

CO 342: Introduction to Graph Theory

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Theorems and more reference sheet.

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1 The Basics

1.1 The Degree of a Vertex

Proposition 1.1.1. The number of vertices in a graph is always even.

The number $\delta(G) = \min\{d(v) | v \in V\}$ is the **minimum degree** of G .

The number $\Delta(G) = \max\{d(v) | v \in V\}$ is the **maximum degree** of G .

The number

$$d(G) = \frac{1}{|V|} \sum_{v \in V} d(v)$$

is the **average degree** of G .

The **average degree ratio** of G is expressed as $\epsilon(G) = |E|/|V|$.

Proposition 1.1.2. Every graph G with at least one edge has a subgraph H with $\delta(H) > \epsilon(H) \geq \epsilon(G)$.

1.2 Paths and Cycles

The *minimum* length of a cycle (contained) in a graph G is the **girth** $g(G)$ of G ; the *maximum* length of a cycle in G is its **circumference**.

Proposition 1.2.1. Every graph G contains a path of length $\delta(G)$ and a cycle of at least $\delta(G) + 1$ (provided that $\delta(G) \geq 2$).

The greatest distance between any two vertices in G is the **diameter** of G , denoted by $\text{diam}G$.

Proposition 1.2.2. Every graph G containing a cycle satisfies $g(G) \leq 2\text{diam}G + 1$.

A vertex is **central** in G if its greatest distance from any other vertex is as small as possible. This distance is the **radius** of G , denoted by $\text{rad}G$.

Proposition 1.2.3. A graph G is radius at most K and maximum degree at most $d \geq 3$ has fewer than $\frac{d}{d-2}(d-1)^K$ vertices.

Theorem 1.2.4. Let G be a graph. If $d(G) \geq d \geq 2$ and $g(G) \geq g \in \mathbb{N}$ then $|G| \geq n_0(d, g)$.

Corollary 1.2.5. If $\delta(G) \geq 3$ then $g(G) < 2 \log |G|$.

1.3 Connectivity

Proposition 1.3.1. The vertices of a connected graph G can always be enumerated, say v_1, \dots, v_n , so that $G_i = G[v_1, \dots, v_i]$ is connected for every i .

G is called **k-connected** (for $k \in \mathbb{N}$) if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. That is, no two vertices of G are separated by fewer than k other vertices.

The greatest integer k such that G is k -connected is the **connectivity** $\kappa(G)$ of G .
 $\kappa(G) = 0 \iff G$ is disconnected or a K^1 , and $\kappa(K^n) = n - 1 \quad \forall n \geq 1$.

If $|G| > 1$ and $G - F$ is connected for every set $F \subseteq E$ of fewer than ℓ edges, then G is called **ℓ -edge-connected**.

The greatest integer ℓ such that G is ℓ -edge connected is the **edge-connectivity** $\lambda(G)$ of G .

$\lambda(G) = 0$ if G is disconnected.

Proposition 1.3.2. If G is non-trivial then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

By Proposition 1.4.2, high connectivity requires a large minimum degree.

Conversely, large minimum degree does not ensure high connectivity, not even high edge-connectivity.

A large minimum degree does imply the existence of a highly connected subgraph.

Theorem 1.3.3. Let $0 \neq k \in \mathbb{N}$. Every graph G with $d(G) \geq 4k$ has a $(k + 1)$ -connected subgraph H such that $\epsilon(H) > \epsilon(G) - k$.

1.4 Trees and Forests

Theorem 1.4.1. The following assertions are equivalent for a graph T :

- (i) T is a tree
- (ii) Any two vertices of T are linked by a unique path in T
- (iii) T is minimally, i.e. T is connected but $T - e$ is disconnected for every edge $e \in T$
- (iv) T is maximally acyclic, i.e. T contains no cycle but $T + xy$ does for any two non-adjacent vertices $x, y \in T$

Corollary 1.4.2. The vertices of a tree can be enumerated, say as v_1, \dots, v_n , so that every v_i with $i \geq 2$ has a unique neighbour in $\{v_1, \dots, v_{i-1}\}$.

Corollary 1.4.3. A connected graph with n vertices is a tree \iff it has $n - 1$ edges.

Corollary 1.4.4. If T is a tree and G is any graph with $\delta(G) \geq |T| - 1$, then $T \subseteq G$, i.e. G has a subgraph isomorphic to T .

Lemma 1.4.5. Let T be a normal tree in G .

- (i) Any two vertices $x, y \in T$ are separated in G by the set $[x] \cap [y]$.
- (ii) If $S \subseteq V(T) = V(G)$ and S is down-closed, then the components of $G - S$ are spanned by the sets $[x]$ with x minimal in $T - S$.

Proposition 1.4.6. Every connected graph contains a normal spanning tree, with any specified vertex as its root.

1.5 Bipartite Graphs

Proposition 1.5.1. A graph is bipartite \iff it contains no odd cycle.

1.6 Contraction and Minors

Proposition 1.6.1. The minor relation \preceq and the topological-minor relation are partial orderings on the class of finite graphs, i.e. they are reflexive, antisymmetric and transitive.

Proposition 1.6.2. A finite graph G is an IX \iff X can be obtained from G by a sequence of edge contractions, i.e. \iff there are graphs G_0, \dots, G_n and edges $e_i \in G_i$ such that $G_0 = G$, $G_n \simeq X$, and $G_{i+1} = G_i/e_i \quad \forall i < n$.

Corollary 1.6.3. Let X and Y be finite graphs, X is a minor of Y \iff there are graphs G_0, \dots, G_n such that $G_0 = Y$ and $G_n = X$ and each G_{i+1} arises from G_i by deleting an edge, contracting an edge, or deleting a vertex.

Proposition 1.6.4.

- (i) Every TX is also an IX ; thus, every topological minor of a graph is also its (ordinary) minor
- (ii) If $\Delta(X) \leq 3$, then every IX contains a TX ; thus, every minor with maximum degree at most 3 of a graph is also its topological minor.

1.7 Euler Tours

A closed walk in a graph is an **Euler tour** if it traverses every edge of the graph exactly once.

A graph is **Eulerian** if it admits an Euler tour.

Theorem 1.7.1. *A connected graph is Eulerian \iff every vertex has even degrees.*

2 Connectivity

2.1 2-Connected Graphs and Subgraphs

Proposition 2.1.1. A graph is 2-connected \iff it can be constructed from a cycle by successively adding H -paths to graphs H already constructed.

Lemma 2.1.2. Let G be any graph.

- (i) The cycles of G are precisely the cycles of its blocks
- (ii) The bonds of G are precisely the minimal cuts of its blocks.

Indices

ℓ -edge-connected, [2](#)

average degree, [1](#)

average degree ratio, [1](#)

central, [1](#)

circumference, [1](#)

connectivity, [2](#)

diameter, [1](#)

edge-connectivity, [2](#)

Euler tour, [4](#)

Eulerian, [4](#)

girth, [1](#)

k-connected, [2](#)

maximum degree, [1](#)

minimum degree, [1](#)

radius, [1](#)