Enriched Lawvere Theories for Operational Semantics

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

University of California, Riverside

SYCO 4, May 22 2019

riched Lawvere Theories or Operational

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

theories

Lawvere theories theories and monad

enrichm

enriched categories enriched limits

enriched theories

enriched Lawvere theories enriched monadicity

atural number

arities
arity subcategory

equivalences change of

change of base

change of base preserving theories

application combinators



Introduction

Enriched Lawver Theories for Operational

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

neories

Lawvere theories theories and monads

enrichme

enriched categories enriched limits

enriched theories

enriched Lawvere theorie enriched monadicity

atural number

arity subcategory equivalences

hange of emantics

change of base preserving theories

combinators

change of base

How do we integrate syntax and semantics?

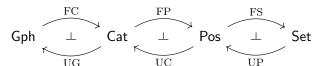
type object term morphism rewrite 2-morphism

$$(\lambda x.x + x \ 2) \xrightarrow{\beta} 2 + 2 \xrightarrow{+} 4$$

Change of semantics

Theories for Operational Semantics

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same



 $\begin{array}{ll} {\rm FC}_* & {\rm maps~small\text{-}step~to~big\text{-}step~operational~semantics}. \\ {\rm FP}_* & {\rm maps~big\text{-}step~to~full\text{-}step~operational~semantics}. \\ {\rm FS}_* & {\rm maps~full\text{-}step~to~denotational~semantics}. \end{array}$

heories

Lawvere theories theories and monads

enrichm

enriched categories enriched limits

enriched theories

enriched Lawvere theories enriched monadicity examples

atural number rities

arity subcategory equivalences

hange of

change of base

preserving theories

application



Lawvere theories

Enriched Lawver Theories for Operational

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

theories

Lawvere theories theories and monads

enrichmen

enriched categories enriched limits

enriched theories

enriched Lawvere theorie enriched monadicity examples

atural number

arity subcategory equivalences

hange of emantics

change of base preserving theories

applications combinators change of base

A **Lawvere theory** is a category T whose objects are finite powers of a distinguished object $t = \tau(1)$.

$$\tau \colon \mathbb{N}^{\mathrm{op}} \to \mathsf{T}$$

Morphisms $t^n \to t$ are *n*-ary operations, and commuting diagrams are equations.

Classical algebra is represented and internalized in any category with finite products.

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

Lawvere theories

enriched categories

enriched monadicity

arity subcategory equivalences

change of base

preserving theories

combinators change of base

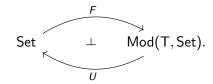
Let C be a category with finite products.

A **model** of a Lawvere theory T in C is a product-preserving functor $\mu \colon T \to C$. A morphism of models is a natural transformation

These form a category Mod(T, C). For example, Mod(Th(Grp), Top) is the category of topological groups.

Monadicity

Another way to describe algebraic structure is by *monads*. Theories induce "free-forgetful" adjunctions:



$$U :: \mu \mapsto \mu(1)$$
 $F :: n \mapsto \mathsf{T}(n, -)$ extends to Set by colimit.

This induces a monad T on Set, and we have that:

$$T$$
-Alg $\simeq Mod(T, Set)$.

Theories
for Operational
Semantics

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

neories

Lawvere theories theories and monads

nrichme

enriched categories enriched limits

enriched theories

enriched Lawvere theories enriched monadicity examples

atural number rities

arity subcategory equivalences

hange of emantics

change of base

applicatio

combinators change of base



Enriched categories

composition

Let V be a monoidal category. A V-enriched category is:

object set Ob(C)hom function

 $C(-,-): Ob(C) \times Ob(C) \rightarrow Ob(V)$ $\circ_{a,b,c} : \mathsf{C}(b,c) \times \mathsf{C}(a,b) \to \mathsf{C}(a,c)$

identity $i_a \colon 1_V \to \mathsf{C}(a,a)$

with composition associative and unital.

A V-functor $F: C \to D$ is:

object function $F: \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D})$ hom morphisms $F_{ab}: C(a,b) \to D(F(a),F(b))$

preserving composition and identity.

A V-natural transformation $\alpha: F \Rightarrow G$ is:

components $\alpha_a : 1_V \to D(F(a), G(a))$

which is "natural" in a. These form the 2-category VCat.

