Optimal Incentive Design for Decentralized Dynamic Matching Markets

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Abstract

In decentralized dynamic matching markets, agents can benefit from sharing resources, but they will collaborate only if properly incentivized. Motivated by applications such as multi-hospital kidney exchanges and job-hunting markets, this paper proposes novel monetary and non-monetary mechanisms to incentivize collaboration in such decentralized dynamic markets. We study a model in which multiple self-interested agents manage local multi-way dynamic matching problems. Jobs of different types arrive stochastically at each agent, expire after a limited time, and yield rewards when matched. An agent's job backlog and actions are her private information, and each agent aims to maximize her long-run average matching reward.

We design simple mechanisms that incentivize agents to submit all jobs upon arrival, thereby enabling centralized matching. Our first mechanism, the Marginal-Value (`MV`) mechanism, reimburses agents based on the marginal value of their submitted jobs. This can be implemented in a non-monetary way by randomly selecting an agent (with a specified probability) to perform a match and collect the associated matching reward. We show that under the `MV` mechanism, full job submission constitutes an approximate Nash equilibrium—that is, the gain from unilateral deviation vanishes as the number of agents grows. To further eliminate incentives for deviation, we propose a refined mechanism, the Marginal-Value-plus-Credit (`MVC`) mechanism, and show that when the number of agents exceeds a constant threshold, full job submission constitutes a stronger oblivious equilibrium. Numerical experiments based on kidney exchange data demonstrate that the gains from deviation under our mechanisms are small even in moderately sized markets.

Subject classifications: Decentralized matching, incentive design, market design, dynamic matching, dynamic games.

1 Introduction

Matching markets play a crucial role in connecting diverse items to form mutually beneficial relationships, with applications across both for-profit and non-profit platforms. A well-studied setting is the *centralized matching*, where a central planner has full control over all items, and the goal is to design algorithms that maximize total match value as different types of items enter and exit the marketplace. In this paper, we shift focus to the *decentralized setting*, which introduces challenges beyond algorithm design. Specifically, we explore how to incentivize self-interested entities, each managing their own streams of items, to participate in the marketplace in a way that maximizes overall efficiency. This requires handling strategic behavior and preventing free-riding. We begin by presenting two motivating examples.

Example 1: Multi-Hospital Kidney Exchange. Kidney exchange enables two or more patients with willing but incompatible donors to swap donors, so that each patient receives a compatible organ (Ashlagi and Roth 2021). Each kidney transplant saves the high cost of lifelong dialysis and results in a welfare gain exceeding one million US dollars (Held et al. 2016). In 2019, more than 1,500 kidney exchange transplants were performed in the U.S., and the number is rapidly increasing (Agarwal et al. 2021). However, donor-patient pairs are typically registered at individual hospitals or transplant centers, leading to a decentralized matching process in which hospitals may prioritize their own interests. Specifically, hospitals often withhold easy-to-match pairs for internal matches while submitting hard-to-match pairs to the national exchange, creating a *free-riding* problem (Ashlagi and Roth 2014). This behavior persists despite the potential for a greater total number of matches under full participation in a nationwide exchange, highlighting the need for mechanisms that incentivize hospitals to share all pairs and maximize overall system efficiency.

Example 2: Collaboration among Matchmakers. Matchmakers, such as real estate brokers and headhunting agents, facilitate the matching of supply and demand in two-sided markets. These markets often exhibit strong network effects: As the user base grows, the likelihood of achieving fast and high-quality matches increases, thereby enhancing the value of the intermediary to users (Farrell and Klemperer 2007). This suggests that even competing intermediaries may benefit from granting each other access to their respective pools of customers and resources. For example, in job-hunting markets, both vacancies and candidates typically remain available for only a limited time, due to evolving outside options or changing market conditions. Headhunters can share information about job openings and candidates to facilitate matching, but strategic incentives to withhold easy-toplace candidates may persist and hinder full cooperation. The critical challenge, therefore, lies in designing reward mechanisms that encourage intermediaries to collaborate fully in equilibrium.

These applications exhibit several common features. First, the market is fragmented, with resources siloed across different agents, but greater aggregate value can be achieved by pooling these resources to create a thicker market. Second, decisions are dynamic—items such as donor-patient pairs, job candidates, or vacancies arrive stochastically over time and expire if left unmatched for too long. Finally, participants, such as hospitals or matchmakers, are strategic and possess private information about their resources, and will participate (e.g., by sharing items) only when it aligns with their own interests. In this paper, we model a decentralized dynamic matching market that incorporates these key features. In our model, multiple strategic agents control local multi-way matching problems,[1] where items (or jobs) of different types arrive over time and remain available for matching for a limited time. Information about job arrivals is private to each agent. Different types of matches require different combinations of jobs and yield different rewards, although the reward structure is the same for all agents. Each agent must decide between making local matches or submitting jobs to a shared pool, with the reward being determined by a designed mechanism. The goal of each agent is to maximize her long-run average payoff from a combination of local matching and submissions. Our goal is to design mechanisms that maximize social welfare, defined as the total matching reward accumulated across all agents.

As nicely summarized by Ashlagi and Roth (2021), the decentralized matching problem consists of two key components: submission and matching. Extensive literature examines each of these aspects in isolation. Studies on the submission side, primarily from the market design literature, focus on incentivizing agents (such as hospitals) to submit all items (see, e.g., Sönmez and Ünver 2013, Ashlagi and Roth 2014, and Agarwal et al. 2019). However, these works often rely on static models, which fail to capture real-world market frictions related to limited item lifespans (e.g., donor-patient pairs in kidney exchange) and market dynamics. On the other hand, works on (centralized) dynamic matching (e.g., Aouad and Sarıtaç 2022 and Aveklouris et al. 2025) develop algorithms with theoretical performance guarantees but generally overlook the strategic behavior of market participants. Our work advances the literature by jointly considering both submission and matching in a decentralized dynamic setting. Tackling this market design problem is challenging yet important. The primary challenge lies in analyzing a dynamic game with a large strategy space and incomplete information. This problem is important because, as highlighted by Ashlagi and Roth (2021), mechanisms that offer strong incentives in static models may prove inefficient in practice. Hence, there is

[1] That is, a match can connect two or more items.

a need for a deeper understanding of dynamic mechanisms that can achieve high system efficiency.

1.1 Overview of Results and Contributions

Our main contributions include proposing both monetary and non-monetary mechanisms for the decentralized dynamic matching problem and rigorously analyzing their performances in equilibrium. These mechanisms are simple and intuitive. Table 1 summarizes and compares the proposed mechanisms. Below, we elaborate on our main results.

A key observation underlying our mechanisms is as follows (Section 4.1): If each agent is reimbursed for a submitted job based on its marginal value in the aggregate (centralized) matching problem, then it is a dominant strategy for agents to submit all jobs upon arrival, regardless of the strategies adopted by others and even when the agents have heterogeneous job arrival rates. In this way, individual incentives align perfectly with social welfare. Notably, computing marginal values requires solving only the (static) centralized matching problem, which depends solely on aggregate job arrival rates and matching rewards, thereby requiring minimal information about individual agents. Finally, the reimbursement is approximately budget-balanced in large markets (Remark 4.1).

We then demonstrate that the above form of reimbursement can be approximately implemented without monetary transfers through a randomized matching allocation, which we refer to as the Marginal-Value (`MV`) mechanism (Section 4.2). Under this mechanism, when a match is about to be performed, it is assigned to a participant job with a probability proportional to its marginal value. The agent who submitted the selected job then executes the match (e.g., by performing the transplant in the kidney exchange context) and receives the associated matching reward. This randomized allocation guarantees that the expected return of any submitted job equals its marginal value, conditional on being matched, regardless of the specific match in which it participates. To carry out the centralized matching of submitted jobs, we apply a matching algorithm to the shared pool that is asymptotically optimal as job arrival rates grow large. As shown in Section 3, a simple periodic matching suffices.

We analyze the performance of the `MV` mechanism in a large market regime with many agents (Section 4.3). Intuitively, when all agents submit their jobs, the job arrival rates at the shared pool grow to infinity in the large market regime. Consequently, the shared pool becomes congested, and the fluid relaxation of the centralized matching problem approximates the dynamics of the shared pool well. Since the shared pool operates under an asymptotically optimal matching policy, the probability that a job with positive marginal value (indicating it is over-demanded) is matched before departing approaches one. As a result, if an agent submits a job, it yields an expected payoff close to its marginal value, which incentivizes an agent to submit all her jobs. We prove that under the `MV` mechanism, all agents fully submitting their jobs is an approximate Nash equilibrium, meaning that the benefit of unilaterally deviating to other strategies vanishes to zero as the number of agents grows.

Although the `MV` mechanism provides strong incentives for agents to submit over-demanded jobs (i.e., those with positive marginal values),[2] agents may still have incentives to withhold under-demanded jobs (i.e., those with zero marginal values). Intuitively, submitting an under-demanded job yields no benefit under the `MV` mechanism, whereas holding it may enable an internal match with a future over-demanded job. To address this issue, we introduce a refinement of the `MV` mechanism that restores incentives for agents to fully submit all jobs, including those with zero marginal value. We show that under this refinement, full job submission constitutes an equilibrium that is stronger than an approximate Nash equilibrium: The incentive to deviate from full submission drops to zero once the number of participating agents exceeds a certain finite threshold, under mild assumptions on agents' policy spaces.

Specifically, we consider an oblivious equilibrium (also known as mean-field equilibrium; see, e.g., Weintraub et al. 2008) framework (Section 5.1). Unlike approximate Nash equilibrium, which compromises agents' optimality by requiring only approximate optimality, oblivious equilibrium preserves agents' optimality and instead relaxes the informational requirements of agents: Agents are assumed to make decisions based only on its own state and knowledge of the long-run system state,[3] but ignores current information about other agents. A strategy profile leading to an oblivious equilibrium often also leads to an approximate Nash equilibrium.[4] In our mechanism, since agents interact with each other through the shared pool, whose dynamics stabilize around a steady state as the number of agents grows, each agent's decision problem effectively decouples from others when the number of agents is sufficiently large.

We illustrate with a simple example (Section 5.2) that, due to the lack of incentives for submitting under-demanded jobs, full job submission does not constitute an oblivious equilibrium under the `MV` mechanism. Furthermore, simply tweaking the allocation probabilities in the `MV` mechanism cannot resolve this incentive issue. To address this, we propose the Marginal-Value-plus-Credit (`MVC`) mechanism as a simple yet careful refinement of the `MV` mechanism. In the `MVC` mechanism,

[2]Specifically, under the `MV` mechanism, agents never withhold over-demanded jobs in anticipation of future internal matches. In other words, an over-demanded job is either submitted or internally matched with a previously withheld under-demanded job upon arrival (Proposition 5.1).

[3]In our case, the steady-state probabilities of each job being matched before departing at the share pool.

[4]See, for example, Iyer et al. (2014), Balseiro et al. (2015), Arnosti et al. (2021), and Chen et al. (2023).

agents earn credits for submitting under-demanded jobs. These credits expire at the same rate as their corresponding jobs and can be redeemed for additional matches in the future. Further details on the `MVC` mechanism are provided in Section 5.3. We demonstrate that the `MVC` mechanism provides sufficient incentives for full submission of under-demanded jobs. Furthermore, under the `MVC` mechanism, full job submission by all agents constitutes an oblivious equilibrium when the number of agents exceeds a finite threshold, and it forms an approximate Nash equilibrium in the original problem.

Finally, in Section 6, we numerically evaluate the performance of our mechanisms on both a simple synthetic example (Section 6.1) and a more realistic example using kidney exchange data (Section 6.2). We demonstrate that full job submission is an approximate equilibrium within our mechanisms even in moderately sized markets.

Mechanism	**Equilibrium Notion**	**Budget Balance**	**Monetary Transfers**
Marginal-value-based reimbursement (Section 4.1)	Dominant strategy (Proposition 4.2)	Approximate	Yes
`MV` mechanism (Section 4.2)	Approximate Nash (Theorem 4.5)	Exact	No
`MVC` mechanism (Section 5.3)	Oblivious equilibrium (Theorem 5.4) and approximate Nash (Proposition 5.5)	Exact	No

Table 1: Comparison of the proposed mechanisms.

The remainder of the paper is organized as follows. Section 1.2 reviews the related literature. Section 2 formulates the problem. Section 3 investigates the centralized setting as a benchmark. Section 4 examines the decentralized setting and introduces our `MV` mechanism. Specifically, Section 4.1 presents the marginal-value-based monetary mechanism as a warm-up, Section 4.2 introduces our non-monetary `MV` mechanism, and Section 4.3 shows that full job submission constitutes an approximate Nash equilibrium under the `MV` mechanism. Section 5 introduces the oblivious equilibrium concept and presents our `MVC` mechanism, a refinement of the `MV` mechanism. Section 5.3.2 shows that, under the `MVC` mechanism, full job submission constitutes a stronger oblivious equilibrium when the number of agents is large, as well as an approximate equilibrium in the original problem. Section 6 numerically evaluates the performance of our mechanisms. Finally, Section 7 concludes.

1.2 Related Literature

Our paper relates to several strands of fast-growing literature spanning operations research, economics, and computer science.

Incentive Issues in Multi-Hospital Kidney Exchange Since the early days of kidney exchange, researchers have observed that hospitals participating in multi-hospital kidney exchanges may benefit from strategically reserving easy-to-match donor-patient pairs for internal matches (e.g., Sönmez and Ünver 2013 and Ashlagi and Roth 2014). Subsequent works have proposed various mechanisms to encourage hospitals to submit all their pairs and have provided theoretical guarantees on the effectiveness of these mechanisms. This stream of literature predominantly focus on static models, where donor-patient pairs do not arrive or depart over time (e.g., Ashlagi and Roth 2014, Toulis and Parkes 2015, and Ashlagi et al. 2015). We contribute to this literature by studying a general dynamic game-theoretical model in which agents act strategically under private information, and we design mechanisms that incentivize full submission in a stochastic, dynamic setting.

It is worth noting that our mechanism shares similarities with the one introduced in Agarwal et al. (2019). Agarwal et al. (2019) study submission incentives in kidney exchange through the lens of neoclassical producer theory, modeling a platform that procures submissions from hospitals and produces transplants to reward submissions. Their model investigates a static (or steady-state) setting that abstracts away from stochastic job arrivals and departures and the timing of submissions; thus, it does not specify an extensive-form game. In contrast, we study a stochastic, dynamic setting with limited information and design mechanisms that incentivize full submission. The main theorem of Agarwal et al. (2019) shows that hospitals should be rewarded based on the marginal contribution of their submissions (plus a small adjustment). We demonstrate that marginal-value rewards can leave incentives to withhold under-demanded jobs in a dynamic setting, and we show that the `MVC` mechanism's expiring credits patch precisely this gap. Finally, Agarwal et al. (2019) propose a point system in which all points are homogeneous. By contrast, we employ credits (or points) only as a refinement of our `MV` mechanism, which rewards submissions based on randomized match allocation. Moreover, our credits have multiple types, with each credit corresponding to a virtual copy of the associated under-demanded submission.

Centralized Dynamic Matching While we focus on incentivizing agents in a decentralized dynamic matching market to submit all items, a related question is how to design algorithms to match these submitted items. Our mechanism is agnostic to the specific matching algorithm used in the shared pool (see Remark 3.2), although the algorithmic choice affects our performance guarantees. Many studies examine dynamic matching problems in which items exit the system if they remain unmatched for a certain duration, closely related to our setting (e.g., Aouad and Sarıtaç 2022, Aveklouris et al. 2025, Akbarpour et al. 2020, and Ashlagi et al. 2023). More recently, a growing literature has investigated any-time low-regret dynamic matching algorithms in finite-horizon settings in which items never expire (e.g., Kerimov et al. 2024, Kerimov et al. 2025, Gupta 2024, and Wei et al. 2023). Although the contexts differ, these studies highlight the connection between algorithm performance (i.e., regret) and the structural properties of the fluid approximation of the matching problem, such as dual degeneracy, which we also observe in our setting.

Cooperation Mechanism in Other Setups Several studies have examined mechanisms to incentivize cooperation among firms and platforms in various contexts beyond dynamic matching, including airline alliances (Netessine and Shumsky 2005, Wright et al. 2010, and Hu et al. 2013), cooperation among ride-sharing companies (Cohen and Zhang 2022), and cooperation among manufacturing firms (Roels and Tang 2017).

In addition, some papers use a cooperative game model to allocate total costs or profits among agents to sustain cooperation (e.g., Anily and Haviv 2010, Karsten et al. 2015, and Liu and Yu 2022). However, applying this approach to our setting would require agents to have complete information about one another to ensure all jobs are submitted. Given our focus on a private information setting, we instead adopt a non-cooperative game approach.

Chen et al. (2023) also employ a non-cooperative game approach to incentivize resource pooling in a distinct decentralized queueing setting. Specifically, Chen et al. (2023) study the mechanism design problem of incentivizing cooperation among multiple servers, each managing an $M/M/1$ queue and minimizing its own job holding and processing costs. In contrast, we study the problem of incentivizing cooperation among matchmakers, each of whom manages jobs of multiple types that arrive with limited lifespans and seeks to maximize the long-run matching rewards. Thus, the problem setups are fundamentally different. Second, although both papers design mechanisms that incentivize full cooperation and achieve the first-best performance in the large market regime, the mechanisms differ significantly. Specifically, Chen et al. (2023) propose a token-based mechanism, in which a server can spend tokens to request help from others and earn tokens (with a pre-determined probability ϕ) by offering help. The key design part is calibrating the difficulty of earning tokens (characterized by ϕ) to induce complete resource pooling. In contrast, our mechanisms are based on marginal values, with credits (or tokens) used only as a refinement. Finally, although both of us employ an oblivious equilibrium analytical framework, the theoretical analyses differ substantially. In Chen et al. (2023), since jobs and tokens are homogeneous, each server faces a one-dimensional decision problem under a fluid mean-field approximation.[5] The steady state of

[5]Specifically, a server's optimal policy depends only on the number of jobs it holds and follows a cutoff structure.

the oblivious equilibrium is also one-dimensional, corresponding to the expected waiting time in the shared pool. In contrast, in our work, since jobs and credits involve multiple types, each agent faces a more complex multi-dimensional decision problem. The steady state of the oblivious equilibrium is multi-dimensional, corresponding to the matching probabilities for jobs of different types.

1.3 Notation and Terminology

We let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{N}_+$ the set of strictly positive integers. For any two integers $a, b \in \mathbb{N}$ with $a \leq b$, we let $[a:b] = \{a, a+1, \ldots, b-1, b\}$ denote a sequence of integers starting from a and ending with b, and we denote $[n] = [1:n]$ for any $n \in \mathbb{N}_+$. For any real number $x \in \mathbb{R}$, we let $(x)^+ \triangleq \max\{x, 0\}$ denote the maximum of x and 0. Given a vector $\mathbf{x} = (x_i)_{i\in[n]} \in \mathbb{R}^n$, we let $\mathbf{x} \geq \mathbf{0}$ denote $x_i \geq 0$ for any entry $i \in [n]$.

2 Model Formulation

We consider a continuous-time model with N strategic agents. Each agent $i \in [N]$ manages a local dynamic multi-way matching problem.[6] The market designer seeks to develop a mechanism that incentivizes agents to submit jobs to the shared pool—a virtual repository of submitted but unmatched jobs—and manages matchings in the shared pool.

Specifically, there are J job types. Jobs of type $j \in [J]$ arrive at agent i following an independent Poisson process with a rate of $\lambda_{ij} \geq 0$. Let $\boldsymbol{\lambda}_i = (\lambda_{ij})_{j\in[J]} \in \mathbb{R}_+^J$ denote the vector of arrival rates for agent i. Each job expires if not matched within a certain waiting time, with type-j jobs' waiting time following an exponential distribution with rate $\theta_j > 0$ for any $j \in [J]$.

There are K matching types. A match of type $m \in [K]$ requires $M_{jm} \in \mathbb{N}$ units of type-j jobs for each $j \in [J]$ and generates a reward of $r_m > 0$ for any agent.[7] Each match involves at least two jobs; that is, $\sum_{j\in[J]} M_{jm} \geq 2$ for all $m \in [K]$.[8] Matched jobs leave the system immediately. We let $\mathbf{r} = (r_m)_{m\in[K]} \in \mathbb{R}_+^K$ denote the vector of matching rewards, $r_{\max} \triangleq \max_{m\in[K]} r_m < \infty$ the maximum matching reward, and $\mathbf{M} = (M_{jm})_{j\in[J],m\in[K]} \in \mathbb{N}^{J\times K}$ the matching matrix.

We will analyze the system in the large market regime, where the number of agents N grows large, and we impose the following regularity assumption.

Assumption 2.1. The total job arrival rates satisfy $\sum_{i\in[N]} \boldsymbol{\lambda}_i = N\boldsymbol{\lambda} \in \mathbb{R}_+^J$ for some vector $\boldsymbol{\lambda} =$

[6]For example, an agent may represent a local hospital in the multi-hospital kidney exchange context.

[7]We discuss a more general setting where agents have heterogeneous matching rewards in Remark 4.2.

[8]The assumption that each match requires at least two jobs can be relaxed without affecting our results in Section 4.

$(\lambda_j)_{j\in[J]}$, where $\lambda_j > 0$ for all $j \in [J]$. Moreover, there exist constants $C_j > 0$ such that, for any agent $i \in [N]$ and job type $j \in [J]$, $\lambda_{ij}/\lambda_j \le C_j$.

Assumption 2.1 requires that agents' job arrival rates do not differ substantially across individuals. This assumption is mild and hold, for instance, in a high-multiplicity model where agents are partitioned into a fixed number L of types. All agents of the same type ℓ share the same arrival rate vector $\boldsymbol{\lambda}_\ell = (\lambda_{\ell j})_{j\in[J]} \in \mathbb{R}_+^J$ and constitute a fixed proportion α_ℓ of the total population. In this case, we have $\boldsymbol{\lambda} \triangleq \sum_{\ell\in[L]} \alpha_\ell \, \boldsymbol{\lambda}_\ell$ and $C_j \triangleq \max_{\ell\in[L]} \lambda_{\ell j}/\lambda_j$.

System Dynamics The state of the system evolves based on the actions of the agents and the rules of the shared pool. For any agent $i \in [N]$ and job type $j \in [J]$, let $A_{ij}(t)$ denote the number of type-j jobs that have arrived at agent i by time t. By assumption, $A_{ij}(t)$ follows a Poisson process with rate λ_{ij}. Once a job arrives, agent i must decide on an action: match the job internally with other jobs in her inventory, submit it to the shared pool, or hold it for a potential future action. These decisions form the agent's strategy. Let $N_{im}(t)$ denote the number of type-m internal matchings performed by agent i and $T_{ij}(t)$ the number of type-j jobs agent i submits to the shared pool, both by time t. The values of $N_{im}(t)$ and $T_{ij}(t)$ depend on the strategy of agent i. Finally, let $X_{ij}(t)$ be the number of type-j jobs possessed by agent i at time t, and $D_{ij}(t)$ the number of type-j jobs that have expired at agent i by time t. The process $D_{ij}(t)$ depends on $X_{ij}(t)$, as each job in $X_{ij}(t)$ expires according to an exponential clock with rate θ_j. The inventory of jobs held by agent i evolves according to the following balance equation:

$$X_{ij}(t) = X_{ij}(0) + A_{ij}(t) - T_{ij}(t) - \sum_{m\in[K]} M_{jm} N_{im}(t) - D_{ij}(t), \; \forall\, j \in [J].$$

Additionally, let $X_{0j}(t)$ denote the number of type-j jobs in the shared pool at time t. The arrivals to the shared pool correspond to the jobs submitted by all agents, $\sum_{i\in[N]} T_{ij}(t)$. Let $N_{0m}(t)$ be the number of type-m matchings performed in the shared pool by time t, which is determined by the designer's matching policy. Jobs in the shared pool also expire, with $D_{0j}(t)$ representing the number of type-j jobs that have expired by time t. The job dynamics in the shared pool can be described as follows:

$$X_{0j}(t) = X_{0j}(0) + \sum_{i\in[N]} T_{ij}(t) - \sum_{m\in[K]} M_{jm} N_{0m}(t) - D_{0j}(t), \; \forall\, j \in [J].$$

Mechanism A mechanism defines the "rules of the game" that the designer establishes to govern the shared pool. The designer's actions are to choose these rules. Specifically, a mechanism consists of two components that the designer controls: (i) a matching policy the designer employs to decide which matches are performed in the shared pool based on its current job inventory, and (ii) a reward rule specifies how the rewards generated from matches within the shared pool are allocated among the agents who submitted the constituent jobs, which is the designer's primary tool for creating incentives for participation.

Agents' Problem Agents operate in self-interest, and each maximizes her own expected long-run average payoff over an infinite time horizon under a given mechanism. An agent's strategy is a dynamic policy that specifies what action to take (match internally, submit, or hold) for any job in their inventory at any point in time. The objective is to maximize the long-run average payoff, defined as follows:

$$\liminf_{t\to\infty} \frac{1}{t} V_i(t)\,, \qquad \text{where } V_i(t) \triangleq \mathbb{E}\left[\sum_{m\in[K]} r_m N_{im}(t) + \sum_{j\in[J]} \sum_{\xi\in[T_{ij}(t)]} \omega_{j\xi}\right],$$

where the first term represents rewards from internal matches, and $\{\omega_{j\xi}\}$ in the second term denote the rewards from submitting type-j jobs, as determined by the designer's reward rule. Note that the second term captures that the submission rewards can be state-dependent, highlighting the complexity of both the mechanism's design space and the agents' decision space in the dynamic setting.

We assume that each agent holds private information about her job arrival process and actions. As a result, the number of jobs of different types she currently holds and the internal matching she performs are unobservable to other agents and the designer.

Designer's Problem On the other hand, the designer's long-run average payoff under a given mechanism is

$$\liminf_{t\to\infty} \frac{1}{t} V_0(t)\,, \qquad \text{where } V_0(t) \triangleq \mathbb{E}\left[\sum_{m\in[K]} r_m N_{0m}(t) - \sum_{i\in[N]} \sum_{j\in[J]} \sum_{\xi\in[T_{ij}(t)]} \omega_{j\xi}\right].$$

Here, the first term represents the cumulative rewards from matchings performed in the shared pool, while the second term accounts for the designer's costs in rewarding agents for their job submissions.

