1 CSE103 Final Practice Problems, set 1

1. Alice and Bob each choose at random a real number between -2 and 2. We assume a uniform probability law under which the probability of an event is proportional to its area. Consider the following events:

   A : The numbers that they have chosen have different signs

   B : The sum of the absolute of the numbers chosen by them is less than or equal to 1.

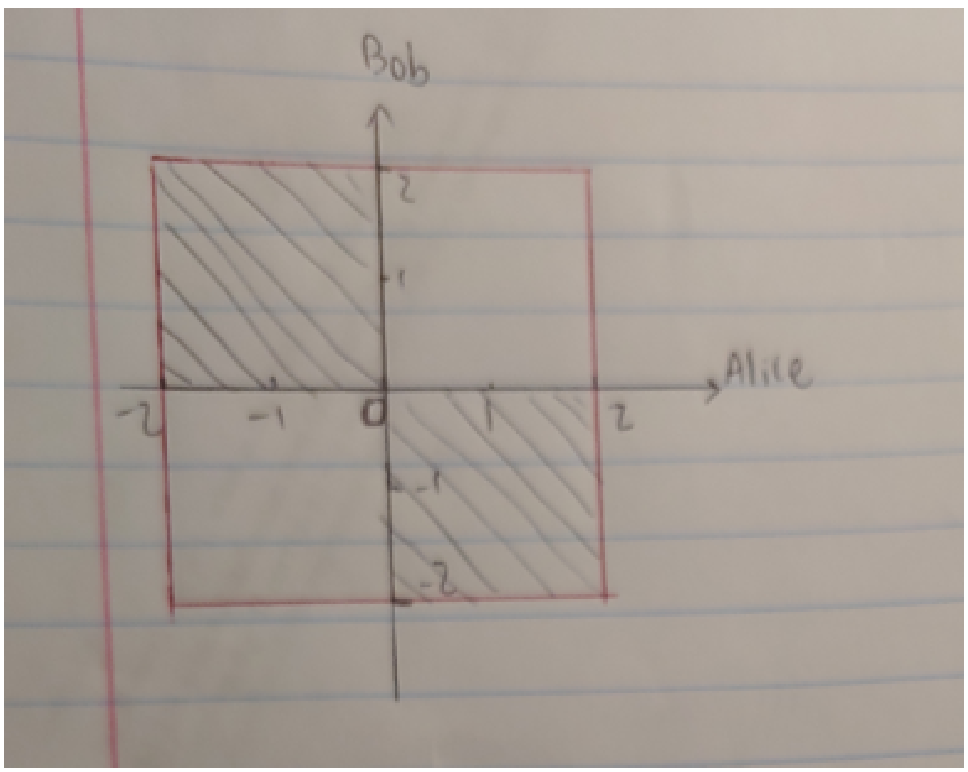
   C : The absolute difference between the two numbers is less than or equal to 1/2.

   D : The minimum of the two numbers is equal to -1

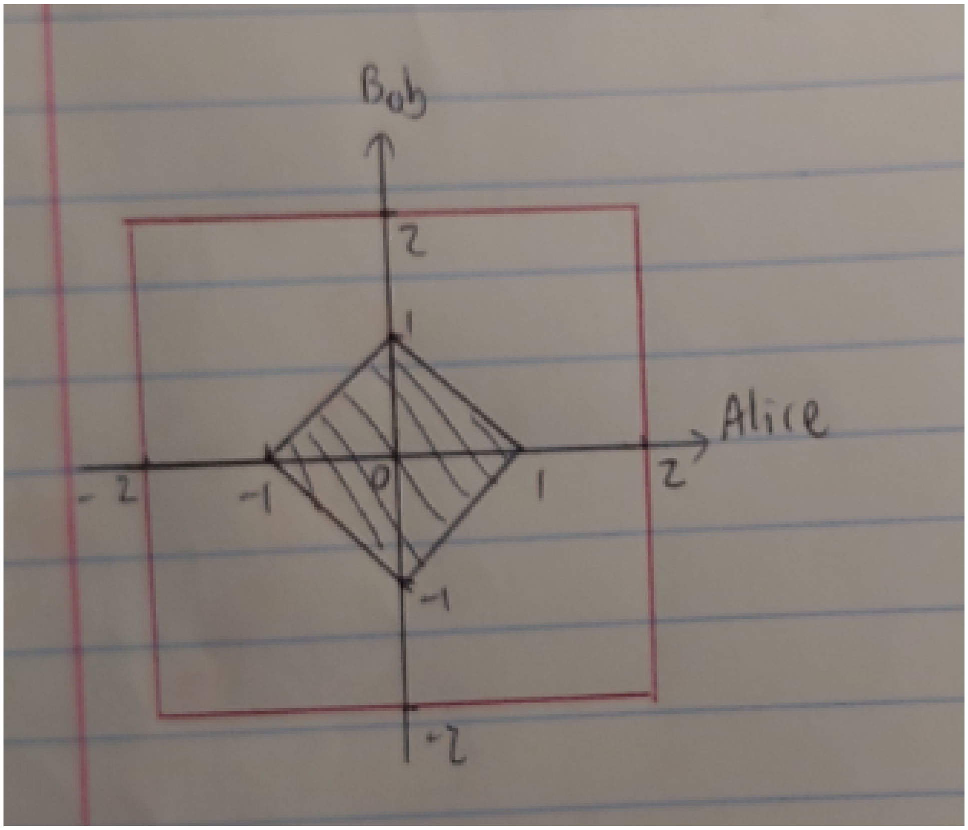
   E : Alice's number is greater than 1/2 and greater than Bob's

