i\}} \tilde{\epsilon}(e,c)E(e)(h(f_e^+) - h(f_e^-))\end{aligned}$$

Here $\tilde{\epsilon}(e,a)$ is the relative orientation number for $e$ with bipartite orientation on an oriented curve $a$. Note that $e \subset a$ and $\tilde{\epsilon}(e,a)$ is not the same as the oriented intersection number $\epsilon(e,\gamma)$.

**Lemma 8.1.1.** *For a closed nonseparating simple path $C \subset \Gamma \subset \Sigma$, the difference*

$$\Delta_c h = h(f_e^+) - h(f_e^-)$$

*does not depend on $e$.*

*Proof.* Let $e, e' \subset C$ be two different edges. Connect $f_e^+$ to $f_e^-$ by a path and use the definition of height functioon. Connect $f_e^+$ with $f_{e'}^+$ by a path following $C$ immediately to the left and $f_e^-$ with $f_{e'}^-$ by a path following $C$ immediately to the right and $f_{e'}^+$ to $f_{e'}^-$. The change of the height function along these paths is 0 because height function does not depend the path we select. □

Thus, we have:

$$-\ln w(\mathcal{D}) = \sum_{E(\Gamma)} \omega(e)E(e) + \sum_f h(f)E(f) + \sum_{c\in\{A_i\}\cup\{B_i\}} \sum_c \tilde{\epsilon}(e,c)E(e)\Delta_c$$

Define the space of height functions on $X_\Gamma$ as

$$\mathcal{H}_{\Gamma,\omega,\{A\},\{B\}} = \{h | h(f^+) - h(f^-) = \sigma_{\mathcal{D}}(e) - \omega(e), \quad \forall e \in \{A\}\cup\{B\}, \text{for some dimer config.}\mathcal{D}\}$$

We proved the following theorem:

**Theorem 8.1.2.** *The dimer partition function can be written as the partition function for random height functions as*

$$Z^{\text{dimer}}_{\Gamma\subset\Sigma} = \sum_{\mathcal{D}\subset\Gamma}\prod_{e\in\mathcal{D}} w(e) = \prod w(e)^{\omega(e)} \sum_h \prod_f q_f^{h(f)} \prod_{c\in\{A_i\}\cup\{B_i\}} \prod_{e\in c} w(e)^{\tilde{\epsilon}(e,c)\Delta_c(h)}$$

*where $q_f = \prod_{\partial f} w(e)^{\tilde{\epsilon}(e,\partial f)}$.*

Note that the space of height functions is characterized without referring to dimers.

## 8.2 The partition function in terms of height functions on surfaces with boundary

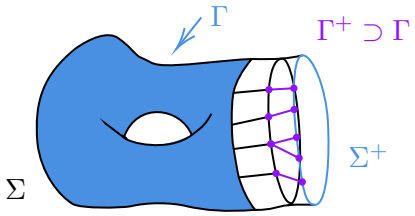
First, let us extend the conclusions from the previous section to graphs on surfaces with boundary. In this lecture we set $\Gamma$ to be a bipartite graph.

Let $\Sigma$ be an oriented compact surface with boundary. We define a surface graph $\Gamma \subset \Sigma$ to be a graph embedded into $\Sigma$ such that it defines a cell complex $X\Gamma$ of $\Sigma$ and $\Gamma \cap \partial\Sigma$ be a cell compplex for $\partial\Sigma$.

### 8.2.1 Collar extension of a surface graph

As in the planar case denote by $\Sigma^+$ a collar extension of $\Sigma$ and $\Gamma^+$ an extension of $\Gamma$ to $\Sigma^+$ by semi open edges.

As in the planar case, The graph $\Gamma^+ \subset \Sigma 6+$ defines the cell complex $X^+_\Gamma$ completely, similarly to when $\Sigma$ is a disk.




Denote by $b(\Gamma^+)$ the border edges of $\Gamma^+$:

$$b(\Gamma^+) = \Gamma^+ \setminus \Gamma$$

Border edges shown in purple on Figure above.

Recall: Border edges are semi open and boundary vertices of $\Gamma$ are not matched border edges of $\Gamma_+$. Such dimer configuration can be regarded as a 1-chain

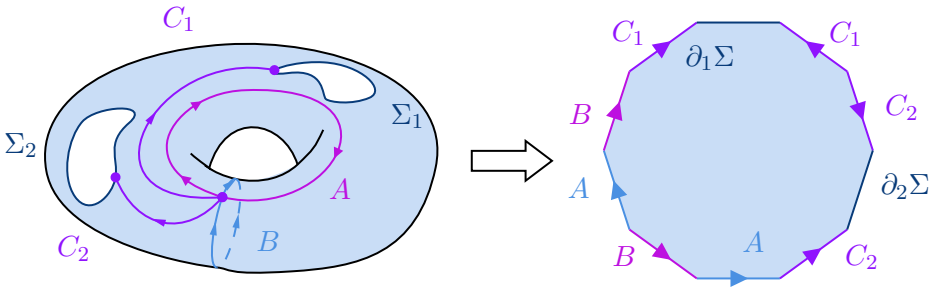
$$\hat{\mathcal{D}}^+ = \sum_{e \in \mathcal{D}^+} e \in C_1(X_{\Gamma^+;\mathbb{Z}})$$

with $\partial\hat{\mathcal{D}}^+ = \sum b - \sum w$.

Let $\omega \in C_1(X_{\Gamma^+}, \mathbb{R})$ be another 1-chain with the same boundary as before, making there difference be a cycle.

For closed surfaces we can define the height function by cut along canonical intersections. The result of the cut is a cell complex for a disc $\Sigma'$. We do the same for a surface with boundary. Choose simple paths $A_1, ..., A_g, B_1, ...B_g, C_1, ..., C_k$, where $k$ is the number of boundary components of $\Sigma$. We can choose them as paths $A_i, B_i$ we used for closed surfaces and $C_i$ connecting the $i$-th component of $\partial\Sigma$ from $p_i$ to one reference point $p$.

Note $c_{i,i+1} = (-C_i) \circ C_{i+1}$, $i < k$ and $A_i, B_i$ form a basis of $H_1(X_{\Gamma^+}, \mathbb{Z}_2)$.Homology classes of $(-C_i)(\partial_i\Sigma)(C_i)$ and $A_i, B_i$ form a basis in $H_1(X_\Gamma, \mathbb{Z}_2)$. One example of result of cutting along a manifold is shown below:

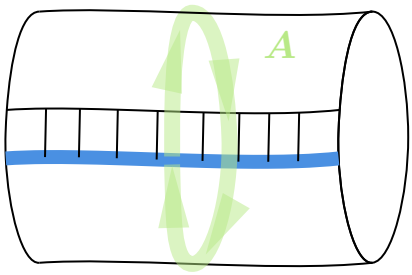



for a dimer configuration on $\Gamma^+$ we can represent the cycle $\hat{\mathcal{D}}^+ - \omega$ as

$$\hat{\mathcal{D}}^+ - \omega = \sum_i (\Delta_{A_i} h\hat{A}_i + \Delta_{B_i} h\hat{B}_i) + \sum \Delta_{C_i} h\hat{C}_i + \partial h_{\mathcal{D}^+,\omega}$$

Here $h_{\mathcal{D}^+,\omega}$ is the height function on $X'_{\Gamma^+}$ defined by equation $(\star)$.

Sometimes it is convenient to represent the jump of $h_{\mathcal{D}^+,\omega}$ along a simple path $A \subset (\Gamma^+)^\vee$. An example is shown below:




We calculate by features for plane regions and closed surfaces:

$$-\ln w(\mathcal{D}) = \sum_{e\in\Gamma}(\sigma_{\mathcal{D}}(e)-\omega(e))E(e) = \sum E(e)w(e) + \sum_{e\in\Gamma} E(e)\Delta_e(h)$$
$$= \sum E(e)w(e) + \sum_f(\sum_{\partial f}\tilde{\epsilon}(e,\partial f)E(e))h(f) + \sum_{i=1}^{g}\sum_{e\in A_i}\tilde{\epsilon}(e,A_i)E(e)\Delta_e h$$
$$+\sum_{i=1}^{g}\sum_{e\in B_i}\tilde{\epsilon}(e,B_i)E(e)\Delta_e h + \sum_{f_b} h(f_b)\sum_{e\in\partial f_b\cap\Gamma}\epsilon(e,f)E(e)$$

Let:

$$\mathcal{H}_{\Gamma,\omega,\{A\},\{B\},C} = \left\{\begin{array}{l} h: F(\Gamma)\to\mathbb{R} | h(f^+)-h(f^-) = \pm(1-\omega(e)) \text{or} \pm\omega(e), \\ \text{exactly one } \pm(1-\omega) \text{ in each Star}(v), \\ \text{signs depend on the orientation of the edge } e, \\ \text{for edges of } \Gamma^+ \text{ except those belong to } A_i, B_i, C_{i,i+1} \end{array}\right\}$$

