Lecture 8. Linear Algebra I

Bowen Chen

Kansas State University

August, 2018

Concepts to know

1. Matrix operations: $\hat{\beta} = (X'X)^{-1}X'Y$
2. Rank;
3. Inverse matrix;
4. Determinant.

Introduction

1. Linear algebra plays important roles in applied economics research, because it is useful when dealing with system of equations. And it has been used as the main toolkit in econometrics;

2. The goal is to make you understand how to use matrix algebra to solve system of equations problems. This is the focus of this session. Besides, I will introduce some basic concepts that are building blocks of the econometrics classes.

Getting started

Here is what a matrix looks like:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

where a_{11} is called elements of the matrix $\mathbf{A}$. The numbers in horizontal lines are called *rows*; the numbers in vertical lines are called *columns*. The two numbers define the dimensions of the matrix, which is "3 by 3" in this case.

Matrix operations

You need to know the followings to proceed:

1. Addition and subtraction;
2. Scalar multiplication;
3. Vector multiplication;
4. Matrix multiplication;
5. Operation laws;

*6. Special kinds of matrices;

7. Matrix basics.

Re-expressing problems using matrices

1. Sums of values;

2. System of equations

3. A system of equations expressed in matrix forms can be solved through elementary row operation (or Gauss -Jordan elimination) or matrix operation. This is a naive way.

Elementary row operations

1. The three elementary equation operations:
   (1) Interchange two rows of matrix;
   (2) Change a row by adding to it a multiple of another row;
   (3) Multiply each element in a row by the same non-zero number.

2. The idea is to simplify the expression through the operations without changing the solutions.

3. There are cases that a equation of system might not have a unique solution.

Existence and uniqueness of solution: the rank criteria

1. The rank of a matrix is the number of nonzero rows in its row echelon form.

2. A system of linear equations having A as its coefficient matrix will have a solution for every choice of right-hand side values if and only if

$$\text{rank } A = \text{number of rows of A}$$

3. Any system of linear equations having A as its coefficient matrix will have at most one solution for every choice of right-hand side values if and only if

$$\text{rank } A = \text{number of columns of A}$$

Existence and uniqueness of solution: the rank criteria

4. Now, we combine 3 and 4. A system of linear equations having A as its coefficient matrix will have one and only one solution for every choice of right-hand side values if and only if

$$\text{rank } A = \text{number of rows of A} = \text{number of columns of A}$$

5. In this case, the coefficient matrix A is called **nonsingular** (remember the word). A necessary condition is that the matrix A must have the same number of rows as columns.

Facts about system of equations

1. A linear system of equations must have either no solution, one solution, or infinitely many solutions. A system has more than one solution has infinitely many;

2. The coefficient matrix A has at least as many rows as columns, if a system has exactly one solution (more observations than variables);

3. If a system has more unknowns than equations, it must have either no solution or infinitely many solutions;

(Next, we discuss how to solve system of equations using matrix operations, or matrix algebra).

Solving system of equations by taking matrix inverses

1. By imagination, if $\boldsymbol{AX} = \boldsymbol{B}$, then $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$.

2. If a square matrix $\boldsymbol{A}$ is nonsingular, then it is invertible.

3. The next question is how to derive for the matrix inverse. Again, we can use row operations to get inverse of a square matrices.

4. Note that only square matrix could have inverse (or invertible) and it can have at most one inverse. Non-square matrix does not have inverse, through they might have left or right inverse matrix.

More on matrix inverses

1. If a square matrix $\boldsymbol{A}$ is invertible, then it is nonsingular, and the unique solution to the system of linear equations $\boldsymbol{AX} = \boldsymbol{B}$ is $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$.

2. $\boldsymbol{A}^{-1}$ is called inverse of matrix $\boldsymbol{A}$. And by definition,$\boldsymbol{AA}^{-1} = \boldsymbol{I}$.

3. As a summary, for any square matrix $\boldsymbol{A}$, the following statements are equivalents:

   (1) $\boldsymbol{A}$ is invertible;

   (2) Every system $\boldsymbol{AX} = \boldsymbol{B}$ has unique solution.

   (3) $\boldsymbol{A}$ is nonsingular;

   (4) $\boldsymbol{A}$ has maximal rank n.

Properties of inverse matrix

1. Let A and B be square invertible matrices. Then,

   (1) $(A^{-1})^{-1} = A$,

   (2) $(A^T)^{-1} = A^{-1})^T$,

   (3) AB is intervible, $and (AB)^{-1} = B^{-1}A^{-1}$.

2. If A is invertible:

   (1) A^m is invertible for any integer m and $(A^m)^{-1} = (A^{-1})^m$ $= A^{-m}$;

   (2) for any integer r and s, $A^r A^s = A^{r+s}$;

   (3) for any scalar $r \neq 0$, rA is invertible and $(rA)^{-1} = r^{-1}A^{-1}$.

Determinants

1. The most frequently used matrix is the square matrix, and most important square matrix is the nonsingular matrix, which gives unique solution.

2. Now, we introduce a way of determining whether the square matrix is nonsingular (or invertible) or not. The key concept is "determinant".

3. So, what is determinant? How can we compute it?

Determinants

1. Determinants can easily be computed with low-dimension matrix;

2. For higher-order matrix, we follow the definition. Let A be an n by n matrix. Let A_{ij} be the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j from A. Then, the scalar

$$M_{ij} = det\ A_{ij},$$

is called the (i, j)th minor of A and the scalar

$$C_{ij} = (-1)^{i+j} M_{ij},$$

is called the (i, j)th cofactor of A. The determinant of an n by n matrix A is:

$$det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdot + a_{1n}C_{1n}$$

$$= a_{11}M_{11} - a_{12}M_{12} + \cdot + (-1)^{n+1}a_{1n}M_{1n}.$$

Determinants

1. Again, a square matrix is nonsingular if and only if it's determinant is nonzero;

*2. Some useful propertities. Let A be a square matrix. Then,

(1) $det(A^T) = det(A)$;

(2) $det(A \cdot B) = det(A)det(B)$;

(3) $det(A + B) \neq detA + detB$;

Solving for inverse matrix: an application of determinant

Let A be a nonsingular matrix. Then the inverse of the matrix is

$$A^{-1} = \frac{1}{|A|} Adj\ A$$

where $Adj\ A$ is called the adjoint matrix, which is the transpose of a cofactor matrix.

End.