logic in color

The Yoneda Lemma

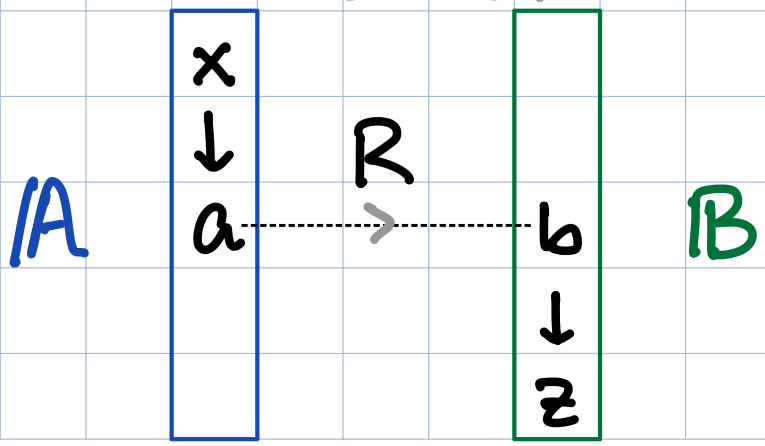
(& free discussion)

A predicate on a set $\mathbb{A}$
is a function $\phi: \mathbb{A} \to \{0,1\}$.
This is the basis of predicate logic.

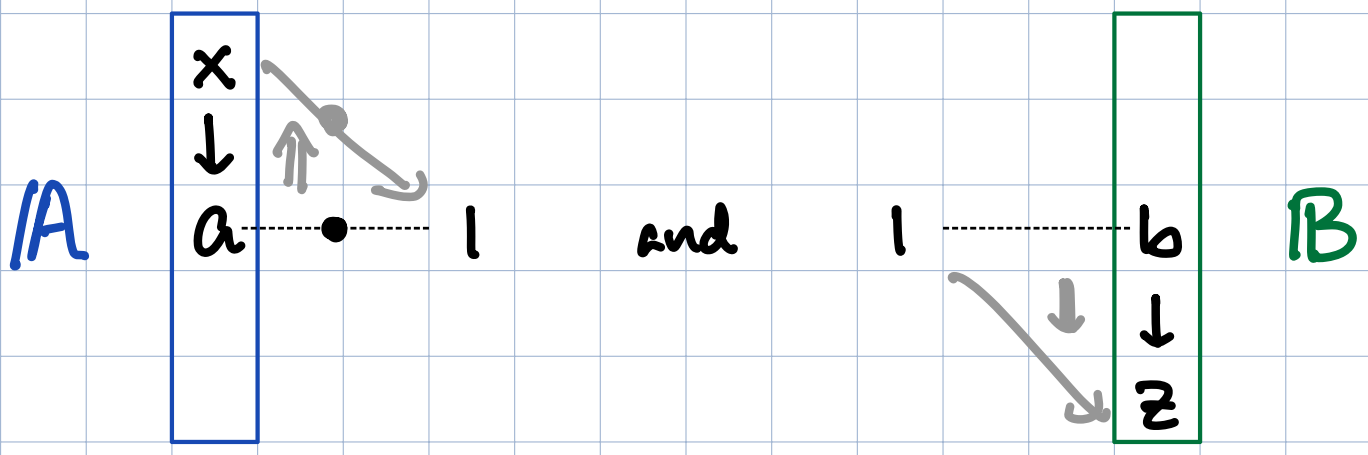
A relation from $\mathbb{A}$ to $\mathbb{B}$
is a predicate $R: \mathbb{A} \times \mathbb{B} \to \{0,1\}$
and these form $\mathbb{R}$el.

We saw that relations of sets
generalize to profunctors of categories.

"bimodule" "monad"



So – what's a "predicate" now?
Well, now there are two kinds!



$\mathbb{A}^{op} \to \mathbf{Set}$ *

"negative"

$\mathbb{A} \to \mathbf{Set}$

"positive"

These are left & right modules,
aka "presheaves" & "copresheaves".

example. Let $\mathbb{G}$ be a category with one object & all morphisms invertible. (group)

what is a positive predicate on $\mathbb{G}$?

