Enriched Lawvere Theories for Operational Semantics

John C. Baez
Christian Williams

University of California, Riverside

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Introduction

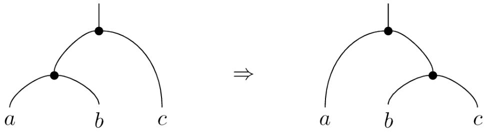
How do we integrate syntax and semantics?

type	object
term	morphism
$*$ rewrite	2-morphism $*$

algebraic theories : *denotational* semantics

$$(ab)c = a(bc)$$

enriched theories : *operational* semantics



Lawvere theories

Th(Mon)

type		M		monoid
operations	$m\colon$	M^2	$\to M$	multiplication
	$e\colon$	1	$\to M$	identity

equations

$$
\begin{tikzcd}
M^3 \arrow[r] \arrow[d] & M^2 \arrow[d] \\
M^2 \arrow[r] & M
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & M^2 \arrow[dr] & \\
1 \times M \arrow[ur] \arrow[rr] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M^2 \arrow[dl] & \\
M & & M \times 1 \arrow[ul] \arrow[ll]
\end{tikzcd}
$$

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Christian Williams

Enriched theories

$$\mathsf{Th}(\mathsf{PsMon})$$

type	M		pseudomonoid
operations	$\otimes:$	$M^2 \to M$	multiplication
	$I:$	$1 \to M$	identity

rewrites

$$\begin{tikzcd}
M^3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \alpha"] & M^2 \arrow[d] \\
M^2 \arrow[r] & M
\end{tikzcd}$$

$$\begin{tikzcd}
& M^2 \arrow[dr] \arrow[d, phantom, "\Downarrow \lambda"] & \\
1 \times M \arrow[ur] \arrow[rr] & {} & M
\end{tikzcd}
\qquad
\begin{tikzcd}
& M^2 \arrow[dl] \arrow[d, phantom, "\rho \Downarrow"] & \\
M & {} & M \times 1 \arrow[ul] \arrow[ll]
\end{tikzcd}$$

equations pentagon, triangle identities

Enriched categories

Let V be monoidal (cartesian).

V-**category**	$C(a, b)$	$\in V$
V-**functor**	$F_{ab}\colon C(a, b) \to D(F(a), F(b))$	$\in V$
V-**transformation**	$\alpha_a\colon 1_V \to D(F(a), G(a))$	$\in V$

These form the 2-category VCat.

Our enriching category

Let V be a cartesian closed category:

$$\mathsf{V}(a \times b, c) \cong \mathsf{V}(a, [b, c]).$$

Then $\underline{\mathsf{V}} \in \mathsf{VCat}$.

Let $\mathsf{V} \in \mathsf{CCC}_{fc(1)}$, meaning assume and choose:

$$n_{\mathsf{V}} := \sum_{n} 1_{\mathsf{V}}.$$

$$\text{Let} \quad \mathsf{N}_{\mathsf{V}} := \{n_{\mathsf{V}} | n \in \mathsf{N}\} \subset_{full} \mathsf{V}$$

$$\text{and} \quad \mathsf{A}_{\mathsf{V}} := \underline{\mathsf{N}}_{\mathsf{V}}^{\mathrm{op}} \quad \text{– our “arities”}.$$

Enriched products

The V-**product** of $(a_i) \in \mathsf{C}$ is an object $\prod_i a_i \in \mathsf{C}$ equipped with a V-natural isomorphism

$$\mathsf{C}(-, \prod_i a_i) \cong \prod_i \mathsf{C}(-, a_i).$$

A V-functor $F\colon \mathsf{C} \to \mathsf{D}$ **preserves** V-products if the "projections" induce a V-natural isomorphism:

$$\mathsf{D}(-, F(\prod_i a_i)) \cong \prod_i \mathsf{D}(-, F(a_i)).$$

Let VCat_{fp} be the 2-category of V-categories with finite V-products and V-functors preserving them.

Enriched Lawvere theories

Definition

A V-**theory** is a V-category $\mathsf{T} \in \mathsf{VCat}_{fp}$ equipped with:

$$\tau \colon \mathsf{A_V} \to \mathsf{T}$$

i.e., whose objects are finite V-products of $t := \tau(1_\mathsf{V})$.

