CSE 103

Homework #5 Solution
Fall 2019

Due: Monday, November 4, 2019 at 11:00PM on Gradescope

1 Directions

You may work with one other student. If working with a partner, **submit only one submission per pair** : one partner uploads the submission and adds the other partner to the Gradescope submission. You can post public questions about the assignment to Piazza, discuss the questions and their answers with at most one other student, and ask questions in office hours

Your answers have to be typeset, not handwritten. This is for two reasons: (a) to reduce ambiguity of the answers, and (b) to be kind to the TA's eyesight. We recommend you use latex, but you can also use word-processors that support mathematical formulas. More directions are available here: `https://tinyurl.com/y2gv9bn9`.

You will submit this assignment via Gradescope (`https://www.gradescope.com`) in the assignment called "Homework 5". You can submit each question as many times as you like. You should solve the problems and ask questions about them offline first, then try submitting once you are confident in your answers.

No late submissions are accepted.

2 Problems

1. Let random variable X be uniformly distributed in the unit interval [0,1]. Consider the random variable $Y = g(X)$, where $g(x)$ is defined as:

$$\begin{cases} 1 & x \leq 1/3 \\ 2 & x \geq 1/3 \end{cases}$$

(a) Find the expected value of Y by first deriving its PMF. [8 points]
Solution
Y takes on only two values: 1 and 2. The probability of a 1 is 1/3 and the probability of 2 is 2/3. Writing this as a PMF:

$$P(Y = k) = \begin{cases} 1/3 & \text{if } k = 1 \\ 2/3 & \text{if } k = 2 \\ 0 & \text{otherwise} \end{cases}$$

Using this PMFe we can calculate the txpected value of Y

$$E(Y) = 1 \times \frac{1}{3} + 2 \times \frac{2}{3} = \frac{5}{3}$$

(b) Verify the result obtained in (a) using the expected value rule on function of a random variable X. [8 points]
Solution
X is a uniformly distributed continuous random variable whose PDF is

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

therefor the expected value of $Y = g(X)$ is an integral:

$$E(Y) = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

As $f_X(x)$ is zero outside of $0 \leq x \leq 1$ we can limit the integral to the range $[0, 1]$. Further, as $g(x)$ is 1 i the range $[0, 1/3]$ and 2 in the range $[1/3, 1]$ we can rewrite the last integral as follows:

$$\int_{-\infty}^{\infty} g(x) f_X(x) dx = \int_{0}^{1/3} 1 dx + \int_{1/3}^{1} 2 dx = 1/3 + (2/3) \times 2 = \frac{5}{3}$$

and we recover the same answer as in part a

2. Consider a triangle and a point chosen within the triangle according to the uniform probability law. Let X be the distance from the point to the base of the triangle. Given the height of the triangle, find the CDF and the PDF of X [16 points]

Solution
Consider a randomly chosen point inside the triangle which is at distance X from the base. Then consider the term $P(X > x)$ for some $0 < x < h$ where h is height of the triangle. The term $P(X > x)$ denotes the probability that the distance of the randomly chosen point from base is greater than x. This probability can be calculated by constructing a triangle with same vertex as original but a base parallel to the original base and at distance x from the original base. $P(X > x)$ then is just the ratio of this triangle and the original triangle. Using the concept of similar triangles, this ratio comes out to be : $(h - x)^2/h^2$

Hence, $P(X \leq x)$ can be computed using :

$$P(X \leq x) = 1 - P(X > x)$$
$$P(X \leq x) = 1 - (h - x)^2/h^2$$
$$CDF(X) = 1 - (h - x)^2/h^2$$

We will now differentiate the CDF that we got above w.r.t to x to get the pdf. The pdf is thus :

$$pdf_X = \frac{d(1 - (h - x)^2/h^2)}{dx} (for\ 0 < x < h)$$

$$pdf_X = \frac{2(h - x)}{x^2} (for\ 0 < x < h)$$

3. Calamity Jane goes to the bank to make a withdrawal, and is equally likely to find 0 or 1 customers ahead of her. The service of the customer ahead, if present, is exponentially distributed with parameter λ. What is the CDF of Jane's waiting time? [18 points]

Solution
Let X denote the waiting time for Jane. Then, we are supposed to find out the CDF of X. i.e. $P(X \leq x)$ where $x \geq 0$. Also, let Y denote the random variable denoting the number of customers Jane finds ahead of her when she arrives at the bank. Y can only be 0 or 1. Thus,

$$P(X \leq x) = P(X \leq x \cap Y = 0) + P(X \leq x \cap Y = 1)$$

$$P(X \leq x) = P(X \leq x|Y = 0)P(Y = 0) + P(X \leq x|Y = 1)P(Y = 1)$$

It is given that $P(Y = 0) = P(Y = 1) = 0.5$

Computing $P(X \leq x|Y = 0)$: This is always 1 because if there is no customer ahead of Jane, then, waiting time is always 0 and for any $x \geq 0$, we are certain that Jane's waiting time given Y=0 is 0.

Computing $P(X \leq x|Y = 1)$: If there is one customer ahead of her, the waiting time of Jane is equal to the service time of the customer in front. Since the customer's service time is exponentially distributed, so is Jane's waiting time in this case. This term thus just becomes the CDF of exponential random variable : $1 - e^{-x\lambda}$

Final Answer Substituting all the terms in the calculated above :

$$P(X \leq x) = 0.5(1 + (1 - e^{-x\lambda}))$$

$$P(X \leq x) = 1 - 0.5e^{-\lambda x}$$

4. A point is chosen at random (according to a uniform PDF) within a semicircle of the form $\{(x,y)|x^2+y^2 <= r^2, y >= 0\}$, for some given $r > 0$.

   (a) Find the joint PDF of the coordinates X and Y of the chosen point. [6 points]
   Solution Area of half of a circle: $\frac{1}{2}\pi r^2$

$$f(x,y) = \frac{1}{\frac{1}{2}\pi r^2}$$

$$f(x,y) = \begin{cases} \frac{2}{\pi r^2} & x^2+y^2 \leq, y \geq 0 \\ 0 & \text{Otherwise} \end{cases}$$

   (b) Find the marginal PDF of Y and use it to find $E[Y]$. [7 points]
   Solution
   $x^2+y^2 \leq, y$, solve for x

$$-\sqrt{r^2-y^2} \leq x \leq \sqrt{r^2-y^2}$$

$$f(y) = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{2}{\pi r^2} dx$$

$$f(y) = \begin{cases} \frac{4}{\pi r^2}\sqrt{r^2-y^2} & 0 \leq y \leq r \\ 0 & \text{Otherwise} \end{cases}$$

$$E[Y] = \int_{-\infty}^{+\infty} y f(y) dx = \int_0^r \frac{4}{\pi r^2}\sqrt{r^2-y^2} dx = \frac{4\pi}{3r}$$

   (c) Check your answer in (b) by computing $E[Y]$ directly without using the marginal PDF of Y. [7 points]

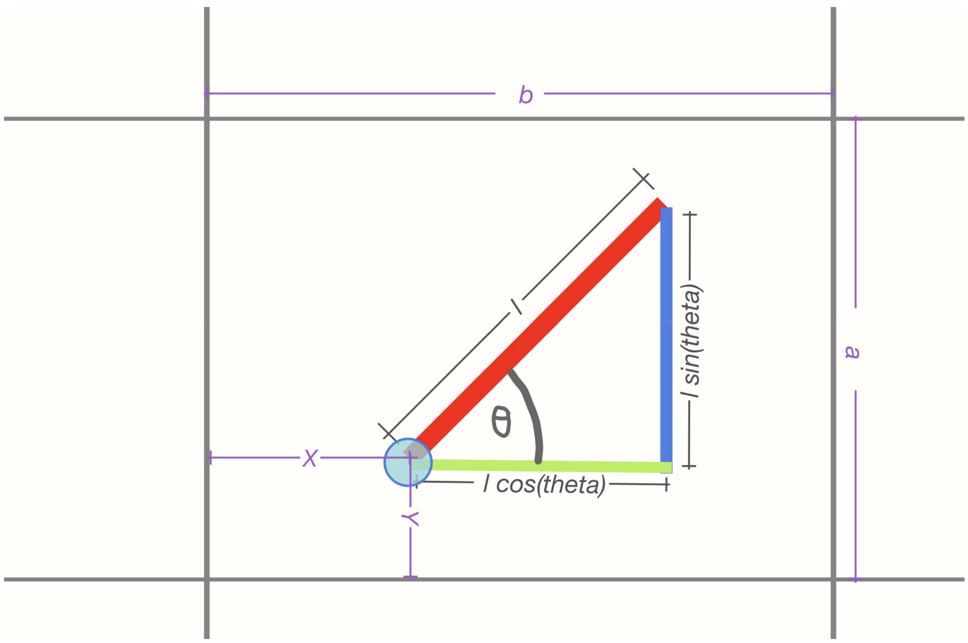
Solution

$$-\sqrt{r^2-y^2} \leq x \leq \sqrt{r^2-y^2}, 0 \leq y \leq r$$

$$E[Y] = \int\int y f(x,y) dxdy = \int_0^r \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} y \frac{2}{\pi r^2} dxdy = \frac{4\pi}{3r}$$

5. A needle of length l is dropped on an infinite plane surface that is partitioned into rectangles by horizontal lines that are a apart and vertical lines that are b apart. Suppose that the needle's length l satisfies $l < a$ and $l < b$. What is the expected number of rectangle sides crossed by the rectangle? What is the probability that the needle will cross at least one side of some rectangle? [30 points]

Solution



The figure shows a needle (red segment) inside one of the rectangles defined by the grid. The height of the rectange is a and the width is b. lower left end of the needle is marked with a blue circle. We will define the sample space with respect to the lower left point (LL).

