Enriched Lawvere Theories for Operational Semantics

John C. Baez and Christian Williams

University of California, Riverside

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1 Introduction

Formal systems are often defined without intrinsic connection to how they actually operate in practice. In everyday computation, the *structure* of the program is separate from the *dynamics* — but their potential disparity is the primary source of error. If these are unified as one mathematical object, the system can be proven *correct by construction*. **Operational semantics** [14] is an essential tool in language design and verification, which formally specifies program behavior by labelled transition systems, or labelled directed graphs:

$$(\lambda x.x + x\ \ 2) \xrightarrow{\beta} 2 + 2 \xrightarrow{+} 4$$

The idea is to reify operational semantics via **enrichment** [6]. In the categorical representation of an algebraic theory, the objects are types, and the morphisms are terms; hence to represent the actual process of computation, we need the higher-level notion of rewriting one term into another: the hom-object or "thing of morphisms" between two terms should be not a set but a *category*-like structure, in which the morphisms represent rewrites. For example, the SK-combinator calculus is the "abstraction-free" λ-calculus, generated by two basic rewrite rules (see §8):

$$(((Sx)y)z) \overset{t^3}{\underset{t}{\Big(\Rightarrow\Big)}} ((xz)(yz)) \qquad\qquad ((Kx)y) \overset{t^2}{\underset{t}{\Big(\Rightarrow\Big)}} x$$

A **Lawvere theory** [7] defines an algebraic structure abstractly, as a category $\mathcal{T}$ generated by powers of a single object t, for "type", and morphisms $t^n \to t$ representing n-ary operations, with commutative diagrams specifying equations. This presents the "theory" of a kind of algebra, which can be modelled in a category $\mathcal{C}$ by a power-preserving functor $\mu \colon \mathcal{T} \to \mathcal{C}$. This is a very general notion of "algebra" — computational formalisms are also presented by generators and relations: in particular, a **term calculus** represents a formal language by sorts, term constructors, and congruence rules.

Using enriched Lawvere theories for operational semantics has been explored in the past. This was studied in the case of categories by Seely [18], posets by Ghani and Lüth [10], and others, for various related purposes. Here we allow quite general enrichments, to incorporate these approaches in a common framework; but we focus attention on graph-enriched Lawvere theories, which have a clear connection to the original idea of operational semantics:

sorts	:	generating object t
term constructors	:	generating morphisms $t^n \to t$
structural congruence	:	commuting diagrams
$*$ rewrite rules	:	generating hom-edges $*$

There are many other useful enriching categories. Better yet, there are functors between them that allow the seamless transition between different kinds of operational semantics. There is a *spectrum* of enriching categories which forms a gradient of resolution for the semantics of term calculi. For an enriching category $\mathcal{V}$, a **$\mathcal{V}$-theory** is a $\mathcal{V}$-enriched Lawvere theory with natural number arities (see §4):

Graphs: Gph-theories represent "small-step" operational semantics *
— a hom-graph edge represents a *single* term rewrite.
Categories: Cat-theories represent "big-step" operational semantics:
— identity and composition represent the *reflexive-transitive* closure of the rewrite relation.
Posets: Pos-theories represent "full-step" operational semantics:
— a hom-poset boolean represents the *existence* of a big-step rewrite.
Sets: Set-theories represent denotational semantics (provided the calculus is confluent, see [19]):
— a hom-set element represents an *equivalence class* of the symmetric closure of the big-step relation.

(We will use reflexive graphs, meaning every vertex has a distinguished self-loop; this is not the convention, but it is needed for the "free category" functor to be a change-of-semantics (§6). Also, these can be labelled transition systems by a simple augmentation of this theory; we left this implicit for simplicity.)

Operational semantics is unified by enriched Lawvere theories and canonical functors between enriching categories. This provides a more systematic categorical representation of computation.

A motivating example is "logic as a distributive law" [20], an algorithm for deriving a spatial-behavioral type system from a formal presentation of a computational calculus. Essential properties such as soundness are made provable, a powerful query language is generated, and modalities can be "built in" to express principles of the system.

In section §2, we review Lawvere theories as a more explicit, but equivalent, presentation of finitary monads. In §3, we establish the basics of enrichment, and understand the arities of a theory as an action of the enriching category on the enriched category. In §4 we give the central definition of $\mathcal{V}$-theory, from Lucyshyn-Wright [9], which allows us to parameterize our theory by a monoidal subcategory of arities.

In §5 we discuss how functors between enriching categories induce change-of-base 2-functors between their categories of enriched categories, and in §6 we show that product-preserving functors induce *change-of-semantics*, i.e. preserve theories and models. In §7 we make a somewhat tangential remark of future interest, that theories of all different enrichments and models can be assimilated into one category using the iterated Grothendieck construction.

Finally in §8 we bring it all together by demonstrating these concepts with the SKI-combinator calculus, introducing the idea of developing actual programming languages with enriched Lawvere theories.

This paper builds upon the ideas of Mike Stay and Greg Meredith, presented in "Representing operational semantics with enriched Lawvere theories." We appreciate the opportunity to develop this research for use in the innovative distributed computing system RChain, and we gratefully acknowledge the support of Pyrofex Incorporated. [21]

2 Lawvere Theories

Computer science loves monads, but they are widely regarded as somewhat mysterious. They are almost too elegant; it is difficult to "grok" how they work before working with them extensively. This is ironic, because most are equivalent to something more intuitive: Lawvere theories.

The "theory of monoids" can be defined without any reference to sets:

$$\mathrm{Th}(\mathrm{Mon})$$

$$\begin{array}{rl} \text{an object} & M \\ \text{an identity element} & e\colon 1 \to M \\ \text{and multiplication} & m\colon M^2 \to M \\ \text{with associativity} & m \circ (m \times M) = m \circ (M \times m) \\ \text{and unitality} & e \circ M = M = M \circ e \end{array}$$

Lawvere theories formalize this idea. They were originally called "finite product" theories: a skeleton N of the category of finite sets FinSet is the free category with finite coproducts on 1 — every finite set is equal to the disjoint union of copies of $\{*\}$; conversely, N^{op} is the free category with finite products on 1. So, a category with finite products $\mathcal{T}$ equipped with a strictly (finite) product-preserving bijective-on-objects functor $\tau\colon \mathrm{N}^{\mathrm{op}} \to \mathcal{T}$ is essentially a category generated by one object $\tau(1) = M$ and n-ary operations $M^n \to M$, as well as the projection and diagonal morphisms of finite products. Lawvere theories form a category Law, with finite-product functors $f\colon \mathcal{T} \to \mathcal{T}'$ such that $f\tau = \tau'$.

The abstraction of this definition is powerful: the syntax encapsulates the algebraic theory, independently of semantics, and then M can be realized as almost any formal object. For another category with finite products $\mathcal{C}$, a **model** of the Lawvere theory in $\mathcal{C}$ is a product-preserving functor $\mu\colon \mathcal{T} \to \mathcal{C}$. By the "free" property above, this functor is determined by $\mu(\tau(1)) = \mu(M) = X \in \mathcal{C}$. The models of $\mathcal{T}$ in $\mathcal{C}$ form a category $[\mathcal{T}, \mathcal{C}]_{fp}$, in which the morphisms are natural transformations. The general theory can be thereby modelled in many useful ways. For example, ordinary groups are models $\mathcal{T}_{\mathrm{Grp}} \to \mathrm{Set}$, while models $\mathcal{T}_{\mathrm{Grp}} \to \mathrm{Top}$ are topological groups.

Lawvere theories and *finitary monads* provide complementary representations of algebraic structures and computation, as discussed by Hyland and Power in [5], and they were proven to be equivalent by Linton in [8]. For every Lawvere theory $\mathcal{T}$, there is an adjunction:

$$\mathrm{Set} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} [\mathcal{T}, \mathrm{Set}]_{fp}$$

There is the underlying set functor

$$U\colon [\mathcal{T}, \mathrm{Set}]_{fp} \to \mathrm{Set}$$

which sends each model μ to the image of the generating object, $\mu(\tau(1)) = X$ in Set. There is the free model functor

$$F\colon \mathrm{Set} \to [\mathcal{T}, \mathrm{Set}]_{fp}$$

which sends each finite set n to the representable functor $\mathcal{T}(|n|, -)\colon \mathcal{T} \to \mathrm{Set}$, and in general a set X to the functor which sends $t^n \in \mathcal{T}$ to the set of all n-ary operations on X: $\{f(x_1, ..., x_n) | f \in \mathcal{T}(n, 1), x_i \in X\}$ — this is the filtered colimit of representables indexed by the poset of finite subsets of X [17](see §3 and §4). These form the adjunction:

$$\mathrm{Mod}(F(n), \mu) = \mathrm{Mod}(\mathcal{T}(|n|, -), \mu) \cong \mu(n) \cong \mu(1)^n = \mathrm{Set}(n, U(\mu))$$

The left isomorphism is by the Yoneda lemma, and the right isomorphism is by the product-preservation of μ. Essentially, these are opposite ways of representing the n-ary operations of a model.

