

Enriched Lawvere Theories for Operational Semantics

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theories and monads

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enriched limits

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How do we integrate syntax and semantics?

| | |
|---------|------------|
| type | object |
| term | morphism |
| rewrite | 2-morphism |

$$(\lambda x. x + x \ 2) \xrightarrow{\beta} 2 + 2 \xrightarrow{+} 4$$

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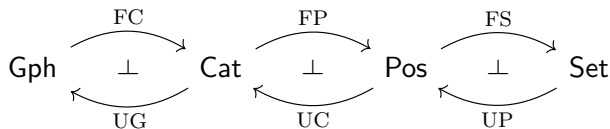
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- FC_* maps small-step to big-step operational semantics.
 FP_* maps big-step to full-step operational semantics.
 FS_* maps full-step to denotational semantics.

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A **Lawvere theory** is a category \mathbb{T} whose objects are finite powers of a distinguished object $t = \tau(1)$.

$$\tau: \mathbb{N}^{\text{op}} \rightarrow \mathbb{T}$$

Morphisms $t^n \rightarrow t$ are n -ary operations, and commuting diagrams are equations.

Classical algebra is represented and internalized in any category with finite products.

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Let C be a category with finite products.

A **model** of a Lawvere theory T in C is a product-preserving functor $\mu: T \rightarrow C$. A morphism of models is a natural transformation.

These form a category $\text{Mod}(T, C)$. For example, $\text{Mod}(\text{Th}(\text{Grp}), \text{Top})$ is the category of topological groups.

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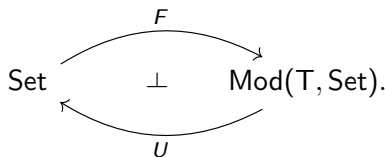
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Another way to describe algebraic structure is by *monads*.
Theories induce “free-forgetful” adjunctions:



$$U :: \mu \mapsto \mu(1) \quad F :: n \mapsto T(n, -)$$

extends to \mathbf{Set} by colimit.

This induces a monad T on \mathbf{Set} , and we have that:

$$T\text{-Alg} \simeq \mathbf{Mod}(T, \mathbf{Set}).$$

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Enriched categories

Let V be a monoidal category. A **V -enriched category** is:

| | |
|--------------|--|
| object set | $\text{Ob}(C)$ |
| hom function | $C(-, -): \text{Ob}(C) \times \text{Ob}(C) \rightarrow \text{Ob}(V)$ |
| composition | $\circ_{a,b,c}: C(b, c) \times C(a, b) \rightarrow C(a, c)$ |
| identity | $i_a: 1_V \rightarrow C(a, a)$ |

with composition associative and unital.

A **V -functor** $F: C \rightarrow D$ is:

| | |
|-----------------|---|
| object function | $F: \text{Ob}(C) \rightarrow \text{Ob}(D)$ |
| hom morphisms | $F_{ab}: C(a, b) \rightarrow D(F(a), F(b))$ |

preserving composition and identity.

A **V -natural transformation** $\alpha: F \Rightarrow G$ is:

$$\text{components } \alpha_a: 1_V \rightarrow D(F(a), G(a))$$

which is “natural” in a . These form the 2-category $VCat$.

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Preservation

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Generalized arities

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Example: pseudomonoid

$\text{Th}(\text{PsMon})$

type

M

pseudomonoid

operations

$m: M^2 \rightarrow M$

multiplication

$e: 1 \rightarrow M$

identity

rewrites

$\alpha: m(m \times \text{id}_M) \cong m(\text{id}_M \times m)$

associator

$\lambda: m(e \times \text{id}_M) \cong \text{id}_M$

left unitor

$\rho: m(\text{id}_M \times e) \cong \text{id}_M$

right unitor

equations

pentagon, triangle

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Example: cartesian object

$\text{Th}(\text{Cart})$

type

X

cartesian object

operations

$m: X^2 \rightarrow X$

product

$e: 1 \rightarrow X$

terminal element

rewrites

$\Delta: \text{id}_X \Longrightarrow m \circ \Delta_X$

unit of $m \vdash \Delta_X$

$\pi: \Delta_X \circ m \Longrightarrow \text{id}_{X^2}$

counit of $m \vdash \Delta_X$

$\top: \text{id}_X \Longrightarrow e \circ !_X$

unit of $e \vdash !_X$

$\epsilon: !_X \circ e \Longrightarrow \text{id}_1$

counit of $e \vdash !_X$

equations

triangle identities

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Natural numbers in V

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Equivalences

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Preservation of theories

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The SKI-combinator calculus

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The theory of SKI

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Bisimulation

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Conclusion

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