Lecture 5. Calculus IV

Bowen Chen

Kansas State University

August, 2018

Today's plan

1. So far, we focus on derivatives of functions with one variable only;

2. Today, we explore derivatives of functions with multiple variables;

3. Meanwhile, we introduce some theorems that are commonly used in economics research.

Partial derivatives

1. **Definition** Given a multivariate function, $y = f(x_1, ..., x_n)$, the partial derivative of the function f with respect to x_i at point $\boldsymbol{x}^0$ is

$$\frac{\partial f}{\partial x_i}(x_1^0, ..., x_n^0) = \lim_{h \to 0} \frac{f(x_1^0, ..., x_i^0 + h, ..., x_1^n) - f(x_1^0, ..., x_i^0, ..., x_1^n)}{h}$$

2. Partial derivative measures the changes in y in response to a change in x_i;

3. In economics, we are often interested in the effect of certain variable on another variable, while holding other variables constant (ceteris paribus);

4. You can treat other variables as constants when deriving for partial derivatives.

Second-order partial derivatives

1. Given a function $z = f(x, y)$, the second-order partial derivatives are

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\Big(\frac{\partial z}{\partial x}\Big) = \frac{\partial^2 z}{\partial x^2}.$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\Big(\frac{\partial z}{\partial y}\Big) = \frac{\partial^2 z}{\partial y^2}.$$

2. The cross partial derivatives are

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\Big(\frac{\partial z}{\partial x}\Big) = \frac{\partial^2 z}{\partial y \partial x}.$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\Big(\frac{\partial z}{\partial y}\Big) = \frac{\partial^2 z}{\partial x \partial y}.$$

*3. Symmetry of cross partial derivative (or Young's theorem)

$$f_{yx} = f_{xy}.$$

Total differentials

1. For a function with multiple independent variables, total differential measures changes in the dependent variable in response to **small** changes in each of the independent variables.

2. For instance, if $z = f(x, y)$, the total differential dz is

$$dz = z_x dx + z_y dy.$$

3. We can use it to approximate values of a function with multiple variables. R examples here.

Partial differentials

1. Partial differential measures changes in the dependent variable in one of the variables, and ceteris paribus. For instance,

$$dz = z_x dx.$$

2. Note that if x and y are correlated, or y is a function of x, then the partial differential should be

$$dz = z_x dx + z_y y_z dx$$

3*. The maths with multivariate functions are usually not very difficult, and the difficult thing is to figure out the interconnections between variables.

4*. Which variable is exogenous, and which variable is endogenous?

*A special note: hat algebra

1. Totally differentiating a model in logarithms of variables yields a linear system relating small proportional changes via elasticities and shares (Jones and Neary, 1980);
2. Formally,
$$\hat{x} = \frac{dx}{x}.$$
3. Recall
$$d \log x = \frac{dx}{x}.$$
4. Hence,
$$\hat{x} = d \log x.$$
5. Hat algebra is a handy tool for comparative static analysis, popular for computer general equilibrium modeling.

Gradient Vector

1. The gradient vector is a column matrix of the partial derivatives;

$$Df = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{bmatrix}.$$

2^*. Gradient vector is commonly used in gradient-based optimization algorithms.

3^*. It tells us which direction to go at each step of optimization.

Hessian matrix

1. The Hessian matrix is a matrix with the second order partial derivatives as elements. In a case with two variables x and y, the Hessian matrix is

$$D^2 f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

2^*. Hessian matrix tells us about the local curvature (max. or min.);

3^*. Hessian matrix is used for generating the asymptotic covariance matrix for maximum likelihood estimator (do not be surprised when you get there).

Euler's theorem

1. **Euler's theorem** If a function f is homogenous of degree k, then

$$x_1 \frac{\partial f}{\partial x_1}(\boldsymbol{x}) + x_2 \frac{\partial f}{\partial x_2}(\boldsymbol{x}) + \cdots + x_n \frac{\partial f}{\partial x_n}(\boldsymbol{x}) = kf(\boldsymbol{x}).$$

2. The Euler's theorem enables us to decompose changes in y into its its derivatives.

More on homogenous functions

1. If a function f is homogenous of degree k, then its first order partial derivatives are homogenous of degree $k-1$;

2^*. The tangent planes to the level sets of a homogenous function have constant slope along each ray from the origin (sort of linear expansion property).

3^*. Consequently, the resource allocation does not change as we relax the constraint.

Implicit function theorem

1. Sometimes, we could not write variable y explicitly as a function of x, like $y = f(x)$. Instead, we only know that $G(x, y) = 0$. How do y changes in response to changes in x?

2. The implicit function theorem says that for the implicit function, if $\frac{\partial G}{\partial y}(x_0, y_0) \neq 0$, then

$$\frac{dy}{dx}(x_0) = -\frac{\partial G/\partial x(x_0, y_0)}{\partial G/\partial y(x_0, y_0)}.$$

3*. Implicit function theorem can be used to derive the slopes of level curves, such as the indifference curve of utility function.

Unconstrained optimization with multiple variables

1. The multivariate optimization problems are analogous to one-dimensional problems.

2. Two-step procedures:

   (1) First-order necessary condition (FONC): take the first-order partial derivatives and set them equal to zero; then, solve the equations;

   (2) Second-order sufficient condition (SOC): check on the second-order direct partial derivatives. It is a max. if the Hessian matrix is negative definite; it is a min. if the Hessian matrix is positive definite.

3. Matrix definiteness will be covered in lecture 9.

End.