Now we proved the following theorem:

**Theorem 8.2.1.** *The dimer partition function on a surface bipartite graph $\Gamma^+\subset\Sigma^+$ with fixed boundary dimer configuration can be written as the partition function for height functions.*

$$Z^{\text{dimer}}_{\Gamma\subset\Sigma} = \sum_{\mathcal{D}}\prod_{e\in\mathcal{D}} w(e) = \prod w(e)^{\omega}(e)\sum_h\prod_f q_f^{h(f)}\prod q(A_i)^{\Delta_{A_i}h}\prod q(B_i)^{\Delta_{B_i}h}\prod_{i<k} q(C_{i,i+1})^{\Delta_{c_{i,i+1}}h}\prod_{f_b} {q_{f_b}}^{hf_b}$$

*where*

$$q_f = \prod_{\partial f} w(e)^{\tilde{\epsilon}(e,\partial f)}$$
$$q(C) = \prod_{e\in C} w(e)^{\tilde{\epsilon}(e,\partial f)}$$

*for $C = A_i, B_i, C_{i,i+1}, \partial_i\Gamma$.*

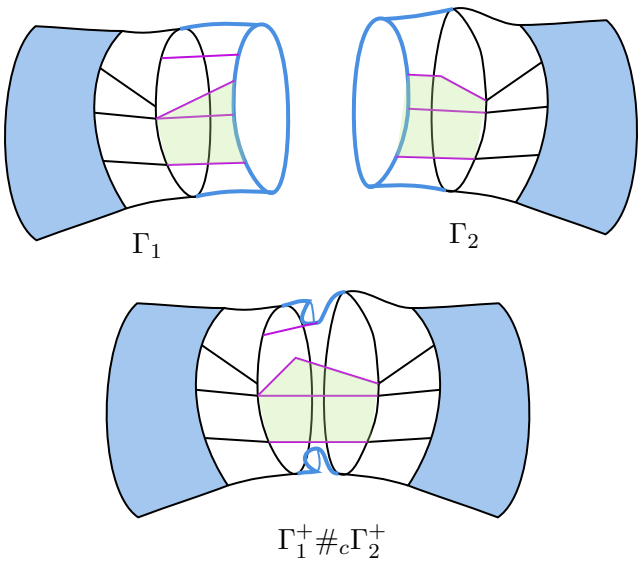
Now we can define the natural prob measure on height functions:

$$\text{Prob}(h) = Z^{-1}\prod_f q_f^{h(f)}\prod q(A_i)^{\Delta_{A_i}h}\prod q(B_i)^{\Delta_{B_i}h}\prod_{i<k} q(C_{i,i+1})^{\Delta_{c_{i,i+1}}h}$$

## 8.3 Gluing two surface graphs along a subset of a collar

Let $\Gamma_1^+\subset\Sigma_1^+$ and $\Gamma_2^+\subset\Sigma_2^+$ be two collar extended surface graphs $\Gamma_1,\Gamma_2$. Assume that collars of $\Sigma_1^+$ and $\Sigma_2^+$ have connected simply connected domains which are identified. Find $C\subset\Sigma_1^+$ and $C\subset\Sigma_2^+$ and will write $\Sigma_1^+\cup_C\Sigma_2^+$ for the result.

Assume $X_{\Gamma_i}^+\cap C$ are isomorphic as open subcomplexes, we can construct $X_{\Gamma_1}\cup_C X_{\Gamma_2}$. The gluing process is illustrated below:




The region $C \subset \Sigma_1^+$ and $C \subset \Sigma_2^+$ is in green. It becomes a region in the int. of the final surface. This gives natural rules for gluing of part. functions.

# Chapter 9

# Lecture 9

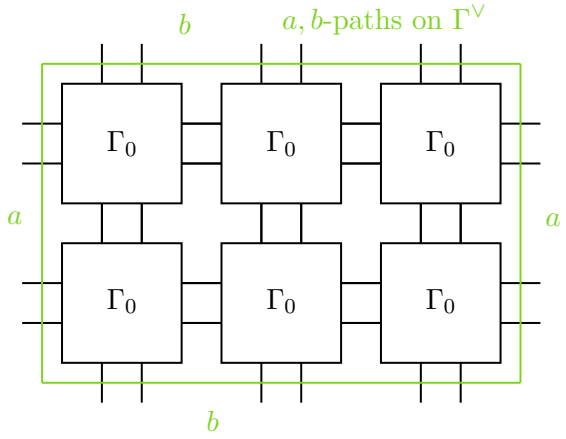
## 9.1 Periodic graphs on a torus

We consider a $Mk \times Nl$ torus graph consisting of $M \times N$ fundamental domains $\Gamma_0$ of the size $k \times l$. Denote it as $\Gamma_0^{NM}$. For convenience, we let $M, N$ be even.

### 9.1.1 local K. orientations

Let $\hat{Gamma}_0$ be a fundamental domain placed on a torus in a natural way: Left edges are identified with right edges, top ones are identified with bottom edges. Give it a K. orientation $K_0$(it exists of course), and we can get a periodic K. orientation of $\Gamma_0^{NM}$, denoted as $K_0^{NM}$.

Now just assume the weight of edges are periodic, we can define $A_{NM}^{K_0}$ be the K. matrix with such periodic weight system.




We still need to define $K_0^{NM}(\epsilon_1, \epsilon_2)$, where $\epsilon_i = 0, 1$. representing times of edges intersecting $a = c_1$ or $b = c_2$ reversing their orientation. The corresponding K. matrices is denoted as $A_{NM}^{K}(\epsilon_1, \epsilon_2)$.

We have the theorem below.

**Theorem 9.1.1.** *The dimer part. function on the torus $\Gamma^M N_0$ is given by:*

$$Z_{MN}^{\Gamma_0} = \frac{1}{2}(-\mathrm{Pf}(A_{MN}^{K_0}(0,0)) + \mathrm{Pf}(A_{MN}^{K_0}(1,0)) + \mathrm{Pf}(A_{MN}^{K_0}(0,1)) + \mathrm{Pf}(A_{MN}^{K_0}(1,1)))$$

*Proof.* Remember our formula derived before:

$$Z = \frac{1}{2^g}(-1)^{Arf(q_{\mathcal{D}}^K)}\epsilon^K(\mathcal{D}) \sum_{\epsilon_i=0,1} (-1)^{\sum \epsilon_i(n_k(c_i)+1)+\epsilon_1\epsilon_2([c_1],[c_2])}\mathrm{Pf}(A_{K_\epsilon})$$

we need two results to finish the proof.

a):

$$(-1)^{Arf(q_{\mathcal{D}}^K)}\epsilon^K(\mathcal{D}) = -1$$

remind Arf can be defined by $q^{-\frac{1}{2}}\sum(-1)^q(\alpha)$, then we can get:

$$(-1)^{Arf(q=q_{\mathcal{D}}^K)} = \frac{1}{2}(1+(-1)^{q(c_1)}+(-1)^{q(c_2)}+(-1)^{q(c_1+c_2)})$$

where the quadratic form is the ine we defined:

$$q_{\mathcal{D}}^K([c_1]) = n_{c_1}^K + l_{\mathcal{D}}(c_1) + 1 = N(n_{c_1^0}^{K_0} + l) + 1$$

since $N$ is even, we can know that

$$q_{\mathcal{D}}^K(c_1) = 1 \bmod 2$$

and $(-1)^{Arf(q)} = \frac{1}{2}(1-1-1-1) = -1$. Since $M, N$ is even, $\epsilon$ won't contribute.
b) Coefficients in front other Pfaffians. Since $n^K(C_1) = Nn^{K_0}(C_1^0) = 0 \bmod 2$, then the signs are $-1, -1, -1$ respectively, and we finished the proof. □

Each Pfaffian is easy to compute using the discrete Fourier transform, assuming we know the spectrum of the K. matrix for the fundamental domain.

Let us first compute $Pf(A_M^{K_0}N(0,0))$. Consider the vector space of vertices on $\Gamma_0^{M,N}$. There is a natural basis $e_{v+ma+nb}, v \in \Gamma_0, n = 0, ..., N-1, m = 0, ..., M-1$, the operator $A_{MN}^{K_0} = A_{MN}^{K_0}(0,0)$ acts on this basis as

$$A_{MN}^{K_0}e_{v+ma+nb} = \sum_{u\in\Gamma_0+ma+nb} A_{MN}^{K_0}e_{u+ma+nb} + \sum_{\pm}\sum_{u\in\Gamma_0+(m\pm1)a+nb} A_{uv}^{K_0}e_{u+(m\pm1)a+nb} + \sum_{\pm}\sum_{u\in\Gamma_0+ma+(n\pm1)b} A_{uv}^{K_0}e_{u+ma+(n\pm1)b}$$

where $A^{K_0}$ is the Kasteleyn matrix for the fundamental domain.