   Find the probabilities $P(A), P(B), P(C), P(D), P(E), P(B \cap C), P(A \cup C)$[Problem 2.4 0.61]

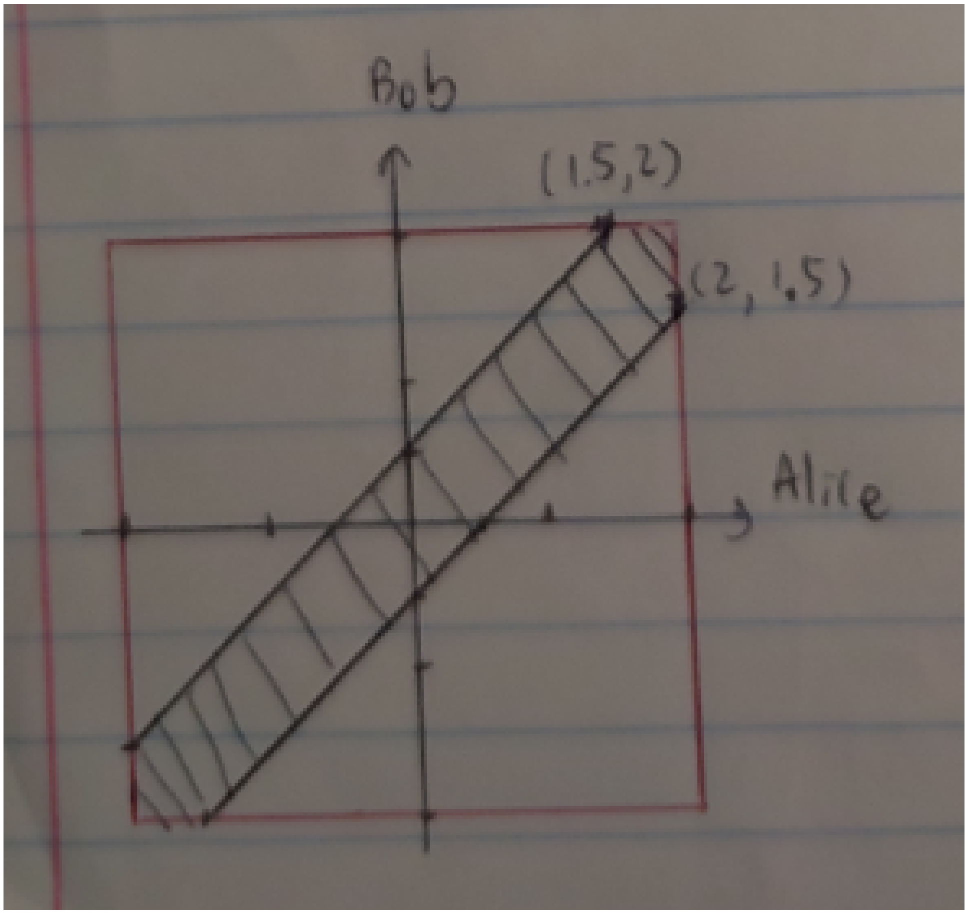
   Solution



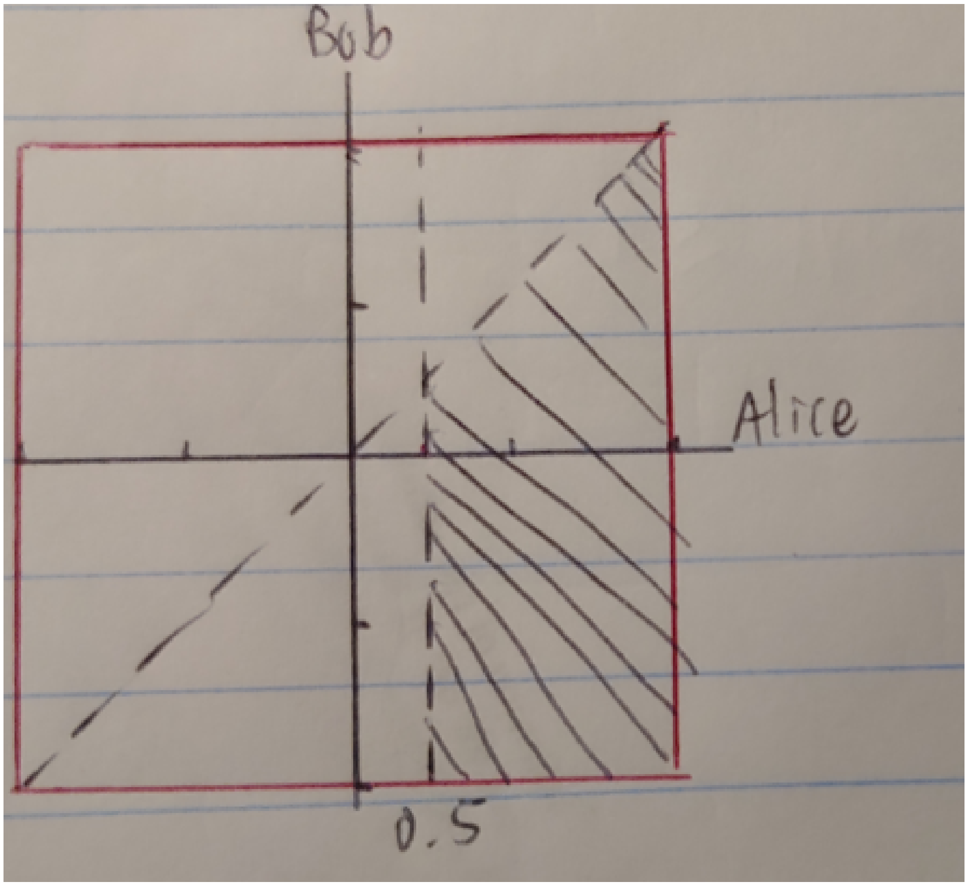
   $P(A) = \frac{\text{shaded area}}{16}$