The designer's objective is to design a mechanism that incentivizes agent cooperation while maximizing the social welfare—that is, the total long-run average payoff from all matches, both internal and in the shared pool:

$$\liminf_{t\to\infty} \frac{1}{t}\left(\sum_{i\in[N]} V_i(t) + V_0(t)\right) = \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\sum_{m\in[K]} r_m \left(\sum_{i\in[N]} N_{im}(t) + N_{0m}(t)\right)\right].$$

We consider mechanisms that satisfy a budget-balance constraint, requiring that the designer's cumulative payoff remain non-negative at all times; that is, $V_0(t) \geq 0, \forall\, t > 0$, so that the mechanism never operates at a deficit. This constraint makes the mechanism more practical, particularly for adoption by non-profit platforms.

The designer has limited information about individual agents and knows only the aggregate job arrival rate $\boldsymbol{\lambda} \in \mathbb{R}_+^J$, the matching reward vector $\mathbf{r} \in \mathbb{R}_+^K$, the matching matrix $\mathbf{M} \in \mathbb{N}^{J\times K}$, and the timing and types of submitted jobs.[9] Instead of describing the set of possible mechanisms in a general form—which would be overly general and uninformative—we focus on rigorously analyzing the games induced by our proposed mechanisms to demonstrate their performance and incentive properties.

3 The Centralized Matching

Before analyzing the decentralized game, we first study an idealized centralized setting to establish a performance benchmark. Specifically, we assume all agents follow a fixed policy of submitting every job to the shared pool upon arrival. This removes the agents' strategic submission decisions and isolates the matching component of the mechanism design: how to match jobs in the shared pool to maximize the expected long-run average payoff of the entire system. Since there is only one decision maker (the designer), we adopt the notation defined for the system dynamics and drop the dependence on the index i throughout this section.

The designer's decision problem in this centralized stochastic system is to determine which matches to perform at any given time, based on the current inventory of jobs. Let V^* denote the optimal long-run average payoff achievable under centralized matching. Solving this optimal payoff directly is intractable due to the curse of dimensionality. To approximate the centralized control problem, we employ a fluid relaxation that simplifies job arrivals and matches as deterministic continuous flows rather than stochastic discrete events. The decision variable $\mathbf{x} = (x_m)_{m\in[K]}$ in the

[9] Knowing only the relative values of the aggregate job arrival rate $(\lambda_j)_{j\in[J]}$ rather than their absolute values will be sufficient for our mechanisms.

fluid model represents the long-run average rate at which each match-type m is performed when the job arrival rates are normalized to be $\boldsymbol{\lambda}$. This static vector $\mathbf{x}$ is the deterministic counterpart to the complex, state-dependent matching decisions in the original stochastic system. This relaxation also ignores job expiration, resulting in the following linear program (LP):

$$
\begin{aligned}
V^{\mathrm{F}}(\boldsymbol{\lambda}) = \max_{\mathbf{x}\in\mathbb{R}_+^K} \quad & \mathbf{r}^{\mathrm{T}}\mathbf{x} \\
\text{s.t.} \quad & \mathbf{M}\mathbf{x} \le \boldsymbol{\lambda}.
\end{aligned}
\tag{1}
$$

Here, the objective maximizes the total rewards from matching. The constraint ensures that the rate at which jobs are used for matching does not exceed the normalized aggregate job arrival rates $\boldsymbol{\lambda}$. The optimal value of (1), denoted as $V^{\mathrm{F}}(\boldsymbol{\lambda})$, represents the maximum payoff achievable under the fluid relaxation for a single player with arrival rate $\boldsymbol{\lambda}$. When accounting for N agents with aggregate job arrival rates $N\boldsymbol{\lambda}$, the following result holds.

Lemma 3.1. *We have $V^* \le V^{\mathrm{F}}(N\boldsymbol{\lambda}) = N \cdot V^{\mathrm{F}}(\boldsymbol{\lambda})$.*

We prove Lemma 3.1 in Appendix A.1. Lemma 3.1 shows that the centralized system's long-run average payoff V^* is upper bounded by $NV^{\mathrm{F}}(\boldsymbol{\lambda})$. Let $V^{\mathrm{F}} \triangleq V^{\mathrm{F}}(\boldsymbol{\lambda})$ and $\mathbf{x}^* = (x_m^*)_{m\in[K]} \in \mathbb{R}_+^K$ denote an optimal solution to (1). The dual problem of (1) is given by the following LP:

$$
\begin{aligned}
\min_{\mathbf{p}\in\mathbb{R}_+^J} \quad & \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{p} \\
\text{s.t.} \quad & \mathbf{M}^{\mathrm{T}}\mathbf{p} \ge \mathbf{r}.
\end{aligned}
\tag{2}
$$

Let $\mathbf{p}^* = (p_j^*)_{j\in[J]} \in \mathbb{R}_+^J$ denote an optimal solution to (2). We can interpret the value of p_j^* as the marginal value of a type-j job to the centralized problem.

3.1 Asymptotically Optimal Matching Policy

In this section, we examine a simple periodic matching policy, denoted by π^{P}, which is asymptotically optimal in the large market regime (i.e., as $N \to \infty$). We establish convergence rates and provide pre-limit performance guarantees, which will be important in analyzing the performance of the proposed mechanisms in subsequent sections.

Let $\mathbf{p}^* = (p_j^*)_{j\in[J]}$ be an optimal solution to (2). Define

$$
\mathcal{M}_0 \triangleq \left\{ m \in [K] : \sum_{j\in[J]} p_j^* M_{jm} > r_m \right\}
$$

as the set of suboptimal matchings in the fluid relaxation (1). To interpret this definition, note that for any optimal solution $\mathbf{x}^* = (x_m^*)_{m\in[K]}$ to (1), complementary slackness implies $x_m^* = 0$ for all $m \in \mathcal{M}_0$. Therefore, no match in $\mathcal{M}_0$ is used by *any* optimal solution to (1). Let

$$\mathcal{M}_+ \triangleq [K] \setminus \mathcal{M}_0 = \left\{ m \in [K] : \sum_{j\in[J]} p_j^* M_{jm} = r_m \right\}$$

represent the set of matchings that may be used in the fluid relaxation (1). Throughout the remaining part of the paper, we impose the constraint $x_m = 0$ for all $m \in \mathcal{M}_0$ into (1) without loss of optimality. We now formally define the periodic matching policy π^{P}.

Definition 3.1 (Periodic Matching Policy). At each time point $t = k\Delta$, where $\Delta \in \mathbb{R}_+$ is a fixed time interval and $k \in \mathbb{N}_+$, given the system state $\mathbf{X}(t) \in \mathbb{N}^J$ (representing the number of jobs of each type at time t), the periodic matching policy π^{P} first solves the fluid relaxation $V^{\mathrm{F}}(\mathbf{X}(t))$ (imposing the constraint $x_m = 0$, $\forall\, m \in \mathcal{M}_0$) to obtain an optimal solution $\mathbf{x}^* \in \mathbb{R}_+^K$, and then performs $\lfloor x_m^* \rfloor$ type-m matches for every m.

In words, the designer solves an optimal matching in every Δ units of time and implements the matching. In the following, we show that this policy is asymptotically optimal in the large market regime with an appropriately selected value of Δ and we analyze the dynamics of the system under this policy.

We first demonstrate in Lemma 3.2 that policy π^{P} is asymptotically optimal in the large market regime under the generic case.

Lemma 3.2 (Asymptotic Optimality). *Let the interval length be $\Delta = N^{-\frac{1}{3}}$. The performance of the periodic matching policy, denoted by V^{P}, satisfies*

$$\frac{V^* - V^{\mathrm{P}}}{V^*} \le \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{NV^{\mathrm{F}}} \le C_1 \cdot N^{-\frac{1}{3}}$$

for some constant $C_1 > 0$.

We prove Lemma 3.2 in Appendix A.2. Lemma 3.2 implies that the fluid relaxation (1) is asymptotically tight and policy π^{P} is asymptotically optimal. In the proof, we analyze the performance gap between policy π^{P} and the fluid relaxation upper bound by decomposing the loss over a time interval of length Δ into three components: the expiration loss, the concavity loss, and the rounding loss. First, expiration loss occurs because jobs that arrive but remain unmatched expire at a certain rate, reducing the system's overall payoff. To mitigate this, the planner would prefer a small Δ to conduct matches more frequently. Second, even in the absence of job expiration, there is concavity loss due to fluctuations in job arrivals. Specifically, the actual system state may deviate from the fluid state, resulting in a loss relative to the optimal fluid solution, because the fluid relaxation $V^{\mathrm{F}}(\boldsymbol{\lambda})$ is concave, as implied by the dual formulation (2) and strong duality. To mitigate this, the planner would prefer a large Δ to accumulate sufficient jobs before matching to reduce the impact of arrival variability. Finally, rounding loss occurs when the planner implements a rounded solution per period. To minimize this loss, the planner would again prefer a large Δ to reduce the frequency of rounding. In the proof, we bound each of these three losses and observe that the concavity loss dominates the rounding loss. To balance the tradeoff between the concavity loss and expiration loss, we determine the optimal value of Δ as $\Theta\left(N^{-\frac{1}{3}}\right)$.

We next demonstrate that the performance guarantee of policy π^{P} can be strengthened if the fluid relaxation (1) satisfies a regularity condition termed non-degeneracy. This condition requires that the optimal dual variable $\mathbf{p}^*$ remains stable under small perturbations of the aggregate job arrival rates $\boldsymbol{\lambda}$. We first define the non-degeneracy condition in Definition 3.2.

Definition 3.2 (Non-Degeneracy Condition)**.** Problem (1) is non-degenerate if there exists a positive constant $\delta > 0$ and a dual variable $\mathbf{p}^* \in \mathbb{R}_+^J$ such that $\mathbf{p}^*$ is the unique optimal dual solution of $V^{\mathrm{F}}(\boldsymbol{\lambda}')$ for any $\boldsymbol{\lambda}'$ such that $||\boldsymbol{\lambda}' - \boldsymbol{\lambda}||_\infty \leq \delta$.

To better understand non-degeneracy, Remark 3.1 presents necessary and sufficient conditions for this property. These conditions follow from the fact that $V^{\mathrm{F}}(\boldsymbol{\lambda})$ is piecewise linear and concave in $\boldsymbol{\lambda}$, and that $\mathbf{p}^* \in \mathbb{R}_+^J$ is an optimal dual variable of $V^{\mathrm{F}}(\boldsymbol{\lambda})$ if and only if it is a subgradient of $V^{\mathrm{F}}(\cdot)$ at the point $\boldsymbol{\lambda}$. We provide more details in Appendix A.3.

Remark 3.1 (Necessary and Sufficient Conditions for Definition 3.2)**.** We have the following:

1. (*Necessary and Sufficient Condition*) Problem (1) is non-degenerate if and only if it has a unique optimal dual solution $\mathbf{p}^*$.
2. (*Sufficient Condition*) Problem (1) is non-degenerate if it has a non-degenerate optimal basic feasible solution.[10]

Under the non-degeneracy condition, the performance of policy π^{P} improves significantly, as demonstrated in Lemma 3.3.

Lemma 3.3. *Suppose that (1) is non-degenerate (i.e., Definition 3.2 holds), and denote by V^{P} the performance of the periodic matching policy π^{P}. The following hold:*

[10]See Section 2.4 of Bertsimas and Tsitsiklis (1997) for the definition of a non-degenerate basic solution.

1. *Let the interval length be* $\Delta = N^{-\frac{1}{2}}$*; then*

$$\frac{V^* - V^{\mathrm{P}}}{V^*} \leq \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{NV^{\mathrm{F}}} \leq C_2 \cdot N^{-\frac{1}{2}}$$

for some constant $C_2 > 0$.

2. *If, in addition, the matching matrix* $\mathbf{M}$ *is totally unimodular,*[11] *and the interval length is chosen as* $\Delta = c \cdot \frac{\ln N}{N}$*, where* $c > 0$ *is a constant depending only on the values of aggregate job arrival rates* $\boldsymbol{\lambda}$ *and non-degeneracy parameter* δ*, then we have*

$$\frac{V^* - V^{\mathrm{P}}}{V^*} \leq \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{NV^{\mathrm{F}}} \leq C_3 \cdot \frac{\ln N}{N}$$

for some constant $C_3 > 0$.

We prove Lemma 3.3 in Appendix A.4. Intuitively, when the non-degeneracy condition holds, $V^{\mathrm{F}}(\boldsymbol{\lambda})$ is linear in $\boldsymbol{\lambda}$ in the vicinity of $\boldsymbol{\lambda}$. As a result, the concavity loss becomes negligible, leaving only the rounding and expiration losses to be balanced. Furthermore, if the matching matrix $\mathbf{M}$ is totally unimodular, the rounding loss is eliminated, as the optimal solution to the matching problem solved at any time point $k\Delta$ is always integral. Consequently, we can simply select a small interval length Δ to minimize the expiration loss.

3.2 Convergence of Dynamics

In this section, we analyze the system dynamics under policy π^{P}. We first show in Lemma 3.4 that the long-run average matching rates under policy π^{P} converge to an optimal solution of (1) in the large market regime.

Lemma 3.4 (Convergence of Dynamics)**.** *Let* $x_m^P \triangleq \lim_{t\to\infty} \mathbb{E}[N_m(t)]/t$ *denote the long-run average matching rate of match* m *under policy* π^{P}*, and let* $\mathbf{x}^P = (x_m^P)_{m\in[K]} \in \mathbb{R}_+^K$*. There exists a constant* $C_4 > 0$ *such that, for any number of agents* N*, there exists an optimal solution* $\mathbf{x}^*$ *of (1) satisfying:*

$$\left\|\frac{\mathbf{x}^P}{N} - \mathbf{x}^*\right\|_\infty \leq C_4 \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}\,.$$

Combined with the performance guarantees established in Lemmas 3.2 and 3.3, this implies the existence of a constant $C_5 > 0$ *such that, for any number of agents* N*, there exists an optimal*

[11] The matching matrix $\mathbf{M}$ is totally unimodular if, for instance, each match requires exactly two jobs of different types.

solution $\mathbf{x}^*$ *of (1) that satisfies the following:*

1. *If the interval length is* $\Delta = N^{-\frac{1}{3}}$*, then* $\left|\left|\frac{\mathbf{x}^P}{N} - \mathbf{x}^*\right|\right|_\infty \leq C_5 \cdot N^{-\frac{1}{3}}$*;*
2. *If (1) is non-degenerate and* $\Delta = N^{-\frac{1}{2}}$*, then* $\left|\left|\frac{\mathbf{x}^P}{N} - \mathbf{x}^*\right|\right|_\infty \leq C_5 \cdot N^{-\frac{1}{2}}$*;*
3. *If (1) is non-degenerate, the matching matrix* $\mathbf{M}$ *is totally unimodular, and* $\Delta = c \cdot \frac{\ln N}{N}$*, where* c *is the constant specified in Lemma 3.3, then* $\left|\left|\frac{\mathbf{x}^P}{N} - \mathbf{x}^*\right|\right|_\infty \leq C_5 \cdot \frac{\ln N}{N}$*.*

We prove Lemma 3.4 in Appendix A.5. The proof is based on two key observations. First, the normalized matching rates $\mathbf{x}^P/N$ are feasible to (1). Second, the optimal solution of the fluid relaxation (1) is Lipschitz continuous with respect to changes in the constraints' right-hand side, allowing us to bound the deviation of $\mathbf{x}^P/N$ from the optimal solution $\mathbf{x}^*$. Finally, we note that Lemma 3.4 is independent of the specific matching algorithm used in the shared pool, as stated in Remark 3.2.

Remark 3.2 (Agnostic to Specific Matching Algorithm)**.** Lemma 3.4 applies to any asymptotically optimal matching policy besides policy π^{P}, where the convergence rate of the dynamics aligns with the respective convergence rate of performance, following the same proof.

Lemma 3.4 implies that the long-run fraction of type-j jobs that participate in the type-m match is approximately $M_{jm}x_m^*/\lambda_j$ in the large market regime, as we formally state in Corollary 3.5 and prove in Appendix A.6.

Corollary 3.5. *Let* $A_j(t)$ *denote the number of type-*j *jobs that have arrived by time* t*, and* $A_{jm}(t)$ *the number of type-*j *jobs that have participated in type-*m *matches by time* t *under policy* π^{P}*. We have:*

$$q_{jm} \triangleq \lim_{t\to\infty} \mathbb{E}\left[\frac{A_{jm}(t)}{A_j(t)}\right] = \mathbb{E}\left[\lim_{t\to\infty} \frac{A_{jm}(t)}{A_j(t)}\right],$$

where q_{jm} *represents the long-run average fraction of type-*j *jobs that participate in type-*m *matches, and the second equality follows from the bounded convergence theorem because* $A_{jm}(t) \leq A_j(t)$*. Moreover, there exists a constant* $C_6 > 0$ *such that, for any number of agents* N*, there exists an optimal solution* $\mathbf{x}^*$ *of (1) satisfying:*

$$\left|q_{jm} - M_{jm}x_m^*/\lambda_j\right| \leq C_6 \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}$$

for any $j \in [J]$ *and* $m \in [K]$*.*

Let $\mathbf{p}^* = (p_j^*)_{j\in[J]}$ denote an optimal dual variable of (1). In the following, we define a job type j as *over-demanded* if $p_j^* > 0$, and *under-demanded* if $p_j^* = 0$. To interpret this definition, let $\mathbf{m}_j^{\mathrm{T}}$ be the j-th row of the matching matrix $\mathbf{M}$, and let $\mathbf{x}^*$ be any optimal solution of (1). By complementary slackness, we have $\mathbf{m}_j^{\mathrm{T}}\mathbf{x}^* = \lambda_j$ for every job type j with $p_j^* > 0$. Therefore, the capacity constraint in (1) is binding for all over-demanded job types. We denote the set of over-demanded job types by $\mathcal{N}_+ \triangleq \left\{j \in [J] : p_j^* > 0\right\}$ and the set of under-demanded job types by $\mathcal{N}_0 \triangleq [J] \setminus \mathcal{N}_+$. According to Corollary 3.5, an over-demanded job is matched before departure with probability one in the large market regime, as illustrated in Remark 3.3.

Remark 3.3 (Over-Demanded Jobs Matched with Probability One in the Large Market Regime)**.** In the long run, a type-j job is matched before departure with probability $\sum_{m\in[K]} q_{jm}$, which converges to $\mathbf{m}_j^{\mathrm{T}}\mathbf{x}^*/\lambda_j$ in the large market regime (i.e., as $N \to \infty$) by Corollary 3.5 and Lemmas 3.2 and 3.3. For an over-demanded type-j job, since $\mathbf{m}_j^{\mathrm{T}}\mathbf{x}^* = \lambda_j$, it is matched with probability one before expiration. In contrast, an under-demanded type-j job remains unmatched before expiration with probability $1 - \mathbf{m}_j^{\mathrm{T}}\mathbf{x}^*/\lambda_j$.

3.3 Random Enrollment Deferral

To conclude Section 3, we introduce a slight modification to the periodic matching policy π^{P} used in the shared pool, termed *random enrollment deferral*. This modification will simplify the analysis in the decentralized setting.

Recall that under policy π^{P}, the shared pool performs centralized matching every $\Delta \in \mathbb{R}_+$ units of time. In the modified policy, when a type-j job is submitted with $x \in [0, \Delta]$ time remaining until the next matching epoch, it is immediately enrolled with probability $p_j(x) \in [0, 1]$, allowing participation in the upcoming matching epoch. Otherwise, with probability $1 - p_j(x)$, the job's enrollment is deferred by Δ units, so that its first matching epoch occurs $x + \Delta$ time units after submission. The enrollment probability $p_j(x)$ is defined as:

$$p_j(x) = \frac{\exp(-\theta_j\Delta) - \exp(-\theta_j(x+\Delta))}{\exp(-\theta_j x) - \exp(-\theta_j(x+\Delta))} \approx \frac{x}{\Delta}\,,$$

and uniquely satisfies:

$$p_j(x)\cdot\big(1-\exp(-\theta_j x)\big) + (1-p_j(x))\cdot\big(1-\exp(-\theta_j(x+\Delta))\big) = 1-\exp(-\theta_j\Delta)\,.$$

This design ensures that the probability of a job expiring before participating in centralized matching remains constant at $1-\exp(-\theta_j\Delta) \approx \theta_j\Delta$, regardless of its submission time. This is particularly useful in the decentralized setting, as it removes agents' incentives to strategically time job submissions given that the schedule of matching epochs is publicly known. Hence, if an agent intends to submit a job, it is optimal to do so immediately to avoid unnecessary expiration risks.

We emphasize that all results in Section 3—including both performance guarantees and convergence of dynamics—remain valid under random enrollment deferral (see Remark A.1 in the Appendix). Intuitively, as the number of agents N increases, the deferral Δ (as given in Lemmas 3.2 and 3.3) vanishes, rendering its effect negligible. For the remainder of the paper, we incorporate random enrollment deferral into our mechanisms in Sections 4 and 5, to preclude agents from strategically timing job submissions and to simplify subsequent analysis.

4 Decentralized Setting: The Marginal-Value (MV) Mechanism

In this section, we introduce our baseline `MV` mechanism for the decentralized, limited information setting. We first present a key observation that serves as the backbone for all our mechanisms (Section 4.1). A direct consequence of this observation is a monetary mechanism that incentivizes agents to submit all their jobs upon arrival. Building on this, we present our `MV` mechanism in Section 4.2 as a non-monetary implementation achieved through randomized matching allocation. Finally, in Section 4.3, we demonstrate that under the `MV` mechanism, full job submission by all agents constitutes an approximate Nash equilibrium, meaning that the benefit of unilaterally deviating from full submission diminishes as the number of agents grows.

4.1 Warm-Up: Mechanism with Monetary Transfer

We begin by presenting a mechanism that uses monetary transfer to incentivize agents to fully submit their jobs, as a preparation for the non-monetary `MV` mechanism introduced in Section 4.2. Let $\mathbf{p}^* = (p_j^*)_{j\in[J]}$ be an optimal dual variable to (1), representing marginal values of jobs of each type. The mechanism rewards an agent with a value of p_j^* whenever she submits a job of type j to the shared pool. All the submissions are irrevocable. The mechanism then performs an asymptotically optimal matching in the shared pool (e.g., implementing the periodic matching policy π^{P} described in Section 3.1). In the following, we show that this mechanism incentivizes all agents to submit their jobs fully.

To do so, we analyze the decision problem faced by an individual agent i under this mechanism. The agent's true problem is a complex stochastic control problem that requires a dynamic policy to determine when to match jobs internally and when to submit them. To analyze this problem and establish an upper bound on the agent's achievable payoff, we formulate a fluid relaxation.

The decision variables in this relaxation represent the long-run average rates of the agent's actions. Specifically, the vector $\mathbf{x} = (x_m)_{m\in[K]}$ represents the rates at which agent i performs internal matches, and the vector $\mathbf{s} = (s_j)_{j\in[J]}$ represents the rates at which agent i submits jobs to the shared pool. The agent's objective is to choose these rates to maximize her long-run average payoff, which is the sum of rewards from internal matches and the monetary rewards from submissions. This leads to the following LP (3):

$$\begin{aligned} \max_{\mathbf{x}\in\mathbb{R}_+^K,\,\mathbf{s}\in\mathbb{R}_+^J} \quad & \mathbf{r}^{\mathrm{T}}\mathbf{x} + \mathbf{p}^{*\mathrm{T}}\mathbf{s} \\ \text{s.t.} \quad & \mathbf{M}\mathbf{x} + \mathbf{s} \le \boldsymbol{\lambda}_i. \end{aligned} \tag{3}$$

The constraint in (3) requires that, for each job type, the combined rate of using jobs for local matching and submitting jobs to the shared pool cannot exceed the job arrival rate. The dual problem of (3) is given in (4).

$$\begin{aligned} \min_{\mathbf{p}\in\mathbb{R}_+^J} \quad & \boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{p} \\ \text{s.t.} \quad & \mathbf{M}^{\mathrm{T}}\mathbf{p} \ge \mathbf{r}, \\ & \mathbf{p} \ge \mathbf{p}^*. \end{aligned} \tag{4}$$

Proposition 4.1 provides optimal solutions to the primal and dual problems.

Proposition 4.1. *$\mathbf{x} = \mathbf{0}$ and $\mathbf{s} = \boldsymbol{\lambda}_i$ is an optimal primal solution and $\mathbf{p} = \mathbf{p}^*$ is an optimal dual solution to (3), with the optimal value being $\boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{p}^*$.*

Proof. Note that $\mathbf{x} = \mathbf{0}$ and $\mathbf{s} = \boldsymbol{\lambda}_i$ is feasible to (3) and $\mathbf{p} = \mathbf{p}^*$ is feasible to (4) because $\mathbf{p} = \mathbf{p}^*$ is feasible to (2). Furthermore, these solutions achieve the same objective value. By weak duality, they must be optimal solutions to the primal and dual problems, respectively. □

Proposition 4.1 shows that submitting all arriving jobs to the shared pool is optimal to an agent's fluid relaxation problem (3). Moreover, under the mechanism, the marginal value of a type-j job to an agent's problem equals the job's marginal value in the centralized problem (1), which is p_j^*, regardless of the agent's individual job arrival rates. Finally, Proposition 4.1 also implies that it is optimal for an agent to submit all jobs to the shared pool in the original problem, as we state in Proposition 4.2.

Proposition 4.2. *For any agent $i \in [N]$, it is a dominant strategy to submit all the jobs to the shared pool upon arrival, and this yields an expected long-run average payoff of $\boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{p}^*$.*

Proof. This is because full submission achieves the fluid relaxation upper bound (3). □

We conclude this section with two remarks. First, we remark that the monetary mechanism described above is approximately budget-balanced in the large market regime, as illustrated in Remark 4.1. In later sections, we propose non-monetary mechanisms that satisfy the budget-balance constraint exactly.

Remark 4.1 (Budget Balance in the Large Market Limit). According to Proposition 4.2, each agent $i \in [N]$ fully submits her jobs to the shared pool and receives a payoff of $\boldsymbol{\lambda}_i^{\mathrm{T}} \mathbf{p}^*$ under the mechanism. Consequently, the total payment to agents is $\sum_{i \in [N]} \boldsymbol{\lambda}_i^{\mathrm{T}} \mathbf{p}^* = N \boldsymbol{\lambda}^{\mathrm{T}} \mathbf{p}^* = N V^{\mathrm{F}}(\boldsymbol{\lambda})$, where the last equality follows from the strong duality between the centralized problem (1) and (2). Meanwhile, since the shared pool implements an asymptotically optimal matching policy, it collects a matching reward of $N\left(V^{\mathrm{F}}(\boldsymbol{\lambda}) - o(1)\right)$ from the submitted jobs according to Lemmas 3.2 and 3.3. As a result, the mechanism is budget-balanced in the large market regime (that is, $N \to \infty$), and it distributes the payoff from centralized matching performed at the shared pool to agents fully and proportionally to each agent's contribution to centralized matching.

Second, we remark that the monetary mechanism preserves its properties in a more general setting where agents generate heterogeneous matching rewards.

Remark 4.2 (Heterogeneous Matching Rewards). Consider a more general setting in which agents generate heterogeneous matching rewards. The monetary mechanism with appropriate marginal values continues to make full submission a dominant strategy, thereby achieving the first-best performance, and remains approximately budget-balanced in large markets. However, no non-monetary mechanism can attain the first-best performance, unlike the case with homogeneous matching rewards. We provide further details in Appendix C.

4.2 Mechanism without Money Transfer: The `MV` Mechanism

The key design principle in Section 4.1 is to ensure that submitting a type-j job to the shared pool is rewarded with its marginal value, p_j^*, in the centralized setting. In this section, we show that this can be nearly achieved without relying on monetary transfer, by assigning a match to an agent with a specified probability and letting the agent perform the match and collect the corresponding reward. We refer to this non-monetary mechanism as the Marginal-Value (`MV`) mechanism.

Let $\mathbf{p}^* = (p_j^*)_{j \in [J]}$ be an optimal solution to the dual problem (2). Recall from Section 3.1 that we define $\mathcal{M}_+ \triangleq \left\{m \in [K] : \sum_{j \in [J]} p_j^* M_{jm} = r_m\right\}$ as the set of matches that may be used in the fluid relaxation (1) and we impose the constraint $x_m = 0$ for any $m \in \mathcal{M}_0$ whenever the periodic matching policy π^{P} solves a matching problem (Definition 3.1). Therefore, policy π^{P} only performs matches in $\mathcal{M}_+$.

Note that for any match $m \in \mathcal{M}_+$, its reward r_m equals the sum of optimal dual variables p_j^* associated with the participating jobs. Leveraging this fact, instead of reimbursing a job with its marginal value p_j^* as done in Section 4.1, the non-monetary `MV` mechanism approximately achieves the same outcome through a random matching allocation when there are many agents. We formally describe the non-monetary mechanism in Definition 4.1.

Definition 4.1 (The `MV` Mechanism)**.** The non-monetary `MV` mechanism proceeds as follows:

1. Implement an asymptotically optimal matching policy at the shared pool (e.g., the periodic matching policy π^{P} described in Definition 3.1).
2. Whenever a type-m match is being performed at the shared pool, assign the match to a participant job with a probability proportional to its marginal value (i.e., with probability p_j^*/r_m) and let the agent of that job perform the match and collect the matching reward r_m. This can be implemented by an automated algorithm such as a smart contract.

Note that the allocation probabilities in Step 2 are well-defined because $\sum_{j\in[J]} M_{jm} \cdot p_j^*/r_m = 1$ by the definition of set $\mathcal{M}_+$. Furthermore, a type-j job submitted to the shared pool yields an expected payoff of p_j^* upon being matched, regardless of the specific match type. As a result, from an agent's perspective, the specific match type does not influence the expected payoff, and an agent's decision of job submission depends solely on the probability that the job will be matched, rather than the type of match it will participate in.

4.3 Performance Analysis: Approximate Nash Equilibrium

In this section, we analyze the `MV` mechanism and demonstrate that full job submission by all agents constitutes an approximate equilibrium. Note that the `MV` mechanism induces a stochastic game among the agents. We first introduce the notion of an approximate Nash equilibrium for the induced game.

Let $\mathbf{x}_i(t)$ denote the state of agent i at time t, representing the number of her jobs across all types. A strategy for agent i, denoted by δ_i, specifies an action as a function of her current state $\mathbf{x}_i(t)$ and, possibly, the entire history of past interactions with the shared pool and her beliefs about the states of other agents at time t. The action includes whether to perform an internal match (and which one to perform) and whether to submit jobs to the shared pool (and which ones to submit).

Let $\boldsymbol{\delta} = (\delta_i, \delta_{-i})$ denote the strategy profile of all agents, where δ_{-i} represents the $N-1$ strategies of all agents other than i. Let $V_i(\delta_i, \delta_{-i})$ denote the expected time-average payoff of agent i.

Definition 4.2 (Approximate Nash Equilibrium)**.** For each agent $i \in [N]$, let Δ_i denote the set of all admissible strategies. A strategy profile $\boldsymbol{\delta}^*$ is an ϵ-approximate Nash equilibrium if, for every agent $i \in [N]$ and strategy $\delta_i \in \Delta_i$, we have $V_i(\delta_i^*, \delta_{-i}^*) \geq V_i(\delta_i, \delta_{-i}^*) - \epsilon$.

Intuitively, when all agents fully submit their jobs and the number of agents increases, the probability that an over-demanded job is matched before expiration approaches one (Remark 3.3). As a result, each submitted type-j job yields an expected payoff of p_j^* under the randomized matching allocation. This, in turn, incentivizes agents to fully submit jobs to maximize their expected payoffs according to Proposition 4.2.

To formally establish this result, we first show in Lemma 4.3 that the optimal value of (3) is an upper bound on an agent's long-run average payoff under the `MV` mechanism, regardless of the strategies of other agents and even when the agent has complete information about the system. We prove Lemma 4.3 in Appendix A.7.

Lemma 4.3. *Under the `MV` mechanism, the long-run average payoff of any agent i is at most the optimal value of* (3)*, given by $\boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{p}^*$, regardless of the strategies chosen by other agents, and even if agent i has complete information about the system.*

Next, suppose that all agents submit their jobs fully to the shared pool. Since the dynamics of the shared pool converge to the fluid relaxation (1) for the centralized setting (Lemma 3.4), the long-run average payoff of agent i is at least $\boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{p}^*$ minus a diminishing term; we state this formally in Lemma 4.4.

Lemma 4.4. *Suppose all agents submit their jobs fully to the shared pool. Then, under the `MV` mechanism, the long-run average payoff of agent i is at least $\boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{p}^* - C_7 \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}$, where $C_7 \triangleq r_{\max} \cdot C_4 \sum_{j \in [J]} C_j \sum_{m \in [K]} M_{jm} > 0$ is a constant (with C_j and C_4 positive constants specified in Assumption 2.1 and Lemma 3.4, respectively), V^{P} denotes the long-run average matching rewards collected at the shared pool, and the term $\frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}$ diminishes to zero at a rate characterized by Lemmas 3.2 and 3.3.*

We prove Lemma 4.4 in Appendix A.8. According to Lemmas 4.3 and 4.4, the benefit of unilaterally deviating from full submission is only a negligible term that is $O\big(N^{-\frac{1}{3}}\big)$ in the general case and $O\big(N^{-\frac{1}{2}}\big)$ or $O\left(\frac{\ln N}{N}\right)$ when (1) is non-degenerate, even if agent i has complete information about the system. Consequently, full job submission by all agents constitutes an approximate Nash equilibrium, as formally stated in Theorem 4.5.

Theorem 4.5. *Under the* `MV` *mechanism, full job submission by all agents constitutes an* $O\left(\frac{NV^{\mathrm{F}}-V^{\mathrm{P}}}{N}\right)$*-approximate Nash equilibrium.*

We remark that the `MV` mechanism provides sufficient incentives for submission of over-demanded jobs in the large market regime. Specifically, as we demonstrate later (see Proposition 5.1), under the `MV` mechanism, when the number of agents is large and all other agents fully submit their jobs, an agent never strategically withholds over-demanded jobs to perform internal matching with future arrivals.

However, this does not mean all over-demanded jobs are submitted: an agent may strategically retain under-demanded jobs (i.e., those with zero marginal value) in anticipation of over-demanded jobs. Thus, when an over-demanded job arrives, it can be matched internally rather than submitted to the shared pool, allowing the agent to capture a vanishingly small additional payoff. Intuitively, submitting an under-demanded job under the `MV` mechanism yields no benefit, whereas retaining it may facilitate internal matching with future over-demanded jobs, as illustrated later in Example 5.1.

In Section 5, we leverage a mean-field analysis to illustrate these points. Furthermore, we refine the `MV` mechanism to provide proper incentives for full submission of under-demanded jobs and demonstrate that full job submission forms a stronger oblivious equilibrium (than the approximate Nash equilibrium) under the refinement.

5 Refinement: The Marginal-Value-plus-Credit (MVC) Mechanism

Although Theorem 4.5 establishes that full job submission by all agents constitutes an approximate Nash equilibrium under the `MV` mechanism, an incentive misalignment persists: agents may still deviate by strategically withholding under-demanded jobs to gain a vanishingly small additional benefit. Such behavior can result in over-demanded jobs being matched internally, thus potentially suboptimally, rather than being submitted to the shared pool.