After the discrete Fourier transform

$$e_u(z,w) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} z^m w^n e_{u+ma+nb}$$

the matrix $A_{MN}^{K_0}$ becomes block diagonal:

$$A_{MN}^{K_0}e_v(z,w) = \sum_{u\in\Gamma_0} A_{uv}^{K_0}(z,w)e_u(z,w)$$

where $A_{uv}^{K_0}(z,w)$ is the K. matrix for the fundamental domain $\Gamma_0$ on a torus with the extra weight $z$ assigned to each edge crossing the $B$-cycle and the extra weight $w$ assigned to each edge crossing the $A$-cycle.

Then, in the new basis, we get $\mathrm{Pf}(A_{MN}^{K_0}) = \prod_{z^N=w^M=1}\mathrm{Pf}(A^{K_0})$

The Pfaffian $\mathrm{Pf}(A^{K_0}(z,w))$ is called the spectral polynomial of the fund. region

$$P_{\Gamma_0}(z,w) = \mathrm{Pf}(A^{K_0}(z,w))$$

For Pfaffians of $A_{MN}^{K_0(\epsilon_1,\epsilon_2)}$ we have similar formulae:

$$\mathrm{Pf}(A_{M,N}^{K_0}) = \prod_{z^N=(-1)^{\epsilon_1}, w^M=(-1)^{\epsilon_2}} P_{\Gamma_0}(z,w)$$

where $P_{\Gamma_0}(z,w)$ is the characteristic polynomial.

Note that

$$\epsilon^{K_0}(\mathcal{D}_0)\mathrm{Pf}(A^{K_0}(z,w)) = \sum_{\alpha\in H_1(T,\mathbb{Z}_2)} (-1)^{q^{K_0}_{\mathcal{D}_0}(\alpha)} Z_{\alpha,\mathcal{D}_0}(z,w)$$

where $Z_{\alpha,\mathcal{D}_0}(z,w)$ is the partition function on a torus with extra weights for edges intersecting cycles $A, B$, for $\mathcal{D}\Delta\mathcal{D}_0 = \alpha$.

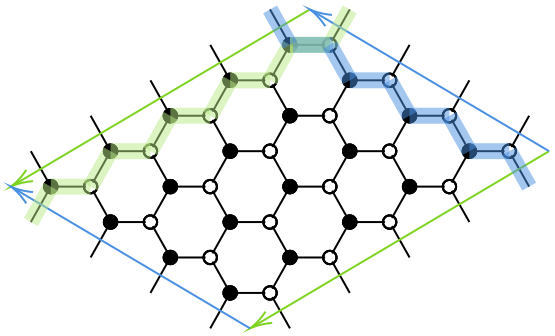
now, we can obtain the following formula for the partition function:

$$Z_{\Gamma_0^{MN}} = \frac{1}{2}(-\prod_{z^M=w^N=1} P_{\Gamma_0}(z,w) + \prod_{z^M=-1,w^N=1} P_{\Gamma_0}(z,w) + \prod_{z^M=1,w^N=-1} P_{\Gamma_0}(z,w) + \prod_{z^M=-1,w^N=-1} P_{\Gamma_0}(z,w))$$

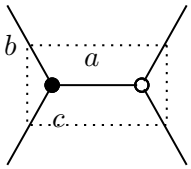
From here it is easy to pass to limit $N, M \to \infty$.

## 9.2 Dimers on a hexagonal lattice on a torus

Let $\Gamma_{N,M}$ be the hexagonal lattice on a torus embedded as on the figure that follows. We name the blue and green lines $B^\vee$ and $C^\vee$, and lines next to it $B$ and $C$:



In this model, we assume the weight of edges do not depend on the position of the edge but only the type (direction) of an edge. We will consider fragments




as fundamental domains.

We select the standard K.orientation by let every edge have orientation to white points, and $K_{\epsilon_1,\epsilon_2}$ are defined as reserving non-horizontal edges on $B$ and $C$ by $\epsilon_1$ or $\epsilon_2$ times.

We have the following result

$$Z^{\mathrm{dimer}}_{T_{N,M}} = \frac{1}{2}(-P_{00} + P_{01} + P_{10} + P_{11})$$

where $P$ is defined as before.

Let us consider the fundamental domain on the torus. There are 2 pts; $B$ and $W$, and the matrix elements of $A^{K_0}(z,w)$ is:

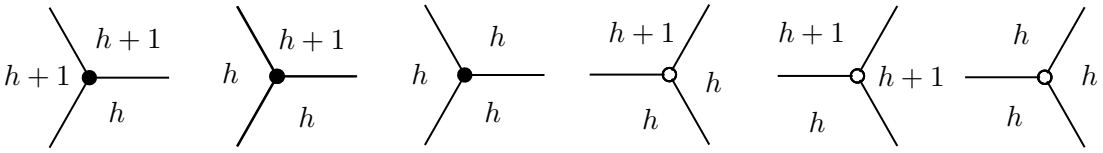
$$A_B = (a + bz + cw)W, A_W = -(a + bz + cw)B$$

Thus the Pffafian is:

$$\mathrm{Pf}(A) = a + bz + cw$$

And this gives the partition function.

Choose the dimer configuration where all horizontal edges are occupied $\mathcal{D}_0$, we can define height functions on $w = \sigma_{\mathcal{D}_0}$. Locally, those functions should look like these:




Globally. these rules define possible height functions on $\Gamma \setminus \{B, C\}$.

$$Z^{dimer} = a^{NM} \sum_{\Delta B, \Delta C} \sum_{h \in H_{\Gamma_{B,C}}(\Delta_B, \Delta_C)} \prod_f (\frac{b}{a})^{M\Delta_B} (c/a)^{N\Delta_C}$$

the sum is taken over all possible height functions with the local properties of all possible $\Delta_B, \Delta_C$. Fix any $\Delta_B$, $\Delta_C$,$H_{\Gamma_{B,C}}(\Delta_B, \Delta_C)$ is defined as all functions satisfying the local conditions above with gain $\Delta_x$ along $x^\vee$.

denote $b = e^H$,$c = e^V$. Changing the weight by an overall factor does not change anything, so just set $a = 1$, and have:

$$Z^{dimer}_\Gamma = \sum_{\Delta_B, \Delta_C} |H_{\Gamma_{B,C}}(\Delta_B, \Delta_C)| e^{HM\Delta_B + VN\Delta_C}$$

this is just the part. function. So:

$$\sum_{\Delta_B, \Delta_C} |H_{\Gamma_{B,C}}(\Delta_B, \Delta_C)| e^{HM\Delta_B + VN\Delta_C} = \frac{1}{2}(-P_{00} + P_{01} + P_{10} + P_{11})$$

# Chapter 10

# Lecture 10

## 10.1 Thermodynamic limit

First, we discuss about Thermodynamic limit for hexagonal lattice on a torus. Denote $b = e^H, c = e^V$ as before. We have the following theorem describing the asymptotic of each term in the formula for the partition function.

**Theorem 10.1.1.** *As $N, M$ go to infinity, we have*

$$Z_{\Gamma_{N,M}}(H,V) = \exp(-NMf(H,V)(1+o(1)))$$

*where*

$$f(H,V) := -(\frac{1}{2\pi i})^2 \iint_{|z|=e^H,|w|=e^V} \ln(1+z+w)\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}$$

*is called free energy per site.*

*Proof.* From our formula, we only need to deal with those Pfaffians:

$$\frac{1}{NM}\ln P_{ab} = \frac{1}{NM}\sum_{z^N=(-1)^a, w^M=(-1)^b} \ln(1+ze^H+we^V)$$

This equals to

$$\frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} \ln(1+e^{\frac{i\pi}{N}(2n+a)+H}+e^{\frac{i\pi}{M}(2m+b)+V})$$

which will be described by an integral

$$\frac{1}{(2\pi i)^2}\iint_{|z|=|w|=1} \ln(1+ze^H+we^V)\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}$$

then, since $|P_{ab}|$ are also sum of different states but with $+$ or $-$, so we have the following inequality:

$$\max|P_{ab}| <= Z <= 2\max|P_{ab}|$$

and we will arrive the final result. □

Now, let $N_{n,m}$ be the number of states with the height function gaining $n$ along $B$-cycle and gaining $m$ along $C$-cycle. Assume $N, M, n, m$ go to infinity as

$$N = L/\epsilon, M = T/\epsilon, n = sN, m = tM$$

then:

$$N_{n,m} = \exp(-NM\sigma(s,t) + o(\frac{1}{\epsilon^2}))$$

Here $\sigma$ is known as the conditioned free energy where we fixed $t = m/M$ and $s = n/N$; also, it is known as entropy per site, and as the surface tension.