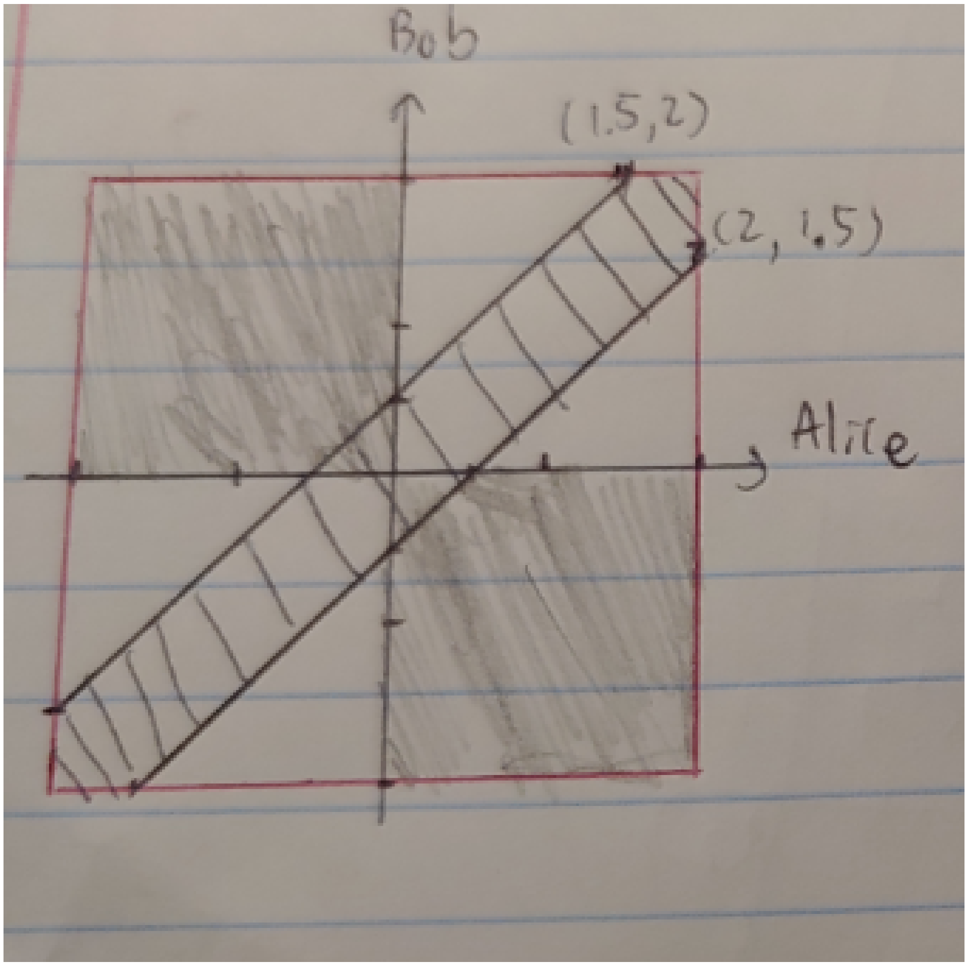
$P(B) = \frac{\text{shaded area}}{16}$



