Homework #0

DUE DATE: NONE

1 Probability and Statistics

(1) (combinatorics)
Let $C(N,K) = 1$ for $K = 0$ or $K = N$, and $C(N,K) = C(N-1,K) + C(N-1,K-1)$ for $N \geq 1$. Prove that $C(N,K) = \frac{N!}{K!(N-K)!}$ for $N \geq 1$ and $0 \leq K \leq N$.

(2) (counting)
What is the probability of getting exactly 6 heads when flipping 10 fair coins?

What is the probability of getting a full house (XXXYY) when randomly drawing 5 cards out of a deck of 52 cards?

(3) (conditional probability)
If your friend flipped a fair coin three times, and tell you that one of the tosses resulted in head, what is the probability that all three tosses resulted in heads?

(4) (Bayes theorem)
A program selects a random integer X like this: a random bit is first generated uniformly. If the bit is 0, X is drawn uniformly from $\{0, 1, \ldots, 7\}$; otherwise, X is drawn uniformly from $\{0, -1, -2, -3\}$. If we get an X from the program with $|X| = 1$, what is the probability that X is negative?

(5) (union/intersection)
If $P(A) = 0.3$ and $P(B) = 0.4$,
what is the maximum possible value of $P(A \cap B)$?
what is the minimum possible value of $P(A \cap B)$?
what is the maximum possible value of $P(A \cup B)$?
what is the minimum possible value of $P(A \cup B)$?

(6) (mean/variance)
Let mean $\overline{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$ and variance $\sigma_X^2 = \frac{1}{N-1}\sum_{n=1}^{N}(X_n - \overline{X})^2$. Prove that

$$\sigma_X^2 = \frac{N}{N-1}\left(\frac{1}{N}\sum_{n=1}^{N} X_n^2 - \overline{X}^2\right).$$

(7) (Gaussian distribution)
If X_1 and X_2 are independent random variables, where $p(X_1)$ is Gaussian with mean 2 and variance 1, $p(X_2)$ is Gaussian with mean -3 and variance 4. Let $Z = X_1 + X_2$. Prove $p(Z)$ is Gaussian, and determine its mean and variance.

2 Linear Algebra

(1) (rank)

What is the rank of $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix}$?

(2) (inverse)

What is the inverse of $\begin{pmatrix} 0 & 2 & 4 \\ 2 & 4 & 2 \\ 3 & 3 & 1 \end{pmatrix}$?

(3) (eigenvalues/eigenvectors)

What are the eigenvalues and eigenvectors of $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$?

(4) (singular value decomposition)

(a) For a real matrix M, let $\mathrm{M} = \mathrm{U}\Sigma\mathrm{V}^T$ be its singular value decomposition. Define $\mathrm{M}^\dagger = \mathrm{V}\Sigma^\dagger\mathrm{U}^T$, where $\Sigma^\dagger[i][j] = \frac{1}{\Sigma[i][j]}$ when $\Sigma[i][j]$ is nonzero, and 0 otherwise. Prove that $\mathrm{M}\mathrm{M}^\dagger\mathrm{M} = \mathrm{M}$.

(b) If M is invertible, prove that $\mathrm{M}^\dagger = \mathrm{M}^{-1}$.

(5) (PD/PSD)
A symmetric real matrix A is positive definite (PD) *iff* $\mathbf{x}^T\mathrm{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$, and positive semi-definite (PSD) if "$>$" is changed to "$\geq$". Prove:

(a) For any real matrix Z, $\mathrm{Z}\mathrm{Z}^T$ is PSD.

(b) A symmetric A is PD *iff* all eigenvalues of A are strictly positive.

