MATH 136: Linear Algebra I

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Quick reference sheet.

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Contents

1 Matrices

Definition. An **augmented matrix** has the form $\left[A \mid \vec{b}\right]$

Definition. A **homogeneous matrix** has the form $\left[A \mid \vec{0}\right]$
A homogeneous system is always consistent, $\vec{x} = \vec{0}$ is always a solution.

1.1 Elementary Row Operations

1. Add a multiple of one row to another: $R_i + cR_j$
2. Multiple a row by a non-zero constant: cR_i
3. Exchange two rows: $R_i \leftrightarrow R_j$

1.2 Reduced Row Echelon Form

Definition. A matrix R is said to be in **Reduced Row Echelon Form** (RREF) if:

1. All rows containing a non-zero entry are above rows which only contains zeros
2. The first non-zero entry in each non-zero row is 1, called a **leading one** (or a pivot)
3. The leading one in each non-zero row is to the right of the leading one in any row above it
4. A leading one is the only non-zero entry in its column

Definition. The **rank** of a matrix A is the number of leading ones in the RREF of the matrix.

1.3 System Rank Theorem

Theorem. Let A be the coefficient matrix of a system of m linear equations in n unknowns $\left[A \mid \vec{b}\right]$

1. If the rank of A is less than the rank of the augmented matrix $\left[A \mid \vec{b}\right]$, then the system is inconsistent

2. If the system $\begin{bmatrix} A \mid \vec{b} \end{bmatrix}$ is consistent, then the system contains $(n - rank(A))$ free variables

3. $rank(A) = m$ iff the system $\begin{bmatrix} A \mid \vec{b} \end{bmatrix}$ is consistent for every $\vec{b} \in \mathbb{R}^m$

2 Identities

2.1 Transpose

Definition. The **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix $A^\intercal$ whose ij-th entry is the ji-th entry of A; that is:

$$(A^\intercal)_{ij} = (A)_{ji}$$

Theorem. If A and B are $m \times n$ matrices and $c \in \mathbb{R}$, then:

1. $(A^\intercal)^\intercal = A$
2. $(A + B)^\intercal = A^\intercal + B^\intercal$
3. $(cA)^\intercal = cA^\intercal$
4. $(AB)^\intercal = B^\intercal A^\intercal$

3 Inverses

3.1 Inverse of a 2-by-2 Matrix

$$\text{Suppose } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\text{Then } A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

3.2 Invertible Matrix Theorem

Theorem. For any $n \times n$ matrix A, the following are equivalent:

1. A is invertible
2. $rank(A) = n$

3. RREF of $A = I$
4. $A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$
5. The columns of A terms are a linearly independent set
6. The columns of A spans $\mathbb{R}^n$
7. The columns of A forms a basis for $\mathbb{R}^n$
8. The rows of A forms a basis for $\mathbb{R}^n$
9. $Col(A) = \mathbb{R}^n$
10. $Null(A) = \{\vec{0}\}$
11. $A^\intercal$ is invertible
12. $Row(A) = \mathbb{R}^n$
13. $Null(A^\intercal) = \{\vec{0}\}$