CHAPTER 9
Relationships Between Points, Lines, and Planes

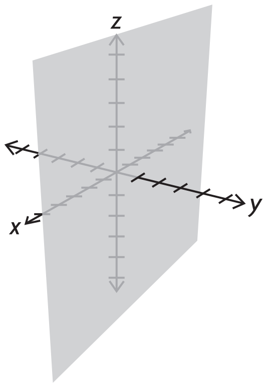
Review of Prerequisite Skills, p. 487

1. a. Yes; $(2, -5) = (10, -12) + t(8, -7)$
$(2, -5) = (10, -12) + 1(8, -7)$
b. No; $12(1) + 5(2) - 13 = 9 \neq 0$
c. Yes; $(7, -3, 8) = (1, 0, -4) + t(2, -1, 4)$
$(7, -3, 8) = (1, 0, -4) + 3(2, -1, 4)$
d. No; $(1, 0, 5) = (2, 1, -2) + t(4, -1, 2)$
$(-1, -1, 7) \neq t(4, -1, 2)$
There is no value of t that satisfies the equation.
2. Answers may vary. For example:
a. Vector: $\vec{m} = (7, 3) - (2, 5) = (5, -2)$
$\vec{r} = (2, 5) + t(5, -2), t \in \mathbf{R}$
Parametric: $x = 2 + 5t, y = 5 - 2t, t \in \mathbf{R}$
b. Vector: $\vec{m} = (4, -7) - (-3, 7) = (7, -14)$
$\vec{r} = (-3, 7) + t(7, -14), t \in \mathbf{R}$
Parametric: $x = -3 + 7t, y = 7 - 14t, t \in \mathbf{R}$
c. Vector: $\vec{m} = (-3, -11) - (-1, 0)$
$= (-2, -11)$
$\vec{r} = (-1, 0) + t(-2, -11), t \in \mathbf{R}$
Parametric: $x = -1 + -2t, y = -11t, t \in \mathbf{R}$
d. Vector: $\vec{m} = (6, -7, 0) - (1, 3, 5)$
$= (5, -10, -5)$
$\vec{r} = (1, 3, 5) + t(5, -10, -5), t \in \mathbf{R}$
Parametric: $x = 1 + 5t, y = 3 - 10t, z = 5 - 5t, t \in \mathbf{R}$
e. Vector: $\vec{m} = (-1, 5, 2) - (2, 0, -1)$
$= (-3, 5, 3)$
$\vec{r} = (2, 0, -1) + t(-3, 5, 3), t \in \mathbf{R}$
Parametric: $x = 2 - 3t, y = -5t, z = -1 + 3t, t \in \mathbf{R}$
f. Vector: $\vec{m} = (12, -5, -7) - (2, 5, -1)$
$= (10, -10, -6)$
$\vec{r} = (2, 5, -1) + t(10, -10, -6), t \in \mathbf{R}$
Parametric: $x = 2 + 10t, y = 5 - 10t, z = -1 - 6t, t \in \mathbf{R}$
3. a. Since $\vec{n} = (2, 6, -1)$, the Cartesian equation of the plane is of the form $2x + 6y - z + D = 0$, where D is to be determined. Since $P_0(4, 1, -3)$ is on the plane, it must satisfy the equation. So $2(4) + 6(1) - 1(-3) + D = 8 + 6 + 3 + D$
$= 17 + D = 0$. $D = -17$, and the equation of the plane is $2x + 6y - z - 17 = 0$.
b. Since $\vec{n} = (0, 7, 0)$, the Cartesian equation of the plane is of the form $7y + D = 0$, where D is to be determined. Since $P_0(-2, 0, 5)$ is on the plane, it must satisfy the equation. So $7(0) + D = 0 + D = 0$ thus $D = 0$. The equation of the plane is $7y = 0$, or $y = 0$.
c. Since $\vec{n} = (4, -3, 0)$, the Cartesian equation of the plane is of the form $4x - 3y + D = 0$, where D is to be determined. Since $P_0(3, -1, -2)$ is on the plane, it must satisfy the equation. So $4(3) - 3(-1) + D = 12 + 3 + D = 15 + D = 0$. $D = -15$, and the equation of the plane is $4x + 3y - 15 = 0$.
d. Since $\vec{n} = (6, 5, -3)$, the Cartesian equation of the plane is of the form $6x - 5y + 3z + D = 0$, where D is to be determined. Since $P_0(0, 0, 0)$ is on the plane, it must satisfy the equation. So $6(0) - 5(0) + 3(0) + D = 0$, or $D = 0$. The equation of the plane is $6x - 5y + 3z = 0$.
e. Since $\vec{n} = (11, -6, 0)$, the Cartesian equation of the plane is of the form $11x - 6y + D = 0$, where D is to be determined. Since $P_0(4, 1, 8)$ is on the plane, it must satisfy the equation. So $11(4) - 6(1) + D = 44 - 6 + D = 38 + D = 0$. $D = -38$, and the equation of the plane is $11x - 6y - 38 = 0$.
f. Since $\vec{n} = (1, 1, -1)$, the Cartesian equation of the plane is of the form $x + y + -z + D = 0$, where D is to be determined. Since $P_0(2, 5, 1)$ is on the plane, it must satisfy the equation. So $2 + 5 - 1 + D = 6 + D = 0$. $D = -6$, and the equation of the plane is $x + y - z - 6 = 0$.
4. Start by writing the given line in parametric form: $(x, y, z) = (2 + s + 2t, 1 - s, 3s - 5t)$, so $x = 2 + s + 2t$, $y = 1 - s$, and $z = 3s - 5t$. Solving for s in each component, we get $s = 1 - y$ and substituting this into $z = 3s - 5t$ gives $z = 3(1 - y) - 5t = 3 - 3y - 5t$.

So now $-3 + 3y + z = -5t$ and $t = \frac{3 - 3y - z}{5}$.
Finally, substituting both equations for s and t into $x = 2 + s + 2t$, we get
$x = 2 + (1 - y) + 2\left(\frac{3 - 3y - z}{5}\right)$.
Rearranging, we get
$$5x = 10 + 5 - 5y + 6 - 6y - 2z$$
$$5x + 11y + 2z - 21 = 0.$$
5. L_1 is not parallel to the plane because (3, 0, 2) is a point on the line and the plane. Substitute the expressions for the components of the parametric equation of the line into the equation of the plane.
$$4(3 + t) + (-2t) - (2 + 2t) - 10 = 0$$
$$12 + 4t - 2t - 2 - 2t - 10 = 0$$
$$0 = 0$$
This last statement is always true. So every point on the line is also in the plane. Therefore, the line lies on the plane.
For L_2 substitute the expressions for the components of the parametric equation of the line into the equation of the plane.
$$4(-3t) + (-5 + 2t) - (-10t) - 10 = 0$$
$$-12t - 5 + 2t + 10t - 10 = 0$$
$$-15 = 0$$
This last statement is never true. So the line and the plane have no points in common. Therefore, L_2 is parallel to the plane. The line cannot lie on the plane.
For L_3 use the symmetric equation to rewrite x and z in terms of y.
$x = -4y - 23$
$z = -y - 6$
Substitute into the equation of the plane.
$$4(-4y - 23) + y - (-y - 6) - 10 = 0$$
$$-16y - 92 + y + y + 6 - 10 = 0$$
$$-14y - 96 = 0$$
This equation has a solution. Therefore, L_3 and the plane have a point in common and are not parallel. However, $(5, -7, 1)$ is a point that lies on the line that does not lie on the plane. Therefore, L_3 does not lie in the plane.
6. a. A normal vector to this plane is determined by calculating the cross product of the position vectors, $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
$$\overrightarrow{AB} = (2, 0, 0) - (1, 0, -1) = (1, 0, 1)$$
$$\overrightarrow{AC} = (6, -1, 5) - (1, 0, -1) = (5, -1, 6)$$
$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= ((0 \cdot 6) - (1 \cdot -1), (1 \cdot 5) \\ &\quad - (1 \cdot 6), (1 \cdot -1) - (0 \cdot 5)) \\ &= (0 + 1, 5 - 6, -1 - 0) \\ &= (1, -1, -1) = \vec{n}.\end{aligned}$$
If $P(x, y, z)$ is any point on the plane, then $\overrightarrow{AP} = (x - 1, y, z + 1)$, and if the normal to the plane is $(1, -1, -1)$, then
$(x - 1, y, z + 1) \cdot (1, -1, -1) = 0$, so
$x - 1 - y - z - 1 = 0$ and thus,
$x - y - z - 2 = 0$
b. $\overrightarrow{PQ} = (6, 4, 0) - (4, 1, -2) = (2, 3, 2)$
$\overrightarrow{PR} = (0, 0, -3) - (4, 1, -2) = (-4, -1, -1)$
$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$
$= (3(-1) - 2(-1)), 2(-4) - 2(-1),$
$2(-1) - 3(4))$
$= (-3 + 2, -8 + 2, -2 + 12) = (-1, -6, 10)$
Since $(-1, -6, 10) = -1(1, 6, -10)$, we will use $(1, 6, -10)$ as the normal vector so that the coefficient of x is positive. If $P(x, y, z)$ is any point on the plane, then $\overrightarrow{AP} = (x - 4, y - 1, z + 2)$, and if the normal to the plane is $(1, 6, -10)$, then
$(x - 4, y - 1, z + 2) \cdot (1, 6, -10) = 0$,
so $x - 4 + 6y - 6 - 10z - 20 = 0$,
and thus $x + 6y - 10z - 30 = 0$.
7. Answers may vary. For example: One direction vector is $\vec{m} = (2, -1, 6) - (1, -4, 3) = (1, 3, 3)$. Now we need to find a normal to the plane such that $\vec{n} \cdot \vec{m} = 0$. So $(1, 3, 3) \cdot (a, 0, c) = 0$. Now we have $a + 3c = 0$. A possible solution to this is $a = 3, c = -1$. So $\vec{n} = (3, 0, -1)$ and the Cartesian equation of the plane is $3x - z = 0$.
Since the plane is parallel to the y-axis, (0, 1, 0) is another direction vector for the plane. Therefore, a vector equation for the plane is
$\vec{r} = (1, -4, 3) + t(1, 3, 3) + s(0, 1, 0), s, t \in \mathbf{R}$.
8. We are given the point $A(-1, 3, 4)$. We need to find a normal vector $\vec{n} = (a, b, c)$ such that
$a(x + 1) + b(y - 3) + c(z - 4) + d = 0$.
The normal vector also must be perpendicular to the two planes and their normals, $(2, -1, 3)$ and $(5, 1, -3)$. One possible solution for the normal is $\vec{n} = (0, 3, 1)$. So we have
$3(y - 3) + z - 4 = 0$
$3y + z - 9 - 4 = 0$
And the equation of the plane is $3y + z = 13$.

9.1 The Intersection of a Line with a Plane and the Intersection of Two Lines, pp. 496–498

1. a. First, show the parametric equations as $x = 1 + 5s$, $y = 2 + s$, $z = -3 + s$. Then the plane can be written as $\pi: x - 2y - 3z = 6$, and the vector equation of the line is $\vec{r} = (1, 2, -3) + s(5, 1, 1), s \in \mathbf{R}$.
b. When we substitute the parametric equations into the Cartesian equation for the plane, we get
$(1 + 5s) - 2(2 + s) - 3(-3 + s) = 6$
$1 - 4 + 9 + 5s - 2s - 3s = 6 - 0s = 6$
Note that by finishing the solution, we get $0s = 0$. Since any real number will satisfy this equation, we have an infinite number of solutions, and this line lies on the plane.
2. a. A line and a plane can intersect in three ways:
Case 1: The line and the plane have zero points of intersection. This occurs when the lines are not incidental, meaning they do not intersect.
Case 2: The line and the plane have only one point of intersection. This occurs when the line crosses the plane at a single point.
Case 3: The line and the plane have an infinite number of intersections. This occurs when the line is coincident with the plane, meaning the line lies on the plane.
b. Assume that the line and the plane have more than one intersection, but not an infinite number. For simplicity, assume two intersections. At the first intersection, the line crosses the plane. In order for the line to continue on, it must have the same direction vector. If the line has already crossed the plane, then it continues to move away from the plane, and can not intersect again. So the line and the plane can only intersect zero, one, or infinitely many times.
3. a. The line $\vec{r} = s(1, 0, 0)$ is the x-axis.
b. The plane $y = 1$ has the form $\vec{r} = (x, 1, z)$, where x, and z are any values in $\mathbf{R}$. So the plane is parallel to the xz-plane, but just one unit away to the right.
c.



d. There are no intersections between the line and the plane.
4. a. For $x + 4y + z - 4 = 0$, if we substitute the parametric equations, we have
$(-2 + t) + 4(1 - t) + (2 + 3t) + 4$
$= -2 + 4 + 2 + t - 4t + 3t - 4$
$= 0t + 0$
$= 0$. All values of t give a solution to the equation, so all points on the line are also on the plane.
b. For the plane $2x - 3y + 4z - 11 = 0$, we can substitute the parametric equations derived from $\vec{r} = (1, 5, 6) + t(1, -2, -2)$:
$x = 1 + t$, $y = 5 - 2t$, $z = 6 - 2t$.
So we have $2(1 + t) - 3(5 - 2t) + 4(6 - 2t) - 11$
$= 2 - 15 + 24 - 11 + 2t + 6t - 8t$
$= 0t + 0$
$= 0$
Similar to part a., all values of t give a solution to this equation, so all points on the line are also on the plane.
5. a. First, we should determine the parametric equations from the vector form: $x = -1 - s$, $y = 1 + 2s$, $z = 2s$. Substituting these into the equation of the plane, we get
$2(-1 - s) - 2(1 + 2s) + 3(2s) - 1$
$= -2 - 2 - 1 - 2s - 4s + 6s$
$= -5 + 0s$
Since there are no values of s such that $-5 = 0$, this line and plane do not intersect.
b. Substituting the parametric equations into the equation of the plane, we get
$2(1 + 2t) - 4(-2 + 5t) + 4(1 + 4t) - 13$
$2 + 8 + 4 - 13 + 4t - 20t + 16t$
$= 1 + 0t$
Since there are no values of t such that $1 = 0$, there are no solutions, and the plane and the line do not intersect.

6. a. The direction vector is $\vec{m} = (-1, 2, 2)$ and the normal is $\vec{n} = (2, -2, 3)$, so if the line and the plane meet at right angles, $\vec{m} \cdot \vec{n} = 0$. So $(-1 \cdot 2) + (2 \cdot -2) + (2 \cdot 3) = -2 - 4 + 6 = 0$, but $2(-1) - 2(1) + 3(0) - 1 = -5 \neq 0$. So the point on the line is not on the plane.

b. The direction vector is $\vec{m} = (2, 5, 4)$ and the normal is $\vec{n} = (2, -4, 4)$, so if the line and the plane meet at right angles, $\vec{m} \cdot \vec{n} = 0$. So $(2 \cdot 2) + (5 \cdot -4) + (4 \cdot 4) = 4 - 20 + 16 = 0$, but $2(1) - 4(-2) + 4(1) - 13 = 1 \neq 0$. So the point on the line is not on the plane.

7. a. If the line and the plane intersect, then they are equal at a particular point p. So we must substitute the parametric equations into the equation of the plane, and then solve for p.

$(-1 + 6p) + 2(3 + p) - (4 - 2p) + 29$
$= -1 + 6 - 4 + 6p + 2p + 2p + 29$
$= 30 + 10p$
$= 0$. So now $-10p = 30$ and $p = -3$.

Now we must find the point at which the line and plane intersect. To do this, just substitute $p = -3$ into the vector form of the line: $(-1, 3, 4) + -3(6, 1, -2) = (-19, 0, 10)$.

b. If the line and the plane intersect, then they are equal at a particular point p. So we must substitute the parametric equations into the equation of the plane, and then solve for p.

$x = 1 + 4s, y = -2 - s, z = 3 + s$
$2(1 + 4s) + 7(-2 - s) + (3 + s) + 15$
$= 2 - 14 + 3 + 15 + 8s - 7s + s$
$= 6 + 2s$
$= 0$. So now $-2s = 6$ and $s = -3$. Now we must find the point at which the line and plane intersect. To do this, just substitute $s = -3$ into the vector form of the line:

$(1, -2, 3) + -3(4, -1, 1) = (-11, 1, 0)$

8. a. Comparing the x and y components in L_1 and L_2, we have

$3 + 4s = 4 + 13t$
$1 - s = 1 - 5t$

We can easily solve for one of the variables by using the second equation: $s = 5t$. Substituting this back into the first equation: $3 + 20t = 4 + 13t$ so $1 = 7t$ and thus $t = \frac{1}{7}$. So now we must solve for s: $3 + 4s = 4 + \frac{13}{7}$ and $s = \frac{20}{28} = \frac{5}{7}$. Placing these back into the equations for L_1 and L_2:

$$L_1: (3, 1, 5) + \frac{5}{7}(4, -1, 2) = \left(\frac{41}{7}, \frac{2}{7}, \frac{45}{7}\right)$$
$$L_2: \left(4 + \frac{13}{7}, 1 - \frac{5}{7}, \frac{5}{7}\right) = \left(\frac{41}{7}, \frac{2}{7}, \frac{5}{7}\right)$$

The points must be equal for intersection to occur, so there is no intersection and the lines are skew.

b. If we compare the z components of the two lines, we see $2 = 8 - 6s$ or $s = 1$. Substituting this back into the x component (the y component would work just as well), we have $3 + m = -3 + 7(1) = 4$, or $m = 1$. So now we can substitute m and s back into the equations for the line, and we get

$L_3: (3, 7, 2) + (1, -6, 0) = (4, 1, 2)$
$L_4: (-3, 2, 8) + (7, -1, -6) = (4, 1, 2)$

So $(4, 1, 2)$ is the only point of intersection between these two lines.

9. a. Comparing the y and z components of each vector equation, we get the system of equations:

$3 - 2p = 3 - 2q$
$4 + 3p = -4 + 11q$

Note that from the first equation, $p = q$. So the second equation becomes $4 + 3q = -4 + 11q$. Solving for q, we get $q = 1$. So from the earlier relation, $p = 1$. Placing these two values back into the vector equations, we get

$(-2, 3, 4) + (6, -2, 3) = (4, 1, 7)$
$(-2, 3, -4) + (6, -2, 11) = (4, 1, 7)$

This shows that these two lines intersect at $(4, 1, 7)$.

b. Comparing the x and z components of each vector equation, we get the system of equations:

$4 + r = 2 + s$
$6 + 4r = -8 + 5s$

Note that from the first equation, $s = 2 + r$. So the second equation becomes $6 + 4r = 2 + 5r$. Solving for r, we get $r = 4$. So from the earlier relation, $s = 6$. Placing these two values back into the vector equations, we get

$(4, 1, 6) + 4(1, 0, 4) = (8, 1, 22)$
$(2, 1, -8) + 6(1, 0, 5) = (8, 1, 22)$

This shows that these two lines intersect at $(8, 1, 22)$.

c. Comparing the x and z components of each vector equation, we get the system of equations:

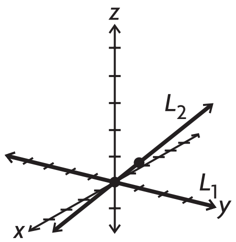
$2 + m = -2 + 3p$
$1 + m = 1 - p$

Note that from the second equation, $m = -p$. So the first equation becomes $2 - p = -2 + 3p$. Solving for p, we get $p = 1$. So from the earlier relation, $m = -1$. Placing these two values back into the vector equations, we get

$$(2, 2, 1) - (1, 1, 1) = (1, 1, 0)$$
$$(-2, 2, 1) + (3, -1, -1) = (1, 1, 0)$$
This shows that these two lines intersect at $(1, 1, 0)$.
d. Comparing the x and y components of each vector equation, we get the system of equations:
$1 + 0m = 2 + s$
$2 + 4m = 3 - 2s$
Note that from the first equation, $s = -1$. So the second equation becomes $2 + 4m = 5$.
Solving for m, we get $m = \frac{3}{4}$. Placing these two values back into the vector equations, we get
$$(9, 1, 2) - \frac{3}{4}(5, 0, 4) = \left(\frac{21}{4}, 1, -1\right)$$
$$(8, 2, 3) - (4, 1, -2) = (4, 1, 5)$$
The two lines do not intersect, so they are skew.
10. At the point where the line intersects the z-axis, the point $Q(0, 0, q)$ equals the vector equation. So for the x component, $-3 + 3s = 0$ or $s = 1$. Substituting this into the vector equation, we get $(-3, 2, 1) + (3, -2, 7) = (0, 0, 8)$. So $q = 8$.
11. a. Comparing the x components, we get $-2 + 7s = -30 + 7t$, which can be reduced to $28 + 7s = 7t$ or $s - t = 4$. Comparing the other components, the same equation results.
b. From L_1, we see that at $(-2, 3, 4)$, $s = 0$. When this occurs, $t = 4$. Substituting this into L_2, we get $(-30, 11, -4) + 4(7, -2, 2) = (-2, 3, 4)$. Since both of these lines have the same direction vector and a common point, the lines are coincidental.
12. a. First, we must determine the values of s and t. So comparing the x and z components, we get
$-3 + s = 1 - 3t$
$1 + s = 2 + 8t$
From the second equation, $s = 1 + 8t$. Substituting this back into the first equation,
$-3 + 1 + 8t = 1 - 3t$ or $t = \frac{3}{11}$.
Substituting back into the second equation,
$-3 + s = 1 - \frac{9}{11} = \frac{2}{11}$, and solving for s,
$s = \frac{2}{11} + 3 = \frac{35}{11}$. Now we can solve for k. Compare the y components after substituting s and t.
$$8 - \frac{35}{11} = 4 + \frac{3}{11}k$$
$$53 = 44 + 3k$$
or $k = 3$.
b. The lines intersect when $s = \frac{35}{11}$. The point of intersection is $\left(-3 + \frac{35}{11}, 8 - \frac{35}{11}, 1 + \frac{35}{11}\right)$ or $\left(\frac{2}{11}, \frac{53}{11}, \frac{46}{11}\right)$.
13. On the xz-plane, the point A has the coordinates $(x, 0, z)$, for any x, z. Similarly, on the yz-plane, the point B has the coordinates $(0, y, z)$ for any y, z. Now the task is to find the required values of s for these points. Starting with the x component of point B, we have $0 = -8 + 2s$ or $s = 4$. So point B is $(-8, -6, -1) + 4(2, 2, 1) = (0, 2, 3)$. For point A, we need the y coordinate to equal 0. So $0 = -6 + 2s$ or $s = 3$. So point A is
$(-8, -6, -1) + 3(2, 2, 1) = (-2, 0, 2)$.
Now we need to find the distance.
$$d = \sqrt{(0 - (-2))^2 + (2 - 0)^2 + (3 - 2)^2}$$
$$= \sqrt{4 + 4 + 1}$$
$$= \sqrt{9}$$
$$= 3$$
14. a. Comparing the y and z components of each vector equation, we get the system of equations:
$1 + 0p = -1 - 2q$
$1 - p = 1 - 2q$
Note that from the first equation, $2 = -2q$ or $q = -1$. So the second equation becomes $1 - p = 1 + 2$ or $p = -2$.
Placing these two values back into the vector equations to find the intersection point A, we get
$$(2, 1, 1) - 2(4, 0, -1) = (-6, 1, 3)$$
$$(3, -1, 1) - (9, -2, -2) = (-6, 1, 3)$$
Thus, the intersection point is $(-6, 1, 3)$.
b. A point on the xy plane has the form $(x, y, 0)$. If such a point is $(-6, 1, 0)$ then the distance from this point is $d = \sqrt{0 + 0 + 3^2} = 3$.
15. a. Comparing the x and y components of each vector equation, we get the system of equations:
$-1 + 5s = 4 + 0t$
$3 - 2s = -1 + 2t$
Note that from the first equation, $5 = 5s$ or $s = 1$. So the second equation becomes $3 - 2 = -1 + 2t$ or $t = 1$. Placing these two values back into the vector equations to find the intersection point A, we get
$$(-1, 3, 2) + (5, -2, 10) = (4, 1, 12)$$
$$(4, -1, 1) + (0, 2, 11) = (4, 1, 12)$$
Thus, the intersection point is $(4, 1, 12)$.
b. We need to find a vector (a, b, c) such that
$5a - 2b + 10c = 0$
$2b + 11c = 0$
A possible solution to the second equation is $(a, 11, -2)$. If we substitute this into the first equation, we get $5a - 22 - 20 = 0 \rightarrow 5a = 42$.

We can use this to get a solution of $\left(\frac{42}{5}, 11, -2\right)$. To eliminate the fraction, we get $(42, 55, -10)$. So the vector equation is $\vec{r} = (4, 1, 12) + t(42, 55, -10)$, $t \in \mathbf{R}$.

16. a.



b. The only point of intersection is at the origin $(0, 0, 0)$.
c. If $p = 0$ and $q = 0$, the intersection occurs at $(0, 0, 0)$.
17. a. Represent the lines parametrically, and then substitute into the equation for the plane.
For the first equation, $x = t$, $y = 7 - 8t$, $z = 1 + 2t$. Substituting into the plane equation, $2t + 7 - 8t + 3 + 6t - 10 = 0$. Simplifying, $0t = 0$. So the line lies on the plane.
For the second line, $x = 4 + 3s$, $y = -1$, $z = 1 - 2s$
Substituting into the plane equation, $8 + 6s - 1 + 3 - 6s - 10 = 0$. Simplifying, $0s = 0$. This line also lies on the plane.
b. Compare the x and y components:
$4 + 3s = t$
$7 - 8t = -1$
From the second equation, $t = 1$. Substituting back into the first equation, $4 + 3s = 1$, or $s = -1$.
Determine the point of intersection:
$(1, 7 - 8, 1 + 2) = (1, -1, 3)$
$(4 - 3, -1, 1 + 2) = (1, -1, 3)$
The point of intersection is $(1, -1, 3)$.
18. Answers may vary. For example:
$\vec{r} = (2, 0, 0) + p(2, 0, 1)$, $p \in \mathbf{R}$

9.2 Systems of Equations, pp. 507–509

1. a. linear
b. not linear
c. linear
d. not linear
2. Answers may vary. For example:
$x + y + 2z = -15$
a. $x + 2y + z = -3$
$2x + y + z = -10$
b. Subtract the first equation from the second, and subtract twice the first equation from the third.
$x + y + 2z = -15$
$0x - y + z = -12$
$0x - y - 3z = 20$
Add the second and third equation.
$x + y + 2z = -15$
$0x - y + z = -12$
$0x + 0y - 4z = 32$
From the third equation, $z = -8$.
Substitute z back into the second equation,
$-y - 8 = -12$
$-y = -12 + 8 = -4$
So $y = 4$. Now substitute y and z back into the first equation.
$x + 4 + 2(-8) = x - 12 = -15$
And so $x = -3$. Thus the solution is $(-3, 4, -8)$ as expected.

3. a. $-7 - 3(5) + 4\left(\frac{3}{4}\right) = -7 - 15 + 3 = -19$

$-7 - 8\left(\frac{3}{4}\right) = -7 - 6 = -13$

$-7 + 2(5) = 3$

Yes, $\left(-7, 5, \frac{3}{4}\right)$ is a solution.
b.

$3(-7) - 2(5) + 16\left(\frac{3}{4}\right) = -21 - 10 + 12 = -19$

$3(-7) - 2(5) = -21 - 10 = -31$
$\neq -23$

$8(-7) - 5 + 4\left(\frac{3}{4}\right) = -56 - 5 + 3 = -58$

Because the second equation fails to produce an equality, $\left(-7, 5, \frac{3}{4}\right)$ is not a solution.
4. a. Solve for y. $y = -3$
The solution is $(-2, -3)$.
b. Multiply the second equation by 6
$3x + 5y = -21$
$x - 3y = 7$
Add 3 times the first equation to 5 times the second equation.
$3x + 5y = -21$
$14x = -28$
From the second equation, $x = -2$.
Substituting x back into the first equation,
$3(-2) + 5y = -21$
$5y = -15$
So $y = -3$.
The two systems are equivalent because they have the same solution.
5. a. Add the second equation to 5 times the first equation.

$2x - y = 11$
$11x = 66$
Solve for x in the second equation, $x = 6$. Substitute x back into the first equation
$2(6) - y = 11$
$-y = 11 - 12 = -1$
So $y = 1$
Therefore, the solution is $(6, 1)$.
b. Subtract three times the first equation from twice the second equation.
$2x + 5y = 19$
$0x - 7y = -35$
From the second equation, $y = 5$.
Substitute y back into the first equation.
$2x + 5(5) = 19$
$2x = 19 - 25 = -6$
So $x = -3$
Therefore, the solution is $(-3, 5)$.
c. Add the second equation to 3 times the first equation to the second equation
$-x + 2y = 10$
$0x + 11y = 33$
From the second equation, $y = 3$.
Substitute y back into the first equation.
$-x + 2(3) = 10$
$-x = 4$
So $x = -4$.
Therefore the solution is $(-4, 3)$.
6. a. These two lines are parallel, and therefore cannot have an intersection.
b. The second equation is five times the first, therefore the lines are coincident.
7. a. Let $x = t$. So $2t - y = 3$ then $y = 2t - 3$.
b. Let $x = t$, $y = s$. Then $t - 2s + z = 0$ and $z = 2s + t$.
8. a. If $x = t$, $y = -2t - 11$, then $y = -2x - 11$ and so $2x + y = -11$ is the required linear equation.
b. $2x + y = -11$
$2(3t + 3) + (-6t - 17) = 6t - 6t + 6 - 17$
$= -11$
9. a. The two equations will have no solutions when $k \neq 12$, since they will be parallel should this occur.
b. It is impossible to have only one solution for these two equations. They have exactly the same direction vector. They will never intersect at exactly one place.
c. The two equations will have infinitely many solutions when $k = 12$. When this occurs, the two equations are coincident.
10. a. There are infinitely many solutions to this equation. This is reason why it is represented graphically as a line.
b. Let $x = t$. So $2t + 4y = 11$, then $4y = 11 - 2t$ and $y = \frac{11}{4} - \frac{1}{2}t$, $t \in \mathbf{R}$
c. This equation will not have any integer solutions because the left hand side is an even function and the right side is an odd function.
11. a. Add the second equation to -2 times the first.
$x + 3y = a$
$0x - 3y = b - 2a$
Divide the second equation by -3 to get $y = -\frac{1}{3}b + \frac{2}{3}a$. Now substitute this back into the first equation.
$$x + 3\left(-\frac{1}{3}b + \frac{2}{3}a\right) = a$$
$$x - b + 2a = a$$
$$x = -a + b$$
b. Since they have different direction vectors, these two equations are not parallel or coincident and will intersect somewhere.
12. a. Add the third equation to the first to eliminate z.
$x + y + z = 0$
$x - y + 0z = 1$
$x + 2y + 0z = -5$
Add twice the second equation to the third equation to eliminate
Add twice the second equation to the third equation to eliminate y.
$x + y + z = 0$
$x - y + 0z = 1$
$3x + 0y + 0z = -3$
Divide the third equation by -3 to get $x = -1$.
Now substitute into the second equation.
$-1 - y = 1$
$y = -2$
Finally, substitute x and y to get
$-1 + -2 + z = 0$
So $z = 3$. Therefore, the solution is $(-1, -2, 3)$.
b. Add the first equation to -2 times the second, and add the first equation to -2 times the third.
$2x - 3y + z = 6$
$0x - 5y - 3z = -56$
$0x - y + 3z = 40$
Now add the second equation to -1 times the third.
$2x - 3y + z = 6$
$0x - 5y - 3z = -56$
$0x - 4y + 0z = -16$

From the third equation, $y = 4$.
Now substitute this into the second equation.

$$\begin{aligned} -5(4) - 3z &= -56 \\ -3z &= -36 \\ z &= 12 \end{aligned}$$

Now substitute these two values back into the first equation.

$$\begin{aligned} 2x - 3(4) + 12 &= 6 \\ 2x &= 6, x = 3 \end{aligned}$$

So the solution is $(3, 4, 12)$.

c. Add the second equation to -1 times the third.

$$\begin{aligned} x + y + 0z &= 10 \\ 0x + y + z &= -2 \\ -x + y + 0z &= 2 \end{aligned}$$

Add the third equation to the first equation.

$$\begin{aligned} x + y + 0z &= 10 \\ 0x + y + z &= -2 \\ 0x + 2y + 0z &= 12 \end{aligned}$$

So $y = 6$. Now substitute into the other two equations.

$x + 6 = 10 \rightarrow x = 4$
$6 + z = -2 \rightarrow z = -8$

So the solution is $(4, 6, -8)$.

d. To eliminate fractions, multiply each of the equations by 60.

$$\begin{aligned} 20x + 15y + 12z &= 840 \\ 15x + 12y + 20z &= -1260 \\ 12x + 20y + 15z &= 420 \end{aligned}$$

Add 3 times the first equation to -4 times the second, and add 3 times the first equation to -5 times the third.

$$\begin{aligned} 20x + 15y + 12z &= 840 \\ 0x - 3y - 44z &= 7560 \\ 0x - 55y - 39z &= 420 \end{aligned}$$

Now add 55 times the second equation to -3 times the third equation.

$$\begin{aligned} 20x + 15y + 12z &= 840 \\ 0x - 3y - 44z &= 7560 \\ 0x + 0y - 2303z &= 414\,540 \end{aligned}$$

Divide the third equation through by -2303 to get $z = -180$. Substituting z back into the second equation.

$-3y - 44(-180) = 7560 \rightarrow -3y = -360$

So $y = 120$. Now substitute these two values back into the first equation.

$$\begin{aligned} 20x + 15(120) + 12(-180) &= 840 \\ 20x = 840 - 1800 + 2160 &= 1200 \end{aligned}$$

So $x = 60$. Therefore the solution is $(60, 120, -180)$.

e. Note that if $2x - y = 0 \rightarrow y = 2x$, and $2z - x = 0 \rightarrow z = \frac{1}{2}x$. So we substitute these two relations into the second equation.

$$2(2x) - \frac{1}{2}x = \frac{7}{2}x = 7 \rightarrow x = 2$$

So now $z = 1$, $y = 4$, and the solution is $(2, 4, 1)$.

f. Add the first equation to -2 times the second equation.

$$\begin{aligned} x + y + 2z &= 13 \\ -2x + 0y - 7z &= -38 \\ 2x + 0y + 6z &= 32 \end{aligned}$$

Add the second and third equations.

$$\begin{aligned} x + y + 2z &= 13 \\ -2x + 0y - 7z &= -38 \\ 0x + 0y - z &= -6 \end{aligned}$$

So from the third equation, $z = 6$.
Substituting into the second equation,

$$\begin{aligned} -2x - 42 &= -38 \\ -2x = 4 &\rightarrow x = -2 \end{aligned}$$

Finally, substituting both values into the first equation,

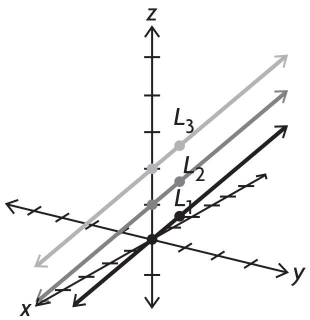
$-2 + y + 12 = 13 \rightarrow y = 3$.

So the final solution is $(-2, 3, 6)$.

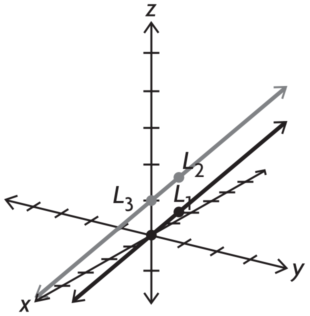
13. Answers may vary. For example:

a.

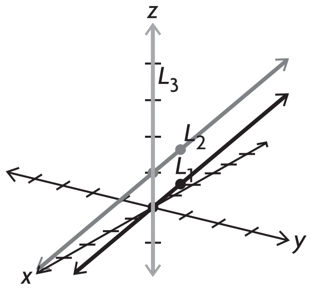
Three lines parallel



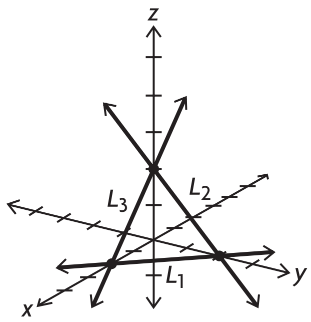
Two lines coincident and the third parallel



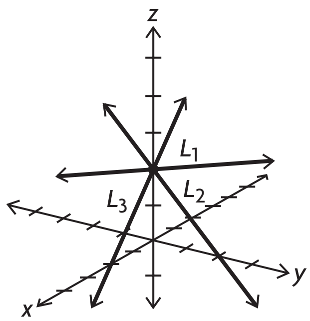
Two parallel lines cut by the third line



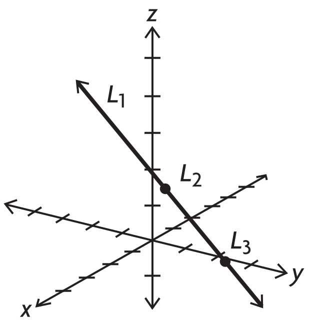
The lines form a triangle



b. Lines meet in a point



c. Three coincident lines



14. a. Add -1 times the first equation and the second equation. Add -1 times the first equation and the third equation.

$$x + y + z = a$$
$$0x + 0y - z = b - a$$
$$-x + 0y + 0z = c - a$$

So $z = a - b$, $x = a - c$. Then substitute into the first equation.

$$a - c + y + a - b = a$$
$$y = -a + b + c$$

So the final solution is $(a - c, -a + b + c, a - b)$.

15. a. For two equations to have no solutions, they must be parallel—meaning it must have a the same direction vector. So if $k = 2$, then the lines are parallel.

b. If two equations have an infinite number of solutions, then the lines must be coincident. One way to do this is if the second equation is a multiple of the first equation. To achieve this, $k = -2$.

c. For two equations to have a unique solution, they must have only one intersection. From a., we saw that if $k = 2$, the lines are parallel, and if $k = -2$, then they are coincident. Since the only other option is for the lines to have a unique solution, $k \neq \pm 2$.

9.3 The Intersection of Two Planes, pp. 516–517

1. a. This means that the two equations represent planes that are parallel and not coincident.

b. Answers may vary. For example: $x - y + z = 1$, $x - y + z = -2$

2. a. The solution to the system of equations is: $x = \frac{1}{2} + \frac{1}{2}s - t$, $y = s$, $z = t$, $s, t \in \mathbf{R}$. The two planes are coincident.

b. Answers may vary. For example:

$x - y + z \Rightarrow -1$; $2x - 2y + 2z = -2$

3. a. $2z = -4 \Rightarrow z = -2$.

$$x - y + (-2) \Rightarrow -1$$
$$x - y \Rightarrow 1.$$

$x = 1 + s$, $y = s$, $z = -2$, $s \in \mathbf{R}$

The two planes intersect in a line.

b. Answers may vary. For example:

$x - y + z = -1$; $x - y - z = 3$.

4. a. ① $2x + y + 6z = p$; ② $x + my + 3z = q$

For the planes to be coincident equation ② must be a multiple of equation ①. Since the coefficients of x and z in equation ① are twice that of the x and z coefficients in equation ② all of the coefficients and constants in equation ② must be half of the corresponding coefficients in equation ①. So:

$m = \dfrac{1}{2}$, $p = 2q$. $q = 1$, and $p = 2$.

The value for m is unique, but p just has to be twice q and arbitrary values can be chosen.

b. For parallel planes all of the coefficients of the variables must be multiples of each other, but the constant terms must differ by a different constant. So a possible solution is:

$m = \frac{1}{2}$, $q = 1$, and $p = 3$.

The value for m is again unique but p and q can be arbitrarily chosen as long as $p \neq 2q$.

c. For the two planes to intersect at right angles the two normal vectors, $\overrightarrow{n_1} = (2, 1, 6)$ and $\overrightarrow{n_2} = (1, m, 3)$, must satisfy:

$\overrightarrow{n_1} \cdot \overrightarrow{n_2} = 0.$

$\overrightarrow{n_1} \cdot \overrightarrow{n_2} = 2 + m + 18 = 0$

$m = -20$. This value is unique, since only one value was found to satisfy the given conditions.

d. From **c.** we know that in order to intersect in right angles $m = -20$. Choose $p = 1$, $q = 1$.

The value for m is unique from the solution to **c.**, but the values for p and q can be arbitrary since the only value which can change the angle between the planes is m.

5. a. Letting $z = s$:

$$y = -3s.$$
$$x + 2(-3s) - 3s = 0.$$
$$x = 9s$$

The solution is:

$x = 9s, y = -3s, z = s, s \in \mathbf{R}$

b. Letting $y = t$.

$$t + 3z = 0$$
$$3z = -t$$
$$z = -\frac{1}{3}t.$$
$$x + 2t - 3\left(-\frac{1}{3}\right)t = 0$$
$$x + 3t = 0$$
$$x = -3t.$$

The solution is:

$x = -3t, y = t, z = -\frac{1}{3}t; t \in \mathbf{R}.$

c. Since t is an arbitrary real number we can express t as:

$t = -3s; s \in \mathbf{R}.$

Substituting this into the solution for **b.** shows that the two solutions are equivalent.

6. a. Equation ② is twice that of equation ①, so they represent intersecting coincident planes.

b. The coefficients of each variable are the same, but the constant terms are different, so the equations represent non-intersecting parallel planes.

c. The coefficients of the x and z variables are the same but the y coefficients are different. So the equations represent planes that intersect in a line.

d. The coefficients of each variable from equation ① to ② are not the same multiple. Therefore the equations represent planes that intersect in a line.

e. The intersection is a line by the same reasoning as **d.**

f. The intersection is a line by the same reasoning as **d.**

7. a. $x = 1 - s - t, y = s, z = t, s, t \in \mathbf{R}$

b. There is no solution since the planes are parallel.

c. ① − ②:

$$-2y = 4$$
$$y = -2.$$
$$x - 2 + 2z = -2$$
$$x + 2z = 0$$
$$x = -2z.$$

$x = -2s, y = -2, z = s, s \in \mathbf{R}.$

d. Let $z = s; s \in \mathbf{R}$.

From ②:

$$x = y + 6.$$
$$(y + 6) + y + 2s = 4$$
$$2y + 2s = -2$$
$$y = -s - 1.$$

$x = -s + 5, y = -s - 1, z = s, s \in \mathbf{R}.$

e. $-2 \cdot$ ②: $2x - 4y - 2z = -2$

Adding ①:

$4x - 5y = 0.$

$$x = \frac{5}{4}y.$$

Let $y = s, s \in \mathbf{R}$.

$$2\left(\frac{5}{4}s\right) - s + 2z = 2$$
$$\frac{3}{2}s + 2z = 2$$
$$z = 1 - \frac{3}{4}s.$$

$x = \frac{5}{4}s, y = s, z = 1 - \frac{3}{4}s, s \in \mathbf{R}$

f. $x - y + 2(4) = 0$

$$x = y - 8.$$

$x = s - 8, y = s, z = 4, s \in \mathbf{R}.$

8. a. The system will have an infinite number of solutions for any value of k. When $k = 2$ equation ② will be twice that of ① so the solution is a plane:

$x = 1 - s - 2t, y = s, z = t, s, t \in \mathbf{R}.$

For any other value of k the solution will be a line.

For example $k = 0$:

$$2y = -4z$$
$$y = -2z.$$

$x + (-2z) + 2z = 1$
$x = 1.$
$x = 1, y = -2s, z = s, s \in \mathbf{R}.$
b. No there is no value of k for which the system will not have a solution. The only time when there is no solution is when the corresponding coefficients for each variable differ by a common multiple between equations, and the constant terms differ by a different multiple. The only way the first condition is satisfied is when $k = 2$, but when this happens the constant terms differ by the same factor as the variables, namely 2.
9. The line of intersection of the two planes:
π_1: $2x - y + z = 0$, π_2: $y + 4z = 0$ is:
$y = -4z$
$2x - (-4z) + z = 0$
$2x = -5z$
$x = -\frac{5}{2}z.$
$x = -\frac{5}{2}s, y = -4s, z = s, s \in \mathbf{R}.$
The direction vector is $\left(-\frac{5}{2}, -4, 1\right)$ or $(-5, -8, 2)$.
$\vec{r_1} = s(-5, -8, 2), s \in \mathbf{R}$. Since the line we are looking for is parallel to this line, we know that the direction vector must be the same. The line passes through $(-2, 3, 6)$ and has direction vector $(-5, -8, 2)$. The equation of the line is
$\vec{r_2} = (-2, 3, 6) + s(-5, -8, 2), s \in \mathbf{R}.$
10. The line of intersection of the two planes, $2x - y + 2z = 0$ and $2x + y + 6z = 4$ is:
$4x + 8z = 4$
$x = 1 - 2z.$
$2(1 - 2z) - y + 2z = 0$
$2 - y - 2z = 0$
$y = 2 - 2z.$
$x = 1 - 2s, y = 2 - 2s, z = s, s \in \mathbf{R}.$
In order for the a line to be contained in the plane we need to check that the values for x, y, and z always satisfy the plane equation:
$5x + 3y + 16z - 11 = 0.$
$5(1 - 2s) + 3(2 - 2s) + 16(s) - 11 = 0$
$5 + 6 - 11 - 10s - 6s + 16s = 0$
$0 = 0$. Since this is true the line is contained in the plane.
11. a. π_1: $2x + y - 3z = 3$, π_2: $x - 2y + z = -1$.
$\pi_1 - 2\pi_2$: $5y - 5z = 5$
$y = 1 + z.$
$2x + (1 + z) - 3z = 3$
$2x - 2z = 2$
$x = 1 + z.$
$x = 1 + s, y = 1 + s, z = s, s \in \mathbf{R}.$
b. L meets the xy-plane when $z = 0$.
$x = 1, y = 1. A = (1, 1, 0).$
L meets the z-axis when both x and y are zero:
$s = -1.$
$z = -1.$
$B = (0, 0, -1)$
The length of AB is therefore:
$\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ or about 1.73.
12. The line with equation $x = -2y = 3z$ has parametric equations: $x = s, y = -\frac{1}{2}s, z = \frac{1}{3}s, s \in \mathbf{R}$.
This has the equivalent vector form:
$\vec{r} = s\left(1, -\frac{1}{2}, \frac{1}{3}\right), s \in \mathbf{R}.$
The line of intersection of the two planes
$x - y + z = 1$ and $2y - z = 0$ is:
$y = \frac{1}{2}z$
$x - \frac{1}{2}z + z = 1$
$x = 1 - \frac{1}{2}z.$
$x = 1 - \frac{1}{2}t, y = \frac{1}{2}t, z = t, t \in \mathbf{R}$. Which has a vector equation of:
$\vec{r} = (1, 0, 0) + t\left(-\frac{1}{2}, \frac{1}{2}, 1\right), t \in \mathbf{R}$. The vector equation of the plane with the given properties is thus:
$$\vec{r} = (1, 0, 0) + t\left(-\frac{1}{2}, \frac{1}{2}, 1\right) + s\left(1, -\frac{1}{2}, \frac{1}{3}\right), s, t \in \mathbf{R}.$$
The normal vector for the plane is then:
$$\left(-\frac{1}{2}, \frac{1}{2}, 1\right) \times \left(1, -\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2} \cdot \frac{1}{3}\right) - \left(1 \cdot -\frac{1}{2}\right),$$
$$1 \cdot 1 - \left(-\frac{1}{2} \cdot \frac{1}{3}\right), -\frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2} \cdot 1 = \left(\frac{2}{3}, \frac{7}{6}, -\frac{1}{4}\right).$$
Or equivalently $(8, 14, -3)$.
The Cartesian equation is then:
$8x + 14y - 3z + D = 0$, and must contain the point $(1, 0, 0)$.
$8(1) + D = 0.$
$D = -8.$
$8x + 14y - 3z - 8 = 0.$

Mid-Chapter Review, pp. 518–519

1. a. $\vec{r} = (4, -3, 15) + t(2, -3, 5), t \in \mathbf{R}$
$x = 4 + 2t, y = -3 - 3t, z = 15 + 5t$
$0 = 15 + 5t$
$t = -3$

$x = 4 + 2(-3), y = -3 - 3(-3),$
$z = 15 + 5(-3)$
$x = -2, y = 6, z = 0$
$(-2, 6, 0)$
b. $\vec{r} = (4, -3, 15) + t(2, -3, 5), t \in \mathbf{R}$
$x = 4 + 2t, y = -3 - 3t, z = 15 + 5t$
$0 = -3 - 3t$
$t = -1$
$x = 4 + 2(-1), y = -3 - 3(-1),$
$z = 15 + 5(-1)$
$x = 2, y = 0, z = 10$
$(2, 0, 10)$
c. $\vec{r} = (4, -3, 15) + t(2, -3, 5), t \in \mathbf{R}$
$x = 4 + 2t, y = -3 - 3t, z = 15 + 5t$
$0 = 4 + 2t$
$t = -2$
$x = 4 + 2(-2), y = -3 - 3(-2),$
$z = 15 + 5(-2)$
$x = 0, y = 3, z = 5$
$(0, 3, 5)$
2. a.–e. Answers may vary. For example:
$A(2, 1, 3), B(3, -2, 5), C(-8, -5, 7)$
$a = (-2.5, -3.5, 6)$
$b = (-3, -2, 5)$
$c = (2.5, -0.5, 4)$
$m_1 = (Aa) = (-4.5, -4.5, 3) = (3, 3, -2)$
$m_2 = (Bb) = (-6, 0, 0) = (1, 0, 0)$
$m_3 = (Cc) = (10.5, 4.5, -3) = (7, 3, -2)$
Then substitute in the point and the direction vector to find the equation of the line.
$A(2, 1, 3), B(3, -2, 5), C(-8, -5, 7)$
$m_1 = (Aa) = (-4.5, -4.5, 3) = (3, 3, -2)$
$m_2 = (Bb) = (-6, 0, 0) = (1, 0, 0)$
$m_3 = (Cc) = (10.5, 4.5, -3) = (7, 3, -2)$
$A: \vec{r} = (2, 1, 3) + t(3, 3, -2), t \in \mathbf{R}$
$x = 2 + 3t, y = 1 + 3t, z = 3 - 2t, t \in \mathbf{R}$
$B: \vec{r} = (3, -2, 5) + t(1, 0, 0), t \in \mathbf{R}$
$x = 3 + t, y = -2, z = 5, t \in \mathbf{R}$
$C: \vec{r} = (-8, -5, 7) + t(7, 3, -2), t \in \mathbf{R}$
$x = -8 + 7t, y = -5 + 3t, z = 7 - 2t, t \in \mathbf{R}$
$A: x = 2 + 3t, y = 1 + 3t, z = 3 - 2t, t \in \mathbf{R}$
$B: x = 3 + t, y = -2, z = 5, t \in \mathbf{R}$
$C: x = -8 + 7t, y = -5 + 3t, z = 7 - 2t, t \in \mathbf{R}$
$y = -2 = 1 + 3t$
$t = -1$
$x = 2 + 3(-1), y = 1 + 3(-1),$
$z = 3 - 2(-1)$
$x = -1, y = -2, z = 5$
$(-1, -2, 5)$
$A: x = 2 + 3t, y = 1 + 3t, z = 3 - 2t, t \in \mathbf{R}$
$B: x = 3 + t, y = -2, z = 5, t \in \mathbf{R}$
$C: x = -8 + 7t, y = -5 + 3t, z = 7 - 2t, t \in \mathbf{R}$
$y = -2 = -5 + 3t$
$t = 1$
$x = -8 + 7(1), y = -5 + 3(1), z = 7 - 2(1)$
$x = -1, y = -2, z = 5$
$(-1, -2, 5)$
The three medians meet at $(-1, -2, 5)$.
3. a. $L_1: 5x + y + 2z + 15 = 0$
$L_2: 4x + y + 2z + 8 = 0$
$L_1 - L_2: x + 7 = 0$
So $x = -7$.
$L_1: y + 2z - 20 = 0$
$L_2: y + 2z - 20 = 0$
$z = t,$
$y + 2(t) - 20 = 0$
$y = 20 - 2t$
$\vec{r} = (-7, 20, 0) + t(0, -2, 1), t \in \mathbf{R}$
b. $L_1: 4x + 3y + 3z - 2 = 0$
$L_2: 5x + 2y + 3z + 5 = 0$
$2L_1 - 3L_2: -7x - 3z - 19 = 0$
$z = 7t,$
$-7x - 3(7t) - 19 = 0,$
$$x = -3t - \frac{19}{7}$$
$$4\left(-3t - \frac{19}{7}\right) + 3y + 3(7t) - 2 = 0$$
$$y = -3t + \frac{30}{7}$$
$$\vec{r} = \left(-\frac{19}{7}, \frac{30}{7}, 0\right) + t(3, 3, -7), t \in \mathbf{R}$$
c. $L_1: \vec{r} = (-7, 20, 0) + t(0, -2, 1), t \in \mathbf{R}$
$$L_2: \vec{r} = \left(-\frac{19}{7}, \frac{30}{7}, 0\right) + t(3, 3, -7), t \in \mathbf{R}$$
$L_1: x = -7, y = 20 - 2t, z = t$
$$L_2: x = -\frac{19}{7} + 3t, y = \frac{30}{7} + 3t, z = -7t$$
$$-\frac{19}{7} + 3t = -7, t = -\frac{30}{21}$$
$$x = -\frac{19}{7} + 3\left(-\frac{30}{21}\right), y = \frac{30}{7} + 3\left(-\frac{30}{21}\right),$$
$$z = -7\left(-\frac{30}{21}\right)$$
$x = -7, y = 0, z = 10$
$(-7, 0, 10)$
4. a. $\pi_1: 3x + y + 7z + 3 = 0$
$\pi_2: x - 13y - 3z - 38 = 0$

$13\pi_1 + \pi_2$: $40x + 88z + 1 = 0$

$z = t,$

$40x + 88(t) + 1 = 0$

$x = -\frac{11t}{5} - \frac{1}{40}$

$3\left(-\frac{11t}{5} - \frac{1}{40}\right) + y + 7(t) + 3 = 0$

$y = -\frac{2t}{5} - \frac{117}{40}$

$x = -\frac{11t}{5} - \frac{1}{40}, y = -\frac{2t}{5} - \frac{117}{40}, z = t, t \in \mathbf{R}$

b. π_1: $x - 3y + z + 11 = 0$

π_2: $6x - 13y + 8z - 28 = 0$

$-6\pi_1 + \pi_2$: $5y + 2z - 94 = 0$

$z = s,$

$5y + 2(s) - 94 = 0$

$y = -\frac{2}{5}s + \frac{94}{5}$

$x - 3\left(-\frac{2}{5}s + \frac{94}{5}\right) + (s) + 11 = 0$

$x = -\frac{11}{5}s + \frac{227}{5}$

$x = -\frac{11}{5}s + \frac{227}{5}, y = -\frac{2}{5}s + \frac{94}{5}, z = s, s \in \mathbf{R}$

c. The lines found in 4. a. and 4. b. do not intersect, because they are in parallel planes.

5. a. For there to be no solution the lines must be inconsistent with each other.

L_1: $x + ay = 9$

L_2: $ax + 9y = -27$

$\frac{1}{a} = \frac{a}{9}$

$a = \pm 3$

For $a = 3$:

L_1: $x + 3y = 9$

L_2: $3x + 9y = -27$

For $a = -3$, the equations are equivalent.

So there is no solution when $a = 3$.

b. To have an infinite number of solutions, the lines must be proportional.

