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EARTH ENERGY

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Abstract

Ashwin and von der Heydt's Earth energy balance model presents the sensitivity and tipping points of the Earth's climate with respect to the Earth's albedo, carbon and black body emissivity. By linearising the albedo term in the model and incorporating water vapour feedback, a new model is created where the values of the parameters affecting the Earth's climate can be solved and found. Substituting the values found into the new equation gives the final equation for this model, which is a fourth order differential equation with respect to the temperature T. By setting the value of this final equation to zero, the real solutions for the equilibrium temperature can be found. Once the equilibrium temperature has been found, sensitivity and stability analysis are then used to determine the dominant system parameters for the model. From this, the carbon is found to be the most sensitive of the three parameters affecting the Earth's energy balance. However, as this model is a simplification of the Earth's actual change in climate, the results from this model may be more difficult to validate due to its limitations.

Contents

1 Introduction

Understanding the balance of energy within the Earth's atmosphere is important. In an age where an emphasis on regulating the temperature of the planet is being highlighted with greater urgency, developing knowledge about the fundamental causes and effects of global warming is currently seen as a global priority. The energy that allows life on this planet to flourish can be attributed to the cosmic radiation emitted by the sun. Once entering the Earth's atmosphere, this energy can be distributed in a number of ways.

Energy entering the atmosphere can be reflected from objects with high albedo or absorbed by the atmosphere itself. However, the majority of the energy is absorbed by the Earth's surface [1]. This heat and light energy is used by organisms and released back into the atmosphere. This demonstrates a pattern of radiation absorption and emission that can lead to heat being trapped in the planet's atmosphere.

A trend of rising global temperatures [2] is correlated to an increase in the burning of fossil fuels (coal, oil, and gas) that, when burned, release carbon dioxide (CO_2) and other greenhouse gases into the air. These particles absorb heat energy and therefore increase the average temperature of the planet. This phenomenon can lead to extreme weather conditions such as droughts, floods and hurricanes due to the changes in atmospheric temperature and pressure. Therefore it is vitally important to understand the balance of energy within the atmosphere and form models that can translate into useful predictions and results.

The modelling of these systems can be difficult. Many variables and parameters can be integrated into the model as there are many processes that play a role in affecting the earth's global temperature. However, without access to large computational devices and specific data records, it is hard to model these accurately. By focusing on the main important factors, not only will this simplify the model, but it has the added benefit of making the model more accessible to the wider public as it omits longer, confusing terms from a potential equation. This is practical as the importance of this topic means that it should be easily translatable to those without mathematical backgrounds.

Energy balance models have been developed and modified by multiple mathematicians, climatologists, and scientists. In 1969, Mikhail I. Budyko and William D. Sellers [3][4] both produced separate models to detail how the Earth's energy input and output change over time. These models utilise albedo, incoming and outgoing radiation plus natural land transportation (plate tectonics) as parameters affecting the planet's climate. Peter Ashwin and Anna S. von der Heydt [1] detailed a new model in 2019 that avoided natural land transportation. This stems from the fact that this process is reasonably unpredictable and difficult to retain results for.

2 Background

The industrial revolution is a significant event that impacted the Earth's environment as a consequence of the increase in the concentration of greenhouse gases. This steady increase in fossil fuel consumption resulted in a rise in atmospheric CO_2, the levels of which have not been observed for at least 400,000 years [5]. This higher concentration of greenhouse gases (including water vapour, methane and carbon dioxide) reduces the amount of solar radiation that can escape our atmosphere as more is now being absorbed. This retention of radiation assists in global warming (rise in temperature). This highlights the importance of this field of study as the potential impact humans have is extremely damaging to the Earth's climate, especially if CO_2 is the most dominant parameter in the Earth's energy balance.

Another significant climate the Earth experienced is the Last Glacial Maximum (LGM), approximately 21,000 years ago [6]. This is the most recent period global ice sheets had maximum integrated volume according to sea-level records [7]. The expansion in ice sheets consequently increased albedo and contributed significantly to the radiative forcing (earth's cooling) during the LGM. This increase in surface ice affected wind flow causing decreased temperatures and encouraging more sea ice growth, increasing albedo and causing a cycle of increased cooling. This has the opposite impact on the earth's climate in comparison to the greenhouse gas effect; it is important to find a balance between all the processes that impact the earth's climate so the environment is livable and beneficial to the organisms residing in it.

2.1 An Existing Model - Extreme sensitivity and climate tipping points

Ashwin and Heydt analysed the sensitivity and tipping points of the climate using equilibrium climate sensitivity (ECS) [1]. ECS is used as a measure of future global warming. ECS is not well constrained due to uncertainty in temperature change due to atmospheric CO_2. The uncertainties in parameters (such as CO_2, albedo, permafrost etc.) affecting temperature change means sensitivity analysis must be used to determine the dominant parameters.

The standard energy balance model consists of three terms to describe the Earth's temperature and climate balance; the albedo, the radiative forcing from atmospheric greenhouse gases and the temperature-dependent emissivity (black body objects). In the following subsections, these terms are pieced together from assumptions to form the energy balance model.

2.1.1 Assumptions

The Earth's climate is a complex system that needs many simplifications and assumptions to be modelled. General assumptions of the climate are explained in this section, however more specific assumptions regarding the parameters are discussed in their relevant subsections.

The primary assumption is that the Earth is in thermal equilibrium given radiative balance. This means the amount of radiation absorbed from space (from solar waves) is the same amount that is emitted back (like albedo which reflects radiation). The model also accounts for the radiation that cannot be emitted into space due to entrapment caused by greenhouse gases. Seasonal changes account for local fluctuations in temperature however climate changes are determined by fluctuations in the Earth's global mean temperature.

2.1.2 Incoming Solar Radiation

The Earth receives incoming solar radiation in the form of short waves, and its climate is largely dependent on this. The distribution of waves must be balanced with the emission of thermal radiation from the earth's atmosphere and surface. Currently, the incoming solar irradiance, Q_0, is 340 Wm^{-2} [8].

2.1.3 Albedo

Albedo is a measure of reflectivity of a surface, with 1 meaning full reflection of incoming radiation and 0 relating to a black body that fully absorbs all incoming radiation [9]. The Earth's average albedo is measured at around 0.3, meaning 30% of sunlight is reflected back into space. Letting $\alpha(T)$ be the albedo in terms of temperature, the total energy being absorbed by the earth's surface be modelled by

$$Q_0(1 - \alpha(T)).$$

Global albedo is subtly effected by changes in cloudiness, airborne pollution, ice cover and land cover. Albedo is assumed to vary with temperature due to changes in land-ice feedback processes. Thus, this model assumes a higher albedo in the presence of more polar regions (or ice cover), and a lower albedo in the presence of more ocean and land surface. Realistically, albedo will not remain constant and thus a linear approximation is adapted to highlight the temperature dependency as seen in section 3.

2.1.4 Temperature-Dependent Emissivity

Emissivity is a measure of the efficiency of a body's ability to emit infrared radiation [10]. Emissivity ranges from 0 to 1, with 0 meaning the body has perfect reflectiveness and 1 describing a black body that emits no radiation. The highest emissivity values are seen from vegetation, water, and ice. Most of Earth's natural surfaces have an emissivity between 0.1 - 0.6. This model uses temperature-dependent emissivity and does not account for other influencing factors such as cloud coverage and wavelength. Emissivity is denoted as $\epsilon(T)$, and when multiplied with the Stefan-Boltzmann constant σ and temperature T^4 it is representative of outgoing long-wave radiation.

2.1.5 Radiative Forcing

All the factors described above lead to an energy balance. The amount by which this energy is imbalanced is known as radiative forcing [11]. This forcing is used to describe global warming and the greenhouse effect, as an imbalance causes a temperature change. The imbalance can be found by subtracting the energy flowing out from the energy flowing in, if this value does not equal 0 then there must be some warming or cooling. The imbalance is difficult to estimate due to many impacting parameters, each with their uncertainties which make measurement difficult; for example, some greenhouse gases have different effects on their combined warming effect, but estimating their impact is complicated. The energy balance model used here is, therefore, a huge simplification of all these processes.

2.1.6 Completing the Picture

Piecing it all together, the energy balance model is obtained in the form of a differential equation [1],

$$c_T\frac{dT}{dt} = Q_0(1 - \alpha(T)) + A\ln(\frac{C}{C_0}) - \epsilon(T)\sigma T^4. \tag{1}$$

Where c_T represents the thermal inertia, T is the temperature in Kelvin, t represents the time (years), Q_0 is the solar constant (energy at the surface of the Sun) and α is the temperature-dependent albedo. The second term represents the change in radiative forcing as a result of atmospheric greenhouse gases, where C is the atmospheric CO_2 concentration and C_0 represents pre-industrial CO_2. Here, CO_2 is considered as purely a forcing term that ignores the effect of temperature on the CO_2 balance from land surface and ocean processes. The last term consists of $\epsilon(T)$, which is based on black body objects (temperature-dependent emissivity) and the Stefan-Boltzmann constant (5.67×10^{-8}) represented by σ, which when multiplied by T^4 becomes representative of the outgoing long-wave radiation.

The energy balance model obtained illustrates the dependency on the sensitivity on each of the variables, and can also be extended into the past (for example the glacial period).

3 Methodology

The energy balance model seen in Equation 1 explored by Ashwin and Heydt provides the basis for the model. Initially, it is assumed that albedo is linear with respect to temperature. As well as this, we want to incorporate water vapour feedback into the model. Water vapour feedback adds water to the atmosphere thus enhancing the greenhouse effect. Hence, a constant D is assigned to represent this, where $D = 2A$ as vapour feedback usually amplifies by a factor of 2 [1]. $A = 5.35Wm^{-2}$, therefore $D = 10.7$. As the albedo term has been linearised, ϵ is made to be a constant. This is because if ϵ was linearised there would at a minimum be a polynomial of order 5, defeating the aim to make as simple yet viable a model as possible. The new model is therefore

$$c_T\frac{dT}{dt} = Q_0(1 - (X_\alpha + Y_\alpha T)) + D\ln(\frac{C}{C_0}) - \epsilon\sigma T^4 \tag{2}$$

where $\alpha = X_\alpha + Y_\alpha T$, and the constants X and Y are to be found.

Equation 2 can be solved for the remaining parameters by substituting in the Pre-Industrial and Last Glacial Maximum (LGM) values for the temperature, CO_2 concentration and albedo, along with the already known values for Q_0, σ and D discussed earlier. The Pre-Industrial and LGM data values are used as data is the most reliable from these times, due to countless research and these were massive events relative to the earth's climate change. From previous papers [1], Pre-Industrial values for carbon and temperature were found to be: $C_0 = 280ppm$, $T_0 = 288K$. The Pre-Industrial albedo was estimated from the global annual mean albedo, $\alpha_0 = 0.29$ [12]. The LGM values for carbon and temperature were found to be: $C_{LGM} = 180ppm$ [13] and $T_{LGM} = 15°C = 282K$ [14]. Assuming Equation 2 is in equilibrium ($c_T \frac{dT}{dt} = 0$), values for Pre-Industrial times can be substituted to determine some of the unknown constants.

$$340(1 - 0.29) = \epsilon(T)\sigma(288)^4 \tag{3}$$

as $\ln(\frac{280}{280}) = 0$. Rearranging this gives a value of $\epsilon = 0.619$ for the emissivity, which is assumed to be constant for all time t. Substituting ϵ and LGM values into Equation (2) gives

$$340(1 - \alpha_{LGM}(T)) + 10.7\ln(\frac{180}{280}) = 0.619\sigma(282)^4 \tag{4}$$

which rearranges to give a value of $\alpha_{LGM} = 0.333$. This leaves two simultaneous equations

$$X_\alpha + 288Y_\alpha = 0.29 \tag{5}$$

$$X_\alpha + 282Y_\alpha = 0.33 \tag{6}$$

which are solved to find the X and Y components for α: $X_\alpha = 2.21$ and $Y_\alpha = -0.0067$.
This gives the final energy balance equation of

$$c_T\frac{dT}{dt} = 340(1 - (2.21 - 0.0067T)) + 10.7\ln(\frac{C}{280}) - (0.619)(5.67 \times 10^{-8})T^4 \tag{7}$$

where C is the variable atmospheric CO_2 concentration and T is the temperature to be found from solving the differential equation.

3.1 Equilibrium Solutions

One of the goals of this paper was to predict future changes in overall temperature. The simplest way of doing this would involve finding equilibria of the model 7. In the context of this report, an equilibrium would represent a value that the earth's temperature would tend to given no changes in the earth's natural processes (i.e albedo, carbon levels and emissivity remaining the same). Evidently, this is a primary conclusion as it assumes no changes in atmosphere and due to the sensitivity, the equilibrium would not be reached for hundreds of years.

$$f(T^*) = 340(1 - (2.21 - 0.0067T^*)) + 10.7\ln(\frac{C}{280}) - (0.619)(5.67 \times 10^{-8})T^{*4} \tag{8}$$

The value of the equilibrium temperature T^* can be found by setting $f(T^*) = 0$ in Equation (8), with $C = 400ppm$ (the most recent atmospheric concentrations of $CO2$) [15]. The polynomial is of order 4 and thus has four solutions: two complex and two real values. The real values are $T_1^* = 207.4642K(4d.p)$ and $T_2^* = 293.9458K(4d.p)$. However, $T^* = 207.4642K$ is not a feasible solution for the equilibrium temperature of the Earth, due to the fact that the black body temperature of the earth is -23°C [16]. The infeasible equilibrium would suggest that the earth would eventually tend to a temperature colder than this, which is physically impossible. Therefore, our most viable solution for the equilibrium temperature is $T^* = 293.9458K$.

3.2 Stability Conditions

Stability analysis can be used to determine how the equilibria of the system behaves under small changes to the root. The stability of the equilibrium solutions can be found using the derivative of $f(T^*)$, where f if the quartic in Equation 8. If $f(T^*) < 0$ then the point is stable, meaning that if the temperature deviated from its equilibrium state, it would return to that state. If the point is unstable then any change would cause the temperature to deviate from its original state.

3.3 Stability Analysis

Using the equilibrium values of T^* found earlier, we can determine the stability of each point. Differentiating Equation 8 gives

$$f'(T^*) = 340(0.0067) - 4(0.619)(5.67 \times 10^{-8})T^{*3} \tag{9}$$

By substituting the equilibria values we see that only one of the points is stable. Substituting $T_1^* = 207.4642K$ gives a value of $f'(T_1^*) = 1.0244(4d.p)$, and substituting $T_2^* = 293.9458K$ gives a value of $f'(T_2^*) = -1.2876(4d.p)$. Therefore T_2^* is stable as it satisfies stability conditions of $f'(T_2^*) < 0$. This supports the argument made earlier which says that T_1^* is not a feasible solution.

3.4 Sensitivity Analysis

In this section we aim to investigate the effect parameters involved in the model (Equation 2) have on the earth's temperature. More specifically analysis will be conducted to provide an indicator into which parameter the temperature is most sensitive to. As there is uncertainty in which values these parameters will take in the future, it is important to understand how these translate and impact the resultant future temperature of the earth. To do this, a parameter, one at a time, has been modelled as a random variable. From this an inverse transform is applied to find the resultant temperature probability distribution. The resulting first 4 moments are found and thus a sensitivity analysis is conducted. In this paper, this analysis has been completed for the variable parameters carbon (C), albedo constant coefficient (X_α), and albedo gradient (Y_α). The sensitivity of emissivity has not been measured as emissivity is partially dependant on albedo. Also, the value of ϵ cannot be changed or impacted drastically by humans due to it being representative of the radiation that the planet naturally emits, so future predictions of this value aren't needed as it has always been approximately the same.

3.4.1 Carbon PDF to Temperature PDF

To begin with, the equilibrium of equation 2 is rearranged to derive a function for Carbon (C) in terms of temperature (t), as seen in equation 10. Note that in the work and derivations to follow, standard notion will be followed and thus temperature will appear as a lower case t, as the temperature distribution and thus random variable temperature T will assume the upper case.

$$C(t) = C_0 \exp\left(\frac{\epsilon\sigma t^4 - Q_0(1 - (2.22 - 0.0067t)}{D}\right) \tag{10}$$

Carbon has been modelled as a random variable to account for the inherent uncertainty in the specific value it will assume in the future. In this work all parameters, in this case carbon, will be modelled as Gaussian distributions with mean and variance values that will be selected to best estimate future values given current data and trends.

That is to say,

$$C(t) \sim \mathcal{N}(\mu,\, \sigma^2) \tag{11}$$

with probability density function (PDF),

$$f_C(C(t)) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(C(t)-\mu)^2/2\sigma^2} \tag{12}$$

Taking the derivative results in this equation:

$$\frac{dC(t)}{dt} = C_0 \left(\frac{4\epsilon\sigma t^3 + 0.0067Q_0}{D}\right) \exp\left(\frac{\epsilon\sigma t^4 - Q_0(1-(2.22-0.0067t)}{D}\right) \tag{13}$$

Using this information, the transformation is now taken to find the temperature distribution. The standard formula utilised for this calculation can be seen below.

$$f_T(t) = f_C(C(t)) \left|\frac{dC(t)}{dt}\right| \tag{14}$$

Note that the function that relates the two random variables, C and T in this instance, must be invertible for equation 29 to hold. This condition is, however, not met when considering an open interval for C(t), i.e. equation 10. This is something to be mindful of when the solution is calculated and graphed. Thus by putting the pieces together by multiplying equation 10 by the absolute value of equation 13, the temperature distribution can be expressed and graphed. The results for this can be seen in Figure 1 in Appendix B.

3.4.2 Albedo PDF to Temperature PDF

The same analysis has been completed for each of the two albedo coefficients when considering albedo as a linear function dependent on temperature as in our appended model 2. Note this analysis has been completed separately. Both terms have been modelled, again, as Gaussian distributions, as seen below in equations 15a and 15b.

$$X_\alpha(t) \sim \mathcal{N}(\mu,\, \sigma^2) \tag{15a}$$

$$Y_\alpha(t) \sim \mathcal{N}(\mu,\, \sigma^2) \tag{15b}$$

The calculations for $X_\alpha(t)$, and $Y_\alpha(t)$, as well as the first derivatives w.r.t temperature have been completed and can be seen in Appendix A. Again transforming these random variables results in two addition probability density functions for Temperature. The results for this can be seen in Figure 2 and Figure 3 in Appendix B.

It should be noted that in these two derivations carbon has been treated as a non-varying parameter. In other words, carbon's dependence on temperature has been ignored to simplify the derivations. The value carbon has been attributed with is, similarly to the mean value attributed in 11, an estimation of what the future carbon value will be according to currents trends. As we are looking at equilibrium solutions and are more keenly interested in the resulting temperature distribution's characteristics at a fixed time in the future, this step does not retract from the results. From this point the moments can be calculated by performing integrals over the relevant domain of the resulting temperature distributions.

3.4.3 Calculating Moments

For each individual temperature probability density function the first four moments were calculated. The general form of the n^{th} moment centered about c is shown below, where c is any chosen point:

$$E\left[(X-c)^n\right] = \int_b^a (x-c)^n f_X(x)dx \tag{16}$$

The 0^{th} moment is the integral of the PDF, which should equal to 1 when integrated over the relevant domain. In our case, the PDF is the temperature distributions derived in the above sections. As the inverse functions that relates the chosen parameters to temperature are not monotonic for all three cases (carbon, X_α and Y_α), the resultant output from the transformation 29 will not sum to 1 unless the domain is restricted so that the input function is monotonic over that specified range. Once the interval of integration is specified over the relevant domain the sum is approximately equal to 1, and thus the temperature's PDF is used to calculate the next 3 moments. Using equation 16, the three temperature distribution's mean, variance and skewness are calculated.

1. Mean(T): $E[T] = \mu_T$, uncentered (c = 0)
2. Variance(T): $E[(T-\mu_T)^2] = \sigma^2$, centered (c = μ_T)
3. Skewness(T): $E[(T-\mu_T)^3]$, centered (c = μ_T)

The skewness is standardised by dividing through by σ_T^3. The results have been summarised in the table below.

Table 1: *Table of Moments derived for each of the three temperature distributions*

Moments of Temperature PDF with Input PDF			
n^{th} Moment	Carbon	X_α	Y_α
0 - Total Mass	0.999	0.999	0.999
1 - Mean	294.183	294.298	294.304
2 - Variance	2.016	1.916	1.985
3 - Skewness	-0.703	-0.119	-0.092

3.4.4 Sensitivity Coefficients

By modelling the input parameters one at a time as random variables the resulting uncertainty in the temperature can be visualised in temperature distributions. Further analysis can be undertaken in which 'sensitivity' coefficients can be derived from the moments in Table 1. and a comparison can be made. The coefficients are as follows.

Carbon Sensitivity Coefficient:

$$\frac{\sigma_t}{\sigma_c/\mu_c} = \frac{\sqrt{2.016}}{\frac{\sqrt{80}}{500}} \simeq 79.4 \tag{17}$$

Albedo X_α Sensitivity Coefficient:

$$\frac{\sigma_t}{\sigma_{X_\alpha}/X_\alpha + Y_\alpha t_0} = \frac{\sqrt{1.916}}{\frac{\sqrt{0.0052}}{2.22-0.0067\times 294}} \simeq 4.8 \tag{18}$$

Albedo Y_α Sensitivity Coefficient:

$$\frac{\sigma_t}{\sigma_{Y_\alpha} t_0 / X_\alpha + Y_\alpha t_0} = \frac{\sqrt{1.985}}{\frac{\sqrt{0.000018 \times 294}}{2.22 - 0.0067 \times 294}} \simeq 65 \tag{19}$$

These coefficients are a way of representing the sensitivity of temperature with respect to a chosen parameter of the model. In this analysis the ratio of the temperature variance and the parameters variance have been calculated. Note the denominators in each expression have been normalised. These terms can be viewed as proportional changes, and is an attempt to make each sensitivity coefficient comparable. The results, as seen directly above, indicate that temperature is a lot more sensitive to carbon and the gradient term in albedo Y_α compared with the constant albedo term X_α. In addition it can be seen that temperature is slightly more sensitive to carbon than Y_α.

4 Discussions

The implications of this project can be understood easily. Socially it can create a discussion about the earth's energy balance as it simplifies a model which is intellectually inaccessible to a large proportion of people. By describing a basic model, the general topic of global warming can engage a wider range of people; a desirable reaction since the issue of global warming is a worldwide problem. Whilst, the negative effects of global warming are evident, there are some positives. A higher global energy budget allows plants to photosynthesize faster and therefore grow quicker. This is useful in a modern day context as the demand for agriculture is growing with a global population that is still rising.

Generating an understanding into what effects the earths global temperature and how we can adapt to reduce negatively impacting the global temperature is a current day issue of unparalleled importance. There are a myriad of papers and current day scientific work that focus on this issue. Modelling the earth's energy balance and generating predictive models of future temperatures can become complicated quite easily. There are many factors that are of relevance to this issue that are hard to accurately capture, that tend to result in highly technical models that are hard to understand. In this paper there has been a focus to reduce the complexity most models tend to exhibit and to provide a general understanding into the main physics that underlines the earths energy balance, so as to make it accessible to a wider audience and in doing so raising more awareness surrounding this issue. A first order differential equation describing the change in temperature with respect to time forms the basis of the model analysed throughout this paper. Key areas of analysis involve solving equilibrium solutions, analysing stability, as well as a sensitivity analysis that attempts to demonstrate the relative impact each parameter has on the global temperature.

The Earth's stable temperature equilibrium was found to be approximately 294K. Another method of stability analysis would be via graphical analysis of bifurcations. Qualitative changes in a systems stability occurs at bifurcation points. Bifurcation points are calculated by finding a value for which T^* satisfies the following conditions: $f(T^*) = 0$ and $f'(T^*) = 0$, where f is Equation 8. These bifurcation points can be found for many of the parameters, but it would be useful to analyse it for the parameters that have the most impact on the changing temperature.

In the model that has been outlined in this paper the key parameters are Carbon (ppm), Albedo, and constants incoming solar radiation, emissivity and Stefan-Boltzmann, which describe the incoming and outgoing radiation respectively. Understanding the relative impacts these parameters have on the global climate is of crucial importance, so as to allow us to direct our efforts to stop the increase in global temperature in a constructive and meaningful way.

The values these parameters take in the future are uncertain. In modelling the the parameters as random variables this uncertainty has been captured. The interesting results come, however, when undertaking a transformation and analysing the resulting temperature distributions. In doing this analysis and taking the moments, the key characteristics of the three temperature distributions can be compared and provide an insight into the relative difference in impact each parameter has on the temperature. In this paper one parameter has been taken at a time and modelled as a random variable. For each case the parameter of relevance has assumed a normal distribution. The mean and variance have been assigned using current day data and trends as discussed throughout the methods. When taking the transformations to solve for the temperature distributions it should be noted that the functions C(t), X(t), Y(t) are not strictly invertible. This is as expected as the equilibrium equation is of order 4. This results in two complex solutions and two real. The equilibrium solution that is of relevance is the higher of the two temperatures. The second solution is, however, still observable in the resulting temperature transformations. To address this issue of multiple equilibria, that arises when computing the moments by integrating, a specified range was identified which then does make the functions C(t), X(t), Y(t) invertible over that range, and thus after transforming does indeed result in a probability distribution function for temperature, confirmed by total mass integrals equaling to approximately one for each case.

After standardising and producing comparable coefficients using the moments it can be seen that the constant term in our linearised albedo, X_α, is of trivial importance when compared to the gradient term in albedo, Y_α, as can be seen by their sensitivity values of 4.8 and 65 respectively. This result is hard to interpret as albedo, modelled here linearly dependent on temperature, is a coupling of these two terms. Ultimately implying as expected albedo as a whole is a crucial parameter when considering modeling earths global temperature. Also of interest is the fact that carbons sensitivity coefficient (79.4) is greater than Y_α.

These results demonstrate that temperature is sensitive to albedo and carbon, but more interestingly, carbon seems to be more sensitive. Understanding the dynamics of nonlinear systems is a crucial field in modern day scientific work. These systems are notoriously sensitive to both initial conditions and parameters and as such leads to the study of chaos. Our results, therefore, seem perfectly sensible. Another interesting observation in the results is that the skewness of each temperature distribution is negative. This is a preferable result as it implies that the temperature will tend to the lower side of our predictions rather than the higher. When applied to the context of this project, the skew suggests that the temperature has a greater chance of falling under the 294°K value, which is desirable when looking to slow global warming.

The desirable Earth equilibrium temperature means that land and ocean processes need to remain relatively similar to how they are currently. Carbon being the most sensitive parameter means that future levels needs to be carefully monitored to ensure the Earth does not become uninhabitable and natural disasters do not become more frequent. However, carbon dioxide levels are extremely unstable and can change easily and thus are difficult to predict due to other natural causes such as volcanic activity and permafrost thawing. Nonetheless, it is clear that fossil fuel consumption should be reduced, and thankfully many governments have already implemented schemes to reduce the amount of greenhouse gas emissions they release annually [17].

The implications of the results derived from the sensitivity analysis shown in this paper overall seem sensible as well as informative. The work, however, could be expanded to deepen more understanding about the effects these various parameters have on the earths temperature. Further work could be done in modelling the uncertainty in the parameters more accurately by using non-standard distributions, or more data orientated models. In addition two or more parameters could be modeled as joint distributions to see the effects when in combinations with one another.

In general, this energy balance model is hard to parameterise due to the volume of physical processes that contribute to albedo and emissivity. Furthermore land and ocean processes have not been considered. An example of these processes is permafrost thawing. This process involves old carbon which is currently inert and frozen being released into the atmosphere, thus affecting the temperature, as can be seen by the research done by Burke, Hartley and Jones [18]. Their report is however based on small-scale samples explored at limited sites, and large-scale extrapolation of these results is hard to validate. The report also discusses how methane had a larger effect on rising global temperatures when compared to the impact of CO_2 in the early 21st century.

References

[1] Peter Ashwin and Anna S. von der Heydt. *Extreme Sensitivity and Climate Tipping Points*. Springer, https://link.springer.com/article/10.1007/s10955-019-02425-x, 2019. [Last Accessed: 12/02/2020].

[2] NASA. *Global Temperature*. NASA, https://climate.nasa.gov/vital-signs/global-temperature/, Live. [Last Accessed: 28 Jan 2020].

[3] M. I. Budyko. *The effect of solar radiation variations on the climate of the Earth*. Tellus, 1969. [Last Accessed: 01 Feb 2020].

[4] William D. Sellers. *A global climatic model based on the energy balance of the earth-atmosphere system*. Journal of Applied Meteorology, 1969. [Last Accessed: 01 Feb 2020].

[5] NASA. *The relentless rise of carbon dioxide*. https://climate.nasa.gov/climate_resources/24/graphic-the-relentless-rise-of-carbon-dioxide/. [Last Accessed: 13/02/2020].

[6] Alan C. Mix Peter U. Clark. *Ice sheets and sea level of the Last Glacial Maximum*. https://www.sciencedirect.com/science/article/pii/S0277379101001184, 2002. [Last Accessed: 17/02/2020].

[7] Peter U. Clark et al. *The Last Glacial Maximum*. https://s3.amazonaws.com/academia.edu.documents/42032421/Paleoclimate._Superlakes_megafloods_and_20160204-30748-tfbse7.pdf?response-content-disposition=inline%3B%20filename%3DPaleoclimate._Superlakes_megafloods_and.pdf&X-Amz-Algorithm=AWS4-HMAC-SHA256&X-Amz-Credential=AKIAIWOWYYGZ2Y53UL3A%2F20200216%2Fus-east-1%2Fs3%2Faws4_request&X-Amz-Date=20200216T154219Z&X-Amz-Expires=3600&X-Amz-SignedHeaders=host&X-Amz-Signature=d15ff7a8793148156ccab2ae7c9c868cbf17164f049fe4af26025e9c902ba931, 2017. [Last Accessed: 16/02/2020].

[8] B. Charnay S. Chelvaniththilan P.G.J. Irwin S. Knight S. Lebonnois S.R. Lewis J. Mendonça L. Montabone P. L. Read, J. Barstow. *Global energy budgets and 'Trenberth diagrams' for the climates of terrestrial and gas giant planets*. Royal Meteorological Society, https://rmets.onlinelibrary.wiley.com/doi/full/10.1002/qj.2704, 2014. [Last Accessed: 05/02/2020].

[9] NASA Earth Observatory. *Measuring Earth's Albedo*. NASA, https://earthobservatory.nasa.gov/images/84499/measuring-earths-albedo, 2014. [Last Accessed: 04/02/2020].

[10] NASA. *NASA Spacecraft Maps Earth's Global Emissivity*. NASA, https://www.jpl.nasa.gov/spaceimages/details.php?id=PIA18833, 2014. [Last Accessed: 11/02/2020].

[11] MIT. *Explained: Radiative forcing*. http://news.mit.edu/2010/explained-radforce-0309. [Last Accessed: 12/02/2020].

[12] Frida A-M. Bender. *Planetary albedo in strongly forced climate, as simulated by the CMIP3 models*. https://link.springer.com/article/10.1007/s00704-011-0411-2, 2011. [Last Accessed: 14/02/2020].

[13] Mark Floyd. *Solving the mystery of why atmospheric carbon dioxide was lower during ice ages*. https://phys.org/news/2019-06-mystery-atmospheric-carbon-dioxide-ice.html, 2019. [Last Accessed: 12/02/2020].

[14] Wikipedia. *Last Glacial Maximum*. https://en.wikipedia.org/wiki/Last_Glacial_Maximum. [Last Accessed: 12/02/2020].

[15] Nicola Jones. *How the World Passed a Carbon Threshold and Why It Matters*. https://e360.yale.edu/features/how-the-world-passed-a-carbon-threshold-400ppm-and-why-it-matters, 2017. [Last Accessed: 14/02/2020].

[16] Chris Park. *A Dictionary of Environment and Conservation.* Oxford University Press, https://www.oxfordreference.com/view/10.1093/acref/9780198609957.001.0001/acref-9780198609957-e-875, 2007. [Last Accessed: 17 Feb 2020].

[17] Committee on Climate Change. *A global climatic model based on the energy balance of the earth-atmosphere system.* Committee on Climate Change, https://www.theccc.org.uk/our-impact/reducing-the-uks-emissions/, 2019. [Last Accessed: 01 Feb 2020].

[18] C. D. Jones E. J. Burke, I. P. Hartley. *Uncertainties in the global temperature change caused by carbon release from permafrost thawing.* The Cryosphere, the-cryosphere.net/6/1063/2012/tc-6-1063-2012.pdf, 27 September 2012. [Last Accessed: 26 Jan 2020].

5 Appendix A

Albedo X_α PDF Derivations:

$$X_\alpha(t) = \frac{1}{Q_0}\left(Q_0 - Q_0 Y_\alpha t + Dln\left(\frac{C}{C_0}\right) - \epsilon\sigma t^4\right) \tag{20}$$

$$\frac{dX_\alpha(t)}{dt} = -Y_\alpha - \frac{4\epsilon\sigma t^3}{Q_0} \tag{21}$$

Modeling X_α as Normal Random Variable gives,

$$X_\alpha(t) \sim \mathcal{N}(\mu,\, \sigma^2) \tag{22}$$

Which implies,

$$f_{X_\alpha}(X_\alpha(t)) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(X_\alpha(t)-\mu)^2/2\sigma^2} \tag{23}$$

Again putting the pieces together results in the transformation and resultant temperature pdf.

$$f_T(t) = f_{X_\alpha}(X_\alpha(t))\left|\frac{dX_\alpha(t)}{dt}\right| \tag{24}$$

Albedo Y_α PDF Derivations:

$$Y_\alpha = \frac{1}{Q_0 t}\left(Q_0 - Q_0 X_\alpha + Dln\left(\frac{C}{C_0}\right) - \epsilon\sigma t^4\right) \tag{25}$$

$$\frac{dY_\alpha}{dt} = \frac{-1}{Q_0 t^2}\left(Q_0 - Q_0 X_\alpha + Dln\left(\frac{C}{C_0}\right) - \epsilon\sigma t^4\right) - \frac{4\epsilon\sigma t^3}{Q_0 t} \tag{26}$$

Again, by modelling Y_α as Normal Random Variable gives,

$$Y_\alpha(t) \sim \mathcal{N}(\mu,\, \sigma^2) \tag{27}$$

Which implies,

$$f_{Y_\alpha}(Y_\alpha(t)) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(Y_\alpha(t)-\mu)^2/2\sigma^2} \tag{28}$$