Take into account that

$$Z_{\Gamma_{N,M}} = \sigma_{\Delta_B,\Delta_C} N_{\Delta_B,\Delta_C} e^{HN\Delta_B + VM\Delta_C} = e^{-NM(\sigma(s,t)+Hs+Vt)}$$

and

$$Z_{\Gamma_{N,M}} = \exp(-NMf(H,V))$$

we conclude that the free energy and the Legendre trans of $\sigma(s,t)$:

$$-f(H,V) = \max_{s,t}(sH + tV - \sigma(s,t))$$

since function $f$ strictly convex, the surface tension is givem by inverse Legendre transform:

$$-\sigma(s,t) = \max_{H,V}(Hs + Vt - f(H,V))$$

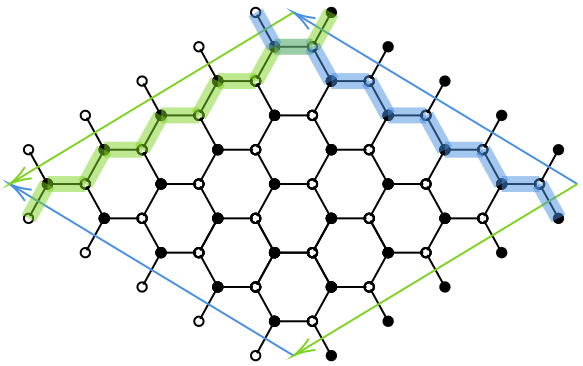
this give the result:

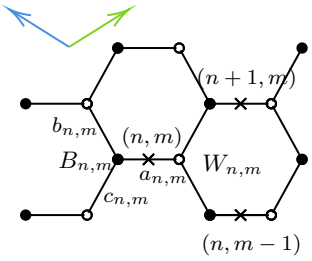
$$\sigma(s,t) = -\frac{1}{\pi}(L(\pi s) + L(\pi t) + L(\pi(1-s-t)))$$

where $L(\theta) = -\int_{[0,\theta]} \ln(2\sin(t))\mathrm{d}t$.

### 10.1.1 Computing the inverse of K.matrices

Before doing our calculation, we will first name those points and weights of the edges as below:






weights and points

Now, we can get those equalities:

$$A^{K_{\epsilon_1,\epsilon_2}} B_{n,m} = a_{n,m}W_{n,m} + c_{n,m}W_{n-1,m} + b_{n,m}W_{n,m+1}$$

$$A^{K_{\epsilon_1,\epsilon_2}} W_{n,m} = -(a_{n,m}B_{n,m} + c_{n+1,m}B_{n+1,m} + b_{n,m-1}B_{n,m-1})$$

The dependence of the K.matrix on $\epsilon$ can be absorbed into periodic condition for $B_{n,m}$:

$$B_{n+N,m} = \epsilon_1 B_{n,m}, B_{n,m+M} = \epsilon_2 B_{n,m}$$

and the same for white vertices.

We will use Fourier transform to calculate the inverse since the weights are homogeneous. Set $z^N = (-1)^{\epsilon_1}, w^M = (-1)^{\epsilon_2}$, then:

$$B(z,w)=\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}z^m w^n B_{n,m}$$

$$W(z,w)=\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}z^m w^n W_{n,m}$$

now, we get:

$$A_{\epsilon_1\epsilon_2}B(z,w)=(a+bz+cw^{-1})W(z,w)$$

$$A_{\epsilon_1\epsilon_2}W(z,w)=-(a+bz^{-1}+cw)B(z,w)$$

We can always multiply a number on the partition function, and therefore we can reset $a=1$. then we can get the inverse by:

$$A^{-1}_{\epsilon_1\epsilon_2}W(z,w)=(1+az+bw^{-1})B(z,w)$$

$$A^{-1}_{\epsilon_1\epsilon_2}B(z,w)=-(1+az^{-1}+bw)W(z,w)$$

Note $1+bz+cw$ is the part. function of a 2-edges **twisted** $1\times 1$ torus hexagonal graph. Define:

$$G_{n,m}(\epsilon_1,\epsilon_2)=\frac{1}{NM}\sum_{z^N=(-1)^{\epsilon_1},w^M=(-1)^{\epsilon_2}}\frac{z^n w^m}{1+az+bw^{-1}}$$

so we can write:

$$A^{-1}B_{n,m}=\sum\sum G_{n-k,m-l}(\epsilon_1,\epsilon_2)W_{l,k}$$

$$A^{-1}W_{n,m}=-\sum\sum G_{k-n,l-m}(\epsilon_1,\epsilon_2)B_{l,k}$$

When taking limit $N\to\infty$, we will get:

$$G_{n,m}=\lim_{N,M\to\infty}G_{n,m}(\epsilon_1,\epsilon_2)=\frac{1}{(2\pi i)^2}\iint_{|z|=|w|=1}\frac{z^m w^n}{1+az+bw^{-1}}\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}$$

exists and independent of $\epsilon$.

Replace the space $B\otimes W$ by $\ell_2(B+W)$, consider $A$ act on this space; In other words, $A=\begin{pmatrix}0 & A^{WB}\\ A^{BW} & 0\end{pmatrix}$ acting in $\ell_2(B+W)$ has an inverse

$$A=\begin{pmatrix}0 & (A^{BW})^{-1}\\ (A^{WB})^{-1} & 0\end{pmatrix}$$

with:

$$(A^{BW})^{-1}_{(l,k)(n,m)}=G_{n-k,m-l}$$

$$(A^{WB})^{-1}_{(l,k)(n,m)}=-G_{k-n,l-m}$$

These expressions have translation invariance:

$$(A^{-1})_{(k,l)(n,m)}=(A^{-1})_{(k+a,l+b)(n+a,m+b)}$$

We can calculate characteristic functions for an edge from $B_x$ to $W_y$, denoted as $\sigma^{BW}_{x,y}$. Since edges occur only when $y-x=(0,0)/(-1,0)/(0,1)$, we can obtain densities of horizonted and tilted dimers by

$$\langle\sigma^{BW}_{(0,0)(0,0)}\rangle NM = a\partial_a \ln(Z[a,b,c])$$
$$\langle\sigma^{BW}_{(0,0)(0,1)}\rangle NM = b\partial_b \ln(Z[a,b,c])$$
$$\langle\sigma^{BW}_{(0,0)(-1,0)}\rangle NM = c\partial_c \ln(Z[a,b,c])$$

and get:

$$\langle\sigma^{BW}_{(0,0)(0,0)}\rangle = G_{0,0} = \frac{1}{(2\pi i)^2}\iint_{|z|=|w|=1}\frac{1}{1+az+bw^{-1}}\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}$$
$$\langle\sigma^{BW}_{(0,0)(0,1)}\rangle = G_{0,-1} = \frac{1}{(2\pi i)^2}\iint_{|z|=|w|=1}\frac{w^{-1}}{1+az+bw^{-1}}\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}$$
$$\langle\sigma^{BW}_{(0,0)(-1,0)}\rangle = G_{0,-1} = \frac{1}{(2\pi i)^2}\iint_{|z|=|w|=1}\frac{z}{1+az+bw^{-1}}\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}$$

For 2-pts. correlation function of horizontal dimers we will get

$$\langle\sigma^{BW}_{(n,m)(n,m)}\sigma^{BW}_{(0,0)(0,0)}\rangle = -A_{nm,nm}A_{00,00}(G^{BW}_{n,m}G^{WB}_{n,m} - G^{BW}_{0,0}G^{WB}_{0,0})$$

## 10.2 Periodic graphs on a torus

Remember the partition function on a periodic graph we obtained the following formula:

$$Z_{\Gamma_0^{MN}} = \frac{1}{2}(-\prod_{z^M=w^N=1}P_{\Gamma_0}(z,w)+\prod_{z^M=-w^N=1}P_{\Gamma_0}(z,w)+\prod_{z^M=-w^N=-1}P_{\Gamma_0}(z,w)+\prod_{z^M=w^N=-1}P_{\Gamma_0}(z,w))$$

We need to fix a colour(black or white) on each vertex, and set a dimer configuration $\mathcal{D}_0$. Denote by $V$ those edges in $\mathcal{D}_0$, we can find explicit bijection $V\simeq B\simeq W$.

The K.matrix acting in $B\otimes W$ gives a linear operator:

$$A^K_{\Gamma_0}(z,w) = \begin{pmatrix}0 & K(z,w)\\ -K^t(z,w) & 0\end{pmatrix}$$

where $K(z,w)=\sum_{e'-e}s(e,e')$, $s(e,e')=s(e'')$ if the connecting edge goes from black to white, otherwise $s(e'')^{-1}$.

