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Modelling the synchronisation of metronomes using a Kuramoto model

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Abstract

The synchronisation of mechanical metronomes is an interesting phenomenon and can provide insight into the synchronisation of other mechanical systems. This report details a simplified model to demonstrate the synchronisation of mechanical metronomes over time using a Kuramoto model. The methodology is outlined to model any number of metronomes synchronising but analysis is performed on the simpler cases of two directly coupled metronomes and then two coupled through a board. The analysis done allows for some insight to be gained as to the conditions required for the synchronisation of the metronomes in terms of the parameters of the system that have been selected. The report then goes on to explore the effect of changing the main parameter k, the coupling strength between the metronomes to see what effect this has on synchronisation. After finding some conditions for the synchronisation of this mechanical system the report goes on to detail an investigation made into the stability of the steady states within the system. Conclusions are then drawn with regard to the conditions enforced upon the parameters to give a linearly stable steady state.

1 Introduction

Motivation

A metronome is musical tool that clicks regularly, at a given pace, to help musicians to keep in time. Originally a mechanical instrument, patented by Johann Maelzel in 1815 [Staff, 2019], the metronome has since been developed and electrical options are now available. Additionally, there are metronomes which use quartz crystals and operate similarly to a watch. This report is concerned with the mechanical version and the phenomenon which sees multiple metronomes synchronise when placed on a free moving platform.

The mechanical metronome consists of a fixed weight attached to a rod, and a moveable weight that controls the tempo of the beats. Generally, the fixed weight is concealed within a pyramid shaped structure and the moveable weight is able to slide up and down the exposed section of the rod. This effectively changes the length of the rod and thus controls the speed of the oscillations. When the device is set in motion, it acts like a pendulum weighted on both ends. The system is run by an escapement mechanism and wound up by the user. As the mechanism unwinds, energy is transferred to the rod and converted between kinetic and gravitational potential as it changes position. [Staff, 2019]

When multiple metronomes are placed on a light-weight platform that is free to move, they begin to synchronise. This happens because the light-weight platform is able to react to the motion of the pendulum and they become coupled [Porter, 2016]. The synchronisation of metronomes has been a subject of great interest and has been explored in many scientific reports, for example 'Synchronisation of Metronomes' by James Pantaleone [Pantaleone, 2002]. These ideas can be applied to other systems of oscillators. For example, the synchronisation of chemical and biological oscillators has applications in neuroscience.

Report aims

This report aims to model the motion of metronomes as they synchronise and explore the necessary conditions for synchronisation. Two models will be considered. Firstly, the Lagrangian Pendulums model, which uses Lagrangian mechanics to obtain equations of motion for a pendulum on a moving cart. A relationship between the movement of the pendulum and the cart is expected to arise. This system is expected to be non-linear and hence difficult to analyse meaning that a simpler model should be considered. The model, is linearised about the upright position of the metronome to provide some insight however this proves unhelpful as coupling terms fall out when considering very small oscillations about the $\theta = 0$ point.

Secondly, the Kuramoto Oscillators model, which sees a coupling force between two pendulums and a surface. Kuramoto equations will be solved in MATLAB, and it is expected that the pendulums will synchronise under certain conditions. Ideally a model would give an understanding of these conditions to allow for predictions to be made as to the synchronisation of other mechanical systems.

Previous literature

In the paper [Pantaleone, 2002] an experimental approach is used to understand the phenomenon that is synchronisation in mechanical systems. There is similar analysis in this paper to what is attempted in this report, in that they both attempt to derive the equations of motion for a metronome and further one on a moving base. In the case of [Pantaleone, 2002] the equations for the metronomes are then used to show the same form as would be expected in a Kuramoto model. In contrast this report derives these equations of motion and then in parallel derives the coupled equations using the Kuramoto equation with some minor adaptations. This report attempts to provide a simple model for synchronisation within mechanical systems whereas [Pantaleone, 2002] provides a more comprehensive but perhaps more computationally intensive solution. Another piece of work that focuses on the Kuramoto model is [Breakspear et al., 2010]. This report takes the simple Kuramoto model and adapts it to show the synchronisation of metronomes where as [Breakspear et al., 2010] uses it to model cortical oscillations within the brain. Whilst this report keeps the model very simple as to cut down on the parmeters needed to describe the system [Breakspear et al., 2010] develops the Kuramoto model to demonstrate the time delay between the firing of different neurons and hence requires a more detailed parameter set to output a more complex solution. It is however a good source for the form of the Kuramoto equation that is used in this report.

2 Method

2.1 Lagrangian Pendulums

The metronome system of interest can be modelled as a number of simple pendulums on a moving cart. This allows a relationship between the movement of the cart and the movement of the pendulum to be observed. Beginning with a single pendulum with a light and in-extensible rod, it can be assumed that the both the pendulum and cart move freely. The length of the pendulum is L, and the pivot is a height l above ground. The mass of pendulum and cart is m_1 and m_2 respectively. Taking the angle of the pendulum from the vertical as θ and the horizontal displacement of the cart as x, the kinetic energy, T and the potential energy, U, of the system can be calculated using the following equations:

$$T = \frac{1}{2}mv^2 \qquad and \qquad U = mgh. \tag{1}$$

For the kinetic energy, the horizontal velocity of the cart must be considered, along with the horizontal and vertical velocities of the pendulum. The following is obtained:

$$T = \frac{1}{2}\dot{x}^2(m_1 + m_2) + \frac{1}{2}m_1 L^2\dot{\theta}^2 + m_1\dot{x}\dot{\theta}L\cos(\theta). \tag{2}$$

For the potential energy, the vertical position if the pendulum is used and the following is obtained:

$$U = m_1 g(l - L\cos(\theta)). \tag{3}$$

Using these two equations, the Lagrangian, L, is calculated using the equation

$$L = T - U. \tag{4}$$

The Euler-Lagrange equations of motion are then calculated using

$$\frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = 0. \tag{5}$$

The subsequent equations of motion are

$$m_1 L(L\ddot{\theta} + \ddot{x}\cos(\theta) + g\sin(\theta)) = 0 \qquad \text{and} \qquad \ddot{x}(m_1 + m_2) + m_1 L(\ddot{\theta}\theta - \dot{\theta}^2\sin(\theta)) = 0. \tag{6}$$

In order to simplify equations (6), it is useful to consider when θ is small. Replacing θ with $\epsilon\,\theta$, the terms including cos and sin can be expanded using the Maclaurin expansion. Terms with higher orders of ϵ can then be disregarded and the equations can be linearised as follows:

$$m_1 L(L\epsilon\ddot{\theta} + \ddot{x}\epsilon + g\epsilon\theta) = 0 \qquad \text{and} \qquad \ddot{x}(m_1 + m_2) = 0. \tag{7}$$

An expected size of x and a format for the values, can then be estimated. This, however, does not allow us gain enough understanding of the problem. This is because the equations (7) do not seem to have a direct coupling term (one in terms of the difference in phase between the metronome and the cart). The equations are obtained by assuming an extremely small oscillation about the $\theta = 0$ rads point so a conclusion can be drawn that there is no real coupling effect unless the movement of the metronome is above a certain threshold.

Also, equations (7) are difficult to work with since they are non-linear. Due to this any analysis done would require advanced techniques that this project does not allow for within its time-span. This means a different approach was needed to obtain any meaningful results from the simulations. To simplify the model an approximation was made to model the metronomes as simple oscillators which allows the use of the Kuramoto model to demonstrate the coupling between the metronomes.

2.2 Kuramoto oscillators

Two directly coupled oscillators

As the metronomes are modelled as simple oscillators the Kuramoto model, as detailed in [Porter, 2016] can be used to demonstrate the coupling force between them. The model takes inspiration from network theory in that the oscillators(metronomes) are shown as nodes on a graph and if an edge is present between the nodes this signifies a coupled pair of oscillators. Each edge in this case is non-directional as if one oscillator is coupled to another then the inverse is implied. In the simplest case this give a graph with two nodes and one edge as seen below.

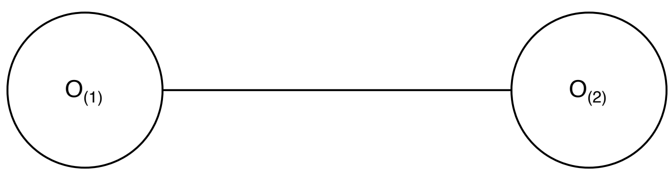

Figure 1: Graph showing the two coupled oscillators

In this case the equations are very simple each oscillator has two properties, its natural frequency ω and its angle θ. Then using the Kuramoto equation for rate of change of θ_i:

$$\dot{\theta}_i = \omega_i + \frac{k}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \tag{8}$$

this equation is then used for the simple two oscillator example yielding two equations:

$$\dot{\theta_1} = \omega_1 + \frac{k}{2}\sin(\theta_2 - \theta_1) \qquad \dot{\theta_2} = \omega_2 + \frac{k}{2}\sin(\theta_1 - \theta_2) \tag{9}$$

These can then be put into MATLAB and using the in built ODE solver *ode23s* the graph of each oscillator's angle with time can be plotted.

Since the synchronisation should be independent of the initial phases of the oscillators they are irrelevant in the analysis of the system meaning that the parameters of this model are:

Parameters	
ω_1	The natural frequency of oscillator one.
ω_2	The natural frequency of oscillator two.
k	The coupling strength between the oscillators.

Table 1: Table of model parameters

Naturally this method can be extended to show the synchronisation of N oscillators by simply changing the ode file in MATLAB to represent equation (8) then for each oscillator summing the coupling force between it and all other oscillators to get the full expression for its $\dot{\theta}$. Similar plots to those that are shown below in figure 3 can then be generated to show the angles of all N oscillators and hence show the synchronisation effect modelled in this report works on a generalised N degree of freedom system.

Two oscillators coupled through a board

To more accurately model the synchronisation phenomenon represented in the 'Synchronisation of Metronomes' paper [Pantaleone, 2002] the Kuramoto model can still be used but an extra element must be added which is the 'coupling element'. In the more accurate case the oscillators are not directly coupled, instead they are all couple to a single entity in physical terms the board on which the metronomes sit. Starting with the simple case of two oscillators coupled through a board a graph representing the system can be found.

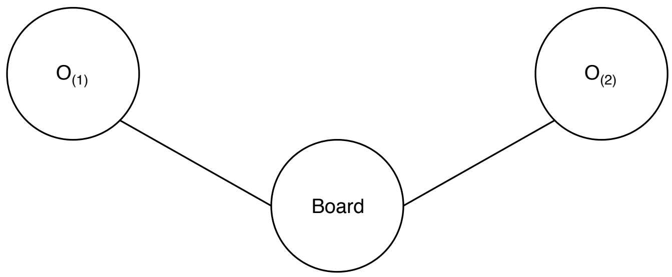

Figure 2: Graph representing the two oscillators coupled through a board system

This version of the Kuramoto model requires a defined θ_b which represents the position of the board and a slight change to the equations in terms of the parameters in table 1 in that for the i^{th} oscillator

$$\dot{\theta}_i = \omega_i + \frac{k}{N+1}\sin(\theta_b - \theta_i). \tag{10}$$

This means an equation that describes the movement of the board becomes

$$\dot{\theta}_b = \frac{k}{N+1}\sum_{k=1}^{N}\sin(\theta_k - \theta_b). \tag{11}$$

This equation does not contain a parmeter ω_b as the board is assumed to have a natural frequency of 0 as it is only driven by the movement of the metronomes.

Once again using MATLAB's ode solving functionality a vector containing the θ's of all the oscillators over time can be produced.

In the analysis done for this report the simplest case of two oscillators on the board, as shown above, was considered. This means that any analysis can be performed on a two degree of freedom system.

3 Analysis of results

Two directly couple oscillators

Phase plots

By integrating the $\dot{\theta}$'s of both oscillators over time using MATLAB's in-build function *ode23s* it is possible to plot a graph showing the change of the angles over time which, when angles are wrapped to 2π shows the initial coupling and the resulting synchronised oscillators.

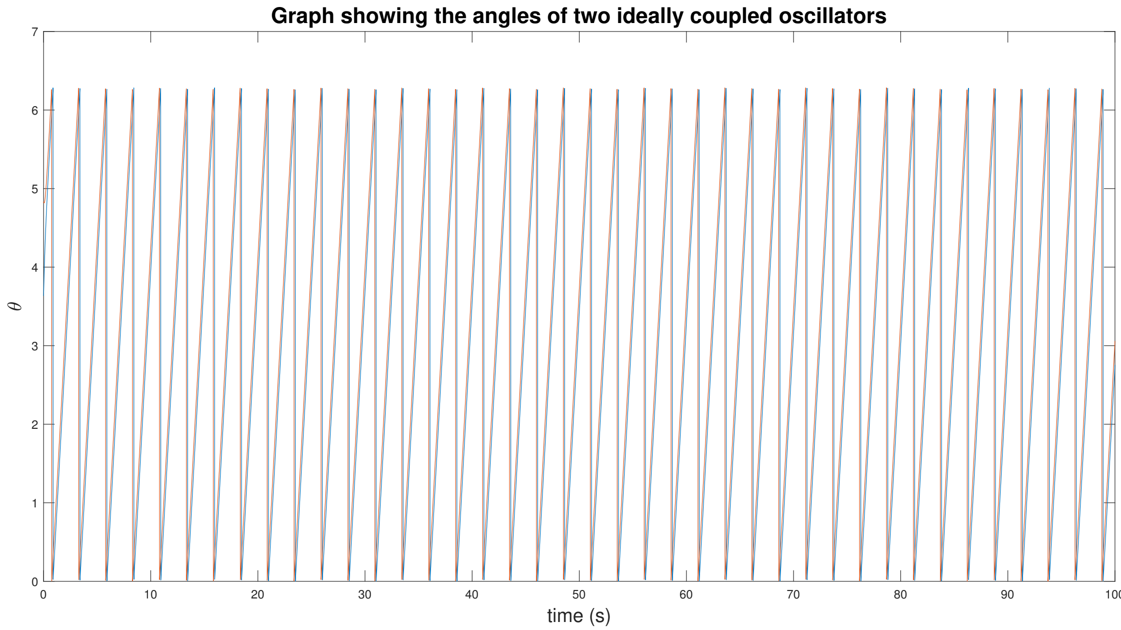

Figure 3: Graph showing the two coupled oscillators angles with time

the graph above shows a constant difference $\Delta\theta$ for the majority of the time period which seems to demonstrate the synchronisation effect however when zoomed in the oscillators have a period of time in which they synchronise which shows some interesting trends.

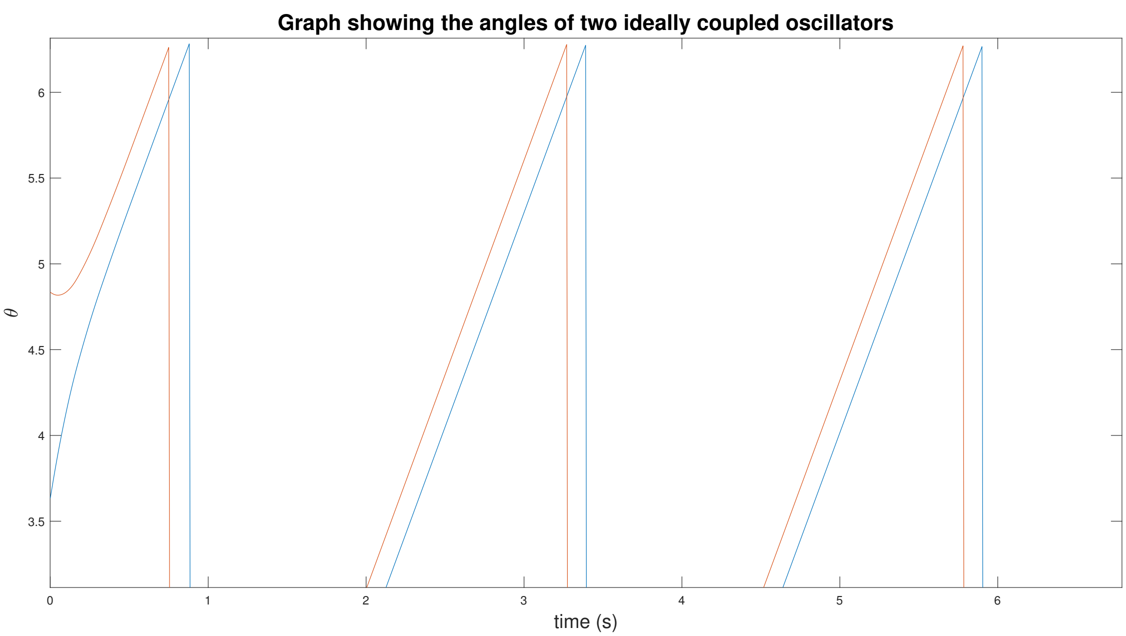

Figure 4: Graph showing the coupling period of the two oscillator and their constant $\Delta\theta$ after synchronisation

In the figure above the synchronisation period is clearly shown. The oscillators start out of phase with one another and within a few milliseconds the oscillators reach what seems to be a steady state. This is a point at which both oscillators move towards each others phase but do not quite match due to their different natural frequencies. Some analysis on this system is required to find out the conditions for synchronisation, the position of this steady state and to find out if it is a stable equilibrium or not.