John C. Baez: baez@math.ucr.edu Christian Williams williams@same

theories and monads

enriched categories

enriched monadicity

arity subcategory equivalences

change of base

combinators

Enriched limits

Theories for Operational

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

heories

Lawvere theories theories and monads

enrichn

enriched categories enriched limits

enriched theories

enriched Lawvere theories enriched monadicity

natural number

arities
arity subcategory

equivalences change of

emantics

change of base preserving theories

application

change of base

d

Preservation

Enriched Lawvere Theories for Operational

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

theories

Lawvere theories theories and monads

enrichi

enriched categories enriched limits

enriched theories

enriched Lawvere theories enriched monadicity

natural number

arity subcategory equivalences

hange of

change of base preserving theories

combinators

change of base

a

Enriched Lawvere theories

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

Lawvere theories theories and monads

enriched categories enriched limits

enriched Lawvere theories enriched monadicity

arity subcategory

equivalences

change of base preserving theories

combinators change of base

Generalized arities

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

Lawvere theories theories and monads

enriched categories enriched limits

enriched Lawvere theories enriched monadicity

arity subcategory

equivalences

change of base preserving theories

combinators

change of base

Enriched monadicity

Enriched Lawvere Theories for Operational

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

heories

Lawvere theories theories and monads

enrichn

enriched categories enriched limits

nriched theor

enriched Lawvere theories enriched monadicity

natural number

arities
arity subcategory

equivalences

change of base

preserving theories

combinators

change of base

a

Example: pseudomonoid

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

theories and monads

enriched categories enriched limits

enriched monadicity examples

arity subcategory equivalences

change of base

combinators change of base

pentagon, triangle

Th(PsMon)

type

Μ pseudomonoid

operations

 $m \cdot M^2 \rightarrow M$ multiplication $e: 1 \rightarrow M$ identity

rewrites

 $\alpha : m(m \times id_M) \cong m(id_M \times m)$ associator $\lambda : m(e \times id_M) \cong id_M$ left unitor $\rho \colon m(\mathrm{id}_M \times e) \cong \mathrm{id}_M$ right unitor

equations

イロト イ押ト イヨト イヨト 一臣 一

Example: cartesian object

Theories
for Operational

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

heories

Lawvere theories theories and monads

enrichment enriched categories

enriched limits

enriched Lawvere theories enriched monadicity

tural number

arity subcategory equivalences

examples

change of semantics

change of base preserving theorie

combinators change of base

Th(Cart)

type

X cartesian object

operations

 $m: X^2 \to X$ product

 $e \colon 1 \to X$ terminal element

rewrites

 $\begin{array}{ll} \triangle \colon \mathrm{id}_{\mathsf{X}} \Longrightarrow m \circ \Delta_{\mathsf{X}} & \text{unit of } m \vdash \Delta_{\mathsf{X}} \\ \pi \colon \Delta_{\mathsf{X}} \circ m \Longrightarrow \mathrm{id}_{\mathsf{X}^2} & \text{counit of } m \vdash \Delta_{\mathsf{X}} \\ \top \colon \mathrm{id}_{\mathsf{X}} \Longrightarrow e \circ !_{\mathsf{X}} & \text{unit of } e \vdash !_{\mathsf{X}} \\ \epsilon \colon !_{\mathsf{X}} \circ e \Longrightarrow \mathrm{id}_1 & \text{counit of } e \vdash !_{\mathsf{X}} \end{array}$

equations triangle identities

◆□▶ ◆□▶ ◆■▶ ● めぬぐ

Natural numbers in V

Enriched Lawvere Theories for Operational

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

heories

Lawvere theories theories and monads

enrichi

enriched categories enriched limits

enriched theories

enriched Lawvere theories enriched monadicity

natural number

arities arity subcategory

equivalences

change of

change of base preserving theories

applicatio combinators

change of base

change of

а

Equivalences

Theories for Operational

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

cheories

Lawvere theories theories and monads

enrichi

enriched categories enriched limits

enriched theories

enriched Lawvere theories enriched monadicity

natural number

arity subcategory

equivalences

change of base

preserving theories

applicatio

change of base

a

Change of base

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

Lawvere theories theories and monads

enriched categories enriched limits

enriched Lawvere theories enriched monadicity

arity subcategory

equivalences

change of base preserving theories

combinators

Preservation of theories

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

Lawvere theories theories and monads

enriched categories enriched limits

enriched Lawvere theories enriched monadicity

arity subcategory

equivalences

change of base preserving theories

change of base

combinators

The SKI-combinator calculus

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

theories and monads

enriched categories enriched limits

enriched monadicity

arity subcategory

equivalences

change of base preserving theories

combinators change of base

The theory of SKI

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

Lawvere theories theories and monads

enriched categories enriched limits

enriched Lawvere theories enriched monadicity

arity subcategory

equivalences

change of base preserving theories

combinators change of base

Change of semantics

Enriched Lawvere Theories for Operational

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

heories

theories and monads

enrichn

enriched categories enriched limits

enriched theories

enriched Lawvere theories enriched monadicity

natural number

arity subcategory

hange of

change of base

preserving theories

combinators

change of base

a

Bisimulation

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

Lawvere theories theories and monads

enriched categories enriched limits

enriched Lawvere theories enriched monadicity

arity subcategory

equivalences

change of base preserving theories

combinators

Conclusion

Enriched Lawvere Theories for Operational

John C. Baez: baez@math.ucr.edu Christian Williams: williams@same

heories

Lawvere theories theories and monads

enrich

enriched categories enriched limits

enriched theories

enriched Lawvere theories enriched monadicity

natural number

arity subcategory

equivalences change of

semantics change of base

preserving theories

applicatio

combinators