Thus, for the characteristic polynomial we have

$$P_{\Gamma_0}(z,w) = \mathrm{Pf}(A^{K_0}(z,w)) = \mathrm{Pf}\begin{pmatrix}0 & K(z,w)\\ -K^t(z,w) & 0\end{pmatrix}$$

The Pfaffian we can compute in terms partial partition functions for the torus graph $\hat{\Gamma_0}$:

$$\epsilon^(\mathcal{D}_0)\mathrm{Pf}(A^{K_0}_{\hat{\Gamma_0}}(z,w)) = \sum_{\alpha\in H_1(T,\mathbb{Z}_2)}(-1)^{q^{K_0}_{\mathcal{D}_0}(\alpha)}Z_{\alpha,\mathcal{D}_0}(z,w)$$

Here, $\hat{\mathcal{D}_0}$ the reference dimer configuration on $\hat{Gamma_0}$ which we can choose to be the same as we need for identification of black and white vertices (or different). $Z$ is the partial partition function as defined:

$$Z_{\alpha,\mathcal{D}_0}(z,w)=\sum_{\mathcal{D},[\mathcal{D}-\mathcal{D}_0]=\alpha}\prod w(e)$$

Here $z,w$ are extra weight for edges intersecting $B,A$.

As we know, this partition function can be written in terms of height functions. We can choose the chain $\omega$ in the definition of height function as $\hat{\mathcal{D}}_0$. Then:

$$Z_{\alpha,\mathcal{D}_0}(z,w)=\prod_{e\in\mathcal{D}_0}w(e)\sum_{h\in\mathscr{H}_{\Gamma_0,\hat{\mathcal{D}_0}}}\prod_f q_f^{h(f)}(wq_A)^{\Delta_A h}(zq_B)^{\Delta_B h}$$

### 10.2.1 Holonomy weights or magnetic fields

Now, let's introduce extra weights in the partition function $Z_{\Gamma_0^{NM}}$ is bipartite and define the height function using a cochain $\omega$ as earlier. The extra weights we introduce as $e^{HN}$ for each edge interesting cycle $B$ on $\Gamma_0^{MN}$ and $e^{VM}$ for each edge intersecting cycle $A$ on $\gamma_0^{MN}$.

In the Pfaffian formula; this results in replacing $z\mapsto ze^H$, $w\mapsto we^V$. $H,V$ have physical meaning of external 'magnetic fields' interacting with dimers.

### 10.2.2 The thermodynamic limit

Now consider $N,M\to\infty$ as limit. Using same arguments as for the hexagonal lattice we have:

$$\lim_{N,M\to\infty}\frac{1}{NM}\ln Z_{NM}(H,V)=\iint\limits_{|z|=e^H,|w|=e^V}\ln|P_{\Gamma_0}(z,w)|\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}$$

In order to compute correlation functions when $N,M\to\infty$ we should find the asymptotic of the inverse to the Kasteleyn operator. We can do this by Fourier transform as before. Let $V_a^{(n,m)}$ be a basis in the space of vertices where $a\in\mathrm{Vert}(\Gamma_0)$, $n=0,...,N-1, m=0,1,...,M-1$. Assume we should have $V^(n,M)=(-1)^\delta V^{(n,0)}$, and $V^(N,m)=(-1)^\delta V^{(0,m)}$.

We have:

$$A^{\epsilon,\delta}V_a^{(n,m)}=\sum_{(\Gamma_0)_{\mathrm{int}}}A_{ba}V_b^{(n,m)}+\sum_{b(\Gamma_0)}A_{ba}^{(\epsilon,\delta)}V_b^{(n,m+1)}+...$$

Denote the Fourier Trans:

$$V_a^{(\epsilon,\delta)}(z,w)=\sum_0^{N-1}\sum_0^{M-1}V_a^{(n,m)}z^nw^m$$

where, $z^N=(-1)^\epsilon, w^M=(-1)^\delta$.

We can then rewrite:

$$A^{\epsilon,\delta}V_a^(\epsilon,\delta)(z,w)=\sum_{b\in\hat{\Gamma}_0}A_{ba}(z,w)V_b^{(\epsilon,\delta)}(z,w)$$

where $A(z,w)$ is the Kasteleyn matrix for the fundamental domain on a torus with extra weights $z$ and $w$ assigned to edges crossing cycles $B$ and $A$ respectively. Then we have the following formula for $A^{-1}$:

$$A_{a,b}^{-1}(n,m)(k,l)=\frac{1}{(2\pi i)^2}\iint_{|z|=|w|=1}\frac{Q_{a,b}(z,w)}{D(z,w)}z^{n-k}w^{m-l}\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}$$

Here: $D(z,w)$ is determinant of $A(z,w)$ and $Q_{a,b}$ is the matrix of minors. So: $A(z,w)_{ab}^{-1}D(z,w)=Q_{ab}(z,w)$.

### 10.2.3 Asymptotic of correlation functions

An important characteristic of the system is the asymptotical behaviour of correlation functions of large distances. We will analyse the behaviour of correlations at large distances.

$$G_{ab}(n,m)=\frac{1}{(2\pi i)^2}\iint_{|z|=e^H,|w|=e^V}\frac{Q_{ab}(z,w)}{D(z,w)}z^n w^m\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}$$

We will use the fact that follows:

**Lemma 10.2.1.** *Assume that a smooth function $P(z,w)$ on $\mathbb{T}^2=\{(z,w)||z|=1,|w|=1\}$ has a single 0 at $(z_0,w_0)$, then $P(z,w)=\alpha(z-z_0)+\beta(w-w_0)+O(|z-z_0|^2+|w-w_0|^2)$ and $x,y\to+\infty$ such that $x/y$ finite. Also,*

$$I_{x,y}=\frac{1}{(2\pi i)^2}\iint_{|z|=|w|=1}\frac{z^x w^y}{P(x,y)}\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}=\frac{-z_0^x w_0^y}{2\pi i(y_\alpha z_0-x\beta w_0)}+O(\frac{1}{x^2+y^2})$$

*Proof.* Clearly the leading contribution comes from the point $(z_0,w_0)$, Assume that $x,y\to\infty$ such that $x=\lambda X$, $y=\lambda Y$, and $\lambda\to\infty$. Consider an $\epsilon$-neighbourhood of $(z_0,w_0)$, where $z=e^{ia}z_0, w=e^{ib}w_0$, and $-\epsilon<|a|,|b|<\epsilon$.

Then, $P(z,w)=\alpha z_0 ia+\beta w_0 ib+O(\epsilon^2)$.

We then calculate:

$$I_{x,y}=\frac{-z_0^x w_0^y}{(2\pi i)^2}\iint_{-\epsilon}^{\epsilon}\frac{e^{iax+iby}}{z_0 i\alpha a+w_0\epsilon\beta b}\mathrm{d}a\mathrm{d}b+\ldots=z_0^x w_0^y\frac{\lambda^{-1}}{(2\pi)^2 i}\iint_{-\lambda\epsilon}^{\lambda\epsilon}\frac{e^{isX+itY}}{z_0 i\alpha s+w_0\epsilon\beta t}\mathrm{d}s\mathrm{d}t+O(\lambda^{-2})$$

After taking limit $\lambda\to\infty$, what we wonder is the value:

$$\iint_{-\infty}^{\infty}\frac{e^{isX+itY}}{z_0 i\alpha s+w_0\epsilon\beta t}\mathrm{d}s\mathrm{d}t+O(\lambda^{-2})=\int_{-\infty}^{\infty}e^{itY}\left(\int_{-\infty}^{\infty}\frac{e^{isX}}{z_0 i\alpha s+w_0\epsilon\beta t}\mathrm{d}s\right)\mathrm{d}t+O(\lambda^{-2})$$

Here the variables are changes as $a=\lambda^{-1}s, b=\lambda^{-1}t$. In the integral over $s$ we deform integral to the upper half plane when $\mathrm{Im}(-\frac{\beta w_0}{\alpha z_0})>0$.

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{isX}}{z_0\alpha s+\beta w_0 t}\mathrm{d}s=\begin{cases}\frac{e^{-\frac{\beta w_0}{\alpha z_0}tX}}{\alpha z_0}, & t>0\\ 0, & t<0\end{cases}$$

and therefore

$$\frac{\lambda^{-1}}{4\pi^2 i}\int_{-\infty}^{\infty}e^{itY}\left(\int_{-\infty}^{\infty}\frac{e^{isX}}{z_0 i\alpha s+w_0\epsilon\beta t}\mathrm{d}s\right)\mathrm{d}t=\frac{\lambda^{-1}}{2\pi}\int_0^{\infty}\frac{e^{it(Y-\frac{\beta w_0}{\alpha z_0}X)}}{\alpha z_0}\mathrm{d}t=-\frac{1}{2\pi i(\alpha z_0 y-\beta w_0 x)}$$

Similarly, when $\mathrm{Im}(-\frac{\beta w_0}{\alpha z_0})<0$.