L_1: $x + ay = 9$

L_2: $ax + 9y = -27$

$-3(x + ay = 9) = -3x - 3ay = -27$

L_1: $-3x - 3ay = -27$

L_2: $ax + 9y = -27$

$a = -3$

c. The system has one solution when $a \neq 3$ or $a \neq -3$, because other values lead to an infinite number of solutions or no solution.

6. L_1: $\frac{x - 11}{2} = \frac{y - 4}{-4} = \frac{z - 27}{5} = s$

L_2: $x = 0, y = 1 - 3t, z = 3 + 2t, t \in \mathbf{R}$

L_1: $x = 2s + 11, y = -4s + 4, z = 27 + 5s$

$x = 0 = 2s + 11,$

$s = -5.5$

$y = -4(-5.5) + 4, z = 27 + 5(-5.5)$

$x = 0, y = 26, z = -0.5$

$y = 26 = 1 - 3t, t = -\frac{25}{3}$

$z = -0.5 = 3 + 2t, t = -\frac{7}{4}$

Since there is no t-value that satisfies the equations, there is no intersection, and these lines are skew.

7. a. L_1: $\frac{x - 5}{2} = y - 2 = \frac{z + 4}{-3} = s$

L_2: $(x - 3, y - 20, z - 7) = t(2, -4, 5), t \in \mathbf{R}$

L_1: $x = 2s + 5, y = s + 2, z = -3s - 4$

L_2: $x = 2t + 3, y = -4t + 20, z = 5t + 7$

$x = 2t + 3 = 2s + 5$

$y = s + 2 = -4t + 20$

$z = -3s - 4 = 5t + 7$

L_3: $2t - 2s - 2 = 0$

L_4: $4t + s - 18 = 0$

L_5: $5t + 3s + 11 = 0$

$L_3 + 2L_4$: $10t - 38 = 0, t = 3.8$

$3L_3 + 2L_5$: $16t + 16 = 0, t = -1$

b. Since there is no t-value that satisfies the equations, there is no intersection, and these lines are skew.

8. L_1: $x = 1 + 2s, y = 4 - s, z = -3s, s \in \mathbf{R}$

L_2: $x = -3, y = t + 3, z = 2t, t \in \mathbf{R}$

$x = -3 = 1 + 2s$

$s = -2$

$x = -3, y = 6, z = 6$

$(-3, 6, 6)$

9. a. L_1: $\vec{r} = (5, 1, 7) + s(2, 0, 5), s \in \mathbf{R}$

L_2: $\vec{r} = (-1, -1, 3) + t(4, 2, -1), t \in \mathbf{R}$

L_1: $x = 5 + 2s, y = 1, z = 7 + 5s$

L_2: $x = -1 + 4t, y = -1 + 2t, z = 3 - t$

$y = 1 = -1 + 2t,$

$t = 1$

$x = -1 + 4(1), y = -1 + 2(1),$

$z = 3 - (1)$

$x = 3, y = 1, z = 2$

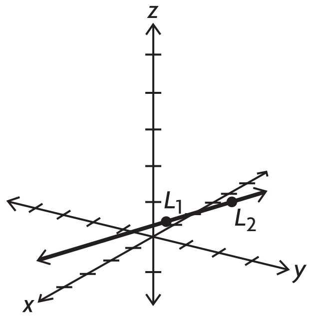
$(3, 1, 2)$

b. L_1: $\vec{r} = (2, -1, 3) + s(5, -1, 6), s \in \mathbf{R}$

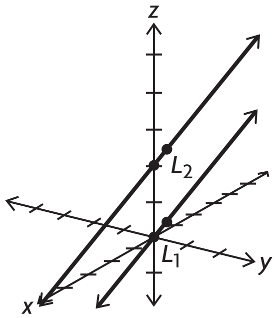
L_2: $\vec{r} = (-8, 1, -9) + t(5, -1, 6), t \in \mathbf{R}$

These lines are the same, so either one of these lines can be used as their intersection.

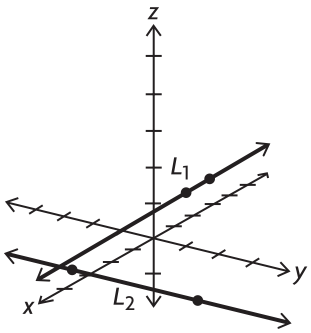
10. a. Answers may vary. For example:
i. coincident



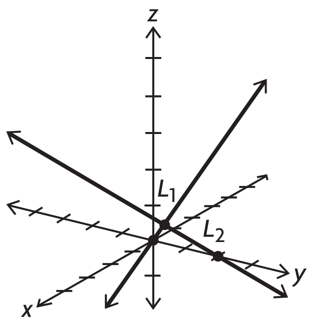
ii. parallel and distinct



iii. skew



iv. intersect in a point



b. i. When lines are the same, they are a multiple of each other.
ii. When lines are parallel, one equation is a multiple of the other equation, except for the constant term.
iii. When lines are skew, there are no common solutions to make each equation consistent.
iv. When the solution meets in a point, there is only one unique solution for the system.

11. a. A line and plane have an infinite number of points of intersection when the line lies in the plane.
b. Answers may vary. For example:
$\vec{r} = t(3, -5, -3), t \in \mathbf{R}$
$\vec{r} = t(3, -5, -3) + s(1, 1, 1), t, s \in \mathbf{R}$

12. a. ① $2x + 3y = 30$
② $x - 2y = -13$
Equation ① − (2 × equation ②): $7y = 56$
$y = 8$
$2x + 24 = 30$
$x = 3$
$(3, 8)$
b. ① $x + 4y - 3z + 6 = 0$
② $2x + 8y - 6z + 11 = 0$
There is no solution to this system, because the planes are parallel, but one plane lies above the other.
c. ① $x - 3y - 2z = -9$
② $2x - 5y + z = 3$
③ $-3x + 6y + 2z = 8$
Equation ① + (2 × equation ②): $5x - 13y = -3$
Equation ② + (equation ③): $-2x + 3y = -1$

$$\begin{array}{r} 2(5x - 13y = -3) \\ +\ 5(-2x + 3y = -1) \\ \hline -11y = -11 \end{array}$$

$y = 1$
$5x - 13(1) = -3$
$x = 2$
$(2) - 3(1) - 2z = -9$
$z = 4$
$(2, 1, 4)$

13. a. The two lines intersect at a point.
b. The two planes are parallel and do not meet.
c. The three planes intersect at a point.

14. a. $L{:}(x - y = 1) + (y + z = -3)$
$= x + z = -2$

$$L_1{:}\ y - z = 0, x = -\frac{1}{2}$$

$x + z = -2$

$$\left(-\frac{1}{2}\right) + z = -2$$

$$z = -\frac{3}{2}$$

$y - z = 0$

$$y - \left(-\frac{3}{2}\right) = 0$$
$$y = -\frac{3}{2}$$
$$\left(-\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}\right)$$

b. $\cos\theta = \dfrac{|n \cdot n_1|}{|n||n_1|}$
$$n = (1, 1, -1)$$
$$n_1 = (0, 1, 1)$$
$$\cos\theta = \frac{0}{|\sqrt{3}||\sqrt{2}|}$$
$$\theta = 90°$$

c. $(0, 1, 1) \times (1, 1, -1) = (-2, 1, -1)$
$$= (2, -1, 1)$$
$$Ax + By + Cz + D = 0$$
$$2x - y + z + D = 0$$
$$2\left(\frac{-1}{2}\right) - \left(\frac{-3}{2}\right) + \left(\frac{-3}{2}\right) + D = 0$$
$$D = 1$$
$$2x - y + z + 1 = 0$$

9.4 The Intersection of Three Planes, pp. 531–533

1. a. ① $x - 3y + z = 2$
② $0x + y - z = -1$
③ $0x + 0y + 3z = -12$
The system can be solved by first solving equation ③ for z. Thus,
$3z = -12$
$z = -4$
If we use the method of back substitution, we can substitute $z = -4$ into equation ② and solve for y.
$y - (-4) = -1$
$y = -5$
If we substitute $y = -5$ and $z = -4$ into equation ① we obtain the value of x.
$x - 3(-5) - 4 = 2$ or $x = -9$
The three planes intersect at the point with coordinates $(-9, -5, -4)$
Check:
Substituting into equation ①:
$x - 3y + z = -9 + 15 - 4 = 2$
Substituting into equation ②:
$0x + y - z = -5 + 4 = -1$
Substituting into equation ③: $0x + 0y + 3z = -12$

b. This solution is the point at which all three planes meet.

2. a. ① $x - y + z = 4$
② $0x + 0y + 0z = 0$
③ $0x + 0y + 0z = 0$
The answer may vary depending upon the constant you multiply the equations by. For example,
$2 \times (x - y + z = 4) = 2x - 2y + 2z = 8$
$3 \times (x - y + z = 4) = 3x - 3y + 3z = 12$
$3x - 3y + 3z = 12$ and $2x - 2y + 2z = 8$ are equations that could work.
b. These three planes are intersecting in one single plane, because all three equations can be changed into one equivalent equation. They are coincident planes.
c. Setting $x = t$ and $y = s$ leads to
$t - s + z = 4$ or $z = s - t + 4, s, t \in \mathbf{R}$
d. Setting $y = t$ and $z = s$ leads to
$x - t + s = 4$ or $x = t - s + 4, s, t \in \mathbf{R}$

3. a. ① $2x - y + 3z = -2$
② $x - y + 4z = 3$
③ $0x + 0y + 0z = 1$
The answer may vary depending upon the constants and equations you use to determine your answer. For example,
Equation ① + equation ② + equation ③ =
$$\begin{array}{r} (2x - y + 3z = -2) \\ +\ (x - y + 4z = 3) \\ +\ (0x + 0y + 0z = 1) \\ \hline 3x - 2y + 7z = 2 \end{array}$$
or
$2 \times$ equation ② $-$ equation ③ =
$$\begin{array}{r} (2x - 2y + 8z = 6) \\ -\ (0x + 0y + 0z = 1) \\ \hline 2x - 2y + 8z = 5 \end{array}$$
$2x - y + 3z = -2, x - y + 4z = 3$, and $3x - 2y + 7z = 2$ is one system of equations that could produce the original system composed of equations ①, ②, and ③.
$2x - y + 3z = -2, x - y + 4z = 3$, and $2x - 2y + 8z = 5$ is another system of equations that could produce the original system composed of equations ①, ②, and ③.
b. The systems have no solutions.

4. a. ① $x + 2y - z = 4$
② $x + 0y - 2z = 0$
③ $2x + 0y + 0z = -6$
The system can be solved by first solving equation ③ for x. So,

$2x = -6$
$x = -3$
If we use the method of back substitution, we can substitute $x = -3$ into equation ② and solve for z.
$-3 - 2z = 0$
$z = -\frac{3}{2}$
If we substitute $x = -3$ and $z = -\frac{3}{2}$ into equation ① we obtain the value of y.
$-3 + 2y + \frac{3}{2} = 4$ or $y = \frac{11}{4}$
The equations intersect at the point with coordinates $(-3, \frac{11}{4}, -\frac{3}{2})$
Check:
Substituting into equation ①:
$x + 2y - z = -3 + \frac{22}{4} + \frac{3}{2} = 4$
Substituting into equation ②:
$x + 0y - 2z = -3 + 3 = 0$
Substituting into equation ③: $2x + 0y + 0z = -6$
b. This solution is the point at which all three planes meet.
5. a. ① $2x - y + z = 1$
② $x + y - z = -1$
③ $-3x - 3y + 3z = 3$
Since equation ③ = −equation ②, equation ② and equation ③ are consistent or lie in the same plane. Equation ① meets this plane in a line.
b. Adding equation ② and equation ① creates an equivalent equation, $3x = 0$ or $x = 0$. Substituting $x = 0$ into equation ① and equation ② gives equation ④ $z - y = 1$ and equation ⑤ $y - z = -1$. Equations ④ and ⑤ indicate the problem has infinite solutions. Substituting $y = t$ into equation ④ or ⑤ leads to
$x = 0$, $y = t$, and $z = 1 + t$, $t \in \mathbf{R}$
Check:
$2(0) - s + (s + 1) = 1$
$0 + s - (s + 1) = -1$
$-3(0) - 3(s) + 3(s + 1) = 3$
6. ① $2x + 3y - 4z = -5$
② $x - y + 3z = -201$
③ $5x - 5y + 15z = -1004$
There is no solution to this system of equations, because if you multiply equation ② by 5 you obtain a new equation, $5x - 5y + 15z = -1005$, which is inconsistent with equation ③.
7. a. Yes when this equation is alone, this is true, because any constants can be substituted into the variables in the equation $0x + 0y + 0z = 0$ and the equation will always be consistent.
b. Answers may vary. For example: To obtain a no solution and an equation with $0x + 0y + 0z = 0$, you must have two equal planes and one parallel distinct plane. For example one solution is:
$x + y + z = 2$
$2x + 2y + 2z = 4$
$3x + 3y + 3z = 12$
8. a. ① $2x + y - z = -3$
② $x - y + 2z = 0$
③ $3x + 2y - z = -5$
2 × equation ② + equation ③ = $5x + 0y + 0z = -5$ which gives $x = -1$.
Equation ① + equation ② = $3x + 0y + 1z = -3$. Substituting $x = 1$ into this equation leads to: $3(-1) + z = -3$ or $z = 0$.
Substituting $z = 0$ and $x = -1$ into equation ① gives: $2(-1)y - 0 = -3$ or $y = -1$. $(-1, -1, 0)$ is the point at which the three planes meet.
Check:
Substituting into equation ①:
$2x + y - z = -2 - 1 + 0 = -3$
Substituting into equation ②:
$x - y + 2z = -1 + 1 + 0 = 0$
Substituting into equation ③:
$3x + 2y - z = -3 - 2 + 0 = -5$
b. ① $\frac{x}{3} - \frac{y}{4} + z = \frac{7}{8}$
② $2x + 2y - 3z = -20$
③ $x - 2y + 3z = 2$
Equation ② + equation ③ = $3x + 0y + 0z = -18$ which gives $x = -6$.
Equation ③ − 3 × Equation ① = $-\frac{5}{4}y = -\frac{5}{8}$ or $y = \frac{1}{2}$. Substituting $x = -6$ and $y = \frac{1}{2}$ into equation ③ leads to:
$-6 - 2\left(\frac{1}{2}\right) + 3z = 2$ or $z = 3$.
$(-6, \frac{1}{2}, 3)$ is the point at which the three planes meet.
Check:
Substituting into equation ①:
$\frac{x}{3} - \frac{y}{4} + z = -2 - \frac{1}{8} + 3 = \frac{7}{8}$
Substituting into equation ②:
$2x + 2y - 3z = -12 + 1 - 9 = -20$
Substituting into equation ③:
$x - 2y + 3z = -6 - 1 + 9 = 2$
c. ① $x - y = -199$
② $x + z = -200$
③ $y - z = 201$
Equation ② + equation ③ = equation ④ $= x + y = 1$

Equation ④ + equation ① = $2x = -198$ or $x = -99$. Substituting $x = -99$ into equation ① leads to:
$-99 - y = -199$ or $y = 100$. Substituting $x = -99$ into equation ②, you obtain:
$-99 + z = -200$ or $z = -101$
$(-99, 100, -101)$ is the point at which the three planes meet.
Check:
Substituting into equation ①:
$x - y = -99 - (100) = -199$
Substituting into equation ②:
$x + z = -99 - 101 = -200$
Substituting into equation ③:
$y - z = 100 - (-101) = 201$
d. ① $x - y - z = -1$
② $y - 2 = 0$
③ $x + 1 = 5$
Rearranging equation ② gives $y = 2$. Solving for x in equation ③ gives $x = 4$.
Substituting $x = 4$ and $y = 2$ into equation ① leads to:
$4 - 2 - z = -1$ or $z = 3$.
(4, 2, 3) is the point at which all three planes meet.
9. a. ① $x - 2y + z = 3$
② $2x + 3y - z = -9$
③ $5x - 3y + 2z = 0$
Equation ③ + equation ② = equation ④
$= 7x + 1z = -9$.
Setting $z = t$, $x = -\frac{1}{7}t - \frac{9}{7}$
Equation ② $- 2 \times$ equation ① = equation ⑤
$= 7y + -3z = -15$.
Setting $z = t$, $y = -\frac{15}{7} + \frac{3}{7}t$
$x = -\frac{1}{7}t - \frac{9}{7}$, $y = -\frac{15}{7} + \frac{3}{7}t$, and $z = t$, $t \in \mathbf{R}$ The planes intersect in a line.
b. ① $x - 2y + z = 3$
② $x + y + z = 2$
③ $x - 3y + z = -6$
Equation ③ − equation ② = $-4y = -8$ or $y = 2$
Equation ③ − equation ① = $-1y = -9$ or $y = 9$
Since the solutions for y are different from these two equations, there is no solution to this system of equations.
c. ① $x - y + z = -2$
② $x + y + z = 2$
③ $x - 3y + z = -6$
Equation ① + equation ② = equation ④
$= 2x + 2y = 0$.
Setting $z = t$, $x = -t$
Using $z = t$ and $x = -t$, Solve equation ①
$-t - y + t = -2$ or $y = 2$
$x = -t$, $y = 2$, and $z = t$, $t \in \mathbf{R}$
The planes intersect in a line.
10. a. ① $x - y + z = 2$
② $2x - 2y + 2z = 4$
③ $x + y - z = -2$
Equation ① + equation ③ = equation ④
$= 2x = 0$ or $x = 0$.
Setting $z = t$,
Equation ①: $0 - y + t = 2$ or $y = t - 2$
$x = 0$, $y = t - 2$, and $z = t$, $t \in \mathbf{R}$
b. ① $2x - y + 3z = 0$
② $4x - 2y + 6z = 0$
③ $-2x + y - 3z = 0$
Equation ① + equation ③ = equation ④
$= 2x = 0$ or $x = 0$.
Setting $y = t$ and $z = s$, equation ①:
$2x - t + 3s = 0$ or $x = \dfrac{t - 3s}{2}$
$x = \dfrac{t - 3s}{2}$, $y = t$, and $z = s$, $s, t \in \mathbf{R}$
11. a. ① $x + y + z = 1$
② $x - 2y + z = 0$
③ $x - y + z = 0$
Equation ① − equation ③ = equation ④
$= 2y = 1$ or $y = \frac{1}{2}$
Equation ② − equation ③ = equation ⑤
$= -y = 0$ or $y = 0$
Since the y-variable is different in equation ④ and equation ⑤, the system is inconsistent and has no solution.
b. Answers may vary. For example: If you use the normals from equations ①, ②, and ③, you can determine the direction vectors from the equations' coefficients.
$\overrightarrow{n_1} = (1, 1, 1)$
$\overrightarrow{n_2} = (1, -2, 1)$
$\overrightarrow{n_3} = (1, -1, 1)$
$m_1 = \overrightarrow{n_1} \times \overrightarrow{n_2} = (3, 0, -3)$
$m_2 = \overrightarrow{n_1} \times \overrightarrow{n_3} = (2, 0, -2)$
$m_3 = \overrightarrow{n_2} \times \overrightarrow{n_3} = (-1, 0, 1)$
c. The three lines of intersection are parallel and are pairwise coplanar, so they form a triangular prism.
d. $\overrightarrow{n_1} \times \overrightarrow{n_2}$ is perpendicular to $\overrightarrow{n_3}$. So since, $(\overrightarrow{n_1} \times \overrightarrow{n_2}) \cdot \overrightarrow{n_3} = 0$, a triangular prism forms.
12. a. ① $x - y + 3z = 3$
② $x - y + 3z = 6$
③ $3x - 5z = 0$

Equation ① and equation ② have the same set of coefficients and variables, however, equations ① equals 3 while equation ② equals 6, which means there is no possible solution.

b. ① $5x - 2y + 3z = 1$
② $5x - 2y + 3z = -1$
③ $5x - 2y + 3z = 13$

All three equations equal different numbers so there is no possible solution.

c. ① $x - y + z = 9$
② $2x - 2y + 2z = 18$
③ $2x - 2y + 2z = 17$

Equation ② equals 18 while equation ③ equals 17, which means there is no possible solution.

d. The coefficients of equation ① are half the coefficients of equation ②, but the constant term is not half the other constant term.

13. a. ① $2x - y - z = 10$
② $x + y + 0z = 7$
③ $0x + y - z = 8$

Equation ① − 2 × equation ② − equation ③: $-4y = -12$ or $y = 3$. Substituting $y = 3$ into equation ② and equation ③ gives:
$x + 3 + 0z = 7$ or $x = 4$
$0x + 3 - z = 8$ or $z = -5$
$(4, 3, -5)$

b. ① $2x - y + z = -3$
② $x + y - 2z = 1$
③ $5x + 2y - 5z = 0$

Equation ① + equation ②: $3x - z = -2$.

Setting $z = t$, $x = \frac{t - 2}{3}$

Equation ① − 2 × equation ②: $-3y + 5z = -5$.

Setting $z = t$, $y = \frac{5t + 5}{3}$

$x = \frac{t - 2}{3}, y = \frac{5t + 5}{3}, z = t, t \in \mathbf{R}$

c. ① $x + y - z = 0$
② $2x - y + z = 0$
③ $4x - 5y + 5z = 0$

Equation ① + equation ②: $3x = 0$ or $x = 0$
Setting $x = 0$ and $z = t$ in equation ② gives,
$2(0) - y + t = 0$ or $y = t$
$x = 0, y = t, z = t, t \in \mathbf{R}$

d. ① $x - 10y + 13z = -4$
② $2x - 20y + 26z = -8$
③ $x - 10y + 13z = -8$

If you multiply equation ② by two, you obtain $2x - 20y + 26z = -16$. Since equation ② and equation ③ equal different numbers, there is no solution to this system.

e. ① $x - y + z = -2$
② $x + y + z = 2$
③ $3x + y + 3z = 2$

Equation ① + equation ②: $-2y = -4$ or $y = 2$
Setting $y = 2$ and $z = t$ in equation ①,
$x - 2 + t = -2$ or $x = -t$
$x = -t, y = 2, z = t, t \in \mathbf{R}$

f. ① $x + y + z = 0$
② $x - 2y + 3z = 0$
③ $2x - y + 3z = 0$

Equation ① − equation ② = equation ④
$= 3y - 2z = 0$
Equation ③ − 2 × equation ② − equation ⑤
$= 3y - 3z = 0$
Equation ④ − equation ⑤: $z = 0$
Setting $z = 0$ in equation ① and equation ②,
Equation ⑥ $= x + y = 0$
Equation ⑦ $= x - 2y = 0$
Equation ⑥ − equation ⑦: $3y = 0$ or $y = 0$
Setting $y = 0$ and $z = 0$ in equation ① leads to $x = 0$
(0, 0, 0)

14. a. First, reorder these equations so that equation ② is first, equation ③ is second, and equation ① last.
① $x - y + z = p$
② $4x + qy + z = 2$
③ $2x + y + z = 4$

To eliminate x from the last two equations, subtract 4 times equation ① from equation ②, and subtract 2 times equation ① from equation ③.
① $x - y + z = p$
② $(q + 4)y - 3z = 2 - 4p$
③ $3y - z = 4 - 2p$

There will be an infinite number of solutions if $q + 4 = 9$ and $3(4 - 2p) = 2 - 4p$ because then equation ② will be 3 times equation ③. This means that $p = q = 5$.

b. Based on what was found in part **a.**, substituting in $p = q = 5$ we will arrive at the equivalent system
① $x - y + z = 5$
② $9y - 3z = -18$
③ $3y - z = -6$

which is really the same as
① $x - y + z = 5$
② $3y - z = -6$

Letting $z = t$, we see that equation ② delivers

$y = \frac{1}{3}(t - 6)$

$= \frac{1}{3}t - 2$

and so equation ① gives

$x = \frac{1}{3}(t - 6) - t + 5$

$= -\frac{2}{3}t + 3$

So the parametric equation of the line of intersection is

$x = -\frac{2}{3}t + 3,\ y = \frac{1}{3}t - 2,\ z = t,\ t \in \mathbf{R}.$

15. a. First, eliminate x from two of these equations. To make things easier, switch equation ① with equation ②, and multiply equation ③ by 2.

① $2x + y + z = -4$

② $4x + 3y + 3z = -8$

③ $6x - 4y + (2m^2 - 12)z = 2m - 8$

Now eliminate x from the last two equations by using proper multiples of the first equation.

① $2x + y + z = -4$

② $y + z = 0$

③ $-7y + (2m^2 - 15)z = 2m + 4$

Now eliminate y from the third equation by using a proper multiple of the second equation.

① $2x + y + z = -4$

② $y + z = 0$

③ $(2m^2 - 8)z = 2m + 4$

If $2m^2 - 8 = 0$ (the coefficient of z in the third equation), then $m = \pm 2$. However, if $m = 2$, the third equation would become $0z = 8$, which has no solutions. So there is no solution if $m = 2$.

b. Working with what was found in part **a.**, if $m \neq \pm 2$, then the third equation in the equivalent system found there will have a unique solution for z, namely

$z = \frac{2m + 4}{2m^2 - 8},$

and back-substituting into the other two equations will give unique solutions for x and y also. So there is a unique solution if $m \neq \pm 2$.

c. Again using the equivalent system found in part **a.**, setting $m = -2$ will deliver the third equation $0z = 0$, which allows for z to be anything at all. So $m = -2$ will give an infinite number of solutions.

16. a. ① $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0$

② $\frac{2}{a} + \frac{3}{b} + \frac{2}{c} = \frac{13}{6}$

③ $\frac{4}{a} - \frac{2}{b} + \frac{3}{c} = \frac{5}{2}$

Equation ② $-\ 2 \times$ equation ①:

$\frac{1}{b} + \frac{4}{c} = \frac{13}{6}$ = equation ④

Equation ③ $-\ 4 \times$ equation ①: $-\frac{6}{b} + \frac{7}{c}$

$m_3 = \vec{n} \times \vec{n_1} = (-1, 0, 1) = \frac{5}{2}$ = equation ⑤

Equation ⑤ $+\ 6 \times$ equation ④:

$\frac{31}{c} = 15.5$ or $c = 2$

Substituting $c = 2$ into equation ④:

$\frac{1}{b} + 2 = \frac{13}{6}$ or $b = 6$

Substituting $c = 2$ and $b = 6$ into equation ①:

$\frac{1}{a} + \frac{1}{6} - \frac{1}{2} = 0$ or $a = 3$

(3, 6, 2)

9.5 The Distance from a Point to a Line in R^2 and R^3, pp. 540–541

1. a. $3x + 4y - 5 = 0$

$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

$d = \frac{|3(-4) + 4(5) - 5|}{\sqrt{3^2 + 4^2}}$

$= \frac{3}{5}$

b. $5x - 12y + 24 = 0$

$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

$d = \frac{|5(-4) - 12(5) + 24|}{\sqrt{5^2 + (-12)^2}}$

$= \frac{56}{13}$ or 4.31

c. $9x - 40y = 0$

$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

$d = \frac{|9(-4) - 40(5)|}{\sqrt{9^2 + (40)^2}}$

$= \frac{236}{\sqrt{1681}}$ or 5.76

2. a. $2x - y + 1 = 0$ and $2x - y + 6 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of

the lines. It is easiest to find a point where the line crosses the x or y-axis.

$2(0) - y + 1 = 0$ or $y = 1$ which corresponds to the point $(0, 1)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|2(0) - 1(1) + 6|}{\sqrt{2^2 + (-1)^2}}$$

$$= \frac{5}{\sqrt{5}} \text{ or } 2.24$$

b. $7x - 24y + 168 = 0$ and $7x - 24y - 336 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of the lines. It is easiest to find a point where the line crosses the x or y-axis.

$7(0) - 24y + 168 = 0$ or $y = 7$ which corresponds to the point $(0, 7)$

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|7(0) - 24(7) - 336|}{\sqrt{7^2 + (-24)^2}}$$

$$= \frac{504}{25} \text{ or } 20.16$$

3. a. $\vec{r} = (-1, 2) + s(3, 4), s \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = -1 + 3s$, $y = 2 + 4s$. We construct a vector from $R(-2, 3)$ to a general point on the line.

$$\vec{a} = [-2 - (-1 + 3s), 3 - (2 + 4s)]$$
$$= (-1 - 3s, 1 - 4s).$$
$$(3, 4) \cdot (-1 - 3s, 1 - 4s) = 0$$
$$(-3 - 9s) + (4 - 16s) = 0$$
$$s = \frac{1}{25}$$

This means that the minimal distance between $R(-2, 3)$ and the line occurs when $s = \frac{1}{25}$. This point corresponds to $\left(-\frac{22}{25}, \frac{54}{25}\right)$. The distance between this point and $(-2, 3)$ is 1.4.

b. $\vec{r} = (1, 0) + t(5, 12), t \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 1 + 5t$, $y = 12t$. We construct a vector from $R(-2, 3)$ to a general point on the line.

$$\vec{a} = [-2 - (1 + 5t), 3 - (12t)]$$
$$= (-3 - 5t, 3 - 12t).$$
$$(5, 12) \cdot (-3 - 5t, 3 - 12t) = 0$$
$$(-15 - 25t) + (36 - 144t) = 0$$
$$t = \frac{21}{169}$$

This means that the minimal distance between $R(-2, 3)$ and the line occurs when $t = \frac{21}{169}$. This point corresponds to $\left(\frac{274}{169}, \frac{252}{169}\right)$. The distance between this point and $(-2, 3)$ is about 3.92.

c. $\vec{r} = (1, 3) + p(7, -24), p \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 1 + 7p$, $y = 3 - 24p$. We construct a vector from $R(-2, 3)$ to a general point on the line.

$$\vec{a} = [-2 - (1 + 7p), 3 - (3 - 24p)]$$
$$= (-3 - 7p, 24p).$$
$$(7, -24) \cdot (-3 - 7p, 24p) = 0$$
$$(-21 - 49p) + (-576p) = 0$$
$$p = -\frac{21}{625}$$

This means that the minimal distance between $R(-2, 3)$ and the line occurs when $p = -\frac{21}{625}$. This point corresponds to $\left(\frac{478}{625}, \frac{2379}{625}\right)$. The distance between this point and $(-2, 3)$ is about 2.88.

4. a. $d = \dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

If you substitute in the coordinates $(0, 0)$, the formula changes to $d = \dfrac{|A(0) + B(0) + C|}{\sqrt{A^2 + B^2}}$, which reduces to $d = \dfrac{|C|}{\sqrt{A^2 + B^2}}$.

b. $3x - 4y - 12 = 0$ and $3x - 4y + 12 = 0$

$$d(L_1) = \frac{|C|}{\sqrt{A^2 + B^2}} = \frac{|-12|}{\sqrt{3^2 + (-4)^2}}$$
$$= \frac{12}{5}$$

$$d(L_2) = \frac{|C|}{\sqrt{A^2 + B^2}} = \frac{|12|}{\sqrt{3^2 + (-4)^2}}$$
$$= \frac{12}{5}$$

The distance between these parallel lines is $\frac{12}{5} + \frac{12}{5} = \frac{24}{5}$, because one of the lines is below the origin and the other is above the origin.

c. $3x - 4y - 12 = 0$ and $3x - 4y + 12 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of the lines. It is easiest to find a point where the line crosses the x or y-axis.

$3(0) - 4y - 12 = 0$ or $y = -3$ which corresponds to the point $(0, 3)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|3(0) - 4(-3) + 12|}{\sqrt{3^2 + (-4)^2}}$$

$$= \frac{24}{5}$$

Both the answers to 4.b. and 4.c. are the same.

5. a. $\vec{r} = (-2, 1) + s(3, 4), s \in \mathbf{R}$

$\vec{r} = (1, 0) + t(3, 4), t \in \mathbf{R}$

First find a random point on one of the lines. We will use $(-2, 1)$ from the first equation. We start by writing the second equation in parametric form. Doing so gives $x = 1 + 3t$, $y = 4t$. We construct a vector from $P(-2, 1)$ to a general point on the line.

$$\vec{a} = [-2 - (1 + 3t), 1 - (4t)]$$
$$= (-3 - 3t, 1 - 4t).$$
$$(3, 4) \cdot (-3 - 3t, 1 - 4t) = 0$$
$$(-9 - 9t) + (4 - 16t) = 0$$
$$t = -\frac{1}{5}$$

This means that the minimal distance between $P(-2, 1)$ and line occurs when $t = -\frac{1}{5}$. This point corresponds to $\left(\frac{2}{5}, -\frac{4}{5}\right)$. The distance between this point and $(-2, 1)$ is 3

b. $\frac{x - 1}{4} = \frac{y}{-3}$ and $\frac{x}{4} = \frac{y + 1}{-3}$

First change one equation into a Cartesian equation, which leads to $3x + 4y - 3 = 0$ and take a point from the other equation such as $(4, -4)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|3(4) + 4(-4) - 3|}{\sqrt{3^2 + 4^2}}$$

$$= \frac{7}{5} \text{ or } 1.4$$

c. $2x - 3y + 1 = 0$ and $2x - 3y - 3 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of the lines. It is easiest to find a point where the line crosses the x or y-axis.

$2(0) - 3y - 3 = 0$ or $y = -1$ which corresponds to the point $(0, -1)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|2(0) - 3(-1) + 1|}{\sqrt{2^2 + (-3)^2}}$$

$$= \frac{4}{\sqrt{13}} \text{ or } 1.11$$

d. $5x + 12y = 120$ and $5x + 12y + 120 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of the lines. It is easiest to find a point where the line crosses the x or y-axis.

$5(0) + 12y = 120$ or $y = 10$ which corresponds to the point $(0, 10)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|5(0) + 12(10) + 120|}{\sqrt{5^2 + 12^2}}$$

$$= \frac{240}{13} \text{ or } 18.46$$

6. a. $P(1, 2, -1)$ $\vec{r} = (1, 0, 0) + s(2, -1, 2), s \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 1 + 2s$, $y = -s$, and $z = 2s$. We construct a vector from $P(1, 2, -1)$ to a general point on the line.

$$\vec{a} = [1 - (1 + 2s), 2 - (-s), -1 - (2s)]$$
$$= (-2s, 2 + s, -1 - 2s).$$
$$(2, -1, 2) \cdot (-2s, 2 + s, -1 - 2s) = 0$$
$$(-4s) + (-2 - s) + (-2 - 4s) = 0$$
$$s = -\frac{4}{9}$$

This means that the minimal distance between $P(1, 2, -1)$ and the line occurs when $s = -\frac{4}{9}$. This point corresponds to $\left(\frac{1}{9}, \frac{4}{9}, -\frac{8}{9}\right)$. The distance between this point and $P(1, 2, -1)$ is 1.80.

b. $P(0, -1, 0)$ $\vec{r} = (2, 1, 0) + t(-4, 5, 20), t \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 2 - 4t$, $y = 1 + 5t$, and $z = 20t$. We construct a vector from $P(0, -1, 0)$ to a general point on the line.

$$\vec{a} = [0 - (2 - 4t), -1 - (1 + 5t), 0 - (20t)]$$
$$= (-2 + 4t, -2 - 5t, 20t).$$
$$(-4, 5, 20) \cdot (-2 + 4t, -2 - 5t, -20t) = 0$$
$$(8 - 16t) + (-10 - 25t) + (-400t) = 0$$
$$t = -\frac{2}{441}$$

This means that the minimal distance between $P(0, -1, 0)$ and the line occurs when $t = -\frac{2}{441}$.

This point corresponds to $\left(\frac{890}{441}, \frac{431}{441}, -\frac{40}{441}\right)$. The distance between this point and $P(0, -1, 0)$ is 2.83.

c. $P(2, 3, 1)$ $\vec{r} = p(12, -3, 4), p \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 12p$, $y = -3p$, and $z = 4p$. We construct a vector from $P(2, 3, 1)$ to a general point on the line.

$$\begin{aligned}\vec{a} &= [2 - (12p), 3 - (-3p), 1 - (4p)]\\ &= (2 - 12p, 3 + 3p, 1 - 4p).\end{aligned}$$

$$(12, -3, 4) \cdot (2 - 12p, 3 + 3p, 1 - 4p) = 0$$
$$(24 - 144p) + (-9 - 9p) + (4 - 16p) = 0$$
$$p = \frac{19}{169}$$

This means that the minimal distance between $P(2, 3, 1)$ and the line occurs when $p = \frac{19}{169}$. This point corresponds to $\left(\frac{228}{169}, -\frac{57}{169}, \frac{76}{169}\right)$. The distance between this point and $P(2, 3, 1)$ is 3.44.

7. a. $\vec{r} = (1, 1, 0) + s(2, 1, 2), s \in \mathbf{R}$

$\vec{r} = (-1, 1, 2) + t(2, 1, 2), t \in \mathbf{R}$

First find a random point on one of the lines. We will use $P(-1, 1, 2)$ from the second equation. We then write the first equation in parametric form. Doing so gives $x = 1 + 2s$, $y = 1 + s$, and $z = 0 + 2s$. We construct a vector from $P(-1, 1, 2)$ to a general point on the line.

$$\begin{aligned}\vec{a} &= [-1 - (1 + 2s), 1 - (1 + s), 2 - 2s]\\ &= (-2 - 2s, 2 - 2s).\end{aligned}$$

$$(2, 1, 2) \cdot (-2 - 2s, -s, 2 - 2s) = 0$$
$$(-4 - 4s) + (-s) + (4 - 4s) = 0$$
$$s = 0$$

This means that the minimal distance between $P(-1, 1, 2)$ and line occurs when $s = 0$. This point corresponds to $(1, 1, 0)$. The distance between this point and $(-1, 1, 2)$ is 2.83

b. $\vec{r} = (3, 1, -2) + m(1, 1, 3), m \in \mathbf{R}$

$\vec{r} = (1, 0, 1) + n(1, 1, 3), n \in \mathbf{R}$

First find a random point on one of the lines. We will use $P(1, 0, 1)$ from the second equation. We then write the first equation in parametric form. Doing so gives $x = 3 + m$, $y = 1 + m$, and $z = -2 + 3m$. We construct a vector from $P(1, 0, 1)$ to a general point on the line.

$$\begin{aligned}\vec{a} &= [1 - (3 + m), 0 - (1 + m), 1 - (-2 + 3m)]\\ &= (-2 - 3m, -1 - m, 3 - 3m).\end{aligned}$$

$$(1, 1, 3) \cdot (-2 - 3m, -1 - m, 3 - 3m) = 0$$
$$(-2 - 3m) + (-1 - m) + (9 - 9m) = 0$$
$$m = \frac{6}{13}$$

This means that the minimal distance between $P(1, 0, 1)$ and line occurs when $m = \frac{6}{13}$. This point corresponds to $\left(\frac{45}{13}, \frac{19}{13}, -\frac{6}{13}\right)$. The distance between this point and $(1, 0, 1)$ is 3.28

8. a. $\vec{r} = (1, -1, 2) + s(1, 3, -1), s \in \mathbf{R}$

First we write the equation in parametric form. Doing so gives $x = 1 + s$, $y = -1 + 3s$, and $z = 2 - s$. We construct a vector from $P(2, 1, 3)$ to a general point on the line.

$$\begin{aligned}\vec{a} &= [2 - (1 + s), 1 - (-1 + 3s), 3 - (2 - s)]\\ &= (1 - s, 2 - 3s, 1 + s).\end{aligned}$$

$$(1, 3, -1) \cdot (1 - s, 2 - 3s, 1 + s) = 0$$
$$(1 - s) + (6 - 9s) + (1 + s) = 0$$
$$s = \frac{6}{11}$$

This means that the minimal distance between $P(2, 1, 3)$ and line occurs when $s = \frac{6}{11}$. This point corresponds to $\left(\frac{17}{11}, \frac{7}{11}, \frac{16}{11}\right)$.

b. The distance between $\left(\frac{17}{11}, \frac{7}{11}, \frac{16}{11}\right)$ and $(2, 1, 3)$ is 1.65.

9. First, find the line L of intersection between the planes

① $x - y + 2z = 2$

② $x + y - z = -2$

Subtract the first equation from the second to eliminate x and get the equivalent system

① $x - y + 2z = 2$

② $2y - 3z = -4$

Let $z = t$. Then the second equation gives

$$2y = 3t - 4$$
$$y = \frac{3}{2}t - 2$$

So substituting these into the first equation gives

$$\begin{aligned}x &= y - 2z + 2\\ &= \left(\frac{3}{2}t - 2\right) - 2t + 2\\ &= -\frac{1}{2}t\end{aligned}$$

So the equation of the line of intersection for these two planes in parametric form is

$x = -\frac{1}{2}t$, $y = \frac{3}{2}t - 2$, $z = t$, $t \in \mathbf{R}$.

The direction vector for this line is $\left(-\frac{1}{2}, \frac{3}{2}, 1\right)$, which is parallel to $(-1, 3, 2)$. So, to make things easier, the parametric form of this line of intersection could also be expressed as

$x = -t$, $y = 3t - 2$, $z = 2t$, $t \in \mathbf{R}$

In vector form, this is the same as

$\vec{r} = (0, -2, 0) + t(-1, 3, 2), t \in \mathbf{R}$.
Since $Q(0, -2, 0)$ is on this line,

$$\overrightarrow{QP} = (-1, 2, -1) - (0, -2, 0)$$
$$= (-1, 4, -1)$$

So the distance from $P(-1, 2, -1)$ to the line of intersection is

$$d = \frac{|(-1, 3, 2) \times (-1, 4, -1)|}{|(-1, 3, 2)|}$$
$$= \frac{|(-11, -3, -1)|}{|(-1, 3, 2)|}$$
$$= \sqrt{\frac{131}{14}}$$
$$\doteq 3.06$$

To find the point on the line that gives this minimal distance, let (x, y, z) be a point on the line. Then, using the parametric equations,
$(x, y, z) = (-t, 3t - 2, 2t)$
So the distance from P to this point is

$$\sqrt{(x + 1)^2 + (y - 2)^2 + (z + 1)^2}$$
$$= \sqrt{(1 - t)^2 + (3t - 4)^2 + (2t + 1)^2}$$
$$= \sqrt{14t^2 - 22t + 18}$$

To get the minimal distance, set this quantity equal to $\sqrt{\frac{131}{14}}$.

$$\sqrt{14t^2 - 22t + 18} = \sqrt{\frac{131}{14}}$$
$$14t^2 - 22t + 18 = \frac{131}{14}$$
$$196t^2 - 308t + 252 = 131$$
$$196t^2 - 308t + 121 = 0$$
$$t = \frac{308 \pm \sqrt{0}}{392}$$
$$= \frac{11}{14}$$

So the point on the line at minimal distance from P is

$$(x, y, z) = (-t, 3t, -2, 2t)$$
$$= \left(-\frac{11}{14}, 3\left(\frac{11}{14}\right) - 2, 2\left(\frac{11}{14}\right)\right)$$
$$= \left(-\frac{11}{14}, \frac{5}{14}, \frac{22}{14}\right)$$

10. A point on the line
$\vec{r} = (0, 0, 1) + s(4, 2, 1), s \in \mathbf{R}$.
has parametric equations
$x = 4s, y = 2s, z = 1 + s, s \in \mathbf{R}$.
Let this point be called
$Q(4s, 2s, 1 + s)$. Then

$$\overrightarrow{QA} = (2, 4, -5) - (4s, 2s, 1 + s)$$
$$= (2 - 4s, 4 - 2s, -6 - s)$$

If Q is at minimal distance from A, then this vector will be perpendicular to the direction vector for the line, $(4, 2, 1)$. This means that

$$0 = (2 - 4s, 4 - 2s, -6 - s) \cdot (4, 2, 1)$$
$$= 10 - 21s$$
$$s = \frac{10}{21}$$

So the point Q on the line at minimal distance from A is

$$Q(4s, 2s, 1 + s) = Q\left(4\left(\frac{10}{21}\right), 2\left(\frac{10}{21}\right), 1 + \frac{10}{21}\right)$$
$$= Q\left(\frac{40}{21}, \frac{20}{21}, \frac{31}{21}\right)$$

Also

$$\overrightarrow{QA} = \left(2 - \frac{40}{21}, 4 - \frac{20}{21}, -5 - \frac{31}{21}\right)$$
$$= \left(\frac{2}{21}, \frac{64}{21}, -\frac{136}{21}\right)$$

So the point A' will satisfy

$$\overrightarrow{QA'} = -\overrightarrow{QA}$$
$$= \left(-\frac{2}{21}, -\frac{64}{21}, \frac{136}{21}\right)$$
$$= A'(a, b, c) - Q$$
$$= \left(a - \frac{40}{21}, b - \frac{20}{21}, c - \frac{31}{21}\right)$$

So $a = \frac{38}{21}$, $b = -\frac{44}{21}$, and $c = \frac{167}{21}$. That is, $A'(\frac{38}{21}, -\frac{44}{21}, \frac{167}{21})$.

11. a. Think of H as being the origin, E as being on the x-axis, D as being on the z-axis, and G as being on the y-axis. That is,
$H(0, 0, 0)$
$E(3, 0, 0)$
$G(0, 2, 0)$
$D(0, 0, 2)$
and so on for the other points as well. Then line segment HB has direction vector
$B(3, 2, 2) - H(0, 0, 0) = (3, 2, 2)$.
Also, $\overrightarrow{HA} = (3, 0, 2)$. So the distance formula says that the distance between A and line segment HB is

$$d = \frac{|(3, 2, 2) \times (3, 0, 2)|}{|(3, 2, 2)|}$$
$$= \frac{|4, 0, -6|}{|(3, 2, 2)|}$$
$$= \sqrt{\frac{52}{17}}$$
$$\doteq 1.75$$

b. Vertices D and G will give the same distance to HB because they are equidistant to the segment HB. (This is easy to check with the distance formula used similarly to part **a.** The vertices C, E, and F give different distances than those found in part **a.**)
c. The height of triangle AHB was found in part **a.**, and was $\sqrt{\frac{52}{17}}$. The base length of this triangle is the magnitude of $\overrightarrow{HB} = (3, 2, 2)$, which is $\sqrt{52}$. So the area of this triangle is

$$\frac{1}{2}\left(\sqrt{\frac{52}{17}}\right)(\sqrt{17}) = \frac{1}{2}(\sqrt{52})$$
$$\doteq 3.6 \text{ units}^2$$

9.6 The Distance from a Point to a Plane, pp. 549–550

1. a. Yes the calculations are correct, Point A lies in the plane.
b. The answer 0 means that the point lies in the plane.
2. Use the distance formula.

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

a. The distance from $A(3, 1, 0)$ to the plane $20x - 4y + 5z + 7 = 0$ is

$$d = \frac{|20(3) + -4(1) + 5(0) + 7|}{\sqrt{20^2 + (-4)^2 + 5^2}}$$
$$= 3$$

b. The distance from $B(0, -1, 0)$ to the plane $2x + y + 2z - 8 = 0$ is

$$d = \frac{|2(0) + 1(-1) + 2(0) - 8|}{\sqrt{2^2 + 1^2 + 2^2}}$$
$$= 3$$

c. The distance from $C(5, 1, 4)$ to the plane $3x - 4y - 1 = 0$ is

$$d = \frac{|3(5) + -4(1) + 0(4) - 1|}{\sqrt{3^2 + (-4)^2 + 0^2}}$$
$$= 2$$

d. The distance from $D(1, 0, 0)$ to the plane $5x - 12y = 0$ is

$$d = \frac{|5(1) - 12(0) + 0(0) + 0|}{\sqrt{5^2 + (-12)^2 + 0^2}}$$
$$= \frac{5}{13} \text{ or } 0.38$$

e. The distance from $E(-1, 0, 1)$ to the plane $18x - 9y + 18z - 11 = 0$ is

$$d = \frac{|18(-1) - 9(0) + 18(1) - 11|}{\sqrt{18^2 + (-9)^2 + 18^2}}$$
$$= \frac{11}{27} \text{ or } 0.41$$

3. a. $3x + 4y - 12z - 26 = 0$ and
$3x + 4y - 12z + 39 = 0$
First find a point in the second plane such as $(-3, 0, 0)$. Then use $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$ to solve.

$$d = \frac{|3(-13) + 4(0) - 12(0) - 26|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$
$$= 5$$

b.
$$\begin{aligned} &3x + 4y - 12z - 26 = 0 \\ +\ &3x + 4y - 12z + 39 = 0 \\ &6x + 8y - 24z + 13 = 0 \end{aligned}$$

c. Answers may vary. Any point on the plane $6x + 8y - 24z + 13 = 0$ will work, for example $\left(-\frac{1}{6}, 0, \frac{1}{2}\right)$.

4. a. The distance from $P(1, 1, -3)$ to the plane $y + 3 = 0$ is

$$d = \frac{|0(1) + 1(1) + 0(-3) + 3|}{\sqrt{0^2 + (1)^2 + 0^2}}$$
$$= 4$$

b. The distance from $Q(-1, 1, 4)$ to the plane $x - 3 = 0$ is

$$d = \frac{|1(-1) + 0(1) + 0(4) - 3|}{\sqrt{1^2 + 0^2 + 0^2}}$$
$$= 4$$

c. The distance from $R(1, 0, 1)$ to the plane $z + 1 = 0$ is

$$d = \frac{|0(1) + 0(0) + 1(1) + 1|}{\sqrt{0^2 + 0^2 + 1^2}}$$
$$= 2$$

5. First you have to find an equation of a plane to the three points. The equation to this plane is $14x - 28y + 28z - 42 = 0$. Then use $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$ to solve for the distance.

$$d = \frac{|14(1) - 28(-1) + 28(1) - 42|}{\sqrt{14^2 + (-28)^2 + 28^2}}$$
$$= \frac{2}{3} \text{ or } 0.67$$

6. $3 = \frac{|A(3) - 2(-3) + 6(1) + 0|}{\sqrt{A^2 + (-2)^2 + 6^2}}$

$$3\sqrt{(A^2 + 40)} = |3A + 12|$$
$$\sqrt{(A^2 + 40)} = |A + 4|$$
$$A^2 + 40 = A^2 + 8A + 16$$
$$24 = 8A$$
$$3 = A$$

$A = 3$ is the only solution to this equation.

7. These lines are skew lines, and the plane containing the second line, $\vec{r} = (0, 0, 1) + t(1, 1, 0), t \in \mathbf{R}$, that is parallel to the first line will have direction vectors $(1, 1, 0)$ and $(3, 0, 1)$. So a normal to this plane is $(1, 1, 0) \times (3, 0, 1) = (1, -1, -3)$.
So the equation of this plane will be of the form $x - y - 3z + D = 0$. We want the point $(0, 0, 1)$ to be on this plane, and substituting this into the equation above gives $D = 3$. So the equation of the plane containing $\vec{r} = (0, 0, 1) + t(1, 1, 0), t \in \mathbf{R}$ and parallel to the first line is
$x - y - 3z + 3 = 0$.
Since $(0, 1, -1)$ is on the first line, the distance between these skew lines is the same as the distance between this point and the plane just determined. By the distance formula, this distance is

$$d = \frac{|(0) - (1) - 3(-1) + 3|}{\sqrt{1^2 + (-1)^2 + (-3)^2}}$$
$$= \frac{5}{\sqrt{11}}$$
$$\doteq 1.51.$$

8. a.–b. We will do both of these parts at once.
The two given lines are
$\vec{r} = (1, -2, 5) + s(0, 1, -1), s \in \mathbf{R}$,
$\vec{r} = (1, -1, -2) + t(1, 0, -1), t \in \mathbf{R}$.
By converting to parametric form, a general point on the first line is
$U(1, s - 2, 5 - s)$,
and on the second line is
$V(1 + t, -1, -2 - t)$.
So the vector
$\overrightarrow{UV} = (t, 1 - s, s - t - 7)$.
If the points U and V are those that produce the minimal distance between these two lines, then $\overrightarrow{UV}$ will be perpendicular to both direction vectors, $(0, 1, -1)$ and $(1, 0, -1)$. In the first case, we get

$$0 = (t, 1 - s, s - t - 7) \cdot (0, 1, -1)$$
$$= 8 - 2s + t$$
$$t = 2s - 8$$

In the second case, we get

$$0 = (t, 1 - s, s - t - 7) \cdot (1, 0, -1)$$
$$= 2t - s + 7$$

Substituting $t = 2s - 8$ into this second equation, we get

$$2(2s - 8) - s + 7 = 0$$
$$s = 3$$
$$t = 2s - 8$$
$$t = -2$$

Substituting these values for s and t into U and V, we get
$U(1, 1, 2)$
$V(-1, -1, 0)$
So $U(1, 1, 2)$ is the point on the first line that produces the minimal distance to the second line at point $V(-1, -1, 0)$. This minimal distance is given by

$$|\overrightarrow{UV}| = |(-2, -2, -2)|$$
$$= \sqrt{12}$$
$$\doteq 3.46$$

Review Exercise, pp. 552–555

1. $2x - y = 31, x + 8y = -34, 3x + ky = 38$

$$(2x - y = 31) - 2(x + 8y = -34)$$
$$= 0x - 17y = 99$$
$$y = -\frac{99}{17}, x = \frac{214}{17}$$
$$3\left(\frac{214}{17}\right) + k\left(\frac{-99}{17}\right) = 38$$
$$k = -\frac{4}{99}$$

2. ① $x - y = 13$
② $3x + 2y = -6$
③ $x + 2y = -19$

(2 × Equation ①) + equation ② $= 5x + 0y = 20$ or $x = 4$. Substituting $x = 4$ into equation ① gives $(4) - y = 13$ or $y = -9$. However, when you substitute this coordinates into the third equation, the third equation is not consistent, so there is no solution to this problem.

3. a. ① $x - y + 2z = 3$
② $2x - 2y + 3z = 1$
③ $2x - 2y + z = 11$

Equation ② − equation ③ $= 5z = -10$ or $z = -2$. Substituting $z = -2$ into all of the equations gives
④ $x - y - 4 = 3$
⑤ $2x - 2y - 6 = 1$
⑥ $2x - 2y - 2 = 11$

There are no x and y variables that satisfy these equations, so the answer is no solution.

b. ① $x + y + z = 300$
② $x + y - z = 98$
③ $x - y + z = 100$

Equation ② + equation ③ $= 2x = 198$ $x = 99$.
Substituting $x = 99$ into all three equations gives:
④ $y + z = 201$
⑤ $y - z = -1$
⑥ $-y + z = 1$

Equation ④ + equation ⑤ $= 2y = 200$ or $y = 100$. You then get $z = 101$ after substituting both x and y into equation ①.
$(99, 100, 101)$
Check:
① $99 + 100 + 101 = 300$
② $99 + 100 - 101 = 98$
③ $99 - 100 + 101 = 100$

4. a. These four points will lie in the same plane if and only if the line determined by the first two points intersects the line determined by the last two points. The direction vector determined by the first two is

$$\vec{a} = (7, -5, 1) - (1, 2, 6) = (6, -7, -5)$$

So these first two points determine the line with vector equation
$\vec{r} = (1, 2, 6) + s(6,-7,-5), s \in \mathbf{R}$.
The direction vector determined by the last two points is

$$\vec{b} = (-3, 5, 6) - (1, 1, 4) = (-4, 4, 2)$$

So these first two points determine the line with vector equation
$\vec{r} = (1, 1, 4) + t(-4, 4, 2), t \in \mathbf{R}$.
Converting these two lines to parametric form, we obtain the equations
① $1 + 6s = 1 - 4t$
② $2 - 7s = 1 + 4t$
③ $6 - 5s = 4 + 2t$

Adding the first and second equations gives $3 - s = 2$, so $s = 1$. Substituting this into the third equation, we get

$$1 = 4 + 2t$$
$$-3 = 2t$$

So $t = -\frac{3}{2}$. We need to check this s and t for consistency. Substituting $s = 1$ into the vector equation for the first line gives

$$\vec{r} = (1, 2, 6) + (1)(6, -7, -5) = (7, -5, 1)$$

as a point on this line. Substituting $t = -\frac{3}{2}$ into the vector equation for the second line gives

$$\vec{r} = (1, 1, 4) + \left(-\frac{3}{2}\right)(-4, 4, 2) = (1, 1, 4) + (6, -6, -3) = (7, -5, 1)$$

as a point on this line. This means the two lines intersect, and so the four points given lie in the same plane.

b. Direction vectors for the plane containing the four points in part **a.** are $(6, -7, -5)$ and $(-4, 4, 2)$. So a normal to this plane is $(6, -7, -5) \times (-4, 4, 2) = (6, 8, -4)$.
We will use the parallel normal $(3, 4, -2)$. So the equation of this plane is of the form
$3x + 4y - 2z + D = 0$.
Substitute in the point $(1, 2, 6)$ to find D.

$$3(1) + 4(2) - 2(6) + D = 0$$
$$D = 1$$

The equation of the plane is
$3x + 4y - 2z + 1 = 0$.
So, using the distance formula, this plane is distance

$$d = \frac{|3(0) + 4(0) - 2(0) + 1|}{|(3, 4, -2)|} = \frac{1}{\sqrt{29}} \doteq 0.19$$

from the origin.

