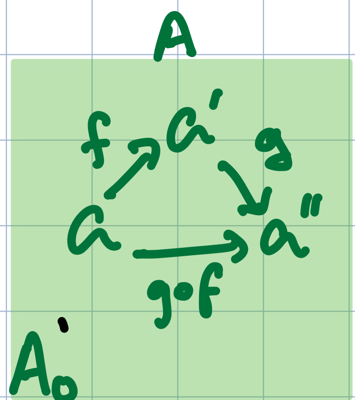
logic in color

Category Theory

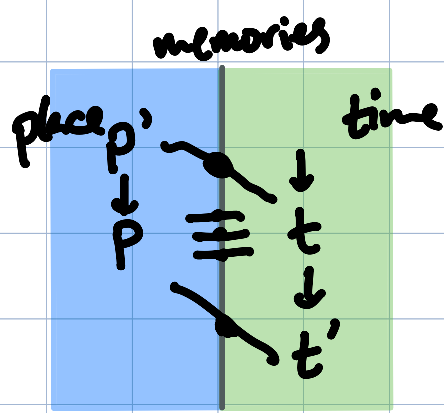
(slow down, back to basics)

let $\mathcal{V}$ = Set.

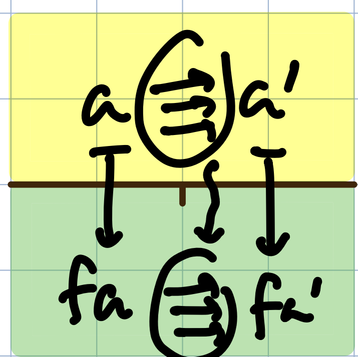
then $\underline{\mathbb{V}}$ = Mod(Mat(Set)) = $\mathbb{C}$at.



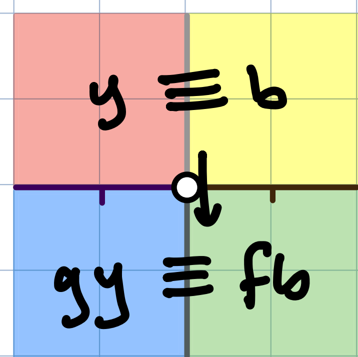
categories



profunctor



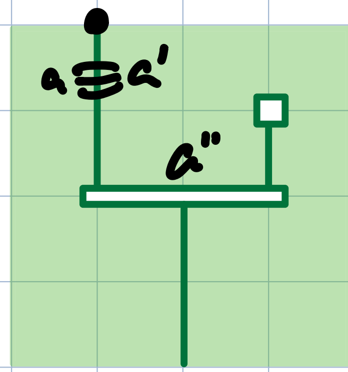
functor



transformations

we now have a language to explore CT.

a <u>category</u> A



- $A_0 : \text{Set}$ (objects)
- $A : \prod a, a' : A_0 . \, \text{Set}$

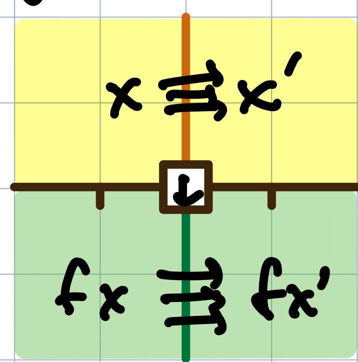
  $A(a,a') =$ set of morphisms

- $- \circ_A - : \prod a, a' : A_0 . \, \sum a'' . \, A(a,a'') \times A(a'',a')$

  composition $\to A(a,a')$

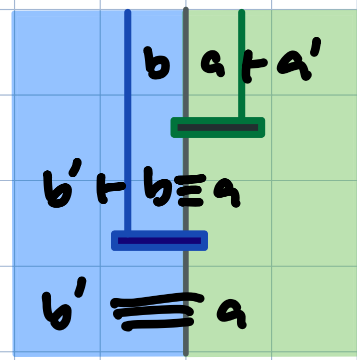
- $1_A : \prod a . \, A(a,a)$ identity (plus axioms)

a <u>functor</u> $f : X \to A$



- $\underline{f} : \prod x . \, A$ function on objects
- $f : \prod x x' . \, X(x,x') \to A(fx, fx')$
- plus axioms (preserving comp & id)

a <u>profunctor</u> $P : B \mid A$

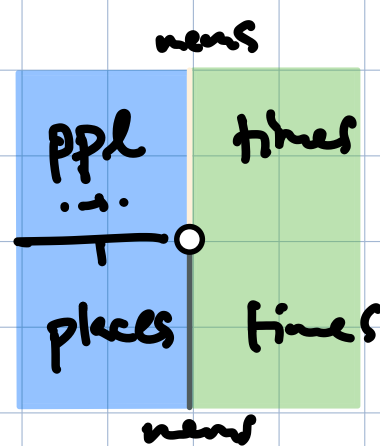


- $P : \Pi ba.\,\mathrm{Set}$
- $l : BP \to P$

  $\Pi b'a.\,\Sigma b.\,B(b';b) \times P(b,a) \to P(b',a)$
- $r : PA \to P$

  $\Pi ba'.\,\Sigma a.\,P(b,a) \times A(a,a') \to P(b,a')$
- (plus axioms)

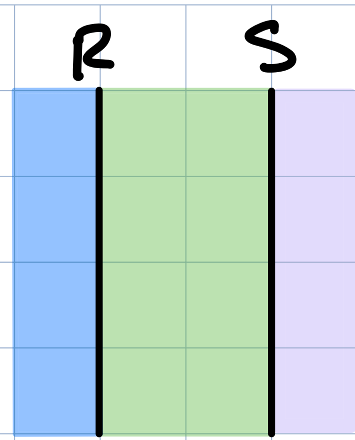
a <u>transformation</u> $\gamma : P \to Q$