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{isX}}{z_0\alpha s+\beta w_0 t}\mathrm{d}s=\begin{cases}0, & t>0\\ -\frac{e^{-\frac{\beta w_0}{\alpha z_0}tX}}{\alpha z_0}, & t<0\end{cases}$$

then we derive the integral should be $-\frac{1}{2\pi i(\alpha z_0 y-\beta w_0 x)}$ too. □

Notice: for functions with more zeros, each one will contribute to the asymptotic.

This lemma implies the result:

$$\iint_{|z|=|w|=1} \frac{Q_{ab}(z,w)}{D(z,w)} \frac{\mathrm{d}z}{z} \frac{\mathrm{d}w}{w} = -2\mathrm{Re}(\frac{z_0^x w_0^y}{2\pi i(az_0 y - \beta w_0 x)}) + O(\frac{1}{x^2+y^2})$$

It is possible that $D(z,w)$ does not have zero on $\mathbb{T}^2$ away from $|z| = |w| = 1$, then the correlation functions decay exponentially if $|z_0|, |w_0| < 1$, which is the ordered or gasous phase. When $|z_0|, |w_0| > 1$ we have a frozen phase when correlation functions are vanishing.

### 10.2.4 Phases of the dimer model on a torus

From the analysis of correlation functions we see that phases of the system are determined py the structure of zeros of $P_{\Gamma_0}(z,w)$. In other words, they are encoded in the structure of algebraic curve:

$$P_{\Gamma_0}(z,w) = 0$$

which is also called the spectral curve.

We will look at examples of hexgonal, triangular and square lattices, then analyse the phases.

## 10.3 Examples

### 10.3.1 Dimers on hexagon lattice

In the case the free energy is

$$f = \frac{1}{(2\pi i)^2} \iint_{|z|=|w|=1} \ln(|1 + az + bw|) \frac{\mathrm{d}z}{z} \frac{\mathrm{d}w}{w}$$

The inverse Kasteleyn matrix and correlation functions are then determined by integrals:

$$G^{BW}_{n,m} = \frac{1}{(2\pi i)^2} \iint_{|z|=|w|=1} \frac{z^n w^m}{1 + bz + cw} \frac{\mathrm{d}z}{z} \frac{\mathrm{d}w}{w}$$

First, let us find when there exist zeroes of the denominator on the unit torus. By:$z_0 = e^{i\theta}, w_0 = e^{i\alpha}$

$$b\sin\theta + c\sin\alpha = 1 + b\cos\theta + c\cos\alpha = 0$$

Then:

$$\cos\theta = \frac{c^2 - b^2 - 1}{2b}, \cos\alpha = \frac{b^2 - c^2 - 1}{2c}$$

possible iff

$$(1-b)^2 < c^2 < (b+1)^2, (1-c)^2 < b^2 < (1+c)^2$$

Since $b, c$ positive, we derived a region where such solution exists. This is what we called a "disordered region", where local correlation functions dacay as the inverse square of the distance.

For other regions:

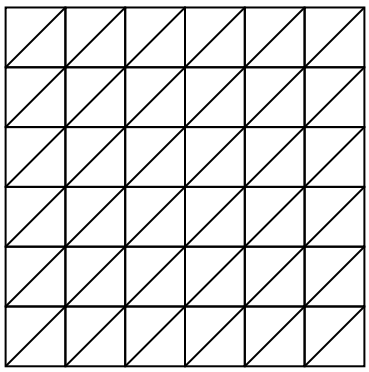
1) $c > b + 1$, In this case solution should be $z_0 = e^t, w_0 = e^{-s}$, then we have frozen phase $f = \ln c$

2) $b > c + 1$, In this case we have frozen phase $f = \ln b$

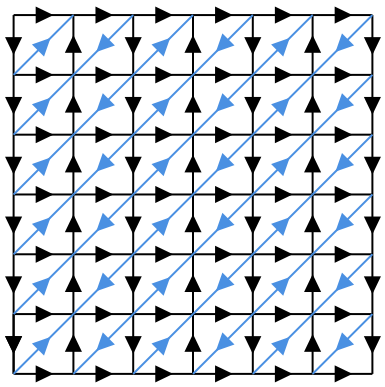
3) $b + c < 1$, this is frozen phase with $f = 0$.

### 10.3.2 Dimers on triangular lattice

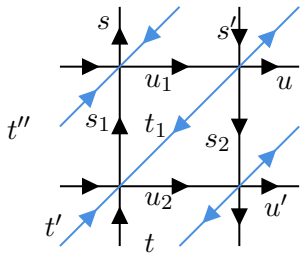
We look at the example of dimers on a triangular lattice. We will represent it as a square lattice with extra diagonal edges as shown below:

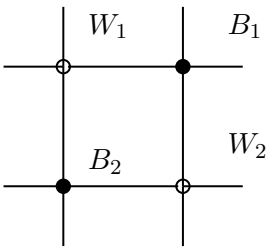


We define each edge's weight: vertical be $s$, horizontal be $u$, diagonal be $t$. After that, we select a Kasteleyn orientation which is periodic:



We can draw it's fundemental domain as:




After that, we get that:

$$^tA^K W_1(n,m) = -sB_2(n,m+1) - tW_2(n,m+1) + u_1 B_1(n,m) - uB_1(n-1,m) + t''W_2(n-1,m) - s_1 B_2(n,m)$$

$$^tA^K B_1(n,m) = -s'W_2(n,m+1) - t'B_2(n+1,m+1) + uw_1(n+1,m) + s_2 w_2(n,m) - t_1 B_2(n,m) - u_1 W_1(n,m)$$

$$^tA^K W_2(n,m) = -s_2 B_1(n,m) - t''W_1(n+1,m) + u'B_2(n+1,m) + s'B_1(n,m-1) - tW_1(n,m-1) - u_2 B_2(n,m)$$

$$^tA^K B_2(n,m) = s_1 W_1(n,m) + t_1 B_1(n,m) + u_2 w_2(n,m) - sw_1(n,m-1) + t'B_1(n-1,m-1) - u'W_2(n-1,m)$$

Now, we wants to do fourier transform:

$$\tilde{X}(z,w) = \sum_{n,m=1}^{N,M} z^n w^m X(n,m)$$

then:

$$
\begin{aligned}
{}^tA(z,w)W_1 &= (u_1 - u/z)B_1 + (sw - s_1)B_2 + (t/z + tw)W_2 \\
{}^tA(z,w)B_1 &= (uz - u_1)W_1 + (-t'zw - t_1)B_2 + (s_2 - s'w)W_2 \\
{}^tA(z,w)W_2 &= (s'/w - s_2)B_1 + (-t''z - t/w)W_1 + (u'z - u_2)B_2 \\
{}^tA(z,w)B_2 &= (s_1 - s/w)W_1 + (t_1 + t'/zw)B_1 + (u_2 - u'/z)W_2
\end{aligned}
$$

Use basis $[W_1, B_1, W_2, B_2]^t$,

$$
{}^tA(z,w) = \begin{bmatrix}
0 & (u_1 - u/z) & (t/z + tw) & (sw - s_1) \\
(uz - u_1) & 0 & (s_2 - s'w) & (-t'zw - t_1) \\
(-t''z - t/w) & (s'/w - s_2) & 0 & (u'z - u_2) \\
(s_1 - s/w) & (t_1 + t'/zw) & (u_2 - u'/z) & 0
\end{bmatrix}
$$

for determinants and Pfaffians we have:

$$\det({}^tA) = \mathrm{Pf}(A^t)^2$$

on the other hand, when K-orientations are periodic, $z^N = w^M = 1$, we have(assume $N, M$ even):

$$\det A = \prod \det A(z,w) = \prod_{k,l=0}^{N/2-1, M/2-1} \det(A(e^{\frac{2\pi ik}{N}}, e^{\frac{2\pi il}{M}}))\det(A(e^{-\frac{2\pi ik}{N}}, e^{-\frac{2\pi il}{M}}))$$

If $A(z,w) = P(z,w)P(z^{-1}, w^{-1})$, then

$$\mathrm{Pf} A = \prod_{z^N = w^M = 1} P(z,w)$$

We now discuss some special cases:

1) $t = 0$, This is the square lattice model.

$$\det A(z,w) = P(z^{-1}, w^{-1})P(z,w)$$

Where $P(z,w) = (uz - u_1)(u_2 - u'/z) - (s_2 - s'w)(s_1 - sw^{-1})$

Free energy per site:

$$f = -\frac{1}{(2\pi i)^2} \iint_{|z|=|w|=1} \ln|P(e^H z, e^V w)| \frac{\mathrm{d}z}{z} \frac{\mathrm{d}w}{w}$$

now let us assume $a = b, c = d$,

$$f(H,V) = \ln(\frac{e}{4}) - \frac{1}{(2\pi i)^2} \iint_{|z|=|w|=1} \ln|Q(e^H z, e^V w)| \frac{\mathrm{d}z}{z} \frac{\mathrm{d}w}{w}$$

Where $Q(z,w) = -4 + p(z + 1/z) + q(w + 1/w), p = \frac{4\sqrt{ab}}{e}, q = \frac{4\sqrt{cd}}{e}$.

