

Given a symmetric, Cartesian, monoidal category $(\mathcal{C}, \circ, id, \times, 1, \pi_1, \pi_2, \top, \sigma)$ with a strong monad (P, μ, η, τ) the natural transformations have the following types:

$$\begin{aligned}
id &: a \rightarrow a \\
\pi_1 &: a_1 \times a_2 \rightarrow a_1 \\
\pi_2 &: a_1 \times a_2 \rightarrow a_2 \\
\top &: a \rightarrow 1 \\
\sigma &: a_1 \times a_2 \rightarrow a_2 \times a_1 \\
\mu &: P(P(a)) \rightarrow P(a) \\
\eta &: a \rightarrow P(a) \\
\tau &: a_1 \times P(a_2) \rightarrow P(a_1 \times a_2)
\end{aligned}$$

Define *comprehensions*, morphisms $[\dots] : a \rightarrow P(b)$, as

- Given $t : a \rightarrow b$,

$$[t] := \eta \circ t : a \rightarrow P(b)$$

- Given $t : b \rightarrow c$ and $u : a \rightarrow P(b)$,

$$[t|b \leftarrow u] := P(t) \circ u : a \rightarrow P(c)$$

- Given $t : a_n \rightarrow b$ and $u_i : a_{i-1} \rightarrow P(a_i)$ for $i = 1, \dots, n$,

$$\begin{aligned}
&[t|a_1 \leftarrow u_1, \dots, a_n \leftarrow u_n] \\
&\quad := \\
&\mu \circ [[t|a_2 \leftarrow u_2, \dots, a_n \leftarrow u_n] | a_1 \leftarrow u_1] : a_0 \rightarrow P(b)
\end{aligned}$$

- Given $u : a \rightarrow P(b)$

$$[b|b \leftarrow u] := u : a \rightarrow P(b)$$

- Given $u : a \rightarrow P(b_1 \times b_2)$,

$$[b_1|b_1 \times b_2 \leftarrow u] := P(\pi_1) \circ u : a \rightarrow P(b_1)$$

$$[b_2|b_1 \times b_2 \leftarrow u] := P(\pi_2) \circ u : a \rightarrow P(b_2)$$

$$[b_2 \times b_1|b_1 \times b_2 \leftarrow u] := P(\sigma) \circ u : a \rightarrow P(b_2 \times b_1)$$

- Given $t : c \rightarrow d$ and $u : a \rightarrow P(b)$,

$$[t \times b|b \leftarrow u] := \tau \circ (t \times u) : c \times a \rightarrow P(d \times b)$$

$$[b \times t|b \leftarrow u] := P(\sigma) \circ \tau \circ (t \times u) \circ \sigma : a \times c \rightarrow P(b \times d)$$

Conjecture. *Extending to co-Cartesian categories will extend the comprehension calculus by pattern matching on sum types, much like π_1, π_2 perform pattern matching on product types.*