CSE 21 Fall 2018
Homework 7
Due: **Saturday, November 17**, 2018 at 10:00pm

INSTRUCTIONS

Homework should be done in groups of **one to three** people. You are free to change group members at any time throughout the quarter. Problems should be solved together, not divided up between partners. Homework must be submitted through **Gradescope** by **a single representative**. Submissions must be received by **10:00pm** on the due date, and there are no exceptions to this rule.

You will be able to look at your scanned work before submitting it. Please ensure that your submission is **legible** (neatly written and not too faint) or your homework may not be graded.

Students should consult their textbook, class notes, lecture slides, instructors, TAs, and tutors when they need help with homework. Students should not look for answers to homework problems in other texts or sources, including the Internet. You may ask questions about the homework in office hours, but **not on Piazza**.

Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should **always explain** how you arrived at your conclusions and **justify your answers** with mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to **convince the reader** that your results and methods are sound.

For questions that require pseudocode, you can follow the same format as the textbook, or you can write pseudocode in your own style, as long as you specify what your notation means. For example, are you using "=" to mean assignment or to check equality? You are welcome to use any algorithm from class as a subroutine in your pseudocode. For example, if you want to sort list A using InsertionSort, you can call InsertionSort(A) instead of writing out the pseudocode for InsertionSort.

REQUIRED READING Rosen 6.1, 6.3, 6.4, 6.5, 8.5

KEY CONCEPTS: basic counting principles (sum and product rules), counting with categories (division rule), inclusion-exclusion, permutations and combinations, binomial coefficient and its identities

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1. In class, we saw that there is only one way to connect three different-colored pipe cleaners together to form a triangle, accounting for physical symmetries of the object. That is, any way to construct a triangle out of the three pipe cleaners is really the same as any other because physical objects can be rotated and turned over in space.

   (a) (5 points) Suppose you have four pipe cleaners of different colors. How many ways can you connect them end-to-end to form a colored square, accounting for object symmetries? That is, how many different physical objects can you create?

   Solution: The number of squares you can create, not accounting for symmetries, is 4!, since you have 4 options for which pipe cleaner you put on the bottom, 3 options for which you put on the right, 2 for which you put on the top, and 1 for which you put on the left. A square has two kinds of symmetries: rotating and flipping. There are 4 options for how much we rotate a square (0 degrees, 90 degrees, 180 degrees, 270 degrees). There are 2 options for whether or not we flip a square. This means each square is the same as $4 * 2 = 8$ other squares if we allow the square to be rotated and turned over in space. So the total number of physical squares we can create is $\frac{4!}{8} = 3$. We can think of this using the "categories" formula. Here, the objects are the squares without accounting for symmetries, the categories are the squares including symmetries (this is what we want to find), and the size of each category is the number of squares that are all considered the same if we can rotate and flip squares. That is,

   $$\#\text{categories} = \frac{\#\text{objects}}{\text{size of each category}} = \frac{4!}{8} = 3.$$

   We can confirm that our answer is correct by noticing that the only way that squares can be truly different (meaning you can't just move one square around in space to make it look like another) is if they have different colors on opposite sides. So if your colors are red, blue, green, and yellow, there are three options for what color could be across from red. Each such option gives a truly different square.

   (b) (5 points) Suppose you have four different colors of paint that you can use to paint the sides of a tetrahedron. Each color should be used once so that the four sides each have a different color when painted. How many ways can you paint the tetrahedron, accounting for object symmetries? That is, how many different physical objects can you create?

