logic in color

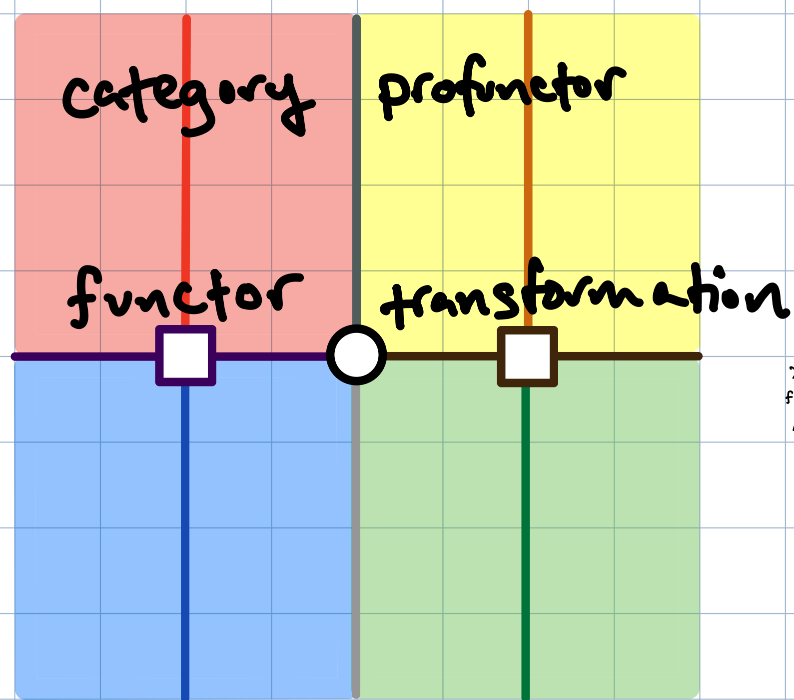
Higher-Order Logic

Recap

$$X : x \xrightarrow{R} y : Y$$

$$A : fx \xrightarrow{S} gy : B$$

Cat



$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ f\downarrow & \Downarrow\psi & \downarrow g \\ A & \xrightarrow[\Psi]{} & B \end{array}$$

From sets + functions,
we built the language of categories.

$$R \vdash T \;=\; \tilde{\prod} xy.\; xRy \vdash xTy$$

$$R \circ U \;=\; \tilde{\Sigma} x.\; -Rx \circ xU-$$

First, we used the language to verify the basic structure of $\mathbb{C}$at.

- seq & par comp : double category
- functor → profunctor : fibrant dc (equipment)

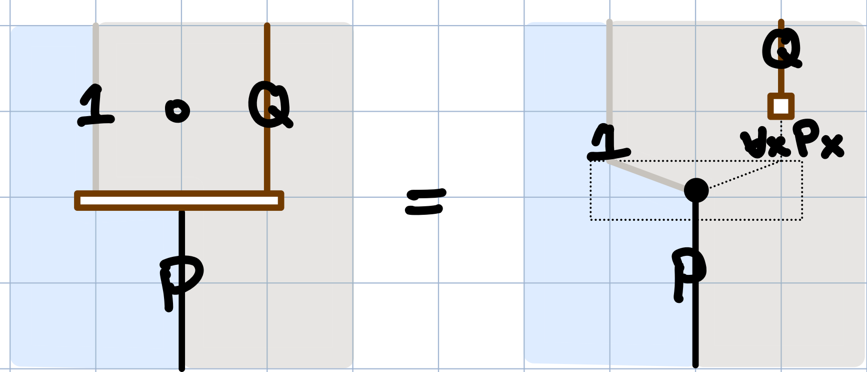
Now, let's actually <u>do</u> logic.

Predicate logic uses $\exists$ + $\forall$.
We used these to <u>form</u> Rel,
but how do we use them <u>in</u> Rel?