5. Use the distance formula.

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

a. The distance from $A(-1, 1, 2)$ to $3x - 4y - 12z - 8 = 0$

$$d = \frac{|3(-1) - 4(1) - 12(2) - 8|}{\sqrt{3^2 + (-4)^2 + (-12)^2}} = 3$$

b. The distance from $B(3, 1, -2)$ to $8x - 8y + 4z - 7 = 0$

$$d = \frac{|8(3) - 8(1) + 4(-2) - 7|}{\sqrt{8^2 + (-8)^2 + (4)^2}} = \frac{1}{12} \text{ or } 0.08$$

6. $\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$
$3x - 4y - 5z = 0$
Find the parametric equations from the first equation, then substitute those equations into the second equation. Solve for t. Substitute that t-value into the first equation.
$\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$
$x = 3 + 2t$

$y = 1 - t$
$z = 1 + 2t$
$3(3 + 2t) - 4(1 - t) - 5(1 + 2t) = 0$
t can be any value to satisfy this value, so the two equations intersect along
$\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$.

7. a. ① $3x - 4y + 5z = 9$
② $6x - 9y + 10z = 9$
③ $9x - 12y + 15z = 9$
$3 \times (3x - 4y + 5z = 9) = 9x - 12y + 15z = 27$
There is no solution because the first and third equations are inconsistent.

b. ① $2x + 3y + 4z = 3$
② $4x + 6y + 8z = 4$
③ $5x + y - z = 1$
$2 \times (2x + 3y + 4z = 3) = 4x + 6y + 8z = 6$
There is no solution because the first and second equations are inconsistent.

c. ① $4x - 3y + 2z = 2$
② $8x - 6y + 4z = 4$
③ $12x - 9y + 6z = 1$
$3 \times (4x - 3y + 2z = 2) = 12x - 9y + 6z = 6$
There is no solution because the first and third equations are inconsistent.

8. a. ① $3x + 4y + z = 4$
② $5x + 2y + 3z = 2$
③ $6x + 8y + 2z = 8$
(Equation ①) − (2 × equation ②)
$= -7x - 5z = 0$
Letting $z = t$, then $x = -\frac{5}{7}t$ and $y = 1 + \frac{2}{7}t$.
$x = -\frac{5}{7}t, y = 1 + \frac{2}{7}t, z = t, t \in \mathbf{R}$

b. ① $4x - 8y + 12z = 4$
② $2x + 4y + 6z = 4$
③ $x - 2y - 3z = 4$
(Equation ①) + (4 × equation ③)
$= 24z = -12$ or $z = -\frac{1}{2}$. Letting $z = -\frac{1}{2}$ creates:

④ $4x - 8y = 10$
⑤ $2x + 4y = 7$
(Equation ①) + (2 × equation ②) $= 8x = 24$ or $x = 3$. Substituting in $x = 3$ and $z = -\frac{1}{2}$ gives $y = \frac{1}{4}$
$x = 3, y = \frac{1}{4}, z = -\frac{1}{2}$

c. ① $x - 3y + 3z = 7$
② $2x - 6y + 6z = 14$
③ $-x + 3y - 3z = -7$
Letting $z = s$, then $y = t$ gives $x - 3t + 3s = 7$ or $x = -3s + 3t + 7$
$x = 3t - 3s + 7, y = t, z = s, s, t \in \mathbf{R}$

9. a. ① $3x - 5y + 2z = 4$
① $6x + 2y - z = 2$
① $6x - 3y + 8z = 6$
(Equation ②) − (2 × equation ①) $= 12y - 5z = -6$
Setting $z = t$,
$12y - 5t = -6$ or $y = -\frac{1}{2} + \frac{5}{12}t$
Substituting these two values into the first equation gives $x = \frac{1}{2} + \frac{1}{36}t$
$x = \frac{1}{2} + \frac{1}{36}t, y = \frac{-1}{2} + \frac{5}{12}t, z = t, t \in \mathbf{R}$

b. ① $2x - 5y + 3z = 1$
② $4x + 2y + 5z = 5$
③ $2x + 7y + 2z = 4$
(Equation ②) − (2 × equation ①)
$= 12y - z = 3$
Setting $z = t$,
$12y - t = 3$ or $y = \frac{1}{4} + \frac{1}{12}t$
Substituting these two values into the first equation gives $x = \frac{9}{8} - \frac{31}{24}t$
$x = \frac{9}{8} - \frac{31}{24}t, y = \frac{1}{4} + \frac{1}{12}t, z = t, t \in \mathbf{R}$

10. a. $2x + y + z = 6$
$x - y - z = -9$
$3x + y = 2$
The first equation + the second equation gives $3x = -3$ or $x = -1$. Substituting $x = -1$ into the third equation, $3(-1) + y = 2$ or $y = 5$.
Substituting these two values into the first equation, $2(-1) + 5 + z = 6$ or $z = 3$
These three planes meet at the point $(-1, 5, 3)$.

b. ① $2x - y + 2z = 2$
② $3x + y - z = 1$
③ $x - 3y + 5z = 4$
Equation ① + equation ② $= 5x + z = 3$
Equation ③ − (3 × equation ①) $= -5x - z = -2$.
These two equations are inconsistent, so the planes do not intersect at any point. Geometrically the planes form a triangular prism.

c. ① $2x + y - z = 0$
② $x - 2y + 3z = 0$
③ $9x + 2y - z = 0$

$2 \times$ equation ① + equation ② $= 5x + z = 0$, so $z = -5x$.
Equation ③ − equation ① $= 7x + y = 0$, so $y = -7x$.
Let $x = t$. The intersection of the planes is a line through the origin with equation $x = t$, $y = -7t$, $z = -5t$, $t \in \mathbf{R}$.

11. $\vec{r} = (2, -1, -2) + s(1, 1, -2), s \in \mathbf{R}$
By substituting in different s-values, you can find when the plane intersects the xz-plane when $y = 0$ and the xy-plane when $z = 0$.
The plane intersects the xz-plane at $(3, 0, -4)$ and the xy-plane at $(1, -2, 0)$. Then find the distance between these two points using the distance formula. The distance between these two points is 4.90.

12. a. $x - 2y + z + 4 = 0$
$\vec{r} = (3, 1, -5) + s(2, 1, 0), s \in \mathbf{R}$
$\vec{m} \cdot \vec{n} = (2, 1, 0) \cdot (1, -2, 1) = 0$ Since the line's direction vector is perpendicular to the normal of the plane and the point $(3, 1, -5)$ lies on both the line and the plane, the line is in the plane.
b. $\vec{r} = (7, 5, -1) + t(4, 3, 2), t \in \mathbf{R}$
$\vec{r} = (3, 1, -5) + s(2, 1, 0), s \in \mathbf{R}$
Solve for the parametric equations of both equations and then set them equal to each other.
$L_1: x = 7 + 4t, y = 5 + 3t, z = -1 + 2t$
$L_2: x = 3 + 2s, y = 1 + s, z = -5$
$z = -5 = -1 + 2t, t = -2$
$t = -2, x = -1, y = -1, z = -5$
$t = -2$ corresponds to the point $(-1, -1, -5)$
c. $x - 2y + z + 4 = 0$
$-1 - 2(-1) + (-5) + 4 = 0$
The point $(-1, -1, -5)$ is on the plane since it satisfies the equation of the plane.
d. $\vec{r} = (7, 5, -1) + t(4, 3, 2), t \in \mathbf{R}$
$(A, B, C) \cdot (4, 3, 2) = 0$
$A = 7, B = -2, C = -11$
$7x - 2y - 11z + D = 0$
$D = -50$
$7x - 2y - 11z - 50 = 0$

13. a. $\vec{r} = (3, 0, -1) + t(1, 1, 2), t \in \mathbf{R}$
$A(-2, 1, 1)$
$x = 3 + t, y = t, z = -1 + 2t$
$0 = 3 + t - x, 0 = t - y, 0 = -1 + 2t - z$
$\sqrt{(3 + t - x)^2 + (t - y)^2 + (-1 + 2t - z)^2}$
$\sqrt{(3 + t + 2)^2 + (t - 1)^2 + (-1 + 2t - 1)^2}$
$\sqrt{6t^2 + 30}$
$t = 0$ gives the lowest distance of 5.48
b. $t = 0$ corresponds to the point $(3, 0, -1)$

14. a. $\vec{r} = (1, -1, 1) + t(3, 2, 1), t \in \mathbf{R}$
$\vec{r} = (-2, -3, 0) + s(1, 2, 3), s \in \mathbf{R}$
Set the equations parametric equations equal to each other, and determine either the s or t-value. Find the point that corresponds to this value.
$L_1: x = 1 + 3t, y = -1 + 2t, z = 1 + t$
$L_2: x = -2 + s, y = -3 + 2s, z = 3s$
$x = 1 + 3t = -2 + s$
$y = -1 + 2t = -3 + 2s$
$z = 1 + t = 3s$
$s = 0, t = -1$
$s = 0$ corresponds to the point $(-2, -3, 0)$.
b. $\vec{r} = (1, -1, 1) + t(3, 2, 1), t \in \mathbf{R}$
$\vec{r} = (-2, -3, 0) + s(1, 2, 3), s \in \mathbf{R}$
$P(-2, -3, 0)$
$\vec{n}_1 \times \vec{n}_2 = (3, 2, 1) \times (1, 2, 3)$
$= (4, -8, 4) = (1, -2, 1)$
$\vec{r} = (-2, -3, 0) + t(1, -2, 1), t \in \mathbf{R}$

15. a. Since the plane we want contains L, we can use the direction vector for L, $(1, 2, -1)$, as one of the plane's direction vectors. Since the plane contains the point $(1, 2, -3)$ (which is on L) and the point $K(3, -2, 4)$, it will contain the direction vector $(3, -2, 4) - (1, 2, -3) = (2, -4, 7)$
To find a normal vector for the plane we want, take the cross product of these two direction vectors.
$(2, -4, 7) \times (1, 2, -1) = (-10, 9, 8)$
So the plane we seek will be of the form $-10x + 9y + 8z + D = 0$.
To determine the value of D, substitute in the point $(1, 2, -3)$ that is to be on this plane.
$-10(1) + 9(2) + 8(-3) + D = 0$
$D = 16$
The equation of the plane we seek is $-10x + 9y + 8z + 16 = 0$.
b. Using the distance formula, the distance from $S(1, 1, -1)$ to the plane $-10x + 9y + 8z + 16 = 0$ is

$$d = \frac{|-10(1) + 9(1) + 8(-1) + 16|}{|(-10, 9, 8)|}$$
$$= \frac{7}{\sqrt{245}}$$
$$\doteq 0.45$$

16. a. ① $x + y - z = 1$
② $2x - 5y + z = -1$
③ $7x - 7y - z = k$
Equation ① + equation ② = equation ④ $= 3x - 4y = 0$

Equation ② + equation ③ = equation ⑤
$= 9x - 12y = -1 + k$
For the solution to this system to be a line, equation ④ and equation ⑤ must be the proportional. $k = 1$ makes these two line proportional and the solution to this system a line.
b. In part a., we found that $k = 1$ by arriving at the equivalent system

① $3x - 4y = 0$
② $2x - 5y + z = -1$
③ $9x - 12y = 0$

As the first and third equations are proportional, this is really the same system as

① $3x - 4y = 0$
② $2x - 5y + z = -1$

Letting $x = t$ in the first equation, we see that $y = \frac{3}{4}t$. Substituting these values for x and y into the second equation, we find that

$$z = 5\left(\frac{3}{4}t\right) - 2t - 1$$
$$= \frac{7}{4}t - 1.$$

So the direction vector for the line that solves this system is $(1, \frac{3}{4}, \frac{7}{4})$, which is parallel to $(4, 3, 7)$.
So equivalent parametric equations of this line are
$x = 4t$
$y = 3t$
$z = -1 + 7t, t \in \mathbf{R}$.
So one possible vector equation of this line is
$\vec{r} = (0, 0, -1) + t(4, 3, 7), t \in \mathbf{R}$.
b. In part **a.**, we found that $k = 1$ by arriving at the equivalent system

① $3x - 4y = 0$
② $2x - 5y + z = -1$
③ $9x - 12y = 0$

As the first and third equations are proportional, this is really the same system as

① $3x - 4y = 0$
② $2x - 5y + z = -1$

Letting $x = t$ in the first equation, we see that $y = \frac{3}{4}t$. Substituting these values for x and y into the second equation, we find that

$$z = 5\left(\frac{3}{4}t\right) - 2t - 1$$
$$= \frac{7}{4}t - 1.$$

So the direction vector for the line that solves this system is $\left(1, \frac{3}{4}, \frac{7}{4}\right)$, which is parallel to (4, 3, 7).
So equivalent parametric equations of this line are
$x = 4t$
$y = 3t$
$z = -1 + 7t, t \in \mathbf{R}$.
So one possible vector equation of this line is
$\vec{r} = (0, 0, -1) + t(4, 3, 7), t \in \mathbf{R}$.

17. a.
① $x + 2y + z = 1$
② $2x - 3y - z = 6$
③ $3x + 5y + 4z = 5$
④ $4x + y + z = 8$

Equation ① + equation ② = equation ⑤
$= 3x - y = 7$
(4 × equation ②) + equation ③ = equation ⑥
$= 11x - 7y = 29$
(7 × equation ⑤) + equation ⑥
= equation ⑦ $= -10x = -20y$ or $x = 2$
Substituting into equation ⑤: $6 - y = 7y = -1$.
Substituting into equation ①: $2 + -2 + z = 1$ or $z = 1$.
$(2, -1, 1)$

b.
① $x - 2y + z = 1$
② $2x - 5y + z = -1$
③ $3x - 7y + 2z = 0$
④ $6x - 14y + 4z = 0$

Equation ② − (2 × equation ①)
= equation ⑤ $= -y - z = -3$,
Setting $z = t$,
$-y - t = -3$ or $y = 3 - t$
Substituting $y = 3 - t$ and $z = t$ into equation ①:
$x - 2(3 - t) + t = 1$ or $x = 7 - 3t$
$x = 7 - 3t, y = 3 - t, z = t, t \in \mathbf{R}$

18. ① $\frac{9a}{b} - 8b + \frac{3c}{b} = 4$

② $-\frac{3a}{b} + 4b + \frac{4c}{b} = 3$

③ $\frac{3a}{b} + 4b - \frac{4c}{b} = 3$

$x = \frac{a}{b}, y = b, z = \frac{c}{b}$
① $9x - 8y + 3z = 4$
② $-3x + 4y + 4z = 3$
③ $3x + 4y - 4z = 3$
③ + ② $= 8y = 6$
$y = \frac{3}{4}$
① $9x + 3z = 10$
② $-3x + 4z = 0$
③ $3x - 4z = 0$
① + 3② $= 15z = 10$

$z = \frac{2}{3}, x = \frac{8}{9}$

$y = \frac{3}{4} = b,$

$x = \frac{8}{9} = \frac{a}{b} = \frac{a}{\frac{3}{4}}, a = \frac{2}{3}$

$z = \frac{2}{3} = \frac{c}{b} = \frac{c}{\frac{3}{4}}, c = \frac{1}{2}$

$\left(\frac{2}{3}, \frac{3}{4}, \frac{1}{2}\right)$

19. First put the equation into parametric form. Then substitute the *x, y,* and *z*-values into $x + 2y - z + 10 = 0$ to determine t. Then substitute t back into the parametric equations to determine the coordinates.

$\frac{x+1}{-4} = \frac{y-2}{3} = \frac{z-1}{-2} = t$

$x = -4t - 1, y = 3t + 2, z = -2t + 1$

$x + 2y - 3z + 10 = 0$

$(-4t - 1) + 2(3t + 2) - 3(-2t + 1) + 10 = 0$

$t = -\frac{5}{4}$

$x = -4\left(-\frac{5}{4}\right) - 1, y = 3\left(-\frac{5}{4}\right) + 2,$

$z = -2\left(-\frac{5}{4}\right) + 1$

$\left(4, -\frac{7}{4}, \frac{7}{2}\right)$

20. Let $A'(a, b, c)$ denote the image point under this reflection. We want to find a, b, and c. The equation of the plane is $x - y + z - 1 = 0$, so letting $y = s$ and $z = t$, we get $x = 1 - t + s, s, t \in \mathbf{R}$. These are the parametric equations of this plane, so a general point on this plane has coordinates $P(1 - t + s, s, t)$.

So $\overrightarrow{PA} = (1, 0, 4) - (1 - t + s, s, t)$
$= (t - s, -s, 4 - t)$

The normal vector to this plane is $(1, -1, 1)$, and in order for $\overrightarrow{PA}$ to be perpendicular to the plane, it must be parallel to this normal. This means that $\overrightarrow{PA}$ and $(1, -1, 1)$ will have a cross product equal to the zero vector.

$(t - s, -s, 4 - t) \times (1, -1, 1)$
$= (4 - s - t, 4 + s - 2t, 2s - t)$
$= (0, 0, 0)$

So we get the system of equations

① $4 - s - t = 0$
② $4 + s - 2t = 0$
③ $2s - t = 0$

Adding the first two equations gives

$8 - 3t = 0$

$t = \frac{8}{3}$

Substituting this value for t into the third equation gives

$0 = 2s - t$
$= 2s - \frac{8}{3}$

$s = \frac{4}{3}$

Substituting these values for s and t into the equation for $\overrightarrow{PA}$, we get

$\overrightarrow{PA} = (t - s, -s, 4 - t)$
$= \left(\frac{8}{3} - \frac{4}{3}, -\frac{4}{3}, 4 - \frac{8}{3}\right)$
$= \left(\frac{4}{3}, -\frac{4}{3}, \frac{4}{3}\right)$

This is the vector that is normal to the plane, with its head at point $A(1, 0, 4)$ and tail at the point in the plane

$P(1 - t + s, s, t) = P\left(1 - \frac{8}{3} + \frac{4}{3}, \frac{4}{3}, \frac{8}{3}\right)$
$= \left(-\frac{1}{3}, \frac{4}{3}, \frac{8}{3}\right)$

So the vector

$\overrightarrow{PA'} = -\overrightarrow{PA}$
$= \left(-\frac{4}{3}, \frac{4}{3}, -\frac{4}{3}\right)$
$= (a, b, c) - \left(-\frac{1}{3}, \frac{4}{3}, \frac{8}{3}\right)$
$= \left(a + \frac{1}{3}, b - \frac{4}{3}, c - \frac{8}{3}\right)$

This means that $a = -\frac{5}{3}$, $b = -\frac{8}{3}$, and $c = -\frac{4}{3}$, That is, the reflected point is $A'\left(-\frac{5}{3}, \frac{8}{3}, \frac{4}{3}\right)$.

21. a. The first plane has normal $(3, 1, 7)$ and the second has normal $(4, -12, 4)$. Their line of intersection will be perpendicular to both of these normals. So we can take as direction vector the cross product of these two normals.

$(3, 1, 7) \times (4, -12, 4) = (88, 16, -40)$
$= 8(11, 2, -5)$

So let's use $(11, 2, -5)$ as the direction vector for this line of intersection. To find a point on both of these planes, solve for z in the second plane, and substitute this into the equation for the first plane.

$$4x - 12y + 4z - 24 = 0$$
$$\begin{aligned}4z &= 24 - 4x + 12y\\ z &= 6 - x + 3y\\ 0 &= 3x + y + 7z + 3\\ &= 3x + y + 7(6 - x + 3y)\\ &\quad + 3\\ &= -4x + 22y + 45\end{aligned}$$

If $y = 0$ in this last equation, then $x = \frac{45}{4}$ and

$$\begin{aligned}z &= 6 - x + 3y\\ &= 6 - \frac{45}{4} + 3(0)\\ &= -\frac{21}{4}\end{aligned}$$

The point $\left(\frac{45}{4}, 0, \frac{21}{4}\right)$, lies on both planes. So the vector equation of the line of intersection for the first two planes is

$$\vec{r} = \left(\frac{45}{4}, 0, -\frac{21}{4}\right) + t(11, 2, -5), t \in \mathbf{R}.$$

The corresponding parametric form is

$$x = \frac{45}{4} + 11t$$
$$y = 2t$$
$$z = -\frac{21}{4} - 5t, t \in \mathbf{R}.$$

We will use a similar procedure for the other two lines of intersection. For the third plane, the normal vector is (1, 2, 3). So a direction vector for the line of intersection between the first and third planes is

$$\begin{aligned}(3, 1, 7) \times (1, 2, 3) &= (-11, -2, 5)\\ &= -(11, 2, -5)\end{aligned}$$

We may use $(11, 2, -5)$ as the direction vector for this line of intersection. We find a point on both of these planes in the same way as before.

$$x + 2y + 3z - 4 = 0$$
$$x = 4 - 2y - 3z$$
$$\begin{aligned}0 &= 3x + y + 7z + 3\\ &= 3(4 - 2y - 3z) + y + 7z + 3\\ &= -6y - 2z + 15\end{aligned}$$

Taking $y = 0$ in this last equation, we get $z = \frac{15}{2}$ and

$$\begin{aligned}x &= 4 - 2y - 3z\\ &= 4 - 2(0) - 3\left(\frac{15}{2}\right)\\ &= -\frac{37}{2}\end{aligned}$$

A point on both the first and third planes is $\left(-\frac{37}{2}, 0, \frac{15}{2}\right)$. So the vector equation for this line of intersection is

$$\vec{r} = \left(-\frac{37}{2}, 0, \frac{15}{2}\right) + t(11, 2, -5), t \in \mathbf{R},$$

and the corresponding parametric equations are

$$x = -\frac{37}{2} + 11t$$
$$y = 2t$$
$$z = \frac{15}{2} - 5t, t \in \mathbf{R}.$$

Finally, we consider the line of intersection between the second and third planes. In this case, a direction vector is

$$\begin{aligned}(4, -12, 4) \times (1, 2, 3) &= (-44, -8, 20)\\ &= -4(11, 2, -5)\end{aligned}$$

We may use $(11, 2, -5)$ as the direction vector for this line of intersection. We find a point on both of these planes in the same way as before.

$$x + 2y + 3z - 4 = 0$$
$$x = 4 - 2y - 3z$$
$$\begin{aligned}0 &= 4x - 12y + 4z - 24\\ &= 4(4 - 2y - 3z) - 12y + 4z - 24\\ &= -20y - 8z - 8\end{aligned}$$

Taking $y = 0$ in this last equation, we get $z = -1$ and

$$\begin{aligned}x &= 4 - 2y - 3z\\ &= 4 - 2(0) - 3(-1)\\ &= 7\end{aligned}$$

A point on both the second and third planes is $(7, 0, -1)$. So the vector equation for this line of intersection is

$$\vec{r} = (7, 0, -1) + t(11, 2, -5), t \in \mathbf{R},$$

and the corresponding parametric equations are

$$x = 7 + 11t$$
$$y = 2t$$
$$z = -1 - 5t, \in \mathbf{R}.$$

b. All three lines of intersection found in part a. have direction vector $(11, 2, -5)$, and so they are all parallel. Since no pair of normal vectors for these three planes is parallel, no pair of these planes is coincident.

22. ① $\frac{2}{a^2} + \frac{5}{b^2} + \frac{3}{c^2} = 40$

② $\frac{3}{a^2} - \frac{6}{b^2} - \frac{1}{c^2} = -3$

③ $\frac{9}{a^2} - \frac{5}{b^2} + \frac{4}{c^2} = 67$

$① + 3② = ④ = \dfrac{11}{a^2} + \dfrac{-13}{b^2} = 31$

$③ + 4② = ⑤ = \dfrac{21}{a^2} + \dfrac{-29}{b^2} = 55$

$21④ - 11⑤ = \dfrac{46}{b^2} = 46, b = +1, b = -1$

$\dfrac{21}{a^2} + \dfrac{-29}{1} = 55, a = \dfrac{1}{2}, a = -\dfrac{1}{2}$

$\dfrac{2}{0.25} + \dfrac{5}{1} + \dfrac{3}{c^2} = 40, c = \dfrac{1}{3}, c = -\dfrac{1}{3}$

$a = \dfrac{1}{2}, a = -\dfrac{1}{2}, b = 1, b = -1, c = \dfrac{1}{3}, c = -\dfrac{1}{3}$

Because each equation has each of a^2, b^2, and c^2, the possible solutions are all combinations of the positive and negative values for *a, b*, and *c*: $\left(\frac{1}{2}, 1, \frac{1}{3}\right)$, $\left(\frac{1}{2}, 1, -\frac{1}{3}\right)$, $\left(\frac{1}{2}, -1, \frac{1}{3}\right)$, $\left(\frac{1}{2}, -1, -\frac{1}{3}\right)$, $\left(-\frac{1}{2}, 1, \frac{1}{3}\right)$, $\left(-\frac{1}{2}, 1, -\frac{1}{3}\right)$, $\left(-\frac{1}{2}, -1, \frac{1}{3}\right)$, and $\left(\frac{1}{2}, -1, -\frac{1}{3}\right)$.

23. The general form of such a parabola is $y = ax^2 + bx + c$. We need to determine *a*, *b*, and *c*. Since $(-1, 2)$, $(1, -1)$, and $(2, 1)$ all lie on the parabola, we get the system of equations

$$\begin{aligned} ① &\quad a - b + c = 2 \\ ② &\quad a + b + c = -1 \\ ③ &\quad 4a + 2b + c = 1 \end{aligned}$$

Adding the first and second equations gives

$$a + c = \frac{1}{2}$$

Subtracting the first from the second equation gives

$$2b = -3$$
$$b = -\frac{3}{2}$$

Using the fact that $a + c = \frac{1}{2}$ and $b = -\frac{3}{2}$ in the third equation gives

$$\begin{aligned} 1 &= 4a + 2b + c \\ &= 3a + 2b + (a + c) \\ &= 3a + 2\left(-\frac{3}{2}\right) + \frac{1}{2} \\ &= 3a - \frac{5}{2} \end{aligned}$$

$$\frac{7}{2} = 3a$$

$$a = \frac{7}{6}$$

So using once more that $a + c = \frac{1}{2}$, we substitute this value in for *a* and get

$$\begin{aligned} \frac{1}{2} &= a + c \\ &= \frac{7}{6} + c \end{aligned}$$

$$c = -\frac{2}{3}$$

So the equation of the parabola we seek is

$$y = \frac{7}{6}x^2 - \frac{3}{2}x - \frac{2}{3}.$$

24. The equation of the plane is $4x - 5y + z - 9 = 0$, which has normal $(4, -5, 1)$. Converting this plane to parametric form gives

$$x = s$$
$$y = t$$
$$z = 9 - 4s + 5t, s, t \in \mathbf{R}.$$

So for any point $Y(s, t, 9 - 4s + 5t)$ on this plane, we can form the vector

$$\begin{aligned} \overrightarrow{XY} &= (s, t, 9 - 4s + 5t) - (3, 2, -5) \\ &= (s - 3, t - 2, 14 - 4s + 5t) \end{aligned}$$

This vector is perpendicular to the plane when it is parallel to the normal vector $(4, -5, 1)$. Two vectors are parallel precisely when their cross product is the zero vector.

$$\begin{aligned} &(s - 3, t - 2, 14 - 4s + 5t) \times (4, -5, 1) \\ &= (68 + 26t - 20s, 59 + 20t - 17s, 23 - 4t - 5s) \\ &= (0, 0, 0) \end{aligned}$$

So we get the system of equations

$$\begin{aligned} ① &\quad 68 + 26t - 20s = 0 \\ ② &\quad 59 + 20t - 17s = 0 \\ ③ &\quad 23 - 4t - 5s = 0 \end{aligned}$$

Subtracting four times the third equation from the first equation gives

$$42t - 24 = 0$$
$$t = \frac{4}{7}$$

Substituting this value for *t* into the second equation gives

$$\begin{aligned} 0 &= 59 + 20t - 17s \\ &= 59 + 20\left(\frac{4}{7}\right) - 17s \end{aligned}$$

$$17s = \frac{493}{7}$$

$$s = \frac{29}{7}$$

Substituting these values for *s* and *t* into the equation for *Y* gives

$$Y(s, t, 9 - 4s + 5t) = Y\left(\frac{29}{7}, \frac{4}{7}, 9 - 4\left(\frac{29}{7}\right) + 5\left(\frac{4}{7}\right)\right) = \left(\frac{29}{7}, \frac{4}{7}, -\frac{33}{7}\right)$$

So the point *M* we wanted is $M\left(\frac{29}{7}, \frac{4}{7}, -\frac{33}{7}\right)$.

25. $\dfrac{11x^2 - 14x + 9}{(3x - 1)(x^2 + 1)} = \dfrac{A}{3x - 1} + \dfrac{Bx + C}{x^2 + 1}$

$\dfrac{11x^2 - 14x + 9}{(3x - 1)(x^2 + 1)}$

$= \dfrac{A(x^2 + 1) + (Bx + C)\,3x - 1}{(3x - 1)(x^2 + 1)}$

$11x^2 - 14x + 9 = (A + 3B)x^2 + (3C - B)x + (A - C)$

$A - C = 9, 3C - B = -14, A + 3B = 11$

$B = 3C + 14, A = C + 9$

$A + 3(3C + 14) = 11, A + 9C = -31$

$(C + 9) + 9C = -31$

$10C = -40, C = -4$

$B = 3(-4) + 14 = 2, A = (-4) + 9 = 5$

$A = 5, B = 2, C = -4$

26. a. The vector

$\overrightarrow{EF} = (-1, -4, -6) - (4, 0, 3)$

$= (-5, -4, -3)$

This is a direction vector for the line containing the segment EF. The point $E(-1, -4, -6)$ is on this line, so the vector equation of this line is

$\vec{r} = (-1, -4, -6) + t(-5, -4, -3), t \in \mathbf{R}$.

b. Based on the equation of the line found in part a., a general point on this line is of the form

$J(-1 - 5t, -4 - 4t, -6 - 3t), t \in \mathbf{R}$.

For this general point, the vector

$\overrightarrow{JD} = (3, 0, 7) - (-1 - 5t, -4 - 4t, -6 - 3t)$

$= (4 + 5t, 4 + 4t, 13 + 3t)$

This vector will be perpendicular to the direction vector for the line found in part a. at the point J we seek. This means that

$0 = (4 + 5t, 4 + 4t, 13 + 3t) \cdot (-5, -4, -3)$

$= -5(4 + 5t) - 4(4 + 4t) - 3(13 + 3t)$

$= -75 - 50t$

$t = -\dfrac{3}{2}$

Substituting this value of t into the equation for the general point on the line in part a.,

$J(-1 - 5t, -4 - 4t, -6 - 3t)$

$= J\left(-1 - 5\left(-\dfrac{3}{2}\right), -4 - 4\left(-\dfrac{3}{2}\right), -6 - 3\left(-\dfrac{3}{2}\right)\right)$

$= \left(\dfrac{13}{2}, 2, -\dfrac{3}{2}\right)$

These are the coordinates for the point J we wanted.

c. Using the coordinates for J found in part b.,

$\overrightarrow{JD} = (3, 0, 7) - \left(\dfrac{13}{2}, 2, -\dfrac{3}{2}\right)$

$= \left(-\dfrac{7}{2}, -2, \dfrac{17}{2}\right)$

This vector forms the height of $\triangle DEF$, and the length of this vector is

$|\overrightarrow{JD}| = \left|\left(-\dfrac{7}{2}, -2, \dfrac{17}{2}\right)\right|$

$= \sqrt{\left(-\dfrac{7}{2}\right)^2 + (-2)^2 + \left(\dfrac{17}{2}\right)^2}$

$= \sqrt{\dfrac{177}{72}}$

$\doteq 9.41$

The length of the base of $\triangle DEF$ is

$|\overrightarrow{EF}| = |(-5, -4, -3)|$

$= \sqrt{(-5)^2 + (-4)^2 + (-3)^2}$

$= \sqrt{50}$

$\doteq 7.07$

So the area of $\triangle DEF$ equals

$\dfrac{1}{2}(\sqrt{50})\left(\sqrt{\dfrac{177}{2}}\right) = \dfrac{5}{2}\sqrt{177}$

$\doteq 33.26 \text{ units}^2$

27. $3x - 2z + 1 = 0$

$4x + 3y + 7 = 0$

$(5, -5, 5)$

$\overrightarrow{n_1} \times \overrightarrow{n_2} = (3, 0, -2) \times (4, 3, 0) = (6, -8, 9)$

$6x - 8y + 9z + D = 0$

$D = -115$

$6x - 8y + 9z - 115 = 0$

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1. a. $\vec{r}_1 = (4, 2, 6) + s(1, 3, 11), s \in \mathbf{R}$,

$\vec{r}_2 = (5, -1, 4) + t(2, 0, 9), t \in \mathbf{R}$

L_1: $x = 4 + s, y = 2 + 3s, z = 6 + 11s$

L_2: $x = 5 + 2t, y = -1, z = 4 + 9t$

$y = -1 = 2 + 3s$

$s = -1$

L_1: $x = 4 + (-1), y = 2 + 3(-1),$

$z = 6 + 11(-1)$

$x = 3, y = -1, z = -5$

$(3, -1, -5)$

b. $x - y + z + 1 = 0$

$3 - (-1) + (-5) + 1 = 0$

$3 + 1 - 5 + 1 = 0$

$0 = 0$

2. Use the distance equation.

$d = \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

a. $A(3, 2, 3)$

$8x - 8y + 4z - 7 = 0$

$$d = \frac{|8x_0 - 8y_0 + 4z_0 - 7|}{\sqrt{(8)^2 + (-8)^2 + (4)^2}}$$
$$= \frac{|8(3) - 8(2) + 4(3) - 7|}{\sqrt{(8)^2 + (-8)^2 + (4)^2}}$$
$$= \frac{13}{12} \text{ or } 1.08$$

b. First, find any point on one of the planes, then use the other plane equation with the distance formula.

$$2x - y + 2z - 16 = 0$$
$$2x - y + 2z + 24 = 0$$
$$2(8) - (0) + 2(0) - 16 = 0$$
$$A(8, 0, 0)$$
$$d = \frac{|2x_0 - 1y_0 + 2z_0 + 24|}{\sqrt{(2)^2 + (-1)^2 + (2)^2}}$$
$$= \frac{|2(8) - 1(0) + 2(0) + 24|}{\sqrt{(2)^2 + (-1)^2 + (2)^2}}$$
$$= \frac{40}{3} \text{ or } 13.33$$

3. a. L_1: $2x + 3y - z = 3$

L_2: $-x + y + z = 1$

$L_1 + 2L_2$: $5y + z = 5$

$$z = t,$$
$$5y + (t) = 5$$
$$y = 1 - \frac{t}{5}$$
$$-x + y + z = 1$$
$$-x + \left(1 - \frac{t}{5}\right) + (t) = 1$$
$$x = \frac{4t}{5}$$
$$x = \frac{4t}{5}, y = 1 - \frac{t}{5}, z = t, t \in \mathbf{R}$$

b. To determine the point of intersection with the xz-plane, set the above y parametric equation equal to 0 and solve for the t. This t corresponds to the point of intersection.

$$x = \frac{4t}{5}, y = 1 - \frac{t}{5}, z = t, t \in \mathbf{R}$$
$$0 = 1 - \frac{t}{5}$$
$$t = 5$$
$$x = \frac{4(5)}{5}, y = 1 - \frac{(5)}{5}, z = (5), t \in \mathbf{R}$$
$$(4, 0, 5)$$

4. a. ① $x - y + z = 10$

② $2x + 3y - 2z = -21$

③ $\frac{1}{2}x + \frac{2}{5}y + \frac{1}{4}z = -\frac{1}{2}$

Equation ② + (2 × equation ①) =

$4x + y = -1$

Equation ② + (8 × equation ③) =

$$6x + \frac{31}{5}y = -25$$
$$-\frac{31}{5}(4x + y = -1)$$
$$+\left(6x + \frac{31}{5}y = -25\right)$$
$$-18.8x = -18.8$$
$$x = 1$$
$$4(1) + y = -1$$
$$y = -5$$
$$(1) - (-5) + z = 10$$
$$z = 4$$
$$(1, -5, 4)$$

b. The three planes intersect at this point.

5. a. ① $x - y + z = -1$

② $2x + 2y - z = 0$

③ $x - 5y + 4z = -3$

Equation ② + (2 × equation ①) =

$4x + z = -2$

$$4x + z = -2$$
$$z = t$$
$$4x + (t) = -2$$
$$x = -\frac{1}{2} - \frac{t}{4}$$
$$x - y + z = -1$$
$$\left(-\frac{1}{2} - \frac{t}{4}\right) - y + (t) = -1$$
$$y = \frac{3t}{4} + \frac{1}{2}$$
$$x = -\frac{1}{2} - \frac{t}{4}, y = \frac{3t}{4} + \frac{1}{2}, z = t, t \in \mathbf{R}$$

b. The three planes intersect at this line.

6. a. L_1: $x + y + z = 0$

L_2: $x + 2y + 2z = 1$

L_3: $2x - y + mz = n$

$L_2 + 2L_3$: $5x + 0y + (2m + 2)z = 2n + 1$

$L_1 + L_3$: $3x + 0y + (m + 1)z = n$

$$\frac{5}{3}(3x + 0y + (m + 1)z = n)$$
$$= 5x + 0y + \frac{5}{3}(m + 1)z = \frac{5}{3}n$$

Then set the two new equations to each other and solve for a m and n value that would give equivalent equations.

$$5x + 0y + \frac{5}{3}(m + 1)z = \frac{5}{3}n$$
$$5x + 0y + (2m + 2)z = 2n + 1$$
$$2m + 2 = \frac{5}{3}(m + 1)$$
$$m = -1$$
$$\frac{5}{3}n = 2n + 1$$
$$n = -3$$

b. L_1: $x + y + z = 0$
L_2: $x + 2y + 2z = 1$
L_3: $2x - y - z = -3$
$L_1 + L_2$: $3x = -3, x = -1$
$(-1) + y + z = 0$
$z = t$
$(-1) + y + (t) = 0$
$y = 1 - t$
$x = -1, y = 1 - t, z = t, t \in \mathbf{R}$

7. First find the parametric equations of each line. Then set these equations equal to each other to find a set of new equations. Use the dot product to determine another set of equations that you will solve for t and s. Find the corresponding points to these values and the distance between them, which is the distance between the two lines.

L_1: $\vec{r} = (-1, -3, 0) + s(1, 1, 1), s \in \mathbf{R}$
L_2: $\vec{r} = (-5, 5, -8) + t(1, 2, 5), t \in \mathbf{R}$
L_1: $x = -1 + s, y = -3 + s, z = s$
L_2: $x = -5 + t, y = 5 + 2t, z = -8 + 5t$
$$\overrightarrow{UV} = [(-1 + s) - (-5 + t), (-3 + s) - (5 + 2t), s - (-8 + 5t)]$$
$$\overrightarrow{UV} = (4 + s - t, -8 + s - 2t, s + 8 - 5t)$$
$$m_1 \cdot \overrightarrow{UV} = 0$$
$$(1, 1, 1) \cdot (4 + s - t, -8 + s - 2t, s + 8 - 5t) = 0$$
$$(1, 2, 5) \cdot (4 + s - t, -8 + s - 2t, s + 8 - 5t) = 0$$
L_4: $4 + 3s - 8t = 0$,
L_5: $28 + 8s - 30t = 0$
$8 \times L_1 + (-3) \times L_2$ yields
$32 + 24s - 64t - 84 - 24s + 90t = 0$, so $t = 2$.
Then $s = 4$. The points corresponding to these values of s and t are $(-1, 3, 0) + 4(1, 1, 1)$ $= (3, 1, 4)$ and $(-5, 5, -8) + 2(1, 2, 5)$ $= (-3, 9, 2)$.
$$d = \sqrt{(3 - (-3))^2 + (1 - 9)^2 + (4 - 2)^2}$$
$$= \sqrt{(6)^2 + (-8)^2 + (2)^2}$$
$$= \sqrt{36 + 64 + 4}$$
$$= \sqrt{104} \text{ or } 10.20$$

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1. a. The angle, θ, between the two vectors is found from the equation $\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.
$$\vec{a} \cdot \vec{b} = (2, -1, -2) \cdot (3, -4, 12)$$
$$= 2(3) - 1(-4) - 2(12)$$
$$= -14$$
$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2}$$
$$= 3$$
$$|\vec{b}| = \sqrt{3^2 + (-4)^2 + 12^2}$$
$$= 13$$
So $\theta = \cos^{-1}\left(\frac{-14}{3 \times 13}\right)$
$\doteq 111.0°$

b. The scalar projection of $\vec{a}$ on $\vec{b}$ is equal to $|\vec{a}|\cos(\theta)$, where θ is the angle between the two vectors. So from the above work, $\cos(\theta) = \frac{-14}{3 \times 13}$ and $|\vec{a}| = 3$, so the scalar projection of $\vec{a}$ on $\vec{b}$ is $\frac{-14}{3 \times 13} \times 3 = -\frac{14}{13}$. The vector projection of $\vec{a}$ on $\vec{b}$ is equal to the scalar projection multiplied by the unit vector in the direction of $\vec{b}$. So the vector projection is $-\frac{14}{13} \times \frac{1}{13}(3, -4, 12) = \left(-\frac{52}{169}, \frac{56}{169}, -\frac{168}{169}\right)$.

c. The scalar projection of $\vec{b}$ on $\vec{a}$ is equal to $|\vec{b}|\cos(\theta)$, where θ is the angle between the two vectors. So from the above work, $\cos(\theta) = \frac{-14}{3 \times 13}$ and $|\vec{b}| = 13$, so the scalar projection of $\vec{a}$ on $\vec{b}$ is $\frac{-14}{3 \times 13} \times 13 = -\frac{14}{3}$. The vector projection of $\vec{b}$ on $\vec{a}$ is equal to the scalar projection multiplied by the unit vector in the direction of $\vec{a}$. So the vector projection is $-\frac{14}{3} \times \frac{1}{3}(2, -1, -2) = \left(-\frac{28}{9}, \frac{14}{9}, \frac{28}{9}\right)$.

2. a. Since the normal of the first plane is $(4, 2, 6)$ and the normal of the second is $(1, -1, 1)$, which are not scalar multiples of each other, there is a line of intersection between the planes.
The next step is to use the first and second equations to find an equation with a zero for the coefficient of x. The first equation minus four times the second equation yields $0x + 6y + 2z + 6 = 0$. We may divide by two to simplify, so $3y + z + 3 = 0$. If we let $y = t$, then $3t + z + 3 = 0$, or $z = -3 - 3t$. Substituting these into the second equation yields $x - (t) + (-3 - 3t) - 5 = 0$ or $x = 8 + 4t$.
So the equation of the line in parametric form is $x = 8 + 4t, y = t, z = -3 - 3t, t \in \mathbf{R}$.

To check that this is correct, we substitute in the solution to both initial equations

$$4x + 2y + 6z - 14 = 4(8 + 4t) + 2(t) + 6(-3 - 3t) - 14 = 0$$

and $x - y + z - 5$
$= (8 + 4t) - (t) + (-3 - 3t) - 5$
$= 0.$

Hence the line given by the parametric equation above is the line of intersection for the planes.

b. The angle between two planes is the same as the angle between their corresponding normal vectors.

$$|(4, 2, 6)| = \sqrt{4^2 + 2^2 + 6^2} = \sqrt{56}$$

$$|(1, -1, 1)| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$(4, 2, 6) \cdot (1, -1, 1) = 8$, so the angle between the planes is $\cos^{-1}\left(\frac{8}{\sqrt{3}\sqrt{56}}\right) \doteq 51.9°$.

3. a. We have that $\cos(60°) = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}$. Also since $\vec{x}$ and $\vec{y}$ are unit vectors, $|\vec{x}| = |\vec{y}| = 1$, and moreover $\cos(60°) = \frac{1}{2}$. So $\vec{x} \cdot \vec{y} = \frac{\vec{x} \cdot \vec{y}}{1 \times 1} = \frac{1}{2}$.

b. Scalar multiples can be brought out to the front of dot products. Hence $2\vec{x} \cdot 3\vec{y} = (2)(3)(\vec{x} \cdot \vec{y})$, and so by part **a.**, $2\vec{x} \cdot 3\vec{y} = 2 \times 3 \times \frac{1}{2} = 3$.

c. The dot product is distributive,
so $(2\vec{x} - \vec{y}) \cdot (\vec{x} + 3\vec{y})$
$= 2\vec{x} \cdot (\vec{x} + 3\vec{y}) - \vec{y} \cdot (\vec{x} + 3\vec{y})$
$= 2\vec{x} \cdot \vec{x} + 2\vec{x} \cdot 3\vec{y} - \vec{y} \cdot \vec{x} - \vec{y} \cdot 3\vec{y}$
$= 2\vec{x} \cdot \vec{x} + 2\vec{x} \cdot 3\vec{y} - \vec{x} \cdot \vec{y} - 3\vec{y} \cdot \vec{y}$

Since $\vec{x}$ and $\vec{y}$ are unit vectors, $\vec{x} \cdot \vec{x} = \vec{y} \cdot \vec{y} = 1$, and so by using the values found in part **a.** and **b.**,

$$(2\vec{x} - \vec{y}) \cdot (\vec{x} + 3\vec{y}) = 2(1) + (3) - \left(\tfrac{1}{2}\right) - 3(1) = \frac{3}{2}$$

4. a. $2(\vec{i} - 2\vec{j} + 3\vec{k}) - 4(2\vec{i} + 4\vec{j} + 5\vec{k}) - (\vec{i} - \vec{j})$
$= 2\vec{i} - 4\vec{j} + 6\vec{k} - 8\vec{i} - 16\vec{j} - 20\vec{k} - \vec{i} + \vec{j}$
$= -7\vec{i} - 19\vec{j} - 14\vec{k}$

b. $-2(3\vec{i} - 4\vec{j} - 5\vec{k}) \cdot (2\vec{i} + 3\vec{k}) + 2\vec{i} \cdot (3\vec{j} - 2\vec{k})$
$= -2(3\vec{i} - 4\vec{j} - 5\vec{k}) \cdot (2\vec{i} + 0\vec{j} + 3\vec{k}) + 2(\vec{i} + 0\vec{j} + 0\vec{k}) \cdot (0\vec{i} + 3\vec{j} - 2\vec{k})$
$= -2(3(2) - 4(0) - 5(3)) + 2(1(0) + 0(3) + 0(-2))$
$= -2(-9) + 2(0)$
$= 18$

5. The direction vectors for the positive x-axis, y-axis, and z-axis are (1, 0, 0), (0, 1, 0), and (0, 0, 1), respectively.

$$|(4, -2, -3)| = \sqrt{4^2 + (-2)^2 + (-3)^2} = \sqrt{29},$$

and $|(1, 0, 0)| = |(0, 1, 0)| = |(0, 0, 1)| = \sqrt{1} = 1.$

$(4, -2, -3) \cdot (1, 0, 0) = 4$, so the angle the vector makes with the x-axis is $\cos^{-1}\left(\frac{4}{1\sqrt{29}}\right) \doteq 42.0°$.

$(4, -2, -3) \cdot (0, 1, 0) = -2$, so the angle the vector makes with the y-axis is $\cos^{-1}\left(\frac{-2}{1\sqrt{29}}\right) = 111.8°$.

$(4,-2,-3) \cdot (0, 0, 1) = -3$, hence the angle the vector makes with the z-axis is $\cos^{-1}\left(\frac{-3}{1\sqrt{29}}\right) \doteq 123.9°$.

6. a. $\vec{a} \times \vec{b} = (1, -2, 3) \times (-1, 1, 2)$
$= (-2(2) - 3(1), 3(-1) - 1(2), 1(1) - (-2)(-1))$
$= (-7, -5, -1)$

b. By the scalar law for vector multiplication,
$2\vec{a} \times 3\vec{b} = 2(3)(\vec{a} \times \vec{b})$
$= 6(\vec{a} \times \vec{b})$
$= 6(-7, -5, -1) = (-42, -30, -6)$

c. The area of a parallelogram determined by $\vec{a}$ and $\vec{b}$ is equal to the magnitude of the cross product of $\vec{a}$ and $\vec{b}$.

A = area of parallelogram
$= |\vec{a} \times \vec{b}|$
$= |(-7, -5, -1)|$
$= \sqrt{(-7)^2 + (-5)^2 + (-1)^2}$
$\doteq 8.66$ square units

d. $(\vec{b} \times \vec{a}) = -(\vec{a} \times \vec{b})$
$= -(-7, -5, -1)$
$= (7, 5, 1)$

So $\vec{c} \cdot (\vec{b} \times \vec{a}) = (3, -4, -1) \cdot (7, 5, 1)$
$= 3(7) - 4(5) - 1(1)$
$= 0$

7. A unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ can be determined from any vector perpendicular to both $\vec{a}$ and $\vec{b}$. $\vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

$\vec{a} \times \vec{b} = (1, -1, 1) \times (2, -2, 3)$
$= (-1(3) - 1(-2), 1(2) - 1(3), 1(-2) - (-1)(2))$
$= (-1, -1, 0)$

$$\begin{aligned}|\vec{a} \times \vec{b}| &= |(-1, -1, 0)|\\ &= \sqrt{(-1)^2 + (-1)^2 + 0^2}\\ &= \sqrt{2}\end{aligned}$$

So $\frac{1}{\sqrt{2}}(-1, -1, 0) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ is an unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is another.

8. a. Answers may vary. For example:
A direction vector for the line is $\overrightarrow{AB}$.

$$\begin{aligned}\overrightarrow{AB} &= (1, 2, 3) - (2, -3, 1)\\ &= (-1, 5, 2)\end{aligned}$$

Since $A(2, -3, 1)$ is a point on the line, $\vec{r} = (2, -3, 1) + t(-1, 5, 2), t \in \mathbf{R}$, is a vector equation for a line and the corresponding parametric equation is $x = 2 - t, y = -3 + 5t, z = 1 + 2t, t \in \mathbf{R}$.

b. If the x-coordinate of a point on the line is 4, then $2 - t = 4$, or $t = -2$. At $t = -2$, the point on the line is $(2, -3, 1) - 2(-1, 5, 2) = (4, -13, -3)$. Hence $C(4, -13, -3)$ is a point on the line.

9. The direction vector of the first line is $(-1, 5, 2)$, while the direction vector for the second line is $(1, -5, -2) = -(-1, 5, 2)$. So the direction vectors for the line are collinear. Hence the lines are parallel. The lines coincide if and only if for any point on the first line and any point on the second line, the vector connecting the two points is a multiple of the direction vector for the lines.
(2, 0, 9) is a point on the first line and $(3, -5, 10)$ is a point on the second line.
$(2, 0, 9) - (3, -5, 10) = (-1, 5, -1) \neq k(-1, 5, 2)$ for any $k \in \mathbf{R}$. Hence the lines are parallel and distinct.

10. The direction vector for the parallel line is (0, 1, 1). Since parallel lines have collinear direction vectors, (0, 1, 1) can be used as a direction vector for the line. Since (0, 0, 4) is a point on the line, $\vec{r} = (0, 0, 4) + t(0, 1, 1), t \in \mathbf{R}$, is a vector equation for a line and the corresponding parametric equation is $x = 0, y = t, z = 4 + t, t \in \mathbf{R}$.

11. The line is parallel to the plane if and only if the direction vector for the line is perpendicular to the normal vector for the plane. The normal vector for the plane is (2, 3, c). The direction vector for the line is (2, 3, 1). The vectors are perpendicular if and only if the dot product between the two is zero.

$$\begin{aligned}(2, 3, c) \cdot (2, 3, 1) &= 2(2) + 3(3) + c(1)\\ &= 13 + c\end{aligned}$$

So if $c = -13$, then the dot product of normal vector and the direction vector is zero. Hence for $c = -13$, the line and plane are parallel.

12. First put the line in its corresponding parametric form. (3, 1, 5) is a direction vector and $(2, -5, 3)$ is the origin point, so a parametric equation for the line is $x = 2 + 3s, y = -5 + s, z = 3 + 5s, s \in \mathbf{R}$. If we substitute these coordinates into the equation of the plane, we may find the s value where the line intersects the plane.

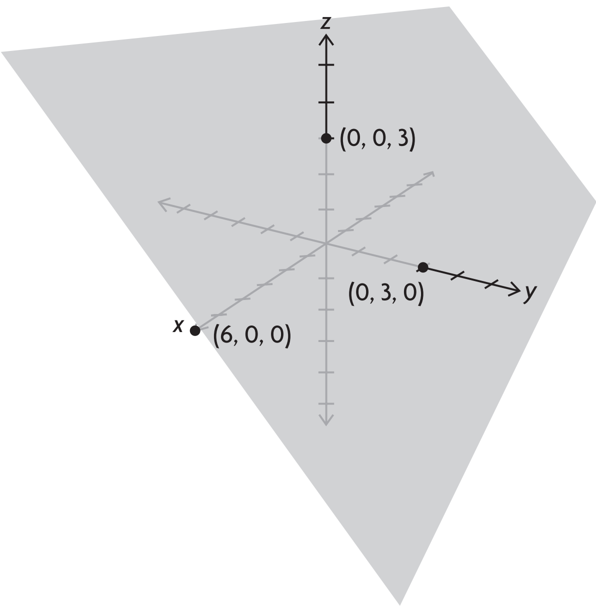
$$\begin{aligned}&5x + y - 2z + 2\\ &= 5(2 + 3s) + (-5 + s) - 2(3 + 5s) + 2\\ &= 10 + 15s + \; -5 + s - 6 - 10s + 2\\ &= 1 + 6s\end{aligned}$$

So if $5x + y - 2z + 2 = 0$, then $1 + 6s = 0$ or $s = -\frac{1}{6}$. At $s = -\frac{1}{6}$, the point on the line is $\left(\frac{3}{2}, -\frac{31}{6}, \frac{13}{6}\right)$. To check that this point is also on the plane, we substitute the x, y, z values into the plane equation and check that it equals zero.

$$\begin{aligned}5x + y - 2z + 2 &= 5\left(\frac{3}{2}\right) + \left(-\frac{31}{6}\right) - 2\left(\frac{13}{6}\right) + 2\\ &= 0\end{aligned}$$

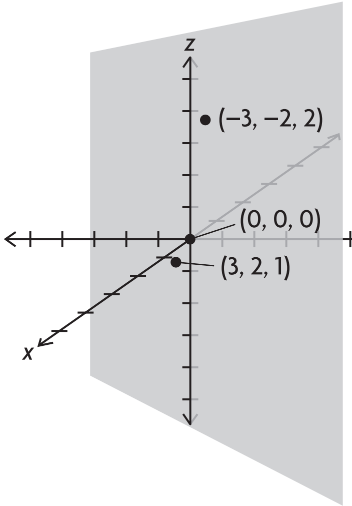
Hence $\left(\frac{3}{2}, -\frac{31}{6}, \frac{13}{6}\right)$ is the point of intersection between the line and the plane.

13. a.



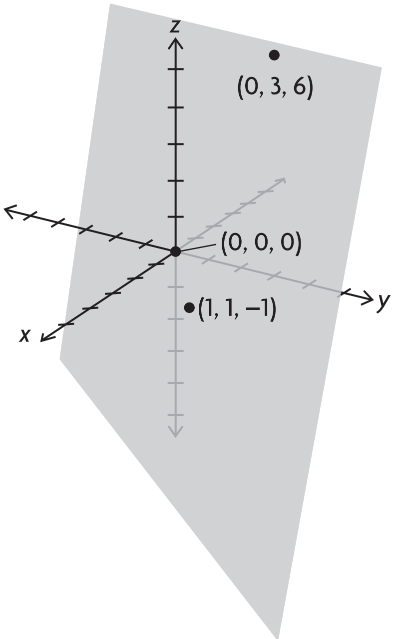
Two direction vectors are:
$(0, 3, 0) - (0, 0, 3) = (0, 3, -3)$
and
$(6, 0, 0) - (0, 0, 3) = (6, 0, -3)$.

b.



Two direction vectors are:
$(-3, -2, 2) - (0, 0, 0) = (-3, -2, 2)$
and
$(3, 2, 1) - (0, 0, 0) = (3, 2, 1)$.

c.



Two direction vectors are:
$(0, 3, 6) - (0, 0, 0) = (0, 3, 6)$
and
$(1, 1, -1) - (0, 0, 0) = (1, 1, -1)$.

14. The plane is the right bisector joining $P(1, -2, 4)$ and its image. The line connecting the two points has a direction vector equal to that of the normal vector for the plane. The normal vector for the plane is $(2, -3, -4)$. So the line connecting the two points is $(1, -2, 4) + t(2, -3, -4), t \in \mathbf{R}$, or in corresponding parametric form is $x = 1 + 2t$, $y = -2 - 3t$, $z = 4 - 4t$, $t \in \mathbf{R}$.
The intersection of this line and the plane is the bisector between P and its image. To find this point we substitute the parametric equation into the plane equation and solve for t.
$2x - 3y - 4z + 66$
$= 2(1 + 2t) - 3(-2 - 3t) - 4(4 - 4t) + 66$
$= 2 + 4t + 6 + 9t - 16 + 16t + 66$
$= 58 + 29t$
So if $2x - 3y - 4z + 66 = 0$, then $58 + 29t = 0$, or $t = -2$.
So the point of intersection is occurs at $t = -2$, since the origin point is P and the intersection occurs at the midpoint of the line connecting P and its image, the image point occurs at $t = 2 \times (-2) = -4$.
So the image point is at $x = 1 + 2(-4) = -7$, $y = -2 - 3(-4) = 10$, $z = 4 - 4(-4) = 20$.
So the image point is $(-7, 10, 20)$.

15. Let (a, b, c) be the direction vector for this line. So a line equation is $\vec{r} = (1, 0, 2) + t(a, b, c), t \in \mathbf{R}$. Since (1, 0, 2) is not on the other line, we may choose a, b, and c such that the intersection occurs at $t = 1$. Since the line is supposed to intersect the given line at a right angle, the direction vectors should be perpendicular. The direction vectors are perpendicular if and only if their dot product is zero. The direction vector for the given line is (1, 1, 2).
$(a, b, c) \cdot (1, 1, 2) = a + b + 2c = 0$, so
$b = -a - 2c$.
Also $(1, 0, 2) + (a, b, c) = (1 + a, b, 2 + c)$ is the point of intersection.
By substituting for b,
$(1 + a, b, 2 + c) = (1 + a, -a - 2c, 2 + c)$.
So for some s value,
$x = -2 + s = 1 + a$
$y = 3 + s = -a - 2c$
$z = 4 + 2s = 2 + c$
Subtracting the first equation from the second yields the equation, $5 + 0s = -2a - 2c - 1$.
Simplifying this gives $6 = -2a - 2c$ or just $a + c = -3$.
Subtracting twice the first equation from the third yields the equation, $8 = -2a + c$.
So $a + c = -3$ and $-2a + c = 8$, which is two equations with two unknowns. Twice the first plus the second equations gives $0a + 3c = 2$ or $c = \frac{2}{3}$.
Solving back for a gives $-\frac{11}{3}$ and since $b = -a - 2c$, $b = \frac{7}{3}$. Since $a + b + 2c = 0$, the direction vectors,

$(1, 1, 2)$ and (a, b, c) are perpendicular. A direction vector for the line is $(-11, 7, 2)$.
We need to check that
$(1, 0, 2) + (a, b, c) = \left(\frac{-8}{3}, \frac{7}{3}, \frac{8}{3}\right)$ is a point on the given line.
$x = -2 + s = -\frac{8}{3}$, at $s = -\frac{2}{3}$. The point on the given line at $s = -\frac{2}{3}$ is $\left(\frac{-8}{3}, \frac{7}{3}, \frac{8}{3}\right)$. Hence
$\vec{q} = (1, 0, 2) + t(-11, 7, 2), t \in \mathbf{R}$, is a line that intersects the given line at a right angle.

16. a. The Cartesian equation is found by taking the cross product of the two direction vectors, $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\begin{aligned}\overrightarrow{AB} &= (-2, 0, 0) - (1, 2, 3)\\ &= (-3, -2, -3)\\ \overrightarrow{AC} &= (1, 4, 0) - (1, 2, 3) = (0, 2, -3)\\ \overrightarrow{AB} \times \overrightarrow{AC} &= (-2(-3) - (-3)(2),\\ &\quad -3(0) - (-3)(-3),\\ &\quad -3(2) - (-2)(0))\\ &= (12, -9, -6)\end{aligned}$$

So $= (12, -9, -6)$ is a normal vector for the plane, so the plane has the form
$12x - 9y - 6z + D = 0$, for some constant D. To find D, we know that $A(1, 2, 3)$ is a point on the plane, so $12(1) - 9(2) - 6(3) + D = 0$. So $-24 + D = 0$, or $D = 24$. So the Cartesian equation for the plane is $12x - 9y - 6z + 24 = 0$.

b. Substitute into the formula to determine distance between a point and a plane. So the distance, d, of $(0, 0, 0)$ to the plane $12x - 9y - 6z + 24 = 0$ is equal to $\frac{|12(0) - 9(0) - 6(0) + 24|}{\sqrt{12^2 + (-9)^2 + (-6)^2}}$.
So $d = \frac{24}{\sqrt{261}} \doteq 1.49$.

17. a. $(3, -5, 4)$ is a normal vector for the plane, so the plane has the form $3x - 5y + 4z + D = 0$, for some constant D. To find D, we know that $A(-1, 2, 5)$ is a point on the plane, so
$3(-1) - 5(2) + 4(5) + D = 0$. So $7 + D = 0$, or $D = -7$. So the Cartesian equation for the plane is $3x - 5y + 4z - 7 = 0$.

b. Since the plane is perpendicular to the line connecting $(2, 1, 8)$ and $(1, 2, -4)$, a direction vector for the line acts as a normal vector for the plane. So $(2, 1, 8) - (1, 2, -4) = (1, -1, 12)$ is a normal vector for the plane. So the plane has the form $x - y + 12z + D = 0$, for some constant D. To find D, we know that $K(4, 1, 2)$ is a point on the plane, so $(4) - (1) + 12(2) + D = 0$. So $27 + D = 0$, or $D = -27$. So the Cartesian equation for the plane is $x - y + 12z - 27 = 0$.

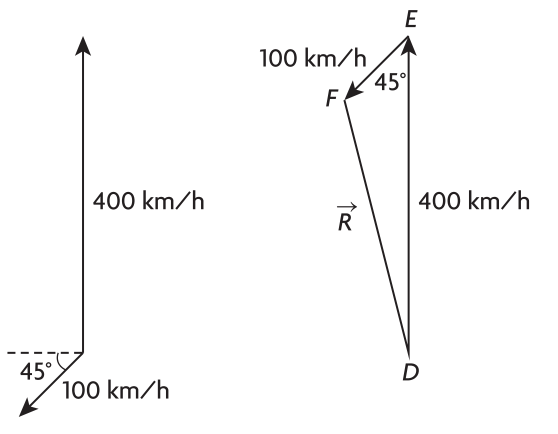
c. Since the plane is perpendicular to the z-axis, a direction vector for the z-axis acts as a normal vector for the plane. Hence $(0, 0, 1)$ is a normal vector for the plane. So the plane has the form $z + D = 0$, for some constant D. To find D, we know that $(3, -1, 3)$ is a point on the plane, so
$0(3) + 0(-1) + (3) + D = 0$. So $3 + D = 0$, or $D = -3$. So the Cartesian equation for the plane is $z - 3 = 0$.

d. The Cartesian equation can be found by taking the cross product of the two direction vectors for the plane. Since $(3, 1, -2)$ and $(1, 3, -1)$ are two points on the plane
$(3, 1, -2) - (1, 3, -1) = (2, -2, -1)$ is a direction vector for the plane. Since the plane is parallel to the y-axis, $(0, 1, 0)$ is also a direction vector for the plane.

$$\begin{aligned}&(2, -2, -1) \times (0, 1, 0) = (-2(0) -\\ &(-1)(1), (-1)(0) - (2)(0), 2(1) - (-2)(0))\\ &\qquad = (1, 0, 2)\end{aligned}$$

So $(1, 0, 2)$ is a normal vector for the plane, so the plane has the form $x + 0y + 2z + D = 0$, for some constant D. To find D, we know that $(3, 1, -2)$ is a point on the plane, so
$(3) + 0(1) + 2(-2) + D = 0$. So $-1 + D = 0$, or $D = 1$. So the Cartesian equation for the plane is $x + 2z + 1 = 0$.

18.



Position Diagram Vector Diagram

From the triangle DEF and the cosine law, we have
$|\vec{R}|^2 = 400^2 + 100^2 - 2(400)(100)\cos(45°)$
$\doteq 336.80$ km/h.
To find the direction of the vector, the sine law is applied.

$$\frac{\sin\angle DEF}{|\vec{R}|} = \frac{\sin\angle EDF}{100}$$

$$\frac{\sin 45°}{336.80} \doteq \frac{\sin\angle EDF}{100}.$$

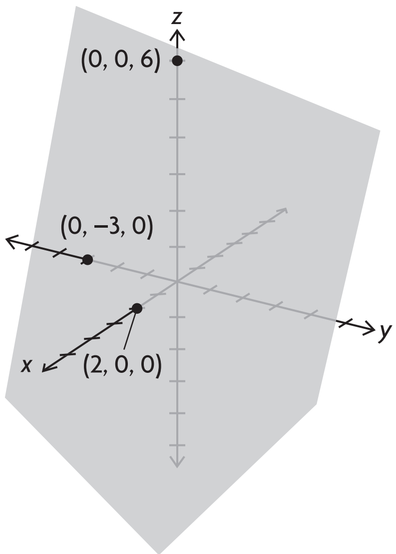
$\sin \angle EDF \doteq \frac{\sin 45°}{336.80} \times 100.$
$\sin \angle EDF \doteq 0.2100.$
Thus $\angle EDF \doteq 12.1°$, so the resultant velocity is 336.80 km/h, N 12.1° W.

19. a. The simplest way is to find the parametric equation, then find the corresponding vector equation. If we substitute $x = s$ and $y = t$ and solve for z, we obtain $3s - 2t + z - 6 = 0$ or $z = 6 - 3s + 2t$. This yields the parametric equations $x = s$, $y = t$, and $z = 6 - 3s + 2t$. So the corresponding vector equation is $\vec{r} = (0, 0, 6) + s(1, 0, -3) + t(0, 1, 2)$, $s, t \in \mathbf{R}$. To check that this is correct, find the Cartesian equation corresponding to the above vector equation and see if it is equivalent to the Cartesian equation given in the problem. A normal vector to this plane is the cross product of the two directional vectors.

$$\begin{aligned}\vec{n} &= (1, 0, -3) \times (0, 1, 2) = (0(2) - (-3)(1),\\ &\quad -3(0) - 1(2), 1(1) - 0(0))\\ &= (3, -2, 1)\end{aligned}$$

So $(3, -2, 1)$ is a normal vector for the plane, so the plane has the form $3x - 2y + z + D = 0$, for some constant D. To find D, we know that (0, 0, 6) is a point on the plane, so $3(0) - 2(0) + (6) + D = 0$. So $6 + D = 0$, or $D = -6$. So the Cartesian equation for the plane is $3x - 2y + z - 6 = 0$. Since this is the same as the initial Cartesian equation, the vector equation for the plane is correct.

b.



20. a. The angle, θ, between the plane and the line is the complementary angle of the angle between the direction vector of the line and the normal vector for the plane. The direction vector of the line is $(2, -1, 2)$ and the normal vector for the plane is (1, 2, 1).

$$\begin{aligned}|(2, -1, 2)| &= \sqrt{2^2 + (-1)^2 + 2^2}\\ &= \sqrt{9}\\ &= 3.\end{aligned}$$

$$\begin{aligned}|(1, 2, 1)| &= \sqrt{1^2 + 2^2 + 1^2}\\ &= \sqrt{6}\end{aligned}$$

$(2, -1, 2) \cdot (1, 2, 1) = 2(1) - 1(2) + 2(1) = 2$

So the angle between the normal vector and the direction vector is $\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right) \doteq 74.21°$. So $\theta \doteq 90° - 74.21° = 15.79°$.
To the nearest degree, $\theta = 16°$.

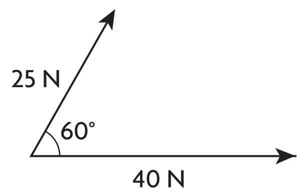
b. The two planes are perpendicular if and only if their normal vectors are also perpendicular.
A normal vector for the first plane is $(2, -3, 1)$ and a normal vector for the second plane is $(4, -3, -17)$. The two vectors are perpendicular if and only if their dot product is zero.

$$\begin{aligned}(2, -3, 1) \cdot (4, -3, -17) &= 2(4) - 3(-3)\\ &\quad + 1(-17)\\ &= 0.\end{aligned}$$

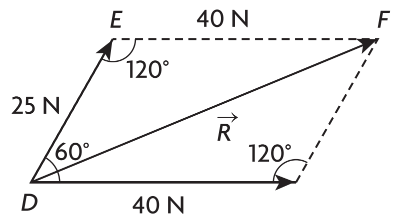
Hence the normal vectors are perpendicular. Thus the planes are perpendicular.

c. The two planes are parallel if and only if their normal vectors are also parallel. A normal vector for the first plane is $(2, -3, 2)$ and a normal vector for the second plane is $(2, -3, 2)$. Since both normal vectors are the same, the planes are parallel. Since $2(0) - 3(-1) + 2(0) - 3 = 0$, the point $(0, -1, 0)$ is on the second plane. Yet since $2(0) - 3(-1) + 2(0) - 1 = 2 \neq 0$, $(0, -1, 0)$ is not on the first plane. Thus the two planes are parallel but not coincident.

21.



Position diagram



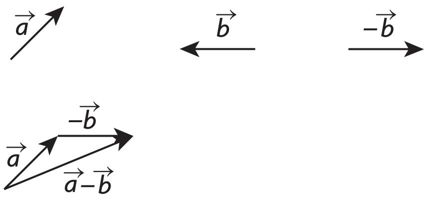
Vector diagram

From the triangle DEF and the cosine law, we have
$|\vec{R}|^2 = 40^2 + 25^2 - 2(40)(25)\cos(120°)$
$\doteq 56.79$ N.
To find the direction of the vector, the sine law is applied.

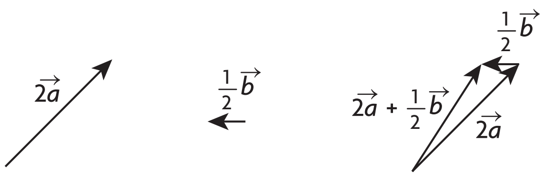
$$\frac{\sin \angle DEF}{|\vec{R}|} = \frac{\sin \angle EDF}{100}$$
$$\frac{\sin 120°}{56.79} \doteq \frac{\sin \angle EDF}{40}.$$
$$\sin \angle EDF \doteq \frac{\sin 120°}{56.79} \times 40.$$
$$\sin \angle EDF \doteq 0.610.$$

Thus $\angle EDF \doteq 37.6°$, so the resultant force is approximately 56.79 N, 37.6° from the 25 N force towards the 40 N force. The equilibrant force has the same magnitude as the resultant, but it is in the opposite direction. So the equilibrant is approximately 56.79 N, $180° - 37.6° = 142.4°$ from the 25 N force away from the 40 N force.

22.



b.



23. a. The unit vector in the same direction of $\vec{a}$ is simply $\vec{a}$ divided by the magnitude of $\vec{a}$.

$$|\vec{a}| = \sqrt{6^2 + 2^2 + (-3)^2}$$
$$= \sqrt{49}$$
$$= 7$$

So the unit vector in the same direction of $\vec{a}$ is
$\frac{1}{|\vec{a}|}\vec{a} = \frac{1}{7}(6, 2, -3) = \left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right)$.

b. The unit vector in the opposite direction of $\vec{a}$ is simply the negative of the unit vector found in part a. So the vector is $-\left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right) = \left(-\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right)$.

24. a. Since $OBCD$ is a parallelogram, the point C occurs at $(-1, 7) + (9, 2) = (8, 9)$. So $\overrightarrow{OC}$ is one vector equivalent to a diagonal and $\overrightarrow{BD}$ is the other.
$\overrightarrow{OC} = (8, 9) - (0, 0) = (8, 9)$
$\overrightarrow{BD} = (9, 2) - (1, 7) = (10, -5)$

b. $|(8, 9)| = \sqrt{8^2 + 9^2}$
$= \sqrt{145}$
$|(10, -5)| = \sqrt{10^2 + (-5)^2}$
$= \sqrt{125}$
$(8, 9) \cdot (10, -5) = 8(10) + 9(-5)$
$= -35$

So the angle between these diagonals is
$\cos^{-1}\left(\frac{-35}{\sqrt{145}\sqrt{125}}\right) \doteq 74.9°$.

c. $\overrightarrow{OB} = (-1, 7)$ and $\overrightarrow{OD} = (9, 2)$
$|(-1, 7)| = \sqrt{(-1)^2 + 7^2}$.
$= \sqrt{50}$
$|(9, 2)| = \sqrt{9^2 + 2^2}$
$= \sqrt{85}$
$(-1, 7) \cdot (9, 2) = -(9) + 7(2)$
$= 5$

So the angle between these diagonals is
$\cos^{-1}\left(\frac{5}{\sqrt{50}\sqrt{85}}\right) \doteq 85.6°$.

25. a. First step is to use the first equation to remove x from the second and third.
① $x - y + z = 2$
② $-x + y + 2z = 1$
③ $x - y + 4z = 5$
So we have
④ $0x + 0y + 3z = 3$, ① + ②
⑤ $0x + 0y + 3z = 3$, $-1 \times$ ① + ③
Hence $3z = 3$, or $z = 1$. Since both equations are the same, this implies that there are infinitely many solutions. Let $x = t$, then by substituting into the equation 2, we obtain
$-t + y + 2(1) = 1$, or $y = -1 + t$.
Hence the solution to these equations is $x = t$,
$y = -1 + t$, $z = 1$, $t \in \mathbf{R}$.

b. First step is to use the first equation to remove x from the second and third.
① $-2x - 3y + z = -11$
② $x + 2y + z = 2$
③ $-x - y + 3z = -12$
So we have
④ $0x + 1y + 3z = -7$, ① + 2 × ②
⑤ $0x - 1y - 5z = 13$, ① − 2 × ③
Now the fourth and fifth equations are used to create a sixth equation where the coefficient of y is zero.
⑥ $0x + 0y - 2z = 6$, ④ + ⑤
So $-2z = 6$ or $z = -3$.

Substituting this into equation ④ yields, $y + 3(-3) = -7$ or $y = 2$. Finally substitute z and y values into equation ② to obtain the x value.
$x + 2(2) + (-3) = 2$ or $x = 1$.
Hence the solution to these three equations is $(1, 2, -3)$.
c. First step is to notice that the second equation is simply twice the first equation.
① $2x - y + z = -1$
② $4x - 2y + 2z = -2$
③ $2x + y - z = 5$
So the solution to these equations is the same as the solution to just the first and third equations. Moreover since this is two equations with three unknowns, there will be infinitely many solutions.
④ $4x + 0y + 0z = 4$, ① + ③
Hence $4x = 4$ or $x = 1$. Let $y = t$ and solve for z using the first equation.
$2(1) - t + z = -1$, so $z = -3 + t$
Hence the solution to these equations is $x = 1$, $y = t$, $z = -3 + t$, $t \in \mathbf{R}$.
d. First step is to notice that the second equations is simply twice the first and the third equation is simply -4 times the first equation.
① $x - y - 3z = 1$
② $2x - 2y - 6z = 2$
③ $-4x + 4y + 12z = -4$
So the solution to these equations is the same as the solution to just the first equation. So the solution to these equations is a plane. To solve this in parametric equation form, simply let $y = t$ and $z = s$ and find the x value.
$x - t - 3s = 1$, or $x = 1 + t + 3s$
So the solution to these equations is $x = 1 + 3s + t$, $y = t$, $z = s$, $s, t \in \mathbf{R}$.
26. a. Since the normal of the first equation is $(1, -1, 1)$ and the normal of the second is $(1, 2, -2)$, which are not scalar multiples of each other, there is a line of intersection between the planes. The next step is to use the first and second equations to find an equation with a zero for the coefficient of x. The second equation minus the first equation yields $0x + 3y - 3z + 3 = 0$. We may divide by three to simplify, so $y - z + 1 = 0$. If we let $z = t$, then $y - t + 1 = 0$, or $y = -1 + t$. Substituting these into the first equation yields $x - (-1 + t) + t - 1 = 0$ or $x = 0$. So the equation of the line in parametric form is $x = 0$, $y = -1 + t$, $z = t$, $t \in \mathbf{R}$.
To check that this is correct, we substitute in the solution to both initial equations
$$\begin{aligned} x - y + z - 1 &= (0) - (-1 + t) + (t) - 1 \\ &= 0 \end{aligned}$$
and
$$\begin{aligned} x + 2y - 2z + 2 &= (0) + 2(-1 + t) - 2(t) + 2 \\ &= 0. \end{aligned}$$
Hence the line given by the parametric equation above is the line of intersection for the planes.
b. The normal vector for the first plane is $(1, -4, 7)$, while the normal vector for the second plane is $(2, -8, 14) = 2(1, -4, 7)$. Hence the planes have collinear normal vectors, and so are parallel.
The second equation is equivalent to $x - 4y + 7z = 30$, since we may divide the equation by two. Since the constant on the right in the first equation is 28, while the constant on the right in the second equivalent equation is 30, these planes are parallel and not coincident. So there is no intersection.
c. The normal vector for the first equation is $(1, -1, 1)$, while the normal vector for the second equation is $(2, 1, 1)$. Since the normal vectors are not scalar multiples of each other, there is a line of intersection between the planes.
The next step is to use the first and second equations to find an equation with a zero for the coefficient of x. The second equation minus twice the first equation yields $0x + 3y - z + 0 = 0$.
Solving for z yields, $z = 3y$. If we let $y = t$, then $z = 3(t) = 3t$.
Substituting these into the first equation yields $x - (t) + (3t) - 2 = 0$ or $x = 2 - 2t$. So the equation of the line in parametric form is $x = 2 - 2t$, $y = t$, $z = 3t$, $t \in \mathbf{R}$.
To check that this is correct, we substitute in the solution to both initial equations
$$\begin{aligned} x - y + z - 2 &= (2 - 2t) - (t) + (3t) - 2 \\ &= 0 \end{aligned}$$
and
$$\begin{aligned} 2x + y + z - 4 &= 2(2 - 2t) + (t) + (3t) - 4 \\ &= 0. \end{aligned}$$
Hence the line given by the parametric equation above is the line of intersection for the planes.
27. The angle, θ, between the plane and the line is the complementary angle of the angle between the direction vector of the line and the normal vector for the plane. The direction vector of the line is

$(1, -1, 0)$ and the normal vector for the plane is $(2, 0, -2)$.

$$|(1, -1, 0)| = \sqrt{1^2 + (-1)^2 + 0^2}$$
$$= \sqrt{2}$$
$$|(2, 0, -2)| = \sqrt{2^2 + 0^2 + (-2)^2} = \sqrt{8}$$

$(1, -1, 0) \cdot (2, 0, -2) = 1(2) - 1(0) + 0(-2) = 2$

So the angle between the normal vector and the direction vector is $\cos^{-1}\left(\frac{2}{\sqrt{2}\sqrt{8}}\right) = 60°$. So $\theta = 90 - 60° = 30°$.

28. a. We have that $\cos(60°) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$. Also since $\vec{a}$ and $\vec{b}$ are unit vectors, $|\vec{a}| = |\vec{b}| = 1$ and $\vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b} = 1$, and moreover $\cos(60°) = \frac{1}{2}$. So $\vec{a} \cdot \vec{b} = \frac{\vec{a} \cdot \vec{b}}{1 \times 1} = \frac{1}{2}$.

The dot product is distributive, so

$$\begin{aligned}(6\vec{a} + \vec{b}) \cdot (\vec{a} - 2\vec{b}) &= 6\vec{a} \cdot (\vec{a} - 2\vec{b}) \\ &\quad + \vec{b} \cdot (\vec{a} - 2\vec{b}) \\ &= 6\vec{a} \cdot \vec{a} + 6\vec{a} \cdot (-2\vec{b}) \\ &\quad + \vec{b} \cdot \vec{a} + \vec{b} \cdot (-2\vec{b}) \\ &= 6\vec{a} \cdot \vec{a} - 12\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} \\ &\quad - 2\vec{b} \cdot \vec{b} \\ &= 6(1) - 12\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) \\ &\quad - 2(1) \\ &= -\frac{3}{2}\end{aligned}$$

b. We have that $\cos(60°) = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}$. Also since $|\vec{x}| = 3$, $|\vec{y}| = 4$, and $\cos(60°) = \frac{1}{2}$, $\vec{x} \cdot \vec{y} = \frac{1}{2}(4)(3) = 6$. Also $\vec{x} \cdot \vec{x} = |\vec{x}|^2 = 9$ and $\vec{y} \cdot \vec{y} = |\vec{y}|^2 = 16$.

The dot product is distributive, so

$$\begin{aligned}(4\vec{x} - \vec{y}) \cdot (2\vec{x} + 3\vec{y}) &= 4\vec{x} \cdot (2\vec{x} + 3\vec{y}) \\ &\quad - \vec{y} \cdot (2\vec{x} + 3\vec{y}) \\ &= 8\vec{x} \cdot \vec{x} + 12\vec{x} \cdot \vec{y} - 2\vec{y} \cdot \vec{x} \\ &\quad - 3\vec{y} \cdot \vec{y} \\ &= 8(9) + 12(6) - 2(6) \\ &\quad - 3(16) \\ &= 84\end{aligned}$$

29. The origin, (0, 0, 0), and $(-1, 3, 1)$ are two points on this line. So $(-1, 3, 1)$ is a direction vector for this line and since the origin is on the line, a possible vector equation is $\vec{r} = t(-1, 3, 1), t \in \mathbf{R}$. $(-1, 3, 1)$ is a normal vector for the plane. So the equation of the plane is $-x + 3y + z + D = 0$. $(-1, 3, 1)$ is a point on the plane. Substitute the coordinates to determine the value of D.

$1 + 9 + 1 + D = 0$
$D = -11$

The equation of the plane is $-x + 3y + z - 11 = 0$.

30. The plane is the right bisector joining $P(-1, 0, 1)$ and its image. The line connecting the two points has a direction vector equal to that of the normal vector for the plane. The normal vector for the plane is $(0, 1, -1)$. So the line connecting the two points is $(-1, 0, 1) + t(0, 1, -1), t \in \mathbf{R}$, or in corresponding parametric form is $x = -1, y = t, z = 1 - t, t \in \mathbf{R}$. The intersection of this line and the plane is the bisector between P and its image. To find this point we plug the parametric equation into the plane equation and solve for t.

$$\begin{aligned}0x + y - z &= 0(-1) + (t) - (1 - t) \\ &= -1 + 2t\end{aligned}$$

So if $y - z = 0$, then $-1 + 2t = 0$, or $t = \frac{1}{2}$. So the point of intersection is occurs at $t = \frac{1}{2}$, since the origin point is P and the intersection occurs at the midpoint of the line connecting P and its image, the image point occurs at $t = 2 \times \frac{1}{2} = 1$. So the image point is at $x = -1, y = 1, z = 1 - (1) = 0$. So the image point is $(-1, 1, 0)$.

31. a. Thinking of the motorboat's velocity vector (without the influence of the current) as starting at the origin and pointing northward toward the opposite side of the river, the motorboat has velocity vector (0, 10) and the river current has velocity vector (4, 0). So the resultant velocity vector of the motorboat is

$(0, 10) + (4, 0) = (4, 10)$

To reach the other side of the river, the motorboat needs to cover a vertical distance of 2 km. So the hypotenuse of the right triangle formed by the marina, the motorboat's initial position, and the motorboat's arrival point on the opposite side of the river is represented by the vector

$$\frac{1}{5}(4, 10) = \left(\frac{4}{5}, 2\right)$$

(We multiplied by $\frac{1}{5}$ to create a vertical component of 2 in the motorboat's resultant velocity vector, the distance needed to cross the river.) Since this new vector has horizontal component equal to $\frac{4}{5}$, this means that the motorboat arrives $\frac{4}{5} = 0.8$ km downstream from the marina.

b. The motorboat is travelling at 10 km/h, and in part a. we found that it will travel along the vector $\left(\frac{4}{5}, 2\right)$. The length of this vector is

$$\left|\left(\frac{4}{5}, 2\right)\right| = \sqrt{\left(\frac{4}{5}\right)^2 + 2^2}$$
$$= \sqrt{4.64}$$

So the motorboat travels a total of $\sqrt{4.64}$ km to cross the river which, at 10 km/h, takes

$$\sqrt{4.64} \div 10 \doteq 0.2 \text{ hours}$$
$$= 12 \text{ minutes.}$$

32. a. Answers may vary. For example:
A direction vector for this line is

$$\overrightarrow{AB} = (6, 3, 4) - (2, -1, 3)$$
$$= (4, 4, 1)$$

So, since the point $B(6, 3, 4)$ is on this line, the vector equation of this line is
$\vec{r} = (6, 3, 4) + t(4, 4, 1), t \in \mathbf{R}$.
The equivalent parametric form is
$x = 6 + 4t$
$y = 3 + 4t$
$z = 4 + t, t \in \mathbf{R}$.

b. The line found in part a. will lie in the plane $x - 2y + 4z - 16 = 0$ if and only if both points $A(2, -1, 3)$ and $B(6, 3, 4)$ lie in this plane.
We verify this by substituting these points into the equation of the plane, and checking for consistency.
For A:
$2 - 2(-1) + 4(3) - 16 = 0$
For B:
$6 - 2(3) + 4(4) - 16 = 0$
Since both points lie on the plane, so does the line found in part a.

33. The wind velocity vector is represented by (16, 0), and the water current velocity vector is represented by (0, 12). So the resultant of these two vectors is $(16, 0) + (0, 12) = (16, 12)$.
Thinking of this vector with tail at the origin and head at point (16, 12), this vector forms a right triangle with vertices at points (0, 0), (0, 12), and (16, 12). Notice that

$$|(16, 12)| = \sqrt{16^2 + 12^2}$$
$$= \sqrt{400}$$
$$= 20$$

This means that the sailboat is moving at a speed of 20 km/h once we account for wind and water velocities. Also the angle, θ, this resultant vector makes with the positive y-axis satisfies

$$\cos\theta = \frac{12}{20}$$
$$\theta = \cos^{-1}\left(\frac{12}{20}\right)$$
$$\doteq 53.1°$$

So the sailboat is travelling in the direction N 53.1° E, or equivalently E 36.9° N.

34. Think of the weight vector for the crane with tail at the origin at head at $(0, -400)$ (we use one unit for every kilogram of mass). We need to express this weight vector as the sum of two vectors: one that is parallel to the inclined plane and pointing down this incline (call this vector $\vec{x} = (a, b)$), and one that is perpendicular to the inclined plane and pointing toward the plane (call this vector $\vec{y} = (c, d)$). The angle between $\vec{x}$ and $(0, -400)$ is 60° and the angle between $\vec{y}$ and $(0, -400)$ is 30°. Of course, $\vec{x}$ and $\vec{y}$ are perpendicular. Using the formula for dot product, we get

$$\vec{y} \cdot (0, -400) = |\vec{y}||(0, -400)|\cos 30°$$
$$-400d = 400\left(\frac{\sqrt{3}}{2}\right)\sqrt{c^2 + d^2}$$
$$-2d = \sqrt{3} \cdot \sqrt{c^2 + d^2}$$
$$4d^2 = 3(c^2 + d^2)$$
$$d^2 = 3c^2$$

So, since c is positive and d is negative (thinking of the inclined plane as moving upward from left to right as we look at it means that $\vec{y}$ points down and to the right), this last equation means that $\frac{d}{c} = -\sqrt{3}$
So a vector in the same direction as $\vec{y}$ is $(1, -\sqrt{3})$.
We can find the length of $\vec{y}$ by computing the scalar projection of $(0, -400)$ on $(1, -\sqrt{3})$, which equals

$$\frac{(0, -400) \cdot (1, -\sqrt{3})}{|(1, -\sqrt{3})|} = \frac{400\sqrt{3}}{2}$$
$$= 200\sqrt{3}$$

That is, $|\vec{y}| = 200\sqrt{3}$. Now we can find the length of $\vec{x}$ as well by using the fact that

$$|\vec{x}|^2 + |\vec{y}|^2 = |(0, -400)|^2$$
$$|\vec{x}|^2 + (200\sqrt{3})^2 = 400^2$$
$$|\vec{x}| = \sqrt{160000 - 120000}$$
$$= \sqrt{40000}$$
$$= 200$$

So we get that
$|\vec{x}| = 200$ and $|\vec{y}| = 200\sqrt{3}$. This means that the component of the weight of the mass parallel to the inclined plane is

$$9.8 \times |\vec{x}| = 9.8 \times 200 = 1960 \text{ N},$$

and the component of the weight of the mass perpendicular to the inclined plane is

$$9.8 \times |\vec{y}| = 9.8 \times 200\sqrt{3} \doteq 3394.82 \text{ N}.$$

35. a. True; all non-parallel pairs of lines intersect in exactly one point in R^2. However, this is not the case for lines in R^3 (skew lines provide a counterexample).
b. True; all non-parallel pairs of planes intersect in a line in R^3.
c. True; the line $x = y = z$ has direction vector $(1, 1, 1)$, which is not perpendicular to the normal vector $(1, -2, 2)$ to the plane $x - 2y + 2z = k$, k any constant. Since these vectors are not perpendicular, the line is not parallel to the plane, and so they will intersect in exactly one point.
d. False; a direction vector for the line $\frac{x}{2} = y - 1 = \frac{z + 1}{2}$ is $(2, 1, 2)$. A direction vector for the line $\frac{x - 1}{-4} = \frac{y - 1}{-2} = \frac{z + 1}{-2}$ is $(-4, -2, -2)$, or $(2, 1, 1)$ (which is parallel to $(-4, -2, -2)$). Since $(2, 1, 2)$ and $(2, 1, 1)$ are obviously not parallel, these two lines are not parallel.
36. a. A direction vector for

$$L_1: x = 2, \frac{y - 2}{3} = z$$

is $(0, 3, 1)$, and a direction vector for

$$L_2: x = y + k = \frac{z + 14}{k}$$

is $(1, 1, k)$. But $(0, 3, 1)$ is not a nonzero scalar multiple of $(1, 1, k)$ for any k since the first component of $(0, 3, 1)$ is 0. This means that the direction vectors for L_1 and L_2 are never parallel, which means that these lines are never parallel for any k.
b. If L_1 and L_2 intersect, in particular their x-coordinates will be equal at this intersection point. But $x = 2$ always in L_1 so we get the equation

$$2 = y + k$$
$$y = 2 - k$$

Also, from L_1 we know that $z = \frac{y - 2}{3}$, so substituting this in for z in L_2 we get

$$2k = z + 14$$
$$2k = \frac{y - 2}{3} + 14$$
$$3(2k - 14) = y - 2$$
$$y = 6k - 40$$

So since we already know that $y = 2 - k$, we now get

$$2 - k = 6k - 40$$
$$7k = 42$$
$$k = 6$$

So these two lines intersect when $k = 6$. We have already found that $x = 2$ at this intersection point, but now we know that

$$y = 6k - 40 = 6(6) - 40 = -4$$
$$z = \frac{y - 2}{3} = \frac{-4 - 2}{3} = -2$$

So the point of intersection of these two lines is $(2, -4, -2)$, and this occurs when $k = 6$.

Chapter 9
Relationships Between Points, Lines, and Planes

Review of Prerequisite Skills, p. 487

1. a. Yes; $(2, -5) = (10, -12) + t(8, -7)$
$(2, -5) = (10, -12) + 1(8, -7)$
b. No; $12(1) + 5(2) - 13 = 9 \neq 0$
c. Yes; $(7, -3, 8) = (1, 0, -4) + t(2, -1, 4)$
$(7, -3, 8) = (1, 0, -4) + 3(2, -1, 4)$
d. No; $(1, 0, 5) = (2, 1, -2) + t(4, -1, 2)$
$(-1, -1, 7) \neq t(4, -1, 2)$
There is no value of t that satisfies the equation.
2. Answers may vary. For example:
a. Vector: $\vec{m} = (7, 3) - (2, 5) = (5, -2)$
$\vec{r} = (2, 5) + t(5, -2), t \in \mathbf{R}$
Parametric: $x = 2 + 5t, y = 5 - 2t, t \in \mathbf{R}$
b. Vector: $\vec{m} = (4, -7) - (-3, 7) = (7, -14)$
$\vec{r} = (-3, 7) + t(7, -14), t \in \mathbf{R}$
Parametric: $x = -3 + 7t, y = 7 - 14t, t \in \mathbf{R}$
c. Vector: $\vec{m} = (-3, -11) - (-1, 0)$
$= (-2, -11)$
$\vec{r} = (-1, 0) + t(-2, -11), t \in \mathbf{R}$
Parametric: $x = -1 + - 2t, y = -11t, t \in \mathbf{R}$
d. Vector: $\vec{m} = (6, -7, 0) - (1, 3, 5)$
$= (5, -10, -5)$
$\vec{r} = (1, 3, 5) + t(5, -10, -5), t \in \mathbf{R}$
Parametric: $x = 1 + 5t, y = 3 - 10t, z = 5 - 5t,$
$t \in \mathbf{R}$
e. Vector: $\vec{m} = (-1, 5, 2) - (2, 0, -1)$
$= (-3, 5, 3)$
$\vec{r} = (2, 0, -1) + t(-3, 5, 3), t \in \mathbf{R}$
Parametric: $x = 2 - 3t, y = -5t, z = -1 + 3t,$
$t \in \mathbf{R}$
f. Vector: $\vec{m} = (12, -5, -7) - (2, 5, -1)$
$= (10, -10, -6)$
$\vec{r} = (2, 5, -1) + t(10, -10, -6), t \in \mathbf{R}$
Parametric: $x = 2 + 10t, y = 5 - 10t, z = -1 - 6t,$
$t \in \mathbf{R}$
3. a. Since $\vec{n} = (2, 6, -1)$, the Cartesian equation of the plane is of the form $2x + 6y - z + D = 0$, where D is to be determined. Since $P_0(4, 1, -3)$ is on the plane, it must satisfy the equation. So
$2(4) + 6(1) - 1(-3) + D = 8 + 6 + 3 + D$
$= 17 + D = 0$. $D = -17$, and the equation of the plane is $2x + 6y - z - 17 = 0$.
b. Since $\vec{n} = (0, 7, 0)$, the Cartesian equation of the plane is of the form $7y + D = 0$, where D is to be determined. Since $P_0(-2, 0, 5)$ is on the plane, it must satisfy the equation. So $7(0) + D = 0 + D = 0$ thus $D = 0$. The equation of the plane is $7y = 0$, or $y = 0$.
c. Since $\vec{n} = (4, -3, 0)$, the Cartesian equation of the plane is of the form $4x - 3y + D = 0$, where D is to be determined. Since $P_0(3, -1, -2)$ is on the plane, it must satisfy the equation. So $4(3) - 3(-1) + D = 12 + 3 + D = 15 + D = 0$. $D = -15$, and the equation of the plane is $4x + 3y - 15 = 0$.
d. Since $\vec{n} = (6, 5, -3)$, the Cartesian equation of the plane is of the form $6x - 5y + 3z + D = 0$, where D is to be determined. Since $P_0(0, 0, 0)$ is on the plane, it must satisfy the equation. So $6(0) - 5(0) + 3(0) + D = 0$, or $D = 0$. The equation of the plane is $6x - 5y + 3z = 0$.
e. Since $\vec{n} = (11, -6, 0)$, the Cartesian equation of the plane is of the form $11x - 6y + D = 0$, where D is to be determined. Since $P_0(4, 1, 8)$ is on the plane, it must satisfy the equation. So $11(4) - 6(1) + D = 44 - 6 + D = 38 + D = 0$. $D = -38$, and the equation of the plane is $11x - 6y - 38 = 0$.
f. Since $\vec{n} = (1, 1, -1)$, the Cartesian equation of the plane is of the form $x + y + - z + D = 0$, where D is to be determined. Since $P_0(2, 5, 1)$ is on the plane, it must satisfy the equation. So $2 + 5 - 1 + D = 6 + D = 0$. $D = -6$, and the equation of the plane is $x + y - z - 6 = 0$.
4. Start by writing the given line in parametric form: $(x, y, z) = (2 + s + 2t, 1 - s, 3s - 5t)$, so $x = 2 + s + 2t, y = 1 - s$, and $z = 3s - 5t$. Solving for s in each component, we get $s = 1 - y$ and substituting this into $z = 3s - 5t$ gives $z = 3(1 - y) - 5t = 3 - 3y - 5t$.

So now $-3 + 3y + z = -5t$ and $t = \frac{3 - 3y - z}{5}$.
Finally, substituting both equations for s and t into $x = 2 + s + 2t$, we get

$$x = 2 + (1 - y) + 2\left(\frac{3 - 3y - z}{5}\right).$$

Rearranging, we get

$$5x = 10 + 5 - 5y + 6 - 6y - 2z$$
$$5x + 11y + 2z - 21 = 0.$$

5. L_1 is not parallel to the plane because (3, 0, 2) is a point on the line and the plane. Substitute the expressions for the components of the parametric equation of the line into the equation of the plane.

$$4(3 + t) + (-2t) - (2 + 2t) - 10 = 0$$
$$12 + 4t - 2t - 2 - 2t - 10 = 0$$
$$0 = 0$$

This last statement is always true. So every point on the line is also in the plane. Therefore, the line lies on the plane.
For L_2 substitute the expressions for the components of the parametric equation of the line into the equation of the plane.

$$4(-3t) + (-5 + 2t) - (-10t) - 10 = 0$$
$$-12t - 5 + 2t + 10t - 10 = 0$$
$$-15 = 0$$

This last statement is never true. So the line and the plane have no points in common. Therefore, L_2 is parallel to the plane. The line cannot lie on the plane.
For L_3 use the symmetric equation to rewrite x and z in terms of y.

$$x = -4y - 23$$
$$z = -y - 6$$

Substitute into the equation of the plane.

$$4(-4y - 23) + y - (-y - 6) - 10 = 0$$
$$-16y - 92 + y + y + 6 - 10 = 0$$
$$-14y - 96 = 0$$

This equation has a solution. Therefore, L_3 and the plane have a point in common and are not parallel. However, $(5, -7, 1)$ is a point that lies on the line that does not lie on the plane. Therefore, L_3 does not lie in the plane.
6. a. A normal vector to this plane is determined by calculating the cross product of the position vectors, $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = (2, 0, 0) - (1, 0, -1) = (1, 0, 1)$$
$$\overrightarrow{AC} = (6, -1, 5) - (1, 0, -1) = (5, -1, 6)$$

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= ((0 \cdot 6) - (1 \cdot -1), (1 \cdot 5) \\ &\quad - (1 \cdot 6), (1 \cdot -1) - (0 \cdot 5)) \\ &= (0 + 1, 5 - 6, -1 - 0) \\ &= (1, -1, -1) = \vec{n}.\end{aligned}$$

If $P(x, y, z)$ is any point on the plane, then $\overrightarrow{AP} = (x - 1, y, z + 1)$, and if the normal to the plane is $(1, -1, -1)$, then
$(x - 1, y, z + 1) \cdot (1, -1, -1) = 0$, so
$x - 1 - y - z - 1 = 0$ and thus,
$x - y - z - 2 = 0$

b. $\overrightarrow{PQ} = (6, 4, 0) - (4, 1, -2) = (2, 3, 2)$
$\overrightarrow{PR} = (0, 0, -3) - (4, 1, -2) = (-4, -1, -1)$
$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$
$= (3(-1) - 2(-1)), 2(-4) - 2(-1),$
$2(-1) - 3(4))$
$= (-3 + 2, -8 + 2, -2 + 12) = (-1, -6, 10)$
Since $(-1, -6, 10) = -1(1, 6, -10)$, we will use $(1, 6, -10)$ as the normal vector so that the coefficient of x is positive. If $P(x, y, z)$ is any point on the plane, then $\overrightarrow{AP} = (x - 4, y - 1, z + 2)$, and if the normal to the plane is $(1, 6, -10)$, then
$(x - 4, y - 1, z + 2) \cdot (1, 6, -10) = 0$,
so $x - 4 + 6y - 6 - 10z - 20 = 0$,
and thus $x + 6y - 10z - 30 = 0$.
7. Answers may vary. For example: One direction vector is $\vec{m} = (2, -1, 6) - (1, -4, 3) = (1, 3, 3)$. Now we need to find a normal to the plane such that $\vec{n} \cdot \vec{m} = 0$. So $(1, 3, 3) \cdot (a, 0, c) = 0$. Now we have $a + 3c = 0$. A possible solution to this is $a = 3, c = -1$. So $\vec{n} = (3, 0, -1)$ and the Cartesian equation of the plane is $3x - z = 0$.
Since the plane is parallel to the y-axis, (0, 1, 0) is another direction vector for the plane. Therefore, a vector equation for the plane is
$\vec{r} = (1, -4, 3) + t(1, 3, 3) + s(0, 1, 0), s, t \in \mathbf{R}$.
8. We are given the point $A(-1, 3, 4)$. We need to find a normal vector $\vec{n} = (a, b, c)$ such that
$a(x + 1) + b(y - 3) + c(z - 4) + d = 0$.
The normal vector also must be perpendicular to the two planes and their normals, $(2, -1, 3)$ and $(5, 1, -3)$. One possible solution for the normal is $\vec{n} = (0, 3, 1)$. So we have

$$3(y - 3) + z - 4 = 0$$
$$3y + z - 9 - 4 = 0$$

And the equation of the plane is $3y + z = 13$.

9.1 The Intersection of a Line with a Plane and the Intersection of Two Lines, pp. 496–498

1. a. First, show the parametric equations as $x = 1 + 5s$, $y = 2 + s$, $z = -3 + s$. Then the plane can be written as $\pi: x - 2y - 3z = 6$, and the vector equation of the line is $\vec{r} = (1, 2, -3) + s(5, 1, 1), s \in \mathbf{R}$.

b. When we substitute the parametric equations into the Cartesian equation for the plane, we get

$$(1 + 5s) - 2(2 + s) - 3(-3 + s) = 6$$
$$1 - 4 + 9 + 5s - 2s - 3s = 6 - 0s = 6$$

Note that by finishing the solution, we get $0s = 0$. Since any real number will satisfy this equation, we have an infinite number of solutions, and this line lies on the plane.

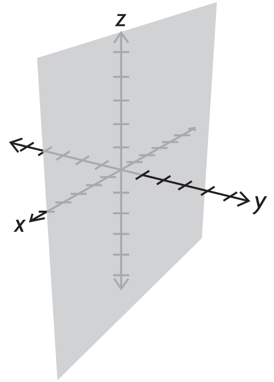
2. a. A line and a plane can intersect in three ways:
Case 1: The line and the plane have zero points of intersection. This occurs when the lines are not incidental, meaning they do not intersect.
Case 2: The line and the plane have only one point of intersection. This occurs when the line crosses the plane at a single point.
Case 3: The line and the plane have an infinite number of intersections. This occurs when the line is coincident with the plane, meaning the line lies on the plane.

b. Assume that the line and the plane have more than one intersection, but not an infinite number. For simplicity, assume two intersections. At the first intersection, the line crosses the plane. In order for the line to continue on, it must have the same direction vector. If the line has already crossed the plane, then it continues to move away from the plane, and can not intersect again. So the line and the plane can only intersect zero, one, or infinitely many times.

3. a. The line $\vec{r} = s(1, 0, 0)$ is the x-axis.

b. The plane $y = 1$ has the form $\vec{r} = (x, 1, z)$, where x, and z are any values in $\mathbf{R}$. So the plane is parallel to the xz-plane, but just one unit away to the right.

c.



d. There are no intersections between the line and the plane.

4. a. For $x + 4y + z - 4 = 0$, if we substitute the parametric equations, we have

$$(-2 + t) + 4(1 - t) + (2 + 3t) + 4$$
$$= -2 + 4 + 2 + t - 4t + 3t - 4$$
$$= 0t + 0$$

$= 0$. All values of t give a solution to the equation, so all points on the line are also on the plane.

b. For the plane $2x - 3y + 4z - 11 = 0$, we can substitute the parametric equations derived from $\vec{r} = (1, 5, 6) + t(1, -2, -2)$: $x = 1 + t$, $y = 5 - 2t$, $z = 6 - 2t$.
So we have $2(1 + t) - 3(5 - 2t) + 4(6 - 2t) - 11$

$$= 2 - 15 + 24 - 11 + 2t + 6t - 8t$$
$$= 0t + 0$$
$$= 0$$

Similar to part a., all values of t give a solution to this equation, so all points on the line are also on the plane.

5. a. First, we should determine the parametric equations from the vector form: $x = -1 - s$, $y = 1 + 2s$, $z = 2s$. Substituting these into the equation of the plane, we get

$$2(-1 - s) - 2(1 + 2s) + 3(2s) - 1$$
$$= -2 - 2 - 1 - 2s - 4s + 6s$$
$$= -5 + 0s$$

Since there are no values of s such that $-5 = 0$, this line and plane do not intersect.

b. Substituting the parametric equations into the equation of the plane, we get

$$2(1 + 2t) - 4(-2 + 5t) + 4(1 + 4t) - 13$$
$$2 + 8 + 4 - 13 + 4t - 20t + 16t$$
$$= 1 + 0t$$

Since there are no values of t such that $1 = 0$, there are no solutions, and the plane and the line do not intersect.

6. a. The direction vector is $\vec{m} = (-1, 2, 2)$ and the normal is $\vec{n} = (2, -2, 3)$, so if the line and the plane meet at right angles, $\vec{m} \cdot \vec{n} = 0$. So $(-1 \cdot 2) + (2 \cdot -2) + (2 \cdot 3) = -2 - 4 + 6 = 0$, but $2(-1) - 2(1) + 3(0) - 1 = -5 \neq 0$. So the point on the line is not on the plane.

b. The direction vector is $\vec{m} = (2, 5, 4)$ and the normal is $\vec{n} = (2, -4, 4)$, so if the line and the plane meet at right angles, $\vec{m} \cdot \vec{n} = 0$. So $(2 \cdot 2) + (5 \cdot -4) + (4 \cdot 4) = 4 - 20 + 16 = 0$, but $2(1) - 4(-2) + 4(1) - 13 = 1 \neq 0$. So the point on the line is not on the plane.

7. a. If the line and the plane intersect, then they are equal at a particular point p. So we must substitute the parametric equations into the equation of the plane, and then solve for p.

$(-1 + 6p) + 2(3 + p) - (4 - 2p) + 29$
$= -1 + 6 - 4 + 6p + 2p + 2p + 29$
$= 30 + 10p$
$= 0$. So now $-10p = 30$ and $p = -3$.

Now we must find the point at which the line and plane intersect. To do this, just substitute $p = -3$ into the vector form of the line:
$(-1, 3, 4) + -3(6, 1, -2) = (-19, 0, 10)$.

b. If the line and the plane intersect, then they are equal at a particular point p. So we must substitute the parametric equations into the equation of the plane, and then solve for p.

$x = 1 + 4s, y = -2 - s, z = 3 + s$
$2(1 + 4s) + 7(-2 - s) + (3 + s) + 15$
$= 2 - 14 + 3 + 15 + 8s - 7s + s$
$= 6 + 2s$
$= 0$. So now $-2s = 6$ and $s = -3$. Now we must find the point at which the line and plane intersect. To do this, just substitute $s = -3$ into the vector form of the line:
$(1, -2, 3) + -3(4, -1, 1) = (-11, 1, 0)$

8. a. Comparing the x and y components in L_1 and L_2, we have
$3 + 4s = 4 + 13t$
$1 - s = 1 - 5t$
We can easily solve for one of the variables by using the second equation: $s = 5t$. Substituting this back into the first equation: $3 + 20t = 4 + 13t$ so $1 = 7t$ and thus $t = \frac{1}{7}$. So now we must solve for s: $3 + 4s = 4 + \frac{13}{7}$ and $s = \frac{20}{28} = \frac{5}{7}$. Placing these back into the equations for L_1 and L_2:

$$L_1: (3, 1, 5) + \frac{5}{7}(4, -1, 2) = \left(\frac{41}{7}, \frac{2}{7}, \frac{45}{7}\right)$$
$$L_2: \left(4 + \frac{13}{7}, 1 - \frac{5}{7}, \frac{5}{7}\right) = \left(\frac{41}{7}, \frac{2}{7}, \frac{5}{7}\right)$$

The points must be equal for intersection to occur, so there is no intersection and the lines are skew.

b. If we compare the z components of the two lines, we see $2 = 8 - 6s$ or $s = 1$. Substituting this back into the x component (the y component would work just as well), we have $3 + m = -3 + 7(1) = 4$, or $m = 1$. So now we can substitute m and s back into the equations for the line, and we get
$L_3: (3, 7, 2) + (1, -6, 0) = (4, 1, 2)$
$L_4: (-3, 2, 8) + (7, -1, -6) = (4, 1, 2)$
So $(4, 1, 2)$ is the only point of intersection between these two lines.

9. a. Comparing the y and z components of each vector equation, we get the system of equations:
$3 - 2p = 3 - 2q$
$4 + 3p = -4 + 11q$
Note that from the first equation, $p = q$. So the second equation becomes $4 + 3q = -4 + 11q$. Solving for q, we get $q = 1$. So from the earlier relation, $p = 1$. Placing these two values back into the vector equations, we get
$(-2, 3, 4) + (6, -2, 3) = (4, 1, 7)$
$(-2, 3, -4) + (6, -2, 11) = (4, 1, 7)$
This shows that these two lines intersect at $(4, 1, 7)$.

b. Comparing the x and z components of each vector equation, we get the system of equations:
$4 + r = 2 + s$
$6 + 4r = -8 + 5s$
Note that from the first equation, $s = 2 + r$. So the second equation becomes $6 + 4r = 2 + 5r$. Solving for r, we get $r = 4$. So from the earlier relation, $s = 6$. Placing these two values back into the vector equations, we get
$(4, 1, 6) + 4(1, 0, 4) = (8, 1, 22)$
$(2, 1, -8) + 6(1, 0, 5) = (8, 1, 22)$
This shows that these two lines intersect at $(8, 1, 22)$.

c. Comparing the x and z components of each vector equation, we get the system of equations:
$2 + m = -2 + 3p$
$1 + m = 1 - p$
Note that from the second equation, $m = -p$. So the first equation becomes $2 - p = -2 + 3p$. Solving for p, we get $p = 1$. So from the earlier relation, $m = -1$. Placing these two values back into the vector equations, we get

$(2, 2, 1) - (1, 1, 1) = (1, 1, 0)$
$(-2, 2, 1) + (3, -1, -1) = (1, 1, 0)$
This shows that these two lines intersect at $(1, 1, 0)$.
d. Comparing the x and y components of each vector equation, we get the system of equations:
$1 + 0m = 2 + s$
$2 + 4m = 3 - 2s$
Note that from the first equation, $s = -1$. So the second equation becomes $2 + 4m = 5$.
Solving for m, we get $m = \frac{3}{4}$. Placing these two values back into the vector equations, we get

$$(9, 1, 2) - \frac{3}{4}(5, 0, 4) = \left(\frac{21}{4}, 1, -1\right)$$

$(8, 2, 3) - (4, 1, -2) = (4, 1, 5)$
The two lines do not intersect, so they are skew.
10. At the point where the line intersects the z-axis, the point $Q(0, 0, q)$ equals the vector equation. So for the x component, $-3 + 3s = 0$ or $s = 1$. Substituting this into the vector equation, we get $(-3, 2, 1) + (3, -2, 7) = (0, 0, 8)$. So $q = 8$.
11. a. Comparing the x components, we get $-2 + 7s = -30 + 7t$, which can be reduced to $28 + 7s = 7t$ or $s - t = 4$. Comparing the other components, the same equation results.
b. From L_1, we see that at $(-2, 3, 4)$, $s = 0$. When this occurs, $t = 4$. Substituting this into L_2, we get $(-30, 11, -4) + 4(7, -2, 2) = (-2, 3, 4)$. Since both of these lines have the same direction vector and a common point, the lines are coincidental.
12. a. First, we must determine the values of s and t. So comparing the x and z components, we get
$-3 + s = 1 - 3t$
$1 + s = 2 + 8t$
From the second equation, $s = 1 + 8t$. Substituting this back into the first equation,
$-3 + 1 + 8t = 1 - 3t$ or $t = \frac{3}{11}$.
Substituting back into the second equation,
$-3 + s = 1 - \frac{9}{11} = \frac{2}{11}$, and solving for s,
$s = \frac{2}{11} + 3 = \frac{35}{11}$. Now we can solve for k. Compare the y components after substituting s and t.

$$8 - \frac{35}{11} = 4 + \frac{3}{11}k$$

$53 = 44 + 3k$
or $k = 3$.
b. The lines intersect when $s = \frac{35}{11}$. The point of intersection is $\left(-3 + \frac{35}{11}, 8 - \frac{35}{11}, 1 + \frac{35}{11}\right)$ or $\left(\frac{2}{11}, \frac{53}{11}, \frac{46}{11}\right)$.

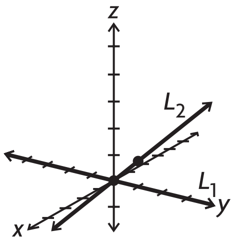
13. On the xz-plane, the point A has the coordinates $(x, 0, z)$, for any x, z. Similarly, on the yz-plane, the point B has the coordinates $(0, y, z)$ for any y, z. Now the task is to find the required values of s for these points. Starting with the x component of point B, we have $0 = -8 + 2s$ or $s = 4$. So point B is $(-8, -6, -1) + 4(2, 2, 1) = (0, 2, 3)$. For point A, we need the y coordinate to equal 0. So $0 = -6 + 2s$ or $s = 3$. So point A is
$(-8, -6, -1) + 3(2, 2, 1) = (-2, 0, 2)$.
Now we need to find the distance.

$$\begin{aligned} d &= \sqrt{(0 - (-2))^2 + (2 - 0)^2 + (3 - 2)^2} \\ &= \sqrt{4 + 4 + 1} \\ &= \sqrt{9} \\ &= 3 \end{aligned}$$

14. a. Comparing the y and z components of each vector equation, we get the system of equations:
$1 + 0p = -1 - 2q$
$1 - p = 1 - 2q$
Note that from the first equation, $2 = -2q$ or $q = -1$. So the second equation becomes $1 - p = 1 + 2$ or $p = -2$.
Placing these two values back into the vector equations to find the intersection point A, we get
$(2, 1, 1) - 2(4, 0, -1) = (-6, 1, 3)$
$(3, -1, 1) - (9, -2, -2) = (-6, 1, 3)$
Thus, the intersection point is $(-6, 1, 3)$.
b. A point on the xy plane has the form $(x, y, 0)$. If such a point is $(-6, 1, 0)$ then the distance from this point is $d = \sqrt{0 + 0 + 3^2} = 3$.
15. a. Comparing the x and y components of each vector equation, we get the system of equations:
$-1 + 5s = 4 + 0t$
$3 - 2s = -1 + 2t$
Note that from the first equation, $5 = 5s$ or $s = 1$. So the second equation becomes $3 - 2 = -1 + 2t$ or $t = 1$. Placing these two values back into the vector equations to find the intersection point A, we get
$(-1, 3, 2) + (5, -2, 10) = (4, 1, 12)$
$(4, -1, 1) + (0, 2, 11) = (4, 1, 12)$
Thus, the intersection point is $(4, 1, 12)$.
b. We need to find a vector (a, b, c) such that
$5a - 2b + 10c = 0$
$2b + 11c = 0$
A possible solution to the second equation is $(a, 11, -2)$. If we substitute this into the first equation, we get $5a - 22 - 20 = 0 \rightarrow 5a = 42$.

We can use this to get a solution of $\left(\frac{42}{5}, 11, -2\right)$. To eliminate the fraction, we get $(42, 55, -10)$. So the vector equation is $\vec{r} = (4, 1, 12) + t(42, 55, -10)$, $t \in \mathbf{R}$.

16. a.



b. The only point of intersection is at the origin $(0, 0, 0)$.

c. If $p = 0$ and $q = 0$, the intersection occurs at $(0, 0, 0)$.

17. a. Represent the lines parametrically, and then substitute into the equation for the plane.
For the first equation, $x = t$, $y = 7 - 8t$, $z = 1 + 2t$. Substituting into the plane equation, $2t + 7 - 8t + 3 + 6t - 10 = 0$. Simplifying, $0t = 0$. So the line lies on the plane.
For the second line, $x = 4 + 3s$, $y = -1$, $z = 1 - 2s$ Substituting into the plane equation, $8 + 6s - 1 + 3 - 6s - 10 = 0$. Simplifying, $0s = 0$. This line also lies on the plane.

b. Compare the x and y components:
$4 + 3s = t$
$7 - 8t = -1$
From the second equation, $t = 1$. Substituting back into the first equation, $4 + 3s = 1$, or $s = -1$.
Determine the point of intersection:
$(1, 7 - 8, 1 + 2) = (1, -1, 3)$
$(4 - 3, -1, 1 + 2) = (1, -1, 3)$
The point of intersection is $(1, -1, 3)$.

18. Answers may vary. For example:
$\vec{r} = (2, 0, 0) + p(2, 0, 1)$, $p \in \mathbf{R}$

9.2 Systems of Equations, pp. 507–509

1. a. linear
b. not linear
c. linear
d. not linear

2. Answers may vary. For example:
$x + y + 2z = -15$
a. $x + 2y + z = -3$
$2x + y + z = -10$

b. Subtract the first equation from the second, and subtract twice the first equation from the third.
$x + y + 2z = -15$
$0x - y + z = -12$
$0x - y - 3z = 20$
Add the second and third equation.
$x + y + 2z = -15$
$0x - y + z = -12$
$0x + 0y - 4z = 32$
From the third equation, $z = -8$.
Substitute z back into the second equation,
$-y - 8 = -12$
$-y = -12 + 8 = -4$
So $y = 4$. Now substitute y and z back into the first equation.
$x + 4 + 2(-8) = x - 12 = -15$
And so $x = -3$. Thus the solution is $(-3, 4, -8)$ as expected.

3. a. $-7 - 3(5) + 4\left(\frac{3}{4}\right) = -7 - 15 + 3 = -19$
$-7 - 8\left(\frac{3}{4}\right) = -7 - 6 = -13$
$-7 + 2(5) = 3$
Yes, $\left(-7, 5, \frac{3}{4}\right)$ is a solution.

b.
$3(-7) - 2(5) + 16\left(\frac{3}{4}\right) = -21 - 10 + 12 = -19$
$3(-7) - 2(5) = -21 - 10 = -31$
$\neq -23$
$8(-7) - 5 + 4\left(\frac{3}{4}\right) = -56 - 5 + 3 = -58$
Because the second equation fails to produce an equality, $\left(-7, 5, \frac{3}{4}\right)$ is not a solution.

4. a. Solve for y. $y = -3$
The solution is $(-2, -3)$.

b. Multiply the second equation by 6
$3x + 5y = -21$
$x - 3y = 7$
Add 3 times the first equation to 5 times the second equation.
$3x + 5y = -21$
$14x = -28$
From the second equation, $x = -2$.
Substituting x back into the first equation,
$3(-2) + 5y = -21$
$5y = -15$
So $y = -3$.
The two systems are equivalent because they have the same solution.

5. a. Add the second equation to 5 times the first equation.

We can use this to get a solution of $\left(\frac{42}{5}, 11, -2\right)$. To eliminate the fraction, we get $(42, 55, -10)$. So the vector equation is $\vec{r} = (4, 1, 12) + t(42, 55, -10)$, $t \in \mathbf{R}$.

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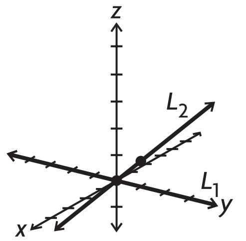

b. The only point of intersection is at the origin $(0, 0, 0)$.

c. If $p = 0$ and $q = 0$, the intersection occurs at $(0, 0, 0)$.

17. a. Represent the lines parametrically, and then substitute into the equation for the plane.
For the first equation, $x = t$, $y = 7 - 8t$, $z = 1 + 2t$. Substituting into the plane equation, $2t + 7 - 8t + 3 + 6t - 10 = 0$. Simplifying, $0t = 0$. So the line lies on the plane.
For the second line, $x = 4 + 3s$, $y = -1$, $z = 1 - 2s$ Substituting into the plane equation, $8 + 6s - 1 + 3 - 6s - 10 = 0$. Simplifying, $0s = 0$. This line also lies on the plane.

b. Compare the x and y components:
$4 + 3s = t$
$7 - 8t = -1$
From the second equation, $t = 1$. Substituting back into the first equation, $4 + 3s = 1$, or $s = -1$.
Determine the point of intersection:
$(1, 7 - 8, 1 + 2) = (1, -1, 3)$
$(4 - 3, -1, 1 + 2) = (1, -1, 3)$
The point of intersection is $(1, -1, 3)$.

18. Answers may vary. For example:
$\vec{r} = (2, 0, 0) + p(2, 0, 1)$, $p \in \mathbf{R}$

9.2 Systems of Equations, pp. 507–509

1. a. linear
b. not linear
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2. Answers may vary. For example:
$x + y + 2z = -15$
a. $x + 2y + z = -3$
$2x + y + z = -10$

b. Subtract the first equation from the second, and subtract twice the first equation from the third.
$x + y + 2z = -15$
$0x - y + z = -12$
$0x - y - 3z = 20$
Add the second and third equation.
$x + y + 2z = -15$
$0x - y + z = -12$
$0x + 0y - 4z = 32$
From the third equation, $z = -8$.
Substitute z back into the second equation,
$-y - 8 = -12$
$-y = -12 + 8 = -4$
So $y = 4$. Now substitute y and z back into the first equation.
$x + 4 + 2(-8) = x - 12 = -15$
And so $x = -3$. Thus the solution is $(-3, 4, -8)$ as expected.

3. a. $-7 - 3(5) + 4\left(\frac{3}{4}\right) = -7 - 15 + 3 = -19$
$-7 - 8\left(\frac{3}{4}\right) = -7 - 6 = -13$
$-7 + 2(5) = 3$
Yes, $\left(-7, 5, \frac{3}{4}\right)$ is a solution.

b.
$3(-7) - 2(5) + 16\left(\frac{3}{4}\right) = -21 - 10 + 12 = -19$
$3(-7) - 2(5) = -21 - 10 = -31$
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$8(-7) - 5 + 4\left(\frac{3}{4}\right) = -56 - 5 + 3 = -58$
Because the second equation fails to produce an equality, $\left(-7, 5, \frac{3}{4}\right)$ is not a solution.

4. a. Solve for y. $y = -3$
The solution is $(-2, -3)$.

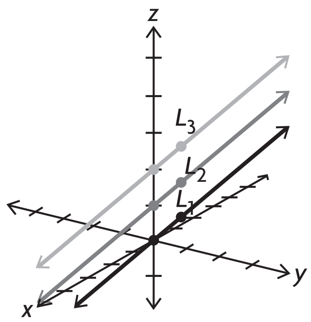
b. Multiply the second equation by 6
$3x + 5y = -21$
$x - 3y = 7$
Add 3 times the first equation to 5 times the second equation.
$3x + 5y = -21$
$14x = -28$
From the second equation, $x = -2$.
Substituting x back into the first equation,
$3(-2) + 5y = -21$
$5y = -15$
So $y = -3$.
The two systems are equivalent because they have the same solution.

5. a. Add the second equation to 5 times the first equation.

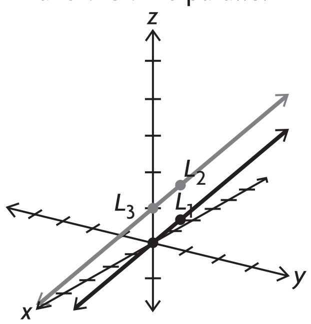
$2x - y = 11$
$11x = 66$
Solve for x in the second equation, $x = 6$. Substitute x back into the first equation
$2(6) - y = 11$
$-y = 11 - 12 = -1$
So $y = 1$
Therefore, the solution is $(6, 1)$.
b. Subtract three times the first equation from twice the second equation.
$2x + 5y = 19$
$0x - 7y = -35$
From the second equation, $y = 5$.
Substitute y back into the first equation.
$2x + 5(5) = 19$
$2x = 19 - 25 = -6$
So $x = -3$
Therefore, the solution is $(-3, 5)$.
c. Add the second equation to 3 times the first equation to the second equation
$-x + 2y = 10$
$0x + 11y = 33$
From the second equation, $y = 3$.
Substitute y back into the first equation.
$-x + 2(3) = 10$
$-x = 4$
So $x = -4$.
Therefore the solution is $(-4, 3)$.
6. a. These two lines are parallel, and therefore cannot have an intersection.
b. The second equation is five times the first, therefore the lines are coincident.
7. a. Let $x = t$. So $2t - y = 3$ then $y = 2t - 3$.
b. Let $x = t$, $y = s$. Then $t - 2s + z = 0$ and $z = 2s + t$.
8. a. If $x = t$, $y = -2t - 11$, then $y = -2x - 11$ and so $2x + y = -11$ is the required linear equation.
b. $2x + y = -11$
$2(3t + 3) + (-6t - 17) = 6t - 6t + 6 - 17$
$= -11$
9. a. The two equations will have no solutions when $k \neq 12$, since they will be parallel should this occur.
b. It is impossible to have only one solution for these two equations. They have exactly the same direction vector. They will never intersect at exactly one place.
c. The two equations will have infinitely many solutions when $k = 12$. When this occurs, the two equations are coincident.
10. a. There are infinitely many solutions to this equation. This is reason why it is represented graphically as a line.
b. Let $x = t$. So $2t + 4y = 11$, then $4y = 11 - 2t$ and $y = \frac{11}{4} - \frac{1}{2}t$, $t \in \mathbf{R}$
c. This equation will not have any integer solutions because the left hand side is an even function and the right side is an odd function.
11. a. Add the second equation to -2 times the first.
$x + 3y = a$
$0x - 3y = b - 2a$
Divide the second equation by -3 to get $y = -\frac{1}{3}b + \frac{2}{3}a$. Now substitute this back into the first equation.
$$x + 3\left(-\frac{1}{3}b + \frac{2}{3}a\right) = a$$
$$x - b + 2a = a$$
$$x = -a + b$$
b. Since they have different direction vectors, these two equations are not parallel or coincident and will intersect somewhere.
12. a. Add the third equation to the first to eliminate z.
$x + y + z = 0$
$x - y + 0z = 1$
$x + 2y + 0z = -5$
Add twice the second equation to the third equation to eliminate
Add twice the second equation to the third equation to eliminate y.
$x + y + z = 0$
$x - y + 0z = 1$
$3x + 0y + 0z = -3$
Divide the third equation by -3 to get $x = -1$.
Now substitute into the second equation.
$-1 - y = 1$
$y = -2$
Finally, substitute x and y to get
$-1 + -2 + z = 0$
So $z = 3$. Therefore, the solution is $(-1, -2, 3)$.
b. Add the first equation to -2 times the second, and add the first equation to -2 times the third.
$2x - 3y + z = 6$
$0x - 5y - 3z = -56$
$0x - y + 3z = 40$
Now add the second equation to -1 times the third.
$2x - 3y + z = 6$
$0x - 5y - 3z = -56$
$0x - 4y + 0z = -16$

From the third equation, $y = 4$.
Now substitute this into the second equation.
$-5(4) - 3z = -56$
$-3z = -36$
$z = 12$
Now substitute these two values back into the first equation.
$2x - 3(4) + 12 = 6$
$2x = 6, x = 3$
So the solution is $(3, 4, 12)$.
c. Add the second equation to -1 times the third.
$x + y + 0z = 10$
$0x + y + z = -2$
$-x + y + 0z = 2$
Add the third equation to the first equation.
$x + y + 0z = 10$
$0x + y + z = -2$
$0x + 2y + 0z = 12$
So $y = 6$. Now substitute into the other two equations.
$x + 6 = 10 \rightarrow x = 4$
$6 + z = -2 \rightarrow z = -8$
So the solution is $(4, 6, -8)$.
d. To eliminate fractions, multiply each of the equations by 60.
$20x + 15y + 12z = 840$
$15x + 12y + 20z = -1260$
$12x + 20y + 15z = 420$
Add 3 times the first equation to -4 times the second, and add 3 times the first equation to -5 times the third.
$20x + 15y + 12z = 840$
$0x - 3y - 44z = 7560$
$0x - 55y - 39z = 420$
Now add 55 times the second equation to -3 times the third equation.
$20x + 15y + 12z = 840$
$0x - 3y - 44z = 7560$
$0x + 0y - 2303z = 414\,540$
Divide the third equation through by -2303 to get $z = -180$. Substituting z back into the second equation.
$-3y - 44(-180) = 7560 \rightarrow -3y = -360$
So $y = 120$. Now substitute these two values back into the first equation.
$20x + 15(120) + 12(-180) = 840$
$20x = 840 - 1800 + 2160 = 1200$
So $x = 60$. Therefore the solution is $(60, 120, -180)$.

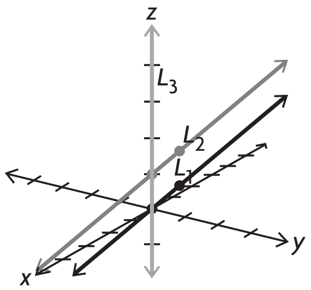
e. Note that if $2x - y = 0 \rightarrow y = 2x$, and $2z - x = 0 \rightarrow z = \frac{1}{2}x$. So we substitute these two relations into the second equation.
$2(2x) - \frac{1}{2}x = \frac{7}{2}x = 7 \rightarrow x = 2$
So now $z = 1$, $y = 4$, and the solution is $(2, 4, 1)$.
f. Add the first equation to -2 times the second equation.
$x + y + 2z = 13$
$-2x + 0y - 7z = -38$
$2x + 0y + 6z = 32$
Add the second and third equations.
$x + y + 2z = 13$
$-2x + 0y - 7z = -38$
$0x + 0y - z = -6$
So from the third equation, $z = 6$.
Substituting into the second equation,
$-2x - 42 = -38$
$-2x = 4 \rightarrow x = -2$
Finally, substituting both values into the first equation,
$-2 + y + 12 = 13 \rightarrow y = 3$.
So the final solution is $(-2, 3, 6)$.
13. Answers may vary. For example:
a. Three lines parallel



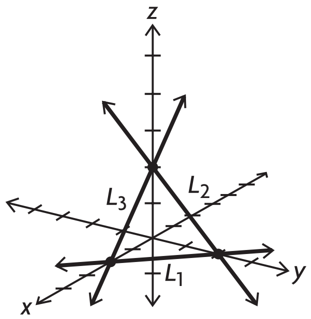
Two lines coincident and the third parallel



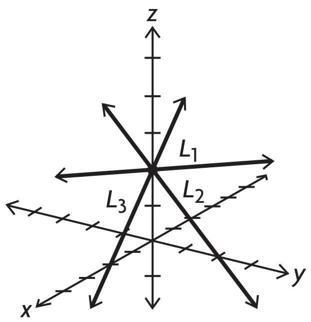
Two parallel lines cut
by the third line



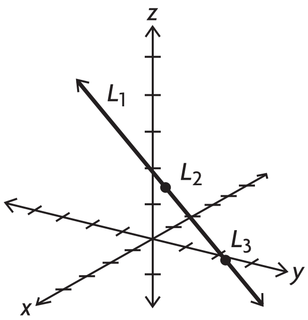
The lines form a triangle



b. Lines meet in a point



c. Three coincident lines



14. a. Add -1 times the first equation and the second equation. Add -1 times the first equation and the third equation.

$$x + y + z = a$$
$$0x + 0y - z = b - a$$
$$-x + 0y + 0z = c - a$$

So $z = a - b$, $x = a - c$. Then substitute into the first equation.

$$a - c + y + a - b = a$$
$$y = -a + b + c$$

So the final solution is $(a - c, -a + b + c, a - b)$.

15. a. For two equations to have no solutions, they must be parallel—meaning it must have a the same direction vector. So if $k = 2$, then the lines are parallel.

b. If two equations have an infinite number of solutions, then the lines must be coincident. One way to do this is if the second equation is a multiple of the first equation. To achieve this, $k = -2$.

c. For two equations to have a unique solution, they must have only one intersection. From a., we saw that if $k = 2$, the lines are parallel, and if $k = -2$, then they are coincident. Since the only other option is for the lines to have a unique solution, $k \neq \pm 2$.

9.3 The Intersection of Two Planes, pp. 516–517

1. a. This means that the two equations represent planes that are parallel and not coincident.

b. Answers may vary. For example: $x - y + z = 1$, $x - y + z = -2$

2. a. The solution to the system of equations is: $x = \frac{1}{2} + \frac{1}{2}s - t$, $y = s$, $z = t$, $s, t \in \mathbf{R}$. The two planes are coincident.

b. Answers may vary. For example: $x - y + z \Rightarrow -1$; $2x - 2y + 2z = -2$

3. a. $2z = -4 \Rightarrow z = -2$.

$$x - y + (-2) \Rightarrow -1$$
$$x - y \Rightarrow 1.$$

$x = 1 + s$, $y = s$, $z = -2$, $s \in \mathbf{R}$

The two planes intersect in a line.

b. Answers may vary. For example: $x - y + z = -1$; $x - y - z = 3$.

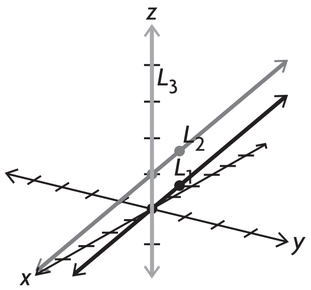
4. a. ① $2x + y + 6z = p$; ② $x + my + 3z = q$

For the planes to be coincident equation ② must be a multiple of equation ①. Since the coefficients of x and z in equation ① are twice that of the x and z coefficients in equation ② all of the coefficients and constants in equation ② must be half of the corresponding coefficients in equation ①. So:

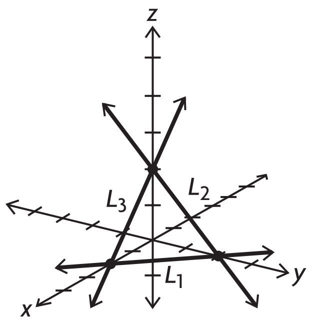
$m = \frac{1}{2}$, $p = 2q$. $q = 1$, and $p = 2$.

The value for m is unique, but p just has to be twice q and arbitrary values can be chosen.

Two parallel lines cut by the third line



The lines form a triangle



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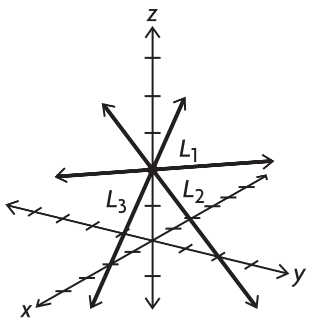
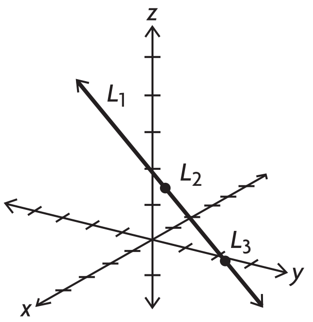


c. Three coincident lines



14. a. Add -1 times the first equation and the second equation. Add -1 times the first equation and the third equation.

$$\begin{aligned} x + y + z &= a \\ 0x + 0y - z &= b - a \\ -x + 0y + 0z &= c - a \end{aligned}$$

So $z = a - b$, $x = a - c$. Then substitute into the first equation.

$$\begin{aligned} a - c + y + a - b &= a \\ y &= -a + b + c \end{aligned}$$

So the final solution is $(a - c, -a + b + c, a - b)$.

15. a. For two equations to have no solutions, they must be parallel—meaning it must have a the same direction vector. So if $k = 2$, then the lines are parallel.

b. If two equations have an infinite number of solutions, then the lines must be coincident. One way to do this is if the second equation is a multiple of the first equation. To achieve this, $k = -2$.

c. For two equations to have a unique solution, they must have only one intersection. From a., we saw that if $k = 2$, the lines are parallel, and if $k = -2$, then they are coincident. Since the only other option is for the lines to have a unique solution, $k \neq \pm 2$.

9.3 The Intersection of Two Planes, pp. 516–517

1. a. This means that the two equations represent planes that are parallel and not coincident.

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b. Answers may vary. For example:
$x - y + z \Rightarrow -1$; $2x - 2y + 2z = -2$

3. a. $2z = -4 \Rightarrow z = -2$.

$$\begin{aligned} x - y + (-2) &\Rightarrow -1 \\ x - y &\Rightarrow 1. \end{aligned}$$

$x = 1 + s$, $y = s$, $z = -2$, $s \in \mathbf{R}$

The two planes intersect in a line.

b. Answers may vary. For example:
$x - y + z = -1$; $x - y - z = 3$.

4. a. ① $2x + y + 6z = p$; ② $x + my + 3z = q$

For the planes to be coincident equation ② must be a multiple of equation ①. Since the coefficients of x and z in equation ① are twice that of the x and z coefficients in equation ② all of the coefficients and constants in equation ② must be half of the corresponding coefficients in equation ①. So:

$m = \dfrac{1}{2}$, $p = 2q$. $q = 1$, and $p = 2$.

The value for m is unique, but p just has to be twice q and arbitrary values can be chosen.

b. For parallel planes all of the coefficients of the variables must be multiples of each other, but the constant terms must differ by a different constant. So a possible solution is:
$m = \frac{1}{2}$, $q = 1$, and $p = 3$.
The value for m is again unique but p and q can be arbitrarily chosen as long as $p \neq 2q$.
c. For the two planes to intersect at right angles the two normal vectors, $\vec{n_1} = (2, 1, 6)$ and $\vec{n_2} = (1, m, 3)$, must satisfy:
$\vec{n_1} \cdot \vec{n_2} = 0.$
$\vec{n_1} \cdot \vec{n_2} = 2 + m + 18 = 0$
$m = -20$. This value is unique, since only one value was found to satisfy the given conditions.
d. From **c.** we know that in order to intersect in right angles $m = -20$. Choose $p = 1$, $q = 1$. The value for m is unique from the solution to **c.**, but the values for p and q can be arbitrary since the only value which can change the angle between the planes is m.
5. a. Letting $z = s$:
$$y = -3s.$$
$$x + 2(-3s) - 3s = 0.$$
$$x = 9s$$
The solution is:
$x = 9s, y = -3s, z = s, s \in \mathbf{R}$
b. Letting $y = t$.
$$t + 3z = 0$$
$$3z = -t$$
$$z = -\frac{1}{3}t.$$
$$x + 2t - 3\left(-\frac{1}{3}\right)t = 0$$
$$x + 3t = 0$$
$$x = -3t.$$
The solution is:
$x = -3t, y = t, z = -\frac{1}{3}t; t \in \mathbf{R}.$
c. Since t is an arbitrary real number we can express t as:
$t = -3s; s \in \mathbf{R}.$
Substituting this into the solution for **b.** shows that the two solutions are equivalent.
6. a. Equation ② is twice that of equation ①, so they represent intersecting coincident planes.
b. The coefficients of each variable are the same, but the constant terms are different, so the equations represent non-intersecting parallel planes.
c. The coefficients of the x and z variables are the same but the y coefficients are different. So the equations represent planes that intersect in a line.
d. The coefficients of each variable from equation ① to ② are not the same multiple. Therefore the equations represent planes that intersect in a line.
e. The intersection is a line by the same reasoning as **d.**
f. The intersection is a line by the same reasoning as **d.**
7. a. $x = 1 - s - t, y = s, z = t, s, t \in \mathbf{R}$
b. There is no solution since the planes are parallel.
c. ① − ②:
$$-2y = 4$$
$$y = -2.$$
$$x - 2 + 2z = -2$$
$$x + 2z = 0$$
$$x = -2z.$$
$x = -2s, y = -2, z = s, s \in \mathbf{R}.$
d. Let $z = s; s \in \mathbf{R}.$
From ②:
$$x = y + 6.$$
$$(y + 6) + y + 2s = 4$$
$$2y + 2s = -2$$
$$y = -s - 1.$$
$x = -s + 5, y = -s - 1, z = s, s \in \mathbf{R}.$
e. $-2 \cdot$ ②: $2x - 4y - 2z = -2$
Adding ①:
$4x - 5y = 0.$
$$x = \frac{5}{4}y.$$
Let $y = s, s \in \mathbf{R}.$
$$2\left(\frac{5}{4}s\right) - s + 2z = 2$$
$$\frac{3}{2}s + 2z = 2$$
$$z = 1 - \frac{3}{4}s.$$
$x = \frac{5}{4}s, y = s, z = 1 - \frac{3}{4}s, s \in \mathbf{R}$
f. $x - y + 2(4) = 0$
$$x = y - 8.$$
$x = s - 8, y = s, z = 4, s \in \mathbf{R}.$
8. a. The system will have an infinite number of solutions for any value of k. When $k = 2$ equation ② will be twice that of ① so the solution is a plane:
$x = 1 - s - 2t, y = s, z = t, s, t \in \mathbf{R}.$
For any other value of k the solution will be a line.
For example $k = 0$:
$$2y = -4z$$
$$y = -2z.$$

$x + (-2z) + 2z = 1$
$x = 1.$
$x = 1, y = -2s, z = s, s \in \mathbf{R}.$
b. No there is no value of k for which the system will not have a solution. The only time when there is no solution is when the corresponding coefficients for each variable differ by a common multiple between equations, and the constant terms differ by a different multiple. The only way the first condition is satisfied is when $k = 2$, but when this happens the constant terms differ by the same factor as the variables, namely 2.
9. The line of intersection of the two planes:
π_1: $2x - y + z = 0$, π_2: $y + 4z = 0$ is:
$y = -4z$
$2x - (-4z) + z = 0$
$2x = -5z$
$x = -\frac{5}{2}z.$
$x = -\frac{5}{2}s, y = -4s, z = s, s \in \mathbf{R}.$
The direction vector is $\left(-\frac{5}{2}, -4, 1\right)$ or $(-5, -8, 2)$.
$\vec{r_1} = s(-5, -8, 2), s \in \mathbf{R}$. Since the line we are looking for is parallel to this line, we know that the direction vector must be the same. The line passes through $(-2, 3, 6)$ and has direction vector $(-5, -8, 2)$. The equation of the line is
$\vec{r_2} = (-2, 3, 6) + s(-5, -8, 2), s \in \mathbf{R}.$
10. The line of intersection of the two planes, $2x - y + 2z = 0$ and $2x + y + 6z = 4$ is:
$4x + 8z = 4$
$x = 1 - 2z.$
$2(1 - 2z) - y + 2z = 0$
$2 - y - 2z = 0$
$y = 2 - 2z.$
$x = 1 - 2s, y = 2 - 2s, z = s, s \in \mathbf{R}.$
In order for the a line to be contained in the plane we need to check that the values for x, y, and z always satisfy the plane equation:
$5x + 3y + 16z - 11 = 0.$
$5(1 - 2s) + 3(2 - 2s) + 16(s) - 11 = 0$
$5 + 6 - 11 - 10s - 6s + 16s = 0$
$0 = 0$. Since this is true the line is contained in the plane.
11. a. π_1: $2x + y - 3z = 3$, π_2: $x - 2y + z = -1.$
$\pi_1 - 2\pi_2$: $5y - 5z = 5$
$y = 1 + z.$
$2x + (1 + z) - 3z = 3$
$2x - 2z = 2$
$x = 1 + z.$
$x = 1 + s, y = 1 + s, z = s, s \in \mathbf{R}.$

b. L meets the xy-plane when $z = 0$.
$x = 1, y = 1. A = (1, 1, 0).$
L meets the z-axis when both x and y are zero:
$s = -1.$
$z = -1.$
$B = (0, 0, -1)$
The length of AB is therefore:
$\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ or about 1.73.
12. The line with equation $x = -2y = 3z$ has parametric equations: $x = s, y = -\frac{1}{2}s, z = \frac{1}{3}s, s \in \mathbf{R}$.
This has the equivalent vector form:
$\vec{r} = s\left(1, -\frac{1}{2}, \frac{1}{3}\right), s \in \mathbf{R}.$
The line of intersection of the two planes
$x - y + z = 1$ and $2y - z = 0$ is:
$y = \frac{1}{2}z$
$x - \frac{1}{2}z + z = 1$
$x = 1 - \frac{1}{2}z.$
$x = 1 - \frac{1}{2}t, y = \frac{1}{2}t, z = t, t \in \mathbf{R}$. Which has a vector equation of:
$\vec{r} = (1, 0, 0) + t(-\frac{1}{2}, \frac{1}{2}, 1), t \in \mathbf{R}$. The vector equation of the plane with the given properties is thus:
$\vec{r} = (1, 0, 0) + t\left(-\frac{1}{2}, \frac{1}{2}, 1\right) + s\left(1, -\frac{1}{2}, \frac{1}{3}\right), s, t \in \mathbf{R}.$
The normal vector for the plane is then:
$\left(-\frac{1}{2}, \frac{1}{2}, 1\right) \times \left(1, -\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2} \cdot \frac{1}{3}\right) - \left(1 \cdot -\frac{1}{2}\right),$
$1 \cdot 1 - \left(-\frac{1}{2} \cdot \frac{1}{3}\right), -\frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2} \cdot 1 = \left(\frac{2}{3}, \frac{7}{6}, -\frac{1}{4}\right).$
Or equivalently $(8, 14, -3)$.
The Cartesian equation is then:
$8x + 14y - 3z + D = 0$, and must contain the point $(1, 0, 0)$.
$8(1) + D = 0.$
$D = -8.$
$8x + 14y - 3z - 8 = 0.$

Mid-Chapter Review, pp. 518–519

1. a. $\vec{r} = (4, -3, 15) + t(2, -3, 5), t \in \mathbf{R}$
$x = 4 + 2t, y = -3 - 3t, z = 15 + 5t$
$0 = 15 + 5t$
$t = -3$

$x + (-2z) + 2z = 1$
$x = 1.$
$x = 1, y = -2s, z = s, s \in \mathbf{R}.$
b. No there is no value of k for which the system will not have a solution. The only time when there is no solution is when the corresponding coefficients for each variable differ by a common multiple between equations, and the constant terms differ by a different multiple. The only way the first condition is satisfied is when $k = 2$, but when this happens the constant terms differ by the same factor as the variables, namely 2.
9. The line of intersection of the two planes:
π_1: $2x - y + z = 0$, π_2: $y + 4z = 0$ is:
$y = -4z$
$2x - (-4z) + z = 0$
$2x = -5z$
$x = -\frac{5}{2}z.$
$x = -\frac{5}{2}s, y = -4s, z = s, s \in \mathbf{R}.$
The direction vector is $\left(-\frac{5}{2}, -4, 1\right)$ or $(-5, -8, 2)$.
$\vec{r_1} = s(-5, -8, 2), s \in \mathbf{R}$. Since the line we are looking for is parallel to this line, we know that the direction vector must be the same. The line passes through $(-2, 3, 6)$ and has direction vector $(-5, -8, 2)$. The equation of the line is
$\vec{r_2} = (-2, 3, 6) + s(-5, -8, 2), s \in \mathbf{R}.$
10. The line of intersection of the two planes, $2x - y + 2z = 0$ and $2x + y + 6z = 4$ is:
$4x + 8z = 4$
$x = 1 - 2z.$
$2(1 - 2z) - y + 2z = 0$
$2 - y - 2z = 0$
$y = 2 - 2z.$
$x = 1 - 2s, y = 2 - 2s, z = s, s \in \mathbf{R}.$
In order for the a line to be contained in the plane we need to check that the values for x, y, and z always satisfy the plane equation:
$5x + 3y + 16z - 11 = 0.$
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$5 + 6 - 11 - 10s - 6s + 16s = 0$
$0 = 0$. Since this is true the line is contained in the plane.
11. a. π_1: $2x + y - 3z = 3$, π_2: $x - 2y + z = -1$.
$\pi_1 - 2\pi_2$: $5y - 5z = 5$
$y = 1 + z.$
$2x + (1 + z) - 3z = 3$
$2x - 2z = 2$
$x = 1 + z.$
$x = 1 + s, y = 1 + s, z = s, s \in \mathbf{R}.$

b. L meets the xy-plane when $z = 0$.
$x = 1, y = 1. A = (1, 1, 0).$
L meets the z-axis when both x and y are zero:
$s = -1.$
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$B = (0, 0, -1)$
The length of AB is therefore:
$\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ or about 1.73.
12. The line with equation $x = -2y = 3z$ has parametric equations: $x = s, y = -\frac{1}{2}s, z = \frac{1}{3}s, s \in \mathbf{R}$.
This has the equivalent vector form:
$\vec{r} = s\left(1, -\frac{1}{2}, \frac{1}{3}\right), s \in \mathbf{R}.$
The line of intersection of the two planes
$x - y + z = 1$ and $2y - z = 0$ is:
$y = \frac{1}{2}z$
$x - \frac{1}{2}z + z = 1$
$x = 1 - \frac{1}{2}z.$
$x = 1 - \frac{1}{2}t, y = \frac{1}{2}t, z = t, t \in \mathbf{R}$. Which has a vector equation of:
$\vec{r} = (1, 0, 0) + t\left(-\frac{1}{2}, \frac{1}{2}, 1\right), t \in \mathbf{R}$. The vector equation of the plane with the given properties is thus:
$\vec{r} = (1, 0, 0) + t\left(-\frac{1}{2}, \frac{1}{2}, 1\right) + s\left(1, -\frac{1}{2}, \frac{1}{3}\right), s, t \in \mathbf{R}.$
The normal vector for the plane is then:
$\left(-\frac{1}{2}, \frac{1}{2}, 1\right) \times \left(1, -\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2} \cdot \frac{1}{3}\right) - \left(1 \cdot -\frac{1}{2}\right),$
$1 \cdot 1 - \left(-\frac{1}{2} \cdot \frac{1}{3}\right), -\frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2} \cdot 1 = \left(\frac{2}{3}, \frac{7}{6}, -\frac{1}{4}\right).$
Or equivalently $(8, 14, -3)$.
The Cartesian equation is then:
$8x + 14y - 3z + D = 0$, and must contain the point $(1, 0, 0)$.
$8(1) + D = 0.$
$D = -8.$
$8x + 14y - 3z - 8 = 0.$

Mid-Chapter Review, pp. 518–519

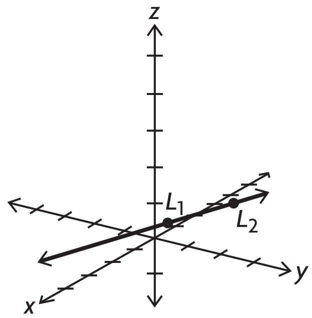
1. a. $\vec{r} = (4, -3, 15) + t(2, -3, 5), t \in \mathbf{R}$
$x = 4 + 2t, y = -3 - 3t, z = 15 + 5t$
$0 = 15 + 5t$
$t = -3$

$x = 4 + 2(-3), y = -3 - 3(-3),$
$z = 15 + 5(-3)$
$x = -2, y = 6, z = 0$
$(-2, 6, 0)$
b. $\vec{r} = (4, -3, 15) + t(2, -3, 5), t \in \mathbf{R}$
$x = 4 + 2t, y = -3 - 3t, z = 15 + 5t$
$0 = -3 - 3t$
$t = -1$
$x = 4 + 2(-1), y = -3 - 3(-1),$
$z = 15 + 5(-1)$
$x = 2, y = 0, z = 10$
$(2, 0, 10)$
c. $\vec{r} = (4, -3, 15) + t(2, -3, 5), t \in \mathbf{R}$
$x = 4 + 2t, y = -3 - 3t, z = 15 + 5t$
$0 = 4 + 2t$
$t = -2$
$x = 4 + 2(-2), y = -3 - 3(-2),$
$z = 15 + 5(-2)$
$x = 0, y = 3, z = 5$
$(0, 3, 5)$
2. a.–e. Answers may vary. For example:
$A(2, 1, 3), B(3, -2, 5), C(-8, -5, 7)$
$a = (-2.5, -3.5, 6)$
$b = (-3, -2, 5)$
$c = (2.5, -0.5, 4)$
$m_1 = (Aa) = (-4.5, -4.5, 3) = (3, 3, -2)$
$m_2 = (Bb) = (-6, 0, 0) = (1, 0, 0)$
$m_3 = (Cc) = (10.5, 4.5, -3) = (7, 3, -2)$
Then substitute in the point and the direction vector to find the equation of the line.
$A(2, 1, 3), B(3, -2, 5), C(-8, -5, 7)$
$m_1 = (Aa) = (-4.5, -4.5, 3) = (3, 3, -2)$
$m_2 = (Bb) = (-6, 0, 0) = (1, 0, 0)$
$m_3 = (Cc) = (10.5, 4.5, -3) = (7, 3, -2)$
$A: \vec{r} = (2, 1, 3) + t(3, 3, -2), t \in \mathbf{R}$
$x = 2 + 3t, y = 1 + 3t, z = 3 - 2t, t \in \mathbf{R}$
$B: \vec{r} = (3, -2, 5) + t(1, 0, 0), t \in \mathbf{R}$
$x = 3 + t, y = -2, z = 5, t \in \mathbf{R}$
$C: \vec{r} = (-8, -5, 7) + t(7, 3, -2), t \in \mathbf{R}$
$x = -8 + 7t, y = -5 + 3t, z = 7 - 2t, t \in \mathbf{R}$
$A: x = 2 + 3t, y = 1 + 3t, z = 3 - 2t, t \in \mathbf{R}$
$B: x = 3 + t, y = -2, z = 5, t \in \mathbf{R}$
$C: x = -8 + 7t, y = -5 + 3t, z = 7 - 2t, t \in \mathbf{R}$
$y = -2 = 1 + 3t$
$t = -1$
$x = 2 + 3(-1), y = 1 + 3(-1),$
$z = 3 - 2(-1)$
$x = -1, y = -2, z = 5$
$(-1, -2, 5)$
$A: x = 2 + 3t, y = 1 + 3t, z = 3 - 2t, t \in \mathbf{R}$
$B: x = 3 + t, y = -2, z = 5, t \in \mathbf{R}$
$C: x = -8 + 7t, y = -5 + 3t, z = 7 - 2t, t \in \mathbf{R}$
$y = -2 = -5 + 3t$
$t = 1$
$x = -8 + 7(1), y = -5 + 3(1), z = 7 - 2(1)$
$x = -1, y = -2, z = 5$
$(-1, -2, 5)$
The three medians meet at $(-1, -2, 5)$.
3. a. $L_1: 5x + y + 2z + 15 = 0$
$L_2: 4x + y + 2z + 8 = 0$
$L_1 - L_2: x + 7 = 0$
So $x = -7$.
$L_1: y + 2z - 20 = 0$
$L_2: y + 2z - 20 = 0$
$z = t,$
$y + 2(t) - 20 = 0$
$y = 20 - 2t$
$\vec{r} = (-7, 20, 0) + t(0, -2, 1), t \in \mathbf{R}$
b. $L_1: 4x + 3y + 3z - 2 = 0$
$L_2: 5x + 2y + 3z + 5 = 0$
$2L_1 - 3L_2: -7x - 3z - 19 = 0$
$z = 7t,$
$-7x - 3(7t) - 19 = 0,$
$x = -3t - \frac{19}{7}$
$4\left(-3t - \frac{19}{7}\right) + 3y + 3(7t) - 2 = 0$
$y = -3t + \frac{30}{7}$
$\vec{r} = \left(-\frac{19}{7}, \frac{30}{7}, 0\right) + t(3, 3, -7), t \in \mathbf{R}$
c. $L_1: \vec{r} = (-7, 20, 0) + t(0, -2, 1), t \in \mathbf{R}$
$L_2: \vec{r} = \left(-\frac{19}{7}, \frac{30}{7}, 0\right) + t(3, 3, -7), t \in \mathbf{R}$
$L_1: x = -7, y = 20 - 2t, z = t$
$L_2: x = -\frac{19}{7} + 3t, y = \frac{30}{7} + 3t, z = -7t$
$-\frac{19}{7} + 3t = -7, t = -\frac{30}{21}$
$x = -\frac{19}{7} + 3\left(-\frac{30}{21}\right), y = \frac{30}{7} + 3\left(-\frac{30}{21}\right),$
$z = -7\left(-\frac{30}{21}\right)$
$x = -7, y = 0, z = 10$
$(-7, 0, 10)$
4. a. $\pi_1: 3x + y + 7z + 3 = 0$
$\pi_2: x - 13y - 3z - 38 = 0$

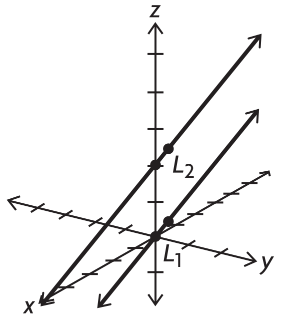
$13\pi_1 + \pi_2$: $40x + 88z + 1 = 0$
$z = t,$
$40x + 88(t) + 1 = 0$
$x = -\frac{11t}{5} - \frac{1}{40}$
$3\left(-\frac{11t}{5} - \frac{1}{40}\right) + y + 7(t) + 3 = 0$
$y = -\frac{2t}{5} - \frac{117}{40}$
$x = -\frac{11t}{5} - \frac{1}{40}, y = -\frac{2t}{5} - \frac{117}{40}, z = t, t \in \mathbf{R}$
b. π_1: $x - 3y + z + 11 = 0$
π_2: $6x - 13y + 8z - 28 = 0$
$-6\pi_1 + \pi_2$: $5y + 2z - 94 = 0$
$z = s,$
$5y + 2(s) - 94 = 0$
$y = -\frac{2}{5}s + \frac{94}{5}$
$x - 3\left(-\frac{2}{5}s + \frac{94}{5}\right) + (s) + 11 = 0$
$x = -\frac{11}{5}s + \frac{227}{5}$
$x = -\frac{11}{5}s + \frac{227}{5}, y = -\frac{2}{5}s + \frac{94}{5}, z = s, s \in \mathbf{R}$
c. The lines found in 4. a. and 4. b. do not intersect, because they are in parallel planes.
5. a. For there to be no solution the lines must be inconsistent with each other.
L_1: $x + ay = 9$
L_2: $ax + 9y = -27$
$\frac{1}{a} = \frac{a}{9}$
$a = \pm 3$
For $a = 3$:
L_1: $x + 3y = 9$
L_2: $3x + 9y = -27$
For $a = -3$, the equations are equivalent.
So there is no solution when $a = 3$.
b. To have an infinite number of solutions, the lines must be proportional.
L_1: $x + ay = 9$
L_2: $ax + 9y = -27$
$-3(x + ay = 9) = -3x - 3ay = -27$
L_1: $-3x - 3ay = -27$
L_2: $ax + 9y = -27$
$a = -3$
c. The system has one solution when $a \neq 3$ or $a \neq -3$, because other values lead to an infinite number of solutions or no solution.

6. L_1: $\frac{x - 11}{2} = \frac{y - 4}{-4} = \frac{z - 27}{5} = s$
L_2: $x = 0, y = 1 - 3t, z = 3 + 2t, t \in \mathbf{R}$
L_1: $x = 2s + 11, y = -4s + 4, z = 27 + 5s$
$x = 0 = 2s + 11,$
$s = -5.5$
$y = -4(-5.5) + 4, z = 27 + 5(-5.5)$
$x = 0, y = 26, z = -0.5$
$y = 26 = 1 - 3t, t = -\frac{25}{3}$
$z = -0.5 = 3 + 2t, t = -\frac{7}{4}$
Since there is no t-value that satisfies the equations, there is no intersection, and these lines are skew.
7. a. L_1: $\frac{x - 5}{2} = y - 2 = \frac{z + 4}{-3} = s$
L_2: $(x - 3, y - 20, z - 7) = t(2, -4, 5), t \in \mathbf{R}$
L_1: $x = 2s + 5, y = s + 2, z = -3s - 4$
L_2: $x = 2t + 3, y = -4t + 20, z = 5t + 7$
$x = 2t + 3 = 2s + 5$
$y = s + 2 = -4t + 20$
$z = -3s - 4 = 5t + 7$
L_3: $2t - 2s - 2 = 0$
L_4: $4t + s - 18 = 0$
L_5: $5t + 3s + 11 = 0$
$L_3 + 2L_4$: $10t - 38 = 0, t = 3.8$
$3L_3 + 2L_5$: $16t + 16 = 0, t = -1$
b. Since there is no t-value that satisfies the equations, there is no intersection, and these lines are skew.
8. L_1: $x = 1 + 2s, y = 4 - s, z = -3s, s \in \mathbf{R}$
L_2: $x = -3, y = t + 3, z = 2t, t \in \mathbf{R}$
$x = -3 = 1 + 2s$
$s = -2$
$x = -3, y = 6, z = 6$
$(-3, 6, 6)$
9. a. L_1: $\vec{r} = (5, 1, 7) + s(2, 0, 5), s \in \mathbf{R}$
L_2: $\vec{r} = (-1, -1, 3) + t(4, 2, -1), t \in \mathbf{R}$
L_1: $x = 5 + 2s, y = 1, z = 7 + 5s$
L_2: $x = -1 + 4t, y = -1 + 2t, z = 3 - t$
$y = 1 = -1 + 2t,$
$t = 1$
$x = -1 + 4(1), y = -1 + 2(1),$
$z = 3 - (1)$
$x = 3, y = 1, z = 2$
$(3, 1, 2)$
b. L_1: $\vec{r} = (2, -1, 3) + s(5, -1, 6), s \in \mathbf{R}$
L_2: $\vec{r} = (-8, 1, -9) + t(5, -1, 6), t \in \mathbf{R}$
These lines are the same, so either one of these lines can be used as their intersection.

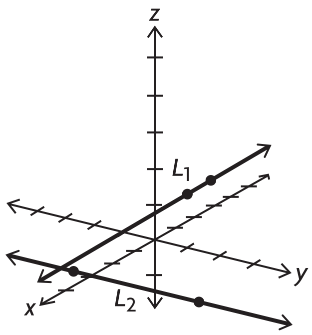
10. a. Answers may vary. For example:
i. coincident



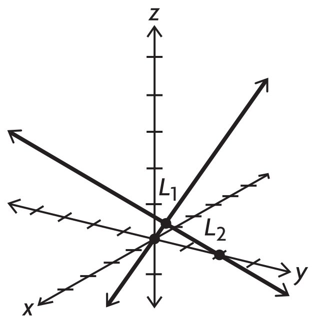
ii. parallel and distinct



iii. skew



iv. intersect in a point



b. i. When lines are the same, they are a multiple of each other.
ii. When lines are parallel, one equation is a multiple of the other equation, except for the constant term.
iii. When lines are skew, there are no common solutions to make each equation consistent.
iv. When the solution meets in a point, there is only one unique solution for the system.

11. a. A line and plane have an infinite number of points of intersection when the line lies in the plane.
b. Answers may vary. For example:
$\vec{r} = t(3, -5, -3), t \in \mathbf{R}$
$\vec{r} = t(3, -5, -3) + s(1, 1, 1), t, s \in \mathbf{R}$

12. a. ① $2x + 3y = 30$
② $x - 2y = -13$
Equation ① − (2 × equation ②): $7y = 56$
$y = 8$
$2x + 24 = 30$
$x = 3$
$(3, 8)$
b. ① $x + 4y - 3z + 6 = 0$
② $2x + 8y - 6z + 11 = 0$
There is no solution to this system, because the planes are parallel, but one plane lies above the other.
c. ① $x - 3y - 2z = -9$
② $2x - 5y + z = 3$
③ $-3x + 6y + 2z = 8$
Equation ① + (2 × equation ②): $5x - 13y = -3$
Equation ② + (equation ③): $-2x + 3y = -1$

$$\begin{array}{r} 2(5x - 13y = -3) \\ +\,5(-2x + 3y = -1) \\ \hline -11y = -11 \end{array}$$

$y = 1$
$5x - 13(1) = -3$
$x = 2$
$(2) - 3(1) - 2z = -9$
$z = 4$
$(2, 1, 4)$

13. a. The two lines intersect at a point.
b. The two planes are parallel and do not meet.
c. The three planes intersect at a point.

14. a. $L{:}(x - y = 1) + (y + z = -3)$
$= x + z = -2$

$L_1{:}\ y - z = 0, x = -\dfrac{1}{2}$

$x + z = -2$

$\left(-\dfrac{1}{2}\right) + z = -2$

$z = -\dfrac{3}{2}$

$y - z = 0$

$$y - \left(-\frac{3}{2}\right) = 0$$
$$y = -\frac{3}{2}$$
$$\left(-\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}\right)$$

b. $\cos\theta = \dfrac{|n \cdot n_1|}{|n||n_1|}$
$$n = (1, 1, -1)$$
$$n_1 = (0, 1, 1)$$
$$\cos\theta = \frac{0}{|\sqrt{3}||\sqrt{2}|}$$
$$\theta = 90°$$

c. $(0, 1, 1) \times (1, 1, -1) = (-2, 1, -1)$
$$= (2, -1, 1)$$
$$Ax + By + Cz + D = 0$$
$$2x - y + z + D = 0$$
$$2\left(\frac{-1}{2}\right) - \left(\frac{-3}{2}\right) + \left(\frac{-3}{2}\right) + D = 0$$
$$D = 1$$
$$2x - y + z + 1 = 0$$

9.4 The Intersection of Three Planes, pp. 531–533

1. a. ① $x - 3y + z = 2$
② $0x + y - z = -1$
③ $0x + 0y + 3z = -12$

The system can be solved by first solving equation ③ for z. Thus,
$3z = -12$
$z = -4$

If we use the method of back substitution, we can substitute $z = -4$ into equation ② and solve for y.
$y - (-4) = -1$
$y = -5$

If we substitute $y = -5$ and $z = -4$ into equation ① we obtain the value of x.
$x - 3(-5) - 4 = 2$ or $x = -9$

The three planes intersect at the point with coordinates $(-9, -5, -4)$

Check:
Substituting into equation ①:
$x - 3y + z = -9 + 15 - 4 = 2$
Substituting into equation ②:
$0x + y - z = -5 + 4 = -1$
Substituting into equation ③: $0x + 0y + 3z = -12$

b. This solution is the point at which all three planes meet.

2. a. ① $x - y + z = 4$
② $0x + 0y + 0z = 0$
③ $0x + 0y + 0z = 0$

The answer may vary depending upon the constant you multiply the equations by. For example,
$2 \times (x - y + z = 4) = 2x - 2y + 2z = 8$
$3 \times (x - y + z = 4) = 3x - 3y + 3z = 12$
$3x - 3y + 3z = 12$ and $2x - 2y + 2z = 8$ are equations that could work.

b. These three planes are intersecting in one single plane, because all three equations can be changed into one equivalent equation. They are coincident planes.

c. Setting $x = t$ and $y = s$ leads to
$t - s + z = 4$ or $z = s - t + 4,\ s, t \in \mathbf{R}$

d. Setting $y = t$ and $z = s$ leads to
$x - t + s = 4$ or $x = t - s + 4,\ s, t \in \mathbf{R}$

3. a. ① $2x - y + 3z = -2$
② $x - y + 4z = 3$
③ $0x + 0y + 0z = 1$

The answer may vary depending upon the constants and equations you use to determine your answer. For example,
Equation ① + equation ② + equation ③ =
$$\begin{array}{r} (2x - y + 3z = -2) \\ +\ (x - y + 4z = 3) \\ +\ (0x + 0y + 0z = 1) \\ \hline 3x - 2y + 7z = 2 \end{array}$$
or
2 × equation ② − equation ③ =
$$\begin{array}{r} (2x - 2y + 8z = 6) \\ -\ (0x + 0y + 0z = 1) \\ \hline 2x - 2y + 8z = 5 \end{array}$$
$2x - y + 3z = -2$, $x - y + 4z = 3$, and $3x - 2y + 7z = 2$ is one system of equations that could produce the original system composed of equations ①, ②, and ③.
$2x - y + 3z = -2$, $x - y + 4z = 3$, and $2x - 2y + 8z = 5$ is another system of equations that could produce the original system composed of equations ①, ②, and ③.

b. The systems have no solutions.

4. a. ① $x + 2y - z = 4$
② $x + 0y - 2z = 0$
③ $2x + 0y + 0z = -6$

The system can be solved by first solving equation ③ for x. So,

$$y - \left(-\frac{3}{2}\right) = 0$$
$$y = -\frac{3}{2}$$
$$\left(-\frac{1}{2}, -\frac{3}{2}, -\frac{3}{2}\right)$$
b. $\cos\theta = \dfrac{|n \cdot n_1|}{|n||n_1|}$
$$n = (1, 1, -1)$$
$$n_1 = (0, 1, 1)$$
$$\cos\theta = \frac{0}{|\sqrt{3}||\sqrt{2}|}$$
$$\theta = 90°$$
c. $(0, 1, 1) \times (1, 1, -1) = (-2, 1, -1)$
$$= (2, -1, 1)$$
$$Ax + By + Cz + D = 0$$
$$2x - y + z + D = 0$$
$$2\left(\frac{-1}{2}\right) - \left(\frac{-3}{2}\right) + \left(\frac{-3}{2}\right) + D = 0$$
$$D = 1$$
$$2x - y + z + 1 = 0$$

9.4 The Intersection of Three Planes, pp. 531–533

1. a. ① $x - 3y + z = 2$
② $0x + y - z = -1$
③ $0x + 0y + 3z = -12$
The system can be solved by first solving equation ③ for z. Thus,
$3z = -12$
$z = -4$
If we use the method of back substitution, we can substitute $z = -4$ into equation ② and solve for y.
$y - (-4) = -1$
$y = -5$
If we substitute $y = -5$ and $z = -4$ into equation ① we obtain the value of x.
$x - 3(-5) - 4 = 2$ or $x = -9$
The three planes intersect at the point with coordinates $(-9, -5, -4)$
Check:
Substituting into equation ①:
$x - 3y + z = -9 + 15 - 4 = 2$
Substituting into equation ②:
$0x + y - z = -5 + 4 = -1$
Substituting into equation ③: $0x + 0y + 3z = -12$
b. This solution is the point at which all three planes meet.
2. a. ① $x - y + z = 4$
② $0x + 0y + 0z = 0$
③ $0x + 0y + 0z = 0$
The answer may vary depending upon the constant you multiply the equations by. For example,
$2 \times (x - y + z = 4) = 2x - 2y + 2z = 8$
$3 \times (x - y + z = 4) = 3x - 3y + 3z = 12$
$3x - 3y + 3z = 12$ and $2x - 2y + 2z = 8$ are equations that could work.
b. These three planes are intersecting in one single plane, because all three equations can be changed into one equivalent equation. They are coincident planes.
c. Setting $x = t$ and $y = s$ leads to
$t - s + z = 4$ or $z = s - t + 4, s, t \in \mathbf{R}$
d. Setting $y = t$ and $z = s$ leads to
$x - t + s = 4$ or $x = t - s + 4, s, t \in \mathbf{R}$
3. a. ① $2x - y + 3z = -2$
② $x - y + 4z = 3$
③ $0x + 0y + 0z = 1$
The answer may vary depending upon the constants and equations you use to determine your answer. For example,
Equation ① + equation ② + equation ③ =
$$\begin{array}{r} (2x - y + 3z = -2) \\ +\ (x - y + 4z = 3) \\ +\ (0x + 0y + 0z = 1) \\ \hline 3x - 2y + 7z = 2 \end{array}$$
or
2 × equation ② − equation ③ =
$$\begin{array}{r} (2x - 2y + 8z = 6) \\ -\ (0x + 0y + 0z = 1) \\ \hline 2x - 2y + 8z = 5 \end{array}$$
$2x - y + 3z = -2$, $x - y + 4z = 3$, and $3x - 2y + 7z = 2$ is one system of equations that could produce the original system composed of equations ①, ②, and ③.
$2x - y + 3z = -2$, $x - y + 4z = 3$, and $2x - 2y + 8z = 5$ is another system of equations that could produce the original system composed of equations ①, ②, and ③.
b. The systems have no solutions.
4. a. ① $x + 2y - z = 4$
② $x + 0y - 2z = 0$
③ $2x + 0y + 0z = -6$
The system can be solved by first solving equation ③ for x. So,

$2x = -6$
$x = -3$
If we use the method of back substitution, we can substitute $x = -3$ into equation ② and solve for z.
$-3 - 2z = 0$
$$z = -\frac{3}{2}$$
If we substitute $x = -3$ and $z = -\frac{3}{2}$ into equation ① we obtain the value of y.
$-3 + 2y + \frac{3}{2} = 4$ or $y = \frac{11}{4}$
The equations intersect at the point with coordinates $(-3, \frac{11}{4}, -\frac{3}{2})$
Check:
Substituting into equation ①:
$x + 2y - z = -3 + \frac{22}{4} + \frac{3}{2} = 4$
Substituting into equation ②:
$x + 0y - 2z = -3 + 3 = 0$
Substituting into equation ③: $2x + 0y + 0z = -6$
b. This solution is the point at which all three planes meet.
5. a. ① $2x - y + z = 1$
② $x + y - z = -1$
③ $-3x - 3y + 3z = 3$
Since equation ③ = −equation ②, equation ② and equation ③ are consistent or lie in the same plane. Equation ① meets this plane in a line.
b. Adding equation ② and equation ① creates an equivalent equation, $3x = 0$ or $x = 0$. Substituting $x = 0$ into equation ① and equation ② gives equation ④ $z - y = 1$ and equation ⑤ $y - z = -1$. Equations ④ and ⑤ indicate the problem has infinite solutions. Substituting $y = t$ into equation ④ or ⑤ leads to
$x = 0$, $y = t$, and $z = 1 + t$, $t \in \mathbf{R}$
Check:
$$2(0) - s + (s + 1) = 1$$
$$0 + s - (s + 1) = -1$$
$$-3(0) - 3(s) + 3(s + 1) = 3$$
6. ① $2x + 3y - 4z = -5$
② $x - y + 3z = -201$
③ $5x - 5y + 15z = -1004$
There is no solution to this system of equations, because if you multiply equation ② by 5 you obtain a new equation, $5x - 5y + 15z = -1005$, which is inconsistent with equation ③.
7. a. Yes when this equation is alone, this is true, because any constants can be substituted into the variables in the equation $0x + 0y + 0z = 0$ and the equation will always be consistent.
b. Answers may vary. For example: To obtain a no solution and an equation with $0x + 0y + 0z = 0$, you must have two equal planes and one parallel distinct plane. For example one solution is:
$x + y + z = 2$
$2x + 2y + 2z = 4$
$3x + 3y + 3z = 12$
8. a. ① $2x + y - z = -3$
② $x - y + 2z = 0$
③ $3x + 2y - z = -5$
$2 \times$ equation ② + equation ③ $= 5x + 0y + 0z = -5$ which gives $x = -1$.
Equation ① + equation ② $= 3x + 0y + 1z = -3$. Substituting $x = 1$ into this equation leads to: $3(-1) + z = -3$ or $z = 0$.
Substituting $z = 0$ and $x = -1$ into equation ① gives: $2(-1)y - 0 = -3$ or $y = -1$. $(-1, -1, 0)$ is the point at which the three planes meet.
Check:
Substituting into equation ①:
$2x + y - z = -2 - 1 + 0 = -3$
Substituting into equation ②:
$x - y + 2z = -1 + 1 + 0 = 0$
Substituting into equation ③:
$3x + 2y - z = -3 - 2 + 0 = -5$
b. ① $\frac{x}{3} - \frac{y}{4} + z = \frac{7}{8}$
② $2x + 2y - 3z = -20$
③ $x - 2y + 3z = 2$
Equation ② + equation ③ $= 3x + 0y + 0z = -18$ which gives $x = -6$.
Equation ③ $- 3 \times$ Equation ① $= -\frac{5}{4}y = -\frac{5}{8}$ or $y = \frac{1}{2}$. Substituting $x = -6$ and $y = \frac{1}{2}$ into equation ③ leads to:
$-6 - 2\left(\frac{1}{2}\right) + 3z = 2$ or $z = 3$.
$(-6, \frac{1}{2}, 3)$ is the point at which the three planes meet.
Check:
Substituting into equation ①:
$\frac{x}{3} - \frac{y}{4} + z = -2 - \frac{1}{8} + 3 = \frac{7}{8}$
Substituting into equation ②:
$2x + 2y - 3z = -12 + 1 - 9 = -20$
Substituting into equation ③:
$x - 2y + 3z = -6 - 1 + 9 = 2$
c. ① $x - y = -199$
② $x + z = -200$
③ $y - z = 201$
Equation ② + equation ③ = equation ④ $= x + y = 1$

Equation ④ + equation ① $= 2x = -198$ or $x = -99$. Substituting $x = -99$ into equation ① leads to:
$-99 - y = -199$ or $y = 100$. Substituting $x = -99$ into equation ②, you obtain:
$-99 + z = -200$ or $z = -101$
$(-99, 100, -101)$ is the point at which the three planes meet.
Check:
Substituting into equation ①:
$x - y = -99 - (100) = -199$
Substituting into equation ②:
$x + z = -99 - 101 = -200$
Substituting into equation ③:
$y - z = 100 - (-101) = 201$

d. ① $x - y - z = -1$
② $y - 2 = 0$
③ $x + 1 = 5$
Rearranging equation ② gives $y = 2$. Solving for x in equation ③ gives $x = 4$.
Substituting $x = 4$ and $y = 2$ into equation ① leads to:
$4 - 2 - z = -1$ or $z = 3$.
(4, 2, 3) is the point at which all three planes meet.

9. a. ① $x - 2y + z = 3$
② $2x + 3y - z = -9$
③ $5x - 3y + 2z = 0$
Equation ③ + equation ② = equation ④
$= 7x + 1z = -9$.
Setting $z = t, x = -\frac{1}{7}t - \frac{9}{7}$
Equation ② $- 2 \times$ equation ① = equation ⑤
$= 7y + -3z = -15$.
Setting $z = t, y = -\frac{15}{7} + \frac{3}{7}t$
$x = -\frac{1}{7}t - \frac{9}{7}, y = -\frac{15}{7} + \frac{3}{7}t$, and $z = t, t \in \mathbf{R}$ The planes intersect in a line.

b. ① $x - 2y + z = 3$
② $x + y + z = 2$
③ $x - 3y + z = -6$
Equation ③ − equation ② $= -4y = -8$ or $y = 2$
Equation ③ − equation ① $= -1y = -9$ or $y = 9$
Since the solutions for y are different from these two equations, there is no solution to this system of equations.

c. ① $x - y + z = -2$
② $x + y + z = 2$
③ $x - 3y + z = -6$
Equation ① + equation ② = equation ④
$= 2x + 2y = 0$.
Setting $z = t, x = -t$
Using $z = t$ and $x = -t$, Solve equation ①
$-t - y + t = -2$ or $y = 2$
$x = -t, y = 2$, and $z = t, t \in \mathbf{R}$
The planes intersect in a line.

10. a. ① $x - y + z = 2$
② $2x - 2y + 2z = 4$
③ $x + y - z = -2$
Equation ① + equation ③ = equation ④
$= 2x = 0$ or $x = 0$.
Setting $z = t$,
Equation ①: $0 - y + t = 2$ or $y = t - 2$
$x = 0, y = t - 2$, and $z = t, t \in \mathbf{R}$

b. ① $2x - y + 3z = 0$
② $4x - 2y + 6z = 0$
③ $-2x + y - 3z = 0$
Equation ① + equation ③ = equation ④
$= 2x = 0$ or $x = 0$.
Setting $y = t$ and $z = s$, equation ①:
$2x - t + 3s = 0$ or $x = \dfrac{t - 3s}{2}$
$x = \dfrac{t - 3s}{2}$, $y = t$, and $z = s, s, t \in \mathbf{R}$

11. a. ① $x + y + z = 1$
② $x - 2y + z = 0$
③ $x - y + z = 0$
Equation ① − equation ③ = equation ④
$= 2y = 1$ or $y = \frac{1}{2}$
Equation ② − equation ③ = equation ⑤
$= -y = 0$ or $y = 0$
Since the y-variable is different in equation ④ and equation ⑤, the system is inconsistent and has no solution.

b. Answers may vary. For example: If you use the normals from equations ①, ②, and ③, you can determine the direction vectors from the equations' coefficients.
$\overrightarrow{n_1} = (1, 1, 1)$
$\overrightarrow{n_2} = (1, -2, 1)$
$\overrightarrow{n_3} = (1, -1, 1)$
$m_1 = \overrightarrow{n_1} \times \overrightarrow{n_2} = (3, 0, -3)$
$m_2 = \overrightarrow{n_1} \times \overrightarrow{n_3} = (2, 0, -2)$
$m_3 = \overrightarrow{n_2} \times \overrightarrow{n_3} = (-1, 0, 1)$

c. The three lines of intersection are parallel and are pairwise coplanar, so they form a triangular prism.

d. $\overrightarrow{n_1} \times \overrightarrow{n_2}$ is perpendicular to $\overrightarrow{n_3}$. So since, $(\overrightarrow{n_1} \times \overrightarrow{n_2}) \cdot \overrightarrow{n_3} = 0$, a triangular prism forms.

12. a. ① $x - y + 3z = 3$
② $x - y + 3z = 6$
③ $3x - 5z = 0$

Equation ① and equation ② have the same set of coefficients and variables, however, equations ① equals 3 while equation ② equals 6, which means there is no possible solution.

b. ① $5x - 2y + 3z = 1$
② $5x - 2y + 3z = -1$
③ $5x - 2y + 3z = 13$

All three equations equal different numbers so there is no possible solution.

c. ① $x - y + z = 9$
② $2x - 2y + 2z = 18$
③ $2x - 2y + 2z = 17$

Equation ② equals 18 while equation ③ equals 17, which means there is no possible solution.

d. The coefficients of equation ① are half the coefficients of equation ②, but the constant term is not half the other constant term.

13. a. ① $2x - y - z = 10$
② $x + y + 0z = 7$
③ $0x + y - z = 8$

Equation ① − 2 × equation ② − equation ③: $-4y = -12$ or $y = 3$. Substituting $y = 3$ into equation ② and equation ③ gives:
$x + 3 + 0z = 7$ or $x = 4$
$0x + 3 - z = 8$ or $z = -5$
$(4, 3, -5)$

b. ① $2x - y + z = -3$
② $x + y - 2z = 1$
③ $5x + 2y - 5z = 0$

Equation ① + equation ②: $3x - z = -2$.
Setting $z = t$, $x = \frac{t-2}{3}$
Equation ① − 2 × equation ②: $-3y + 5z = -5$.
Setting $z = t$, $y = \frac{5t+5}{3}$
$x = \frac{t-2}{3}, y = \frac{5t+5}{3}, z = t, t \in \mathbf{R}$

c. ① $x + y - z = 0$
② $2x - y + z = 0$
③ $4x - 5y + 5z = 0$

Equation ① + equation ②: $3x = 0$ or $x = 0$
Setting $x = 0$ and $z = t$ in equation ② gives,
$2(0) - y + t = 0$ or $y = t$
$x = 0, y = t, z = t, t \in \mathbf{R}$

d. ① $x - 10y + 13z = -4$
② $2x - 20y + 26z = -8$
③ $x - 10y + 13z = -8$

If you multiply equation ② by two, you obtain $2x - 20y + 26z = -16$. Since equation ② and equation ③ equal different numbers, there is no solution to this system.

e. ① $x - y + z = -2$
② $x + y + z = 2$
③ $3x + y + 3z = 2$

Equation ① + equation ②: $-2y = -4$ or $y = 2$
Setting $y = 2$ and $z = t$ in equation ①,
$x - 2 + t = -2$ or $x = -t$
$x = -t, y = 2, z = t, t \in \mathbf{R}$

f. ① $x + y + z = 0$
② $x - 2y + 3z = 0$
③ $2x - y + 3z = 0$

Equation ① − equation ② = equation ④
$= 3y - 2z = 0$
Equation ③ − 2 × equation ② − equation ⑤
$= 3y - 3z = 0$
Equation ④ − equation ⑤: $z = 0$
Setting $z = 0$ in equation ① and equation ②,
Equation ⑥ $= x + y = 0$
Equation ⑦ $= x - 2y = 0$
Equation ⑥ − equation ⑦: $3y = 0$ or $y = 0$
Setting $y = 0$ and $z = 0$ in equation ① leads to $x = 0$
$(0, 0, 0)$

14. a. First, reorder these equations so that equation ② is first, equation ③ is second, and equation ① last.
① $x - y + z = p$
② $4x + qy + z = 2$
③ $2x + y + z = 4$

To eliminate x from the last two equations, subtract 4 times equation ① from equation ②, and subtract 2 times equation ① from equation ③.
① $x - y + z = p$
② $(q + 4)y - 3z = 2 - 4p$
③ $3y - z = 4 - 2p$

There will be an infinite number of solutions if $q + 4 = 9$ and $3(4 - 2p) = 2 - 4p$ because then equation ② will be 3 times equation ③. This means that $p = q = 5$.

b. Based on what was found in part **a.**, substituting in $p = q = 5$ we will arrive at the equivalent system
① $x - y + z = 5$
② $9y - 3z = -18$
③ $3y - z = -6$

which is really the same as
① $x - y + z = 5$
② $3y - z = -6$

Letting $z = t$, we see that equation ② delivers

$y = \frac{1}{3}(t - 6)$

$= \frac{1}{3}t - 2$

and so equation ① gives

$x = \frac{1}{3}(t - 6) - t + 5$

$= -\frac{2}{3}t + 3$

So the parametric equation of the line of intersection is

$x = -\frac{2}{3}t + 3, y = \frac{1}{3}t - 2, z = t, t \in \mathbf{R}.$

15. a. First, eliminate x from two of these equations. To make things easier, switch equation ① with equation ②, and multiply equation ③ by 2.

① $2x + y + z = -4$

② $4x + 3y + 3z = -8$

③ $6x - 4y + (2m^2 - 12)z = 2m - 8$

Now eliminate x from the last two equations by using proper multiples of the first equation.

① $2x + y + z = -4$

② $y + z = 0$

③ $-7y + (2m^2 - 15)z = 2m + 4$

Now eliminate y from the third equation by using a proper multiple of the second equation.

① $2x + y + z = -4$

② $y + z = 0$

③ $(2m^2 - 8)z = 2m + 4$

If $2m^2 - 8 = 0$ (the coefficient of z in the third equation), then $m = \pm 2$. However, if $m = 2$, the third equation would become $0z = 8$, which has no solutions. So there is no solution if $m = 2$.

b. Working with what was found in part **a.**, if $m \neq \pm 2$, then the third equation in the equivalent system found there will have a unique solution for z, namely

$z = \frac{2m + 4}{2m^2 - 8}$,

and back-substituting into the other two equations will give unique solutions for x and y also. So there is a unique solution if $m \neq \pm 2$.

c. Again using the equivalent system found in part **a.**, setting $m = -2$ will deliver the third equation $0z = 0$, which allows for z to be anything at all. So $m = -2$ will give an infinite number of solutions.

16. a. ① $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0$

② $\frac{2}{a} + \frac{3}{b} + \frac{2}{c} = \frac{13}{6}$

③ $\frac{4}{a} - \frac{2}{b} + \frac{3}{c} = \frac{5}{2}$

Equation ② − 2 × equation ①:

$\frac{1}{b} + \frac{4}{c} = \frac{13}{6}$ = equation ④

Equation ③ − 4 × equation ①: $-\frac{6}{b} + \frac{7}{c}$

$m_3 = \vec{n} \times \vec{n_1} = (-1, 0, 1) = \frac{5}{2}$ = equation ⑤

Equation ⑤ + 6 × equation ④:

$\frac{31}{c} = 15.5$ or $c = 2$

Substituting $c = 2$ into equation ④:

$\frac{1}{b} + 2 = \frac{13}{6}$ or $b = 6$

Substituting $c = 2$ and $b = 6$ into equation ①:

$\frac{1}{a} + \frac{1}{6} - \frac{1}{2} = 0$ or $a = 3$

(3, 6, 2)

9.5 The Distance from a Point to a Line in R^2 and R^3, pp. 540–541

1. a. $3x + 4y - 5 = 0$

$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

$d = \frac{|3(-4) + 4(5) - 5|}{\sqrt{3^2 + 4^2}}$

$= \frac{3}{5}$

b. $5x - 12y + 24 = 0$

$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

$d = \frac{|5(-4) - 12(5) + 24|}{\sqrt{5^2 + (-12)^2}}$

$= \frac{56}{13}$ or 4.31

c. $9x - 40y = 0$

$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

$d = \frac{|9(-4) - 40(5)|}{\sqrt{9^2 + (40)^2}}$

$= \frac{236}{\sqrt{1681}}$ or 5.76

2. a. $2x - y + 1 = 0$ and $2x - y + 6 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of

$y = \frac{1}{3}(t - 6)$

$= \frac{1}{3}t - 2$

and so equation ① gives

$x = \frac{1}{3}(t - 6) - t + 5$

$= -\frac{2}{3}t + 3$

So the parametric equation of the line of intersection is

$x = -\frac{2}{3}t + 3,\ y = \frac{1}{3}t - 2,\ z = t,\ t \in \mathbf{R}.$

15. a. First, eliminate x from two of these equations. To make things easier, switch equation ① with equation ②, and multiply equation ③ by 2.

① $2x + y + z = -4$

② $4x + 3y + 3z = -8$

③ $6x - 4y + (2m^2 - 12)z = 2m - 8$

Now eliminate x from the last two equations by using proper multiples of the first equation.

① $2x + y + z = -4$

② $y + z = 0$

③ $-7y + (2m^2 - 15)z = 2m + 4$

Now eliminate y from the third equation by using a proper multiple of the second equation.

① $2x + y + z = -4$

② $y + z = 0$

③ $(2m^2 - 8)z = 2m + 4$

If $2m^2 - 8 = 0$ (the coefficient of z in the third equation), then $m = \pm 2$. However, if $m = 2$, the third equation would become $0z = 8$, which has no solutions. So there is no solution if $m = 2$.

b. Working with what was found in part **a.**, if $m \neq \pm 2$, then the third equation in the equivalent system found there will have a unique solution for z, namely

$z = \frac{2m + 4}{2m^2 - 8}$,

and back-substituting into the other two equations will give unique solutions for x and y also. So there is a unique solution if $m \neq \pm 2$.

c. Again using the equivalent system found in part **a.**, setting $m = -2$ will deliver the third equation $0z = 0$, which allows for z to be anything at all. So $m = -2$ will give an infinite number of solutions.

16. a. ① $\frac{1}{a} + \frac{1}{b} - \frac{1}{c} = 0$

② $\frac{2}{a} + \frac{3}{b} + \frac{2}{c} = \frac{13}{6}$

③ $\frac{4}{a} - \frac{2}{b} + \frac{3}{c} = \frac{5}{2}$

Equation ② − 2 × equation ①:

$\frac{1}{b} + \frac{4}{c} = \frac{13}{6}$ = equation ④

Equation ③ − 4 × equation ①: $-\frac{6}{b} + \frac{7}{c}$

$m_3 = \vec{n} \times \vec{n_1} = (-1, 0, 1) = \frac{5}{2}$ = equation ⑤

Equation ⑤ + 6 × equation ④:

$\frac{31}{c} = 15.5$ or $c = 2$

Substituting $c = 2$ into equation ④:

$\frac{1}{b} + 2 = \frac{13}{6}$ or $b = 6$

Substituting $c = 2$ and $b = 6$ into equation ①:

$\frac{1}{a} + \frac{1}{6} - \frac{1}{2} = 0$ or $a = 3$

(3, 6, 2)

9.5 The Distance from a Point to a Line in R^2 and R^3, pp. 540–541

1. a. $3x + 4y - 5 = 0$

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|3(-4) + 4(5) - 5|}{\sqrt{3^2 + 4^2}}$$

$$= \frac{3}{5}$$

b. $5x - 12y + 24 = 0$

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|5(-4) - 12(5) + 24|}{\sqrt{5^2 + (-12)^2}}$$

$$= \frac{56}{13} \text{ or } 4.31$$

c. $9x - 40y = 0$

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|9(-4) - 40(5)|}{\sqrt{9^2 + (40)^2}}$$

$$= \frac{236}{\sqrt{1681}} \text{ or } 5.76$$

2. a. $2x - y + 1 = 0$ and $2x - y + 6 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of

the lines. It is easiest to find a point where the line crosses the x or y-axis.

$2(0) - y + 1 = 0$ or $y = 1$ which corresponds to the point $(0, 1)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|2(0) - 1(1) + 6|}{\sqrt{2^2 + (-1)^2}}$$

$$= \frac{5}{\sqrt{5}} \text{ or } 2.24$$

b. $7x - 24y + 168 = 0$ and $7x - 24y - 336 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of the lines. It is easiest to find a point where the line crosses the x or y-axis.

$7(0) - 24y + 168 = 0$ or $y = 7$ which corresponds to the point $(0, 7)$

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|7(0) - 24(7) - 336|}{\sqrt{7^2 + (-24)^2}}$$

$$= \frac{504}{25} \text{ or } 20.16$$

3. a. $\vec{r} = (-1, 2) + s(3, 4), s \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = -1 + 3s$, $y = 2 + 4s$. We construct a vector from $R(-2, 3)$ to a general point on the line.

$$\vec{a} = [-2 - (-1 + 3s), 3 - (2 + 4s)]$$
$$= (-1 - 3s, 1 - 4s).$$
$$(3, 4) \cdot (-1 - 3s, 1 - 4s) = 0$$
$$(-3 - 9s) + (4 - 16s) = 0$$
$$s = \frac{1}{25}$$

This means that the minimal distance between $R(-2, 3)$ and the line occurs when $s = \frac{1}{25}$.

This point corresponds to $\left(-\frac{22}{25}, \frac{54}{25}\right)$. The distance between this point and $(-2, 3)$ is 1.4.

b. $\vec{r} = (1, 0) + t(5, 12), t \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 1 + 5t$, $y = 12t$. We construct a vector from $R(-2, 3)$ to a general point on the line.

$$\vec{a} = [-2 - (1 + 5t), 3 - (12t)]$$
$$= (-3 - 5t, 3 - 12t).$$
$$(5, 12) \cdot (-3 - 5t, 3 - 12t) = 0$$
$$(-15 - 25t) + (36 - 144t) = 0$$
$$t = \frac{21}{169}$$

This means that the minimal distance between $R(-2, 3)$ and the line occurs when $t = \frac{21}{169}$.

This point corresponds to $\left(\frac{274}{169}, \frac{252}{169}\right)$. The distance between this point and $(-2, 3)$ is about 3.92.

c. $\vec{r} = (1, 3) + p(7, -24), p \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 1 + 7p$, $y = 3 - 24p$. We construct a vector from $R(-2, 3)$ to a general point on the line.

$$\vec{a} = [-2 - (1 + 7p), 3 - (3 - 24p)]$$
$$= (-3 - 7p, 24p).$$
$$(7, -24) \cdot (-3 - 7p, 24p) = 0$$
$$(-21 - 49p) + (-576p) = 0$$
$$p = -\frac{21}{625}$$

This means that the minimal distance between $R(-2, 3)$ and the line occurs when $p = -\frac{21}{625}$.

This point corresponds to $\left(\frac{478}{625}, \frac{2379}{625}\right)$.

The distance between this point and $(-2, 3)$ is about 2.88.

4. a. $d = \dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$

If you substitute in the coordinates $(0, 0)$, the formula changes to $d = \dfrac{|A(0) + B(0) + C|}{\sqrt{A^2 + B^2}}$,

which reduces to $d = \dfrac{|C|}{\sqrt{A^2 + B^2}}$.

b. $3x - 4y - 12 = 0$ and $3x - 4y + 12 = 0$

$$d(L_1) = \frac{|C|}{\sqrt{A^2 + B^2}} = \frac{|-12|}{\sqrt{3^2 + (-4)^2}}$$
$$= \frac{12}{5}$$
$$d(L_2) = \frac{|C|}{\sqrt{A^2 + B^2}} = \frac{|12|}{\sqrt{3^2 + (-4)^2}}$$
$$= \frac{12}{5}$$

The distance between these parallel lines is $\frac{12}{5} + \frac{12}{5} = \frac{24}{5}$, because one of the lines is below the origin and the other is above the origin.

c. $3x - 4y - 12 = 0$ and $3x - 4y + 12 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of the lines. It is easiest to find a point where the line crosses the x or y-axis.

$3(0) - 4y - 12 = 0$ or $y = -3$ which corresponds to the point $(0, 3)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|3(0) - 4(-3) + 12|}{\sqrt{3^2 + (-4)^2}}$$

$$= \frac{24}{5}$$

Both the answers to 4.b. and 4.c. are the same.

5. a. $\vec{r} = (-2, 1) + s(3, 4), s \in \mathbf{R}$

$\vec{r} = (1, 0) + t(3, 4), t \in \mathbf{R}$

First find a random point on one of the lines. We will use $(-2, 1)$ from the first equation. We start by writing the second equation in parametric form. Doing so gives $x = 1 + 3t$, $y = 4t$. We construct a vector from $P(-2, 1)$ to a general point on the line.

$\vec{a} = [-2 - (1 + 3t), 1 - (4t)]$

$= (-3 - 3t, 1 - 4t)$.

$(3, 4) \cdot (-3 - 3t, 1 - 4t) = 0$

$(-9 - 9t) + (4 - 16t) = 0$

$$t = -\frac{1}{5}$$

This means that the minimal distance between $P(-2, 1)$ and line occurs when $t = -\frac{1}{5}$. This point corresponds to $\left(\frac{2}{5}, -\frac{4}{5}\right)$. The distance between this point and $(-2, 1)$ is 3

b. $\dfrac{x - 1}{4} = \dfrac{y}{-3}$ and $\dfrac{x}{4} = \dfrac{y + 1}{-3}$

First change one equation into a Cartesian equation, which leads to $3x + 4y - 3 = 0$ and take a point from the other equation such as $(4, -4)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|3(4) + 4(-4) - 3|}{\sqrt{3^2 + 4^2}}$$

$$= \frac{7}{5} \text{ or } 1.4$$

c. $2x - 3y + 1 = 0$ and $2x - 3y - 3 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of the lines. It is easiest to find a point where the line crosses the x or y-axis.

$2(0) - 3y - 3 = 0$ or $y = -1$ which corresponds to the point $(0, -1)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|2(0) - 3(-1) + 1|}{\sqrt{2^2 + (-3)^2}}$$

$$= \frac{4}{\sqrt{13}} \text{ or } 1.11$$

d. $5x + 12y = 120$ and $5x + 12y + 120 = 0$

In order to find the distance between these two parallel lines, you must first find a point on one of the lines. It is easiest to find a point where the line crosses the x or y-axis.

$5(0) + 12y = 120$ or $y = 10$ which corresponds to the point $(0, 10)$.

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

$$d = \frac{|5(0) + 12(10) + 120|}{\sqrt{5^2 + 12^2}}$$

$$= \frac{240}{13} \text{ or } 18.46$$

6. a. $P(1, 2, -1)$ $\vec{r} = (1, 0, 0) + s(2, -1, 2), s \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 1 + 2s$, $y = -s$, and $z = 2s$. We construct a vector from $P(1, 2, -1)$ to a general point on the line.

$\vec{a} = [1 - (1 + 2s), 2 - (-s), -1 - (2s)]$

$= (-2s, 2 + s, -1 - 2s)$.

$(2, -1, 2) \cdot (-2s, 2 + s, -1 - 2s) = 0$

$(-4s) + (-2 - s) + (-2 - 4s) = 0$

$$s = -\frac{4}{9}$$

This means that the minimal distance between $P(1, 2, -1)$ and the line occurs when $s = -\frac{4}{9}$. This point corresponds to $\left(\frac{1}{9}, \frac{4}{9}, -\frac{8}{9}\right)$. The distance between this point and $P(1, 2, -1)$ is 1.80.

b. $P(0, -1, 0)$ $\vec{r} = (2, 1, 0) + t(-4, 5, 20), t \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 2 - 4t$, $y = 1 + 5t$, and $z = 20t$. We construct a vector from $P(0, -1, 0)$ to a general point on the line.

$\vec{a} = [0 - (2 - 4t), -1 - (1 + 5t), 0 - (20t)]$

$= (-2 + 4t, -2 - 5t, 20t)$.

$(-4, 5, 20) \cdot (-2 + 4t, -2 - 5t, -20t) = 0$

$(8 - 16t) + (-10 - 25t) + (-400t) = 0$

$$t = -\frac{2}{441}$$

This means that the minimal distance between $P(0, -1, 0)$ and the line occurs when $t = -\frac{2}{441}$.

This point corresponds to $\left(\frac{890}{441}, \frac{431}{441}, -\frac{40}{441}\right)$. The distance between this point and $P(0, -1, 0)$ is 2.83.

c. $P(2, 3, 1)$ $\vec{r} = p(12, -3, 4), p \in \mathbf{R}$

We start by writing the given equation of the line in parametric form. Doing so gives $x = 12p$, $y = -3p$, and $z = 4p$. We construct a vector from $P(2, 3, 1)$ to a general point on the line.

$$\vec{a} = [2 - (12p), 3 - (-3p), 1 - (4p)]$$
$$= (2 - 12p, 3 + 3p, 1 - 4p).$$
$$(12, -3, 4) \cdot (2 - 12p, 3 + 3p, 1 - 4p) = 0$$
$$(24 - 144p) + (-9 - 9p) + (4 - 16p) = 0$$
$$p = \frac{19}{169}$$

This means that the minimal distance between $P(2, 3, 1)$ and the line occurs when $p = \frac{19}{169}$. This point corresponds to $\left(\frac{228}{169}, -\frac{57}{169}, \frac{76}{169}\right)$. The distance between this point and $P(2, 3, 1)$ is 3.44.

7. a. $\vec{r} = (1, 1, 0) + s(2, 1, 2), s \in \mathbf{R}$

$\vec{r} = (-1, 1, 2) + t(2, 1, 2), t \in \mathbf{R}$

First find a random point on one of the lines. We will use $P(-1, 1, 2)$ from the second equation. We then write the first equation in parametric form. Doing so gives $x = 1 + 2s$, $y = 1 + s$, and $z = 0 + 2s$. We construct a vector from $P(-1, 1, 2)$ to a general point on the line.

$$\vec{a} = [-1 - (1 + 2s), 1 - (1 + s), 2 - 2s]$$
$$= (-2 - 2s, 2 - 2s).$$
$$(2, 1, 2) \cdot (-2 - 2s, -s, 2 - 2s) = 0$$
$$(-4 - 4s) + (-s) + (4 - 4s) = 0$$
$$s = 0$$

This means that the minimal distance between $P(-1, 1, 2)$ and line occurs when $s = 0$. This point corresponds to $(1, 1, 0)$. The distance between this point and $(-1, 1, 2)$ is 2.83

b. $\vec{r} = (3, 1, -2) + m(1, 1, 3), m \in \mathbf{R}$

$\vec{r} = (1, 0, 1) + n(1, 1, 3), n \in \mathbf{R}$

First find a random point on one of the lines. We will use $P(1, 0, 1)$ from the second equation. We then write the first equation in parametric form. Doing so gives $x = 3 + m$, $y = 1 + m$, and $z = -2 + 3m$. We construct a vector from $P(1, 0, 1)$ to a general point on the line.

$$\vec{a} = [1 - (3 + m), 0 - (1 + m), 1 - (-2 + 3m)]$$
$$= (-2 - 3m, -1 - m, 3 - 3m).$$
$$(1, 1, 3) \cdot (-2 - 3m, -1 - m, 3 - 3m) = 0$$
$$(-2 - 3m) + (-1 - m) + (9 - 9m) = 0$$
$$m = \frac{6}{13}$$

This means that the minimal distance between $P(1, 0, 1)$ and line occurs when $m = \frac{6}{13}$. This point corresponds to $\left(\frac{45}{13}, \frac{19}{13}, -\frac{6}{13}\right)$. The distance between this point and $(1, 0, 1)$ is 3.28

8. a. $\vec{r} = (1, -1, 2) + s(1, 3, -1), s \in \mathbf{R}$

First we write the equation in parametric form. Doing so gives $x = 1 + s$, $y = -1 + 3s$, and $z = 2 - s$. We construct a vector from $P(2, 1, 3)$ to a general point on the line.

$$\vec{a} = [2 - (1 + s), 1 - (-1 + 3s), 3 - (2 - s)]$$
$$= (1 - s, 2 - 3s, 1 + s).$$
$$(1, 3, -1) \cdot (1 - s, 2 - 3s, 1 + s) = 0$$
$$(1 - s) + (6 - 9s) + (1 + s) = 0$$
$$s = \frac{6}{11}$$

This means that the minimal distance between $P(2, 1, 3)$ and line occurs when $s = \frac{6}{11}$. This point corresponds to $\left(\frac{17}{11}, \frac{7}{11}, \frac{16}{11}\right)$.

b. The distance between $\left(\frac{17}{11}, \frac{7}{11}, \frac{16}{11}\right)$ and $(2, 1, 3)$ is 1.65.

9. First, find the line L of intersection between the planes

① $x - y + 2z = 2$

② $x + y - z = -2$

Subtract the first equation from the second to eliminate x and get the equivalent system

① $x - y + 2z = 2$

② $2y - 3z = -4$

Let $z = t$. Then the second equation gives

$$2y = 3t - 4$$
$$y = \frac{3}{2}t - 2$$

So substituting these into the first equation gives

$$x = y - 2z + 2$$
$$= \left(\frac{3}{2}t - 2\right) - 2t + 2$$
$$= -\frac{1}{2}t$$

So the equation of the line of intersection for these two planes in parametric form is

$x = -\frac{1}{2}t$, $y = \frac{3}{2}t - 2$, $z = t$, $t \in \mathbf{R}$.

The direction vector for this line is $\left(-\frac{1}{2}, \frac{3}{2}, 1\right)$, which is parallel to $(-1, 3, 2)$. So, to make things easier, the parametric form of this line of intersection could also be expressed as

$x = -t$, $y = 3t - 2$, $z = 2t$, $t \in \mathbf{R}$

In vector form, this is the same as

$\vec{r} = (0, -2, 0) + t(-1, 3, 2), t \in \mathbf{R}$.
Since $Q(0, -2, 0)$ is on this line,

$$\overrightarrow{QP} = (-1, 2, -1) - (0, -2, 0)$$
$$= (-1, 4, -1)$$

So the distance from $P(-1, 2, -1)$ to the line of intersection is

$$d = \frac{|(-1, 3, 2) \times (-1, 4, -1)|}{|(-1, 3, 2)|}$$
$$= \frac{|(-11, -3, -1)|}{|(-1, 3, 2)|}$$
$$= \sqrt{\frac{131}{14}}$$
$$\doteq 3.06$$

To find the point on the line that gives this minimal distance, let (x, y, z) be a point on the line. Then, using the parametric equations,
$(x, y, z) = (-t, 3t - 2, 2t)$
So the distance from P to this point is

$$\sqrt{(x + 1)^2 + (y - 2)^2 + (z + 1)^2}$$
$$= \sqrt{(1 - t)^2 + (3t - 4)^2 + (2t + 1)^2}$$
$$= \sqrt{14t^2 - 22t + 18}$$

To get the minimal distance, set this quantity equal to $\sqrt{\frac{131}{14}}$.

$$\sqrt{14t^2 - 22t + 18} = \sqrt{\frac{131}{14}}$$
$$14t^2 - 22t + 18 = \frac{131}{14}$$
$$196t^2 - 308t + 252 = 131$$
$$196t^2 - 308t + 121 = 0$$
$$t = \frac{308 \pm \sqrt{0}}{392}$$
$$= \frac{11}{14}$$

So the point on the line at minimal distance from P is

$$(x, y, z) = (-t, 3t, -2, 2t)$$
$$= \left(-\frac{11}{14}, 3\left(\frac{11}{14}\right) - 2, 2\left(\frac{11}{14}\right)\right)$$
$$= \left(-\frac{11}{14}, \frac{5}{14}, \frac{22}{14}\right)$$

10. A point on the line
$\vec{r} = (0, 0, 1) + s(4, 2, 1), s \in \mathbf{R}$.
has parametric equations
$x = 4s, y = 2s, z = 1 + s, s \in \mathbf{R}$.
Let this point be called
$Q(4s, 2s, 1 + s)$. Then

$$\overrightarrow{QA} = (2, 4, -5) - (4s, 2s, 1 + s)$$
$$= (2 - 4s, 4 - 2s, -6 - s)$$

If Q is at minimal distance from A, then this vector will be perpendicular to the direction vector for the line, $(4, 2, 1)$. This means that

$$0 = (2 - 4s, 4 - 2s, -6 - s) \cdot (4, 2, 1)$$
$$= 10 - 21s$$
$$s = \frac{10}{21}$$

So the point Q on the line at minimal distance from A is

$$Q(4s, 2s, 1 + s) = Q\left(4\left(\frac{10}{21}\right), 2\left(\frac{10}{21}\right), 1 + \frac{10}{21}\right)$$
$$= Q\left(\frac{40}{21}, \frac{20}{21}, \frac{31}{21}\right)$$

Also

$$\overrightarrow{QA} = \left(2 - \frac{40}{21}, 4 - \frac{20}{21}, -5 - \frac{31}{21}\right)$$
$$= \left(\frac{2}{21}, \frac{64}{21}, -\frac{136}{21}\right)$$

So the point A' will satisfy

$$\overrightarrow{QA'} = -\overrightarrow{QA}$$
$$= \left(-\frac{2}{21}, -\frac{64}{21}, \frac{136}{21}\right)$$
$$= A'(a, b, c) - Q$$
$$= \left(a - \frac{40}{21}, b - \frac{20}{21}, c - \frac{31}{21}\right)$$

So $a = \frac{38}{21}$, $b = -\frac{44}{21}$, and $c = \frac{167}{21}$. That is, $A'\left(\frac{38}{21}, -\frac{44}{21}, \frac{167}{21}\right)$.

11. a. Think of H as being the origin, E as being on the x-axis, D as being on the z-axis, and G as being on the y-axis. That is,
$H(0, 0, 0)$
$E(3, 0, 0)$
$G(0, 2, 0)$
$D(0, 0, 2)$
and so on for the other points as well. Then line segment HB has direction vector
$B(3, 2, 2) - H(0, 0, 0) = (3, 2, 2)$.
Also, $\overrightarrow{HA} = (3, 0, 2)$. So the distance formula says that the distance between A and line segment HB is

$$d = \frac{|(3, 2, 2) \times (3, 0, 2)|}{|(3, 2, 2)|}$$
$$= \frac{|4, 0, -6|}{|(3, 2, 2)|}$$
$$= \sqrt{\frac{52}{17}}$$
$$\doteq 1.75$$

b. Vertices D and G will give the same distance to HB because they are equidistant to the segment HB. (This is easy to check with the distance formula used similarly to part **a.** The vertices C, E, and F give different distances than those found in part **a.**)

c. The height of triangle AHB was found in part **a.**, and was $\sqrt{\frac{52}{17}}$. The base length of this triangle is the magnitude of $\overrightarrow{HB} = (3, 2, 2)$, which is $\sqrt{52}$. So the area of this triangle is

$$\frac{1}{2}\left(\sqrt{\frac{52}{17}}\right)(\sqrt{17}) = \frac{1}{2}(\sqrt{52})$$
$$\doteq 3.6 \text{ units}^2$$

9.6 The Distance from a Point to a Plane, pp. 549–550

1. a. Yes the calculations are correct, Point A lies in the plane.

b. The answer 0 means that the point lies in the plane.

2. Use the distance formula.

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

a. The distance from $A(3, 1, 0)$ to the plane $20x - 4y + 5z + 7 = 0$ is

$$d = \frac{|20(3) + -4(1) + 5(0) + 7|}{\sqrt{20^2 + (-4)^2 + 5^2}}$$
$$= 3$$

b. The distance from $B(0, -1, 0)$ to the plane $2x + y + 2z - 8 = 0$ is

$$d = \frac{|2(0) + 1(-1) + 2(0) - 8|}{\sqrt{2^2 + 1^2 + 2^2}}$$
$$= 3$$

c. The distance from $C(5, 1, 4)$ to the plane $3x - 4y - 1 = 0$ is

$$d = \frac{|3(5) + -4(1) + 0(4) - 1|}{\sqrt{3^2 + (-4)^2 + 0^2}}$$
$$= 2$$

d. The distance from $D(1, 0, 0)$ to the plane $5x - 12y = 0$ is

$$d = \frac{|5(1) - 12(0) + 0(0) + 0|}{\sqrt{5^2 + (-12)^2 + 0^2}}$$
$$= \frac{5}{13} \text{ or } 0.38$$

e. The distance from $E(-1, 0, 1)$ to the plane $18x - 9y + 18z - 11 = 0$ is

$$d = \frac{|18(-1) - 9(0) + 18(1) - 11|}{\sqrt{18^2 + (-9)^2 + 18^2}}$$
$$= \frac{11}{27} \text{ or } 0.41$$

3. a. $3x + 4y - 12z - 26 = 0$ and
$3x + 4y - 12z + 39 = 0$

First find a point in the second plane such as $(-3, 0, 0)$. Then use $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$ to solve.

$$d = \frac{|3(-13) + 4(0) - 12(0) - 26|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$
$$= 5$$

b.
$$\begin{array}{r} 3x + 4y - 12z - 26 = 0 \\ + \; 3x + 4y - 12z + 39 = 0 \\ \hline 6x + 8y - 24z + 13 = 0 \end{array}$$

c. Answers may vary. Any point on the plane $6x + 8y - 24z + 13 = 0$ will work, for example $\left(-\frac{1}{6}, 0, \frac{1}{2}\right)$.

4. a. The distance from $P(1, 1, -3)$ to the plane $y + 3 = 0$ is

$$d = \frac{|0(1) + 1(1) + 0(-3) + 3|}{\sqrt{0^2 + (1)^2 + 0^2}}$$
$$= 4$$

b. The distance from $Q(-1, 1, 4)$ to the plane $x - 3 = 0$ is

$$d = \frac{|1(-1) + 0(1) + 0(4) - 3|}{\sqrt{1^2 + 0^2 + 0^2}}$$
$$= 4$$

c. The distance from $R(1, 0, 1)$ to the plane $z + 1 = 0$ is

$$d = \frac{|0(1) + 0(0) + 1(1) + 1|}{\sqrt{0^2 + 0^2 + 1^2}}$$
$$= 2$$

5. First you have to find an equation of a plane to the three points. The equation to this plane is $14x - 28y + 28z - 42 = 0$. Then use $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$ to solve for the distance.

$$d = \frac{|14(1) - 28(-1) + 28(1) - 42|}{\sqrt{14^2 + (-28)^2 + 28^2}}$$
$$= \frac{2}{3} \text{ or } 0.67$$

b. Vertices D and G will give the same distance to HB because they are equidistant to the segment HB. (This is easy to check with the distance formula used similarly to part **a.** The vertices C, E, and F give different distances than those found in part **a.**)

c. The height of triangle AHB was found in part **a.**, and was $\sqrt{\frac{52}{17}}$. The base length of this triangle is the magnitude of $\overrightarrow{HB} = (3, 2, 2)$, which is $\sqrt{52}$. So the area of this triangle is

$$\frac{1}{2}\left(\sqrt{\frac{52}{17}}\right)(\sqrt{17}) = \frac{1}{2}(\sqrt{52})$$
$$\doteq 3.6 \text{ units}^2$$

9.6 The Distance from a Point to a Plane, pp. 549–550

1. a. Yes the calculations are correct, Point A lies in the plane.

b. The answer 0 means that the point lies in the plane.

2. Use the distance formula.

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

a. The distance from $A(3, 1, 0)$ to the plane $20x - 4y + 5z + 7 = 0$ is

$$d = \frac{|20(3) + -4(1) + 5(0) + 7|}{\sqrt{20^2 + (-4)^2 + 5^2}}$$
$$= 3$$

b. The distance from $B(0, -1, 0)$ to the plane $2x + y + 2z - 8 = 0$ is

$$d = \frac{|2(0) + 1(-1) + 2(0) - 8|}{\sqrt{2^2 + 1^2 + 2^2}}$$
$$= 3$$

c. The distance from $C(5, 1, 4)$ to the plane $3x - 4y - 1 = 0$ is

$$d = \frac{|3(5) + -4(1) + 0(4) - 1|}{\sqrt{3^2 + (-4)^2 + 0^2}}$$
$$= 2$$

d. The distance from $D(1, 0, 0)$ to the plane $5x - 12y = 0$ is

$$d = \frac{|5(1) - 12(0) + 0(0) + 0|}{\sqrt{5^2 + (-12)^2 + 0^2}}$$
$$= \frac{5}{13} \text{ or } 0.38$$

e. The distance from $E(-1, 0, 1)$ to the plane $18x - 9y + 18z - 11 = 0$ is

$$d = \frac{|18(-1) - 9(0) + 18(1) - 11|}{\sqrt{18^2 + (-9)^2 + 18^2}}$$
$$= \frac{11}{27} \text{ or } 0.41$$

3. a. $3x + 4y - 12z - 26 = 0$ and
$3x + 4y - 12z + 39 = 0$

First find a point in the second plane such as $(-3, 0, 0)$. Then use $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$ to solve.

$$d = \frac{|3(-13) + 4(0) - 12(0) - 26|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$
$$= 5$$

b.
$$\begin{aligned} &3x + 4y - 12z - 26 = 0 \\ +\ &3x + 4y - 12z + 39 = 0 \\ &6x + 8y - 24z + 13 = 0 \end{aligned}$$

c. Answers may vary. Any point on the plane $6x + 8y - 24z + 13 = 0$ will work, for example $\left(-\frac{1}{6}, 0, \frac{1}{2}\right)$.

4. a. The distance from $P(1, 1, -3)$ to the plane $y + 3 = 0$ is

$$d = \frac{|0(1) + 1(1) + 0(-3) + 3|}{\sqrt{0^2 + (1)^2 + 0^2}}$$
$$= 4$$

b. The distance from $Q(-1, 1, 4)$ to the plane $x - 3 = 0$ is

$$d = \frac{|1(-1) + 0(1) + 0(4) - 3|}{\sqrt{1^2 + 0^2 + 0^2}}$$
$$= 4$$

c. The distance from $R(1, 0, 1)$ to the plane $z + 1 = 0$ is

$$d = \frac{|0(1) + 0(0) + 1(1) + 1|}{\sqrt{0^2 + 0^2 + 1^2}}$$
$$= 2$$

5. First you have to find an equation of a plane to the three points. The equation to this plane is $14x - 28y + 28z - 42 = 0$. Then use $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$ to solve for the distance.

$$d = \frac{|14(1) - 28(-1) + 28(1) - 42|}{\sqrt{14^2 + (-28)^2 + 28^2}}$$
$$= \frac{2}{3} \text{ or } 0.67$$

6. $3 = \dfrac{|A(3) - 2(-3) + 6(1) + 0|}{\sqrt{A^2 + (-2)^2 + 6^2}}$

$$3\sqrt{(A^2 + 40)} = |3A + 12|$$
$$\sqrt{(A^2 + 40)} = |A + 4|$$
$$A^2 + 40 = A^2 + 8A + 16$$
$$24 = 8A$$
$$3 = A$$

$A = 3$ is the only solution to this equation.

7. These lines are skew lines, and the plane containing the second line, $\vec{r} = (0, 0, 1) + t(1, 1, 0), t \in \mathbf{R}$, that is parallel to the first line will have direction vectors $(1, 1, 0)$ and $(3, 0, 1)$. So a normal to this plane is $(1, 1, 0) \times (3, 0, 1) = (1, -1, -3)$.
So the equation of this plane will be of the form $x - y - 3z + D = 0$. We want the point $(0, 0, 1)$ to be on this plane, and substituting this into the equation above gives $D = 3$. So the equation of the plane containing $\vec{r} = (0, 0, 1) + t(1, 1, 0), t \in \mathbf{R}$ and parallel to the first line is
$x - y - 3z + 3 = 0$.
Since $(0, 1, -1)$ is on the first line, the distance between these skew lines is the same as the distance between this point and the plane just determined.
By the distance formula, this distance is

$$d = \frac{|(0) - (1) - 3(-1) + 3|}{\sqrt{1^2 + (-1)^2 + (-3)^2}}$$
$$= \frac{5}{\sqrt{11}}$$
$$\doteq 1.51.$$

8. a.–b. We will do both of these parts at once.
The two given lines are
$\vec{r} = (1, -2, 5) + s(0, 1, -1), s \in \mathbf{R}$,
$\vec{r} = (1, -1, -2) + t(1, 0, -1), t \in \mathbf{R}$.
By converting to parametric form, a general point on the first line is
$U(1, s - 2, 5 - s)$,
and on the second line is
$V(1 + t, -1, -2 - t)$.
So the vector
$\overrightarrow{UV} = (t, 1 - s, s - t - 7)$.
If the points U and V are those that produce the minimal distance between these two lines, then $\overrightarrow{UV}$ will be perpendicular to both direction vectors, $(0, 1, -1)$ and $(1, 0, -1)$. In the first case, we get

$$0 = (t, 1 - s, s - t - 7) \cdot (0, 1, -1)$$
$$= 8 - 2s + t$$
$$t = 2s - 8$$

In the second case, we get

$$0 = (t, 1 - s, s - t - 7) \cdot (1, 0, -1)$$
$$= 2t - s + 7$$

Substituting $t = 2s - 8$ into this second equation, we get

$$2(2s - 8) - s + 7 = 0$$
$$s = 3$$
$$t = 2s - 8$$
$$t = -2$$

Substituting these values for s and t into U and V, we get
$U(1, 1, 2)$
$V(-1, -1, 0)$
So $U(1, 1, 2)$ is the point on the first line that produces the minimal distance to the second line at point $V(-1, -1, 0)$. This minimal distance is given by

$$|\overrightarrow{UV}| = |(-2, -2, -2)|$$
$$= \sqrt{12}$$
$$\doteq 3.46$$

Review Exercise, pp. 552–555

1. $2x - y = 31, x + 8y = -34, 3x + ky = 38$

$$(2x - y = 31) - 2(x + 8y = -34)$$
$$= 0x - 17y = 99$$
$$y = -\frac{99}{17}, x = \frac{214}{17}$$
$$3\left(\frac{214}{17}\right) + k\left(\frac{-99}{17}\right) = 38$$
$$k = -\frac{4}{99}$$

2.
① $x - y = 13$
② $3x + 2y = -6$
③ $x + 2y = -19$

$(2 \times$ Equation ①$)$ + equation ② $= 5x + 0y = 20$ or $x = 4$. Substituting $x = 4$ into equation ① gives $(4) - y = 13$ or $y = -9$. However, when you substitute this coordinates into the third equation, the third equation is not consistent, so there is no solution to this problem.

3. a.
① $x - y + 2z = 3$
② $2x - 2y + 3z = 1$
③ $2x - 2y + z = 11$

Equation ② $-$ equation ③ $= 5z = -10$ or $z = -2$. Substituting $z = -2$ into all of the equations gives
④ $x - y - 4 = 3$
⑤ $2x - 2y - 6 = 1$
⑥ $2x - 2y - 2 = 11$

There are no x and y variables that satisfy these equations, so the answer is no solution.

6. $3 = \dfrac{|A(3) - 2(-3) + 6(1) + 0|}{\sqrt{A^2 + (-2)^2 + 6^2}}$

$3\sqrt{(A^2 + 40)} = |3A + 12|$

$\sqrt{(A^2 + 40)} = |A + 4|$

$A^2 + 40 = A^2 + 8A + 16$

$24 = 8A$

$3 = A$

$A = 3$ is the only solution to this equation.

7. These lines are skew lines, and the plane containing the second line, $\vec{r} = (0, 0, 1) + t(1, 1, 0), t \in \mathbf{R}$, that is parallel to the first line will have direction vectors $(1, 1, 0)$ and $(3, 0, 1)$. So a normal to this plane is $(1, 1, 0) \times (3, 0, 1) = (1, -1, -3)$.

So the equation of this plane will be of the form $x - y - 3z + D = 0$. We want the point $(0, 0, 1)$ to be on this plane, and substituting this into the equation above gives $D = 3$. So the equation of the plane containing $\vec{r} = (0, 0, 1) + t(1, 1, 0), t \in \mathbf{R}$ and parallel to the first line is

$x - y - 3z + 3 = 0$.

Since $(0, 1, -1)$ is on the first line, the distance between these skew lines is the same as the distance between this point and the plane just determined.

By the distance formula, this distance is

$d = \dfrac{|(0) - (1) - 3(-1) + 3|}{\sqrt{1^2 + (-1)^2 + (-3)^2}}$

$= \dfrac{5}{\sqrt{11}}$

$\doteq 1.51.$

8. a.–b. We will do both of these parts at once.

The two given lines are

$\vec{r} = (1, -2, 5) + s(0, 1, -1), s \in \mathbf{R}$,

$\vec{r} = (1, -1, -2) + t(1, 0, -1), t \in \mathbf{R}$.

By converting to parametric form, a general point on the first line is

$U(1, s - 2, 5 - s)$,

and on the second line is

$V(1 + t, -1, -2 - t)$.

So the vector

$\overrightarrow{UV} = (t, 1 - s, s - t - 7)$.

If the points U and V are those that produce the minimal distance between these two lines, then $\overrightarrow{UV}$ will be perpendicular to both direction vectors, $(0, 1, -1)$ and $(1, 0, -1)$. In the first case, we get

$0 = (t, 1 - s, s - t - 7) \cdot (0, 1, -1)$

$= 8 - 2s + t$

$t = 2s - 8$

In the second case, we get

$0 = (t, 1 - s, s - t - 7) \cdot (1, 0, -1)$

$= 2t - s + 7$

Substituting $t = 2s - 8$ into this second equation, we get

$2(2s - 8) - s + 7 = 0$

$s = 3$

$t = 2s - 8$

$t = -2$

Substituting these values for s and t into U and V, we get

$U(1, 1, 2)$

$V(-1, -1, 0)$

So $U(1, 1, 2)$ is the point on the first line that produces the minimal distance to the second line at point $V(-1, -1, 0)$. This minimal distance is given by

$|\overrightarrow{UV}| = |(-2, -2, -2)|$

$= \sqrt{12}$

$\doteq 3.46$

Review Exercise, pp. 552–555

1. $2x - y = 31, x + 8y = -34, 3x + ky = 38$

$(2x - y = 31) - 2(x + 8y = -34)$

$= 0x - 17y = 99$

$y = -\dfrac{99}{17}, x = \dfrac{214}{17}$

$3\left(\dfrac{214}{17}\right) + k\left(\dfrac{-99}{17}\right) = 38$

$k = -\dfrac{4}{99}$

2. ① $x - y = 13$

② $3x + 2y = -6$

③ $x + 2y = -19$

$(2 \times$ Equation ①$)$ + equation ② $= 5x + 0y = 20$ or $x = 4$. Substituting $x = 4$ into equation ① gives $(4) - y = 13$ or $y = -9$. However, when you substitute this coordinates into the third equation, the third equation is not consistent, so there is no solution to this problem.

3. a. ① $x - y + 2z = 3$

② $2x - 2y + 3z = 1$

③ $2x - 2y + z = 11$

Equation ② − equation ③ $= 5z = -10$ or $z = -2$. Substituting $z = -2$ into all of the equations gives

④ $x - y - 4 = 3$

⑤ $2x - 2y - 6 = 1$

⑥ $2x - 2y - 2 = 11$

There are no x and y variables that satisfy these equations, so the answer is no solution.

b. ① $x + y + z = 300$
② $x + y - z = 98$
③ $x - y + z = 100$
Equation ② + equation ③ $= 2x = 198$ $x = 99$.
Substituting $x = 99$ into all three equations gives:
④ $y + z = 201$
⑤ $y - z = -1$
⑥ $-y + z = 1$
Equation ④ + equation ⑤ $= 2y = 200$ or
$y = 100$. You then get $z = 101$ after substituting both x and y into equation ①.
$(99, 100, 101)$
Check:
① $99 + 100 + 101 = 300$
② $99 + 100 - 101 = 98$
③ $99 - 100 + 101 = 100$

4. a. These four points will lie in the same plane if and only if the line determined by the first two points intersects the line determined by the last two points. The direction vector determined by the first two is

$$\vec{a} = (7, -5, 1) - (1, 2, 6)$$
$$= (6, -7, -5)$$

So these first two points determine the line with vector equation
$\vec{r} = (1, 2, 6) + s(6, -7, -5), s \in \mathbf{R}$.
The direction vector determined by the last two points is

$$\vec{b} = (-3, 5, 6) - (1, 1, 4)$$
$$= (-4, 4, 2)$$

So these first two points determine the line with vector equation
$\vec{r} = (1, 1, 4) + t(-4, 4, 2), t \in \mathbf{R}$.
Converting these two lines to parametric form, we obtain the equations
① $1 + 6s = 1 - 4t$
② $2 - 7s = 1 + 4t$
③ $6 - 5s = 4 + 2t$
Adding the first and second equations gives
$3 - s = 2$, so $s = 1$. Substituting this into the third equation, we get

$$1 = 4 + 2t$$
$$-3 = 2t$$

So $t = -\frac{3}{2}$. We need to check this s and t for consistency. Substituting $s = 1$ into the vector equation for the first line gives

$$\vec{r} = (1, 2, 6) + (1)(6, -7, -5)$$
$$= (7, -5, 1)$$

as a point on this line. Substituting $t = -\frac{3}{2}$ into the vector equation for the second line gives

$$\vec{r} = (1, 1, 4) + \left(-\frac{3}{2}\right)(-4, 4, 2)$$
$$= (1, 1, 4) + (6, -6, -3)$$
$$= (7, -5, 1)$$

as a point on this line. This means the two lines intersect, and so the four points given lie in the same plane.

b. Direction vectors for the plane containing the four points in part **a.** are $(6, -7, -5)$ and $(-4, 4, 2)$. So a normal to this plane is
$(6, -7, -5) \times (-4, 4, 2) = (6, 8, -4)$.
We will use the parallel normal $(3, 4, -2)$. So the equation of this plane is of the form
$3x + 4y - 2z + D = 0$.
Substitute in the point $(1, 2, 6)$ to find D.

$$3(1) + 4(2) - 2(6) + D = 0$$
$$D = 1$$

The equation of the plane is
$3x + 4y - 2z + 1 = 0$.
So, using the distance formula, this plane is distance

$$d = \frac{|3(0) + 4(0) - 2(0) + 1|}{|(3, 4, -2)|}$$
$$= \frac{1}{\sqrt{29}}$$
$$\doteq 0.19$$

from the origin.

5. Use the distance formula.

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

a. The distance from $A(-1, 1, 2)$ to
$3x - 4y - 12z - 8 = 0$

$$d = \frac{|3(-1) - 4(1) - 12(2) - 8|}{\sqrt{3^2 + (-4)^2 + (-12)^2}}$$
$$= 3$$

b. The distance from $B(3, 1, -2)$ to
$8x - 8y + 4z - 7 = 0$

$$d = \frac{|8(3) - 8(1) + 4(-2) - 7|}{\sqrt{8^2 + (-8)^2 + (4)^2}}$$
$$= \frac{1}{12} \text{ or } 0.08$$

6. $\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$
$3x - 4y - 5z = 0$
Find the parametric equations from the first equation, then substitute those equations into the second equation. Solve for t. Substitute that t-value into the first equation.
$\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$
$x = 3 + 2t$

$y = 1 - t$
$z = 1 + 2t$
$3(3 + 2t) - 4(1 - t) - 5(1 + 2t) = 0$
t can be any value to satisfy this value, so the two equations intersect along
$\vec{r} = (3, 1, 1) + t(2, -1, 2), t \in \mathbf{R}$.

7. a. ① $3x - 4y + 5z = 9$
② $6x - 9y + 10z = 9$
③ $9x - 12y + 15z = 9$
$3 \times (3x - 4y + 5z = 9) = 9x - 12y + 15z = 27$
There is no solution because the first and third equations are inconsistent.

b. ① $2x + 3y + 4z = 3$
② $4x + 6y + 8z = 4$
③ $5x + y - z = 1$
$2 \times (2x + 3y + 4z = 3) = 4x + 6y + 8z = 6$
There is no solution because the first and second equations are inconsistent.

c. ① $4x - 3y + 2z = 2$
② $8x - 6y + 4z = 4$
③ $12x - 9y + 6z = 1$
$3 \times (4x - 3y + 2z = 2) = 12x - 9y + 6z = 6$
There is no solution because the first and third equations are inconsistent.

8. a. ① $3x + 4y + z = 4$
② $5x + 2y + 3z = 2$
③ $6x + 8y + 2z = 8$
(Equation ①) − (2 × equation ②)
$= -7x - 5z = 0$
Letting $z = t$, then $x = -\frac{5}{7}t$ and $y = 1 + \frac{2}{7}t$.
$x = -\dfrac{5}{7}t, y = 1 + \dfrac{2}{7}t, z = t, t \in \mathbf{R}$

b. ① $4x - 8y + 12z = 4$
② $2x + 4y + 6z = 4$
③ $x - 2y - 3z = 4$
(Equation ①) + (4 × equation ③)
$= 24z = -12$ or $z = -\frac{1}{2}$. Letting $z = -\frac{1}{2}$ creates:

④ $4x - 8y = 10$
⑤ $2x + 4y = 7$

(Equation ④) + (2 × equation ⑤) = $8x = 24$ or $x = 3$. Substituting in $x = 3$ and $z = -\frac{1}{2}$ gives $y = \frac{1}{4}$
$x = 3, y = \dfrac{1}{4}, z = -\dfrac{1}{2}$

c. ① $x - 3y + 3z = 7$
② $2x - 6y + 6z = 14$
③ $-x + 3y - 3z = -7$
Letting $z = s$, then $y = t$ gives $x - 3t + 3s = 7$ or $x = -3s + 3t + 7$
$x = 3t - 3s + 7, y = t, z = s, s, t \in \mathbf{R}$

9. a. ① $3x - 5y + 2z = 4$
① $6x + 2y - z = 2$
① $6x - 3y + 8z = 6$
(Equation ②) − (2 × equation ①) = $12y - 5z = -6$
Setting $z = t$,
$12y - 5t = -6$ or $y = -\dfrac{1}{2} + \dfrac{5}{12}t$
Substituting these two values into the first equation gives $x = \frac{1}{2} + \frac{1}{36}t$
$x = \dfrac{1}{2} + \dfrac{1}{36}t, y = \dfrac{-1}{2} + \dfrac{5}{12}t, z = t, t \in \mathbf{R}$

b. ① $2x - 5y + 3z = 1$
② $4x + 2y + 5z = 5$
③ $2x + 7y + 2z = 4$
(Equation ②) − (2 × equation ①)
$= 12y - z = 3$
Setting $z = t$,
$12y - t = 3$ or $y = \dfrac{1}{4} + \dfrac{1}{12}t$
Substituting these two values into the first equation gives $x = \frac{9}{8} - \frac{31}{24}t$
$x = \dfrac{9}{8} - \dfrac{31}{24}t, y = \dfrac{1}{4} + \dfrac{1}{12}t, z = t, t \in \mathbf{R}$

10. a. $2x + y + z = 6$
$x - y - z = -9$
$3x + y = 2$
The first equation + the second equation gives $3x = -3$ or $x = -1$. Substituting $x = -1$ into the third equation, $3(-1) + y = 2$ or $y = 5$.
Substituting these two values into the first equation, $2(-1) + 5 + z = 6$ or $z = 3$
These three planes meet at the point $(-1, 5, 3)$.

b. ① $2x - y + 2z = 2$
② $3x + y - z = 1$
③ $x - 3y + 5z = 4$
Equation ① + equation ② = $5x + z = 3$
Equation ③ − (3 × equation ①) = $-5x - z = -2$.
These two equations are inconsistent, so the planes do not intersect at any point. Geometrically the planes form a triangular prism.

c. ① $2x + y - z = 0$
② $x - 2y + 3z = 0$
③ $9x + 2y - z = 0$

$2 \times$ equation ① + equation ② $= 5x + z = 0$, so $z = -5x$.
Equation ③ − equation ① $= 7x + y = 0$, so $y = -7x$.
Let $x = t$. The intersection of the planes is a line through the origin with equation $x = t, y = -7t, z = -5t, t \in \mathbf{R}$.

11. $\vec{r} = (2, -1, -2) + s(1, 1, -2), s \in \mathbf{R}$
By substituting in different s-values, you can find when the plane intersects the xz-plane when $y = 0$ and the xy-plane when $z = 0$.
The plane intersects the xz-plane at $(3, 0, -4)$ and the xy-plane at $(1, -2, 0)$. Then find the distance between these two points using the distance formula. The distance between these two points is 4.90.

12. a. $x - 2y + z + 4 = 0$
$\vec{r} = (3, 1, -5) + s(2, 1, 0), s \in \mathbf{R}$
$\vec{m} \cdot \vec{n} = (2, 1, 0) \cdot (1, -2, 1) = 0$ Since the line's direction vector is perpendicular to the normal of the plane and the point $(3, 1, -5)$ lies on both the line and the plane, the line is in the plane.

b. $\vec{r} = (7, 5, -1) + t(4, 3, 2), t \in \mathbf{R}$
$\vec{r} = (3, 1, -5) + s(2, 1, 0), s \in \mathbf{R}$
Solve for the parametric equations of both equations and then set them equal to each other.
$L_1: x = 7 + 4t, y = 5 + 3t, z = -1 + 2t$
$L_2: x = 3 + 2s, y = 1 + s, z = -5$
$z = -5 = -1 + 2t, t = -2$
$t = -2, x = -1, y = -1, z = -5$
$t = -2$ corresponds to the point $(-1, -1, -5)$

c. $x - 2y + z + 4 = 0$
$-1 - 2(-1) + (-5) + 4 = 0$
The point $(-1, -1, -5)$ is on the plane since it satisfies the equation of the plane.

d. $\vec{r} = (7, 5, -1) + t(4, 3, 2), t \in \mathbf{R}$
$(A, B, C) \cdot (4, 3, 2) = 0$
$A = 7, B = -2, C = -11$
$7x - 2y - 11z + D = 0$
$D = -50$
$7x - 2y - 11z - 50 = 0$

13. a. $\vec{r} = (3, 0, -1) + t(1, 1, 2), t \in \mathbf{R}$
$A(-2, 1, 1)$
$x = 3 + t, y = t, z = -1 + 2t$
$0 = 3 + t - x, 0 = t - y, 0 = -1 + 2t - z$
$\sqrt{(3 + t - x)^2 + (t - y)^2 + (-1 + 2t - z)^2}$
$\sqrt{(3 + t + 2)^2 + (t - 1)^2 + (-1 + 2t - 1)^2}$
$\sqrt{6t^2 + 30}$
$t = 0$ gives the lowest distance of 5.48

b. $t = 0$ corresponds to the point $(3, 0, -1)$

14. a. $\vec{r} = (1, -1, 1) + t(3, 2, 1), t \in \mathbf{R}$
$\vec{r} = (-2, -3, 0) + s(1, 2, 3), s \in \mathbf{R}$
Set the equations parametric equations equal to each other, and determine either the s or t-value. Find the point that corresponds to this value.
$L_1: x = 1 + 3t, y = -1 + 2t, z = 1 + t$
$L_2: x = -2 + s, y = -3 + 2s, z = 3s$
$x = 1 + 3t = -2 + s$
$y = -1 + 2t = -3 + 2s$
$z = 1 + t = 3s$
$s = 0, t = -1$
$s = 0$ corresponds to the point $(-2, -3, 0)$.

b. $\vec{r} = (1, -1, 1) + t(3, 2, 1), t \in \mathbf{R}$
$\vec{r} = (-2, -3, 0) + s(1, 2, 3), s \in \mathbf{R}$
$P(-2, -3, 0)$
$\vec{n_1} \times \vec{n_2} = (3, 2, 1) \times (1, 2, 3)$
$= (4, -8, 4) = (1, -2, 1)$
$\vec{r} = (-2, -3, 0) + t(1, -2, 1), t \in \mathbf{R}$

15. a. Since the plane we want contains L, we can use the direction vector for L, $(1, 2, -1)$, as one of the plane's direction vectors. Since the plane contains the point $(1, 2, -3)$ (which is on L) and the point $K(3, -2, 4)$, it will contain the direction vector $(3, -2, 4) - (1, 2, -3) = (2, -4, 7)$
To find a normal vector for the plane we want, take the cross product of these two direction vectors.
$(2, -4, 7) \times (1, 2, -1) = (-10, 9, 8)$
So the plane we seek will be of the form $-10x + 9y + 8z + D = 0$.
To determine the value of D, substitute in the point $(1, 2, -3)$ that is to be on this plane.
$-10(1) + 9(2) + 8(-3) + D = 0$
$D = 16$
The equation of the plane we seek is $-10x + 9y + 8z + 16 = 0$.

b. Using the distance formula, the distance from $S(1, 1, -1)$ to the plane $-10x + 9y + 8z + 16 = 0$ is

$$d = \frac{|-10(1) + 9(1) + 8(-1) + 16|}{|(-10, 9, 8)|}$$
$$= \frac{7}{\sqrt{245}}$$
$$\doteq 0.45$$

16. a. ① $x + y - z = 1$
② $2x - 5y + z = -1$
③ $7x - 7y - z = k$
Equation ① + equation ② = equation ④ $= 3x - 4y = 0$

Equation ② + equation ③ = equation ⑤
$= 9x - 12y = -1 + k$
For the solution to this system to be a line, equation ④ and equation ⑤ must be the proportional. $k = 1$ makes these two line proportional and the solution to this system a line.
b. In part a., we found that $k = 1$ by arriving at the equivalent system
① $3x - 4y = 0$
② $2x - 5y + z = -1$
③ $9x - 12y = 0$
As the first and third equations are proportional, this is really the same system as
① $3x - 4y = 0$
② $2x - 5y + z = -1$
Letting $x = t$ in the first equation, we see that $y = \frac{3}{4}t$. Substituting these values for x and y into the second equation, we find that

$$z = 5\left(\frac{3}{4}t\right) - 2t - 1$$
$$= \frac{7}{4}t - 1.$$

So the direction vector for the line that solves this system is $(1, \frac{3}{4}, \frac{7}{4})$, which is parallel to $(4, 3, 7)$.
So equivalent parametric equations of this line are
$x = 4t$
$y = 3t$
$z = -1 + 7t, t \in \mathbf{R}$.
So one possible vector equation of this line is
$\vec{r} = (0, 0, -1) + t(4, 3, 7), t \in \mathbf{R}$.
b. In part **a.**, we found that $k = 1$ by arriving at the equivalent system
① $3x - 4y = 0$
② $2x - 5y + z = -1$
③ $9x - 12y = 0$
As the first and third equations are proportional, this is really the same system as
① $3x - 4y = 0$
② $2x - 5y + z = -1$
Letting $x = t$ in the first equation, we see that $y = \frac{3}{4}t$. Substituting these values for x and y into the second equation, we find that

$$z = 5\left(\frac{3}{4}t\right) - 2t - 1$$
$$= \frac{7}{4}t - 1.$$

So the direction vector for the line that solves this system is $\left(1, \frac{3}{4}, \frac{7}{4}\right)$, which is parallel to (4, 3, 7).
So equivalent parametric equations of this line are
$x = 4t$
$y = 3t$
$z = -1 + 7t, t \in \mathbf{R}$.
So one possible vector equation of this line is
$\vec{r} = (0, 0, -1) + t(4, 3, 7), t \in \mathbf{R}$.

17. a. ① $x + 2y + z = 1$
② $2x - 3y - z = 6$
③ $3x + 5y + 4z = 5$
④ $4x + y + z = 8$
Equation ① + equation ② = equation ⑤
$= 3x - y = 7$
(4 × equation ②) + equation ③ = equation ⑥
$= 11x - 7y = 29$
(7 × equation ⑤) + equation ⑥
= equation ⑦ $= -10x = -20y$ or $x = 2$
Substituting into equation ⑤: $6 - y = 7y = -1$.
Substituting into equation ①: $2 + -2 + z = 1$ or $z = 1$.
$(2, -1, 1)$
b. ① $x - 2y + z = 1$
② $2x - 5y + z = -1$
③ $3x - 7y + 2z = 0$
④ $6x - 14y + 4z = 0$
Equation ② − (2 × equation ①)
= equation ⑤ $= -y - z = -3$,
Setting $z = t$,
$-y - t = -3$ or $y = 3 - t$
Substituting $y = 3 - t$ and $z = t$ into equation ①:
$x - 2(3 - t) + t = 1$ or $x = 7 - 3t$
$x = 7 - 3t, y = 3 - t, z = t, t \in \mathbf{R}$

18. ① $\frac{9a}{b} - 8b + \frac{3c}{b} = 4$
② $-\frac{3a}{b} + 4b + \frac{4c}{b} = 3$
③ $\frac{3a}{b} + 4b - \frac{4c}{b} = 3$
$x = \frac{a}{b}, y = b, z = \frac{c}{b}$
① $9x - 8y + 3z = 4$
② $-3x + 4y + 4z = 3$
③ $3x + 4y - 4z = 3$
③ + ② $= 8y = 6$
$y = \frac{3}{4}$
① $9x + 3z = 10$
② $-3x + 4z = 0$
③ $3x - 4z = 0$
① + 3② $= 15z = 10$

$z = \frac{2}{3}, x = \frac{8}{9}$

$y = \frac{3}{4} = b,$

$x = \frac{8}{9} = \frac{a}{b} = \frac{a}{\frac{3}{4}}, a = \frac{2}{3}$

$z = \frac{2}{3} = \frac{c}{b} = \frac{c}{\frac{3}{4}}, c = \frac{1}{2}$

$\left(\frac{2}{3}, \frac{3}{4}, \frac{1}{2}\right)$

19. First put the equation into parametric form. Then substitute the *x, y,* and *z*-values into $x + 2y - z + 10 = 0$ to determine t. Then substitute t back into the parametric equations to determine the coordinates.

$\frac{x+1}{-4} = \frac{y-2}{3} = \frac{z-1}{-2} = t$

$x = -4t - 1, y = 3t + 2, z = -2t + 1$

$x + 2y - 3z + 10 = 0$

$(-4t - 1) + 2(3t + 2) - 3(-2t + 1) + 10 = 0$

$t = -\frac{5}{4}$

$x = -4\left(-\frac{5}{4}\right) - 1, y = 3\left(-\frac{5}{4}\right) + 2,$

$z = -2\left(-\frac{5}{4}\right) + 1$

$\left(4, -\frac{7}{4}, \frac{7}{2}\right)$

20. Let $A'(a, b, c)$ denote the image point under this reflection. We want to find a, b, and c. The equation of the plane is $x - y + z - 1 = 0$, so letting $y = s$ and $z = t$, we get $x = 1 - t + s, s, t \in \mathbf{R}$. These are the parametric equations of this plane, so a general point on this plane has coordinates $P(1 - t + s, s, t)$.

So $\overrightarrow{PA} = (1, 0, 4) - (1 - t + s, s, t)$
$= (t - s, -s, 4 - t)$

The normal vector to this plane is $(1, -1, 1)$, and in order for $\overrightarrow{PA}$ to be perpendicular to the plane, it must be parallel to this normal. This means that $\overrightarrow{PA}$ and $(1, -1, 1)$ will have a cross product equal to the zero vector.

$(t - s, -s, 4 - t) \times (1, -1, 1)$
$= (4 - s - t, 4 + s - 2t, 2s - t)$
$= (0, 0, 0)$

So we get the system of equations

① $4 - s - t = 0$
② $4 + s - 2t = 0$
③ $2s - t = 0$

Adding the first two equations gives

$8 - 3t = 0$

$t = \frac{8}{3}$

Substituting this value for t into the third equation gives

$0 = 2s - t$
$= 2s - \frac{8}{3}$

$s = \frac{4}{3}$

Substituting these values for s and t into the equation for $\overrightarrow{PA}$, we get

$\overrightarrow{PA} = (t - s, -s, 4 - t)$
$= \left(\frac{8}{3} - \frac{4}{3}, -\frac{4}{3}, 4 - \frac{8}{3}\right)$
$= \left(\frac{4}{3}, -\frac{4}{3}, \frac{4}{3}\right)$

This is the vector that is normal to the plane, with its head at point $A(1, 0, 4)$ and tail at the point in the plane

$P(1 - t + s, s, t) = P\left(1 - \frac{8}{3} + \frac{4}{3}, \frac{4}{3}, \frac{8}{3}\right)$
$= \left(-\frac{1}{3}, \frac{4}{3}, \frac{8}{3}\right)$

So the vector

$\overrightarrow{PA'} = -\overrightarrow{PA}$
$= \left(-\frac{4}{3}, \frac{4}{3}, -\frac{4}{3}\right)$
$= (a, b, c) - \left(-\frac{1}{3}, \frac{4}{3}, \frac{8}{3}\right)$
$= \left(a + \frac{1}{3}, b - \frac{4}{3}, c - \frac{8}{3}\right)$

This means that $a = -\frac{5}{3}$, $b = -\frac{8}{3}$, and $c = -\frac{4}{3}$,
That is, the reflected point is $A'\left(-\frac{5}{3}, \frac{8}{3}, \frac{4}{3}\right)$.

21. a. The first plane has normal $(3, 1, 7)$ and the second has normal $(4, -12, 4)$. Their line of intersection will be perpendicular to both of these normals. So we can take as direction vector the cross product of these two normals.

$(3, 1, 7) \times (4, -12, 4) = (88, 16, -40)$
$= 8(11, 2, -5)$

So let's use $(11, 2, -5)$ as the direction vector for this line of intersection. To find a point on both of these planes, solve for z in the second plane, and substitute this into the equation for the first plane.

$$4x - 12y + 4z - 24 = 0$$
$$4z = 24 - 4x + 12y$$
$$z = 6 - x + 3y$$
$$\begin{aligned} 0 &= 3x + y + 7z + 3 \\ &= 3x + y + 7(6 - x + 3y) \\ &\quad + 3 \\ &= -4x + 22y + 45 \end{aligned}$$

If $y = 0$ in this last equation, then $x = \frac{45}{4}$ and

$$\begin{aligned} z &= 6 - x + 3y \\ &= 6 - \frac{45}{4} + 3(0) \\ &= -\frac{21}{4} \end{aligned}$$

The point $\left(\frac{45}{4}, 0, \frac{21}{4}\right)$, lies on both planes. So the vector equation of the line of intersection for the first two planes is

$$\vec{r} = \left(\frac{45}{4}, 0, -\frac{21}{4}\right) + t(11, 2, -5), t \in \mathbf{R}.$$

The corresponding parametric form is

$$x = \frac{45}{4} + 11t$$
$$y = 2t$$
$$z = -\frac{21}{4} - 5t, t \in \mathbf{R}.$$

We will use a similar procedure for the other two lines of intersection. For the third plane, the normal vector is (1, 2, 3). So a direction vector for the line of intersection between the first and third planes is

$$\begin{aligned} (3, 1, 7) \times (1, 2, 3) &= (-11, -2, 5) \\ &= -(11, 2, -5) \end{aligned}$$

We may use $(11, 2, -5)$ as the direction vector for this line of intersection. We find a point on both of these planes in the same way as before.

$$x + 2y + 3z - 4 = 0$$
$$x = 4 - 2y - 3z$$
$$\begin{aligned} 0 &= 3x + y + 7z + 3 \\ &= 3(4 - 2y - 3z) + y + 7z + 3 \\ &= -6y - 2z + 15 \end{aligned}$$

Taking $y = 0$ in this last equation, we get $z = \frac{15}{2}$ and

$$\begin{aligned} x &= 4 - 2y - 3z \\ &= 4 - 2(0) - 3\left(\frac{15}{2}\right) \\ &= -\frac{37}{2} \end{aligned}$$

A point on both the first and third planes is $\left(-\frac{37}{2}, 0, \frac{15}{2}\right)$. So the vector equation for this line of intersection is

$$\vec{r} = \left(-\frac{37}{2}, 0, \frac{15}{2}\right) + t(11, 2, -5), t \in \mathbf{R},$$

and the corresponding parametric equations are

$$x = -\frac{37}{2} + 11t$$
$$y = 2t$$
$$z = \frac{15}{2} - 5t, t \in \mathbf{R}.$$

Finally, we consider the line of intersection between the second and third planes. In this case, a direction vector is

$$\begin{aligned} (4, -12, 4) \times (1, 2, 3) &= (-44, -8, 20) \\ &= -4(11, 2, -5) \end{aligned}$$

We may use $(11, 2, -5)$ as the direction vector for this line of intersection. We find a point on both of these planes in the same way as before.

$$x + 2y + 3z - 4 = 0$$
$$x = 4 - 2y - 3z$$
$$\begin{aligned} 0 &= 4x - 12y + 4z - 24 \\ &= 4(4 - 2y - 3z) - 12y + 4z - 24 \\ &= -20y - 8z - 8 \end{aligned}$$

Taking $y = 0$ in this last equation, we get $z = -1$ and

$$\begin{aligned} x &= 4 - 2y - 3z \\ &= 4 - 2(0) - 3(-1) \\ &= 7 \end{aligned}$$

A point on both the second and third planes is $(7, 0, -1)$. So the vector equation for this line of intersection is

$$\vec{r} = (7, 0, -1) + t(11, 2, -5), t \in \mathbf{R},$$

and the corresponding parametric equations are

$$x = 7 + 11t$$
$$y = 2t$$
$$z = -1 - 5t, \in \mathbf{R}.$$

b. All three lines of intersection found in part a. have direction vector $(11, 2, -5)$, and so they are all parallel. Since no pair of normal vectors for these three planes is parallel, no pair of these planes is coincident.

22. ① $\dfrac{2}{a^2} + \dfrac{5}{b^2} + \dfrac{3}{c^2} = 40$

② $\dfrac{3}{a^2} - \dfrac{6}{b^2} - \dfrac{1}{c^2} = -3$

③ $\dfrac{9}{a^2} - \dfrac{5}{b^2} + \dfrac{4}{c^2} = 67$

$① + 3② = ④ = \frac{11}{a^2} + \frac{-13}{b^2} = 31$

$③ + 4② = ⑤ = \frac{21}{a^2} + \frac{-29}{b^2} = 55$

$21④ - 11⑤ = \frac{46}{b^2} = 46, b = +1, b = -1$

$\frac{21}{a^2} + \frac{-29}{1} = 55, a = \frac{1}{2}, a = -\frac{1}{2}$

$\frac{2}{0.25} + \frac{5}{1} + \frac{3}{c^2} = 40, c = \frac{1}{3}, c = -\frac{1}{3}$

$a = \frac{1}{2}, a = -\frac{1}{2}, b = 1, b = -1, c = \frac{1}{3}, c = -\frac{1}{3}$

Because each equation has each of a^2, b^2, and c^2, the possible solutions are all combinations of the positive and negative values for *a, b,* and *c*: $\left(\frac{1}{2}, 1, \frac{1}{3}\right)$, $\left(\frac{1}{2}, 1, -\frac{1}{3}\right)$, $\left(\frac{1}{2}, -1, \frac{1}{3}\right)$, $\left(\frac{1}{2}, -1, -\frac{1}{3}\right)$, $\left(-\frac{1}{2}, 1, \frac{1}{3}\right)$, $\left(-\frac{1}{2}, 1, -\frac{1}{3}\right)$, $\left(-\frac{1}{2}, -1, \frac{1}{3}\right)$, and $\left(\frac{1}{2}, -1, -\frac{1}{3}\right)$.

23. The general form of such a parabola is $y = ax^2 + bx + c$. We need to determine *a*, *b*, and *c*. Since $(-1, 2)$, $(1, -1)$, and $(2, 1)$ all lie on the parabola, we get the system of equations

$① \quad a - b + c = 2$

$② \quad a + b + c = -1$

$③ \quad 4a + 2b + c = 1$

Adding the first and second equations gives

$a + c = \frac{1}{2}$

Subtracting the first from the second equation gives

$2b = -3$

$b = -\frac{3}{2}$

Using the fact that $a + c = \frac{1}{2}$ and $b = -\frac{3}{2}$ in the third equation gives

$$\begin{aligned} 1 &= 4a + 2b + c \\ &= 3a + 2b + (a + c) \\ &= 3a + 2\left(-\frac{3}{2}\right) + \frac{1}{2} \\ &= 3a - \frac{5}{2} \end{aligned}$$

$\frac{7}{2} = 3a$

$a = \frac{7}{6}$

So using once more that $a + c = \frac{1}{2}$, we substitute this value in for *a* and get

$$\begin{aligned} \frac{1}{2} &= a + c \\ &= \frac{7}{6} + c \end{aligned}$$

$c = -\frac{2}{3}$

So the equation of the parabola we seek is

$y = \frac{7}{6}x^2 - \frac{3}{2}x - \frac{2}{3}.$

24. The equation of the plane is $4x - 5y + z - 9 = 0$, which has normal $(4, -5, 1)$. Converting this plane to parametric form gives

$x = s$

$y = t$

$z = 9 - 4s + 5t, s, t \in \mathbf{R}.$

So for any point $Y(s, t, 9 - 4s + 5t)$ on this plane, we can form the vector

$$\begin{aligned} \overrightarrow{XY} &= (s, t, 9 - 4s + 5t) - (3, 2, -5) \\ &= (s - 3, t - 2, 14 - 4s + 5t) \end{aligned}$$

This vector is perpendicular to the plane when it is parallel to the normal vector $(4, -5, 1)$. Two vectors are parallel precisely when their cross product is the zero vector.

$$\begin{aligned} &(s - 3, t - 2, 14 - 4s + 5t) \times (4, -5, 1) \\ &= (68 + 26t - 20s, 59 + 20t - 17s, 23 - 4t - 5s) \\ &= (0, 0, 0) \end{aligned}$$

So we get the system of equations

$① \quad 68 + 26t - 20s = 0$

$② \quad 59 + 20t - 17s = 0$

$③ \quad 23 - 4t - 5s = 0$

Subtracting four times the third equation from the first equation gives

$42t - 24 = 0$

$t = \frac{4}{7}$

Substituting this value for *t* into the second equation gives

$$\begin{aligned} 0 &= 59 + 20t - 17s \\ &= 59 + 20\left(\frac{4}{7}\right) - 17s \end{aligned}$$

$17s = \frac{493}{7}$

$s = \frac{29}{7}$

Substituting these values for *s* and *t* into the equation for *Y* gives

$$Y(s, t, 9 - 4s + 5t) = Y\left(\frac{29}{7}, \frac{4}{7}, 9 - 4\left(\frac{29}{7}\right) + 5\left(\frac{4}{7}\right)\right) = \left(\frac{29}{7}, \frac{4}{7}, -\frac{33}{7}\right)$$

So the point *M* we wanted is $M\left(\frac{29}{7}, \frac{4}{7}, -\frac{33}{7}\right)$.

25. $\dfrac{11x^2 - 14x + 9}{(3x - 1)(x^2 + 1)} = \dfrac{A}{3x - 1} + \dfrac{Bx + C}{x^2 + 1}$

$\dfrac{11x^2 - 14x + 9}{(3x - 1)(x^2 + 1)}$

$= \dfrac{A(x^2 + 1) + (Bx + C)\,3x - 1}{(3x - 1)(x^2 + 1)}$

$11x^2 - 14x + 9 = (A + 3B)x^2 + (3C - B)x + (A - C)$

$A - C = 9, 3C - B = -14, A + 3B = 11$

$B = 3C + 14, A = C + 9$

$A + 3(3C + 14) = 11, A + 9C = -31$

$(C + 9) + 9C = -31$

$10C = -40, C = -4$

$B = 3(-4) + 14 = 2, A = (-4) + 9 = 5$

$A = 5, B = 2, C = -4$

26. a. The vector

$\overrightarrow{EF} = (-1, -4, -6) - (4, 0, 3)$
$= (-5, -4, -3)$

This is a direction vector for the line containing the segment EF. The point $E(-1, -4, -6)$ is on this line, so the vector equation of this line is

$\vec{r} = (-1, -4, -6) + t(-5, -4, -3), t \in \mathbf{R}$.

b. Based on the equation of the line found in part a., a general point on this line is of the form

$J(-1 - 5t, -4 - 4t, -6 - 3t), t \in \mathbf{R}$.

For this general point, the vector

$\overrightarrow{JD} = (3, 0, 7) - (-1 - 5t, -4 - 4t, -6 - 3t)$
$= (4 + 5t, 4 + 4t, 13 + 3t)$

This vector will be perpendicular to the direction vector for the line found in part a. at the point J we seek. This means that

$0 = (4 + 5t, 4 + 4t, 13 + 3t) \cdot (-5, -4, -3)$
$= -5(4 + 5t) - 4(4 + 4t) - 3(13 + 3t)$
$= -75 - 50t$

$t = -\dfrac{3}{2}$

Substituting this value of t into the equation for the general point on the line in part a.,

$J(-1 - 5t, -4 - 4t, -6 - 3t)$

$= J\left(-1 - 5\left(-\dfrac{3}{2}\right), -4 - 4\left(-\dfrac{3}{2}\right), -6 - 3\left(-\dfrac{3}{2}\right)\right)$

$= \left(\dfrac{13}{2}, 2, -\dfrac{3}{2}\right)$

These are the coordinates for the point J we wanted.

c. Using the coordinates for J found in part b.,

$\overrightarrow{JD} = (3, 0, 7) - \left(\dfrac{13}{2}, 2, -\dfrac{3}{2}\right)$

$= \left(-\dfrac{7}{2}, -2, \dfrac{17}{2}\right)$

This vector forms the height of $\triangle DEF$, and the length of this vector is

$|\overrightarrow{JD}| = \left|\left(-\dfrac{7}{2}, -2, \dfrac{17}{2}\right)\right|$

$= \sqrt{\left(-\dfrac{7}{2}\right)^2 + (-2)^2 + \left(\dfrac{17}{2}\right)^2}$

$= \sqrt{\dfrac{177}{72}}$

$\doteq 9.41$

The length of the base of $\triangle DEF$ is

$|\overrightarrow{EF}| = |(-5, -4, -3)|$

$= \sqrt{(-5)^2 + (-4)^2 + (-3)^2}$

$= \sqrt{50}$

$\doteq 7.07$

So the area of $\triangle DEF$ equals

$\dfrac{1}{2}(\sqrt{50})\left(\sqrt{\dfrac{177}{2}}\right) = \dfrac{5}{2}\sqrt{177}$

$\doteq 33.26 \text{ units}^2$

27. $3x - 2z + 1 = 0$
$4x + 3y + 7 = 0$

$(5, -5, 5)$

$\vec{n_1} \times \vec{n_2} = (3, 0, -2) \times (4, 3, 0) = (6, -8, 9)$

$6x - 8y + 9z + D = 0$

$D = -115$

$6x - 8y + 9z - 115 = 0$

Chapter 9 Test, p. 556

1. a. $\vec{r}_1 = (4, 2, 6) + s(1, 3, 11), s \in \mathbf{R}$,

$\vec{r}_2 = (5, -1, 4) + t(2, 0, 9), t \in \mathbf{R}$

$L_1: x = 4 + s, y = 2 + 3s, z = 6 + 11s$

$L_2: x = 5 + 2t, y = -1, z = 4 + 9t$

$y = -1 = 2 + 3s$

$s = -1$

$L_1: x = 4 + (-1), y = 2 + 3(-1),$
$z = 6 + 11(-1)$

$x = 3, y = -1, z = -5$

$(3, -1, -5)$

b. $x - y + z + 1 = 0$

$3 - (-1) + (-5) + 1 = 0$

$3 + 1 - 5 + 1 = 0$

$0 = 0$

2. Use the distance equation.

$d = \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

a. $A(3, 2, 3)$

$8x - 8y + 4z - 7 = 0$

25. $\dfrac{11x^2 - 14x + 9}{(3x - 1)(x^2 + 1)} = \dfrac{A}{3x - 1} + \dfrac{Bx + C}{x^2 + 1}$

$\dfrac{11x^2 - 14x + 9}{(3x - 1)(x^2 + 1)}$

$= \dfrac{A(x^2 + 1) + (Bx + C)\,3x - 1}{(3x - 1)(x^2 + 1)}$

$11x^2 - 14x + 9 = (A + 3B)x^2 + (3C - B)x + (A - C)$

$A - C = 9, 3C - B = -14, A + 3B = 11$

$B = 3C + 14, A = C + 9$

$A + 3(3C + 14) = 11, A + 9C = -31$

$(C + 9) + 9C = -31$

$10C = -40, C = -4$

$B = 3(-4) + 14 = 2, A = (-4) + 9 = 5$

$A = 5, B = 2, C = -4$

26. a. The vector

$\overrightarrow{EF} = (-1, -4, -6) - (4, 0, 3)$

$= (-5, -4, -3)$

This is a direction vector for the line containing the segment EF. The point $E(-1, -4, -6)$ is on this line, so the vector equation of this line is

$\vec{r} = (-1, -4, -6) + t(-5, -4, -3), t \in \mathbf{R}$.

b. Based on the equation of the line found in part a., a general point on this line is of the form

$J(-1 - 5t, -4 - 4t, -6 - 3t), t \in \mathbf{R}$.

For this general point, the vector

$\overrightarrow{JD} = (3, 0, 7) - (-1 - 5t, -4 - 4t, -6 - 3t)$

$= (4 + 5t, 4 + 4t, 13 + 3t)$

This vector will be perpendicular to the direction vector for the line found in part a. at the point J we seek. This means that

$0 = (4 + 5t, 4 + 4t, 13 + 3t) \cdot (-5, -4, -3)$

$= -5(4 + 5t) - 4(4 + 4t) - 3(13 + 3t)$

$= -75 - 50t$

$t = -\dfrac{3}{2}$

Substituting this value of t into the equation for the general point on the line in part a.,

$J(-1 - 5t, -4 - 4t, -6 - 3t)$

$= J\left(-1 - 5\left(-\dfrac{3}{2}\right), -4 - 4\left(-\dfrac{3}{2}\right), -6 - 3\left(-\dfrac{3}{2}\right)\right)$

$= \left(\dfrac{13}{2}, 2, -\dfrac{3}{2}\right)$

These are the coordinates for the point J we wanted.

c. Using the coordinates for J found in part b.,

$\overrightarrow{JD} = (3, 0, 7) - \left(\dfrac{13}{2}, 2, -\dfrac{3}{2}\right)$

$= \left(-\dfrac{7}{2}, -2, \dfrac{17}{2}\right)$

This vector forms the height of $\triangle DEF$, and the length of this vector is

$|\overrightarrow{JD}| = \left|\left(-\dfrac{7}{2}, -2, \dfrac{17}{2}\right)\right|$

$= \sqrt{\left(-\dfrac{7}{2}\right)^2 + (-2)^2 + \left(\dfrac{17}{2}\right)^2}$

$= \sqrt{\dfrac{177}{72}}$

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The length of the base of $\triangle DEF$ is

$|\overrightarrow{EF}| = |(-5, -4, -3)|$

$= \sqrt{(-5)^2 + (-4)^2 + (-3)^2}$

$= \sqrt{50}$

$\doteq 7.07$

So the area of $\triangle DEF$ equals

$\dfrac{1}{2}(\sqrt{50})\left(\sqrt{\dfrac{177}{2}}\right) = \dfrac{5}{2}\sqrt{177}$

$\doteq 33.26 \text{ units}^2$

27. $3x - 2z + 1 = 0$

$4x + 3y + 7 = 0$

$(5, -5, 5)$

$\vec{n_1} \times \vec{n_2} = (3, 0, -2) \times (4, 3, 0) = (6, -8, 9)$

$6x - 8y + 9z + D = 0$

$D = -115$

$6x - 8y + 9z - 115 = 0$

Chapter 9 Test, p. 556

1. a. $\vec{r}_1 = (4, 2, 6) + s(1, 3, 11), s \in \mathbf{R}$,

$\vec{r}_2 = (5, -1, 4) + t(2, 0, 9), t \in \mathbf{R}$

L_1: $x = 4 + s, y = 2 + 3s, z = 6 + 11s$

L_2: $x = 5 + 2t, y = -1, z = 4 + 9t$

$y = -1 = 2 + 3s$

$s = -1$

L_1: $x = 4 + (-1), y = 2 + 3(-1)$,

$z = 6 + 11(-1)$

$x = 3, y = -1, z = -5$

$(3, -1, -5)$

b. $x - y + z + 1 = 0$

$3 - (-1) + (-5) + 1 = 0$

$3 + 1 - 5 + 1 = 0$

$0 = 0$

2. Use the distance equation.

$d = \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$

a. $A(3, 2, 3)$

$8x - 8y + 4z - 7 = 0$

$$d = \frac{|8x_0 - 8y_0 + 4z_0 - 7|}{\sqrt{(8)^2 + (-8)^2 + (4)^2}}$$
$$= \frac{|8(3) - 8(2) + 4(3) - 7|}{\sqrt{(8)^2 + (-8)^2 + (4)^2}}$$
$$= \frac{13}{12} \text{ or } 1.08$$

b. First, find any point on one of the planes, then use the other plane equation with the distance formula.

$2x - y + 2z - 16 = 0$
$2x - y + 2z + 24 = 0$
$2(8) - (0) + 2(0) - 16 = 0$
$A(8, 0, 0)$

$$d = \frac{|2x_0 - 1y_0 + 2z_0 + 24|}{\sqrt{(2)^2 + (-1)^2 + (2)^2}}$$
$$= \frac{|2(8) - 1(0) + 2(0) + 24|}{\sqrt{(2)^2 + (-1)^2 + (2)^2}}$$
$$= \frac{40}{3} \text{ or } 13.33$$

3. a. $L_1: 2x + 3y - z = 3$
$L_2: -x + y + z = 1$
$L_1 + 2L_2: 5y + z = 5$

$z = t,$
$5y + (t) = 5$
$y = 1 - \frac{t}{5}$
$-x + y + z = 1$
$-x + \left(1 - \frac{t}{5}\right) + (t) = 1$
$x = \frac{4t}{5}$

$x = \frac{4t}{5}, y = 1 - \frac{t}{5}, z = t, t \in \mathbf{R}$

b. To determine the point of intersection with the xz-plane, set the above y parametric equation equal to 0 and solve for the t. This t corresponds to the point of intersection.

$x = \frac{4t}{5}, y = 1 - \frac{t}{5}, z = t, t \in \mathbf{R}$
$0 = 1 - \frac{t}{5}$
$t = 5$
$x = \frac{4(5)}{5}, y = 1 - \frac{(5)}{5}, z = (5), t \in \mathbf{R}$
$(4, 0, 5)$

4. a. ① $x - y + z = 10$
② $2x + 3y - 2z = -21$
③ $\frac{1}{2}x + \frac{2}{5}y + \frac{1}{4}z = -\frac{1}{2}$

Equation ② + (2 × equation ①) =
$4x + y = -1$
Equation ② + (8 × equation ③) =
$6x + \frac{31}{5}y = -25$

$$-\frac{31}{5}(4x + y = -1)$$
$$\underline{+\left(6x + \frac{31}{5}y = -25\right)}$$
$-18.8x = -18.8$
$x = 1$
$4(1) + y = -1$
$y = -5$
$(1) - (-5) + z = 10$
$z = 4$
$(1, -5, 4)$

b. The three planes intersect at this point.

5. a. ① $x - y + z = -1$
② $2x + 2y - z = 0$
③ $x - 5y + 4z = -3$

Equation ② + (2 × equation ①) =
$4x + z = -2$

$4x + z = -2$
$z = t$
$4x + (t) = -2$
$x = -\frac{1}{2} - \frac{t}{4}$
$x - y + z = -1$
$\left(-\frac{1}{2} - \frac{t}{4}\right) - y + (t) = -1$
$y = \frac{3t}{4} + \frac{1}{2}$

$x = -\frac{1}{2} - \frac{t}{4}, y = \frac{3t}{4} + \frac{1}{2}, z = t, t \in \mathbf{R}$

b. The three planes intersect at this line.

6. a. $L_1: x + y + z = 0$
$L_2: x + 2y + 2z = 1$
$L_3: 2x - y + mz = n$
$L_2 + 2L_3: 5x + 0y + (2m + 2)z = 2n + 1$
$L_1 + L_3: 3x + 0y + (m + 1)z = n$

$$\frac{5}{3}(3x + 0y + (m + 1)z = n)$$
$$= 5x + 0y + \frac{5}{3}(m + 1)z = \frac{5}{3}n$$

Then set the two new equations to each other and solve for a m and n value that would give equivalent equations.

$$5x + 0y + \frac{5}{3}(m+1)z = \frac{5}{3}n$$
$$5x + 0y + (2m+2)z = 2n + 1$$
$$2m + 2 = \frac{5}{3}(m+1)$$
$$m = -1$$
$$\frac{5}{3}n = 2n + 1$$
$$n = -3$$

b. L_1: $x + y + z = 0$
L_2: $x + 2y + 2z = 1$
L_3: $2x - y - z = -3$
$L_1 + L_2$: $3x = -3, x = -1$
$(-1) + y + z = 0$
$z = t$
$(-1) + y + (t) = 0$
$y = 1 - t$
$x = -1, y = 1 - t, z = t, t \in \mathbf{R}$

7. First find the parametric equations of each line. Then set these equations equal to each other to find a set of new equations. Use the dot product to determine another set of equations that you will solve for t and s. Find the corresponding points to these values and the distance between them, which is the distance between the two lines.

L_1: $\vec{r} = (-1, -3, 0) + s(1, 1, 1), s \in \mathbf{R}$
L_2: $\vec{r} = (-5, 5, -8) + t(1, 2, 5), t \in \mathbf{R}$
L_1: $x = -1 + s, y = -3 + s, z = s$
L_2: $x = -5 + t, y = 5 + 2t, z = -8 + 5t$

$$\overrightarrow{UV} = [(-1+s) - (-5+t), (-3+s) - (5+2t), s - (-8+5t)]$$
$$\overrightarrow{UV} = (4 + s - t, -8 + s - 2t, s + 8 - 5t)$$
$$m_1 \cdot \overrightarrow{UV} = 0$$
$$(1, 1, 1) \cdot (4 + s - t, -8 + s - 2t, s + 8 - 5t) = 0$$
$$(1, 2, 5) \cdot (4 + s - t, -8 + s - 2t, s + 8 - 5t) = 0$$

L_4: $4 + 3s - 8t = 0$,
L_5: $28 + 8s - 30t = 0$
$8 \times L_1 + (-3) \times L_2$ yields
$32 + 24s - 64t - 84 - 24s + 90t = 0$, so $t = 2$.
Then $s = 4$. The points corresponding to these values of s and t are $(-1, 3, 0) + 4(1, 1, 1) = (3, 1, 4)$ and $(-5, 5, -8) + 2(1, 2, 5) = (-3, 9, 2)$.

$$d = \sqrt{(3 - (-3))^2 + (1 - 9)^2 + (4 - 2)^2}$$
$$= \sqrt{(6)^2 + (-8)^2 + (2)^2}$$
$$= \sqrt{36 + 64 + 4}$$
$$= \sqrt{104} \text{ or } 10.20$$

Cumulative Review of Vectors, pp. 557–560

1. a. The angle, θ, between the two vectors is found from the equation $\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.

$$\vec{a} \cdot \vec{b} = (2, -1, -2) \cdot (3, -4, 12)$$
$$= 2(3) - 1(-4) - 2(12)$$
$$= -14$$
$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2}$$
$$= 3$$
$$|\vec{b}| = \sqrt{3^2 + (-4)^2 + 12^2}$$
$$= 13$$
$$\text{So } \theta = \cos^{-1}\left(\tfrac{-14}{3 \times 13}\right)$$
$$\doteq 111.0°$$

b. The scalar projection of $\vec{a}$ on $\vec{b}$ is equal to $|\vec{a}|\cos(\theta)$, where θ is the angle between the two vectors. So from the above work, $\cos(\theta) = \frac{-14}{3 \times 13}$ and $|\vec{a}| = 3$, so the scalar projection of $\vec{a}$ on $\vec{b}$ is $\frac{-14}{3 \times 13} \times 3 = -\frac{14}{13}$. The vector projection of $\vec{a}$ on $\vec{b}$ is equal to the scalar projection multiplied by the unit vector in the direction of $\vec{b}$. So the vector projection is $-\frac{14}{13} \times \frac{1}{13}(3, -4, 12) = (-\frac{52}{169}, \frac{56}{169}, -\frac{168}{169})$.

c. The scalar projection of $\vec{b}$ on $\vec{a}$ is equal to $|\vec{b}|\cos(\theta)$, where θ is the angle between the two vectors. So from the above work, $\cos(\theta) = \frac{-14}{3 \times 13}$ and $|\vec{b}| = 13$, so the scalar projection of $\vec{a}$ on $\vec{b}$ is $\frac{-14}{3 \times 13} \times 13 = -\frac{14}{3}$. The vector projection of $\vec{b}$ on $\vec{a}$ is equal to the scalar projection multiplied by the unit vector in the direction of $\vec{a}$. So the vector projection is $-\frac{14}{3} \times \frac{1}{3}(2, -1, -2) = (-\frac{28}{9}, \frac{14}{9}, \frac{28}{9})$.