   Solution: Again, use the category idea. There are 4! different tetrahedrons, not accounting for symmetries. There are two kinds of physical symmetries, rotating and flipping, in the following sense. If we put the tetrahedron on a table, we can rotate it without lifting it off the table (the axis of rotation goes through the top vertex and the center of the base) in one of 3 ways (0 degrees, 120 degrees, 240 degrees). To flip a tetrahedron, we can change which side we put face-down on the table, and there are 4 ways in which we could choose that face. So each tetrahedron is equivalent to $3 * 4 = 12$ others. Therefore,

   $$\#\text{categories} = \frac{\#\text{objects}}{\text{size of each category}} = \frac{4!}{12} = 2.$$

   Again, we can explain why this makes sense. Say your paint colors are red, blue, green, and yellow. To figure out which tetrahedrons are truly different as physical objects, take any tetrahedron and place it on the table with the red size down. Now rotate it so the green side is facing you. There are only two different ways the tetrahedron could be painted now. It could have yellow on the left and blue on the right, or vice versa. These are the two truly different tetrahedrons, and all others can be obtained from rotating and flipping these.

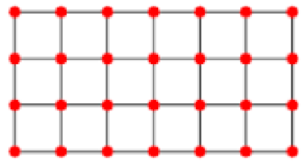
(c) (5 points) Suppose you have six different colors of paint that you can use to paint the sides of a cube. Each color should be used once so that the six sides each have a different color when painted. How many ways can you paint the cube, accounting for object symmetries? That is, how many different physical objects can you create?

Solution: For a cube, there are 6! ways to paint the sides, not accounting for symmetries. The symmetries are rotating and flipping in 3-D. We can put the cube on the table and rotate which side is front-facing. There are 4 options for how much we rotate it. We can also change which side we put face-down. There are 6 options because there are 6 faces. Thus, the total number of symmetries is $4 * 6 = 24$. The formula gives

$$\#\text{categories} = \frac{\#\text{objects}}{\text{size of each category}} = \frac{6!}{24} = 30.$$

We can again explain the answer by thinking about how cubes can be different as physical objects. Say the paint colors are red, blue, green, and yellow, purple, and orange. We can compare two cubes by always putting the red size down. There are then 5 different options for what the top color could be. How do we compare whether two cubes with the same top color, say green, are the same physical object? You can always rotate the cube so the yellow size is facing you. You know your cube has red on the bottom, green on top, and red facing you. Now the cube is fixed and we can't move it any more while keeping those three sides in the positions we want them in. So any coloring of the remaining sides will be different. There are three sides left, and three colors to choose from, so there are $3! = 6$ different cubes associated with each top color. Since there were 5 options for the top color, the total number of cubes is $5 * 3! = 5 * 6 = 30$.

2. Under the standard (x, y)-coordinate system, a grid of points with integer coordinates is given in the picture below. Here, the **bottom-left** point is the origin $(0, 0)$ and the **top-right** point has coordinate $(6, 3)$. The coordinates of the remaining points can easily be computed accordingly.



A walk on a grid of points with integer coordinates that uses the steps $(1, 0)$ (east-step) and $(0, 1)$ (north-step) is called a *north-eastern lattice path*

(a.) (4 points) Let m, n be positive integers. Find the number of north-eastern lattice paths from $(0, 0)$ to (m, n).

Solution: To get from $(0, 0)$ to (m, n) we need to take $m + n$ steps which consist of m east-steps along the x-axis and n north-steps along the y-axis. Thus, a north-eastern lattice path can be encoded as a sequence of length $m+n$ in which there are m E-letters and n N-letters. The number of such sequences is $\binom{m+n}{n}$, or equivalently $\binom{m+n}{m}$, which is also the number of required north-eastern lattice paths.

(b.) (8 points) Use the north-eastern lattice path model to give a combinatorial proof to the following identity

$$\sum_{k=0}^{m} \binom{n+k}{k} = \binom{m+n+1}{m}.$$

That is, you need to specify what each side of the identity is counting, in terms north-eastern lattice paths, and explain why they must have the same enumeration.

Solution. The right-hand side counts the number of north-eastern lattice paths from $(0, 0)$ to $(m, n + 1)$.

For the interpretation of the left-hand side, we shall classify the north-eastern lattice paths from $(0, 0)$ to $(m, n + 1)$ by the location of their last north-step.

