CSE 21 Fall 2018
Homework 5
Due: Sunday, November 3, 2018 at 10:00pm

INSTRUCTIONS

Homework should be done in groups of **one to three** people. You are free to change group members at any time throughout the quarter. Problems should be solved together, not divided up between partners. Homework must be submitted through **Gradescope** by **a single representative**. Submissions must be received by **10:00pm** on the due date, and there are no exceptions to this rule.

You will be able to look at your scanned work before submitting it. Please ensure that your submission is **legible** (neatly written and not too faint) or your homework may not be graded.

Students should consult their textbook, class notes, lecture slides, instructors, TAs, and tutors when they need help with homework. Students should not look for answers to homework problems in other texts or sources, including the Internet. You may ask questions about the homework in office hours, but **not on Piazza**.

Your assignments in this class will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should **always explain** how you arrived at your conclusions and **justify your answers** with mathematically sound reasoning. Whether you use formal proof techniques or write a more informal argument for why something is true, your answers should always be well-supported. Your goal should be to **convince the reader** that your results and methods are sound.

For questions that require pseudocode, you can follow the same format as the textbook, or you can write pseudocode in your own style, as long as you specify what your notation means. For example, are you using "=" to mean assignment or to check equality? You are welcome to use any algorithm from class as a subroutine in your pseudocode. For example, if you want to sort list A using InsertionSort, you can call InsertionSort(A) instead of writing out the pseudocode for InsertionSort.

REQUIRED READING Rosen 10.1, 10.2, 10.3, 10.4 through Theorem 1, 10.5 through Example 7.

KEY CONCEPTS:Graphs (definitions, modeling problems using graphs), Hamiltonian tours, Eulerian tours, Fleury's algorithm.

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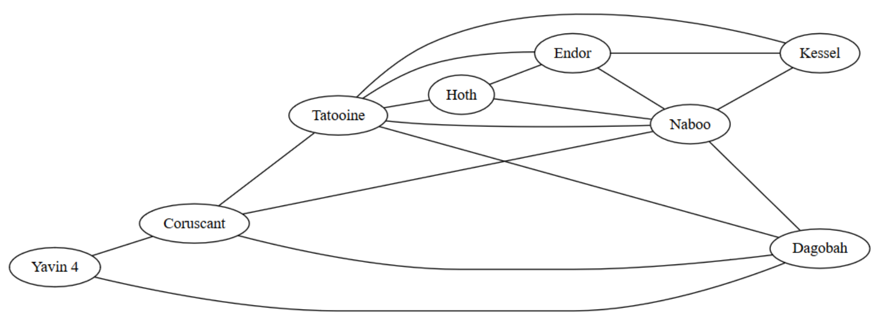
1. Han and Chewie are on the run from the Imperial Army. Starting from the Rebel's base on Yavin 4, in order to shake off Darth Vader's pursuit, they have to make a series of hyperspace jumps to seven different planets then return to the original base. However, the Millennium Falcon cannot make any jump longer than 12 *parsec* (Note: *parsec* is a unit of distance, not time). They looked up their star map and made the following table of the distances between planets. All the entries below are measured in *parsec*.

	Yavin 4	Coruscant	Tatooine	Hoth	Endor	Naboo	Kessel	Dagobah
Yavin 4	0	10	15	15	18	14	16	12
Coruscant	10	0	12	13	16	12	14	10
Tatooine	15	12	0	10	12	9	10	12
Hoth	15	13	10	0	12	12	13	15
Endor	18	16	12	12	0	12	11	16
Naboo	14	12	9	12	12	0	9	12
Kessel	16	14	10	13	11	9	0	14
Dagobah	12	10	12	12	16	12	14	0

(a) (4 points) Draw a graph that will help Han and Chewie plan their escape. Specify whether your graph is directed or undirected. Carefully describe what the vertices of your graph represent, and when two vertices are connected with an edge.

Solution: This is an undirected graph. Here each vertex represents a planet. We put an edge between two vertices if an only if the distance between the two corresponding planets is less than or equal to 12 *parsec*.