This adjunction induces a monad T on Set:

$$T(X) = \int^{n \in \mathrm{N}} \mathcal{T}(n, 1) \times X^n \tag{1}$$

the integral symbol is a "coend", essentially a coproduct quotiented by the equations of the theory and the equations induced by the cartesian structure of the category; hence T sends each set X to the set of all terms in the theory on X up to equality.

Conversely, for a monad T on Set, its Kleisli category $\mathcal{K}(T)$ is the category of all free algebras of the monad, which has all coproducts. There is a "comparison" functor $k \colon \mathrm{Set} \to \mathcal{K}(T)$ which is the identity on objects and preserves coproducts (because it is a right adjoint); so,

$$k^{\mathrm{op}} \colon \mathrm{Set}^{\mathrm{op}} \to \mathcal{K}(T)^{\mathrm{op}}$$

is a product-preserving functor, and restricting its domain to N^{op} forms the canonical Lawvere theory corresponding to the monad. This restriction is what limits the equivalence to finitary monads (§3). There is a good explanation of all this in Milewski's "Category Theory for Programmers" [12], as well as [4].

The correspondence of Lawvere theories and finitary monads forms an equivalence between the category of Lawvere theories and the category of finitary monads on Set, as well as the categories of models and algebras for every corresponding pair $(\mathcal{T}, T)$:

$$\begin{array}{rcl} \mathrm{Law} & \simeq & \mathrm{Mnd}_f \\ \mathrm{Mod}(\mathcal{T}) & \simeq & \mathrm{Alg}(T) \end{array}$$

This generalizes to arbitrary "locally finitely presentable" modelling categories $\mathcal{C}$ (§3). The previous references suffice; we do not need further details.

3 Enrichment

We generalize hom-sets to hom-*objects*, to equip theories of formal languages with higher-dimensional rewrites.

Let $(\mathcal{V}, \otimes, I)$ be a monoidal category [11], the "enriching" category.

A **$\mathcal{V}$-category** or $\mathcal{V}$-enriched category $\mathcal{C}$ is:

$$\begin{array}{rl} \text{a collection of objects} & \mathrm{Obj}(\mathcal{C}) \\ \text{a hom-object function} & \mathcal{C}(-,-) \colon Obj(\mathcal{C}) \times Obj(\mathcal{C}) \to Obj(\mathcal{V}) \\ \text{composition morphisms} & \circ_{a,b,c} \colon \mathcal{C}(b,c) \otimes \mathcal{C}(a,b) \to \mathcal{C}(a,c) \quad \forall a, b, c \in Obj(\mathcal{C}) \\ \text{identity elements} & i_a \colon I \to \mathcal{C}(a,a) \quad \forall a \in Obj(\mathcal{C}) \end{array}$$

such that composition is associative and unital.

A **$\mathcal{V}$-functor** $F \colon \mathcal{C} \to \mathcal{D}$ is:

$$\begin{array}{rl} \text{a function} & F_0 \colon Obj(\mathcal{C}) \to Obj(\mathcal{D}) \\ \text{hom-morphisms} & F_{ab} \colon \mathcal{C}(a,b) \to \mathcal{D}(F_0(a), F_0(b)) \quad \forall a, b \in \mathcal{C} \end{array}$$

such that F is compatible with composition and identity.

A **$\mathcal{V}$-natural transformation** $\alpha \colon F \Rightarrow G$ is:

$$\text{a family} \quad \alpha_a \colon I \to \mathcal{D}(F_0(a), G_0(a)) \quad \forall a \in Obj(\mathcal{C})$$

such that α is "natural" in a. Hence there is a 2-category $\mathcal{V}\mathrm{Cat}$ of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations. See [6] for reference.

Let $\mathcal{V}$ be a *closed symmetric monoidal category*, providing

$$\begin{array}{rl} \text{internal hom} & [-,-]\colon \mathcal{V}^{\mathrm{op}} \otimes \mathcal{V} \to \mathcal{V} \\ \text{symmetry braiding} & \tau_{a,b}\colon a \otimes b \cong b \otimes a \quad \forall a, b \in Obj(\mathcal{C}) \\ \text{tensor-hom adjunction} & \mathcal{V}(a \otimes b, c) \cong \mathcal{V}(a, [b, c]) \quad \forall a, b, c \in Obj(\mathcal{V}) \end{array}$$

Then $\mathcal{V}$ is itself a $\mathcal{V}$-category, denoted $\tilde{\mathcal{V}}$, with internal hom as the hom-object function. The tensor-hom adjunction is called "currying" in computer science; the counit is evaluation.

These adjoints generalize to *actions* of $\mathcal{V}$ on a $\mathcal{V}$-category $\mathcal{C}$: **power** and **copower** are $\mathcal{V}$-functors:

$$\begin{array}{rl} \odot\colon & \mathcal{V} \otimes \mathcal{C} \to \mathcal{C} \\ \pitchfork\colon & \mathcal{V}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{C} \end{array}$$

such that for $x \in \mathrm{Obj}(\mathcal{V})$ and $a, b \in \mathrm{Obj}(\mathcal{C})$, there is the adjunction (provided the objects exist):

$$\mathcal{C}(a \odot x, b) \cong [x, \mathcal{C}(a, b)] \cong \mathcal{C}(a, x \pitchfork b) \tag{2}$$

and $\mathcal{C}$ is $\mathcal{V}$-*powered* or *copowered* if all powers or copowers exist.

These are the two basic forms of *enriched limit* and *colimit*, which are not especially intuitive, but they are direct generalizations of familiar ideas in the category of sets. In Set, the power is the "exponential" function set and the copower is the product. To generalize this to an action on other Set-categories, note that:

$$X \pitchfork Y = \quad Y^X \quad \cong \prod_{x \in X} Y$$

$$X \odot Y = \quad X \times Y \quad \cong \sum_{y \in Y} X$$

So, categories are canonically Set-powered or copowered by indexed products or coproducts of copies of an object, provided that these exist. (We will use exponential notation $b^x := x \pitchfork b$, and denote the unit I by 1, because the enriching categories under consideration are cartesian.)

There are just a few more technicalities. Given a $\mathcal{V}$-category $\mathcal{C}$, one often considers the Yoneda embedding into the $\mathcal{V}$-presheaf category $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$, and it is important to know whether certain subcategories are representable; generally, some properties of $\mathcal{C}$ depend on a condition of "finitude" [1]. A category is **locally finitely presentable** if it is the category of models for a "sketch", a theory with not only products but general limits, and an object is "finite" if its representable functor is **finitary**, or preserves filtered colimits.

A $\mathcal{V}$-category $\mathcal{C}$ is **locally finitely presentable** if its underlying category $\mathcal{C}_0$ is locally finitely presentable, $\mathcal{C}$ has finite powers, and $(-)^x\colon \mathcal{C}_0 \to \mathcal{C}_0$ is finitary for all finitely presentable x. The details are not crucial: all categories to be considered are locally finitely presentable. Denote by $\mathcal{V}_f$ the subcategory of $\mathcal{V}$ of finite objects: in Gph, these are simply graphs with finitely many vertices and edges.

Even though the definition of Lawvere theory seems to be all about products, it is actually about *powers*, because these constitute the *arities* of the operations. These become greatly generalized in the enriched case, because whereas the only finite objects in Set are *finite sets*, there are more complex finite objects in any other enriching $\mathcal{V}$. However in the next section, we will discuss the difficulties of this generality.