Suppose we have a predicate $P : X \mid 1$.
Its universal quantification is $\forall x\, Px : 1 \mid 1$ (a proposition),
characterized by $\dfrac{Q}{\forall x\, Px} \sim \forall x \dfrac{Q[x]}{Px}$

In colors, this looks like:

$\forall x \dfrac{Q[x]}{Px}$



$\dfrac{Q}{\forall x\, Px}$

$(\forall x\, Px)$

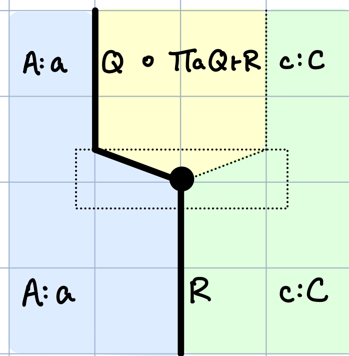
This is the right extension of P along 1.

Let $Q : A|B$ & $R : A|C$.

The <u>right extension</u> of R along Q is:

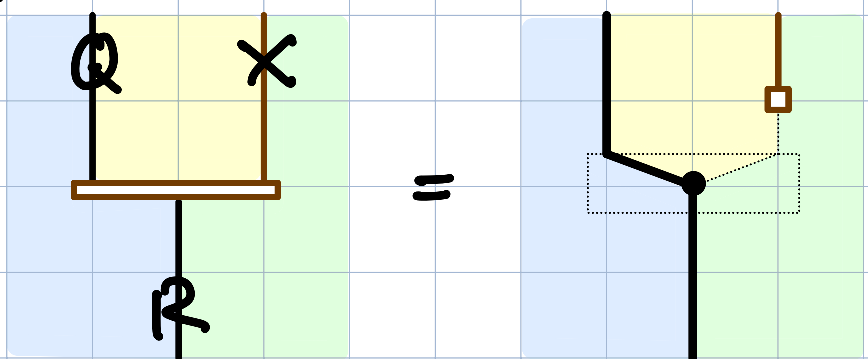
$\Pi a.Q\vdash R : B|C$

$$\frac{A{:}a \quad Q \circ (\Pi a.Q\vdash R) \quad c{:}C}{A{:}a \qquad R \qquad c{:}C}$$



"eval"

so that for every inference $\gamma : Q \circ X \vdash R$ there is a unique $\bar{\gamma} : X \vdash \Pi a.Q\vdash R$ so that



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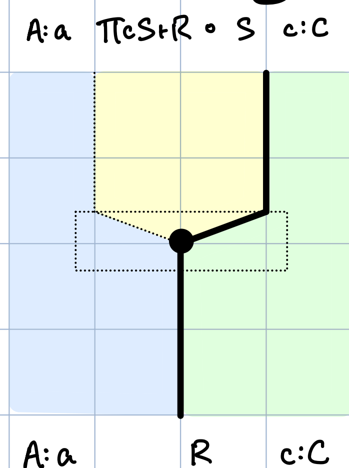
The <u>right lifting</u> of R along S is:

$$\frac{A{:}a \quad (\Pi c.S\vdash R)\circ S \quad c{:}C}{A{:}a \qquad R \qquad c{:}C}$$



such that (same).

How is this useful?

Higher-order logic is based on <u>right adjoints</u>:

props	sets
$\dfrac{X \wedge Y \vdash Z}{X \vdash Y \to Z}$	$\dfrac{A \times B \vdash C}{A \vdash B \to C}$

Right extension along Q is right adjoint to <u>precomposition</u> by Q:

$$
\begin{aligned}
& Q \circ X && \vdash R \\
= \ & \Pi ac.\ (\Sigma b\ aQb \circ bXc) && \vdash aRc \\
\sim \ & \Pi ac.\ \Pi b\ aQb \circ bXc && \vdash aRc \\
\sim \ & \Pi abc. \quad bXc && \vdash (aQb \vdash aRc) \\
\sim \ & \Pi bc. \quad bXc && \vdash \Pi a.\, aQb \vdash aRc \\
& X && \vdash \text{extension}
\end{aligned}
$$

$$
\begin{aligned}
& Q \circ X && \vdash R \\
= \ & \Pi ac\ (\Sigma b\ aQb \circ bXc) && \vdash aRc \\
\sim \ & \Pi ac\ \ \Pi b\ aQb \circ bXc && \vdash aRc \\
\sim \ & \Pi abc \quad bXc && \vdash (aQb \vdash aRc) \\
\sim \ & \Pi bc \quad bXc && \vdash \Pi a.aQb \vdash aRc \\
= \ & X && \vdash \underline{\Pi a.Q \vdash R}
\end{aligned}
$$

(we can prove $[B|C] \underset{\Pi Q}{\overset{Q \circ -}{\rightleftarrows}} [A|C]$.)

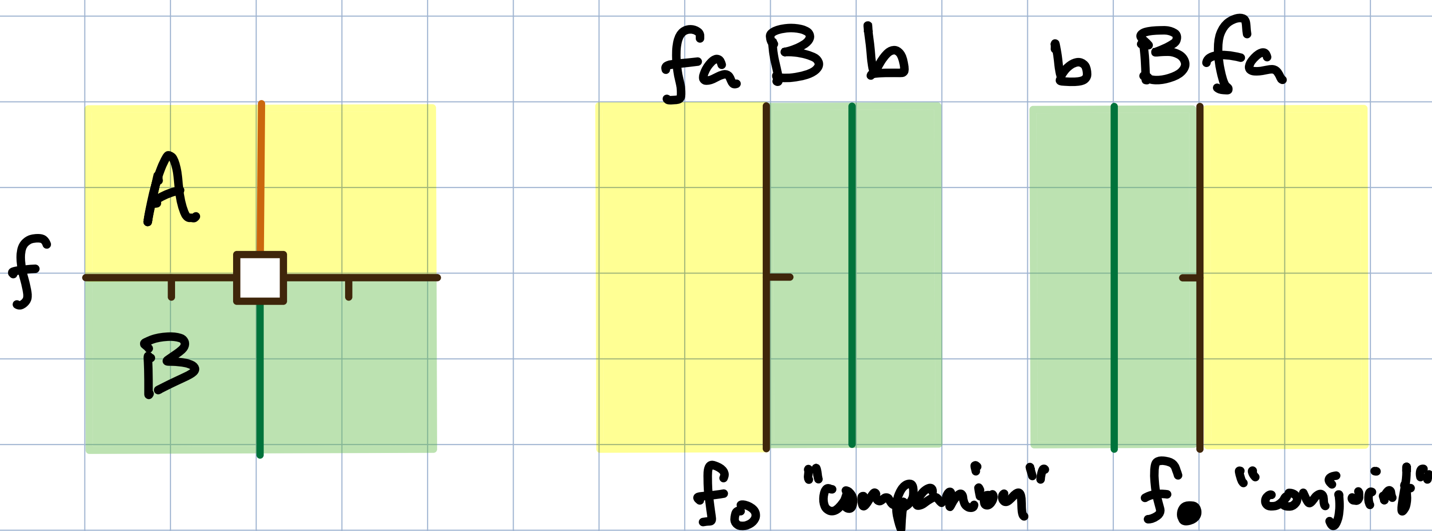
This generalizes $\forall$ to <u>categories</u>.

- exercise: do this for <u>post</u>composition.

So, for any pair $Q : A | B$ & $R : A | C$
there's a "universal judgement" $\Pi a. Q \vdash R : B | C$.

This is great — yet it's a judgement.
Logic is really "about" <u>terms</u>.