Each term of the left-hand side counts the number of north-eastern lattice paths from $(0, 0)$ to (k, n) where $0 \le k \le m$. For each given k, $0 \le k \le m$, let $\mathcal{P}_k$ be the set of north-eastern lattice paths from $(0, 0)$ to (k, n) with $|\mathcal{P}_k| = \binom{n+k}{k}$. Let $P \in \mathcal{P}_k$. We shall append one last north-step followed by $m - k$ east-steps to the end of P to obtain a path from $(0, 0)$ to $(m, n + 1)$.

Since the sets $\mathcal{P}_k$ are all distinct, by the sum rule, we then have the required identity.

(c.) (**Optional - Hard**) Now let $m = n$. Find the number of north-eastern lattice paths from $(0, 0)$ to (n, n) that **do not go below** the $45°$ line $y = x$.

These are called **Dyck paths** and their enumeration is given by the Catalan numbers. Calatan number is one of the most "famous" sequences in Mathematics, being the enumeration for plenty sets of objects.

3. (8 points) The following binomial coefficient identity holds for any integer $n > 0$.

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$$

Prove this identity combinatorially by interpreting both sides in terms of fixed density binary strings.

Solution:

The right side of this identity counts the number of binary string of length $2n$ with n ones. The key observation is that we can divide such a string down the middle, to create two binary strings each of length n. Let k be the number of ones in the first binary string of length n. Then there will be $n-k$ ones in the second string of length n. Notice that k can be as low as 0 if there are no ones in the first string, or as high as n if all the ones are in the first string.

To prove this identity, let S be the set of all binary strings of length $2n$ with n ones and partition S into disjoint sets based on the number of ones in the first half of the string, which we will call k. For $k = 0, 1, \ldots, n$, let S_k be the set of all strings in S with k ones in the first half of the string. Since these sets S_k are all disjoint, we can use the Sum Rule to say

$$|S| = \sum_{k=0}^{n} |S_k|.$$

To find $|S_k|$, we must count all binary strings of length $2n$ with n ones, k of which are in the first n positions. There are $\binom{n}{k}$ ways to place the k ones in the first half of the string, and $\binom{n}{n-k}$ ways to place the remaining ones in the second half of the string. These choices are independent, so the Product Rule gives that $|S_k| = \binom{n}{k}\binom{n}{n-k}$. Also, $|S| = \binom{2n}{n}$ using the formula for fixed-density strings. Replacing these expressions above gives the desired result

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

4. Recall from Problem 1 of Homework 2 that a list of n distinct integers $a_1, a_2, \ldots, a_n$ is called a **mountain list** if the elements reading from left to right first increase and then decrease.

   A permutation is called a **mountain permutation** if the numbers, when reading from left to right, form a mountain list. For example, the permutation 1235764 is a mountain permutation of length 7, and 25431 is a mountain permutation of length 5.

   We also consider increasing permutations of the form $123\ldots n$ and decreasing permutations of the form $n\ldots 321$ to be mountain permutations. For example, 1234 and 4321 are mountain permutations of length 4.

   (a) (**Optional - Easy**) Let $M(n)$ be the number of mountain permutations of length n. Write a recurrence relation that $M(n)$ satisfies, then solve the recurrence to show that the closed-form formula for $M(n)$ is $M(n) = 2^{n-1}$.

   Solution: Given a mountain permutation of length $n-1$, we can turn it into a mountain permutation of length n by adding the largest element n either right before $n-1$ or right after $n-1$. For example, the mountain permutation 235641 can be turned into 2357641 or 2356741. Thus, there are two mountain permutations of length n associated with each mountain permutation of length $n-1$, so $M(n) = 2M(n-1)$. Since there is only one mountiain permutation of length 1 (the permutation 1), we have $M(1) = 1$ as the base case of this recurrence.

