CO 342: Introduction to Graph Theory

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Theorems and more reference sheet.

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Contents

1 The Basics

1.1 The Degree of a Vertex

Proposition 1.1.1. The number of vertices in a graph is always even.

The number $\delta(G) = min\{d(v)|v \in V\}$ is the **minimum degree** of G.
The number $\Delta(G) = max\{d(V)|v \in V\}$ is the **maximum degree** of G.
The number

$$d(G) = \frac{1}{|V|}\sum_{v \in V} d(v)$$

is the **average degree** of G.
The **average degree ratio** of G is expressed as $\epsilon(G) = |E|/|V|$.

Proposition 1.1.2. Every graph G with at least one edge has a subgraph H with $\delta(H) > \epsilon(H) \geq \epsilon(G)$.

1.2 Paths and Cycles

The *minimum* length of a cycle (contained) in a graph G is the **girth** $g(G)$ of G; the *maximum* length of a cycle in G is its **circumference**.

Proposition 1.2.1. Every graph G contains a path of length $\delta(G)$ and a cycle of at least $\delta(G) + 1$ (provided that $\delta(G) \geq 2$).

The **distance** $d_G(x, y)$ in G of two vertices x, y is the length of a shortest x-y path in G; if no such path exists, we exist $d(x, y) = \infty$.

The greatest distance between any two vertices in G is the **diameter** of G, denoted by diam$G$.

Proposition 1.2.2. Every graph G containing a cycle satisfies $g(G) \leq 2\text{diam}G+1$.

A vertex is **central** in G if its greatest distance from any other vertex is as small as possible. This distance is the **radius** of G, denoted by rad$G$.

Proposition 1.2.3. A graph G is radius at most K and maximum degree at most $d \geq 3$ has fewer than $\frac{d}{d-2}(d-1)^k$ vertices.

Theorem 1.2.4. *Let G be a graph. If $d(G) \geq d \geq 2$ and $g(G) \geq g \in \mathbb{N}$ then $|G| \geq n_0(d, g)$.*

Corollary 1.2.5. If $\delta(G) \geq 3$ then $g(G) < 2\log|G|$.

1.3 Connectivity

Proposition 1.3.1. The vertices of a connected graph G can always be enumerated, say $v_1, \dots, v_n$, so that $G_i = G[v_1, \dots, v_i]$ is connected for every i.

G is called **k-connected** (for $k \in \mathbb{N}$) if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. That is, no two vertices of G are separated by fewer than k other vertices.
The greatest integer k such that G is k-connected is the **connectivity** $\kappa(G)$ of G.
$\kappa(G) = 0 \iff G$ is disconnected or a K^1, and $\kappa(K^n) = n - 1 \quad \forall n \geq 1$.

If $|G| > 1$ and $G - F$ is connected for every set $F \subseteq E$ of fewer than ℓ edges, then G is called **ℓ-edge-connected**.
The greatest integer ℓ such that G is ℓ-edge connected is the **edge-connectivity** $\lambda(G)$ of G.
$\lambda(G) = 0$ if G is disconnected.

Proposition 1.3.2. If G is non-trivial then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

By Proposition 1.4.2, high connectivity requires a large minimum degree.
Conversely, large minimum degree does not ensure high connectivity, not even high edge-connectivity.
A large minimum degree does imply the existence of a highly connected subgraph.

Theorem 1.3.3. *Let $0 \neq k \in \mathbb{N}$. Every graph G with $d(G) \geq 4k$ has a $(k+1)$-connected subgraph H such that $\epsilon(H) > \epsilon(G) - k$.*

1.4 Trees and Forests

Theorem 1.4.1. *The following assertions are equivalent for a graph T:*

(i) T is a tree

(ii) Any two vertices of T are linked by a unique path in T

(iii) T is minimally, i.e. T is connected but $T - e$ is disconnected for every edge $e \in T$

(iv) T is maximally acyclic, i.e. T contains no cycle but $T + xy$ does for any two non-adjacent vertices $x, y \in T$

Corollary 1.4.2. The vertices of a tree can be enumerated, say as $v_1, \dots, v_n$, so that every v_i with $i \geq 2$ has a unique neighbour in $\{v_1, \dots, v_{i-1}\}$.

