

# CO 342: Introduction to Graph Theory

Charles Shen

Fall 2016, University of Waterloo

Theorems and more reference sheet.

**Dropped**

# Contents

<b>1</b>	<b>The Basics</b>	<b>1</b>
1.1	The Degree of a Vertex . . . . .	1
1.2	Paths and Cycles . . . . .	1
1.3	Connectivity . . . . .	2
1.4	Trees and Forests . . . . .	2
1.5	Bipartite Graphs . . . . .	3
1.6	Contraction and Minors . . . . .	3
1.7	Euler Tours . . . . .	4
<b>2</b>	<b>Connectivity</b>	<b>4</b>
2.1	2-Connected Graphs and Subgraphs . . . . .	4
2.2	Blocks . . . . .	5
	<b>Indices</b>	<b>6</b>

# 1 The Basics

## 1.1 The Degree of a Vertex

**Proposition 1.1.1.** The number of vertices in a graph is always even.

The number  $\delta(G) = \min\{d(v) | v \in V\}$  is the **minimum degree** of  $G$ .

The number  $\Delta(G) = \max\{d(v) | v \in V\}$  is the **maximum degree** of  $G$ .

The number

$$d(G) = \frac{1}{|V|} \sum_{v \in V} d(v)$$

is the **average degree** of  $G$ .

The **average degree ratio** of  $G$  is expressed as  $\epsilon(G) = |E|/|V|$ .

**Proposition 1.1.2.** Every graph  $G$  with at least one edge has a subgraph  $H$  with  $\delta(H) > \epsilon(H) \geq \epsilon(G)$ .

## 1.2 Paths and Cycles

The *minimum* length of a cycle (contained) in a graph  $G$  is the **girth**  $g(G)$  of  $G$ ; the *maximum* length of a cycle in  $G$  is its **circumference**.

**Proposition 1.2.1.** Every graph  $G$  contains a path of length  $\delta(G)$  and a cycle of at least  $\delta(G) + 1$  (provided that  $\delta(G) \geq 2$ ).

The **distance**  $d_G(x, y)$  in  $G$  of two vertices  $x, y$  is the length of a shortest  $x$ - $y$  path in  $G$ ; if no such path exists, we exist  $d(x, y) = \infty$ .

The greatest distance between any two vertices in  $G$  is the **diameter** of  $G$ , denoted by  $\text{diam}G$ .

**Proposition 1.2.2.** Every graph  $G$  containing a cycle satisfies  $g(G) \leq 2\text{diam}G + 1$ .

A vertex is **central** in  $G$  if its greatest distance from any other vertex is as small as possible. This distance is the **radius** of  $G$ , denoted by  $\text{rad}G$ .

**Proposition 1.2.3.** A graph  $G$  is radius at most  $K$  and maximum degree at most  $d \geq 3$  has fewer than  $\frac{d}{d-2}(d-1)^K$  vertices.

**Theorem 1.2.4.** Let  $G$  be a graph. If  $\delta(G) \geq d \geq 2$  and  $g(G) \geq g \in \mathbb{N}$  then  $|G| \geq n_0(d, g)$ .

**Corollary 1.2.5.** If  $\delta(G) \geq 3$  then  $g(G) < 2 \log |G|$ .

### 1.3 Connectivity

**Proposition 1.3.1.** The vertices of a connected graph  $G$  can always be enumerated, say  $v_1, \dots, v_n$ , so that  $G_i = G[v_1, \dots, v_i]$  is connected for every  $i$ .

$G$  is called **k-connected** (for  $k \in \mathbb{N}$ ) if  $|G| > k$  and  $G - X$  is connected for every set  $X \subseteq V$  with  $|X| < k$ . That is, no two vertices of  $G$  are separated by fewer than  $k$  other vertices.

The greatest integer  $k$  such that  $G$  is  $k$ -connected is the **connectivity**  $\kappa(G)$  of  $G$ .  
 $\kappa(G) = 0 \iff G$  is disconnected or a  $K^1$ , and  $\kappa(K^n) = n - 1 \quad \forall n \geq 1$ .

If  $|G| > 1$  and  $G - F$  is connected for every set  $F \subseteq E$  of fewer than  $\ell$  edges, then  $G$  is called  **$\ell$ -edge-connected**.

The greatest integer  $\ell$  such that  $G$  is  $\ell$ -edge connected is the **edge-connectivity**  $\lambda(G)$  of  $G$ .

$\lambda(G) = 0$  if  $G$  is disconnected.

**Proposition 1.3.2.** If  $G$  is non-trivial then  $\kappa(G) \leq \lambda(G) \leq \delta(G)$ .

By Proposition 1.4.2, high connectivity requires a large minimum degree.

Conversely, large minimum degree does not ensure high connectivity, not even high edge-connectivity.

A large minimum degree does imply the existence of a highly connected subgraph.

**Theorem 1.3.3.** Let  $0 \neq k \in \mathbb{N}$ . Every graph  $G$  with  $d(G) \geq 4k$  has a  $(k + 1)$ -connected subgraph  $H$  such that  $\epsilon(H) > \epsilon(G) - k$ .

