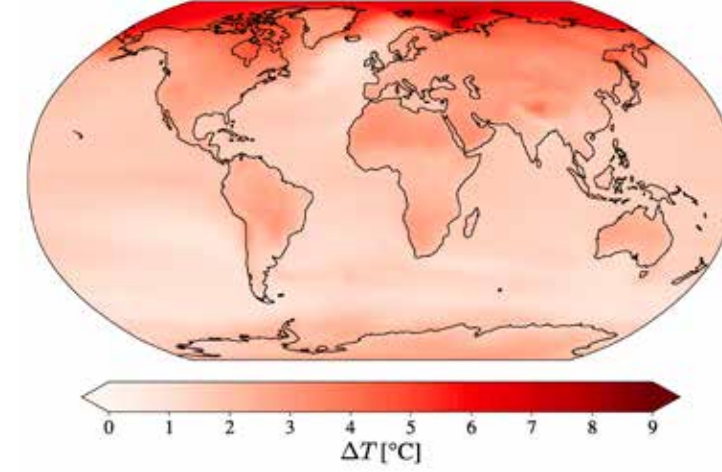


Goal: Emulate the statistics of a chaotic system

1. Select a climate variable of interest



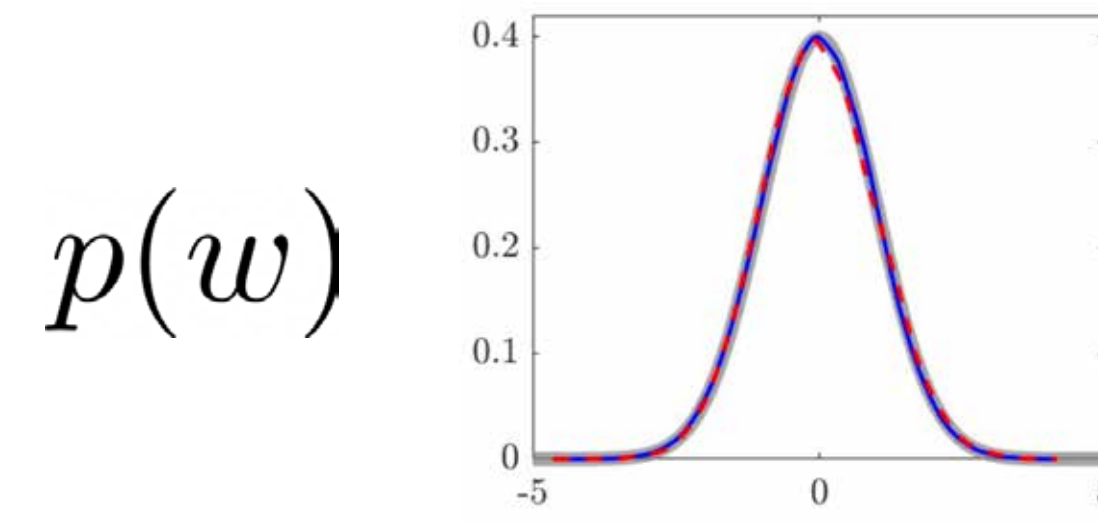
$$\frac{\partial w}{\partial t} = \mathcal{N}(w, F) + \epsilon \xi,$$

Option 1:

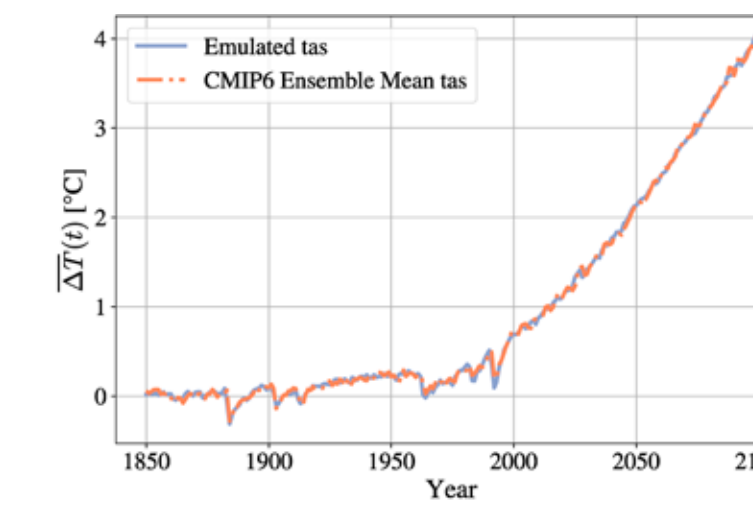
Option 2:

Section 2.2: Operator Framework for Emulators

2a. Emulate the full probability distribution



2b. Emulate a statistical quantity (e.g. mean/variance)



Section 2.2.3:
Conceptual Bridge
(Duality)

3a. Approximate Fokker-Planck Operator

$$\mathcal{F}(\cdot) = \frac{\partial}{\partial w} \left[D \frac{\partial}{\partial w} (\cdot) - \mathcal{N}(w, F)(\cdot) \right]$$

Appendix A2:
Technical Bridge
(Adjoint)

4a. Approximate Koopman Operator

$$\mathcal{K}(\cdot) = \mathcal{N}(w, F) \frac{\partial (\cdot)}{\partial w} + D \frac{\partial^2 (\cdot)}{\partial w^2}$$

Section 2.2.4: Connecting to Linear Response Theory

Appendix A6:
Connecting response theory to the Fokker-Planck operator

Technical Bridge
(GFDT)

Fluctuation Dissipation Theorem (FDT)

$$\langle g(w) \rangle = \langle g(w) \rangle_0 + \int_{-\infty}^t e^{\mathcal{K}s} F(t-s) ds$$

Sections 2.3.1 & 2.3.3: Koopman Emulators

Method 3: EDMD

$$\phi(T_{n+1}) = \tilde{\mathcal{K}} \phi(T_n) + \psi(F_n)$$

Linear Basis

Method 2: DMD

$$T_{n+1} = \mathcal{L} T_n + F_n$$

Quasi-Equilibrium

Method 1: Pattern Scaling

$$T(\mathbf{x}, t) = a(\mathbf{x}) \bar{T}(t)$$

Section 2.3.2: Impulse Response Emulators

Method 4: GFDT

$$R(\mathbf{x}, \mathbf{x}', t) = \frac{\langle T_\delta(\mathbf{x}, t) - T_0(\mathbf{x}, t) \rangle}{|\delta(\mathbf{x}')|}$$

No Access to Governing Eq.

Method 5: Deconvolution

$$\mathbf{R} = \frac{\mathbf{F}^{-1} \mathbf{T}}{\Delta t}$$

Exponential Decay

Method 6: Mode Fitting

$$R(t) = \sum_{i=1}^3 \alpha_i e^{\lambda_i t}$$

Machine Learning Emulators

Deep learning/diffusion models; fitting statistical distributions

Connect to GFDT

Section 2.2.4: Score-based response function

$$R(\mathbf{x}, \mathbf{x}', t) = -\langle T(\mathbf{x}, t) s(T(\mathbf{x}', 0)) \rangle$$