Lecture 6. Calculus IV

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Constrained optimization

1. Economics is a study on optimal allocation of scarce resources;

2. Scarcity, in math terms, is constraint;

3. Hence, many economics problems are actually constrained optimization problems;

4. In this lecture, I expect you to know two things: (1) how these problems can be solved; (2) how to interpret the Langrange multiplier.

5. In reality, you do not need to solve them by hand.

Constrained optimization

1. Compared with unconstrained optimization, constrained optimization is a different animal to deal with. It is also harder to deal with;

2. To make things simpler, I break the constrained optimization problems into two categories: problems with equality constraints and inequality constraints;

3. But note that, in reality, constrained optimization problems are often mixed with equality and inequality constraints.

4^*. GAMS is arguably good at solving constrained optimization problems.

Case 1. Equality constraint

Example. Solve the problem below

$$\max.\ f(x, y)$$

$$\text{s.t. } h(x, y) = c$$

Solution:
Step 1. Set a new function F (Lagrangian function), where

$$F(x, y, \lambda) = f(x, y) + \lambda[c - h(x, y)].$$

Step 2. Take partial derivatives of F with respect to all three variables, including x, y and λ, and set them to be zero;
Step 3. Check with the second-order condition.
(Four examples here).

How come does the Lagrangian function work?

1. See a general example;
2. The optimal solution exists at where the slopes overlap;
3. The Lagrangian function, magically, lead us to that condition;
4. Optimal decision is made at the margin!
5. This intuition also applies to the inequality constraint problems.

*Case 2. Inequality constraint

Example. Solve the problem below

$$\max.\ \ f(x,y)$$

$$\text{s.t.}\ \ h(x,y) \leq c$$

Solution:
Step 1. Set a new function F (Lagrangian function), where

$$F(x,y,\lambda) = f(x,y) + \lambda[c - h(x,y)].$$

Step 2. Take partial derivatives of F with respect to all three variables, including x, y and λ, and then formulate the **Kuhn-Tucker conditions**;
Step 3. Check with the second-order condition.
(You learn more about this from AGEC 712).

The Lagrange multiplier

1. The Lagrange multiplier λ approximates the marginal effect of a small change in the constant of the constraint on the objective function;

2. Economically, the term is often referred to as shadow price (or the market value of relaxing the constraint);

3. See an numerical example;

4. Why it is the case? Because of the envelope theorem.

*Envelope theorem

Let $f(\boldsymbol{x}, a)$ be a function with the scalar a. For each choice of the parameter a, consider the unconstrained maximization problem,

$$\max f(\boldsymbol{x}, a).$$

Let $\boldsymbol{x}^*(a)$ be the solution of the problem, then

$$\frac{d}{da} f(\boldsymbol{x}^*(a); a) = \frac{\partial}{\partial a} f(\boldsymbol{x}^*(a); a).$$

(The Lagrange multiplier is the value of the partial derivative!)

*Envelope theorem

1. The envelope theorem underlies the interpretation of the lagrange multiplier;

2. In a boarder sense, the theorem enables us to derive the marginal effect of parameter on the optimal value of the objective function;

3. Some economic implications can be readily derived from this.

4. We are done with the calculus part until here. We move to linear algebra since tomorrow.

End.