4 Enriched Lawvere theories

Power introduced the notion of enriched Lawvere theory about twenty years ago, "in seeking a general account of what have been called notions of computation", while studying 2-monads on Cat [15]. The original definition is as follows: for a symmetric monoidal closed category $(\mathcal{V}, \otimes, I)$, a $\mathcal{V}$-*enriched*

Lawvere theory is a finitely-powered $\mathcal{V}$-category $\mathcal{T}$, equipped with a strictly power-preserving identity-on-objects $\mathcal{V}$-functor

$$\tau\colon \mathcal{V}_f^{\mathrm{op}} \to \mathcal{T}$$

A *model* of a $\mathcal{V}$-theory is a $\mathcal{V}$-functor $\mu\colon \mathcal{T} \to \mathcal{V}$ which preserves powers by objects of $\mathcal{V}_f$, and models form the $\mathcal{V}$-category $[\mathcal{T}, \mathcal{V}]_{fp}$ with morphisms being $\mathcal{V}$-natural transformations. The monadic adjunction and equivalence of §2 generalize to these theories.

However, this requires $\mathcal{T}$ to have all powers of $\mathcal{V}_f$, i.e. the theory must have arities for every finite object of $\mathcal{V}$. These *generalized arities* may be very powerful — rather than only inputting n-tuples of terms, we can operate on any finite object of terms! But what does it mean for an operation to take in a finite graph of terms? To what subjects does this pertain primarily, and how can we learn to utilize this generality?

Despite its potential, this idea has remained dormant for decades, partly for the difficulty of intuition, but also that of presentation: while it is easy to inductively generate all n-ary operations from binary, unary, nullary ones, it is certainly more subtle to generate powers for all finite objects. We hope that some intrepid researchers bring this subject to light, so that we can begin to expand the notion of "operation" to much more than finite sets.

For this paper, however, we only need *natural number* arities, while still retaining enrichment. A very general and useful definition of enriched algebraic theory was introduced by Lucyshyn-Wright [9], which allows for theories to be parameterized by a **system of arities**, a full monoidal subcategory

$$j\colon \mathcal{J} \hookrightarrow \mathcal{V}.$$

Definition. A $\mathcal{V}$-enriched algebraic theory with j-arities or $\mathcal{J}$-$\mathcal{V}$ **theory** $(\mathcal{T}, \tau, j)$ is a $\mathcal{V}$-category $\mathcal{T}$ equipped with an identity-on-objects $\mathcal{V}$-functor

$$\tau\colon \tilde{\mathcal{J}}^{\mathrm{op}} \to \mathcal{T}$$

which is "$\mathcal{J}$-power preserving", or preserves powers by objects of (the subcategory image of) $\mathcal{J}$.

A **model** of $\mathcal{T}$ in a $\mathcal{V}$-category $\mathcal{C}$ is a $\mathcal{J}$-power preserving $\mathcal{V}$-functor

$$\mu : \mathcal{T} \to \mathcal{C}.$$

In the same way that the objects of a Lawvere theory are N-powers of a generating object, the objects of a $\mathcal{J}$-$\mathcal{V}$ theory are $\mathcal{J}$-powers t^J of a generating object t, for each $J \in \mathcal{J}$ - note that t itself is t^I. Just as every $n \in \mathrm{N}^{\mathrm{op}}$ is a power of $1 \in \mathrm{Set}$, every $J \in \tilde{\mathcal{J}}^{\mathrm{op}}$ is a power of the monoidal unit $I \in \mathcal{V}$, i.e. using equation 2 for $\tilde{\mathcal{J}}^{\mathrm{op}}$, $(\star \mapsto 1_J) \in [I, \tilde{\mathcal{J}}^{\mathrm{op}}(J, J)]$ is sent to the canonical isomorphism:

$$J \cong I^J. \tag{3}$$

This is just the opposite of the usual isomorphism $J \cong J^I$. Then, since τ preserves $\mathcal{J}$-powers, this implies that every object of $\mathcal{T}$ is a power of $t = \tau(I)$.

$\mathcal{J}$-$\mathcal{V}$ theories form the category $\mathcal{V}$Law, the morphisms of which are $\mathcal{J}$-power preserving $\mathcal{V}$-functors $f\colon \mathcal{T} \to \mathcal{T}'$ such that $f\tau = \tau'$. For every $\mathcal{J}$-$\mathcal{V}$ theory $\mathcal{T}$ and every $\mathcal{V}$-category $\mathcal{C}$ with $\mathcal{J}$-powers, the category of models $\mathrm{Mod}(\mathcal{T}, \mathcal{C})$ consists of $\mathcal{J}$-power preserving $\mathcal{V}$-functors $\mathcal{T} \to \mathcal{C}$ and $\mathcal{V}$-natural transformations. (Note: if $\mathcal{V}$ is a *cosmos*, i.e. complete and cocomplete, then the functor categories of $\mathcal{V}$Cat are also $\mathcal{V}$-categories, including $\mathcal{V}$Law and $\mathrm{Mod}(\mathcal{T}, \mathcal{C})$. This is potentially very useful, and the "operational" $\mathcal{V}$'s of this paper are indeed cosmoi.)

Here is an overview of the concepts:

$j\colon$	$\mathcal{J}$	$\hookrightarrow$	$\mathcal{V}$	arities
$---$	$---$	$---$	$---$	enrichment
$\tau\colon$	$\tilde{\mathcal{J}}^{\mathrm{op}}$	$\to$	$\mathcal{T}$	theory
			$\downarrow$	model
			$\mathcal{C}$	semantic $\mathcal{V}$-category

This parameterization is quite general; for example, Power's definition is the case $\mathcal{J} = \mathcal{V}_f$. A system of arities is **eleutheric** if left Kan extensions along j exist and are preserved by $\mathcal{V}(K, -)$ for all $K \in Ob(\mathcal{J})$. This is what is needed to have the essential *monadicity* theorems: Lucyshyn-Wright proved that any $\mathcal{J}$-$\mathcal{V}$ theory for an eleutheric system of arities has a category of models for $\mathcal{C} = \mathcal{V}$ which is monadic over $\mathcal{V}$. The usual kinds of arities are eleutheric: in particular, *finite cardinals*. This will be our $\mathcal{J}$.

Let $(\mathcal{V}, \times, I_\mathcal{V})$ be a cartesian closed category with finite coproducts of $I_\mathcal{V}$. Define $\mathrm{N}_\mathcal{V}$ to be the full subcategory of finite coproducts of the unit object:

$$n_\mathcal{V} = \coprod_{n \in \mathrm{N}} I_\mathcal{V}$$

which is the *copower* of $I_\mathcal{V}$ by a finite set $n \in \mathrm{N}$, characterized by the universal property

$$\mathcal{V}(n_\mathcal{V}, a) = \mathcal{V}(I_\mathcal{V} \odot n, a) \cong \mathrm{Set}(n, \mathcal{V}(I_\mathcal{V}, a)). \tag{4}$$

For $\mathcal{J} = \mathrm{N}_\mathcal{V}$, we will call $\mathcal{J}$-$\mathcal{V}$ theories **$\mathcal{V}$-theories** for simplicity.

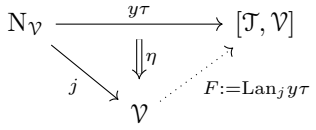
Because $\mathrm{N}_\mathcal{V}$ is eleutheric, $\mathcal{V}$-theories correspond to $\mathcal{V}$-monads on $\mathcal{V}$, just as ordinary Lawvere theories correspond to monads on Set, and §6 will demonstrate that the arities are essentially the same — but now we have the rich "operational" information of $\mathcal{V}$, and this $\mathcal{V}$ is adaptable.

$$\mathcal{T} \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Mod}(\mathcal{T}, \mathcal{V})$$

It is worth explaining, how does this $\mathcal{V}$-adjunction work in detail? As described in [9], section 8, the **free model** of a $\mathcal{V}$-theory is the enriched generalization of the free model described in §2: it is the composite of the theory $\tau \colon \mathrm{N}_\mathcal{V}^{\mathrm{op}} \to \mathcal{T}$ with the covariant Yoneda embedding $y \colon \mathcal{T} \to [\mathcal{T}, \mathcal{V}]$, sending each $n_\mathcal{V} \in \mathrm{N}_\mathcal{V}$ to its representable $\mathcal{V}$-functor, i.e. the n-ary operations of $\mathcal{T}$:

$$\begin{array}{ccccc} \mathrm{N}_\mathcal{V}^{\mathrm{op}} & \xrightarrow{\tau} & \mathcal{T} & \xrightarrow{y} & [\mathcal{T}, \mathcal{V}] \\ n_\mathcal{V} & \mapsto & t^{n_\mathcal{V}} & \mapsto & \mathcal{T}(t^{n_\mathcal{V}}, -) \end{array}$$

Since an object of $\mathcal{V}$ does not necessarily have a "poset of finite subobjects" over which to take a filtered colimit (as in Set), the extension of the free model to all of $\mathcal{V}$ is specified by a somewhat higher-powered generalization: the "free model" functor on $\mathcal{V}$ is the *left Kan extension* of $y\tau$ along j.