   We can solve the recurrence by unraveling or by guess and check. We will use guess and check. The table of values below suggests that a closed-form formula is $M(n) = 2^{n-1}$.

n	1	2	3	4	5
$M(n)$	1	2	4	8	16

   To prove our formula is correct, use induction on n. When $n = 1$, $M(1) = 1$ according to the base case of the recurrence, and this matches the formula $M(1) = 2^{1-1} = 2^0 = 1$. Suppose as the induction hypothesis that for $n > 1$, $M(n-1) = 2^{n-2}$. Then the recurrence says $M(n) = 2M(n-1) = 2 * 2^{n-2} = 2^{n-1}$ by applying the induction hypothesis and simplifying. So the formula holds for n, which means it is correct by induction.

   (b) (8 points) Prove the following identity by interpreting it combinatorially in the context of mountain permutations. That is, what is each side counting in terms of mountain permutations, and why are those the same?

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}$$

   Hint: Let k represent the position of the largest element, n, in the permutation.

   Solution: The right side counts the size of the set of all mountain permutations of length n, since we showed in part (a) that there are 2^{n-1} such permutations.

   The left side is a sum, so it comes from classifying the set of all mountain permutations of length n **based on where the largest element, n, appears in the permutation**.

   Let M represent the set of all mountain permutations of length n, and let M_1 be the subset of these with n in position 1, M_2 the subset with n in position 2, and so on. In general, M_k is the set of all mountain permutations of length n with the largest element n in position k. Here, k can be any value between 1 and n. The sets M_k are disjoint because it's impossible to have the largest element n appear in more than one position.

   To find $|M_k|$, where k is an integer between 1 and n, we will count the number of mountain permutations of length n with the largest element, n, in position k. If n is in position

k, then we must choose $k-1$ of the numbers $\{1, 2, \ldots, n-1\}$ to come before the n in position k. There are $\binom{n-1}{k-1}$ such choices. Each such choice of $k-1$ numbers from $\{1, 2, \ldots, n-1\}$ completely determines a mountain permutation. The numbers we have chosen must appear in increasing order in the first $k-1$ positions, then there must be an n in the kth position, then any numbers we have not chosen must appear in decreasing order. Therefore, the number of mountain permutations of length n with n in position k is given by $\binom{n-1}{k-1}$. That is, $|M_k| = \binom{n-1}{k-1}$.

Therefore, since the sum rule gave

$$2^{n-1} = |M| = \sum_{k=1}^{n} |M_k| = \sum_{k=1}^{n} \binom{n-1}{k-1}$$

as desired.

(c) (2 points) How many bits (0's and 1's) are required to represent a mountain permutation of length n? Simplify your answer.

Solution: Since the number of mountain permutations of length n is given by 2^{n-1}, the number of bits needed to represent a mountain permutation of length n is given by $\lceil \log_2 2^{n-1} \rceil = \lceil n-1 \rceil = n-1$.

(d) (5 points) Describe how to encode a mountain permutation of length n using the number of bits you gave in part (c). Illustrate your description with an example.

Solution: We can encode a mountain permutation using $n-1$ bits by thinking of the mth bit from left to right representing whether the element m comes before or after n in the mountain permutation. We will use a 0 in the mth bit to say that m comes before n in the permutation, and a 1 to say that m comes after n in the permutation. We need to give this information for all values of m between 1 and $n-1$. For example, if $n = 7$, the mountain permutation 1235764 would be encoded as 000101 which says that 1, 2, 3, and 5 come before the 7, whereas 4 and 6 come after.

(e) (5 points) Describe how to decode a mountain permutation of length n, if it has been encoded using the method you described in part (d). Illustrate your description with an example.

Solution: The encoding method described in part (d) gives enough information to decode a string in a unique way. If we are given a string of length ℓ, we know it represents a mountain permutation of length $n = \ell + 1$. Then we can interpret each bit in the string so that a 0 in position m means m comes before n, and a 1 in position m means m comes after n. Knowing which elements come before n and which come after n is enough to reconstruct the string, as we must list the elements before n in increasing order, then put n, then list the elements after n in decreasing order to create a mountain permutation. For example, if we are given the string 11001, we know it encodes a mountain permutation of length 6, and we know that 1, 2, and 5 come before 6, whereas 3 and 4 come after 6. From, this, there is only way to create the mountain permutation, which results in 125643.