Given a symmetric, Cartesian, monoidal category $(\mathcal{C}, \circ, id, \times, \mathtt{1}, \pi_1, \pi_2, \top, \sigma)$ with a strong monad (P, μ, η, τ) the natural transformations have the following types:

$$id : a \to a$$

$$\pi_1 : a_1 \times a_2 \to a_1$$

$$\pi_2 : a_1 \times a_2 \to a_2$$

$$\top : a \to 1$$

$$\sigma : a_1 \times a_2 \to a_2 \times a_1$$

$$\mu : P\left(P\left(a\right)\right) \to P\left(a\right)$$

$$\eta : a \to P\left(a\right)$$

$$\tau : a_1 \times P\left(a_2\right) \to P\left(a_1 \times a_2\right)$$

$$\lambda : 1 \times a \to a$$

$$\rho : a \times 1 \to a$$

By the universal property of $\times$ we may define a unique morphism $\Delta : a \to a \times a$ such that the following diagram commutes.

$$\begin{array}{ccccc} & & a & & \\ & {}^{id}\swarrow & \downarrow\Delta & \searrow^{id} & \\ a & \underset{\pi_1}{\longleftarrow} & a \times a & \underset{\pi_2}{\longrightarrow} & a \end{array}$$

For a morphism $f : a \to P\left(b\right)$ define,

$$\bar{f} : a \to P\left(a \times b\right)$$

$$\bar{f} = \tau \circ \left(id \times f\right) \circ \Delta$$

For a morphism $f : a \to P\left(1\right)$ define,

$$\tilde{f} : a \to P\left(a\right)$$

$$\tilde{f} = P\left(\lambda\right) \circ \bar{f}$$

Now we can define a "calculus of comprehensions" in the Kleisli category of P as follows:

- Given $t : a \to b$,
$$[t] \coloneqq \eta \circ t : a \to P\left(b\right)$$
- Given $t : a \times b \to c$ and $u : a \to P\left(b\right)$,
$$[t | b \leftarrow u] \coloneqq P\left(t\right) \circ \bar{u} : a \to P\left(c\right)$$
- Given $t : a \to c$ and $u : a \to P\left(1\right)$,
$$[t |\text{ if } u] = P\left(t\right) \circ \tilde{u}$$
- Given $t : a_0 \times \cdots \times a_n \to b$ and $u_i : a_0 \times \cdots \times a_{i-1} \to P\left(a_i\right)$ for $i = 1, \ldots, n$,
$$[t | a_1 \leftarrow u_1, \ldots, a_n \leftarrow u_n] \\ \coloneqq \\ \mu \circ \left[[t | a_2 \leftarrow u_2, \ldots, a_n \leftarrow u_n] \,|a_1 \leftarrow u_1\right] : a_0 \times \cdots \times a_n \to P\left(b\right)$$
- Given $u : a \to P\left(b\right)$,
$$[b | b \leftarrow u] \coloneqq [id | b \leftarrow u] : a \to P\left(b\right)$$

- Given $u : a \to P\,(b_1 \times b_2)$,

$$[b_1 | b_1 \times b_2 \leftarrow u] \coloneqq [\pi_1 | b_1 \times b_2 \leftarrow u] : a \to P\,(b_1)$$
$$[b_2 | b_1 \times b_2 \leftarrow u] \coloneqq [\pi_2 | b_1 \times b_2 \leftarrow u] : a \to P\,(b_2)$$
$$[b_2 \times b_1 | b_1 \times b_2 \leftarrow u] \coloneqq [\sigma | b_1 \times b_2 \leftarrow u] : a \to P\,(b_2 \times b_1)$$

- Given $t : c \to d$ and $u : a \to P\,(b)$,

$$[t \times b | b \leftarrow u] \coloneqq \tau \circ (t \times u) : c \times a \to P\,(d \times b)$$
$$[b \times t | b \leftarrow u] \coloneqq P\,(\sigma) \circ \tau \circ (t \times u) \circ \sigma : a \times c \to P\,(b \times d)$$

- Given $t : a \to b$ and $\varphi : a \to P\,(1)$,

$$[t | \text{if } \varphi] \coloneqq P\,(\lambda) \circ \tau \circ (\varphi \times t) \circ \Delta : a \to P\,(b)$$

Conjecture. *Extending to co-Cartesian categories will extend the comprehension calculus by pattern matching on sum types, much like* π_1,π_2 *perform pattern matching on product types.*