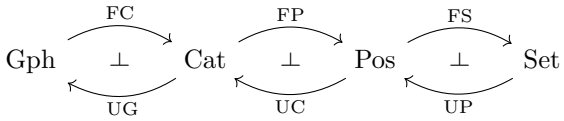
This is essentially the universal "best solution" to this diagram commuting; i.e. for any other functor $G : \mathcal{V} \to [\mathcal{T}, \mathcal{V}]$ and $\mathcal{V}$-natural transformation $y\tau \Rightarrow Gj$, the latter factors uniquely through the unit η.

The forgetful adjoint $U \colon [\mathcal{T}, \mathcal{V}] \to \mathcal{V}$ is still evaluation at the generating object, and hence the $\mathcal{V}$-monad has a more concrete (elementwise) formula as an enriched coend:

$$T(V) = \int^{n_\mathcal{V} \in \mathrm{N}_\mathcal{V}} \mathcal{T}(t^{n_\mathcal{V}}, t) \times V^{n_\mathcal{V}} \tag{5}$$

5 Change of Base

We now have the tools to formulate the main idea: certain $\mathcal{V}$ correspond to certain kinds of *semantics*, and changing enrichments corresponds to a *change of semantics*. We propose a general framework in which one can transition between different forms of semantics: small-step, big-step, full-step operational semantics, and denotational semantics.



This is effected by a (strong) **monoidal functor**: a functor

$$(F, \lambda, \upsilon) \colon (\mathcal{V}, \otimes_{\mathcal{V}}, I_{\mathcal{V}}) \to (\mathcal{W}, \otimes_{\mathcal{W}}, I_{\mathcal{W}})$$

which transfers the tensor and unit via the *laxor* and *unitor*

$$\begin{array}{ll} \lambda\colon & F(a) \otimes_{\mathcal{W}} F(b) \cong F(a \otimes_{\mathcal{V}} b) \\ \upsilon\colon & I_{\mathcal{W}} \cong F(I_{\mathcal{V}}) \end{array}$$

such that λ is natural in a, b and associative, and unital relative to υ.

This induces a **change of base** functor $F_* \colon \mathcal{V}\mathrm{Cat} \to \mathcal{W}\mathrm{Cat}$ [3]. This is the strange but elegant operation on enriched categories, whereby the objects remain unchanged, but the hom-objects are transformed by the functor between enriching categories. The $\mathcal{W}$-category $F_*(\mathcal{C})$ is defined as follows:

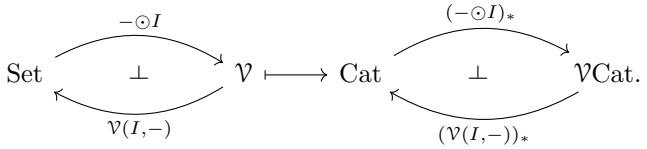
$$\begin{array}{rl} \text{objects} & \mathrm{Obj}(\mathcal{C}) \\ \text{hom-function} & F \circ \mathcal{C}(-,-) \\ \text{composition} & F(\circ_{a,b,c}) \circ \lambda \\ \text{identity} & F(i_a) \circ \upsilon. \end{array}$$

If $f \colon \mathcal{C} \to \mathcal{D} \in \mathcal{V}\mathrm{Cat}$ is a $\mathcal{V}$-functor, then $F_*(f)_{\mathrm{obj}} = f_{\mathrm{obj}}$ and $F_*(f)_{\mathrm{hom}} = F \circ f_{\mathrm{hom}}$. If $\alpha \colon f \Rightarrow g$ is a $\mathcal{V}$-natural transformation and $c \in \mathcal{C}$, then $F_*(\alpha)_c := F(\alpha_c) \circ \upsilon$.

Hence, the change of base operation forms a 2-functor (or "Cat-functor"):

$$\begin{array}{ccc} \mathrm{MonCat} & \xrightarrow{(-)_*} & 2\mathrm{Cat} \\ (F \colon \mathcal{V} \to \mathcal{W}) & \mapsto & (F_* \colon \mathcal{V}\mathrm{Cat} \to \mathcal{W}\mathrm{Cat}) \end{array}$$

In particular, there is an important correspondence of adjunctions (if $\mathcal{V}$ has all coproducts of $I_{\mathcal{V}}$):



Each set X is represented in $\mathcal{V}$ as the X-indexed coproduct of the unit object, and conversely each object v of $\mathcal{V}$ is represented in Set by the hom-set from the unit to v. The latter induces the "underlying (Set-)category" change of base, which forgets the enrichment. The former induces the "free $\mathcal{V}$-enrichment" change of base, whereby ordinary Set-categories are converted to $\mathcal{V}$-categories, denoted $\mathcal{C} \mapsto \tilde{\mathcal{C}}$. These form an adjunction, because 2-functors preserve adjunctions.

This is what we implicitly used in the definition of $\mathcal{V}$-theory: the arity category N "sits inside" many enriching categories under various guises: as finite discrete graphs, categories, posets, etc. For each $\mathcal{V}$ we define the arity subcategory $\mathrm{N}_{\mathcal{V}}$ to be the full subcategory of finite coproducts (copowers) of the unit object, and this remains essentially unchanged by the change-of-base to $\tilde{\mathrm{N}}_{\mathcal{V}}$.

We only need to show that everything is simplified by restricting to this particular $\mathcal{J}$.

6 Simplify with $\mathrm{N}_{\mathcal{V}}$-arities

Most of the enriched algebraic theory literature deals with generalized arities; these will be important in time, but for present applications, we would like the benefits of enrichment with the simplicity of natural number arities. Here we provide some lemmas for this simplification. The idea is that instead of thinking about fancy enriched powers, we are justified in considering ordinary products.

Let $\mathcal{V}$ be a cartesian closed category with finite coproducts of the unit object, and let $\mathrm{N}_{\mathcal{V}}$ be defined as above.

Lemma 1. The functors $[n_{\mathcal{V}}, -] \colon \mathcal{V} \to \mathcal{V}$ and $(-)^n \colon \mathcal{V} \to \mathcal{V}$ are naturally isomorphic, i.e. $n_{\mathcal{V}}$-powers in $\tilde{\mathcal{V}}$ are isomorphic to n-powers (n-fold products) in $\mathcal{V}$.

Proof. If $a, b \in \mathcal{V}$, then

$$\begin{array}{rcll} \mathcal{V}(a, [n_{\mathcal{V}}, b]) & \cong & \mathcal{V}(a \times n_{\mathcal{V}}, b) & \text{hom-tensor adjunction} \\ & = & \mathcal{V}(a \times (\coprod_n I_{\mathcal{V}}), b) & \text{definition of } n_{\mathcal{V}} \\ & \cong & \mathcal{V}(\coprod_n (a \times I_{\mathcal{V}}), b) & \text{distributivity} \\ & \cong & \mathcal{V}(\coprod_n a, b) & \text{unitality} \\ & \cong & \prod_n \mathcal{V}(a, b) & \text{cocontinuity of hom} \\ & \cong & V(a, \prod_n b) & \text{continuity of hom.} \end{array}$$

Each of these isomorphisms is natural in a and b; hence by the Yoneda lemma, $[n_{\mathcal{V}}, -] \cong (-)^n$. □

So, the full sub-$\mathcal{V}$-category $\tilde{\mathrm{N}}_{\mathcal{V}}$ has hom-objects which behave like the exponentiation of N:

$$[n_{\mathcal{V}}, m_{\mathcal{V}}] \cong \prod_n (\coprod_m I_{\mathcal{V}}) \cong (m^n)_{\mathcal{V}}.$$

In $\mathcal{V}$Cat, the objects of the theory $\mathcal{T}$ are $n_{\mathcal{V}}$-powers of a generating object s. Alas, we cannot simply say that " $s^{n_{\mathcal{V}}} \cong \prod_n s$ ", because the latter does not type-check in the $\mathcal{V}$-category $\mathcal{T}$: products are characterized by a Set-enriched universal property. However, we only need:

Lemma 2. Let $\mathcal{T}$ be a $\mathcal{V}$-category with $\mathrm{N}_{\mathcal{V}}$-powers, and let $s \in \mathcal{T}$. Then a hom into $s^{n_{\mathcal{V}}}$ is isomorphic to n homs into s:

$$\mathcal{T}(a, s^{n_{\mathcal{V}}}) \cong [n_{\mathcal{V}}, \mathcal{T}(a, s)] \cong \prod_n \mathcal{T}(a, s)$$

by definition of power, and Lemma 1.

We want to know when the functor $F \colon \mathcal{V} \to \mathcal{W}$ induces a change of base $F_* \colon \mathcal{V}\mathrm{Cat} \to \mathcal{W}\mathrm{Cat}$ which "preserves enriched-theories" — if by F every $\mathcal{V}$-theory $\tau_{\mathcal{V}}$ gives rise to a $\mathcal{W}$-theory $\tau_{\mathcal{W}}$, then F is a *change of semantics*. That is, given a $\mathcal{V}$-theory

$$\tau_{\mathcal{V}} \colon \tilde{\mathrm{N}}_{\mathcal{V}}^{\mathrm{op}} \to \mathcal{T}$$

we want to determine a minimal condition for the base-changed functor

$$F_*(\tau_{\mathcal{V}}) \colon F_*(\tilde{\mathrm{N}}_{\mathcal{V}}^{\mathrm{op}}) \to F_*(\mathcal{T})$$

to induce a $\mathcal{W}$-theory in a canonical way. So, assuming there is a clear identification of $F_*(\tilde{\mathrm{N}}_{\mathcal{V}}^{\mathrm{op}})$ and $\tilde{\mathrm{N}}_{\mathcal{W}}^{\mathrm{op}}$, it suffices to require that $F_*(\tau_{\mathcal{V}})$ preserves $\mathrm{N}_{\mathcal{W}}$-powers.

Because $F_*(-)_{\mathrm{hom}}$ is defined

$$F_*(\mathcal{T})(a, s^{n_{\mathcal{V}}}) = F(\mathcal{T}(a, s^{n_{\mathcal{V}}})),$$

combined with the previous lemmas, the preservation of "$\mathrm{N}_{(-)}$-power preserving functors" by F_* is implied by the preservation of finite products by F: and since our enriching categories and base-change functors are cartesian, this is automatic.

Lemma 3. Let $F\colon \mathcal{V} \to \mathcal{W}$ be a cartesian functor, and let $\mathrm{N}_\mathcal{V}$, $\mathrm{N}_\mathcal{W}$ be defined as above. If $f\colon \mathcal{C} \to \mathcal{D}$ is a $\mathcal{V}$-functor which preserves $\mathrm{N}_\mathcal{V}$-powers, then $F_*(f)\colon F_*(\mathcal{C}) \to F_*(\mathcal{D})$ is a $\mathcal{W}$-functor which preserves $\mathrm{N}_\mathcal{W}$-powers.

Proof.

$$\begin{array}{rcll} F_*(\mathcal{D})(F_*(f)(a), F_*(f)(s^{n_\mathcal{V}})) & = & F(\mathcal{D}(f(a), f(s^{n_\mathcal{V}})) & \text{definition of base change} \\ & \cong & F(\mathcal{D}(f(a), f(s)^{n_\mathcal{V}}) & f \text{ preserves } \mathrm{N}_\mathcal{V}\text{-powers} \\ & \cong & F(\prod_n \mathcal{D}(f(a), f(s))) & \text{Lemma 2 for } \mathcal{V} \\ & \cong & \prod_n F(\mathcal{D}(f(a), f(s))) & F \text{ preserves finite products} \\ & = & \prod_n F_*(\mathcal{D})(f(a), f(s)) & \text{definition of base change} \\ & \cong & F_*(\mathcal{D})(f(a), f(s)^{n_\mathcal{W}}) & \text{Lemma 2 for } \mathcal{W} \end{array}$$

□

Finally, we use $F_*(\tau)$ and the isomorphism $N\colon \mathrm{N}_\mathcal{V} \simeq \mathrm{N}_\mathcal{W}$ — denote by $\tilde{N}\colon \tilde{\mathrm{N}}^{\mathrm{op}}_\mathcal{W} \to F_*(\tilde{\mathrm{N}}^{\mathrm{op}}_\mathcal{V})$ the isomorphism which sends $n_\mathcal{W} \mapsto n_\mathcal{V}$ and is the identity on morphisms - to construct a $\mathcal{W}$-functor which precisely fits the definition of a $\mathcal{W}$-theory:

Theorem 4. Let $\mathcal{V}$, $\mathcal{W}$ be cartesian closed categories with finite coproducts of their unit objects, and let $F\colon \mathcal{V} \to \mathcal{W}$ be a cartesian functor. Then F is a **change of semantics**; i.e. for every $\mathcal{V}$-theory $\tau_\mathcal{V}\colon \tilde{\mathrm{N}}^{\mathrm{op}}_\mathcal{V} \to \mathcal{T}$, the $\mathcal{W}$-functor

$$\tau_\mathcal{W} := F_*(\tau_\mathcal{V}) \circ \tilde{N}\colon \mathrm{N}^{\mathrm{op}}_\mathcal{W} \to F_*(\mathcal{T})$$

is a $\mathcal{W}$-theory. Moreover, F preserves *models*, i.e. for every $\mathrm{N}_\mathcal{V}$-power preserving $\mathcal{V}$-functor $\mu\colon \mathcal{T} \to \mathcal{C}$, the $\mathcal{W}$-functor $F_*(\mu)$ preserves $\mathrm{N}_\mathcal{W}$-powers.

Proof. The $\mathcal{W}$-functor $\tau_\mathcal{W}$ is bijective-on-objects because $\tau_\mathcal{V}$ and $\tilde{N}$ are; and it preserves $\mathrm{N}_\mathcal{W}$-powers because $\tilde{N}$ does and $F_*(\tau_\mathcal{V})$ does by the previous lemma. This preservation is strict because $F_*(\mathcal{T})$ has the same objects as $\mathcal{T}$, so the isomorphism implies that $\tau_\mathcal{W}(I_\mathcal{W}^{n_\mathcal{W}}) = \tau_\mathcal{W}(I_\mathcal{W})^{n_\mathcal{W}}$. The preservation of models follows from the previous lemma. □

Hence, any cartesian functor between cartesian closed categories constitutes a "change of semantics" - this is a simple, ubiquitous condition, which provides for a method of transitioning formal systems between various *modus operandi*.

Moreover, this reasoning generalizes to **multisorted** $\mathcal{V}$-theories, enriched theories which have multiple sorts — given any such cartesian $\mathcal{V}$ and any $n \in \mathrm{N}$, the monoidal subcategory $\prod_n \mathrm{N}_\mathcal{V}$ is also an eleutheric system of arities. At the end of §8, we give an example demonstrating why this is a very useful generalization.

Before exploring applications, we introduce two more useful kinds of transitions, and demonstrate how all of this information be encapsulated in one categorical notion.

7 The category of all $\mathcal{V}$-theories

In addition to change-of-base, there are two other natural and useful transitions for these theories. Let $\mathcal{V}\mathrm{Law}$ be the category of $\mathcal{V}$-theories, and let $f\colon \mathcal{T} \to \mathcal{T}'$ be a morphism of theories; this induces a "change-of-theory" functor between the respective categories of models

$$f^*\colon \mathcal{V}\mathrm{Mod}(\mathcal{T}', \mathcal{C}) \to \mathcal{V}\mathrm{Mod}(\mathcal{T}, \mathcal{C})$$

defined as precomposition by f. Similarly, given a finite-product functor $g\colon \mathcal{C} \to \mathcal{C}'$, this induces a "change-of-model" functor

$$g_*\colon \mathcal{V}\mathrm{Mod}(\mathcal{T}, \mathcal{C}) \to \mathcal{V}\mathrm{Mod}(\mathcal{T}, \mathcal{C}')$$

defined as postcomposition by g.

These transitions, as well as change-of-base, can all be packed up nicely using the **Grothendieck construction**: given a (pseudo)functor $F\colon \mathcal{D} \to \mathrm{Cat}$, there is a fibration $\bar{F}\colon \int F \to \mathcal{D}$ which encapsulates all of the categories in the image of F []: the category $\int F$ consists of

$$\begin{array}{rl} \text{objects} & (d, x) : d \in \mathcal{D},\ x \in F(d) \\ \text{morphisms} & (f\colon d \to d', a\colon F(f)(x) \to x') \\ \text{composition} & (f, a) \circ (f', a') = (f \circ f', a \circ F(f)(a')). \end{array}$$

(Although we noted after Definition 4.1 that $\mathcal{V}\mathrm{Law}$ and $\mathrm{Mod}(\mathcal{T}, \mathcal{C})$ are $\mathcal{V}$-categories when $\mathcal{V}$ is a cosmos, we will focus on the nonenriched case for simplicity and generality.)

This idea allows us to bring together all of the different enrichments, theories, and models into one big category. For every enriching category $\mathcal{V}$, let $\mathcal{V}\mathrm{Cat}_{fp}$ be the subcategory of $\mathcal{V}\mathrm{Cat}$ of $\mathcal{V}$-categories with finite powers and finite-power preserving functors; then there is a functor

$$\mathcal{V}\mathrm{Mod}\colon \mathcal{V}\mathrm{Law}^{\mathrm{op}} \times \mathcal{V}\mathrm{Cat}_{fp} \to \mathrm{Cat}$$

which sends $(\mathcal{T}, \mathcal{C}) \mapsto \mathcal{V}\mathrm{Mod}(\mathcal{T}, \mathcal{C})$. Functoriality characterizes the contravariant change-of-theory and the covariant change-of-model above.

Utilizing the construction, there is a category $\int \mathcal{V}\mathrm{Mod}$ with objects and morphisms

$$((f, g), \alpha)\colon ((\mathcal{T}, \mathcal{C}), \mu) \to ((\mathcal{T}', \mathcal{C}'), \mu')$$

being $\mathcal{V}$-functors $f\colon \mathcal{T} \to \mathcal{T}'$, $g\colon \mathcal{C} \to \mathcal{C}'$, and $\mathcal{V}$-natural transformation $\alpha\colon \mathcal{V}\mathrm{Mod}(f, g)(\mu) \to \mu'$.

Lemma 5. There is a functor $\mathrm{thy}\colon \mathrm{CCC}_{fp} \to \mathrm{Cat}$ which assigns $\mathcal{V} \mapsto \int \mathcal{V}\mathrm{Mod}$ and change-of-semantics $(F\colon \mathcal{V} \to \mathcal{W}) \mapsto (F^*_*\colon \int \mathcal{V}\mathrm{Mod} \to \int \mathcal{W}\mathrm{Mod})$.

Proof. Given $F\colon \mathcal{V} \to \mathcal{W}$, base change $F_*\colon \mathcal{V}\mathrm{Cat} \to \mathcal{W}\mathrm{Cat}$ is a 2-functor, thereby inducing the functor $F^*_*\colon \mathcal{V}\mathrm{Mod} \to \mathcal{W}\mathrm{Mod}$ which sends a morphism $((f, g), \alpha)$ to $((F_*(f), F_*(g)), F_*(\alpha))$. Checking functoriality is left to the reader. □

Thus, we can use the construction once more to encapsulate even the enrichment:

Theorem 6. There is a category $\mathrm{Thy} := \int \mathrm{thy}$ with objects and morphisms

$$(F, ((f, g), \alpha))\colon (\mathcal{V}, ((\mathcal{T}, \mathcal{C}), \mu)) \to (\mathcal{W}, ((\mathcal{T}', \mathcal{C}'), \mu'))$$

being a change-of-semantics F and a morphism $(f, g, \alpha)\colon F^*_*(((\mathcal{T}, \mathcal{C}), \mu)) \to ((\mathcal{T}', \mathcal{C}'), \mu')$ in $\mathcal{W}\mathrm{Mod}$.

This category assimilates a whole lot of useful information. Most importantly, there are morphisms between objects of "different kinds", something we consider often but is normally not possible in category theory. For example, in Thy there is a morphism:

$$(\mathrm{Set}, ((u_{\mathrm{Grp}}, \mathrm{Disc}), e^x))\colon (\mathrm{Set}, ((\mathcal{T}_{\mathrm{Ring}}, \mathrm{Set}), (\mathbb{Z}, \times, +, 1))) \to (\mathrm{Set}, ((\mathcal{T}_{\mathrm{Grp}}, \mathrm{Top}), (\mathbb{R}, \times)))$$

There are many unexplored questions about the large, heterogeneous categories which arise from the Grothendieck construction, regarding what unusual structure may be gained, such as limits and colimits with objects of different types, or identifying "processes" in which the kinds of objects change in an essential way. This is just a remark; for our purposes we need only recognize that enriched Lawvere theories can be assimilated into one category, which provides a unified context for change-of-base, change-of-theory, and change-of-modelling.

8 Applications

In theoretical computer science literature, enriched algebraic theories have primarily been studied in the context of "computational effects". Mike Stay and Greg Meredith have recognized that Lawvere theories can actually be utilized for the design of *programming languages* [20]. This idea comes from caring about an important but underappreciated subject in foundations — *combinatory logic*.

8.1 The SKI-combinator calculus

The λ-calculus is an elegant formal language which is the foundation of functional computation, the model of intuitionistic logic, and the internal logic of cartesian closed categories: this is the Curry-Howard-Lambek correspondence [2].

Terms are constructed recursively by *variables*, *application*, and *abstraction*, and the basic rewrite is *beta reduction*:

$$M, N := x \mid (M\ N) \mid \lambda x.M$$

$$(\lambda x.M\ N) \Rightarrow M[N/x]$$

Despite its simplicity, there are subtle complications regarding *substitution*, or evaluation of functions. Consider the term $M = \lambda x.(\lambda y.(xy))$: if this is applied to the variable y, then $(M\ y) \Rightarrow \lambda y.(y\ y)$ — but this is not intended, because the y in M is just a placeholder, it is "bound" by whatever will be plugged in, while the y being substituted is "free", meaning it can refer to some other value or function in the program. Hence whenever a free variable is to be substituted for a bound variable, we need to rename the bound to prevent "variable capture" (e.g. $(My) \Rightarrow \lambda z.(y\ z)$).

This problem was noticed early in the history of mathematical foundations, even before the λ-calculus, and so Moses Schönfinkel invented **combinatory logic** [16], a basic form of logic without the red tape of variable binding, hence without functions in the usual sense. The SKI-calculus is the "variable-free" representation of the λ-calculus; λ-terms are translated via "abstraction elimination" into strings of combinators and applications. This is an important method for programming languages to minimize the subtleties of variables.

The key insight here is that Lawvere theories are by definition free of variables, and it is precisely through abstraction elimination that a programming language can be made an algebraic object. When representing a computational calculus as an Gph-theory, the general rewrite rules are simply edges in the hom-graphs $t^n \to t$, with the object t serving in place of the variable. Below is the theory of the SKI-combinator calculus:

$\boxed{\mathrm{Th}(SKI)}$

Sorts	t	
Term Constructors	S	$: 1 \to t$
	K	$: 1 \to t$
	I	$: 1 \to t$
	$(- -)$	$: t^2 \to t$
Structural Congruence	n/a	
Rewrites	σ	$: (((S\ x)\ y)\ z) \Rightarrow ((x\ z)\ (y\ z))$
	κ	$: ((K\ y)\ z) \Rightarrow y$
	ι	$: (I\ z) \Rightarrow z$

These denote rewrites for arbitrary subterms x, y, z without any variable binding involved, by using

the cartesian structure of the category. They are simply edges with vertices:

$$\begin{aligned}
(((S\ x)\ y)\ z) &: t^3 \xrightarrow{l^{-1}\times t^3} 1\times t^3 \xrightarrow{S\times t^3} t^4 \xrightarrow{(-\ -)\times t^2} t^3 \xrightarrow{(-\ -)\times t} t^2 \xrightarrow{(-\ -)} t \\
((x\ z)\ (y\ z)) &: t^3 \xrightarrow{t^2\times\Delta} t^4 \xrightarrow{t\times\tau\times t} t^4 \xrightarrow{(-\ -)\times(-\ -)} t^2 \xrightarrow{(-\ -)} t
\end{aligned}$$

$$\begin{aligned}
((K\ y)\ z) &: t^2 \xrightarrow{l^{-1}\times t^2} 1\times t^2 \xrightarrow{K\times t^2} t^3 \xrightarrow{(-\ -)\times t} t^2 \xrightarrow{(-\ -)} t \\
y &: t^2 \xrightarrow{t\times !} t\times 1 \xrightarrow{r} t
\end{aligned}$$

$$\begin{aligned}
(I\ z) &: t \xrightarrow{l^{-1}} 1\times t \xrightarrow{I\times t} t^2 \xrightarrow{(-\ -)} t \\
z &: t \xrightarrow{t} t
\end{aligned}$$