Recall that every functor determines
a dual pair of "profunctors" (relations)



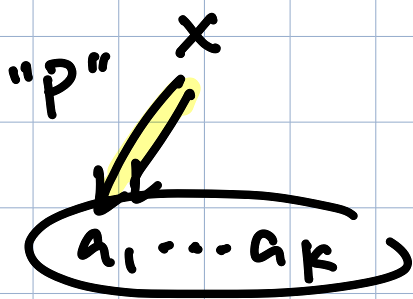
A <u>representation</u> of a profunctor $R : A | B$
is a functor $f : A \to B$ or $g : B \to A$
& an invertible transform

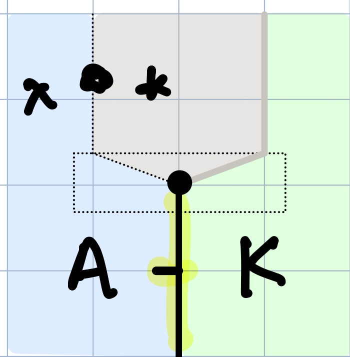
$$R \sim f_\circ \quad \text{or} \quad R \sim g_\bullet .$$

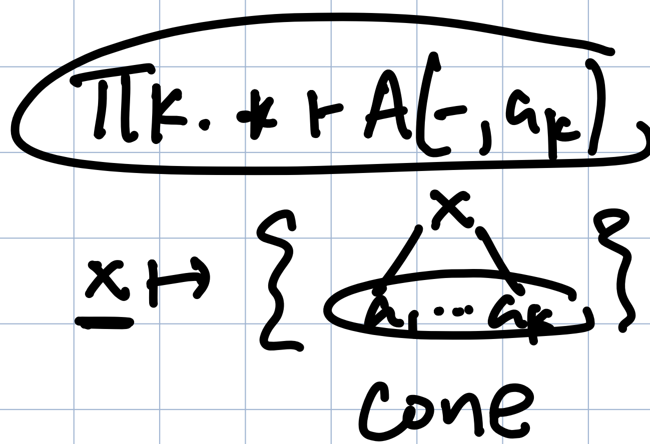
Representations of universal judgements
are extremely useful.

Let $a_{(-)} : K \xrightarrow{\text{set}} A$ be a function. (picks out $\{a_k\} : A$)
This defines a profunctor
"$\forall$" $a_\bullet = A(-, a_{(-)}) : A \mid K$ ("conjoint")

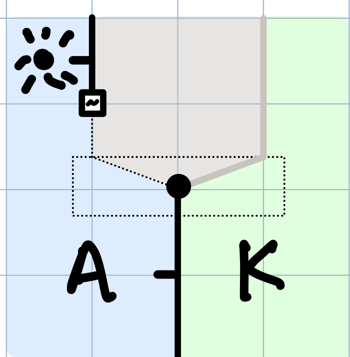
The right lifting along $* : 1 \mid K$
defines $\Pi k. * \vdash a_\bullet : A \mid 1$.

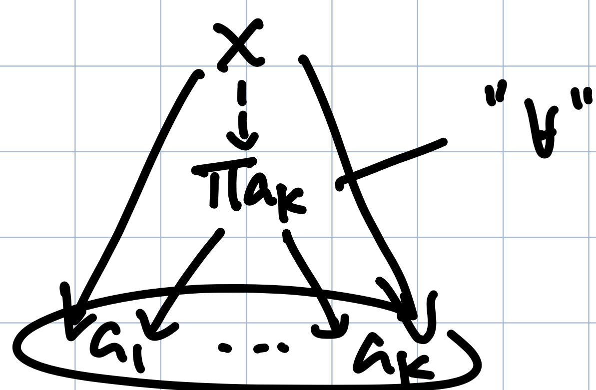






A representation of $\Pi k. A(-, a_k)$
is a *product* $\Pi a_k : A$.



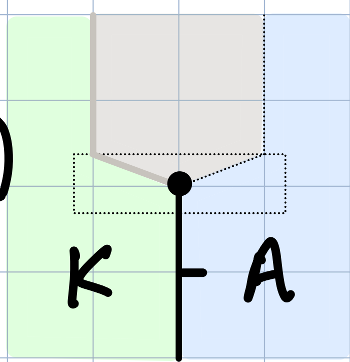


In general, a *limit* of a functor $f : I \to A$
is a representation of $\Pi i. A(-, fi)$.

"∃"

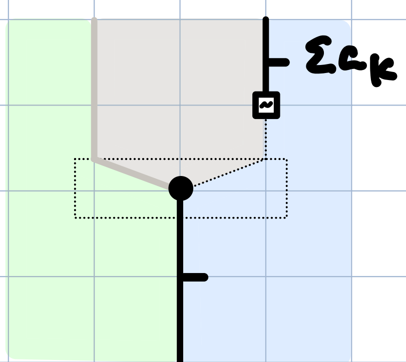
Dually, $a_{(-)}: K \to A$ defines
$a_\circ = A(a_{(-)}, -) : K | A$. ("companion")

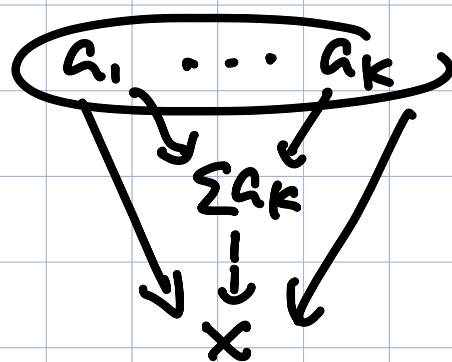
The right extension along $* : K | 1$
defines $\Pi k. * \vdash a_\circ$

$\Pi k. * \vdash A(a_k, -)$ 

$x \mapsto \left\{ \begin{array}{c} (a_1 \cdots a_k) \\ \searrow \quad \swarrow \\ x \end{array} \right\}$

A representation of $\Pi k. A(a_k, -)$
is a coproduct $\Sigma a_k : A$. "∃"





In general, a colimit of $f : I \to A$
is a representation of $\Pi i. A(fi, -)$.

<u>Puzzle</u>: let $I = \boxed{0 \rightrightarrows 1}$ & $f : I \to A$.
What is the limit & colimit of f?

(equations...)

<u>Note</u>: We've only used $! : I \to 1$.
This gives "cones & cocones", which consist of individual morphisms.

Let $f : I \to A$ & $W : C \mid I$.
Then the <u>W-weighted limit</u> of f, if it exists, is a representation

$$(\Pi W.f)_\bullet \sim \Pi i. W \vdash f_\bullet \quad .$$

If $W : I \mid C$, then the <u>W-weighted colimit</u> is a representation

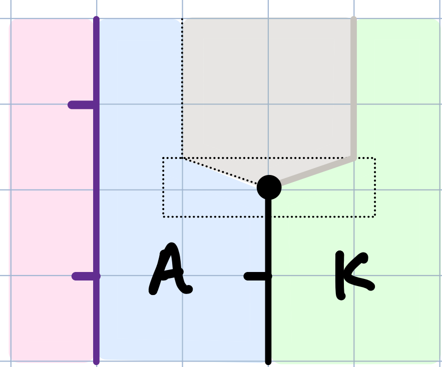
$$(\Sigma W.f)_\circ \sim \Pi i. W \vdash f_\circ \quad .$$

(puzzle: what <u>are</u> these?)

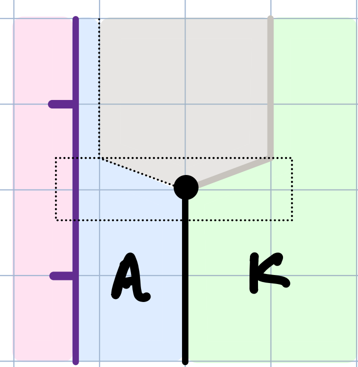
Preservation

Limits & colimits are precious jewels, and we care whether composition <u>preserves</u> them.

For example, given $\prod a_k : A$ & $f : A \to B$, there is a canonical $f(\prod a_k) \to \prod f(a_k)$. If it is invertible, the product is <u>preserved</u>.



$$\overset{?}{=}$$



(formal)

Puzzle: right adjoints preserve weighted limits (right lifts)

& left adjoints preserve weighted colimits (right exts)

Dually, there are left exts & lifts — but they don't always exist.

Try it: what is needed?

Questions / Thoughts?