The 2-category of V-theories is $\mathsf{VLaw} := \mathsf{A_V} \downarrow \mathsf{VCat}_{fp}$.
If V is *complete*, then $\underline{\mathsf{VLaw}} \in \mathsf{VCat}$.

Enriched models

Definition

A **context** is a V-category $\mathsf{C} \in \mathsf{VCat}_{fp}$.
A **model** of T is a V-functor

$$\mu\colon \mathsf{T} \to \mathsf{C} \in \mathsf{VCat}_{fp}.$$

The 2-category of models is $\mathsf{Mod}(\mathsf{T},\mathsf{C}) := \mathsf{VCat}_{fp}(\mathsf{T},\mathsf{C})$.
If V is *complete*, then $\underline{\mathsf{Mod}}(\mathsf{T},\mathsf{C}) \in \mathsf{VCat}$.

Example: cartesian object

Let V = Cat.

Th(Cart)

type		X	cartesian object
operations	m:	$X^2 \to X$	product
	e:	$1 \to X$	terminal element
rewrites	$\triangle$:	$\mathrm{id}_X \Longrightarrow m \circ \Delta_X$	unit of $m \vdash \Delta_X$
	π:	$\Delta_X \circ m \Longrightarrow \mathrm{id}_{X^2}$	counit of $m \vdash \Delta_X$
	$\top$:	$\mathrm{id}_X \Longrightarrow e \circ !_X$	unit of $e \vdash !_X$
	ϵ:	$!_X \circ e \Longrightarrow \mathrm{id}_1$	counit of $e \vdash !_X$
equations			triangle identities

Change of base

Let $F\colon \mathsf{V} \to \mathsf{W}$ preserve finite products, and $\mathsf{C} \in \mathsf{VCat}$.

Then F induces a **change of base**:

$$F_*(\mathsf{C})(a, b) := F(\mathsf{C}(a, b)).$$

This gives a 2-functor

$$F_*\colon \mathsf{VCat} \to \mathsf{WCat}.$$

Enrichment provides semantics, so change of base should *preserve* theories to be a *change of semantics*.

Change of semantics

There is a "spectrum" of semantics:

$$
\begin{tikzcd}
\mathsf{Gph} \arrow[r, bend left, "\mathrm{FC}"] \arrow[r, phantom, "\perp"] & \mathsf{Cat} \arrow[l, bend left, "\mathrm{UG}"] \arrow[r, bend left, "\mathrm{FP}"] \arrow[r, phantom, "\perp"] & \mathsf{Pos} \arrow[l, bend left, "\mathrm{UC}"] \arrow[r, bend left, "\mathrm{FS}"] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, bend left, "\mathrm{UP}"]
\end{tikzcd}
$$

FC_* maps small-step to big-step operational semantics.
FP_* maps big-step to full-step operational semantics.
FS_* maps full-step to denotational semantics.

Preservation of theories

Theorem

Let $F \colon \mathsf{V} \to \mathsf{W} \in \mathsf{CCC}_{fc(1)}$.
Then F *is a* **change of semantics:**

F_* *preserves theories. For every* V*-theory* $\tau_\mathsf{V} \colon \mathsf{A}_\mathsf{V} \to \mathsf{T}$,

$$\tau_\mathsf{W} := \mathsf{A}_\mathsf{W} \xrightarrow{\sim} F_*(\mathsf{A}_\mathsf{V}) \xrightarrow{F_*(\tau_\mathsf{V})} F_*(\mathsf{T}) \quad \text{is a } \mathsf{W}\text{-theory.}$$

F_* *preserves models. For every model* $\mu \colon \mathsf{T} \to \mathsf{C}$,

$$F_*(\mu) \colon F_*(\mathsf{T}) \to F_*(\mathsf{C}) \quad \text{is a model of } (F_*(\mathsf{T}), \tau_\mathsf{W}).$$

Combinatory logic

The λ calculus:

$$M, N := x \mid (M\ N) \mid \lambda x.M$$

$$\beta\colon (\lambda x.M\ N) \Rightarrow M[N/x].$$

Variable binding is subtle:

$$(\lambda x.(\lambda y.(x\ y))\ y) \Rightarrow\ ?$$

Translate to *combinators*:

$$[[-]]\colon \Lambda \to \{\mathsf{SKI}\}.$$

The theory of SKI

Th(SKI)

type	t		
terms	$S:$	$1 \to t$	
	$K:$	$1 \to t$	
	$I:$	$1 \to t$	
	$(-\ -):$	$t^2 \to t$	
rewrites	$\sigma:$	$(((S\ a)\ b)\ c)$	$\Rightarrow ((a\ c)\ (b\ c))$
	$\kappa:$	$((K\ a)\ b)$	$\Rightarrow a$
	$\iota:$	$(I\ a)$	$\Rightarrow a$