"right"

a $\mathbb{G}$-action $(\mathbb{G} \to [X, X])$

$$\alpha : \mathbb{G} \to \mathbf{Set}$$

$$\begin{array}{ccc} \bullet & \mapsto & X \\ g \downarrow & & \downarrow \alpha g \\ \bullet & \mapsto & X \end{array}$$

So if a predicate is a judgement (profunctor) $\mathbb{C}.P.1$, then the "power set" is the category

$$\Omega\mathbb{C} \equiv [\mathbb{C} \mid 1] \qquad P \vdash Q = \prod_c . Pc \vdash Qc$$
$$= [\mathbb{C}^{op} \vdash Set]$$

In $\mathbb{R}el$, power sets are highly structured because $0 \vdash 1$ is.

In $\mathbb{C}at$, the same is true because Set is nice.

The "category of presheaves" $\Omega\mathbb{C}$ (aka $P(\mathbb{C})$) is a topos, a setting for predicate logic.

The great news is that every category embeds into such a rich world.

The Yoneda embedding of $\mathbb{C}$ is defined as follows.

$$Y = \mathbb{C}[-] : \mathbb{C} \vdash \Omega\mathbb{C}$$

$$c \mapsto \mathbb{C}(-,c) : (x \mapsto \mathbb{C}(x,c))$$

$$f \downarrow \qquad \mathbb{C}(-,f) \downarrow \qquad \downarrow f \circ -$$

$$d \mapsto \mathbb{C}(-,d) \qquad \mathbb{C}(x,d)$$

Each $\_\mathbb{C}c$ is a representable* predicate.

example: Cayley's theorem.

$$G \hookrightarrow [G, G]$$
$$g \mapsto g \cdot -$$

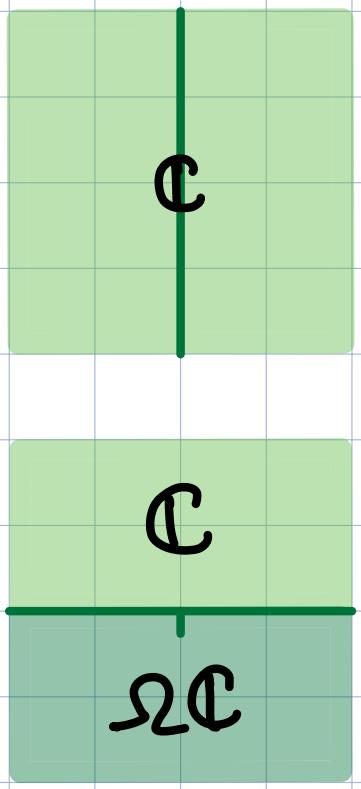
$$G \times G \to G$$
$$\sim$$
$$G \to [G, G]$$

is a nice inclusion.

So, the embedding is simply

$$c : \mathbb{C} \mapsto \_\mathbb{C}c : [\mathbb{C} | 1]$$

this is "currying" the hom of $\mathbb{C}$.



$$\frac{A \times B \to C}{A \to [B \to C]}$$

$$\_\mathbb{C}\_ : \mathbb{C} | \mathbb{C}$$
$$\sim$$
$$\_\mathbb{C}\_ : \mathbb{C}^{op} \times \mathbb{C} \vdash \mathbf{Set}$$
$$c, d \quad c \rightrightarrows d$$

$$\mathbb{C}[-] : \mathbb{C} \vdash [\mathbb{C}^{op} \vdash \mathbf{Set}]$$

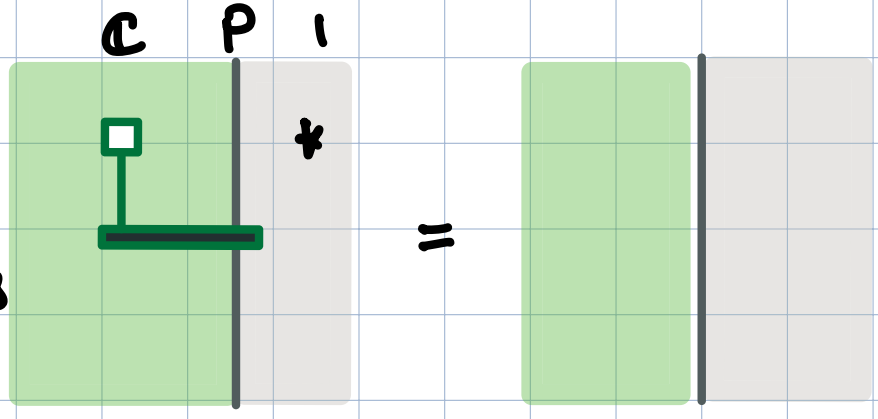
* that's why ppl like (-) preds
embedding is covariant

$\Omega\mathbb{C}$ is the "power category".

