Enriched Lawvere Theories for Operational Semantics

John C. Baez: baez@math.ucr.edu
Christian Williams: williams@same

University of California, Riverside

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Introduction

How do we integrate syntax and semantics?

type	object
term	morphism
rewrite	2-morphism

$$(\lambda x.x + x \;\; 2) \xrightarrow{\beta} 2 + 2 \xrightarrow{+} 4$$

John C. Baez: baez@math.ucr.edu
Christian Williams: williams@same

Change of semantics

$$
\begin{tikzcd}
\mathsf{Gph} \arrow[r, bend left, "\mathrm{FC}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, bend left, "\mathrm{UG}"] \arrow[r, bend left, "\mathrm{FP}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos} \arrow[l, bend left, "\mathrm{UC}"] \arrow[r, bend left, "\mathrm{FS}"] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, bend left, "\mathrm{UP}"]
\end{tikzcd}
$$

FC_* maps small-step to big-step operational semantics.

FP_* maps big-step to full-step operational semantics.

FS_* maps full-step to denotational semantics.

John C. Baez: baez@math.ucr.edu
Christian Williams: williams@same

Lawvere theories

A **Lawvere theory** is a category T whose objects are finite powers of a distinguished object $t = \tau(1)$.

$$\tau \colon \mathbb{N}^{\mathrm{op}} \to \mathsf{T}$$

Morphisms $t^n \to t$ are n-ary operations, and commuting diagrams are equations.

Classical algebra is represented and internalized in any category with finite products.

John C. Baez: baez@math.ucr.edu
Christian Williams: williams@same

Models

Let C be a category with finite products.

A **model** of a Lawvere theory T in C is a product-preserving functor $\mu \colon \mathsf{T} \to \mathsf{C}$. A morphism of models is a natural transformation.

These form a category $\mathrm{Mod}(\mathsf{T}, \mathsf{C})$. For example, $\mathrm{Mod}(\mathrm{Th}(\mathsf{Grp}), \mathsf{Top})$ is the category of topological groups.

John C. Baez: baez@math.ucr.edu
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Monadicity

Another way to describe algebraic structure is by *monads*.
Theories induce "free-forgetful" adjunctions:

$$\begin{tikzcd} \mathrm{Set} \arrow[rr, bend left, "F"] \arrow[rr, phantom, "\perp"] & & \mathrm{Mod}(\mathsf{T}, \mathrm{Set}). \arrow[ll, bend left, "U"] \end{tikzcd}$$

$U :: \mu \mapsto \mu(1)$ $\qquad$ $F :: n \mapsto \mathsf{T}(n, -)$
extends to Set by colimit.

This induces a monad T on Set, and we have that:

$$T\text{-Alg} \simeq \mathrm{Mod}(\mathsf{T}, \mathrm{Set}).$$

John C. Baez: baez@math.ucr.edu
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Enriched categories

Let V be a monoidal category. A V-**enriched category** is:

object set	$\mathrm{Ob}(\mathsf{C})$
hom function	$\mathsf{C}(-,-)\colon \mathrm{Ob}(\mathsf{C}) \times \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{V})$
composition	$\circ_{a,b,c}\colon \mathsf{C}(b,c) \times \mathsf{C}(a,b) \to \mathsf{C}(a,c)$
identity	$i_a\colon 1_{\mathsf{V}} \to \mathsf{C}(a,a)$

with composition associative and unital.
A V-**functor** $F\colon \mathsf{C} \to \mathsf{D}$ is:

object function	$F\colon \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D})$
hom morphisms	$F_{ab}\colon \mathsf{C}(a,b) \to \mathsf{D}(F(a),F(b))$

preserving composition and identity.
A V-**natural transformation** $\alpha\colon F \Rightarrow G$ is:

components $\alpha_a\colon 1_{\mathsf{V}} \to \mathsf{D}(F(a),G(a))$

which is "natural" in a. These form the 2-category VCat.

John C. Baez: baez@math.ucr.edu
Christian Williams: williams@same

Enriched limits

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John C. Baez:
baez@math.ucr.edu
Christian Williams:
williams@same

Preservation

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John C. Baez: baez@math.ucr.edu
Christian Williams: williams@same

Enriched Lawvere theories

a

John C. Baez:
baez@math.ucr.edu
Christian Williams:
williams@same

Generalized arities

a

John C. Baez:
baez@math.ucr.edu
Christian Williams:
williams@same

Enriched monadicity

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Example: pseudomonoid

$$\mathrm{Th}(\mathrm{PsMon})$$

type	
M	pseudomonoid
operations	
$m\colon M^2 \to M$	multiplication
$e\colon 1 \to M$	identity
rewrites	
$\alpha\colon m(m \times \mathrm{id}_M) \cong m(\mathrm{id}_M \times m)$	associator
$\lambda\colon m(e \times \mathrm{id}_M) \cong \mathrm{id}_M$	left unitor
$\rho\colon m(\mathrm{id}_M \times e) \cong \mathrm{id}_M$	right unitor
equations	pentagon, triangle

John C. Baez: baez@math.ucr.edu
Christian Williams: williams@same

Example: cartesian object

Th(Cart)

type	
X	cartesian object
operations	
$m\colon X^2 \to X$	product
$e\colon 1 \to X$	terminal element
rewrites	
$\triangle\colon \mathrm{id}_X \Longrightarrow m \circ \Delta_X$	unit of $m \vdash \Delta_X$
$\pi\colon \Delta_X \circ m \Longrightarrow \mathrm{id}_{X^2}$	counit of $m \vdash \Delta_X$
$\top\colon \mathrm{id}_X \Longrightarrow e \circ !_X$	unit of $e \vdash !_X$
$\epsilon\colon !_X \circ e \Longrightarrow \mathrm{id}_1$	counit of $e \vdash !_X$
equations	triangle identities

Natural numbers in V

a

John C. Baez: baez@math.ucr.edu
Christian Williams: williams@same

Equivalences

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Change of base

a

John C. Baez: baez@math.ucr.edu
Christian Williams: williams@same

Preservation of theories

a

John C. Baez:
baez@math.ucr.edu
Christian Williams:
williams@same

The SKI-combinator calculus

a

John C. Baez: baez@math.ucr.edu
Christian Williams: williams@same

The theory of SKI

a

John C. Baez: baez@math.ucr.edu
Christian Williams: williams@same

Change of semantics

a

John C. Baez:
baez@math.ucr.edu
Christian Williams:
williams@same

Bisimulation

a

John C. Baez:
baez@math.ucr.edu
Christian Williams:
williams@same

Conclusion

John C. Baez: baez@math.ucr.edu
Christian Williams: williams@same