### 1.4 Trees and Forests

**Theorem 1.4.1.** The following assertions are equivalent for a graph  $T$ :

- (i)  $T$  is a tree
- (ii) Any two vertices of  $T$  are linked by a unique path in  $T$
- (iii)  $T$  is minimally, i.e.  $T$  is connected but  $T - e$  is disconnected for every edge  $e \in T$
- (iv)  $T$  is maximally acyclic, i.e.  $T$  contains no cycle but  $T + xy$  does for any two non-adjacent vertices  $x, y \in T$

**Corollary 1.4.2.** The vertices of a tree can be enumerated, say as  $v_1, \dots, v_n$ , so that every  $v_i$  with  $i \geq 2$  has a unique neighbour in  $\{v_1, \dots, v_{i-1}\}$ .

**Corollary 1.4.3.** A connected graph with  $n$  vertices is a tree  $\iff$  it has  $n - 1$  edges.

**Corollary 1.4.4.** If  $T$  is a tree and  $G$  is any graph with  $\delta(G) \geq |T| - 1$ , then  $T \subseteq G$ , i.e.  $G$  has a subgraph isomorphic to  $T$ .

**Lemma 1.4.5.** Let  $T$  be a normal tree in  $G$ .

- (i) Any two vertices  $x, y \in T$  are separated in  $G$  by the set  $[x] \cap [y]$ .
- (ii) If  $S \subseteq V(T) = V(G)$  and  $S$  is down-closed, then the components of  $G - S$  are spanned by the sets  $[x]$  with  $x$  minimal in  $T - S$ .

**Proposition 1.4.6.** Every connected graph contains a normal spanning tree, with any specified vertex as its root.

## 1.5 Bipartite Graphs

**Proposition 1.5.1.** A graph is bipartite  $\iff$  it contains no odd cycle.

## 1.6 Contraction and Minors

**Proposition 1.6.1.** The minor relation  $\preceq$  and the topological-minor relation are partial orderings on the class of finite graphs, i.e. they are reflexive, antisymmetric and transitive.

**Proposition 1.6.2.** A finite graph  $G$  is an  $IX \iff X$  can be obtained from  $G$  by a sequence of edge contractions, i.e.  $\iff$  there are graphs  $G_0, \dots, G_n$  and edges  $e_i \in G_i$  such that  $G_0 = G$ ,  $G_n \simeq X$ , and  $G_{i+1} = G_i/e_i \quad \forall i < n$ .

**Corollary 1.6.3.** Let  $X$  and  $Y$  be finite graphs,  $X$  is a minor of  $Y \iff$  there are graphs  $G_0, \dots, G_n$  such that  $G_0 = Y$  and  $G_n = X$  and each  $G_{i+1}$  arises from  $G_i$  by deleting an edge, contracting an edge, or deleting a vertex.

**Proposition 1.6.4.**

- (i) Every  $TX$  is also an  $IX$ ; thus, every topological minor of a graph is also its (ordinary) minor
- (ii) If  $\Delta(X) \leq 3$ , then every  $IX$  contains a  $TX$ ; thus, every minor with maximum degree at most 3 of a graph is also its topological minor.

## 1.7 Euler Tours

A closed walk in a graph is an **Euler tour** if it traverses every edge of the graph exactly once.

A graph is **Eulerian** if it admits an Euler tour.

**Theorem 1.7.1.** *A connected graph is Eulerian  $\iff$  every vertex has even degrees.*

## 2 Connectivity

### 2.1 2-Connected Graphs and Subgraphs

A relation,  $\approx$  say, on a set  $E$  is **equivalence relation** if

- $x \approx x$  for all  $x$  in  $E$  [reflexive]
- whenever  $x, y \in E$  and  $x \approx y$  we also have  $y \approx x$  [symmetric]
- If  $x \approx y$  and  $y \approx z$  then  $x \approx z$  [transitive]

If  $E$  is the vertex set of a graph and  $\approx$  means “is joined by a path to”, then  $\approx$  is an equivalence relation on  $E$ .

**Lemma 2.1.1.** If  $G$  is a graph then “is equal to or lies in a circuit with” is an equivalence relation on  $E(G)$ .

**Proposition 2.1.2.** A graph is 2-connected  $\iff$  it can be constructed from a cycle by successively adding  $H$ -paths to graphs  $H$  already constructed.

**Lemma 2.1.3.** Let  $G$  be any graph.

- (i) The cycles of  $G$  are precisely the cycles of its blocks
- (ii) The bonds of  $G$  are precisely the minimal cuts of its blocks.

**Theorem 2.1.4.** *For a connected graph  $G$  with at least three vertices, the following properties are equivalent:*

- $G$  is 2-connected
- any two edges of  $G$  lie on a circuit (cycle)
- any two vertices of  $G$  lie on a circuit

## 2.2 Blocks

A connected subgraph  $H$  of  $G$  is a **block** of  $G$  if it has no cut-vertex, but any subgraph of  $G$  that contains  $H$  properly is either not connected or has a cut-vertex.

**Lemma 2.2.1.** Let  $G$  be a graph. The equivalence classes of edges of  $G$  under  $\approx$  are precisely the edge sets of the blocks of  $G$ .

**Lemma 2.2.2.** Any cut vertex of a graph lies in at least two blocks.

**Lemma 2.2.3.** The block-cut vertex graph of a graph is a forest.

## Indices

$\ell$ -edge-connected, [2](#)

average degree, [1](#)

average degree ratio, [1](#)

block, [5](#)

central, [1](#)

circumference, [1](#)

connectivity, [2](#)

diameter, [1](#)

distance, [1](#)

edge-connectivity, [2](#)

equivalence relation, [4](#)

Euler tour, [4](#)

Eulerian, [4](#)

girth, [1](#)

k-connected, [2](#)

maximum degree, [1](#)

minimum degree, [1](#)

radius, [1](#)