

Given a symmetric, Cartesian, monoidal category $(\mathcal{C}, \circ, id, \times, 1, \pi_1, \pi_2, \top, \sigma)$ with a strong monad (P, μ, η, τ) the natural transformations have the following types:

$$\begin{aligned}
id &: a \rightarrow a \\
\pi_1 &: a_1 \times a_2 \rightarrow a_1 \\
\pi_2 &: a_1 \times a_2 \rightarrow a_2 \\
\top &: a \rightarrow 1 \\
\sigma &: a_1 \times a_2 \rightarrow a_2 \times a_1 \\
\mu &: P(P(a)) \rightarrow P(a) \\
\eta &: a \rightarrow P(a) \\
\tau &: a_1 \times P(a_2) \rightarrow P(a_1 \times a_2) \\
\lambda &: 1 \times a \rightarrow a \\
\rho &: a \times 1 \rightarrow a
\end{aligned}$$

By the universal property of \times we may define a unique morphism $\Delta : a \rightarrow a \times a$ such that the following diagram commutes.

$$\begin{array}{ccccc}
& & a & & \\
& id \swarrow & \downarrow \Delta & \searrow id & \\
a & \xleftarrow{\pi_1} & a \times a & \xrightarrow{\pi_2} & a
\end{array}$$

For a morphism $f : a \rightarrow P(b)$ define,

$$\begin{aligned}
\bar{f} &: a \rightarrow P(a \times b) \\
\bar{f} &= \tau \circ (id \times f) \circ \Delta
\end{aligned}$$

For a morphism $f : a \rightarrow P(1)$ define,

$$\begin{aligned}
\tilde{f} &: a \rightarrow P(a) \\
\tilde{f} &= P(\rho) \circ \bar{f}
\end{aligned}$$

Now we can define a “calculus of comprehensions” in the Kleisli category of P as follows:

- Given $t : a \rightarrow b$,

$$[t] := \eta \circ t : a \rightarrow P(b)$$

- Given $t : a \times b \rightarrow c$ and $u : a \rightarrow P(b)$,

$$[t|b \leftarrow u] := P(t) \circ \bar{u} : a \rightarrow P(c)$$

- Given $t : a \rightarrow c$ and $u : a \rightarrow P(1)$,

$$[t| \text{ if } u] = P(t) \circ \tilde{u}$$

- Given $t : a_0 \times \cdots \times a_n \rightarrow b$ and $u_i : a_0 \times \cdots \times a_{i-1} \rightarrow P(a_i)$ for $i = 1, \dots, n$,

$$\begin{aligned}
&[t|a_1 \leftarrow u_1, \dots, a_n \leftarrow u_n] \\
&:= \\
&\mu \circ [[t|a_2 \leftarrow u_2, \dots, a_n \leftarrow u_n] | a_1 \leftarrow u_1] : a_0 \times \cdots \times a_n \rightarrow P(b)
\end{aligned}$$

- Given $u : a \rightarrow P(b)$,

$$[b|b \leftarrow u] := [id|b \leftarrow u] : a \rightarrow P(b)$$

- Given $u : a \rightarrow P(b_1 \times b_2)$,

$$[b_1|b_1 \times b_2 \leftarrow u] := [\pi_1|b_1 \times b_2 \leftarrow u] : a \rightarrow P(b_1)$$

$$[b_2|b_1 \times b_2 \leftarrow u] := [\pi_2|b_1 \times b_2 \leftarrow u] : a \rightarrow P(b_2)$$

$$[b_2 \times b_1|b_1 \times b_2 \leftarrow u] := [\sigma|b_1 \times b_2 \leftarrow u] : a \rightarrow P(b_2 \times b_1)$$
- Given $t : c \rightarrow d$ and $u : a \rightarrow P(b)$,

$$[t \times b|b \leftarrow u] := \tau \circ (t \times u) : c \times a \rightarrow P(d \times b)$$

$$[b \times t|b \leftarrow u] := P(\sigma) \circ \tau \circ (t \times u) \circ \sigma : a \times c \rightarrow P(b \times d)$$
- Given $t : a \rightarrow b$ and $\varphi : a \rightarrow P(1)$,

$$[t|\text{if } \varphi] := P(\lambda) \circ \tau \circ (\varphi \times t) \circ \Delta : a \rightarrow P(b)$$

Conjecture. *Extending to co-Cartesian categories will extend the comprehension calculus by pattern matching on sum types, much like π_1, π_2 perform pattern matching on product types.*