2. a. Since the normal of the first plane is $(4, 2, 6)$ and the normal of the second is $(1, -1, 1)$, which are not scalar multiples of each other, there is a line of intersection between the planes.
The next step is to use the first and second equations to find an equation with a zero for the coefficient of x. The first equation minus four times the second equation yields $0x + 6y + 2z + 6 = 0$. We may divide by two to simplify, so $3y + z + 3 = 0$. If we let $y = t$, then $3t + z + 3 = 0$, or $z = -3 - 3t$. Substituting these into the second equation yields $x - (t) + (-3 - 3t) - 5 = 0$ or $x = 8 + 4t$.
So the equation of the line in parametric form is $x = 8 + 4t, y = t, z = -3 - 3t, t \in \mathbf{R}$.

$$5x + 0y + \frac{5}{3}(m + 1)z = \frac{5}{3}n$$
$$5x + 0y + (2m + 2)z = 2n + 1$$
$$2m + 2 = \frac{5}{3}(m + 1)$$
$$m = -1$$
$$\frac{5}{3}n = 2n + 1$$
$$n = -3$$

b. L_1: $x + y + z = 0$
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L_3: $2x - y - z = -3$
$L_1 + L_2$: $3x = -3, x = -1$
$(-1) + y + z = 0$
$z = t$
$(-1) + y + (t) = 0$
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$x = -1, y = 1 - t, z = t, t \in \mathbf{R}$

7. First find the parametric equations of each line. Then set these equations equal to each other to find a set of new equations. Use the dot product to determine another set of equations that you will solve for t and s. Find the corresponding points to these values and the distance between them, which is the distance between the two lines.
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L_1: $x = -1 + s, y = -3 + s, z = s$
L_2: $x = -5 + t, y = 5 + 2t, z = -8 + 5t$
$\overrightarrow{UV} = [(-1 + s) - (-5 + t), (-3 + s) - (5 + 2t), s - (-8 + 5t)]$
$\overrightarrow{UV} = (4 + s - t, -8 + s - 2t, s + 8 - 5t)$
$m_1 \cdot \overrightarrow{UV} = 0$
$(1, 1, 1) \cdot (4 + s - t, -8 + s - 2t, s + 8 - 5t) = 0$
$(1, 2, 5) \cdot (4 + s - t, -8 + s - 2t, s + 8 - 5t) = 0$
L_4: $4 + 3s - 8t = 0,$
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$$d = \sqrt{(3 - (-3))^2 + (1 - 9)^2 + (4 - 2)^2}$$
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So $\theta = \cos^{-1}\left(\frac{-14}{3 \times 13}\right)$
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b. The scalar projection of $\vec{a}$ on $\vec{b}$ is equal to $|\vec{a}|\cos(\theta)$, where θ is the angle between the two vectors. So from the above work, $\cos(\theta) = \frac{-14}{3 \times 13}$ and $|\vec{a}| = 3$, so the scalar projection of $\vec{a}$ on $\vec{b}$ is $\frac{-14}{3 \times 13} \times 3 = -\frac{14}{13}$. The vector projection of $\vec{a}$ on $\vec{b}$ is equal to the scalar projection multiplied by the unit vector in the direction of $\vec{b}$. So the vector projection is $-\frac{14}{13} \times \frac{1}{13}(3, -4, 12) = (-\frac{52}{169}, \frac{56}{169}, -\frac{168}{169})$.

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2. a. Since the normal of the first plane is (4, 2, 6) and the normal of the second is $(1, -1, 1)$, which are not scalar multiples of each other, there is a line of intersection between the planes.
The next step is to use the first and second equations to find an equation with a zero for the coefficient of x. The first equation minus four times the second equation yields $0x + 6y + 2z + 6 = 0$. We may divide by two to simplify, so $3y + z + 3 = 0$. If we let $y = t$, then $3t + z + 3 = 0$, or $z = -3 - 3t$. Substituting these into the second equation yields $x - (t) + (-3 - 3t) - 5 = 0$ or $x = 8 + 4t$.
So the equation of the line in parametric form is $x = 8 + 4t, y = t, z = -3 - 3t, t \in \mathbf{R}$.

To check that this is correct, we substitute in the solution to both initial equations

$$\begin{aligned}4x + 2y + 6z - 14 &= 4(8 + 4t) + 2(t)\\ &\quad + 6(-3 - 3t) - 14\\ &= 0\end{aligned}$$

and $x - y + z - 5$

$= (8 + 4t) - (t) + (-3 - 3t) - 5$

$= 0.$

Hence the line given by the parametric equation above is the line of intersection for the planes.

b. The angle between two planes is the same as the angle between their corresponding normal vectors.

$$\begin{aligned}|(4, 2, 6)| &= \sqrt{4^2 + 2^2 + 6^2}\\ &= \sqrt{56}\\ |(1, -1, 1)| &= \sqrt{1^2 + 1^2 + 1^2}\\ &= \sqrt{3}\end{aligned}$$

$(4, 2, 6) \cdot (1, -1, 1) = 8$, so the angle between the planes is $\cos^{-1}\left(\frac{8}{\sqrt{3}\sqrt{56}}\right) \doteq 51.9°$.

3. a. We have that $\cos(60°) = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}$. Also since $\vec{x}$ and $\vec{y}$ are unit vectors, $|\vec{x}| = |\vec{y}| = 1$, and moreover $\cos(60°) = \frac{1}{2}$. So $\vec{x} \cdot \vec{y} = \frac{\vec{x} \cdot \vec{y}}{1 \times 1} = \frac{1}{2}$.

b. Scalar multiples can be brought out to the front of dot products. Hence $2\vec{x} \cdot 3\vec{y} = (2)(3)(\vec{x} \cdot \vec{y})$, and so by part **a.**, $2\vec{x} \cdot 3\vec{y} = 2 \times 3 \times \frac{1}{2} = 3$.

c. The dot product is distributive,

so $(2\vec{x} - \vec{y}) \cdot (\vec{x} + 3\vec{y})$

$= 2\vec{x} \cdot (\vec{x} + 3\vec{y}) - \vec{y} \cdot (\vec{x} + 3\vec{y})$

$= 2\vec{x} \cdot \vec{x} + 2\vec{x} \cdot 3\vec{y} - \vec{y} \cdot \vec{x} - \vec{y} \cdot 3\vec{y}$

$= 2\vec{x} \cdot \vec{x} + 2\vec{x} \cdot 3\vec{y} - \vec{x} \cdot \vec{y} - 3\vec{y} \cdot \vec{y}$

Since $\vec{x}$ and $\vec{y}$ are unit vectors, $\vec{x} \cdot \vec{x} = \vec{y} \cdot \vec{y} = 1$, and so by using the values found in part **a.** and **b.**,

$$\begin{aligned}(2\vec{x} - \vec{y}) \cdot (\vec{x} + 3\vec{y}) &= 2(1) + (3) - \left(\tfrac{1}{2}\right) - 3(1)\\ &= \frac{3}{2}\end{aligned}$$

4. a. $2(\vec{i} - 2\vec{j} + 3\vec{k}) - 4(2\vec{i} + 4\vec{j} + 5\vec{k}) - (\vec{i} - \vec{j})$

$= 2\vec{i} - 4\vec{j} + 6\vec{k} - 8\vec{i} - 16\vec{j} - 20\vec{k} - \vec{i} + \vec{j}$

$= -7\vec{i} - 19\vec{j} - 14\vec{k}$

b. $-2(3\vec{i} - 4\vec{j} - 5\vec{k}) \cdot (2\vec{i} + 3\vec{k}) + 2\vec{i} \cdot (3\vec{j} - 2\vec{k})$

$= -2(3\vec{i} - 4\vec{j} - 5\vec{k}) \cdot (2\vec{i} + 0\vec{j} + 3\vec{k})$

$\quad + 2(\vec{i} + 0\vec{j} + 0\vec{k}) \cdot (0\vec{i} + 3\vec{j} - 2\vec{k})$

$= -2(3(2) - 4(0) - 5(3)) + 2(1(0)$

$\quad + 0(3) + 0(-2))$

$= -2(-9) + 2(0)$

$= 18$

5. The direction vectors for the positive x-axis, y-axis, and z-axis are (1, 0, 0), (0, 1, 0), and (0, 0, 1), respectively.

$$\begin{aligned}|(4, -2, -3)| &= \sqrt{4^2 + (-2)^2 + (-3)^2}\\ &= \sqrt{29},\end{aligned}$$

$$\begin{aligned}\text{and } |(1, 0, 0)| &= |(0, 1, 0)|\\ &= |(0, 0, 1)|\\ &= \sqrt{1}\\ &= 1.\end{aligned}$$

$(4, -2, -3) \cdot (1, 0, 0) = 4$, so the angle the vector makes with the x-axis is $\cos^{-1}\left(\frac{4}{1\sqrt{29}}\right) \doteq 42.0°$.

$(4, -2, -3) \cdot (0, 1, 0) = -2$, so the angle the vector makes with the y-axis is $\cos^{-1}\left(\frac{-2}{1\sqrt{29}}\right) = 111.8°$.

$(4,-2,-3) \cdot (0, 0, 1) = -3$, hence the angle the vector makes with the z-axis is $\cos^{-1}\left(\frac{-3}{1\sqrt{29}}\right) \doteq 123.9°$.

6. a.

$$\begin{aligned}\vec{a} \times \vec{b} &= (1, -2, 3) \times (-1, 1, 2)\\ &= (-2(2) - 3(1), 3(-1) - 1(2),\\ &\quad 1(1) - (-2)(-1))\\ &= (-7, -5, -1)\end{aligned}$$

b. By the scalar law for vector multiplication,

$$\begin{aligned}2\vec{a} \times 3\vec{b} &= 2(3)(\vec{a} \times \vec{b})\\ &= 6(\vec{a} \times \vec{b})\\ &= 6(-7, -5, -1) = (-42, -30, -6)\end{aligned}$$

c. The area of a parallelogram determined by $\vec{a}$ and $\vec{b}$ is equal to the magnitude of the cross product of $\vec{a}$ and $\vec{b}$.

$$\begin{aligned}A &= \text{area of parallelogram}\\ &= |\vec{a} \times \vec{b}|\\ &= |(-7, -5, -1)|\\ &= \sqrt{(-7)^2 + (-5)^2 + (-1)^2}\\ &\doteq 8.66 \text{ square units}\end{aligned}$$

d.

$$\begin{aligned}(\vec{b} \times \vec{a}) &= -(\vec{a} \times \vec{b})\\ &= -(-7, -5, -1)\\ &= (7, 5, 1)\end{aligned}$$

$$\begin{aligned}\text{So } \vec{c} \cdot (\vec{b} \times \vec{a}) &= (3, -4, -1) \cdot (7, 5, 1)\\ &= 3(7) - 4(5) - 1(1)\\ &= 0\end{aligned}$$

7. A unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ can be determined from any vector perpendicular to both $\vec{a}$ and $\vec{b}$. $\vec{a} \times \vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$.

$$\begin{aligned}\vec{a} \times \vec{b} &= (1, -1, 1) \times (2, -2, 3)\\ &= (-1(3) - 1(-2), 1(2) - 1(3),\\ &\quad 1(-2) - (-1)(2))\\ &= (-1, -1, 0)\end{aligned}$$

$$|\vec{a} \times \vec{b}| = |(-1, -1, 0)|$$
$$= \sqrt{(-1)^2 + (-1)^2 + 0^2}$$
$$= \sqrt{2}$$

So $\frac{1}{\sqrt{2}}(-1, -1, 0) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ is an unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is another.

8. a. Answers may vary. For example:
A direction vector for the line is $\overrightarrow{AB}$.
$$\overrightarrow{AB} = (1, 2, 3) - (2, -3, 1)$$
$$= (-1, 5, 2)$$
Since $A(2, -3, 1)$ is a point on the line,
$\vec{r} = (2, -3, 1) + t(-1, 5, 2), t \in \mathbf{R}$, is a vector equation for a line and the corresponding parametric equation is $x = 2 - t, y = -3 + 5t, z = 1 + 2t$, $t \in \mathbf{R}$.

b. If the x-coordinate of a point on the line is 4, then $2 - t = 4$, or $t = -2$. At $t = -2$, the point on the line is $(2, -3, 1) - 2(-1, 5, 2) = (4, -13, -3)$. Hence $C(4, -13, -3)$ is a point on the line.

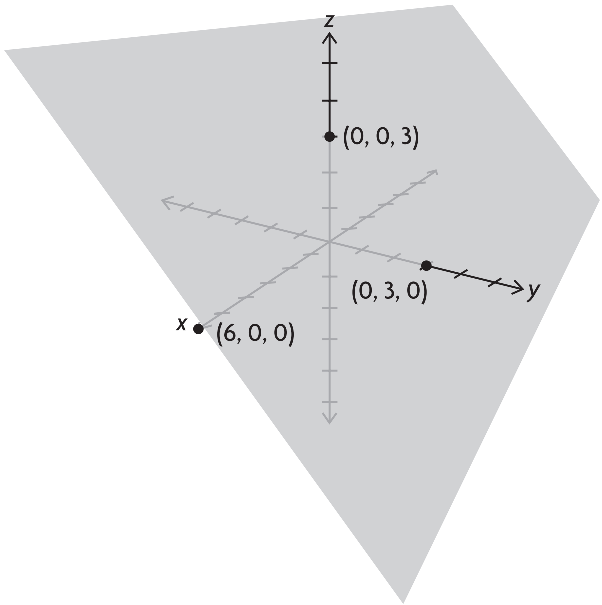
9. The direction vector of the first line is $(-1, 5, 2)$, while the direction vector for the second line is $(1, -5, -2) = -(-1, 5, 2)$. So the direction vectors for the line are collinear. Hence the lines are parallel. The lines coincide if and only if for any point on the first line and any point on the second line, the vector connecting the two points is a multiple of the direction vector for the lines.
(2, 0, 9) is a point on the first line and $(3, -5, 10)$ is a point on the second line.
$(2, 0, 9) - (3, -5, 10) = (-1, 5, -1) \neq k(-1, 5, 2)$ for any $k \in \mathbf{R}$. Hence the lines are parallel and distinct.

10. The direction vector for the parallel line is (0, 1, 1). Since parallel lines have collinear direction vectors, (0, 1, 1) can be used as a direction vector for the line. Since (0, 0, 4) is a point on the line, $\vec{r} = (0, 0, 4) + t(0, 1, 1), t \in \mathbf{R}$, is a vector equation for a line and the corresponding parametric equation is $x = 0, y = t, z = 4 + t, t \in \mathbf{R}$.

11. The line is parallel to the plane if and only if the direction vector for the line is perpendicular to the normal vector for the plane. The normal vector for the plane is $(2, 3, c)$. The direction vector for the line is (2, 3, 1). The vectors are perpendicular if and only if the dot product between the two is zero.
$$(2, 3, c) \cdot (2, 3, 1) = 2(2) + 3(3) + c(1)$$
$$= 13 + c$$
So if $c = -13$, then the dot product of normal vector and the direction vector is zero. Hence for $c = -13$, the line and plane are parallel.

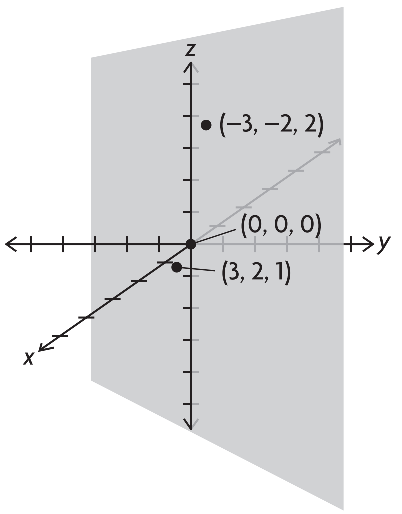
12. First put the line in its corresponding parametric form. (3, 1, 5) is a direction vector and $(2, -5, 3)$ is the origin point, so a parametric equation for the line is $x = 2 + 3s, y = -5 + s, z = 3 + 5s, s \in \mathbf{R}$. If we substitute these coordinates into the equation of the plane, we may find the s value where the line intersects the plane.
$$5x + y - 2z + 2$$
$$= 5(2 + 3s) + (-5 + s) - 2(3 + 5s) + 2$$
$$= 10 + 15s + \ -5 + s - 6 - 10s + 2$$
$$= 1 + 6s$$
So if $5x + y - 2z + 2 = 0$, then $1 + 6s = 0$ or $s = -\frac{1}{6}$. At $s = -\frac{1}{6}$, the point on the line is $\left(\frac{3}{2}, -\frac{31}{6}, \frac{13}{6}\right)$.
To check that this point is also on the plane, we substitute the x, y, z values into the plane equation and check that it equals zero.
$$5x + y - 2z + 2 = 5\left(\frac{3}{2}\right) + \left(-\frac{31}{6}\right) - 2\left(\frac{13}{6}\right) + 2$$
$$= 0$$
Hence $\left(\frac{3}{2}, -\frac{31}{6}, \frac{13}{6}\right)$ is the point of intersection between the line and the plane.

13. a.



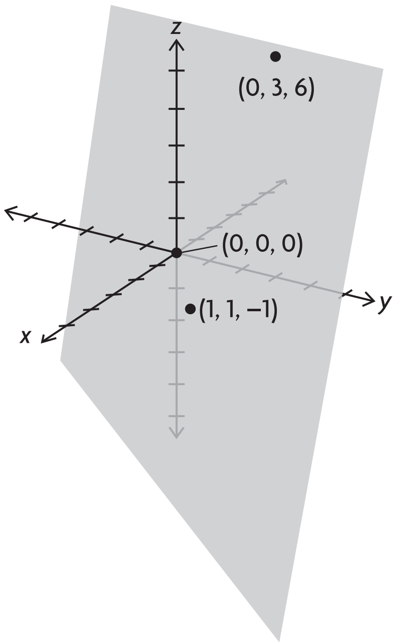
Two direction vectors are:
$(0, 3, 0) - (0, 0, 3) = (0, 3, -3)$
and
$(6, 0, 0) - (0, 0, 3) = (6, 0, -3)$.

b.



Two direction vectors are:
$(-3, -2, 2) - (0, 0, 0) = (-3, -2, 2)$
and
$(3, 2, 1) - (0, 0, 0) = (3, 2, 1)$.

c.



Two direction vectors are:
$(0, 3, 6) - (0, 0, 0) = (0, 3, 6)$
and
$(1, 1, -1) - (0, 0, 0) = (1, 1, -1)$.

14. The plane is the right bisector joining $P(1, -2, 4)$ and its image. The line connecting the two points has a direction vector equal to that of the normal vector for the plane. The normal vector for the plane is $(2, -3, -4)$. So the line connecting the two points is $(1, -2, 4) + t(2, -3, -4), t \in \mathbf{R}$, or in corresponding parametric form is $x = 1 + 2t$, $y = -2 - 3t$, $z = 4 - 4t$, $t \in \mathbf{R}$.
The intersection of this line and the plane is the bisector between P and its image. To find this point we substitute the parametric equation into the plane equation and solve for t.

$$\begin{aligned}&2x - 3y - 4z + 66\\ &= 2(1 + 2t) - 3(-2 - 3t) - 4(4 - 4t) + 66\\ &= 2 + 4t + 6 + 9t - 16 + 16t + 66\\ &= 58 + 29t\end{aligned}$$

So if $2x - 3y - 4z + 66 = 0$, then $58 + 29t = 0$, or $t = -2$.
So the point of intersection is occurs at $t = -2$, since the origin point is P and the intersection occurs at the midpoint of the line connecting P and its image, the image point occurs at $t = 2 \times (-2) = -4$.
So the image point is at $x = 1 + 2(-4) = -7$, $y = -2 - 3(-4) = 10$, $z = 4 - 4(-4) = 20$.
So the image point is $(-7, 10, 20)$.

15. Let (a, b, c) be the direction vector for this line. So a line equation is $\vec{r} = (1, 0, 2) + t(a, b, c), t \in \mathbf{R}$.
Since (1, 0, 2) is not on the other line, we may choose a, b, and c such that the intersection occurs at $t = 1$. Since the line is supposed to intersect the given line at a right angle, the direction vectors should be perpendicular. The direction vectors are perpendicular if and only if their dot product is zero.
The direction vector for the given line is (1, 1, 2).
$(a, b, c) \cdot (1, 1, 2) = a + b + 2c = 0$, so $b = -a - 2c$.
Also $(1, 0, 2) + (a, b, c) = (1 + a, b, 2 + c)$ is the point of intersection.
By substituting for b,
$(1 + a, b, 2 + c) = (1 + a, -a - 2c, 2 + c)$.
So for some s value,

$$\begin{aligned}x &= -2 + s = 1 + a\\ y &= 3 + s = -a - 2c\\ z &= 4 + 2s = 2 + c\end{aligned}$$

Subtracting the first equation from the second yields the equation, $5 + 0s = -2a - 2c - 1$.
Simplifying this gives $6 = -2a - 2c$ or just $a + c = -3$.
Subtracting twice the first equation from the third yields the equation, $8 = -2a + c$.
So $a + c = -3$ and $-2a + c = 8$, which is two equations with two unknowns. Twice the first plus the second equations gives $0a + 3c = 2$ or $c = \frac{2}{3}$.
Solving back for a gives $-\frac{11}{3}$ and since $b = -a - 2c$, $b = \frac{7}{3}$. Since $a + b + 2c = 0$, the direction vectors,

$(1, 1, 2)$ and (a, b, c) are perpendicular. A direction vector for the line is $(-11, 7, 2)$.
We need to check that
$(1, 0, 2) + (a, b, c) = \left(\frac{-8}{3}, \frac{7}{3}, \frac{8}{3}\right)$ is a point on the given line.
$x = -2 + s = -\frac{8}{3}$, at $s = -\frac{2}{3}$. The point on the given line at $s = -\frac{2}{3}$ is $\left(\frac{-8}{3}, \frac{7}{3}, \frac{8}{3}\right)$. Hence
$\vec{q} = (1, 0, 2) + t(-11, 7, 2)$, $t \in \mathbf{R}$, is a line that intersects the given line at a right angle.

16. a. The Cartesian equation is found by taking the cross product of the two direction vectors, $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\begin{aligned}\overrightarrow{AB} &= (-2, 0, 0) - (1, 2, 3)\\ &= (-3, -2, -3)\\ \overrightarrow{AC} &= (1, 4, 0) - (1, 2, 3) = (0, 2, -3)\\ \overrightarrow{AB} \times \overrightarrow{AC} &= (-2(-3) - (-3)(2),\\ &\quad -3(0) - (-3)(-3),\\ &\quad -3(2) - (-2)(0))\\ &= (12, -9, -6)\end{aligned}$$

So $= (12, -9, -6)$ is a normal vector for the plane, so the plane has the form
$12x - 9y - 6z + D = 0$, for some constant D. To find D, we know that $A(1, 2, 3)$ is a point on the plane, so $12(1) - 9(2) - 6(3) + D = 0$. So $-24 + D = 0$, or $D = 24$. So the Cartesian equation for the plane is $12x - 9y - 6z + 24 = 0$.

b. Substitute into the formula to determine distance between a point and a plane. So the distance, d, of $(0, 0, 0)$ to the plane $12x - 9y - 6z + 24 = 0$ is equal to $\frac{|12(0) - 9(0) - 6(0) + 24|}{\sqrt{12^2 + (-9)^2 + (-6)^2}}$.
So $d = \frac{24}{\sqrt{261}} \doteq 1.49$.

17. a. $(3, -5, 4)$ is a normal vector for the plane, so the plane has the form $3x - 5y + 4z + D = 0$, for some constant D. To find D, we know that $A(-1, 2, 5)$ is a point on the plane, so
$3(-1) - 5(2) + 4(5) + D = 0$. So $7 + D = 0$, or $D = -7$. So the Cartesian equation for the plane is $3x - 5y + 4z - 7 = 0$.

b. Since the plane is perpendicular to the line connecting $(2, 1, 8)$ and $(1, 2, -4)$, a direction vector for the line acts as a normal vector for the plane. So $(2, 1, 8) - (1, 2, -4) = (1, -1, 12)$ is a normal vector for the plane. So the plane has the form $x - y + 12z + D = 0$, for some constant D. To find D, we know that $K(4, 1, 2)$ is a point on the plane, so $(4) - (1) + 12(2) + D = 0$. So $27 + D = 0$, or $D = -27$. So the Cartesian equation for the plane is $x - y + 12z - 27 = 0$.

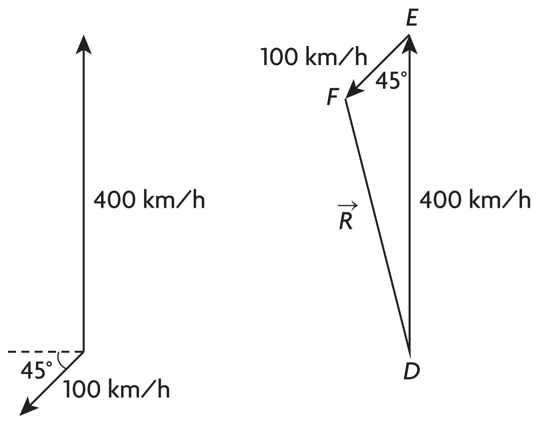
c. Since the plane is perpendicular to the z-axis, a direction vector for the z-axis acts as a normal vector for the plane. Hence $(0, 0, 1)$ is a normal vector for the plane. So the plane has the form $z + D = 0$, for some constant D. To find D, we know that $(3, -1, 3)$ is a point on the plane, so
$0(3) + 0(-1) + (3) + D = 0$. So $3 + D = 0$, or $D = -3$. So the Cartesian equation for the plane is $z - 3 = 0$.

d. The Cartesian equation can be found by taking the cross product of the two direction vectors for the plane. Since $(3, 1, -2)$ and $(1, 3, -1)$ are two points on the plane
$(3, 1, -2) - (1, 3, -1) = (2, -2, -1)$ is a direction vector for the plane. Since the plane is parallel to the y-axis, $(0, 1, 0)$ is also a direction vector for the plane.

$$\begin{aligned}&(2, -2, -1) \times (0, 1, 0) = (-2(0) -\\ &(-1)(1), (-1)(0) - (2)(0), 2(1) - (-2)(0))\\ &\qquad = (1, 0, 2)\end{aligned}$$

So $(1, 0, 2)$ is a normal vector for the plane, so the plane has the form $x + 0y + 2z + D = 0$, for some constant D. To find D, we know that $(3, 1, -2)$ is a point on the plane, so
$(3) + 0(1) + 2(-2) + D = 0$. So $-1 + D = 0$, or $D = 1$. So the Cartesian equation for the plane is $x + 2z + 1 = 0$.

18.



Position Diagram

Vector Diagram

From the triangle DEF and the cosine law, we have
$|\vec{R}|^2 = 400^2 + 100^2 - 2(400)(100)\cos(45°)$
$\doteq 336.80$ km/h.
To find the direction of the vector, the sine law is applied.

$$\frac{\sin \angle DEF}{|\vec{R}|} = \frac{\sin \angle EDF}{100}$$

$$\frac{\sin 45°}{336.80} \doteq \frac{\sin \angle EDF}{100}.$$

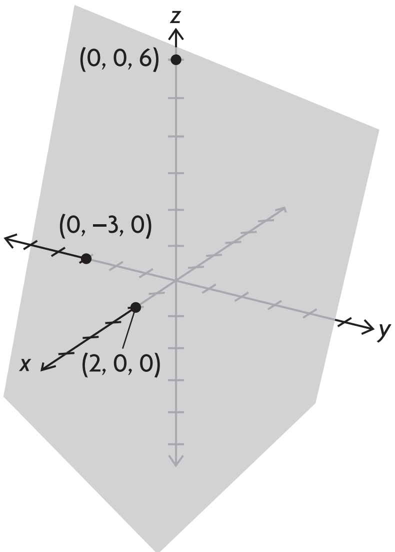
$\sin \angle EDF \doteq \frac{\sin 45°}{336.80} \times 100.$
$\sin \angle EDF \doteq 0.2100.$
Thus $\angle EDF \doteq 12.1°$, so the resultant velocity is 336.80 km/h, N 12.1° W.

19. a. The simplest way is to find the parametric equation, then find the corresponding vector equation. If we substitute $x = s$ and $y = t$ and solve for z, we obtain $3s - 2t + z - 6 = 0$ or $z = 6 - 3s + 2t$. This yields the parametric equations $x = s$, $y = t$, and $z = 6 - 3s + 2t$. So the corresponding vector equation is $\vec{r} = (0, 0, 6) + s(1, 0, -3) + t(0, 1, 2)$, $s, t \in \mathbf{R}$. To check that this is correct, find the Cartesian equation corresponding to the above vector equation and see if it is equivalent to the Cartesian equation given in the problem. A normal vector to this plane is the cross product of the two directional vectors.

$$\begin{aligned}\vec{n} &= (1, 0, -3) \times (0, 1, 2) = (0(2) - (-3)(1),\\ &\quad -3(0) - 1(2), 1(1) - 0(0))\\ &= (3, -2, 1)\end{aligned}$$

So $(3, -2, 1)$ is a normal vector for the plane, so the plane has the form $3x - 2y + z + D = 0$, for some constant D. To find D, we know that (0, 0, 6) is a point on the plane, so $3(0) - 2(0) + (6) + D = 0$. So $6 + D = 0$, or $D = -6$. So the Cartesian equation for the plane is $3x - 2y + z - 6 = 0$. Since this is the same as the initial Cartesian equation, the vector equation for the plane is correct.

b.



20. a. The angle, θ, between the plane and the line is the complementary angle of the angle between the direction vector of the line and the normal vector for the plane. The direction vector of the line is $(2, -1, 2)$ and the normal vector for the plane is (1, 2, 1).

$$\begin{aligned}|(2, -1, 2)| &= \sqrt{2^2 + (-1)^2 + 2^2}\\ &= \sqrt{9}\\ &= 3.\end{aligned}$$

$$\begin{aligned}|(1, 2, 1)| &= \sqrt{1^2 + 2^2 + 1^2}\\ &= \sqrt{6}\end{aligned}$$

$(2, -1, 2) \cdot (1, 2, 1) = 2(1) - 1(2) + 2(1) = 2$

So the angle between the normal vector and the direction vector is $\cos^{-1}\left(\frac{2}{3\sqrt{6}}\right) \doteq 74.21°$. So $\theta \doteq 90° - 74.21° = 15.79°$.
To the nearest degree, $\theta = 16°$.

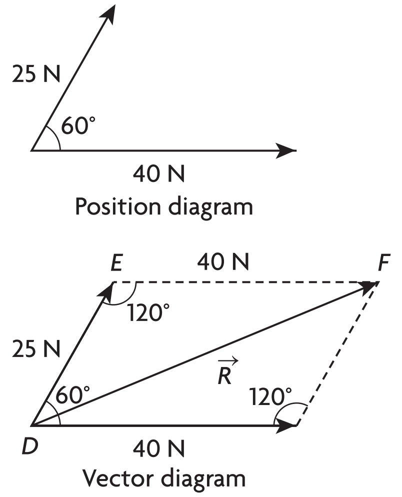
b. The two planes are perpendicular if and only if their normal vectors are also perpendicular. A normal vector for the first plane is $(2, -3, 1)$ and a normal vector for the second plane is $(4, -3, -17)$. The two vectors are perpendicular if and only if their dot product is zero.

$$\begin{aligned}(2, -3, 1) \cdot (4, -3, -17) &= 2(4) - 3(-3)\\ &\quad + 1(-17)\\ &= 0.\end{aligned}$$

Hence the normal vectors are perpendicular. Thus the planes are perpendicular.

c. The two planes are parallel if and only if their normal vectors are also parallel. A normal vector for the first plane is $(2, -3, 2)$ and a normal vector for the second plane is $(2, -3, 2)$. Since both normal vectors are the same, the planes are parallel. Since $2(0) - 3(-1) + 2(0) - 3 = 0$, the point $(0, -1, 0)$ is on the second plane. Yet since $2(0) - 3(-1) + 2(0) - 1 = 2 \neq 0$, $(0, -1, 0)$ is not on the first plane. Thus the two planes are parallel but not coincident.

21.



From the triangle DEF and the cosine law, we have
$|\vec{R}|^2 = 40^2 + 25^2 - 2(40)(25)\cos(120°)$
$\doteq 56.79$ N.
To find the direction of the vector, the sine law is applied.

$$\frac{\sin\angle DEF}{|\vec{R}|} = \frac{\sin\angle EDF}{100}$$

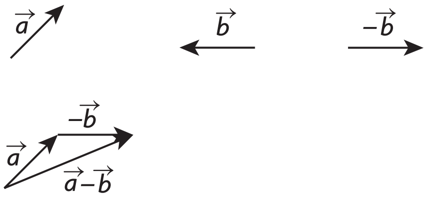
$$\frac{\sin 120°}{56.79} \doteq \frac{\sin\angle EDF}{40}.$$

$$\sin\angle EDF \doteq \frac{\sin 120°}{56.79} \times 40.$$

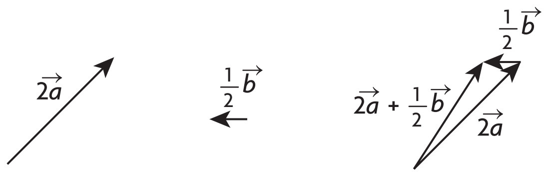
$$\sin\angle EDF \doteq 0.610.$$

Thus $\angle EDF \doteq 37.6°$, so the resultant force is approximately 56.79 N, 37.6° from the 25 N force towards the 40 N force. The equilibrant force has the same magnitude as the resultant, but it is in the opposite direction. So the equilibrant is approximately 56.79 N, $180° - 37.6° = 142.4°$ from the 25 N force away from the 40 N force.

22.



b.



23. a. The unit vector in the same direction of $\vec{a}$ is simply $\vec{a}$ divided by the magnitude of $\vec{a}$.

$$|\vec{a}| = \sqrt{6^2 + 2^2 + (-3)^2}$$
$$= \sqrt{49}$$
$$= 7$$

So the unit vector in the same direction of $\vec{a}$ is
$\frac{1}{|\vec{a}|}\vec{a} = \frac{1}{7}(6, 2, -3) = \left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right)$.

b. The unit vector in the opposite direction of $\vec{a}$ is simply the negative of the unit vector found in part a. So the vector is $-\left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right) = \left(-\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right)$.

24. a. Since $OBCD$ is a parallelogram, the point C occurs at $(-1, 7) + (9, 2) = (8, 9)$. So $\overrightarrow{OC}$ is one vector equivalent to a diagonal and $\overrightarrow{BD}$ is the other.
$\overrightarrow{OC} = (8, 9) - (0, 0) = (8, 9)$
$\overrightarrow{BD} = (9, 2) - (1, 7) = (10, -5)$

b.
$$|(8, 9)| = \sqrt{8^2 + 9^2}$$
$$= \sqrt{145}$$
$$|(10, -5)| = \sqrt{10^2 + (-5)^2}$$
$$= \sqrt{125}$$
$$(8, 9) \cdot (10, -5) = 8(10) + 9(-5)$$
$$= -35$$

So the angle between these diagonals is
$\cos^{-1}\left(\frac{-35}{\sqrt{145}\sqrt{125}}\right) \doteq 74.9°$.

c. $\overrightarrow{OB} = (-1, 7)$ and $\overrightarrow{OD} = (9, 2)$
$$|(-1, 7)| = \sqrt{(-1)^2 + 7^2}.$$
$$= \sqrt{50}$$
$$|(9, 2)| = \sqrt{9^2 + 2^2}$$
$$= \sqrt{85}$$
$$(-1, 7) \cdot (9, 2) = -(9) + 7(2)$$
$$= 5$$

So the angle between these diagonals is
$\cos^{-1}\left(\frac{5}{\sqrt{50}\sqrt{85}}\right) \doteq 85.6°$.

25. a. First step is to use the first equation to remove x from the second and third.
① $x - y + z = 2$
② $-x + y + 2z = 1$
③ $x - y + 4z = 5$
So we have
④ $0x + 0y + 3z = 3$, ① + ②
⑤ $0x + 0y + 3z = 3$, $-1 \times$ ① + ③
Hence $3z = 3$, or $z = 1$. Since both equations are the same, this implies that there are infinitely many solutions. Let $x = t$, then by substituting into the equation 2, we obtain
$-t + y + 2(1) = 1$, or $y = -1 + t$.
Hence the solution to these equations is $x = t$, $y = -1 + t$, $z = 1$, $t \in \mathbf{R}$.

b. First step is to use the first equation to remove x from the second and third.
① $-2x - 3y + z = -11$
② $x + 2y + z = 2$
③ $-x - y + 3z = -12$
So we have
④ $0x + 1y + 3z = -7$, ① + 2 × ②
⑤ $0x - 1y - 5z = 13$, ① − 2 × ③
Now the fourth and fifth equations are used to create a sixth equation where the coefficient of y is zero.
⑥ $0x + 0y - 2z = 6$, ④ + ⑤
So $-2z = 6$ or $z = -3$.

Substituting this into equation ④ yields, $y + 3(-3) = -7$ or $y = 2$. Finally substitute z and y values into equation ② to obtain the x value.
$x + 2(2) + (-3) = 2$ or $x = 1$.
Hence the solution to these three equations is $(1, 2, -3)$.

c. First step is to notice that the second equation is simply twice the first equation.
① $2x - y + z = -1$
② $4x - 2y + 2z = -2$
③ $2x + y - z = 5$
So the solution to these equations is the same as the solution to just the first and third equations. Moreover since this is two equations with three unknowns, there will be infinitely many solutions.
④ $4x + 0y + 0z = 4$, ① + ③
Hence $4x = 4$ or $x = 1$. Let $y = t$ and solve for z using the first equation.
$2(1) - t + z = -1$, so $z = -3 + t$
Hence the solution to these equations is $x = 1$, $y = t$, $z = -3 + t$, $t \in \mathbf{R}$.

d. First step is to notice that the second equations is simply twice the first and the third equation is simply -4 times the first equation.
① $x - y - 3z = 1$
② $2x - 2y - 6z = 2$
③ $-4x + 4y + 12z = -4$
So the solution to these equations is the same as the solution to just the first equation. So the solution to these equations is a plane. To solve this in parametric equation form, simply let $y = t$ and $z = s$ and find the x value.
$x - t - 3s = 1$, or $x = 1 + t + 3s$
So the solution to these equations is $x = 1 + 3s + t$, $y = t$, $z = s$, $s, t \in \mathbf{R}$.

26. a. Since the normal of the first equation is $(1, -1, 1)$ and the normal of the second is $(1, 2, -2)$, which are not scalar multiples of each other, there is a line of intersection between the planes. The next step is to use the first and second equations to find an equation with a zero for the coefficient of x. The second equation minus the first equation yields $0x + 3y - 3z + 3 = 0$. We may divide by three to simplify, so $y - z + 1 = 0$. If we let $z = t$, then $y - t + 1 = 0$, or $y = -1 + t$. Substituting these into the first equation yields $x - (-1 + t) + t - 1 = 0$ or $x = 0$. So the equation of the line in parametric form is $x = 0$, $y = -1 + t$, $z = t$, $t \in \mathbf{R}$.
To check that this is correct, we substitute in the solution to both initial equations
$$\begin{aligned} x - y + z - 1 &= (0) - (-1 + t) + (t) - 1 \\ &= 0 \end{aligned}$$
and
$$\begin{aligned} x + 2y - 2z + 2 &= (0) + 2(-1 + t) - 2(t) + 2 \\ &= 0. \end{aligned}$$
Hence the line given by the parametric equation above is the line of intersection for the planes.

b. The normal vector for the first plane is $(1, -4, 7)$, while the normal vector for the second plane is $(2, -8, 14) = 2(1, -4, 7)$. Hence the planes have collinear normal vectors, and so are parallel.
The second equation is equivalent to $x - 4y + 7z = 30$, since we may divide the equation by two. Since the constant on the right in the first equation is 28, while the constant on the right in the second equivalent equation is 30, these planes are parallel and not coincident. So there is no intersection.

c. The normal vector for the first equation is $(1, -1, 1)$, while the normal vector for the second equation is $(2, 1, 1)$. Since the normal vectors are not scalar multiples of each other, there is a line of intersection between the planes.
The next step is to use the first and second equations to find an equation with a zero for the coefficient of x. The second equation minus twice the first equation yields $0x + 3y - z + 0 = 0$.
Solving for z yields, $z = 3y$. If we let $y = t$, then $z = 3(t) = 3t$.
Substituting these into the first equation yields $x - (t) + (3t) - 2 = 0$ or $x = 2 - 2t$. So the equation of the line in parametric form is $x = 2 - 2t$, $y = t$, $z = 3t$, $t \in \mathbf{R}$.
To check that this is correct, we substitute in the solution to both initial equations
$$\begin{aligned} x - y + z - 2 &= (2 - 2t) - (t) + (3t) - 2 \\ &= 0 \end{aligned}$$
and
$$\begin{aligned} 2x + y + z - 4 &= 2(2 - 2t) + (t) + (3t) - 4 \\ &= 0. \end{aligned}$$
Hence the line given by the parametric equation above is the line of intersection for the planes.

27. The angle, θ, between the plane and the line is the complementary angle of the angle between the direction vector of the line and the normal vector for the plane. The direction vector of the line is

$(1, -1, 0)$ and the normal vector for the plane is $(2, 0, -2)$.

$$|(1, -1, 0)| = \sqrt{1^2 + (-1)^2 + 0^2} = \sqrt{2}$$
$$|(2, 0, -2)| = \sqrt{2^2 + 0^2 + (-2)^2} = \sqrt{8}$$

$(1, -1, 0) \cdot (2, 0, -2) = 1(2) - 1(0) + 0(-2) = 2$

So the angle between the normal vector and the direction vector is $\cos^{-1}\left(\frac{2}{\sqrt{2}\sqrt{8}}\right) = 60°$. So $\theta = 90 - 60° = 30°$.

28. a. We have that $\cos(60°) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$. Also since $\vec{a}$ and $\vec{b}$ are unit vectors, $|\vec{a}| = |\vec{b}| = 1$ and $\vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b} = 1$, and moreover $\cos(60°) = \frac{1}{2}$. So $\vec{a} \cdot \vec{b} = \frac{\vec{a} \cdot \vec{b}}{1 \times 1} = \frac{1}{2}$.

The dot product is distributive, so

$$\begin{aligned}(6\vec{a} + \vec{b}) \cdot (\vec{a} - 2\vec{b}) &= 6\vec{a} \cdot (\vec{a} - 2\vec{b}) \\ &\quad + \vec{b} \cdot (\vec{a} - 2\vec{b}) \\ &= 6\vec{a} \cdot \vec{a} + 6\vec{a} \cdot (-2\vec{b}) \\ &\quad + \vec{b} \cdot \vec{a} + \vec{b} \cdot (-2\vec{b}) \\ &= 6\vec{a} \cdot \vec{a} - 12\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b} \\ &\quad - 2\vec{b} \cdot \vec{b} \\ &= 6(1) - 12\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) \\ &\quad - 2(1) \\ &= -\frac{3}{2}\end{aligned}$$

b. We have that $\cos(60°) = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}||\vec{y}|}$. Also since $|\vec{x}| = 3$, $|\vec{y}| = 4$, and $\cos(60°) = \frac{1}{2}$, $\vec{x} \cdot \vec{y} = \frac{1}{2}(4)(3) = 6$. Also $\vec{x} \cdot \vec{x} = |\vec{x}|^2 = 9$ and $\vec{y} \cdot \vec{y} = |\vec{y}|^2 = 16$.

The dot product is distributive, so

$$\begin{aligned}(4\vec{x} - \vec{y}) \cdot (2\vec{x} + 3\vec{y}) &= 4\vec{x} \cdot (2\vec{x} + 3\vec{y}) \\ &\quad - \vec{y} \cdot (2\vec{x} + 3\vec{y}) \\ &= 8\vec{x} \cdot \vec{x} + 12\vec{x} \cdot \vec{y} - 2\vec{y} \cdot \vec{x} \\ &\quad - 3\vec{y} \cdot \vec{y} \\ &= 8(9) + 12(6) - 2(6) \\ &\quad - 3(16) \\ &= 84\end{aligned}$$

29. The origin, (0, 0, 0), and $(-1, 3, 1)$ are two points on this line. So $(-1, 3, 1)$ is a direction vector for this line and since the origin is on the line, a possible vector equation is $\vec{r} = t(-1, 3, 1), t \in \mathbf{R}$. $(-1, 3, 1)$ is a normal vector for the plane. So the equation of the plane is $-x + 3y + z + D = 0$. $(-1, 3, 1)$ is a point on the plane. Substitute the coordinates to determine the value of D.

$1 + 9 + 1 + D = 0$

$D = -11$

The equation of the plane is $-x + 3y + z - 11 = 0$.

30. The plane is the right bisector joining $P(-1, 0, 1)$ and its image. The line connecting the two points has a direction vector equal to that of the normal vector for the plane. The normal vector for the plane is $(0, 1, -1)$. So the line connecting the two points is $(-1, 0, 1) + t(0, 1, -1), t \in \mathbf{R}$, or in corresponding parametric form is $x = -1, y = t, z = 1 - t, t \in \mathbf{R}$. The intersection of this line and the plane is the bisector between P and its image. To find this point we plug the parametric equation into the plane equation and solve for t.

$$\begin{aligned}0x + y - z &= 0(-1) + (t) - (1 - t) \\ &= -1 + 2t\end{aligned}$$

So if $y - z = 0$, then $-1 + 2t = 0$, or $t = \frac{1}{2}$. So the point of intersection is occurs at $t = \frac{1}{2}$, since the origin point is P and the intersection occurs at the midpoint of the line connecting P and its image, the image point occurs at $t = 2 \times \frac{1}{2} = 1$. So the image point is at $x = -1, y = 1, z = 1 - (1) = 0$. So the image point is $(-1, 1, 0)$.

31. a. Thinking of the motorboat's velocity vector (without the influence of the current) as starting at the origin and pointing northward toward the opposite side of the river, the motorboat has velocity vector (0, 10) and the river current has velocity vector (4, 0). So the resultant velocity vector of the motorboat is

$(0, 10) + (4, 0) = (4, 10)$

To reach the other side of the river, the motorboat needs to cover a vertical distance of 2 km. So the hypotenuse of the right triangle formed by the marina, the motorboat's initial position, and the motorboat's arrival point on the opposite side of the river is represented by the vector

$$\frac{1}{5}(4, 10) = \left(\frac{4}{5}, 2\right)$$

(We multiplied by $\frac{1}{5}$ to create a vertical component of 2 in the motorboat's resultant velocity vector, the distance needed to cross the river.) Since this new vector has horizontal component equal to $\frac{4}{5}$, this means that the motorboat arrives $\frac{4}{5} = 0.8$ km downstream from the marina.

b. The motorboat is travelling at 10 km/h, and in part a. we found that it will travel along the vector $\left(\frac{4}{5}, 2\right)$. The length of this vector is

$$\left|\left(\frac{4}{5}, 2\right)\right| = \sqrt{\left(\frac{4}{5}\right)^2 + 2^2}$$
$$= \sqrt{4.64}$$

So the motorboat travels a total of $\sqrt{4.64}$ km to cross the river which, at 10 km/h, takes

$$\sqrt{4.64} \div 10 \doteq 0.2 \text{ hours}$$
$$= 12 \text{ minutes.}$$

32. a. Answers may vary. For example:
A direction vector for this line is

$$\overrightarrow{AB} = (6, 3, 4) - (2, -1, 3)$$
$$= (4, 4, 1)$$

So, since the point $B(6, 3, 4)$ is on this line, the vector equation of this line is

$\vec{r} = (6, 3, 4) + t(4, 4, 1), t \in \mathbf{R}$.

The equivalent parametric form is

$x = 6 + 4t$
$y = 3 + 4t$
$z = 4 + t, t \in \mathbf{R}$.

b. The line found in part a. will lie in the plane $x - 2y + 4z - 16 = 0$ if and only if both points $A(2, -1, 3)$ and $B(6, 3, 4)$ lie in this plane.
We verify this by substituting these points into the equation of the plane, and checking for consistency.
For A:
$2 - 2(-1) + 4(3) - 16 = 0$
For B:
$6 - 2(3) + 4(4) - 16 = 0$
Since both points lie on the plane, so does the line found in part a.

33. The wind velocity vector is represented by (16, 0), and the water current velocity vector is represented by (0, 12). So the resultant of these two vectors is $(16, 0) + (0, 12) = (16, 12)$.
Thinking of this vector with tail at the origin and head at point (16, 12), this vector forms a right triangle with vertices at points (0, 0), (0, 12), and (16, 12). Notice that

$$|(16, 12)| = \sqrt{16^2 + 12^2}$$
$$= \sqrt{400}$$
$$= 20$$

This means that the sailboat is moving at a speed of 20 km/h once we account for wind and water velocities. Also the angle, θ, this resultant vector makes with the positive y-axis satisfies

$$\cos\theta = \frac{12}{20}$$
$$\theta = \cos^{-1}\left(\frac{12}{20}\right)$$
$$\doteq 53.1°$$

So the sailboat is travelling in the direction N 53.1° E, or equivalently E 36.9° N.

34. Think of the weight vector for the crane with tail at the origin at head at $(0, -400)$ (we use one unit for every kilogram of mass). We need to express this weight vector as the sum of two vectors: one that is parallel to the inclined plane and pointing down this incline (call this vector $\vec{x} = (a, b)$), and one that is perpendicular to the inclined plane and pointing toward the plane (call this vector $\vec{y} = (c, d)$). The angle between $\vec{x}$ and $(0, -400)$ is 60° and the angle between $\vec{y}$ and $(0, -400)$ is 30°. Of course, $\vec{x}$ and $\vec{y}$ are perpendicular. Using the formula for dot product, we get

$$\vec{y} \cdot (0, -400) = |\vec{y}||(0, -400)|\cos 30°$$
$$-400d = 400\left(\frac{\sqrt{3}}{2}\right)\sqrt{c^2 + d^2}$$
$$-2d = \sqrt{3} \cdot \sqrt{c^2 + d^2}$$
$$4d^2 = 3(c^2 + d^2)$$
$$d^2 = 3c^2$$

So, since c is positive and d is negative (thinking of the inclined plane as moving upward from left to right as we look at it means that $\vec{y}$ points down and to the right), this last equation means that $\frac{d}{c} = -\sqrt{3}$
So a vector in the same direction as $\vec{y}$ is $(1, -\sqrt{3})$.
We can find the length of $\vec{y}$ by computing the scalar projection of $(0, -400)$ on $(1, -\sqrt{3})$, which equals

$$\frac{(0, -400) \cdot (1, -\sqrt{3})}{|(1, -\sqrt{3})|} = \frac{400\sqrt{3}}{2}$$
$$= 200\sqrt{3}$$

That is, $|\vec{y}| = 200\sqrt{3}$. Now we can find the length of $\vec{x}$ as well by using the fact that

$$|\vec{x}|^2 + |\vec{y}|^2 = |(0, -400)|^2$$
$$|\vec{x}|^2 + (200\sqrt{3})^2 = 400^2$$
$$|\vec{x}| = \sqrt{160000 - 120000}$$
$$= \sqrt{40000}$$
$$= 200$$

So we get that

$|\vec{x}| = 200$ and $|\vec{y}| = 200\sqrt{3}$. This means that the component of the weight of the mass parallel to the inclined plane is

$$\begin{aligned} 9.8 \times |\vec{x}| &= 9.8 \times 200 \\ &= 1960 \text{ N}, \end{aligned}$$

and the component of the weight of the mass perpendicular to the inclined plane is

$$\begin{aligned} 9.8 \times |\vec{y}| &= 9.8 \times 200\sqrt{3} \\ &\doteq 3394.82 \text{ N}. \end{aligned}$$

35. a. True; all non-parallel pairs of lines intersect in exactly one point in R^2. However, this is not the case for lines in R^3 (skew lines provide a counterexample).

b. True; all non-parallel pairs of planes intersect in a line in R^3.

c. True; the line $x = y = z$ has direction vector $(1, 1, 1)$, which is not perpendicular to the normal vector $(1, -2, 2)$ to the plane $x - 2y + 2z = k$, k any constant. Since these vectors are not perpendicular, the line is not parallel to the plane, and so they will intersect in exactly one point.

d. False; a direction vector for the line $\frac{x}{2} = y - 1 = \frac{z + 1}{2}$ is $(2, 1, 2)$. A direction vector for the line $\frac{x - 1}{-4} = \frac{y - 1}{-2} = \frac{z + 1}{-2}$ is $(-4, -2, -2)$, or $(2, 1, 1)$ (which is parallel to $(-4, -2, -2)$). Since $(2, 1, 2)$ and $(2, 1, 1)$ are obviously not parallel, these two lines are not parallel.

36. a. A direction vector for

$$L_1: x = 2, \frac{y - 2}{3} = z$$

is $(0, 3, 1)$, and a direction vector for

$$L_2: x = y + k = \frac{z + 14}{k}$$

is $(1, 1, k)$. But $(0, 3, 1)$ is not a nonzero scalar multiple of $(1, 1, k)$ for any k since the first component of $(0, 3, 1)$ is 0. This means that the direction vectors for L_1 and L_2 are never parallel, which means that these lines are never parallel for any k.

b. If L_1 and L_2 intersect, in particular their x-coordinates will be equal at this intersection point. But $x = 2$ always in L_1 so we get the equation

$$\begin{aligned} 2 &= y + k \\ y &= 2 - k \end{aligned}$$

Also, from L_1 we know that $z = \frac{y - 2}{3}$, so substituting this in for z in L_2 we get

$$\begin{aligned} 2k &= z + 14 \\ 2k &= \frac{y - 2}{3} + 14 \\ 3(2k - 14) &= y - 2 \\ y &= 6k - 40 \end{aligned}$$

So since we already know that $y = 2 - k$, we now get

$$\begin{aligned} 2 - k &= 6k - 40 \\ 7k &= 42 \\ k &= 6 \end{aligned}$$

So these two lines intersect when $k = 6$. We have already found that $x = 2$ at this intersection point, but now we know that

$$\begin{aligned} y &= 6k - 40 \\ &= 6(6) - 40 \\ &= -4 \\ z &= \frac{y - 2}{3} \\ &= \frac{-4 - 2}{3} \\ &= -2 \end{aligned}$$

So the point of intersection of these two lines is $(2, -4, -2)$, and this occurs when $k = 6$.