Comprehending types

L. G. Meredith
M. Stay

May 24, 2018

Abstract

Abstract.

1 Comprehension signatures

A **comprehension signature** Σ has a collection of sorts; the collection of types is generated inductively:

- There is a **unit type**, denoted by I.
- Every sort is a type.
- Given two types A and B, there is a type $A \otimes B$.
- Given a type A, there is a **power type** PA.

The comprehension signature Σ also has a collection of **function symbols.** To each function symbol is assigned a function signature $A \to B$, where A and B are types.

Terms are generated recursively:

- Every variable x of type A is a term of type A whose only free variable is x.
- If f is a function symbol of signature $A \to B$ and t is a term of type A, then $f(t)$ is a term of type B with the same free variables as t.
- If s is a term of type A and t is a term of type B, then $s \otimes t$ is a term of type $A \otimes B$ with free variables $\mathrm{FV}(s) \sqcup \mathrm{FV}(t)$.
- If s is a term of type $A \otimes B$, t is a variable of type A not appearing in s, u is a variable of type B not appearing in s, and v is a term of type C in which t and u are free, then

  $$\text{let } t \otimes u = s \text{ in } v$$

  is a term of type C with free variables $\mathrm{FV}(s) \sqcup (\mathrm{FV}(v) - \{t, u\})$.

- If
  - $x_1, \ldots, x_n$ are variables of type $A_1 \times \cdots \times A_n$, respectively,
  - $t_1 \ldots, t_n$ are terms of type $PA_1, \ldots, PA_n$, respectively,
  - t_{n+1} is a term of type B,
  - each $\mathrm{FV}(t_i)$ is disjoint,
  - $\mathrm{FV}(t_i)$ does not include any x_j such that $j \geq i$, and
  - $\forall i, x_i \in \bigcup_{j=1}^{n+1} \mathrm{FV}(t_j)$,

  then
  $$[t_{n+1} \mid x_1 \leftarrow t_1 \ ; \ \cdots \ ; \ x_n \leftarrow t_n]$$
  is a term of type PB with free variables $\bigcup_{i=1}^{n+1} FV(t_i) - \bigcup_{i=1}^{n}\{x_i\}$.

A **suitable context** for a term is an ordering of the set of free variables of that term, *i.e.* a list containing each of the free variables with no repetitions.

2 Categorical semantics

An interpretation M of a comprehension signature Σ in a symmetric monoidal category V equipped with a strong monad $(P\colon V \to V, \eta\colon 1 \Rightarrow P, \mu\colon P \circ P \Rightarrow P, \sigma\colon 1 \otimes P \Rightarrow P)$ consists of the following, subject to the conditions on interpretations below:

- MI is the monoidal unit object I in V.
- For each sort A in Σ, a chosen object MA in V.
- If MA and MB have been defined, then $M(A \otimes B)$ is $MA \otimes MB$ in V.
- If MA has been defined, then $M(PA)$ is defined to be $P(MA)$ in V.
- For each function symbol f with function signature $A \to B$, a chosen morphism $Mf\colon MA \to MB$ in V.

Given an interpretation M, the interpretation of a term t of type B in a suitable context $\vec{y}$ is a morphism $[\![\vec{y}.t]\!]_M\colon MA_1 \otimes \cdots \otimes MA_n \to MB$, where $A_1, \ldots, A_n$ are the types of $y_1, \ldots, y_n$, respectively. We define these conditions on interpretations by recursion:

- If x is a variable of type A, then $[\![x.x]\!]_M$ is the identity morphism on MA.
- Given a term t of type A and a function symbol f with function signature $A \to B$, if $[\![\vec{y}.t]\!]_M$ has been defined, then $[\![\vec{y}.f(t)]\!]_M$ is defined to be the composite $Mf \circ [\![\vec{y}.t]\!]_M$.
- If $[\![\vec{y_1}.s]\!]_M$ and $[\![\vec{y_2}.t]\!]_M$ have been defined and $\vec{y_1}$ and $\vec{y_2}$ are disjoint, then $[\![\vec{y}.s \otimes t]\!]_M$ is defined to be $[\![\vec{y_1}.s]\!]_M \otimes [\![\vec{y_2}.t]\!]_M$, where $\vec{y} = \vec{y_1} + \vec{y_2}$ is the concatenation of $\vec{y_1}$ and $\vec{y_2}$.

- If s is a term of type $A \otimes B$, $[\![\vec{y_1}.s]\!]_M$ and $[\![\vec{y_2}.v]\!]_M$ have been defined, t is a variable of type A, u is a variable of type B, $\vec{y_2} = [t, u] + \vec{y_3}$ for some $\vec{y_3}$, and $\vec{y_1}$ and $\vec{y_2}$ are disjoint, then

$$[\![\vec{y_1} + \vec{y_3}.\text{let } t \otimes u = s \text{ in } v]\!]_M$$

is defined to be the composite $[\![\vec{y_2}.v]\!]_M \circ ([\![\vec{y_1}.s]\!]_M \otimes M\vec{y_3})$.

- If
  - $\vec{y} = [y_1, \ldots, y_m]$ is a context and $A_1, \ldots, A_m$ are the types of $y_1, \ldots, y_n$, respectively,
  - $\vec{x} = [x_1, \ldots, x_n]$ is a context and $B_1, \ldots, B_n$ are the types of $x_1, \ldots, x_n$, respectively,
  - t is a term of type PC, and
  - $1 \leq i \leq n$ is such that $[y_1, \ldots, y_m, x_1, \ldots, x_i]$ is a suitable context for t,

  then let $g_{\vec{y},\vec{x},i,t}$ be the composite

$$\mu_{A_1 \times \cdots \times A_n \times B_1 \times \cdots \times B_i} \circ P\left(\sigma_{A_1 \times \cdots \times A_n \times B_1 \times \cdots \times B_i, C} \circ (A_1 \times \cdots \times A_n \times B_1 \times \cdots \times B_i \times [\![\vec{y}.t]\!]_M) \circ \Delta_{A_1 \times \cdots \times A_n \times B_1 \times \cdots \times B_i}\right).$$

  If
  - $\vec{y} = [y_1, \ldots, y_m]$ is a context and $A_1, \ldots, A_m$ are the types of $y_1, \ldots, y_n$, respectively,
  - $\vec{x} = [x_1, \ldots, x_n]$ is a context and $B_1, \ldots, B_n$ are the types of $x_1, \ldots, x_n$, respectively,
  - t is a term of type C,
  - ϕ is a formula,
  - $\vec{y} + \vec{x}$ is a suitable context for t, and
  - $t_1 \ldots, t_n$ are terms of type $PB_1, \ldots, PB_n$, respectively, where $\mathrm{FV}(t_i)$ may include any variable in $\vec{y}$ and any x_j such that $j < i$,

  then $[\![\vec{y}.[t \mid x_1 \leftarrow t_1 \; ; \; \cdots \; ; \; x_n \leftarrow t_n \text{ if } \phi]]\!]_M$ is defined to be the composite

$$M[\![\vec{y}.t]\!]_M \circ g_{\vec{y},\vec{x},n,\phi} \circ \left(\bigcirc_{i=1}^{n} g_{\vec{y},\vec{x},i-1,t_i}\right) \circ \eta_{\vec{y}} \colon MA_1 \times \cdots MA_m \to PC$$

  in V.

- If R is a relation symbol of type B and t is a term of type B, then $[\![\vec{y}.R(t)]\!]_M$ is defined to be the composite $MR \circ [\![\vec{y}.t]\!]_M$ in V.

- If z is a variable of type A, t is a term of type PA, ϕ is a formula, and $\vec{y}, z$ is a suitable context for ϕ and t, then $[\![\vec{y}.\exists z \leftarrow A.\phi]\!]_M$ is defined to be the morphism $[\![\vec{y}.[*|z \leftarrow t \text{ if } \phi]]\!]_M$ in V.

An interpretation M **satisfies** a formula θ with truth value $t \colon 1 \to P1$ if for some suitable context $\vec{y}$, the morphism $[\![\vec{y}.\theta]\!]_M$ factors through t. We write this symbolically as $M \models^{t}_{\vec{y}} \theta$.

3 Equality, inhabitation, and logical connectives

Notably absent from a comprehension signature is any notion of equality, inhabitation, or logical connectives. These concepts all depend on the specifics of the monad, and give us an opportunity to consider broader notions of truth. For instance, when $V = \mathrm{Set}$ and P is the list (or free monoid) monad, $P1$ is isomorphic to the natural numbers $\mathbb{N}$. In this situation, there are infinitely many ways to be "true", each more true than the last, so there's no clear answer to the question "How true is the statement $x = x$?" Neither is there a unique way for an item x to be an inhabitant of the list $[x, y, x]$; we could have $x \in [x, y, x]$ be 2, because it occurs twice; or 3 because it last appears at position 3 in the list; or 1, emulating those programming languages like C that use 1 to mean true and 0 to mean false. Logical connectives have similar problems: $\wedge$ and $\vee$ could just as easily be min and max as bitwise operators. When $V = \mathrm{Set}$ and P is the monad

In the case where P is finitary, we can take the operations of the corresponding Lawvere theory as primitive. For instance, the list monad has a distinguished element $[]$, so we have a distinguished term of type PA for any type A. We can ask whether an interpretation satisfies a formula with truth value $[]$. We can also concatenate lists: given any two terms t, u of type PA for any A, we can form the term $t \otimes u$ of type PA.

4 Comprehensional Mitchell–Bénabou Language

A cartesian category V equipped with a strong monad (P, η, μ, σ) gives rise to a comprehension signature in a fairly straightforward way. The basic sorts of the signature are the objects of V, the function symbols are the morphisms of V, the constant symbols are the morphisms out of 1, and the relation symbols are the morphisms into $P1$. We call this the **comprehensional Mitchell–Bénabou Language** of $(V, P, \eta, \mu, \sigma)$, and it has a tautological interpretation in $(V, P, \eta, \mu, \sigma)$.

5 Kripke–Joyal semantics

???