These implicitly universally quantified rules are applied by precomposing with the terms to be "plugged in": using that a morphism $1 \to t^n$ is equivalent to n morphisms $1 \to t$, terms are constructed from constants, then the abstract rewrite rules evaluate on concrete terms (morphisms $1 \to t$ are the "closed" terms, meaning they have no free variables; in general morphisms $t^n \to t$ are terms with n free variables, and the same reasoning applies):

$$\begin{array}{rl}
((KS)I)\colon & 1 \xrightarrow{S\times I} t^{2} \xrightarrow{((K\ y)\ z)} t \\
{\scriptstyle \kappa\circ(S\times I)}\Downarrow & \\
S\colon & 1 \xrightarrow{S\times I} t^2 \xrightarrow{\ y\ } t
\end{array}$$

A model of this theory is a power-preserving Gph-functor $\mu\colon \mathrm{Th}(SKI) \to \mathrm{Gph}$. This gives a graph $\mu(t)$ of all terms and rewrites in the SKI-calculus, which is generated as follows:

$$1 \cong \mu(1) \xrightarrow{\mu(S)} \mu(t) \xleftarrow{\mu((-\ -))} \mu(t^2) \cong \mu(t)^2$$

The images of the nullary operations S, K, I are distinguished vertices of the graph $\mu(t)$, because μ preserves the terminal object which "points out" vertices. The image of the binary operation $(-\ -)$ gives for every pair of vertices $(u, v) \in \mu(t)^2$, through the isomorphism $\mu(t)^2 \cong \mu(t^2)$, a vertex $(u\ v)$ in $\mu(t)$ which is their application.

In this way all possible terms are generated (writing $\mu(S), \mu(K), \mu(I)$ as S, K, I for sanity):

$$((((S\ (K\ (I\ I)))\ S)\dots$$

The rewrites are transferred by the enrichment of the functor: rather than functions between hom-sets, the morphism component of μ consists of graph homomorphisms between hom-graphs. So,

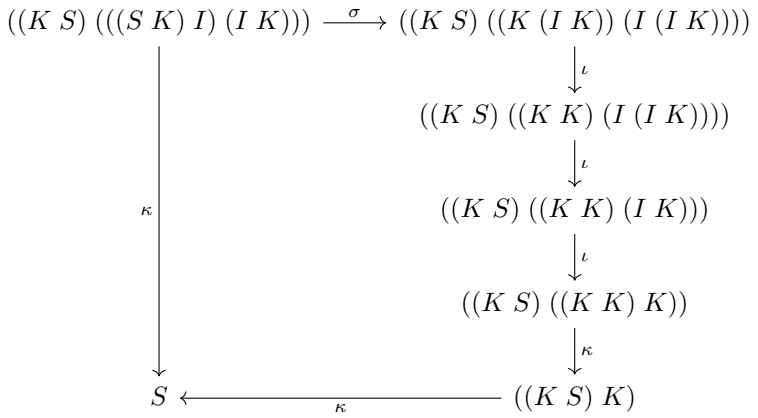
$$\mu_{1,t}\colon \mathrm{Th}(SKI)(1,t) \to \mathrm{Gph}(1, \mu(t))$$

maps the "syntactic" graph of all closed terms and rewrites coherently into the "semantic" graph, meaning a rewrite in the theory $a \Rightarrow b$ is sent to a rewrite in the model $\mu(a) \Rightarrow \mu(b)$, like a functor without composition.

These rewrites in the image of μ are *graph transformations*, like natural transformation without naturality, and this is how the model realizes the Gph-theory as an actual graph of terms and rewrites: in the same way that a transformation between two constant functors $a \Rightarrow b\colon 1 \to \mathcal{C}$ is just a morphism $a(1) \to b(1)$ in $\mathcal{C}$, a rewrite of closed terms $a \Rightarrow b\colon 1 \to \mu(t)$ corresponds to an edge in $\mu(t)$:

$$\mu((I\ S)) \bullet \xrightarrow{\mu(\iota)} \bullet\ \mu(S)$$

Finally, the fact that $\mu((--))$ is not just a function but a graph homomorphism means that pairs of edges (rewrites) $(a \to b, c \to d)$ are sent to rewrites $(a\ b) \to (c\ d)$. This gives the full complexity of the theory: given a large term (program), there are many different ways it can be computed — and some are better than others:



This process is intuitive, but how do we actually define the model, as a functor, to pick out a specific graph? There are many models of $\mathrm{Th}(SKI)$, but in particular we care about the canonical *free* model, which means that $\mu(t)$ is simply the graph of all closed terms and rewrites in the SKI-calculus. This utilizes the enriched adjunction of §4:

$$\mathrm{Gph} \underset{u_{\mathrm{Gph}}}{\overset{f_{\mathrm{Gph}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} [\mathrm{Th}(SKI), \mathrm{Gph}]_{fp}$$

Then the canonical model of closed terms and rewrites is simply the free model on the empty graph, $f_{\mathrm{Gph}}(\emptyset)$, i.e. the $\mathcal{V}$-functor $\mathcal{T}(1,-)\colon \mathcal{T} \to \mathcal{V}$. Hence for us, the syntax and semantics of the SKI combinator calculus, and thus the λ calculus, are unified in the model

$$\mu^{\mathrm{Gph}}_{SKI} := \mathrm{Th}(SKI)(1,-)\colon \mathrm{Th}(SKI) \to \mathrm{Gph}$$

Here we reap the benefits of the abstract construction: the graph $\mu^{\mathrm{Gph}}_{SKI}(t)$ is the *transition system* which represents the **small-step operational semantics** of the SKI-calculus:

$$(\mu(a) \to \mu(b) \in \mu^{\mathrm{Gph}}_{SKI}(t)) \iff (a \Rightarrow b \in \mathrm{Th}(SKI)(1,t))$$

Interestingly, in the free model on a nonempty graph, the vertices represent designated "ground variables", and edges represent rewrites of one variable into another. This is potentially quite useful for "building in" a language with other basic features not intrinsic to the theory.

8.2 Change-of-base

Now we can succinctly characterize the transformation from small-step to **big-step**, which is found throughout the operational semantics literature. The "free category" functor $\mathrm{FC}\colon \mathrm{Gph} \to \mathrm{Cat}$ gives for every graph G the category $\mathrm{FC}(G)$ whose objects are the vertices of G, and whose morphisms are freely generated by the edges of G, i.e. sequences

objects	vertices of G
morphisms	finite sequences of vertices and edges $(v_1, e_1, v_2, e_2, ..., v_n)$
composition	$(v_1, e_1, v_2, e_2, ..., v_n) \circ (v'_1, e'_1, v'_2, e'_2, ..., v'_n) = (v_1, e_1, ..., v_n = v'_1, e'_1, ..., v'_n)$

This functor preserves products, because the definition of graphical product and categorical product are identical except for composition: vertices/objects are pairs of vertices/objects from each component, and same for edges/morphisms; hence the above operation fulfills the preservation isomorphism:

$$\mathrm{FC}(G \times H) \cong \mathrm{FC}(G) \times \mathrm{FC}(H)$$

because they have the same objects, and a morphism of the former is a sequence of pairs, while that of the latter is the corresponding pair of sequences.

Thus FC is the change-of-semantics which induces the transitive closure of the rewrite relation, hence

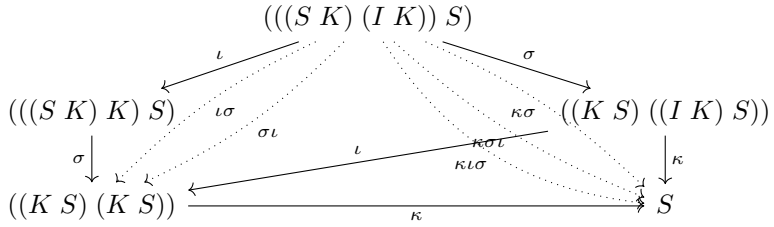
$$\mu^{\mathrm{Cat}}_{SKI} := \mathrm{FC}_*(\mu^{\mathrm{Gph}}_{SKI})$$

is the category which represents the big-step operational semantics of the SKI-calculus.

The same reasoning applies to the "free poset" functor $\mathrm{FP}\colon \mathrm{Cat} \to \mathrm{Pos}$; it is a change-of-semantics because the product of posets is defined in the same way. This induces the lesser-known **full-step semantics**, which collapses hom-sets to subsingletons, simply asserting the existence of a rewrite sequence between terms, without distinguishing between different paths. Since there was no real algebraic information in the free category, this is simply adding the property that all the distinct paths between two terms are equal, while retaining transitivity.

Finally, we can pass to the purely abstract realm where all computation is already complete - the "free set" functor $\mathrm{FS}\colon \mathrm{Pos} \to \mathrm{Set}$ collapses every connected component of the full-step poset to a point, equating every formal expression to its final value (this preserves products because the components of a product is a product of the components). Assuming that the language is *terminating*, meaning every term has a finite sequence of possible rewrites, and *confluent*, meaning every pair of paths which branch from a term eventually rejoin, then this functor gives the **denotational semantics** of the language.

So, from this simple sequence of functors, we can compute the spectrum of semantics for the SKI-calculus. For example, we have the following computation:



The solid arrows are the one-step rewrites of the initial Gph-theory; applying FC_* gives the dotted composites, and FP_* asserts that all composites between any two objects are equal. Finally, FS_* collapses the whole diagram to S, and the calculation is complete. This is a simple demonstration of the basic stages of computation: small-step, big-step, full-step, and denotational semantics.

Of course, most interesting languages are not always terminating, confluent, nor deterministic; the "spectrum" being presented here is simple an initial proof-of-concept. We expect that there are more interesting change-of-base functors which handle these subtleties — they have likely been studied in other contexts.

8.3 Change-of-theory: reduction contexts

We can equip term calculi with *reduction contexts*, which determine when rewrites are valid, thus giving the language a certain **evaluation strategy**. For example, the "weak head normal form" is given by only allowing rewrites on the left-hand side of the term.

We can do this for $\mathrm{Th}(SKI)$ by adding a reduction context marker as a unary operation, and a structural congruence rule which pushes the marker to the left-hand side of an application; lastly we

modify the rewrite rules to be valid only when the marker is present:

$$\boxed{\mathrm{Th}(SKI{+}\mathrm{R})}$$

Sorts	t	
Term Constructors	S, K, I	$: 1 \to t$
	R	$: t \to t$
	$(- \ -)$	$: t^2 \to t$
Structural Congruence	$R(x\ y) = (Rx\ y)$	
Rewrites	σ_r	$: (((RS\ x)\ y)\ z) \Rightarrow ((Rx\ z)\ (y\ z))$
	κ_r	$: ((RK\ y)\ z) \Rightarrow Ry$
	ι_r	$: (RI\ z) \Rightarrow Rz$

The SKI-calculus is thereby equipped with "lazy evaluation", an essential paradigm in modern programming. This represents a broad potential application of equipping theories with computational methods, such as evaluation strategies.

Moreover, these equipments can be added or removed as needed: using change-of-theory, we can utilize a "free reduction" Gph-functor $f_R \colon \mathrm{Th}(SKI) \to \mathrm{Th}(SKI + R)$:

objects	t^n	$\mapsto$	t^n
hom-vertices	S, K, I	$\mapsto$	S, K, I
	$(- \ -)$	$\mapsto$	$R(- \ -)$
hom-edges	σ, κ, ι	$\mapsto$	$\sigma_r, \kappa_r, \iota_r$

This essentially interprets ordinary SKI as having every subterm be a reduction context. This is a Gph-functor because its hom component consists of graph-homomorphisms:

$$f_{n,m} \colon \mathrm{Th}(SKI)(t^n, t^m) \to \mathrm{Th}(SKI + R)(t^n, t^m)$$

which simply send each application to its postcomposition with R, and each rewrite to its "marked" correspondent; and this is all coherent: for example, even though $(((S\ x)\ y)\ z) \mapsto R(R(R(S\ x)\ y)\ z)$, the extra markers are ignored by σ_r, because they are now just a part of the lefthand terms.

So, by precomposition this induces the change of theory on categories of models:

$$f_R^* \colon \mathrm{Mod}(\mathrm{Th}(SKI + R), \mathcal{C}) \to \mathrm{Mod}(\mathrm{Th}(SKI), \mathcal{C})$$

for all semantic categories $\mathcal{C}$, which forgets the reduction contexts.

Similarly, there is a Gph-functor $u_R \colon \mathrm{Th}(SKI + R) \to \mathrm{Th}(SKI)$ which forgets reduction contexts, by sending $\sigma_r, \kappa_r, \iota_r \mapsto \sigma, \kappa, \iota$ and $R \mapsto id_t$; this latter is the only way that the marked reductions can be mapped coherently to the unmarked. However, this means that u_R^* does not give the desired change-of-theory of "freely adjoining contexts", because collapsing R to the identity eliminates the significance of the marker.

This illustrates a key aspect of categorical universal algebra: because change-of-theory is given by precomposition and is thus contravariant, *properties* such as equations can be added or removed, but *structure* can only be removed. This is a necessary limitation, at least in the present setup, but there are ways of working around it. Of course, these abstract theories are not floating in isolation but are implemented in code. One can simply use a "maximal theory" with all pertinent structure, then selectively forget as needed.

8.4 Multisorted: the ρ-calculus

Many algebraic theories involve multiple sorts in an essential way. In concurrency theory, *process calculi* exhibit an ontology which is fundamentally distinct from that of sequential computing — rather than simply expressing a series of terms and rewrites, these calculi represent dynamical systems of communicating processes.

The π-calculus, designed by Milner [13], consists of **names** and **processes**, or *channels* and *agents* which communicate on those channels. Far more than a sequence of instructions on a single machine, computation develops through the interaction of independent participants in a network.

This powerful idea of modern computer science is being utilized by Greg Meredith and Mike Stay to design a deeply cooperative distributive computing system, called RChain. The "R" stands for "reflective higher-order π-calculus", or **ρ-calculus**. It is like Milner's original language, with one crucial difference: "reflection" is a formal system's ability to turn code into data and vice versa.

Utilizing both reflection and combinators in a theory requires special type discipline; there is a designated auxiliary sort "T" for combinatory terms which are analogous to "machine code", as contrasted with the sorts "N" and "P" which are to be thought of as the actual language.

Th(RHO)

Sorts	N		names	T		terms
	P		processes			
Operations	0:	$1 \to P$	null process	S:	$1 \to T$	combinator
	$\&$:	$P \to N$	code to data	K:	$1 \to T$	combinator
	$*$:	$N \to P$	data to code	$(- -)$:	$T^2 \to T$	application
	$!$:	$N \times P \to P$	send			
	$?$:	$N^2 \times P \to P$	receive			
	$- \mid -$:	$P^2 \to P$	parallel			
Equations		$(P, \mid, 0)$	commutative monoid			
Rewrites	γ:	$x?(y).P \mid x!(z) \mid Q \Rightarrow$	$P[z/y] \mid Q$	σ:	$(((S\ a)\ b)\ c) \Rightarrow$	$((a\ c)\ (b\ c))$
	ϵ:	$*(\&(P)) \Rightarrow$	P	κ:	$((K\ a)\ b) \Rightarrow$	a

9 Conclusion

We have established the basics of how enriched Lawvere theories provide a framework for unifying the syntax and semantics, the structure and behavior, the design and study of formal languages. Enriching theories in category-like structures reifies the operational semantics by representing rewrites between terms; and product-preserving functors between enriching categories induce change-of-semantics functors between categories of models — this simplified condition is obtained by using only finite cardinal arities.

This base-change, along with change-of-theory and change-of-modelling, can be assimilated into one category using an iterated Grothendieck construction Thy, which consists of all enriched Lawvere theories. Finally, enriched theories can be used not only for computational effects but the actual design of concrete programming languages, through the use of combinators.

For future work, we plan to build upon these ideas with development of more interesting and useful implementations of change-of-base and change-of-theory, illustrated with other languages and combinators. This approach will be connected with the co-algebra approach to operational semantics, and pertinent concepts such as bisimulation will be comprehended from this perspective.

Also, we will begin research on *distributive laws* for enriched Lawvere theories, as recently developed with Set-enrichment by Eugenia Cheng [4]. We aim to determine necessary and sufficient conditions for the existence of a distributive law between any two enriched Lawvere theories, as well as its construction; this will be utilized in the "logic as a distributive law" algorithm, developed by Greg Meredith and Mike Stay to generate sound type systems and query languages for distributed computing systems. We thank them again for providing us the opportunity and support to develop this research.

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