Stability analysis

Looking at the results of the initial two oscillator model it is a good idea to analyse the equilibrium of the system. It can be seen that the oscillators do not synchronise to the exact same phase but instead there is a constant difference between them $\Delta\theta$. To find this steady state start with the equations for each oscillator's $\dot{\theta}$

$$\dot{\theta_1} = \omega_1 + \frac{k}{2}\sin(\theta_2 - \theta_1) \qquad \dot{\theta_2} = \omega_2 + \frac{k}{2}\sin(\theta_1 - \theta_2). \tag{12}$$

By defining

$$\Delta\theta = \theta_1 - \theta_2$$

and by differentiating

$$(\dot{\Delta\theta}) = \dot{\theta_1} - \dot{\theta_2}$$

it is then possible to write $(\dot{\Delta\theta})$ in terms of $\Delta\theta$ leading to the equation

$$(\dot{\Delta\theta}) = \omega_1 - \omega_2 - k\sin(\Delta\theta). \tag{13}$$

At a steady state $\Delta\theta$ is in equilibrium. This is to say that $(\dot{\Delta\theta}) = 0$. After rearrangement of (13) this gives the equation

$$\sin(\Delta\theta) = \frac{\omega_1 - \omega_2}{k}, \tag{14}$$

which means there is a condition on synchronisation in the system that

$$k > |\omega_1 - \omega_2|.$$

This condition is relevant as it allows for predictions to be made as to whether the system will synchronise given its parameters.

As long as condition (3) above is met it is possible to write the steady state $\Delta\theta$ as

$$\Delta\theta = \arcsin\left(\frac{\omega_1 - \omega_2}{k}\right). \tag{15}$$

Now that the steady state has been found it is necessary to explore the stability of this equilibrium using a technique known as linearisation.

By only considering small oscillations about this steady state it is possible to define the steady state and write $\Delta\theta$ as

$$\Delta\theta \text{ at equilibrium} = \tilde{\Delta\theta} \tag{16}$$

$$\Delta\theta = \tilde{\Delta\theta} + \epsilon x(t). \tag{17}$$

where $x = \mathcal{O}(1)$
Taking the derivative of $\Delta\theta$ removes the steady state as it does not vary with time.

$$\dot{\Delta\theta} = \epsilon\dot{x}(t)$$

The equation for the rate of change of $\Delta\theta$ is described above in equation 13 so by substituting in the new definitions for $\Delta\theta$ and $\dot{\Delta\theta}$, a new equation is found:

$$\epsilon\dot{x} = \omega_1 - \omega_2 - k\sin(\tilde{\Delta\theta} + \epsilon x) \tag{18}$$

it is then possible to replace the sine term with its taylor expansion giving

$$\epsilon\dot{x} = \omega_1 - \omega_2 - k\sin(\tilde{\Delta\theta}) - k\cos(\tilde{\Delta\theta})\epsilon x + H.O.T. \tag{19}$$

From equation 13 and that at $\tilde{\Delta}\theta$ the rate of change of $\Delta\theta$ is zero, it is now possible to reduce this equation down to

$$\dot{x} = -k\cos(\tilde{\Delta}\theta)x. \tag{20}$$

After making the substitution

$$a = -k\cos(\tilde{\Delta}\theta)$$

and solving the differential equation $x(t)$ can be written as

$$x(t) = Ce^{-at}. \tag{21}$$

For the solution to be stable its oscillations must decay to zero and hence a must be positive meaning the stability of this equilibrium is constrained to

$$k\cos(\tilde{\Delta}\theta) > 0.$$

This can in fact be re-written due to

$$k\cos(\tilde{\Delta}\theta) = k\cos\left(\arcsin\left(\frac{\omega_1 - \omega_2}{k}\right)\right). \tag{22}$$

By trigonometry it can be shown that

$$k\cos(\tilde{\Delta}\theta) = \sqrt{k^2 - (\omega_1 - \omega_2)^2}. \tag{23}$$

Using the constraint found above this means that for a stable system

$$k > \omega_1 - \omega_2 \quad \text{and} \tag{24}$$

Taking the values used in the simulations shown above in figure 3

$$k = 10 \quad \omega_1 = 1 \quad \omega_2 = 4$$

the separation between the two oscillators can be found to be

$$\tilde{\Delta}\theta = \arcsin\left(\frac{1-4}{10}\right) = -0.3047 rads. \tag{25}$$

This theoretical result is supported by the MATLAB numerical simulation. Below is a plot of the log of $\Delta\theta - \tilde{\Delta}\theta$ to show how thie distance to the predicted steady state of the system changes over time.

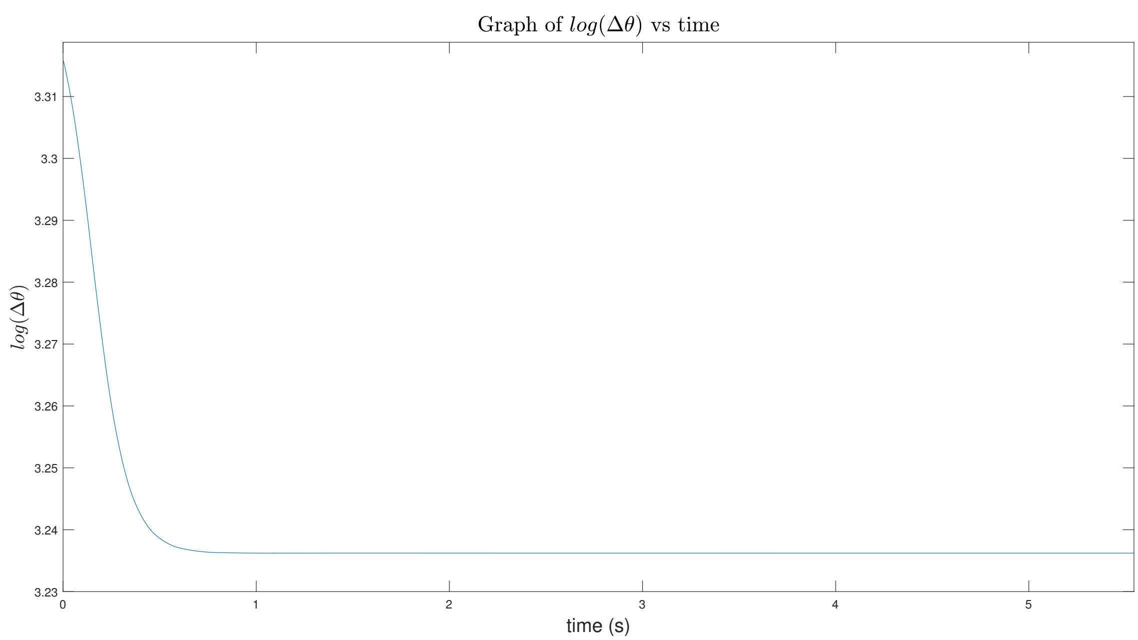

Figure 5: Graph showing the $\Delta\theta$ of the oscillators over time

The plot show the exponential decay that is predicted by the analysis above which would suggest that the numerical approach used in MATLAB does support the analytic method detailed in the stability analysis section of this report.

Two oscillators coupled through a board

Phase plots

By integrating the $\dot{\theta}$'s of the two oscillators and the board once again using MATLAB's *ode23s* and plotting the angle of each against time, again the coupling and synchronisation can be seen.

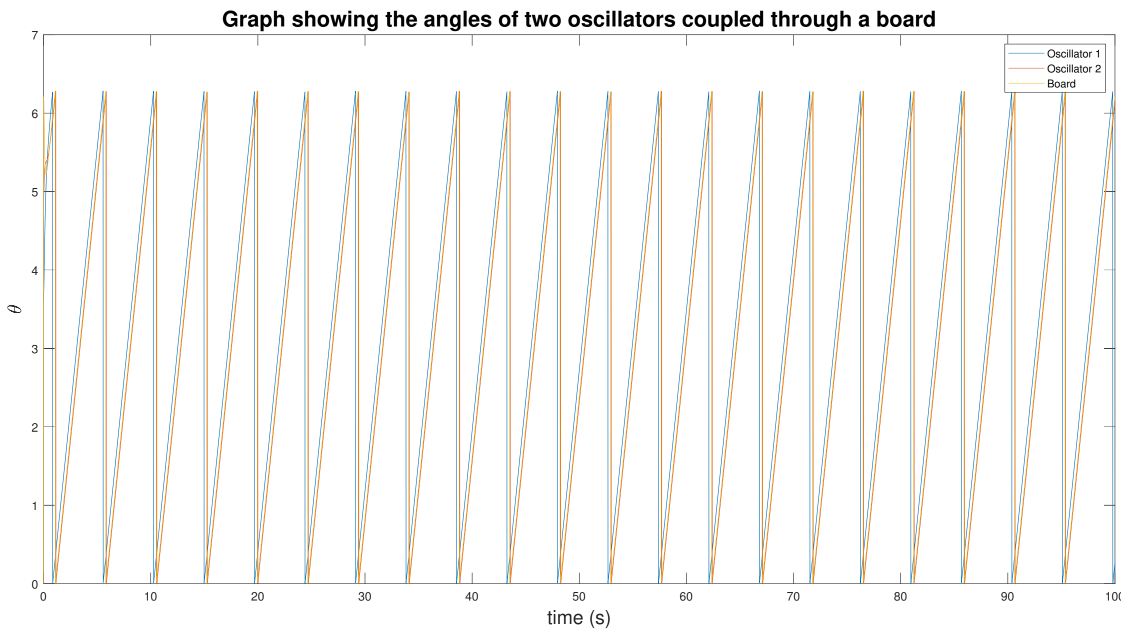

Figure 6: Graph showing the angles of the oscillators and board over time

As with the simpler system before the phase plot shows that the system is synchronised for most of the time period with a slight difference between the board and each of the oscillators from now on defined as $\Delta\theta_1$ and $\Delta\theta_2$. Fortunately, zooming in on the initial time period once again shows the synchronisation period for the oscillators.

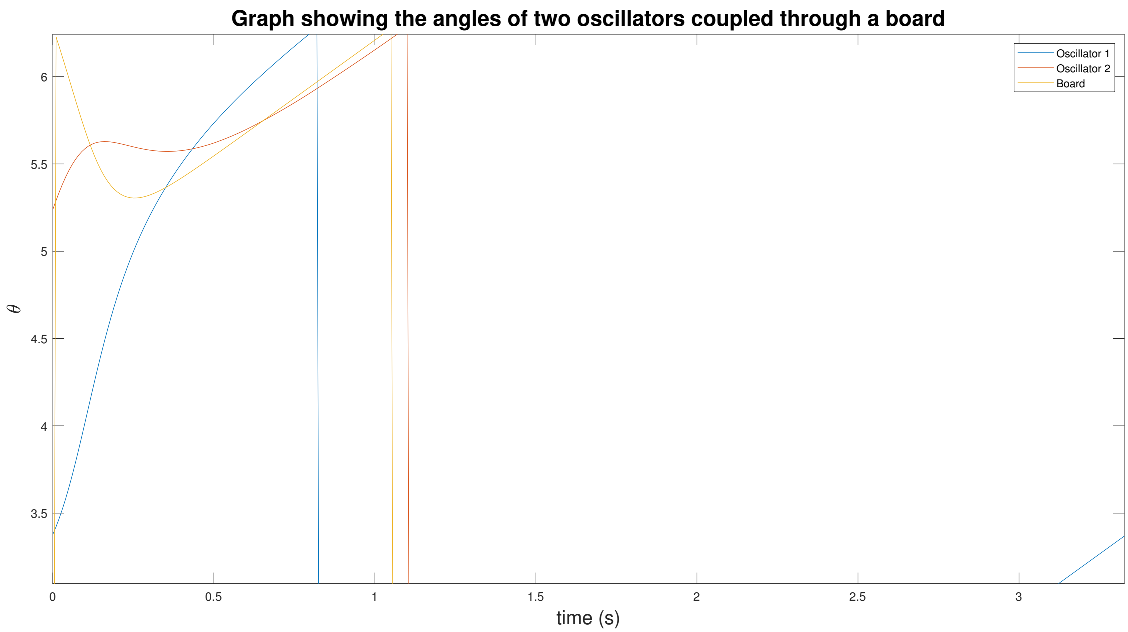

Figure 7: Graph showing the initial synchronisation of the oscillators with the board

In the figure the synchronisation of the two oscillators can be seen. In comparison with the simple direct coupled system the system in figure 7 takes longer to couple due to the reliance on a third entity (The board) to provide the coupling between the oscillators. There is a gap of π radians between the angle of the two oscillators which is an equilibrium as the effects of both oscillators on the board will cancel each other out. This means that the board will have a zero rate of change of θ making this a steady state. It is likely this state will be unstable as when moving slightly away from perfectly out of phase will cause the board to move and then the oscillators to tend to each others phase.

Stability analysis

With this new model for the system described in [Lugo-Cardenas et al., 2012] it is necessary to once again investigate the steady states and the stability of this system.
In this case there are three different objects each with their own natural frequencies and this gives three equations for the rates of change of their respective thetas:

$$\dot{\theta_1} = \omega_1 + \frac{k}{3}\sin(\theta_b - \theta_1) \tag{26}$$

$$\dot{\theta_2} = \omega_2 + \frac{k}{3}\sin(\theta_b - \theta_2) \tag{27}$$

$$\dot{\theta_b} = \frac{k}{3}(\sin(\theta_1 - \theta_b) + \sin(\theta_2 - \theta_b)) \tag{28}$$

Then in the same way as the simpler example above the system can be written in terms of $\Delta\theta_1$ and $\Delta\theta_2$ defined on the difference in θ between the oscillators and the board yielding the equations:

$$(\Delta\dot{\theta}_1) = \omega_1 - \frac{2k}{3}\sin(\Delta\theta_1) - \frac{k}{3}\sin(\Delta\theta_2) \tag{29}$$

$$(\Delta\dot{\theta}_2) = \omega_2 - \frac{2k}{3}\sin(\Delta\theta_2) - \frac{k}{3}\sin(\Delta\theta_1) \tag{30}$$

When analysing at an equilibrium point two conditions for synchronisation can be found:

$$k > |2\omega_2 - \omega_1|$$
$$k > |2\omega_1 - \omega_2|$$

The equations above also lead to the steady states of the system which are:

$$\Delta\theta_1 = \arcsin\left(\frac{2\omega_1 - \omega_2}{k}\right) \tag{31}$$

$$\Delta\theta_2 = \arcsin\left(\frac{2\omega_2 - \omega_1}{k}\right) \tag{32}$$

Using the linearisation technique as before and substituting:

$$\Delta\theta_1 = \tilde{\Delta\theta_1} + \epsilon x_1(t) \tag{33}$$

$$\Delta\theta_2 = \tilde{\Delta\theta_2} + \epsilon x_2(t) \tag{34}$$

By taking the taylor expansions of all sine terms and ignoring terms with powers of ϵ higher than 1 the rate of change of the small perturbations x_1 and x_2 can be written in a matrix form:

$$\begin{pmatrix}\dot{x_1}\\ \dot{x_2}\end{pmatrix} = \frac{-k}{3}\begin{pmatrix}2\cos(\tilde{\Delta\theta_1}) & \cos(\tilde{\Delta\theta_2})\\ \cos(\tilde{\Delta\theta_1}) & 2\cos(\tilde{\Delta\theta_2})\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{35}$$

The solutions to this problem can be written in the form

$$\mathbf{x} = \alpha\mathbf{x_1}e^{\lambda_1 t} + \beta\mathbf{x_2}e^{\lambda_2 t}. \tag{36}$$

Where λ_1 and λ_2 are the eigenvalues of the matrix in (35) and $\mathbf{x_1}$ and $\mathbf{x_2}$ are the corresponding eigenvectors.

It can be seen that for the solution to be stable (decay to 0) both of the eigenvalues of the matrix in equation (35) above must be negative. These values can be found to be

$$\lambda_1 = -\frac{k\sqrt{\frac{k^2-{w_1}^2+4\,w_1\,w_2-4\,{w_2}^2}{k}}}{3} - \frac{k\sqrt{\frac{k^2-4\,{w_1}^2+4\,w_1\,w_2-{w_2}^2}{k}}}{3} - \frac{\sqrt{-\frac{4\,k\left(5\,{w_1}^2-2\,k^2-8\,w_1\,w_2+5\,{w_2}^2+k\sqrt{k-\frac{{w_1}^2}{k}-\frac{4\,{w_2}^2}{k}+\frac{4\,w_1\,w_2}{k}}\sqrt{k-\frac{4\,{w_1}^2}{k}-\frac{{w_2}^2}{k}+\frac{4\,w_1\,w_2}{k}}\right)}{9}}}{2}$$

and

$$\lambda_2 = -\frac{k\sqrt{\frac{k^2-{w_1}^2+4\,w_1\,w_2-4\,{w_2}^2}{k}}}{3} - \frac{k\sqrt{\frac{k^2-4\,{w_1}^2+4\,w_1\,w_2-{w_2}^2}{k}}}{3} + \frac{\sqrt{-\frac{4\,k\left(5\,{w_1}^2-2\,k^2-8\,w_1\,w_2+5\,{w_2}^2+k\sqrt{k-\frac{{w_1}^2}{k}-\frac{4\,{w_2}^2}{k}+\frac{4\,w_1\,w_2}{k}}\sqrt{k-\frac{4\,{w_1}^2}{k}-\frac{{w_2}^2}{k}+\frac{4\,w_1\,w_2}{k}}\right)}{9}}}{2}$$

By taking the simplest case in which ω_1 is equal to ω_2 these eigenvalues can be found:

$$\lambda_1 = -\frac{\sqrt{\frac{4\,k\,(k+\omega)\,(k-\omega)}{9}}}{2} - \frac{2\,k\sqrt{\frac{k^2-\omega^2}{k}}}{3} \tag{37}$$

$$\lambda_2 = \frac{\sqrt{\frac{4\,k\,(k+\omega)\,(k-\omega)}{9}}}{2} - \frac{2\,k\sqrt{\frac{k^2-\omega^2}{k}}}{3} \tag{38}$$

Analysing the magnitude of both halves of the eigenvalues it is found that:

$$\left|\frac{\sqrt{\frac{4\,k\,(k+\omega)\,(k-\omega)}{9}}}{2}\right| < \left|\frac{2\,k\sqrt{\frac{k^2-\omega^2}{k}}}{3}\right|$$

$$12k^2 < 72k^2$$

Since this is the case, the eigenvalues are both negative which means that this steady state is linearly stable and any small perturbation will decay to zero.

Effects of k on the synchronisation

One of the key aspects of this problem is an investigation into decreasing the time required for the synchronisation of the metronomes. The best way to test for this would be to vary the parameters of the model and to see what effect they have on the synchronisation time. The parameter k was chosen as it has a large effect on the synchronisation of the system and maintaining the ability to set all of the metronomes natural frequencies to different values is a must.
A function was created in MATLAB to find the synchronisation time of a single simulation by checking for the time at which the difference between the $\Delta\theta$ for that time and the step before falls below a tolerance. A script can then be written to calculate this 'sync time' for a range of different values of k and a plot is generated plotting each k against its respective sync time.

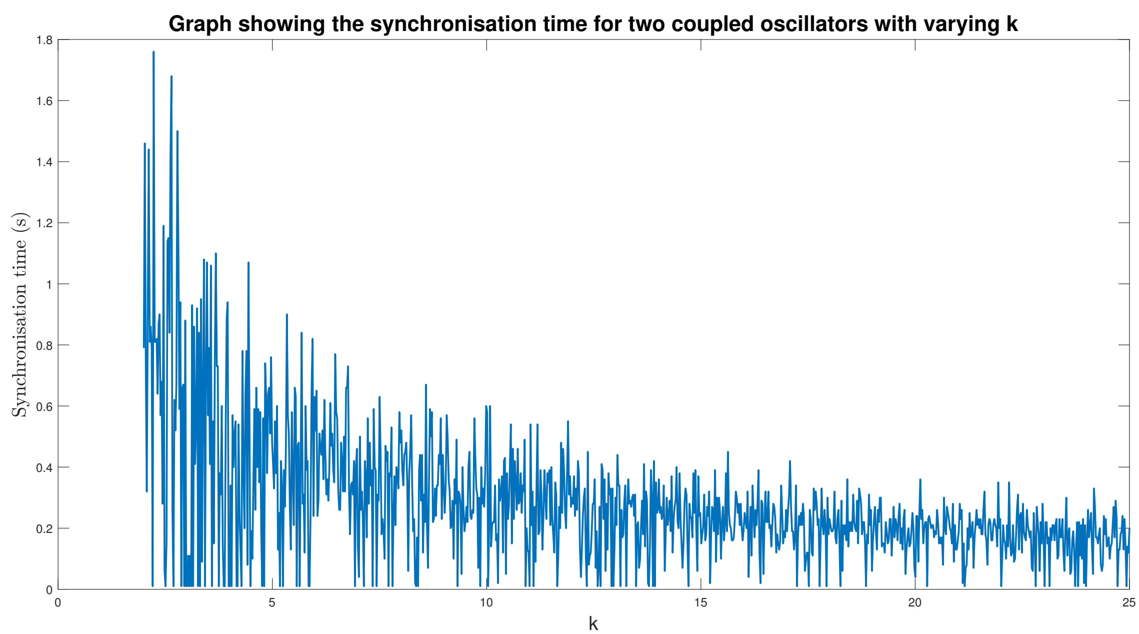

Figure 8: Plot showing sync time for varying k

Figure 8 shows a general trend in that as the value of k is increased the time required for synchronisation drastically decreases.There is however a lot of variance in the synchronisation time as the initial phase of the two oscillators seems to have a larger effect on the synchronisation time than the value of k does.

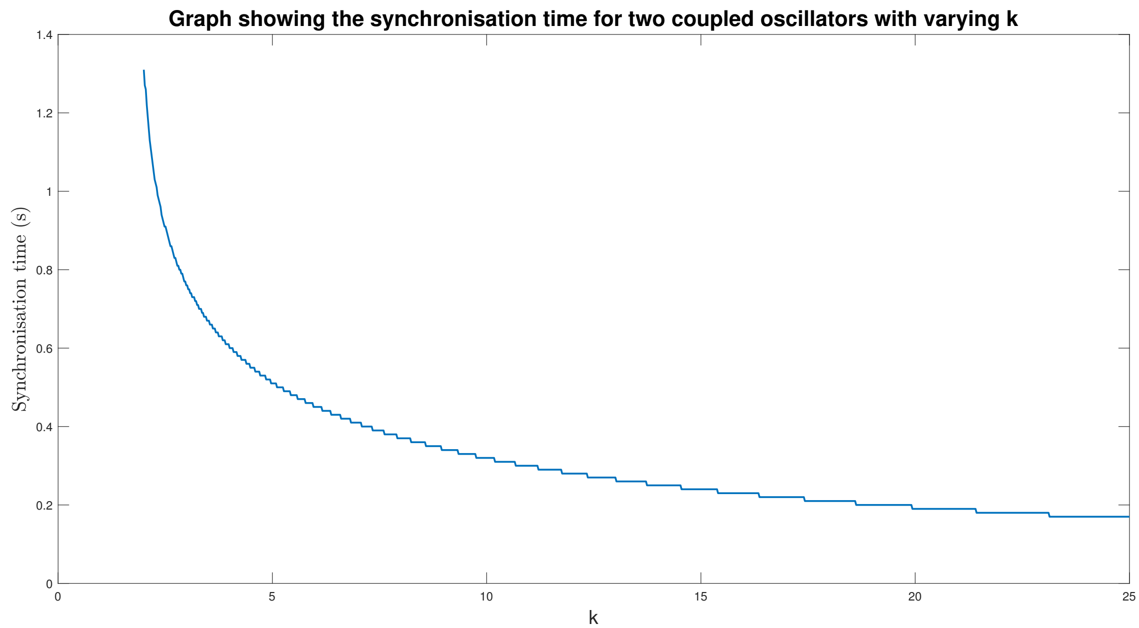

Figure 9: Plot showing sync time for varying k zoomed in after drop

Figure 9 shows that when removing the random element from the starting θ values, a trend can be seen more clearly. As k is increased the synchronisation time is decreased but with diminishing returns as k get significantly large. This is useful in that when designing a system for optimum synchronisation time it is not necessary to increase the value of k too far as this effort will yield an only marginally better synchronisation time.

4 Discussion and Conclusion

Discussion

The main benefit of modelling the synchronisation of metronomes using a simple oscillator model and then coupling using the Kuramoto equation is that this removes the need to deal with the non-linear equations that describe the motion of the full metronome system. This means that analysis of the system can be done much more easily using linearisation and hence the stability of different equilibria within the system can be somewhat concluded upon. From the analysis above conclusions can be drawn on the conditions required within the system to result in synchronisation in terms of the parameters of the model see table 1. This is

$$k > |\omega_1 - \omega_2|$$

in the case of a system of two directly coupled oscillators. Which means that if the coupling strength of the system k can be found then our model can specify the maximum amount of difference between the natural frequencies of the two metronomes. This steady state can also be said to be a stable, and hence consistent synchronisation, given that the following constraint is met:

$$k > |\omega_1 - \omega_2|$$

This will allow for further work to be done that will design systems of coupled oscillators that will not only synchronise but this synchronisation will be linearly stable. One drawback to the this part of the model is that the metronomes are not connected they both rest on the same board. In the case of this report a system in which the oscillators are not directly coupled is needed.

The second model created was one which represents two oscillators coupled through a board. This model still uses the Kuramoto equation as its base but in this case all oscillators are only coupled to the board and the board is coupled to all oscillators. This provides a more accurate representation of the metronomes synchronising on a board that is free to move in the horizontal plane. This model also provides some constraints on synchronisation in terms of k and ω. However, in this case there are two constraints, due to there being two degrees of freedom within the system. They are

$$k > |2\omega_2 - \omega_1|$$
$$k > |2\omega_1 - \omega_2|.$$

These can be used to create a system in which the two metronomes on a free moving board should theoretically synchronise given that the strength of coupling provided by the board is known. It is expected that the k value for this system would be relatively high and hence in most cases the metronomes would synchronise. This expectation is supported by many previous experiments showing the synchronisation of metronomes in a setup similar to the one described in this report.

The main drawback of the model described in this report is that the metronomes are being modelled as simple oscillators whereas in actual fact their motion is described using a set of non-linear equations. Even when the metronome is reduced to a simpler two mass pendulum the equations describing its motion are non-linear as shown above in equation 7. This means that the simple relationships used to show the coupling between the metronomes in the Kuramoto model are most likely an over-simplification of the true effects. One other major drawback is that all of the stability analysis is using linearisation. This means that any conclusions drawn about the stability are true only in the case that the perturbation of the metronome is only infinitesimally small. More work will have to be done to explore the non-linear stability outside of this region.

When analysing the effect of varying k on the synchronisation time of the system the main drawback is the fact that MATLAB only outputs specified time-steps and this means the curve in figure 9 is not smooth, giving less information on the specific trend in the results. However the graph does show a general trend from which a conclusion can be drawn. That is to say that as the value of k is increased the time required for synchronisation tends to drop even with the diminished returns at large values of k.

Further work

Following on from the 'two oscillators on a board' model created in this report it may be worth investigating the effect of a moving board. This would involve creating a new parameter ω_b which could be made a function of time. This investigation would allow for testing with periodic movement of the board and to see the effect this has on the synchronisation of the metronomes. This could then be developed into a control system, in that the driving frequency of the board can be adjusted according to the changes needing to be made on the $\Delta\theta$ between the two oscillators. One potential alternative use for the Kuramoto model used in this report is the modelling of the synchronisation of neurons in the brain see [Breakspear et al., 2010]. In this case the model would have to be adapted to show the time delay between the neurons. This however, may become applicable to the system described in this report as in a case where there are many oscillators and the control input to the board is small there may be a time delay effect on the synchronisation.

Conclusion

Overall the model detailed in this report gives a good picture of the phenomenon that is the synchronisation of metronome. Through mathematical analysis of the model some interesting traits of the system can be found. However it is not a full picture of the effects that can be seen in the real world and any further work should be dedicated to taking the mathematical basis provided in the report and building this up to give a full picture of the non-linear system that is being dealt with.

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