Lecture 7. Calculus V

Bowen Chen

Kansas State University

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Integral calculus

1. We have focused on differential calculus sofar, which measures the rate of change of a function;

2. Integration is the reverse of differentiation. It aims to find the original function given the function derivatives;

3. Alternatively speaking, we use integration to derive for $F(x)$ when we only know $f(x) = F'(x)$;

4. In this department, integration and related concepts are lightly used in the graduate courses;

5. But you shall know what it is, when to use it and how to solve simple problems (important to environmental resource issues).

Integration

1. Formally, if $f(x) = F'(x)$, the integral of $f(x)$ is

$$\int f(x)dx = F(x) + c.$$

where the symbol $\int$ is an integral sign, $f(x)$ is called integrand and c is a constant term;

2. The whole term at left-hand side is called "the indefinite integral of f with respect to x";

3. A graphical illustration.

Rules of integration

1. $\int kdx = kx + c$,
2. $\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$,
3. $\int x^{-1} dx = \log |x| + c \ (x \neq 0)$,
4. $\int a^{kx} dx = \frac{a^{kx}}{k \log a} + c$,
5. $\int e^{kx} dx = \frac{e^{kx}}{k} + c$,
6. $\int kf(x)dx = k \int f(x)dx$,
7. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$.

Initial/boundary conditions

1. In all the examples above, we have the solution that contains an arbitrary constant c;

2. An initial condition or a boundary condition can be used to pin down the value of the constant;

3. It means that we, as economics scientist, can collect data on (x, y) for one time to solve for our model parameters.

More than the simple rules

1. Integration by substitution:
   (1) The idea is to simplify the structure of the integrand through substitution;

   (2) You replace x with another term that is a function of x;

   (2) Substitution might not always work.

2. Integration by parts:
   (1) The idea is to simplify the structure of the integrand by breaking down the integrand into parts;

   (2) You solve for the intergal part by part;

   (2) This trick usually works when an integrand is a product or quotient of differentiable function of x.

3. In general, you want to solve integrals with a simple integrand function.

The Definite Integral

1. The definite integral is the integral within a interval.

2. By definition,

$$\int_a^b f(x)dx = \lim_{x\to\infty} \sum_{i=1}^{\infty} f(x_i)\Delta x_i.$$

where a and b are called the lower limit and upper limit of integration.

3. The fundamental theorem of calculus:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a).$$

Properties of definite integrals

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx.$
2. $\int_a^a f(x)dx = 0.$
3. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$
4. $\int_a^b f(x)dx \pm \int_a^b g(x)dx = \int_a^b [f(x) \pm g(x)]dx.$

Extended problems

1. Areas between curves;

2. Improper integrals.

    1. Improper integrals measure the area under some curves that extend infinitely far along the x axis;

    2. The improper integral converges (or has a solution) when the limit exists.

3. L'Hopital's rule:

If the limit of a function $f(x) = \frac{g(x)}{h(x)}$ as $x \to a$ cannot be evaluated, then we have

$$\lim_{x \to a} \frac{g(x)}{h(x)} = \lim_{x \to a} \frac{g'(x)}{h'(x)}$$

First-order differential equations

1. A differential equation is an equation that expresses an explicit or implicit relationship between a function $y = f(t)$ and one or more of its derivatives or differentials.

2. For instance,

$$\frac{dy}{dt} = 5t + 9, \qquad y' = 12y, \qquad \text{or } y'' - 2y' + 19 = 0.$$

3. We will only cover first-order (with first-order derivative) ordinary differential equations (with single independent variable).

The general formula

For an equation like

$$\frac{dy}{dt} + vy = z.$$

where v and z are constants. The formula for a general solution is

$$y(t) = e^{-\int vdt}\Big(A + \int ze^{\int vdt}\Big).$$

where A is a constant. $e^{-\int vdt}A$ is called the complementaty function, and $e^{-\int vdt}\int ze^{\int vdt}$ is called the particular integral.

A lot more to learn...

1. Partial integration (with multiple independent variables);

2. Phase diagrams for differential equations (stability of the solution);

3. First-order difference equations (a dependent variable is related to a lagged independent variable);

4. Second-order differential equations.

In the following sessions, we will focus on linear algebra.

End.