CS 240: Data Structure and Data Management

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Formulas, run time, and more.

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Contents

1 Order Notation

1.1 Big O

O-Notation (bound from above; worst case):

$$f(n) \in O(g(n))$$

if there exists constants $c, n_0 > 0$ such that

$$0 \leq f(n) \leq cg(n) \quad \forall n \geq n_0$$

1.2 Big Omega

Ω-Notation (bound from below; best case):

$$f(n) \in \Omega(g(n))$$

if there exists constants $c, n_0 > 0$ such that

$$0 \leq cg(n) \leq f(n) \quad \forall n \geq n_0$$

1.3 Theta Bound

θ-Notation (tight bound):

$$f(n) \in \theta(g(n))$$

if there exists constants $c_1, c_2, n_0 > 0$ such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \forall n \geq n_0$$

1.4 Little o

o-Notation (bound from above for all):

$$f(n) \in o(g(n))$$

if for all constants $c > 0$, there exists a constant $n_0 > 0$ such that

$$0 \leq f(n) \leq cg(n) \quad \forall n \geq n_0$$

1.5 Little omega

ω-Notation (bound from below for all):

$$f(n) \in o(g(n))$$

if for all constants $c > 0$, there exists a constant $n_0 > 0$ such that

$$0 \leq cg(n) \leq cf(n) \quad \forall n \geq n_0$$

1.6 Techniques for Order Notation

Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$.
Suppose that

$$L = \lim_{n \to \infty} \frac{f(n)}{g(n)}$$

Then

$$f(n) \in \begin{cases} o(g(n)) & \text{if } L = 0 \\ \theta(g(n)) & \text{if } 0 < L < \infty \\ \omega(g(n)) & \text{if } L = \infty \end{cases}$$

1.7 Relationships Between Order Notations

$$\begin{aligned} f(n) \in \theta(g(n)) &\iff g(n) \in \theta(f(n)) \\ f(n) \in O(g(n)) &\iff g(n) \in \Omega(f(n)) \\ f(n) \in o(g(n)) &\iff g(n) \in \omega(f(n)) \end{aligned}$$

$$\begin{aligned} f(n) \in \theta(g(n)) &\iff f(n) \in O(g(n)) \text{ and } f(n)\Omega(g(n)) \\ f(n) \in o(g(n)) &\iff f(n) \in O(g(n)) \\ f(n) \in o(g(n)) &\iff f(n) \notin \Omega(g(n)) \\ f(n) \in \omega(g(n)) &\iff f(n) \in \Omega(g(n)) \\ f(n) \in \omega(g(n)) &\iff f(n) \notin O(g(n)) \end{aligned}$$

1.8 Algebra of Order Notation

"Maximum" Rules: Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$, then:

$$O(f(n) + g(n)) \; O(max\{f(n), g(n)\})$$

$$\theta(f(n)+g(n))\ \theta(max\{f(n),g(n)\})$$
$$\Omega(f(n)+g(n))\ \Omega(max\{f(n),g(n)\})$$

Transitivity: If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ then $f(n) \in O(h(n))$.

2 Summation Formulas

Arithmetic:

$$\sum_{i=0}^{n-1}(a+di) = na + \frac{dn(n-1)}{2} \in \theta(n^2) \text{ for } d \neq 0$$

Geometric:

$$\sum_{i=0}^{n-1} ar^i = \begin{cases} a\frac{r^n-1}{r-1} & \in \theta(r^n) \text{ if } r > 1 \\ na & \in \theta(n) \text{ if } r = 1 \\ a\frac{1-r^n}{1-r} & \in \theta(1) \text{ if } 0 < r < 1 \end{cases}$$

Harmonic:

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \in \theta(\log n)$$

$$\sum_{i=1}^{n} ir^i = \frac{nr^{n+1}}{r-1} - \frac{r^{n+1}-r}{(r-1)^2}$$

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi}{6}$$

for $k \geq 0, \sum_{i=1}^{n} i^k \in \theta(n^{k+1})$

$n! \in \theta(\frac{n^{\frac{n+1}{2}}}{e^n})$

$\log n! \in \theta(n \log n)$

3 Review

[A] Prove $\frac{1}{n} \in o(1)$

$\frac{1}{n} < c$

$1 < cn$

$n > \frac{1}{c}$

Choose $n_0 = \frac{2}{c}$

Given $c > 0$, set $n_0 = \frac{2}{c}$

$\frac{1}{n} \leq \frac{1}{n_0} \leq \frac{1}{2/c}$

$= \frac{c}{2}$

$< c$ ■

[B] Prove $\frac{1}{n\sqrt{n}} \notin O(\frac{1}{n^2}$

$\exists c, n_0 > 0$ such that

$\frac{1}{n\sqrt{n}} \leq \frac{c}{n^2}$

$\frac{n^2}{n\sqrt{n}} \leq \frac{n^2 c}{n^2}$

$\frac{n}{\sqrt{n}} \leq c$

$\sqrt{n} \leq c$

$n \leq c^2$

Contradiction. Thus $n > c^2$ ■

[C] Suppose you own n electronic devices. You have n charger cables associated with each phone. Each plug is slightly different, but you can't compare plugs with each other. You can only find which charger fits with each phone by plugging it in.
Give a randomized $O(n \log n)$ algorithm: