Lecture 10. Linear Algebra III

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*Eigendecomposition of a matrix

Every n by n symmetric matrix $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \boldsymbol{C}\Lambda\boldsymbol{C}^T = \sum_{j=1}^{N} \lambda_j \gamma_j \gamma_j^T$$

where $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ and $\boldsymbol{C} = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ is an orthogonal matrix consisting of the eigenvectors of $\boldsymbol{A}$.

An example

Suppose that

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

The eigenvalues of the matrix are $\lambda_1 = 2 + \sqrt{5}$, $\lambda_2 = 2 - \sqrt{5}$. The eigenvectors are $\gamma_1 = (0.5257, 0.8506)^T, \gamma_2 = (0.8506, -0.5257)^T$. Then, by the above theorem,

$$\boldsymbol{A} = \begin{bmatrix} 0.5257 & 0.8506 \\ 0.8506 & -0.5257 \end{bmatrix} \begin{bmatrix} 2 + \sqrt{5} & 0 \\ 0 & 2 - \sqrt{5} \end{bmatrix} \begin{bmatrix} 0.5257 & 0.8506 \\ 0.8506 & -0.5257 \end{bmatrix}$$

Determinant and eigenvalues

The eigenvalues and determinant are closely connected. Because

$$\boldsymbol{A} = \boldsymbol{C}\Lambda\boldsymbol{C}^T$$

$$|\boldsymbol{A}| = |\boldsymbol{C}||\Lambda||\boldsymbol{C}^T| = |\boldsymbol{C}||\boldsymbol{C}^T||\Lambda| = |\boldsymbol{C}\boldsymbol{C}^T||\Lambda| = |\boldsymbol{I}||\Lambda| = |\Lambda|$$

The above formula says that the determinant of a matrix equals to the product of its eigenvalues.

*Factoring a matrix

In econometrics classes, you will need a matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{A}^{-1}$$

One choice is

$$\boldsymbol{P} = \Lambda^{-1/2}\boldsymbol{C}^T$$

To prove this,

$$\boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{C}(\Lambda^{-1/2})^T\Lambda^{-1/2}\boldsymbol{C}^T = \boldsymbol{C}\Lambda^{-1}\boldsymbol{C}^T = \boldsymbol{A}^{-1}$$

Trace of a matrix

The trace of a square matrix is the sum of its diagonal elements:

$$tr(\boldsymbol{A}) = \sum_{k=1}^{K} a_{kk}.$$

Some properties:

(1) $tr(c\boldsymbol{A}) = c(tr(\boldsymbol{A}))$,

(2) $tr(\boldsymbol{A}^T) = tr(\boldsymbol{A})$,

(3) $tr(\boldsymbol{A} + \boldsymbol{B}) = tr(\boldsymbol{A}) + tr(\boldsymbol{B})$,

(4) $tr(\boldsymbol{I_k}) = K$,

(5) $tr(\boldsymbol{AB}) = tr(\boldsymbol{BA})$

(6) The trace of a matrix equals to the sum of its eigenvalues.

Derivatives of a matrix

1. Taking derivatives of a matrix is not very different to what we have learnt before. It is only different in the way that we are now dealing them in matrix forms.

2. I find that there is no better way of mastering the knowledge than understanding the formulas first and then keeping them in the bookshelves.

3. I do not expect you to remember all the formulas. But make sure that you know they exist and be able to use them when needed.

Some useful formulas

1. $\frac{\partial x^T a}{\partial x} = \frac{\partial a^T x}{\partial x} = a,$
2. $\frac{\partial a^T X b}{\partial X} = ab^T,$
3. $\frac{\partial a^T X^T b}{\partial X} = ba^T,$
4. $\frac{\partial a^T X a}{\partial X} = \frac{\partial a^T X^T a}{\partial X} = aa^T,$
5. $\frac{\partial a^T X a}{\partial a} = 2Xa,$
6. $\frac{\partial A x}{\partial x^T} = A$

End.