Corollary 1.4.3. A connected graph with n vertices is a tree $\iff$ it has $n-1$ edges.

Corollary 1.4.4. If T is a tree and G is any graph with $\delta(G) \geq |T| - 1$, then $T \subseteq G$, i.e. G has a subgraph isomorphic to T.

Lemma 1.4.5. Let T be a normal tree in G.

(i) Any two vertices $x, y \in T$ are separated in G by the set $\lceil x \rceil \cap \lceil y \rceil$.

(ii) If $S \subseteq V(T) = V(G)$ and S is down-closed, then the components of $G - S$ are spanned by the sets $\lfloor x \rfloor$ with x minimal in $T - S$.

Proposition 1.4.6. Every connected graph contains a normal spanning tree, with any specified vertex as its root.

1.5 Bipartite Graphs

Proposition 1.5.1. A graph is bipartite $\iff$ it contains no odd cycle.

1.6 Contraction and Minors

Proposition 1.6.1. The minor relation $\preceq$ and the topological-minor relation are partial orderings on the class of finite graphs, i.e. they are reflexive, antisymmetric and transitive.

Proposition 1.6.2. A finite graph G is an $IX \iff X$ can be obtained from G by a sequence of edge contractions, i.e. $\iff$ there are graphs $G_0, \ldots, G_n$ and edges $e_i \in G_i$ such that $G_0 = G$, $G_n \simeq X$, and $G_{i+1} = G_i / e_i \quad \forall i < n$.

Corollary 1.6.3. Let X and Y be finite graphs, X is a minor of $Y \iff$ there are graphs $G_0, \ldots, G_n$ such that $G_0 = Y$ and $G_n = X$ and each G_{i+1} arises from G_i by deleting an edge, contracting an edge, or deleting a vertex.

Proposition 1.6.4.

(i) Every TX is also an IX; thus, every topological minor of a graph is also its (ordinary) minor

(ii) If $\Delta(X) \leq 3$, then every IX contains a TX; thus, every minor with maximum degree at most 3 of a graph is also its topological minor.

1.7 Euler Tours

A closed walk in a graph is an **Euler tour** if it traverses every edge of the graph exactly once.
A graph is **Eulerian** if it admits an Euler tour.

Theorem 1.7.1. *A connected graph is Eulerian $\iff$ every vertex has even degrees.*

2 Connectivity

2.1 2-Connected Graphs and Subgraphs

A relation, $\approx$ say, on a set E is **equivalence relation** if

- $x \approx x$ for all x in E [reflexive]
- whenever $x, y \in E$ and $x \approx y$ we also have $y \approx x$ [symmetric]
- If $x \approx y$ and $y \approx z$ then $x \approx z$ [transitive]

If E is the vertex set of a graph and $\approx$ means "is joined by a path to", then $\approx$ is an equivalence relation on E.

Lemma 2.1.1. If G is a graph then "is equal to or lies in a circuit with" is an equivalence relation on $E(G)$.

Proposition 2.1.2. A graph is 2-connected $\iff$ it can be constructed from a cycle by successively adding H-paths to graphs H already constructed.

Lemma 2.1.3. Let G be any graph.

(i) The cycles of G are precisely the cycles of its blocks

(ii) The bonds of G are precisely the minimal cuts of its blocks.

Theorem 2.1.4. *For a connected graph G with at least three vertices, the following properties are equivalent:*

- *G is 2-connected*
- *any two edges of G lie on a circuit (cycle)*
- *any two vertices of G lie on a circuit*

2.2 Blocks

A connected subgraph H of G is a **block** of G if it has no cut-vertex, but any subgraph of G that contains H properly is either not connected or has a cut-vertex.

Lemma 2.2.1. Let G be a graph. The equivalence classes of edges of G under $\approx$ are precisely the edge sets of the blocks of G.

Lemma 2.2.2. Any cut vertex of a graph lies in at least two blocks.

Lemma 2.2.3. The block-cut vertex graph of a graph is a forest.

Indices