A model of Th(SKI)

A Gph-product preserving Gph-functor $\mu\colon \mathrm{Th(SKI)} \to \mathrm{Gph}$ yields a graph $\mu(t)$ of SKI-terms:

$$1 \cong \mu(1) \xrightarrow{\mu(S)} \mu(t) \xleftarrow{\mu((-\ -))} \mu(t^2) \cong \mu(t)^2.$$

The rewrites are transferred by the enrichment of μ:

$$\mu_{1,t}\colon \mathrm{Th(SKI)}(1,t) \to \mathrm{Gph}(1,\mu(t)).$$

The free model of SKI

John C. Baez
Christian Williams

The syntax and semantics of the SKI combinator calculus are given by the free model

$$\mu_{\mathsf{SKI}}^{\mathsf{Gph}} := \mathrm{Th}(\mathsf{SKI})(1,-) \colon \mathrm{Th}(\mathsf{SKI}) \to \mathsf{Gph}.$$

The graph $\mu_{\mathsf{SKI}}^{\mathsf{Gph}}(t)$ is the *transition system* which represents the **small-step operational semantics** of the SKI-calculus:

$$(\mu(a) \to \mu(b) \in \mu_{SKI}^{\mathsf{Gph}}(t)) \iff (a \Rightarrow b \in \mathrm{Th}(\mathsf{SKI})(1,t)).$$

Change of semantics

$\mathrm{FC}\colon \mathsf{Gph} \to \mathsf{Cat}$ preserves products, hence gives a change of semantics from *small-step* to *big-step* operational semantics:

$$
\begin{tikzcd}
 & (((S\ K)\ (I\ K))\ S) \arrow[dl, "\iota"'] \arrow[dr, "\sigma"] \arrow[ddl, dotted, "\iota\sigma"] \arrow[ddl, dotted, "\sigma\iota"] \arrow[ddr, dotted, "\kappa\sigma\iota"] \arrow[ddr, dotted, "\kappa\iota\sigma"] \arrow[ddr, dotted, "\kappa\sigma"] & \\
(((S\ K)\ K)\ S) \arrow[d, "\sigma"'] & & ((K\ S)\ ((I\ K)\ S)) \arrow[d, "\kappa"] \arrow[dll, "\iota"'] \\
((K\ S)\ (K\ S)) \arrow[rr, "\kappa"'] & & S.
\end{tikzcd}
$$

$\mathrm{FP}\colon \mathsf{Cat} \to \mathsf{Pos}$ gives *full-step* (Hasse diagram), and
$\mathrm{FS}\colon \mathsf{Pos} \to \mathsf{Set}$ gives *denotational* semantics, collapsing the connected component to a point.

Conclusion

Enriched theories give a way to unify the structure and behavior of formal languages.

Enriching in category-like structures reifies operational semantics by incorporating rewrites between terms.

Cartesian functors between enriching categories induce change-of-semantics functors between categories of models.

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References I

F. W. Lawvere, Functorial semantics of algebraic theories, reprinted in *Repr. Theory Appl. Categ.* **5** (2004), 1–121.

G. M. Kelly, *Basic Concepts of Enriched Category Theory*, reprinted in *Repr. Theory Appl. Categ.* **10** (2005), 1–136.

G. D. Plotkin, A structural approach to operational semantics, *J. Log. Algebr Program.* **60/61** (2004) 17–139.

M. Hyland and J. Power, Discrete Lawvere theories and computational effects, in *Theoretical Comp. Sci.* **366** (2006), 144–162.

R. B. B. Lucyshyn-Wright, Enriched algebraic theories and monads for a system of arities, *Theory Appl. Categ.* **31** (2016), 101–137.

References II

H.P. Barendregt, The Lambda Calculus, its syntax and semantics, in *Studies in Logic and The Foundations of Mathematics*, Elsevier, London, 1984.

R. Milner, Communicating and Mobile Systems: The Pi Calculus, in *Cambridge University Press*, Cambridge, UK, 1999.

M. Stay and L. G. Meredith, Representing operational semantics with enriched Lawvere theories.