(b) (2 points) Say which graph theory problem they are trying to solve on this graph.

Solution: Han and Chewie are trying the solve the **Hamiltonian Circuit problem**. They wish to find a path which visits every vertex in a graph exactly once and is also a cycle (start and end at the same vertex - Yavin 4).

(c) (2 points) Find a feasible escape route for Han and Chewie that meets the constraints, or explain why no such route exists.

Solution: There are many answers. Here is one such path: Yavin 4 → Coruscant → Tatooine → Kessel → Endor → Hoth → Naboo → Dagobah → Yavin 4.

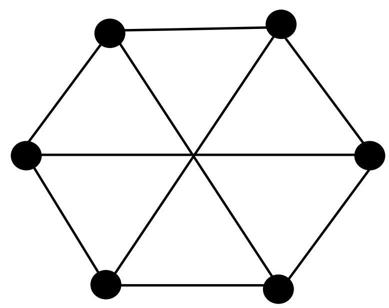
2. A *d-regular graph* is a **simple, undirected** graph in which every vertex has degree d.

   (a) (2 points) Draw a 3-regular graph with 5 vertices, or prove why it is impossible.

   Solution: It is impossible to have a graph with 5 vertices, each of degree 3, because that would make the sum of the degrees of the vertices 15, and we showed that the sum of the degrees of the vertices must be twice the number of edges, which is always an even number.

   (b) (2 points) Draw a 3-regular graph with 6 vertices, or prove why it is impossible.

   Solution:



   (c) (2 points) For which values of $n > 3$ is it possible to draw a 3-regular graph with n vertices?

   Solution: It's possible to draw a 3-regular graph with n vertices for any even value of $n > 3$. One way to do this is arrange the n vertices in a circle and then connect each vertex to two vertices next to it and the one across the circle. For example, the graph given in part (b) does this for $n = 6$. It's impossible to draw a 3-regular graph with n vertices for any odd value of $n > 3$ because that would make the sum of the degrees odd, which cannot happen.

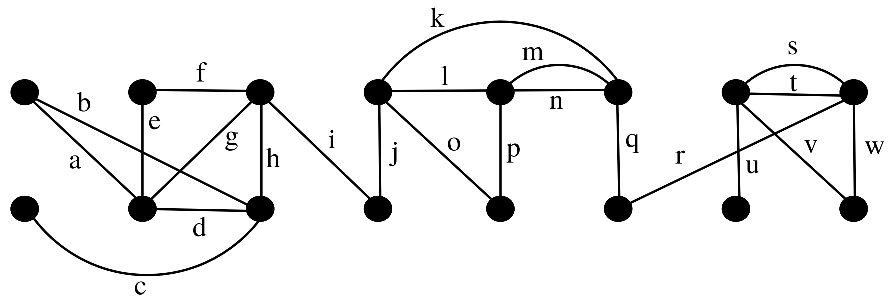
   (d) (2 points) For which values of d does a d-regular graph with 10 vertices have an Eulerian circuit?

   Solution: A d-regular graph with 10 vertices has an Eulerian circuit if and only $d = (0), 2, 4, 6, 8$. First, note that for any value of $d \geq 10$, it is impossible to have a simple graph with 10 vertices of degree d, since the degree of each vertex is more than the number of other vertices in the graph, and simple graphs cannot have loops or multiple edges. Second, we know that for a graph to have an Eulerian circuit, all the vertices must have even degree. We could optionally incude $d \neq 0$, which is just a bunch of isolated vertices, so it could be said to have a trivial Eulerian tour since there are no edges in the graph.

   (e) (2 points) For which values of d does a d-regular graph with 10 vertices have an Eulerian path that starts and ends at **different** vertices?

   Solution: A d-regular graph with 10 vertices can never have an Eulerian path that starts and ends at different vertices. In order to have such a tour, we'd need two of the vertices to have odd degree and the other eight vertices to have even degree. But in a d-regular graph, all the vertices have the same degree, so this is impossible.

