CSE 21 Fall 2018
Bonus Problem 1
Due: Sunday, October 14, 2018 at 10:00pm

Instructions:

- You must **work independently** on this problem and do not consult your the teaching staffs nor your classmates.
- You may use **only the materials covered so far in lecture**. If you decide to apply an outside theorem or lemma, then you need to provide a proof for that result.
- Just like with the regular homework assignments, you will be graded on both the correctness of your answers and your ability to present your ideas clearly and logically. You should **always explain** how you arrived at your conclusions and **justify your answers** with mathematically sound reasoning.
- Having the correct answer to this problem will give you an additional 1% on the first midterm.

Let n be an arbitrary positive integer. Prove that

$$1^k + 2^k + \cdots + n^k \in \Theta(n^{k+1})$$

for all non-negative integers k.

Solution. First we show that $1^k + 2^k + \cdots + n^k \in O(n^{k+1})$. Here, we can "over-estimate" the sum $\sum_{i=1}^{n} i^k$ by replacing each i^k with n^k. This gives

$$1^k + 2^k + \cdots + n^k \leq n^k + n^k + \cdots + n^k = n \cdot n^k = n^{k+1}$$

for all integers $n \leq 1$.

So $1^k + 2^k + \cdots + n^k \in O(n^{k+1})$ by domination property.

Now to show $1^k + 2^k + \cdots + n^k \in \Omega(n^{k+1})$. Let $m = \lceil n/2 \rceil$, we first bound the sum $\sum_{i=1}^{n} i^k$ below with $\sum_{i=m}^{n} i^k$ by discarding the first $n/2$ terms. Next, we "under-estimate" $\sum_{i=m}^{n} i^k$ by replacing each i^k with $(n/2)^k$. This gives

$$1^k + 2^k + \cdots + n^k \geq m^k + (m+1)^k + \cdots + n^k \geq \left(\frac{n}{2}\right) \cdot \left(\frac{n}{2}\right)^k = \left(\frac{n}{2}\right)^{k+1}$$

Here, $\left(\frac{n}{2}\right)^{k+1} \in \Omega(n^{k+1})$ so by domination property, we have $1^k + 2^k + \cdots + n^k \in \Omega(n^{k+1})$.

Hence, $1^k + 2^k + \cdots + n^k \in \Theta(n^{k+1})$.