# Lecture 9. Linear Algebra II

Bowen Chen

Kansas State University

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# Cramer's rule: an application of determinant

Let A be a nonsingular matrix. Then the unique solution of $AX = B$ is

$$x_i = \frac{detB_i}{detA}, \text{for} i = 1, \cdot, n,$$

where $B_i$ is the matrix A with the right-hand side B replacing the $i$th column of A.

# Linear dependence

1. Let $a_1$, $a_2$, ..., $a_n$ be a set of $n$-vectors. We say that they are linearly dependent if the relationship below holds,

$$c_1 a_1 + c_2 a_2 + \cdots + c_n a_n = 0.$$

where $c_i$ are constants, and not all of them are zero. Otherwise, they are linearly independent.

2. If the vectors are linearly dependent, any one of them can be written as a linear combination of others.

3. The vectors are linearly dependent, if $det(A) \neq 0$.

4. The above statement is equal to the statement that $AX = 0$ has an unique solution.

# Functional dependence

Given a system of three equations,

$$y_1 = f_1(x_1, x_2, x_3)$$

$$y_2 = f_2(x_1, x_2, x_3)$$

$$y_3 = f_3(x_1, x_2, x_3)$$

The Jacobian matrix is

$$J = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1 \partial x_2 & \partial y_1 \partial x_3 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \partial y_2/\partial x_3 \\ \partial y_3/\partial x_1 & \partial y_3/\partial x_2 & \partial y_3/\partial x_3 \end{bmatrix}$$

The equations are functionally dependent if $|J| = 0$; the equations are functionally independent if $|J| \neq 0$

# The Hessian matrix

Given a function $z = f(x, y)$ with two independent variables, The Hessian matrix composes of all the second-order partial derivatives, i.e.,

$$H = \begin{bmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{bmatrix}$$

Recall that the sufficient condition for optimum is:
(1) The optimum is a minimum if the Hessian matrix is positive definite;
(2) The optimum is a maximum if the Hessian matrix is negative definite.

*In constrained maximization problem, we will need bordered Hessian matrix for testing on the maximum/minimum

# Matrix definiteness

Consider the Hessian matrix,

$$H = \begin{bmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{bmatrix}$$

When $|H_1| = z_{xx} > 0$ and the second principal minor $|H_2| > 0$, the Hessian matrix is positive definite (or $D_k > 0$, minimum);
When $|H_1| = z_{xx} < 0$ and the second principal minor $|H_2| > 0$, the Hessian matrix is negative definite (or $(-1)^k D_k > 0$, maximum).

# Matrix definiteness

1. Now we introduce another way of testing the matrix definiteness instead of using the principle minors, i.e, the characteristic roots of a matrix.

2. Given a square matrix A, if it is possible to find a vector $V \neq 0$ and a scalar $c$ such that

$$AV = cV.$$

the scalar $c$ is called the characteristic root or eigenvalue; and the vector is called the characteristic vector or eigenvector.

# Eigenvalues and sign definiteness

1. If all eigenvalues are positive, A is positive definite;

2. If all eigenvalues are negative, A is negative definite;

3. If all eigenvalues are nonnegative and at least one equals to zero, A is positive semidefinite;

4. If all eigenvalues are nonpositive and at least one equals to zero, A is negative semidefinite;

5. If some eigenvalues are positive and others negative, A is sign inidefinite.

# Calculating for the eigenvalues and eigenvectors

By definition,

$$AV = cV$$

which can be arranged so that

$$AV - cIV = 0$$

$$(A - cI)V = 0$$

where $A - cI$ is called the characteristic matrix of A. By assumption $V \neq 0$, the characteristic matrix $A - cI$ must be singular, and its determinant must be zero. This is the property that we use to find the eigenvalues.

Once the eigenvalues are found, you can use them to find the eigenvectors. Note that infinite number of solutions would exist for the eigenvector. It is the norm to normalize the solution by requiring the sum of square of them to be 1.

End.