3. Consider the following graph:



(a) (2 points) Which of the edges in the graph above are bridges?

Solution: The following edges are bridges c, i, j, q, r, u

(b) (3 points) Use Fleury's algorithm to find an Eulerian tour of the graph above. Suppose that whenever the algorithm allows you a choice for which edge to take, you always take the edge whose label comes first alphabetically. For example, if you were in a position where you could take edge b, edge c, or edge h, you would take edge b. Write down the Eulerian tour you find by listing the edges of your tour in order.

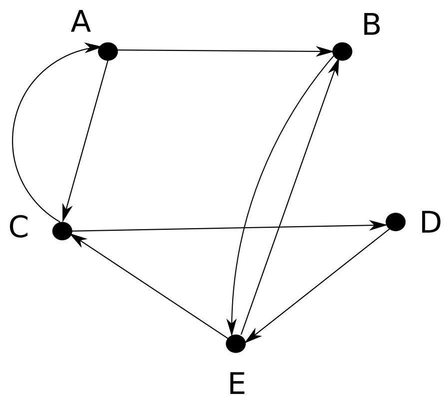
Solution: $c, b, a, d, h, f, e, g, i, j, k, m, l, o, p, n, q, r, s, t, w, v, u$

4. We say a matrix has dimensions $m \times n$ if it has m rows and n columns. If matrix A has dimensions $x \times y$ and matrix B has dimensions $z \times w$, then the product AB exists if and only if $y = z$. In the case where the product exists, AB will have dimensions $x \times w$. In this problem, we are given a list of matrices and their dimensions, and we want to determine if there is an order in which we can multiply all the matrices together, using each matrix exactly once. The list of matrices and their dimensions is as follows:

A is 4×5,
B is 5×7,
C is 5×4,
D is 4×7,
E is 7×5.

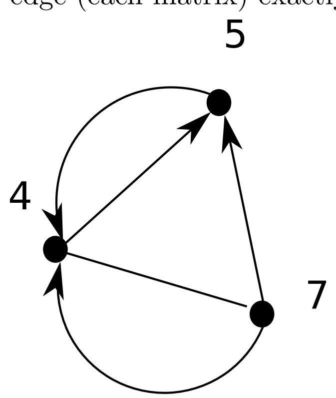
(a) (4 points) Draw a graph that represents this situation in such a way that finding an order we seek corresponds to finding a Hamilton Tour of your graph. Describe what the vertices of your graph represent, and when two vertices are connected with an edge.

Solution: One way is to use each matrix as a vertex, and connect matrix X to matrix Y if the matrix product XY exists. In this setting, we want a Hamilton tour that uses each vertex (each matrix) exactly once.



(b) (4 points) Draw a different graph that represents this situation in such a way that finding an order we seek corresponds to to finding an Euler Tour of your graph. Describe what the vertices of your graph represent, and when two vertices are connected with an edge.

Solution: Another way is to use each dimension that shows up in the list as a vertex, in this case $4, 5, 7$. Connect vertex m to n if there is an $m \times n$ matrix in the list. In this setting, an order that we want corresponds to an Euler Tour because we must use each edge (each matrix) exactly once.



(c) (2 points) Give an order in which we can multiply these matrices, or say that no such order exists.

Solution: DECAB, ABECD, ACDEB are all possible answers.

5. (10 points) We say that two mathematicians are co-authors or collaborators if they have published a research paper together. Show that in any group of six mathematicians, we can either find a group of three such that all pairs in the group are co-authors, or a group of three so that no two in the group are co-authors.

   Solution: We can formulate our answer as a problem of graph theory by constructing an undirected graph as follows. Let the vertex set V be the set of six mathematicians. Connect two mathematicians with an edge if they are co-authors. We must show that one of the following two situations occurs:

   a The graph contains a triangle, i.e. three vertices which are all connected.

   b The graph contains an anti-triangle, i.e. three vertices, none of which are connected.

   Choose one particular vertex, let's say v, and then there are two cases:

   i. $\deg(v) \geq 3$

   ii. $\deg(v) < 3$

   In Case (i), consider the neighbors of v. If no two of them are neighbors of one another, then since there are at least three of them, this forms an anti-triangle, situation (b). If some two of them are neighbor of one another, then those two together with the current vertex v form a triangle, situation (a).

   In Case (ii), if $\deg(v) < 3$ then that means there are at least three vertices that are not neighbors of v. Consider these non-neighbors of the vertex v. If all of them are neighbors of one another, then since there are at least three of them, this forms a triangle, situation (a). If there exists a pair of vertices that are not neighbors of one another, then those two together with $v$form an anti-triangle, situation (b).

   In all cases, we can always be in situation (a) or (b), which is what we were trying to prove.

6. In class, we saw that a necessary condition for an undirected graph $G = (V, E)$ to have an Euler tour is that G must have zero or two odd-degree vertices. Now we want to determine what conditions are necessary for a directed graph to have an Euler tour. For a directed graph, we define

- the *in-degree* of a vertex v, $\deg_{in}(v)$, to be the number of incoming edges into v (with arrows pointing towards v),
- the *out-degree* of v, $\deg_{out}(v)$, to be the number of outgoing edges from v (with arrows pointing away from v), and
- the *balance* of v, $balance(v)$, to be the in-degree of v minus the out-degree of v.

a) (3 points) Prove that for any directed graph $G = (V, E)$,

$$\sum_{v \in V} balance(v) = 0.$$

Solution: We can show that the sum of vertex balances is 0 by splitting the balance into in-degree and out-degree, and rearranging sums

$$\begin{aligned}\sum_{v \in V} balance(v) &= \sum_{v \in V} (\deg_{in}(v) - \deg_{out}(v)) \\ &= \sum_{v \in V} \deg_{in}(v) - \sum_{v \in V} \deg_{out}(v) \\ &= |E| - |E| = 0\end{aligned}$$

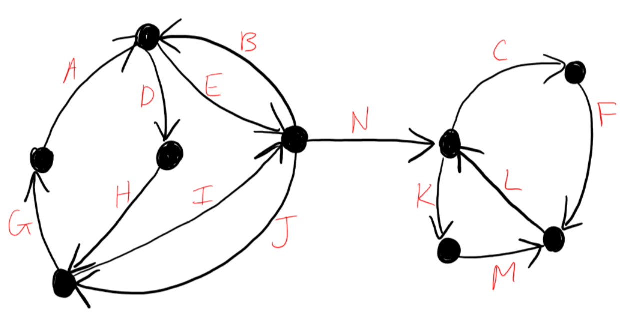
The total in-degree and out-degree should be equal, since there is one in-degree and one out-degree for every edge.

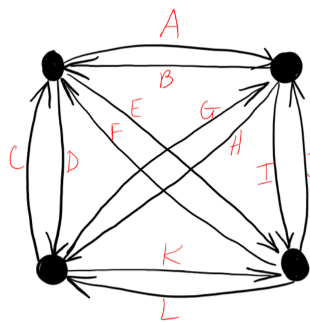
b) (3 points) State (without proof) a necessary condition in terms of balance for a directed graph $G = (V, E)$ to have an Euler path. That is, what condition (involving balance) must be met to ensure that G has an Euler path? Write your condition as an OR statement to encompass the possibility of an Eulerian circuit as well as an Eulerian path that's not a circuit.

Solution: A directed graph has an Eulerian path if the balance of every vertex is 0 or it has one vertex with balance 1 (end-point), one vertex with balance -1 (start-point), and all other vertices have balance 0.

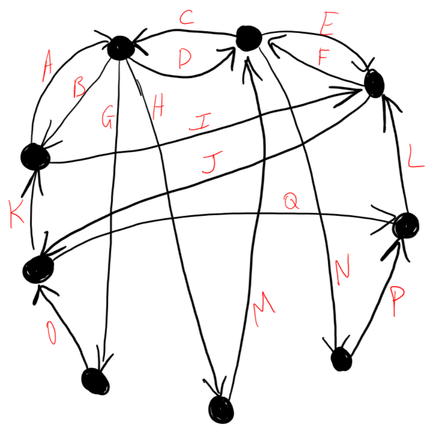
c) (6 points) For each of the following directed graphs, find an Eulerian tour, or say that the graph has no Euler tour.

i.





ii.



iii.

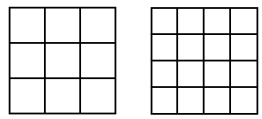
Solution:

i. There is an Eulerian path: $J\ I\ B\ D\ H\ G\ A\ E\ N\ C\ F\ L\ K\ M$
Note that all Eulerian paths must start at the vertex from which J is an outgoing edge and end at the vertex for which M is an incoming edge. This means that all Euler paths for this graph start with edge J or B and end with edge M or F.

ii. There is an Euler tour (in fact a cycle): $K\ L\ G\ H\ C\ E\ F\ A\ I\ J\ B\ D$
Note that this every vertex in this graph has balance 0, so it has Eulerian cycles which can start from any vertex.

iii. This graph has a vertex whose balance is -2 (top left vertex, with edge A incoming). Thus by part (b) the graph cannot have an Eulerian path.

7. You are an intern at Stark Industries working at Avengers Tower. As an eccentric man, Tony Stark designed his building in a peculiar shape where the n-th floor consists of n^2 rooms arranged into an $n \times n$ array. For example, the floor plans of the third and fourth floors are illustrated below. Here you can only travel back and forth between every two rooms if they are adjacent, i.e. sharing a wall.

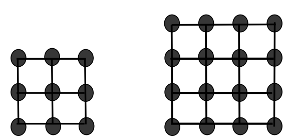


F3 *F4*

One day, there is an intruder inside the tower so you and your fellow intern Peter Parker have to lock down every room to prevent the Avengers' technologies from falling into the wrong hands. Once a room is locked with Peter's special ability, it is no longer available and you cannot revisit it.

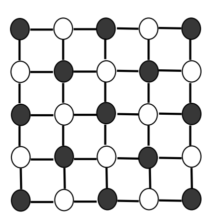
(a.) (9 points) Suppose that you **start at a room adjacent to a corner room**. Prove that when $n \geq 3$ and n is odd, then it is impossible to lock down every room on the n-th floor in this case.

Solution: In terms of graph theory, we turn the rooms into the vertices. We put an undirected edge between any pair of vertices if the corresponding rooms are adjacent. For $n \geq 3$, this process results in a grid graph G_n where n^2 vertices are arranged in an $n \times n$ grid. The graphs G_3 and G_4 are shown below. Under this set up, the question is equivalent to asking for the existence of a Hamiltonian path in G_n, start at a vertex adjacent to a corner, for n odd and $n \geq 3$.



G_3 G_4

We color the graph G_n using two colors, say black and white, in a checkerboard pattern. For example, this coloring of G_5 would look like this:



If n is odd, there will be $\frac{n^2+1}{2}$ black vertices and $\frac{n^2-1}{2}$ white vertices. We can see this because in every pair of two columns, there are the same number of black and white vertices. But the last column will be unpaired, and it has one more black vertex than white. So there is exactly one more black vertex than white vertex, and a total of n vertices, which means there are $\frac{n^2+1}{2}$ black vertices and $\frac{n^2-1}{2}$ white vertices.

Notice that any path through this colored graph must alternate between black and white vertices, since each edge has one endpoint which is black and the other which is white. So a Hamiltonian tour, which is a path using each vertex once, must visit all the vertices in an alternating black-white fashion. Since there is one more black vertex than white vertex, this means a Hamiltonian path must start and end at a black vertex. Since the vertex adjacent to the corner is white, it's impossible to have a Hamiltonian tour starting from here.

(b.) (6 points) When $n \geq 3$ and n is even, is it possible to lock down every room on the n-th floor if you **start at a room adjacent to a corner room**? Illustrate on the map of floors $4, 6$, and 8.

Solution: