Lecture 4. Calculus II

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Today's plan

1. Yesterday, we review the rules for deriving derivatives;

2. Today, we continue on working on derivatives, but in a bit more complex problem setting.

3. Then we introduce the convexity/concavity and relate it with second-order derivatives;

4. Lastly, we show an important application of the derivatives: optimization

Composite function

1. A simple problem setting: assume that $x \Rightarrow y$, i.e., x affect y directly. How y changes when there is a change in x? Then we look at y'.

2. What if $x \Rightarrow z \Rightarrow y$, i.e., x affect y indirectly through z?

3. For instance, weather shocks affect food prices through their effects on food production.

4. This is when we shall bring the composite function into the analysis.

Composite function

1. **Definition** If h and g are two functions, then the function f is called the composite of functions h and g if $f(x) = h[g(x)]$;
2. $x \Rightarrow g(x)$ and then $g(x) \Rightarrow f(x)$;
3. Sometimes, a composite function is written as $f(x) = (h \circ g)(x)$.
4. Composite functions are commonly seen in economics research;
5. Examples.

Chain rule

1. To know how x affects $f(x)$, we will have to use the derivative;
2. But the chain rule has to be used for deriving the derivative;
3. Chain rule for differentiating composite function.

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(h[g(x)]) = h'[g(x)]g'(x).$$

4. Examples;

5*. How would you estimate for the marginal effect of x on y?

Inverse function theorem

1. In economics, you would hear concepts like inverse demand function;

2. The idea is that if we can write demand quantity as a function of market prices, we can also write market price as a function of demand quantity (do not relate this to reverse causality here)

3. Formally, a one-to-one function f has an inverse, noted as f^{-1};

4. Examples

Inverse function theorem

1. To derive for derivatives of an inverse function, you can get it by taking derivatives of the inverse function directly;

2. But there is an alternative, or a shortcut;

3. **Theorem**: The derivative of an inverse function is

$$g'(x) = \frac{1}{f'[g(x)]}$$

4. In other words, you get what you need without explicitly taking the derivatives. Sometimes, this is handy because the inverse function could be hard to differentiate;

5. Proof and examples.

Rest of the class

1. We show focus on using derivatives to solve optimization problems;

2. The concept of convexity/concavity is the building bloc of optimization problems.

Convexity and concavity

Definition A real-valued function f defined on a convex subset U of $\mathbf{R}^n$ is concave if for all $\mathbf{x}$, $\mathbf{y}$ in U and for all t between 0 and 1,

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \geq f(\mathbf{x}) + (1-t)f(\mathbf{y}).$$

The function is convex, if

$$f(t\mathbf{x} + (1-t)\mathbf{y}) \leq f(\mathbf{x}) + (1-t)f(\mathbf{y}).$$

Graphical illustration in R

Second-order derivative test

1. How do we determine convexity/concavity without referring to graphs?

2. **Theorem** A real-valued function f is concave (or convex) on an interval I if and only if $f''(x) \leq 0$ (or $f''(x) \geq 0$) for all x in I;

3. Note that the first-order derivative does not matter. A convex function could be upwards or downwards sloped;

4. Function convexity is different to a set convexity (more on this shortly).

A useful property

1^*. Let f_1, ..., f_k be concave (convex) functions, each defined on the same convex subset U of $\mathbf{R}^n$. Let a_1, ..., a_k be positive numbers. Then, $a_1 f_1 + ... + a_k f_k$ is a concave (convex) function on U.

2. In other words, a linear combination of concave (convex) functions is also a concave (convex) function.

3. Is there a well-behaved social welfare function when each individual maximizes his or her own utility?

*Concavity and convex set

1. A set is convex if the linear combination of any two points are still in the set.
2. Let f be a function defined on a convex subset U in $\mathbf{R}^n$. If f is concave, then for every x_0 in U, the set

$$C^+_{x_0} = \{\mathbf{x} \in U : f(\mathbf{x}) \geq f(\mathbf{x}_0)\}$$

is a convex set. If f is convex, then for every x_0 in U, the set

$$C^-_{x_0} = \{\mathbf{x} \in U : f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$$

is a convex set.
3. This is why the indifference curve of (concave) utility function is convex!

Unconstrained optimization

1. The universal procedures:

   (1) First-order necessary condition (FONC): the first derivative is zero;

   (2) Check on the sign of second derivative.

2. The multivariate optimization problems are analogous to one-dimensional problems.

3. Note that this procedure does not guarantee that the solution is global optimum.

Intuitions

1. A point with zero value of first-order derivative has zero slope. So the curve is flap at that point and it is neither increasing nor decreasing.

2. But this point could give maximum or minimum value. To see what it gives, we look at the curvature.

3. Two R examples here.

Unconstrained optimization

1. First-order necessary condition
   $f'(x) = 0$, x^* is the critical point of f. This is a point that gives max. or min. of f.
2. Whether it is max or min depends on the sign of $f''(x^*)$.
   (1) $f''(x^*) < 0$ corresponds to a max;
   (2) $f''(x^*) > 0$ corresponds to a min;
   (3) $f''(x^*) = 0$, inclusive.
3. The optimum is global if the function is strictly convex or concave.
4. Examples.

End.