If the needle's two ends are the upper left and the lower right, it does not have a lower left end. Doing so does not change whether the needle crosses a line, and so it does not change probabilities.

The sample space for this problem is $\Omega = \{(x, y, \theta)\}$, where $0 \leq x \leq b$ defines the horizontal location of the blue circle relative to the left side of the rectangle, $0 \leq y \leq a$ defines the vertical location of the circle relative to the bottom of the rectangle. Finaly, $\theta \in [0, \pi/2]$ is the angle of the needle relative to the positive direction of the horizontal line. The distribution over Ω is a uniform density, which can be expressed as a product of three uniform distributions. X is distributed according to $U[0, b]$, Y is distributed according to $U[0, a]$, and θ is distributed according to $U[0, \pi/2]$.

The observation that simplifies the analysis is that the needle (red) intersects a horizontal line if and only if the horizontal line intersects the vertical projection of the pin (the blue line). Likewise the needle intersects a vertical line if and only if the vertical line intersects the horizontal projection (the green segment)

In addition note that, if the angle θ is fixed, then the two events:

$$V(\theta) = \{\text{needle at angle } \theta \text{ intersects with a vertical line}\}$$

and

$$H(\theta) = \{\text{needle at angle } \theta \text{ intersects with a horizontal line}\}$$

are independent. To convince yourself that is the case note that the intersection with a vertical line depend only on the horizontal location of the pin, while the intersection with a horizontal line depends only on the vertical position. Dropping the needle in a random location on the plane means that the vertical and horizontal locations are independent.

Now, for some calculations. The length of the vertical projection (the blue line) is $l \sin\theta$, while the length of the horizontal projection (the green line) is $l\cos\theta$. From this we get that

$$P(V(\theta)) = \frac{l\cos\theta}{b}; \quad P(H(\theta)) = \frac{l\sin\theta}{a}$$

.

From this and from the fact that $H(\theta)$ and $V(\theta)$ are independent we can answer the questions.

We now calculate the expected number of intersections. As l is smaller than a and b, there can be 0 or 1 horizontal intersections and 0 or 1 vertical intersections. we calclate the probability of an intersection in each direction an then use linearity of expectation to compute the expected number of intersections.

Let W, Q be the random variables that are the indicator functions of V and H. In other words, W is 1 if there is an intersection with a vertical line and zero otherwise. Similarly for Q and H. We can calculate the expected value of each RV:

$$E[W] = \int_0^{\pi/2} \frac{l\cos\theta}{b}\frac{2}{\pi}d\theta = \frac{2l}{b\pi}$$

$$E[Q] = \int_0^{\pi/2} \frac{l\sin\theta}{a}\frac{2}{\pi}d\theta = \frac{2l}{a\pi}$$

we get from linearity of expectations that the expected number of intersections is

$$E[W+Q] = \frac{2l}{b\pi} + \frac{2l}{a\pi} = \frac{2l}{\pi}\left(\frac{1}{a}+\frac{1}{b}\right)$$

As for crossing at least one side of the rectangle. Fixing the angle θ we can express the probability of crossing at least one side as the complement of not crossing either. Using the independence between vertical intersection and horizontal intersection when the angle is kept fixed.

$$\begin{aligned} P(\text{at least one side is crossed}|\theta) &= 1 - P(\text{no side is crossed}|\theta) \\ &= 1 - \left(1 - \frac{l\sin\theta}{a}\right)\left(1 - \frac{l\cos\theta}{b}\right) \\ &= \frac{l\sin\theta}{a} + \frac{l\cos\theta}{b} - \frac{l\sin\theta}{a}\frac{l\cos\theta}{b} \end{aligned}$$

To compute the overall probability we integrate over the angle:

$$\int_0^{\pi/2} \left(\frac{l\sin\theta}{a} + \frac{l\cos\theta}{b} - \frac{l\sin\theta}{a}\frac{l\cos\theta}{b}\right)\frac{2}{\pi}d\theta = \frac{2l}{\pi}\left(\frac{1}{a}+\frac{1}{b}-\frac{l}{ab}\right)$$