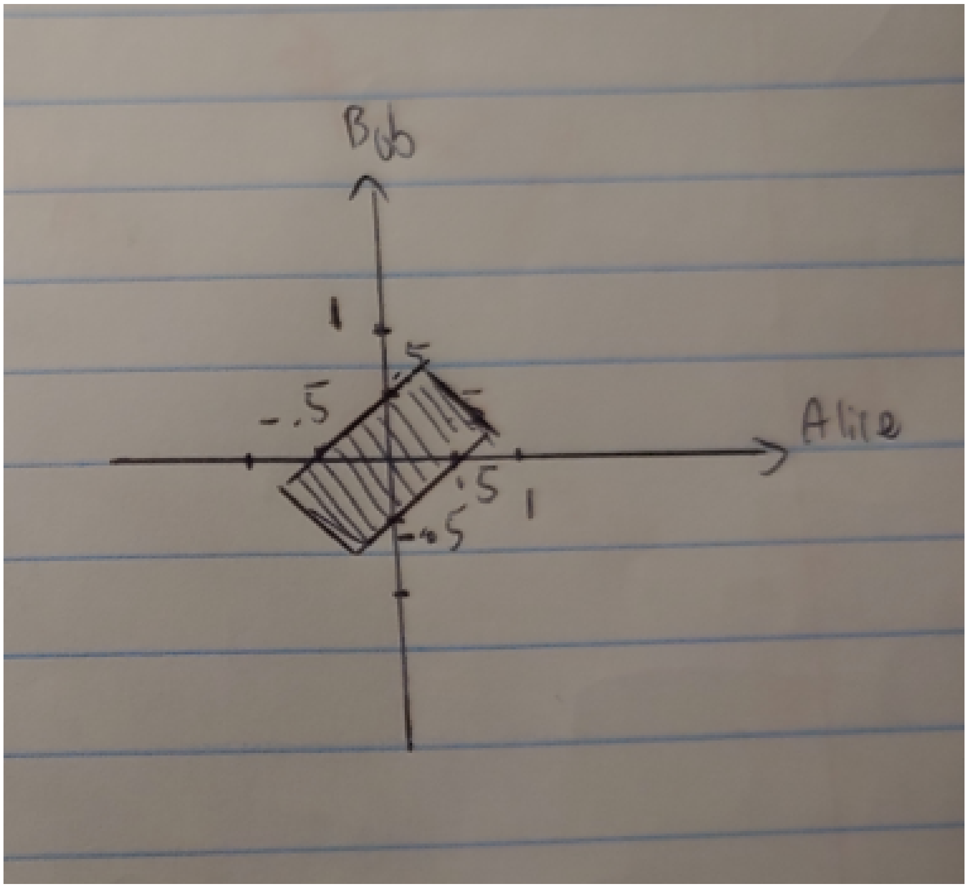
$P(C) = \frac{\text{shaded area}}{16}$
$P(D) = 0$



$P(E) = \frac{\text{shaded area}}{16}$



$P(B \cap C) = \frac{\text{shaded area}}{16}$



$P(A \cup C) = \frac{\text{shaded area}}{16}$

2. A disease D affects 2% of the total population in a city. An individual can get himself tested to see if he has caught disease D. However, the test gives wrong results in 10% of the cases when the individual actually has the disease. The error rate increases to 20% in cases where individual doesn't have the disease. If the test indicates disease for a particular individual, what is the probability that the test results are correct?[Problem 2.3 0.73]

   Solution

   Let D: having disease D

   D': not having disease D

   C: Correctly detecting D

   $P(D) = 0.02$

   $P(D') = 0.98$

   $P(C|D) = 0.9$

   $P(C|D') = 0.2$

$$P(D|C) = \frac{P(D \cap C)}{P(C)} = \frac{P(D)P(C|D)}{P(D)P(C|D) + P(D')P(C|D')}$$

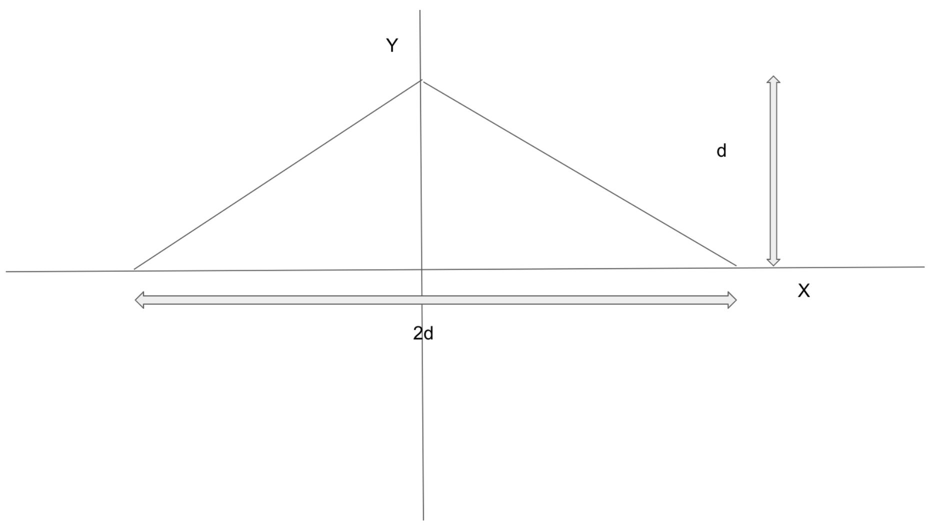
3. A point is chosen at random within an area of the form $\{|x| + |y| <= d, y >= 0\}$, for some given $d > 0$. Consider uniform probability over the area.[Problem 5.4 0.79]

   (a) Find the joint PDF of the coordinates X and Y of the chosen point.

   (b) Find the marginal PDF of Y and use it to find $E[Y]$.

   (c) Check your answer in (b) by computing $E[Y]$ directly without using the marginal PDF of Y.

   (d) Find the expected value of $E[XY]$ and $E[X+Y]$.



(a) Find the joint PDF of the coordinates X and Y of the chosen point. [6 points]

Solution Area of the region : $\frac{1}{2}(d)(2d) = d^2$ Let $f(x, y)$ be the joint pdf of x and y. Since pdf is uniform over area:

$$f(x, y) = \frac{1}{d^2}$$

$$f(x, y) = \begin{cases} \frac{1}{d^2} & |x| + |y| \leq d, y \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

(b) Find the marginal PDF of Y and use it to find $E[Y]$. [7 points]