- $\gamma : \Pi ba.\,P(b,a) \to Q(b,a)$
- (axioms for naturality)

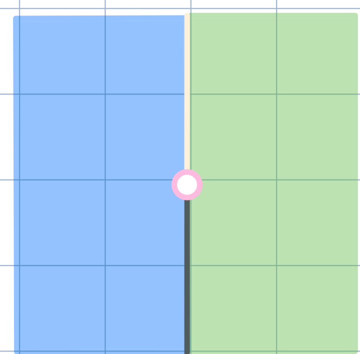
- composition

$$R \circ S = \tilde{\Sigma} b.\, R(a,b) \times S(b,c)$$



inference

$$R \Rightarrow T = \tilde{\Pi}[ab.\, R(a,b) \to T(a,b)]$$



so $[A|B](R,T) = \tilde{\Pi}[ab.\, R(a,b) \to T(a,b)]$

is a <u>category</u>!

& $[A \vdash B](F,G) = \hat{\Pi}[a.\, F(a) \to G(a)]$

is a category.

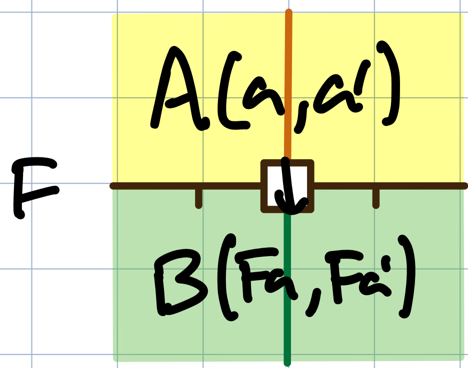
remember, in Rel $\qquad f_\circ(a,b) = [fa=b]$

we can turn a function into a relation.

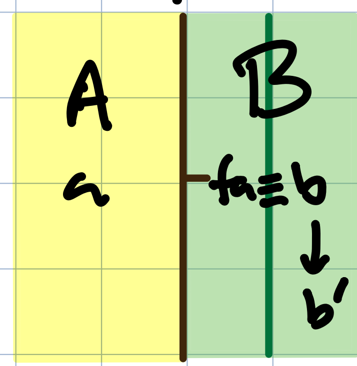
functor $\qquad$ profunctor

puzzle $\qquad ()_\circ : [A \vdash B] \vdash [A | B]$ $\qquad$ "representable"

?



$\rightsquigarrow$



$$F_\circ : \Pi a\, b.\ \mathrm{Set}$$

representable functor : $\qquad (a \mapsto (b \mapsto B(fa,b)))$

"looking out of F"

- show $F_\circ$ is a profunctor (aka module)

$$l : AF_\circ \to F_\circ \qquad r : F_\circ B \to F_\circ$$

$A(a,a') \times B(fa',b) \to B(fa,b)$?

$$B(fa,b) \times B(b,b') \to B(fa,b') \;?$$

composition in B

We can also go back!

judgements $\rightsquigarrow$ terms

How? Think of a relation $R : A \mid B$.

$$R : A \times B \to \{0,1\}$$

We can do this for "relations" of categories too.

$$A \overset{R}{\underset{a \to b}{\nrightarrow}} B \quad \rightsquigarrow \quad \frac{A^{op} \times B}{\mathrm{Set}} \; \overline{R}$$

$\Pi ab.\, \mathrm{Set}$ $\qquad\qquad$ $F : \Pi x.\, Y$

$$A^{op}(a_1, a_2) = A(a_2, a_1)$$

$$R : A \mid B \quad \rightsquigarrow \quad \overline{R} : A^{op} \times B \to \mathrm{Set}$$

$$= (a \mapsto (b \mapsto R(a,b)))$$

Higher-Order Logic

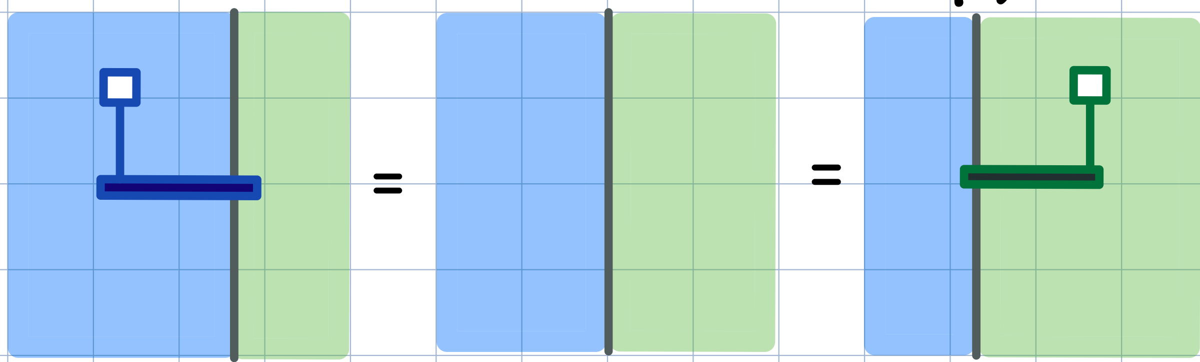
$$A \times B \to C$$
$$\sim A \to [B \to C]$$

$\mathbb{C}at$ is "closed": $$Q \circ X \Rightarrow R$$
$$\sim X \Rightarrow [Q / R]$$

co/Limits

abstraction & realization
(Yoneda & coYoneda)

$R \sim R \circ B$



$$A \circ R \sim R \sim R \circ B = \Sigma \cdots$$

$$A \backslash R \sim R \sim R / B$$

$$A \xrightarrow{R} B$$