There is a pair of theorems called "coYoneda" + "Yoneda" lemma which are fundamental to CT.

Let $P : \Omega\mathbb{C}$, ie $P : \mathbb{C}^{op} \to Set$

$$\lambda \quad \Sigma c.\ \underline{d}\,\mathbb{C}\,c \times c\,P\,1 \approx \underline{P d}$$



predicate P is a sum of representables

simplicial sets $\Delta^{op} \to Set$

$$* \quad \mathbb{C} \circ P \sim P$$

$$Y \quad \pi c.\ c\,\mathbb{C}\,\underline{d} \to c\,P\,1 \approx P\underline{d}$$

$$* \quad \Omega\mathbb{C}(\mathbb{C}(-,d), P) \approx Pd$$

"$P \sim \mathbb{C} \multimap P$"

A predicate on $\mathbb{C}$ is determined by its representable inferences.

This is useful because it gives a method of constructing & proving.

If we apply Yoneda to $\_ \mathbb{C} d$, we have that

"full + faithful"

$$\ast \quad \Omega\mathbb{C}(\mathbb{C}(-,c) \vdash \mathbb{C}(-,d)) \overset{\text{bijection}}{\sim} \mathbb{C}(c,d)$$

— embedding $\mathbb{C}$ into $\Omega\mathbb{C}$ preserves the "logic" of $\mathbb{C}$.

So proving $\dfrac{X \vdash C}{X \vdash d}$ is equivalent to constructing $t : C \vdash d$.

example. what is the product of predicates?

$$\Omega\mathbb{C} : X. \quad \frac{X \vdash P \times Q}{(X \vdash P) \times (X \vdash Q)} \quad . P, Q : \Omega\mathbb{C}$$

$$\frac{\Pi c (Xc \vdash Pc) \times \Pi c (Xc \vdash Qc)}{\Lambda c.\, \lambda x.\, (p, q)}$$

$$\Lambda c.\, \lambda x.\, p \;,\; \Lambda c.\, \lambda x.\, q$$

$$\Pi c.\, Xc \vdash Pc \times Qc$$

The other crucial fact:
$Y : \mathbb{C} \to \Omega\mathbb{C}$ is the <u>free cocompletion</u> of $\mathbb{C}$.

Like the free monoid on a set,
Y turns objects $c : \mathbb{C}$ into rep's $\_\mathbb{C}c$
and "freely forms sums of them".

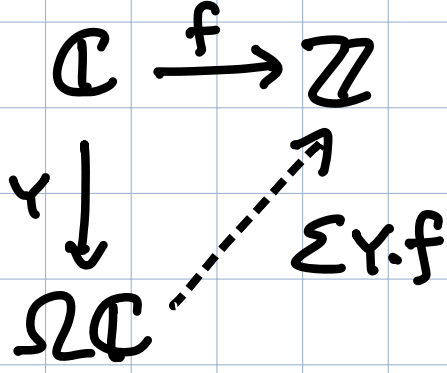
So we saw that every predicate is a sum:

$$P \approx \Sigma c.\ \_\mathbb{C}c \times Pc$$

& this completely characterizes Y.

Let $\mathbb{Z}$ be a category with all colimits
& $f : \mathbb{C} \to \mathbb{Z}$ a functor.
Then $\exists!$ functor which preserves colimits

$$\Sigma Y.f : \Omega\mathbb{C} \to \mathbb{Z}$$



Discussion

What aspects of CT/logic
- do you find interesting/enjoyable?
  confusing/mysterious?
- do you hope to learn?

What do you think of "CT as logic"?
the pictures?
the language?

Always willing to talk!

Linear logic ~ "$\ast$-autonomous categories" + more...

$x, +$ Rel $A \mid B$

* the language of $\mathbb{V}$Cat is made of the language of $\mathbb{V}$

(the language of a topos)

hasn't been done for $(\mathbb{V}, \otimes, \text{coeq}_\Sigma)^\ast$