Again putting the pieces together results in the transformation and resultant temperature pdf.

$$f_T(t) = f_{Y_\alpha}(Y_\alpha(t))\left|\frac{dY_\alpha(t)}{dt}\right| \tag{29}$$

Table 2: *List of parameter value used in pdf derivations*

General Parameters	Numerical Value
Q_0	$340 \quad Wm^{-2}$
ϵ	0.619
σ (Stefan-boltzmann)	$5.67 \times 10^{-8}\ Wm^{-2}K^{-4}$
X_α	2.22
Y_α	-0.0067
Parameters for Input Distributions	**Numerical Value**
Carbon:	
μ	$500 \quad ppm$
σ^2	$80 \quad ppm^2$
X_α :	
μ	2.22
σ^2	0.0052
Y_α :	
μ	-0.0067
σ^2	0.000 018

6 Appendix B

Temperature PDF Graphs:

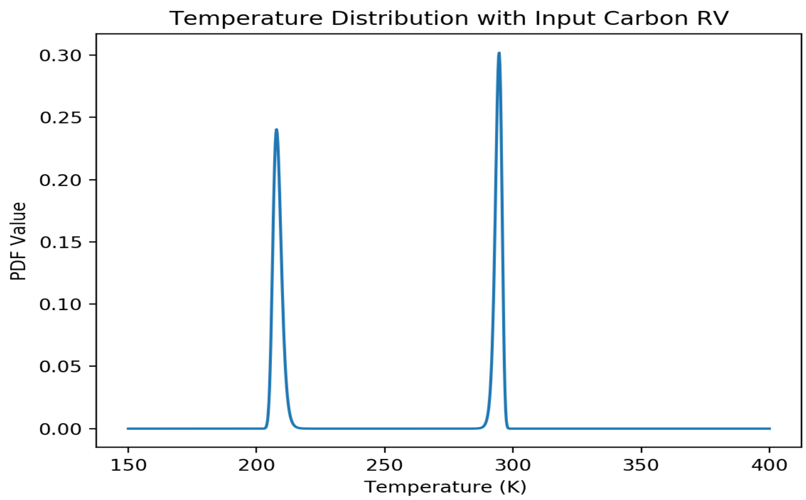

Figure 1: *Temperature Distribution with Carbon as Normal RV.*

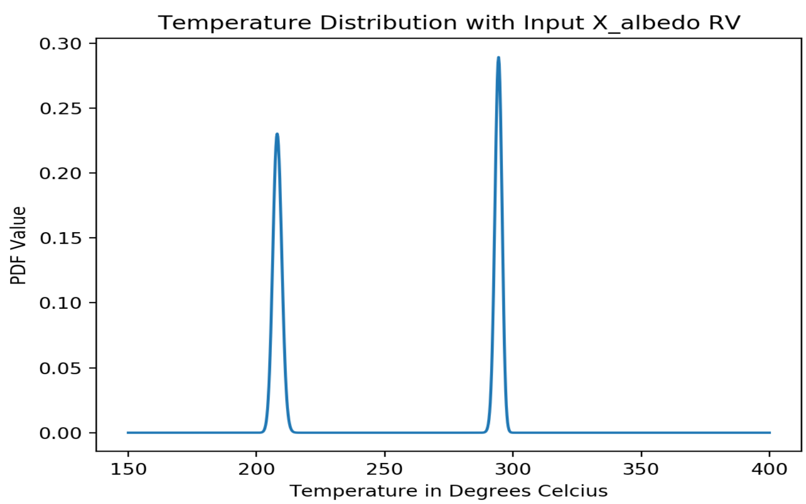

Figure 2: *Temperature Distribution with X_α as Normal RV.*

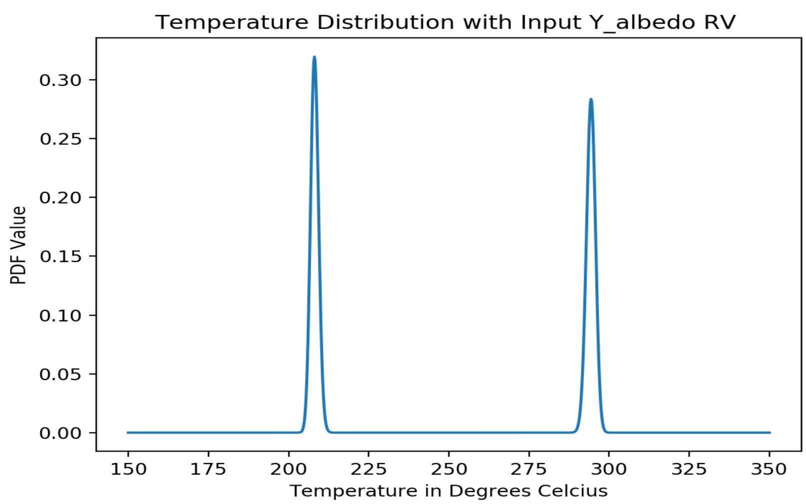

Figure 3: *Temperature Distribution with Y_α as Normal RV.*