Solution

$|x| + |y| \leq d, y \geq 0$, solve for x

$$|x| \leq d - |y|$$
$$-d + y \leq x \leq d - y$$

$$f(y) = \int_{-d+y}^{d-y} \frac{1}{d^2} dx$$

$$f(y) = \begin{cases} \frac{2(d-y)}{d^2} & 0 \leq y \leq d \\ 0 & \text{Otherwise} \end{cases}$$

$$E[Y] = \int_0^d y f(y) dx = \int_0^d \frac{2y(d-y)}{d^2} dx = \frac{d}{3}$$

(c) Check your answer in (b) by computing $E[Y]$ directly without using the marginal PDF of Y. [7 points]

Solution

$$E[Y] = \int\int yf(x,y)dxdy = \int_0^d \int_{-d+y}^{d-y} y\frac{1}{d^2}dxdy = \frac{d}{3}$$

(d) Find E[XY] and E[X+Y] **Solution** Calculating $E[X+Y]$:

$$E[X+Y] = E[X] + E[Y]$$

Since x ranges from -d to +d, it is symmetrical about origin. Hence, x and -x will cancel each other while calculating expectation:

$$E[X] = 0$$

$$E[X+Y] = E[Y] = \frac{d}{3}$$

Calculating $E[XY]$: Again, it can be argued that point (x, y) and point $(-x, y)$ will cancel each other out while calculating expectation of XY. Hence,

$$E[XY] = 0$$

4. Let A and B be two sets. Under what conditions is the set $(A \cup B^c) \cap A$ empty? Explain your answer.[Problem 2.2 0.79]

5. For each of the distributions defined over natural numbers, state whether or not[Problem 1 Midterm 0.59]:
   (a) The distribution is well defined.
   (b) The distribution has a finite expected value.
   (c) The distribution has finite variance.

Circle the correct answer in each of the nine cells:

Distribution	Well Defined	Finite expected value	Finite Variance
$X = i = 1/(Y_3 i^3)$	Yes / No	Yes / No	Yes / No
$X = i = 1/(Y_4 i^4)$	Yes / No	Yes / No	Yes / No
$X = i = i/(Z_1 2^i)$	Yes / No	Yes / No	Yes / No
$X = i = i^2/(Z_2 2^i)$	Yes / No	Yes / No	Yes / No

Where Y_k is a normalization factor such that

$$Y_k = \sum_{i=1}^{\infty} \frac{1}{i^k}$$

It is known that Y_k converges for k being a natural number and k greater than 1.

Where Z_k is a normalization factor such that

$$Z_k = \sum_{i=1}^{\infty} \frac{1}{2^i} i^k$$

It is known that Z_k converges for k being a natural number

Solution

Distribution	Well Defined	Finite expected value	Finite Variance
$X = i = 1/(Y_3 i^3)$	Yes	Yes	No
$X = i = 1/(Y_4 i^4)$	Yes	Yes	Yes
$X = i = i/(Z_1 2^i)$	Yes	Yes	Yes
$X = i = i^2/(Z_2 2^i)$	Yes	Yes	Yes

6. Consider an IID binary sequence $X_1, X_2, \ldots, X_n$ where $X_i = 1$ with probability p and $X_i = 0$ with probability $1 - p$, and the X_i are independent[Problem 2 Midterm 0.75].

   In each of the following questions, show your work, i.e. show how you derived the answer.

   (a) What is the expected sum of the sequence? What is the variance of the sum?

   (b) What is the probability that the **second** 1 in the sequence is in position k? assuming ($n \to \infty$ and relevant position numbers are from 2 to ∞).

   (c) What are the **expected** number of zeros before the first 1 (as in the previous part, $n \to \infty$)?

   (d) Let Y_i be a random variable that is equal to 1 if X_{i-1} and X_{i+1} are same, Let $S = \sum_{i=2}^{n-1} Y_i$. What is the expected value of S (for this part n is finite)?

   Solution

   (a) $E(X) = np$
   $Var(x) = np(1-p)$

   (b) $P(X = k) = (k-1)(1-p)^{k-2}p^2$

   (c) $P(Y = k) = (1-p)^{k-1}p$
   $E(Y) = \frac{1}{p} - 1$

   (d) $E[\sum_{i=2}^{n-1} Y_i] = \sum E[Y_i]$
   $E[Y_i] = 1 * (1 - 2p(1-p)) + 0 * (2p(1-p)$
   $Var(Y_i) = (n-2)(1 - 2p(1-p))$