2) $u = 0$, square(but twisted) lattice model.

$$\det A(z,w) = -P(z^{-1}, w^{-1})P(z,w)$$

$P(z,w) = -(t''z + t/w)(t + t'/zw) - (s'/w - s_2)(s_1 - s/w)$

3) $s = 0$, another square(but twisted) lattice model.

$$\det A(z,w) = P(z^{-1}, w^{-1})P(z,w)$$

$$P(z,w) = (u - u_1/z)(u'z - u_2) - (t'' + t/zw)(t'zw + t_1)$$

4) all $u/t/s$ are the same.

$$\det A(z,w) = P(z,w)^2$$

$$P(z,w) = s^2(\sqrt{w} - 1/\sqrt{w})^2 - t^2(\sqrt{zw} - 1/\sqrt{zw})^2 + u^2(\sqrt{z} - 1/\sqrt{z})^2$$

### 10.3.3 Phase diagram for the dimer model on a square lattice on a torus

Let us look at zeros of $P(z,w)$.

$$P(z,w) = a(z + 1/z) + c(w + 1/w) - 2a - 2c$$

when $z = e^{i\theta+H}, w = e^{i\alpha+V}$,

$$\begin{cases} a\cos\theta \mathrm{ch}H + c\cos\phi \mathrm{ch}V - a - c = 0 \\ a\sin\theta \mathrm{sh}H + c\sin\phi \mathrm{sh}V = 0 \end{cases}$$

The phase diagram:

$$\pm a\mathrm{ch}H \mp c\mathrm{ch}V = a + c$$

gives a region where disordered phase inside and frozen phases outside. Also, $H = V = 0$ is special(it's nodal point of the spectral curve.)

## 10.4 The Variational Principle

### 10.4.1 Convergent sequences of lattice domains

We focus on dimers on domains on a hexagon lattice. We will see later that this analysis extends to other lattices with very little changes.

Recall our settings:

- $\Gamma \subset H$ a domain in $H$, $X_\Gamma$ be a connected simply connected domain with $\partial\Gamma \subset H$ being a simple closed path.
- As in previous lectures: Open extension $\Gamma \subset \Gamma^+ \subset H$. Fix a boundary height function $h_b$ on border faces of $(\Gamma^+)^\vee$.
- $\phi_\epsilon : H \hookrightarrow \mathbb{R}^2$ be the embedding of the hexagonal lattice with mesh $\epsilon$, $\phi_\epsilon(n,m) = (\epsilon n, \epsilon m)$, where $(n,m)$ be coordinates of centers of horizontal tiles.

Now, assume $R \subset \mathbb{R}^2$ ia a connected, simply connected region with piecewise smooth boundary $\partial R$.

Let $\{\epsilon_n\}$ be a positive sequence converges to 0. Assume $\Gamma_n$ is a sequence of graphs$\subset H$ such that $\phi_{\epsilon_n}(\Gamma_n)$ approximates $R$ as $n \to \infty$, i.e. the maximal distance between $R$ and $\phi_{\epsilon_n}(\Gamma_n)$ goes to 0.

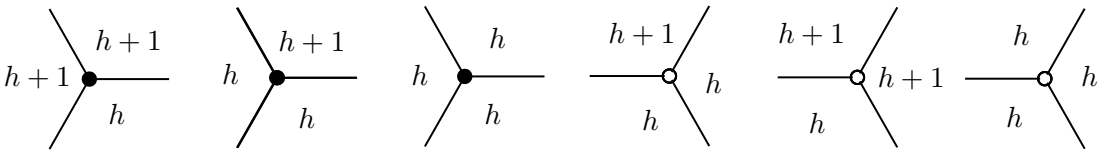
### 10.4.2 Continuous height functions

We are going to define the continuous analog of discrete height functions defined earlier.

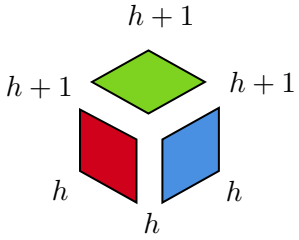
Recall that for dimer model on the hexagonal lattice a convenient choice of height functions is the space:

$$\mathscr{H}_{\Gamma^+,h_b} = \{h : F((\Gamma^+)^\vee) \to \mathbb{Z} | h = h_b \text{ on border faces and } h \text{ satisfy local conditions}\}$$

where local conditions are:




Then we find 3 sides of cubes as lozenges in tiling:




For the horizontal edge corresponds to $(x, y)$.

We can derive from those local rules the properties:

$$0 \leq h_{(x,y)+m(\epsilon,-\frac{-\epsilon}{2})} - h_{(x,y)} \leq m$$

$$0 \leq h_{(x,y)+m(\epsilon,\frac{-\epsilon}{2})} - h_{(x,y)} \leq m$$

$$0 \leq h_{(x,y)+m(0,\epsilon)} - h_{(x,y)} \leq m$$

Now let us define the continuum counterpart of this space:

$$\mathscr{H}_{\mathbb{D},\chi_b} = \{\chi : \mathbb{D} \to \mathbb{R} | \text{ continuous, } \chi|_{\partial\mathbb{D}} = \chi_b, \text{with conditions below}\}$$

$$0 \leq \chi(x + t, y - \frac{t}{2}) - \chi(x, y) \leq t$$

$$0 \leq \chi(x + t, y + \frac{t}{2}) - \chi(x, y) \leq t$$

$$0 \leq \chi(x, y + t) - \chi(x, y) \leq t$$

We assume that $\Gamma_n$ is a sequence of graphs on $H$ approximating region $R$ defined as above and $h_b^{(n)}$ be a sequence of height functions on $b((\Gamma^+)^\vee)$ such that:

$$\epsilon_n h^{(n)} b \to \chi_b$$

as $n \to \infty$. Here the convergence means uniform convergence.

If $h_b^{(n)}$ is such a sequence we can say that spaces $\mathscr{H}_{\Gamma_n^+,h_b^{(n)}}$ converges to $\chi \in \mathscr{H}_{R,\chi_b}$ in a natural way. We will say a sequence $h^{(n)} \in \mathscr{H}_{\Gamma_n^+,h_b^{(n)}}$ converges to $\chi \in \mathscr{H}_{R,\chi_b}$ if

$$\sup_{(x,y)\in\phi_{\epsilon_n}(\Gamma_n^+)_{(x',y')\in R}} |\epsilon_n h^(n)(x,y) - \chi(x',y')| \to 0$$

Now we want to know how random variables $h \in \mathscr{H}_{\Gamma_n^+,h_b^(n)}$ behave when $n \to \infty$.

### 10.4.3 An idea of proving the variational principle

Let $\sigma(s,t)$ be the free energy conditioned to a fixed slope $(s,t)$, or the surface tension function in last section.

Define the following functional on the space of differentiable functions $x \in \mathscr{H}_{R,\chi_b}$:

$$S[x] = -\iint_{\mathcal{D}} \sigma(\nabla h)\mathrm{d}^2 x$$

**Theorem 10.4.1** (Cohn, Kenyon, Propp, 2001)**.** *This functional has unique minimizer in $x \in \mathscr{H}_{R,\chi_b}$.*

*Denote as $\chi_0$.*

*The following theorem shows that height functions $\epsilon_n h \in \mathscr{H}_{\Gamma_n^+,h_b^{(n)}}$ converge to $\chi_0$ as random variables distributed with respect to the uniform measure.*

**Theorem 10.4.2** (Cohn, Kenyon, Propp, 2001)**.** *If $\phi_{\epsilon_n}(\Gamma_n^+) \to R$ and $\epsilon_n h_b^{(n)} \to \chi_b$ as above a random height function, $h \in \mathscr{H}_{\Gamma_n^+,h_b^{(n)}}$ rescaled as $\epsilon_n h$ converges in probability to the limit shape $\chi_0$.*

The convergence in probability here means that for any $\epsilon > 0$,

$$\mathbb{P}(d(\epsilon_n h, \chi_0) \geq \epsilon) \to 0 \text{ as } n \to \infty$$

Here we use the uniform distribution, the distance is defined by uniform metric.