(6) (inner product)
Consider $\mathbf{x} \in R^d$ and some $\mathbf{u} \in R^d$ with $\|\mathbf{u}\| = 1$.
What is the maximum value of $\mathbf{u}^T\mathbf{x}$? What $\mathbf{u}$ results in the maximum value?
What is the minimum value of $\mathbf{u}^T\mathbf{x}$? What $\mathbf{u}$ results in the minimum value?
What is the minimum value of $|\mathbf{u}^T\mathbf{x}|$? What $\mathbf{u}$ results in the minimum value?

(7) (distance)
Consider two parallel hyperplanes in R^d:

$$H_1 : \mathbf{w}^T\mathbf{x} = +3,$$
$$H_2 : \mathbf{w}^T\mathbf{x} = -2,$$

where $\mathbf{w}$ is the norm vector. What is the distance between H_1 and H_2?

3 Calculus

(1) (differential and partial differential)
Let $f(x) = \ln(1 + e^{-2x})$. What is $\frac{df(x)}{dx}$? Let $g(x, y) = e^x + e^{2y} + e^{3xy^2}$. What is $\frac{\partial g(x,y)}{\partial y}$?

(2) (chain rule)
Let $f(x, y) = xy$, $x(u, v) = \cos(u + v)$, $y(u, v) = \sin(u - v)$. What is $\frac{\partial f}{\partial v}$?

(3) (integral)
What is $\int_5^{10} \frac{2}{x-3} dx$?

(4) (gradient and Hessian)
Let $E(u, v) = (ue^v - 2ve^{-u})^2$. Calculate the gradient ∇E and the Hessian $\nabla^2 E$ at $u = 1$ and $v = 1$.

(5) (Taylor's expansion)
Let $E(u, v) = (ue^v - 2ve^{-u})^2$. Write down the second-order Taylor's expansion of E around $u = 1$ and $v = 1$.

(6) (optimization)
For some given $A > 0, B > 0$, solve

$$\min_\alpha Ae^\alpha + Be^{-2\alpha}.$$

(7) (vector calculus)
Let $\mathbf{w}$ be a vector in R^d and $E(\mathbf{w}) = \frac{1}{2}\mathbf{w}^T \mathrm{A}\mathbf{w} + \mathbf{b}^T\mathbf{w}$ for some symmetric matrix A and vector $\mathbf{b}$. Prove that the gradient $\nabla E(\mathbf{w}) = \mathrm{A}\mathbf{w} + \mathbf{b}$ and the Hessian $\nabla^2 E(\mathbf{w}) = \mathrm{A}$.

(8) (quadratic programming)
Following the previous question, if A is not only symmetric but also positive definite (PD), prove that the solution of $\mathrm{argmin}_{\mathbf{w}} E(\mathbf{w})$ is $-\mathrm{A}^{-1}\mathbf{b}$.

(9) (optimization with linear constraint)
Consider

$$\min_{w_1, w_2, w_3} \frac{1}{2}(w_1^2 + 2w_2^2 + 3w_3^2) \text{ subject to } w_1 + w_2 + w_3 = 11.$$

Refresh your memory on "Lagrange multipliers" and show that the optimal solution must happen on $w_1 = \lambda$, $2w_2 = \lambda$, $3w_3 = \lambda$. Use the property to solve the problem.

(10) (optimization with linear constraints)
Let $\mathbf{w}$ be a vector in R^d and $E(\mathbf{w})$ be a convex differentiable function of $\mathbf{w}$. Prove that the optimal solution to

$$\min_{\mathbf{w}} E(\mathbf{w}) \text{ subject to } \mathrm{A}\mathbf{w} + \mathbf{b} = 0.$$

must happen at $\nabla E(\mathbf{w}) + \boldsymbol{\lambda}^T \mathrm{A} = \mathbf{0}$ for some vector $\boldsymbol{\lambda}$. (*Hint: If not, let* $\mathbf{u}$ *be the residual when projecting* $\nabla E(\mathbf{w})$ *to the span of the rows of* A. *Show that for some very small* η, $\mathbf{w} - \eta \cdot \mathbf{u}$ *is a feasible solution that improves* E.)