Preliminary Definitions

Definition 1. A *monad* over a category $\mathcal{C}$ is an endofunctor $P : \mathcal{C} \to \mathcal{C}$ equipped with

- a natural transformation $\eta_A : A \to PA$ (*unit*)
- a natural transformation $\mu_A : PPA \to PA$ (*multiplication*)
- **Diagram**

Definition 2. The *Kleisli category* of a monad (P, μ, η) over a category $\mathcal{C}$ is a category $\mathcal{C}_P$ where

- $Obj(\mathcal{C}) = Obj(\mathcal{C}_\mathcal{P})$
- $Hom_{\mathcal{C}_P}(A, B) = Hom_\mathcal{C}(A, PB)$
- $1_A = \eta_A$
- $g \circ f = \mu_C \circ Tg \circ f : A \to PC$

Definition 3. A *monoidal category* (**mc**) is a category $\mathcal{C}$ equipped with

- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (*tensor product*)
- an object I (*unit object*)
- a natural isomorphism $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ (*associator*)
- a natural isomorphism $\lambda_A : I \otimes A \cong A$ (*left unitor*)
- a natural isomorphism $\rho_A : A \otimes I \cong A$ (*right unitor*)
- **Diagram**

Definition 4. A *symmetric monoidal category* (**smc**) is a monoidal category $(\mathcal{C}, \otimes, I)$ equipped with

- a natural isomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$ (*swap*)
- **Diagram**

Definition 5. A *strong monad* over a monoidal category $(\mathcal{C}, \otimes, I)$ is a monad (P, μ, η) equipped with

- a natural transformation $\tau_{A,B} : A \otimes P(B) \to P(A \otimes B)$ (*strength*)
- **Diagram**

Definition 6. A *cartesian monoidal category* (**cmc**) is a monoidal category $(\mathcal{C}, \otimes, I)$ in which$\otimes$ is the categorical product and I is the terminal object in $\mathcal{C}$. Every object A in $\mathcal{C}$ is equipped with

- a natural transformation $\Delta_A : A \to A \otimes A$ (*duplication*)
- a natural transformation $\delta_A : A \to I$ (*deletion*)
- **Diagram**

that determine a *unique* comonoid on A.

General Comprehensions

Definition 7. A *context* is a finite list $\overrightarrow{x} = x_1x_2...x_n$ of distinct variables, where $n = 0$ is the empty context []. If $\overrightarrow{x}$ is a context and $y \neq x_i$ for all $i \leq n$, then $\overrightarrow{x}y$ denotes the context obtained by appending y to $\overrightarrow{x}$. Similarly $\overrightarrow{x}\overrightarrow{y}$ denotes the context obtained by concatenating $\overrightarrow{x}$ and $\overrightarrow{y}$ ($x_i \neq y_j$ for all x_i in $\overrightarrow{x}$ and y_j in $\overrightarrow{y}$). The *type* of a context $\overrightarrow{x}$ is the string of the types of the variables appearing in it.

In the following definitions, $\mathcal{C}$ is assumed to be a *symmetric cartesian monoidal category* (scmc) $(\mathcal{C}, \times, 1, \sigma)$ with a *strong monad* (P, μ, η, τ). We use the equivalence $MA = M(A_1 \times A_2 \times \ldots \times A_n)$ to denote contexts and $[t \mid x_1 \leftarrow u_1, x_2 \leftarrow u_2, \ldots, x_n \leftarrow u_n] = [t \mid u_1, u_2, ..., u_n]$ to denote comprehensions.

Definition 8. Let$[\![\overrightarrow{x}.u]\!]_M : MA \to P(MB)$ be a morphism in $\mathcal{C}$. The *context sensitive* morphism $[\![\overrightarrow{x}.\bar{u}]\!]_M$ of $[\![\overrightarrow{x}.u]\!]_M$ is the morphism

$$MA \xrightarrow{\Delta_{MA}} MA \times MA \xrightarrow{1_{MA} \times [\![\overrightarrow{x}.u]\!]} MA \times P(MB) \xrightarrow{\tau_{MA,P(MB)}} P(MA \times MB)$$

We can think of a context sensitive morphism as information preserving; the source of the morphism is present in the target.

Definition 9. Let M be a Σ-structure in $\mathcal{C}$. If $\overrightarrow{x}.t$ is a term-in-context over Σ and t is of type PD, then the morphism

$$[\![\overrightarrow{x}.t]\!]_M : MA \to P(MD)$$

is defined inductively by the following clauses:

(1) If $t = [u]$ is of type PD and u is of type D

$$MA \xrightarrow{[\![\overrightarrow{x}.u]\!]_M} MD \xrightarrow{\eta_D} P(MD)$$

(2) If $t = [u \mid v]$ is of type PD, u is of type D, and v is of type PC

$$MA \xrightarrow{[\![\overrightarrow{x}.\bar{v}]\!]_M} P(MA \times MC) \xrightarrow{P([\![\overrightarrow{x}y.u]\!]_M)} P(MD)$$

(3) If $t = [u \mid v, w]$ is of type PD, u is of type D, v is of type PB, and w is of type PC

$$MA \xrightarrow{[\![\overrightarrow{x}.\bar{v}]\!]} P(MA{\times}MB) \xrightarrow{P([\![\overrightarrow{x}y.\bar{w}]\!]_M)} PP(MA{\times}MB{\times}MC) \xrightarrow{P(P([\![\overrightarrow{x}yz.u]\!]_M))} PP(MA{\times}MB{\times}MC{\times}MD)$$
$$\xrightarrow{\mu_{MA\times MB\times MC\times MD}} P(MA \times MB \times MC \times MD$$

Note that (2) and (3) together are sufficient to define comprehensions with non-negative finite qualifiers. The derivation of the composition in (3) is given below.

- $$\begin{aligned} [\![\overrightarrow{x}.t]\!]_M &= [\![\overrightarrow{x}.[u \mid v, w]]\!]_M \\ &= \mu_D \circ [\![[[u \mid w] \mid v]]\!]_M \\ &= \mu_D \circ P([\![\overrightarrow{x}y.[u \mid w]]\!]_M) \circ [\![\overrightarrow{x}.\bar{v}]\!] \\ &= \mu_D \circ P(P([\![\overrightarrow{x}yz.u]\!]_M) \circ [\![\overrightarrow{x}y.\bar{w}]\!]_M) \circ [\![\overrightarrow{x}.\bar{v}]\!] \\ &= \mu_D \circ P(P([\![\overrightarrow{x}yz.u]\!]_M)) \circ P([\![\overrightarrow{x}y.\bar{w}]\!]_M) \circ [\![\overrightarrow{x}.\bar{v}]\!] \end{aligned}$$

Definition 10. Let$[\![\overrightarrow{x}.u]\!]_M : MA \to P(MB)$ and $[\![\overrightarrow{y}.v]\!]_M : MB \to P(MC)$ be morphisms in $\mathcal{C}$. The *Kleisli composition* $[\![\overrightarrow{x}.u]\!]_M \rightsquigarrow [\![\overrightarrow{y}.v]\!]_M$ of $[\![\overrightarrow{x}.u]\!]_M$ and $[\![\overrightarrow{y}.v]\!]_M$ is the morphism

$$MA \xrightarrow{[\![\overrightarrow{x}.u]\!]_M} P(MB) \xrightarrow{P([\![\overrightarrow{y}.v]\!]_M)} PP(MC) \xrightarrow{\mu_{MC}} P(MC)$$