We won't prove these theorem, but we will provide one idea. Since the distribution is uniform, so the part. function:

$$Z^{\text{dimer}}_{\Gamma^+,\mathcal{D}_b} = \#\text{dimer coverings with fixed config. on border edges}$$

equivalently

$$Z^{\text{dimer}}_{\Gamma^+,\mathcal{D}_b} = \#\mathscr{H}_{\Gamma^+,h_b}$$

The probability distribution is uniform, so

$$\mathbb{P}(h) = \frac{1}{Z^{heightfunctions}_{(\Gamma^+)^\vee}, h_b}$$

Let us represent the lattice region $\Gamma^+ \subset H$ as the result of the gluing of many smaller subregions $\Gamma^+ = \bigcup_i \Gamma_i^+$, The partition function for $\Gamma^+$ is given by the gluing formula:"

$$Z_{\Gamma^+}(h_b) = \sum_{h_{\text{int},b}} \prod_i Z_{\Gamma_i^+}(h_{\text{int},b})$$

Here $h_{\text{int},b}$ are height function on inner boundaries, i.e. boundaries between domains $\Gamma_i^+$.

Now, assume that linear size of domains $\Gamma_i$ is much smaller than the one of $\Gamma$ and buch bigger than 1, Then we can approximate height function on $\Gamma_i^+$ by a linear function and

$$Z_{\Gamma_i^+} \simeq \exp(|\Gamma_i|\sigma(\nabla h_i))$$

Here $|\Gamma_i|$ be the number of vertices on $\Gamma_I$, $\nabla h_i$ be the slope of linear function approximating the height function, and $\sigma$ be the so-called surface tension. For the partition function we obtain:

$$Z_{\Gamma_n^+,h_b} \simeq \sum_{h_{\text{int.},b}} \exp(-\sum_i |\Gamma_i|\sigma(\nabla h_i))$$

As $n \to \infty$, $\epsilon_n \to 0$, the exponent is a Riemann sum for the double integral.

$$\sum_i |\Gamma_i|\sigma(\nabla h_i) \to \frac{1}{\epsilon_n^2} \iint_R \sigma(\nabla\chi)\mathrm{d}^2x$$

The asymptotic of $Z_{\Gamma_n^+,h_b^{(n)}}$ as $n \to \infty$ is determined by the collection of internal height function $h_{\text{int},b}$ which approximate $\chi_0$ and we have

$$Z_{\Gamma_n^+,h_b^{(n)}} \to \exp(\frac{1}{\epsilon_n^2} \iint_R \sigma(\nabla\chi)\mathrm{d}^2x)$$

with some pre-exponential factors which we will not discuss now.

Now, let us estimate heuristically the asymptotic of the probability measure. Using the same arguments as above, the probability to find $\epsilon_n h_{\text{int},b}$ approximating a continuous height function $\chi$ is

$$\mathbb{P}(\epsilon_n h_{\text{int},b} \simeq \chi) \sim \frac{\prod_i Z_{\Gamma_i,h_{\text{int},b}}}{Z_{\Gamma^+,h_b}} \sim e^{-\frac{1}{\epsilon_n^2}(S(x)-S(x_0))}$$

This shows that the probability of a height function which is macroscopically different from $\chi_0$ is exponentially suppressed, which indicates the convergence in probability. The rigorous prove goes along the same lines, but custed in rigorous probabilistic framework.

Note the formula also indicates that probability is finite for $\chi = \chi_0 + \epsilon_n\phi$. In this case:

$$\mathbb{P}(h = \frac{1}{\epsilon_n}\chi_0 + \phi) \simeq \exp(S^{(2)}_{\chi_0}\phi)$$

Where:

$$S^{(2)}_{\chi_0}(\phi) = \frac{1}{2} \iint_R \partial_i\partial_j\sigma(\nabla\chi_0)\partial_i\phi\partial_j\phi\mathrm{d}^2x$$

This is a strong indication that when $n \to \infty$ the random height function behave as:

$$h_{k,l} = \epsilon_n^{-1}\chi_0(x,y) + \phi(x,y) + ...$$

where: $x = \epsilon_n k, y = \epsilon_n l$, $\phi$ be a random Gaussian variable with dispersion as $S^{(2)}$. In deed, we will see that the asymptotic of correlation functions is consistent with this. Ina separate lecture we will discuss the convergence $h - \frac{\chi_0}{\epsilon_n}$ to the Gaussian random field.

## 10.5 Euler-Lagrangian and Burgers

### 10.5.1 Limit shapes

Here we will show that the Euler-Langrange equations for the minimization problem can be written as complex Burgers equations. Let $P(z,w)$ be the characteristic polynomial of a dimer model on a bipartite graph. The ccorresponding Ronkin function is defined as

$$R(H,V) = \frac{1}{(2\pi i)^2} \iint_{|z|=e^H,|w|=e^V} \ln|P(z,w)|\frac{\mathrm{d}z}{z}\frac{\mathrm{d}w}{w}$$

Recall that when coefficients of $P(z,w)$ are real the algebraic curve is called a simple Harnack curve.

The image of the curve with respect to the mapping $(z,w) \mapsto (\ln|z|, \ln|w|)$ is called the Amoeba of the curve and denoted by $\mathbb{A}(P) \subset \mathbb{R}^2$.

One of the defining properties of Harnack curves is:

**Proposition 10.5.1.** *if $P(z,w) = 0$, the mapping $(z,w) \mapsto (\ln|z|, \ln|w|)$ is $2 \to 1$ on the interior of $\mathbb{A}(P)$, and $1 \to 1$ on the boundary.*

Let $N(P) \subset \mathbb{R}^2$ be the Newton polygon of the Laurent polynomial $P(z,w) = \sum_{n,m} a_{nm} z^n w^m$, i.e. the convex haul of points $(n,m)$ with $a_{n,m} \neq 0$.

The following is a special property $(\star)$ of Harnack curves:

$$\nabla R(\ln|z|, —\mathrm{w}—) = \frac{1}{\pi}(-\arg w, \arg z) \text{ iff } P = 0$$

Let $\sigma$ be the Legendre transform of $R$ (Also: Free energy as the function of magnetization). From the definition of $\sigma$, $\nabla R \circ \nabla \sigma = \mathrm{id}$ on $N(P)$.

so: $\nabla\sigma : N(P) \simeq \mathbb{A}(P)$, and $\nabla R : \mathbb{A}(P) \simeq N(P)$.

Let $h$ be a smooth function on $\mathbb{R}^2$, define $|z(x,y)|, |w(x,y)|$ such that

$$(\nabla\sigma \circ \nabla h)(x,y) = (\ln|z|, \ln|w|)$$

assuming that $\nabla h(x,y) \in N(P)$.

By the property $(\star)$ there exists $(z(x,y), w(x,y))$ on the curve $P = 0$ such that:

$$\nabla R(\ln|z| \ln|w|) = \frac{1}{\pi}(-\arg w, \arg z)$$

by definition of $|z|, |w|$ and using $(\star)$ we conclude:

$$\arg w = -\pi h_x, \arg z = \pi h_y$$

Thus, we proved the following:

**Theorem 10.5.2.** *For a differentiable function $h$ on $\mathbb{R}^2$ with $\nabla h \in N(P)$ define:*

$$z = \exp(i\pi h_y + \partial_1 \sigma(\nabla h))$$

$$w = \exp(-i\pi h_x + \partial_2 \sigma(\nabla h))$$

*Then $P(z,w) = 0$*

**claim** When $h_x, h_y$ are differentiable,

$$\frac{\partial_x z}{z} + \frac{\partial_y w}{w} = 0$$

holds if $h$ satisfy Euler-Lagrangian equations for $S$.

*Proof.*

$$\partial_x z = (i\pi h_{xy} + h_{xx}\partial_1^2\sigma + h_{xy}\partial_1\partial_2\sigma)z$$

$$\partial_y w = (-i\pi h_{xy} + h_{yy}\partial_2^2\sigma + h_{xy}\partial_1\partial_2\sigma)w$$

The imaginary of our condition is an identity, and the real part is exactly the Euler-Lagrange equation for the limit shape. □

Thus $h$ is a solytion to Euler-Lagange equation iff

$$\partial_x z/z + \partial_y w/w = 0, P(z,w) = 0$$

where $z, w$ are given by above.
Define

$$F = \frac{\partial_x z}{\partial_y z} = \frac{z}{w}\frac{\partial_z P(z,w)}{\partial_w P(z,w)}$$

.

We claim: $\partial_x F = F\partial_y F$