To resolve this, in this section, we refine the `MV` mechanism to ensure that full submission by all agents satisfies a stronger equilibrium notion—the oblivious equilibrium. We begin by introducing and motivating the oblivious equilibrium concept in Section 5.1. We then proceed to analyze the `MV` mechanism within the oblivious equilibrium framework in Section 5.2. A simple example (Example 5.1) illustrates that full job submission does not constitute an oblivious equilibrium under the `MV` mechanism; furthermore, simply adjusting the allocation probabilities in the `MV` mechanism cannot resolve this incentive issue. This motivates our refined `MVC` mechanism, which we introduce in Section 5.3. We show that, under the `MVC` mechanism, full job submission by all agents constitutes an oblivious equilibrium when the number of agents exceeds a finite threshold, and it forms an approximate Nash equilibrium in the original problem.

5.1 Definition of Oblivious Equilibrium

In this section, we introduce the oblivious equilibrium, also known as mean-field equilibrium, as an alternative equilibrium concept for the stochastic game induced by our mechanisms.

In our mechanism, agents interact indirectly through the shared pool. When deciding whether to submit a job, an agent needs to estimate the probability that the job will be matched before it expires if submitted to the shared pool. In the exact equilibrium (that is, the perfect Bayesian equilibrium), the complexity of an agent's decision problem arises from the need to maintain and update beliefs about the current and future states of the shared pool—specifically, the number of jobs of different types at each matching epoch—based on her past interactions with the shared pool and beliefs about the states of other agents. This requires an unrealistic level of rationality assumption and computational capacity on the part of participating agents, especially when the number of agents is large, as argued in Iyer et al. (2014), Balseiro et al. (2015), and other previous works.

However, when there are many agents, it is reasonable to assume that the dynamics of the shared pool stabilize around a steady state due to concentration effects. Moreover, with many agents, the impact of an individual agent on the shared pool becomes negligible. These factors motivate our mean-field (oblivious) approximation, as stated in Assumption 5.1.

Assumption 5.1 (Mean-Field Approximation). Each agent $i \in [N]$ assumes that the number of jobs of each type in the shared pool is in the stationary distribution at any matching epoch $t = k\Delta$, where $k \in \mathbb{N}_+$, independent of her observed history.

The mean-field approximation simplifies an agent's problem as the matching probability of a type-j job in the shared pool becomes a constant $w_j \geq 0$, independent of the observed history. The value of $\mathbf{w} = (w_j)_{j\in[J]}$ will be determined endogenously by an equilibrium. Under Assumption 5.1, each agent's decision problem becomes a Markov decision problem that decouples from those of other agents. Moreover, without loss of optimality, each agent bases her decisions only on her private state (i.e., the number of jobs of each type) and on the stationary distribution of the shared pool through the steady-state matching probabilities $\mathbf{w}$ for different job types. Following Weintraub et al. (2008), we call such agents oblivious, their strategies oblivious strategies, and refer to their decision problems in the mean-field approximation as mean-field problems.

We now introduce the concept of oblivious equilibrium (OE). The OE requires a consistency check: the presumed shared pool job matching probabilities $\mathbf{w} = (w_j)_{j\in[J]}$ must arise from the optimal strategies of all agents in solving their mean-field problems. Formally, we have the following definition.

Definition 5.1 (Oblivious Equilibrium). Let $\mathbf{w} = (w_j)_{j\in[J]}$ be the steady-state matching probabilities for jobs of different types in the shared pool. Let π_i be an oblivious policy of agent i in the mean-field problem, and let $\boldsymbol{\pi} = (\pi_i)_{i\in[N]}$ be the policy profile of all agents. We say that the pair $(\mathbf{w}, \boldsymbol{\pi})$ constitutes an OE if:

1. (Individual Optimality). For each agent i, the policy π_i is optimal for her mean-field problem given the steady-state matching probabilities $\mathbf{w} = (w_j)_{j\in[J]}$ in the shared pool; and
2. (Consistency). The steady-state matching probabilities in the shared pool are $\mathbf{w} = (w_j)_{j\in[J]}$ when all N agents follow the policy profile $\boldsymbol{\pi} = (\pi_i)_{i\in[N]}$.

5.2 Mean-Field Analysis of the `MV` Mechanism

In this section, we analyze the `MV` mechanism within the oblivious equilibrium framework. We first illustrate in Proposition 5.1 that the `MV` mechanism provides sufficient incentives for submitting over-demanded jobs in a large market. Specifically, when the number of agents is large and all other agents fully submit their jobs—rendering the shared pool matching probabilities for over-demanded jobs close to one (Corollary 3.5 and Remark 3.3)—an agent never strategically withholds an over-demanded job to perform internal matching with future arrivals. Therefore, over-demanded jobs are either submitted or internally matched immediately upon arrival.

Proposition 5.1 (Sufficient Incentives for Not Retaining Over-Demanded Jobs). *Let $w_{\min} \triangleq \min_{j\in\mathcal{N}_+} w_j$ denote the minimum steady-state matching probability in the shared pool among over-demanded jobs, and suppose the `MV` mechanism is used. When the value of $w_{\min}$ exceeds a constant that depends only on the problem primitives and not on the number of agents, it is strictly suboptimal for any agent in her mean-field problem to withhold over-demanded jobs for future internal matching.*

We prove Proposition 5.1 in Appendix B.1. Intuitively, if an agent retains some over-demanded jobs to perform matches locally, some of these jobs will inevitably expire without being matched and thus be wasted. Consequently, the long-run average payoff will be strictly less than that under full submission of over-demanded jobs.

According to Proposition 5.1, when submitted over-demanded jobs are matched with high probability, an agent under the `MV` mechanism performs only linking matches (if it performs any internal matches at all)—that is, matches involving exactly one over-demanded job and one or more under-demanded jobs, as defined in Definition 5.2—and executes these matches only upon the arrival of the over-demanded job.

Definition 5.2 (Linking Match). We call a match $m \in \mathcal{M}_+$ a linking match if it requires exactly one over-demanded job and one or more under-demanded jobs. Let $\mathcal{M}_\ell \subseteq \mathcal{M}_+$ denote the set of all linking matches.

5.2.1 Full Submission Is Not an OE Under the `MV` Mechanism

Although the `MV` mechanism provides sufficient incentives for submitting over-demanded jobs, agents may retain under-demanded jobs in anticipation of an over-demanded job, to gain a small additional payoff.[12] Intuitively, under the `MV` mechanism, submitting an under-demanded job yields no benefit, whereas retaining it may facilitate the matching of over-demanded jobs.

In this section, we present a simple example to illustrate that full job submission by all agents does not constitute an OE under the `MV` mechanism, regardless of the number of agents. Furthermore, simply adjusting the allocation probabilities in the `MV` mechanism cannot resolve this incentive issue. We first present the example in Example 5.1.

Example 5.1. Consider a problem instance with two job types and one matching type. A match requires one job from each type and yields a reward of 1. There are N agents that are stochastically identical. Jobs of type one and type two arrive at each agent at rates $\lambda_1 = 1$ and $\lambda_2 = \lambda > 1$, and expire at rates $\theta_1 = 1$ and $\theta_2 = \theta > 0$. Consequently, type-one jobs are over-demanded, and type-two jobs are under-demanded, with marginal values of $p_1^* = 1$ and $p_2^* = 0$, respectively.

We now consider a generalization of the `MV` mechanism for Example 5.1, as defined in Definition 5.3.

Definition 5.3 (The $\texttt{MV}(\epsilon)$ Mechanism for Example 5.1). When the shared pool performs a match, the $\texttt{MV}(\epsilon)$ mechanism, with value $\epsilon \in [0, 1]$, allocates the match to the agent submitting the type-one job with probability $1 - \epsilon$ and to the agent submitting the type-two job with probability ϵ.

[12]This follows from the fact that an over-demanded type-j job submitted to the shared pool may remain unmatched before expiration with probability $1 - w_j$, and thus yield zero reward under the `MV` mechanism.

The `MV` mechanism is a special case of the `MV`(ϵ) mechanism with $\epsilon = 0$. By varying ϵ, we adjust the allocation probabilities within the `MV` mechanism. We now demonstrate that full job submission by all agents is not an OE under the `MV`(ϵ) mechanism for any $\epsilon \in [0, 1]$ and any number of agents N.

Assume all agents in Example 5.1 fully submit their jobs to the shared pool. Let $w_1 = w \in (0, 1)$ denote the steady-state probability that a type-one job is matched before expiration. As the number of agents increases, the probability w approaches one because type-one jobs are over-demanded (Remark 3.3). Furthermore, since the shared pool matches an equal number of type-one and type-two jobs, the probability that a type-two job is matched in the shared pool is $w_2 = w \cdot \lambda_1/\lambda_2 = w/\lambda$.

Full Submission Suppose agent one submits all jobs. Her long-run average expected payoff, denoted by V^{full}, is given by

$$V^{\text{full}} = w_1 \cdot (1 - \epsilon) + \lambda \cdot w_2 \cdot \epsilon = w.$$

An Alternative Policy Since an (over-demanded) type-one job may expire in the shared pool before being matched, resulting in zero value, it may be beneficial to retain some (under-demanded) type-two jobs for internal matching with arriving type-one jobs. Consider the following alternative policy that retains one type-two job:

1. Upon the arrival of a type-two job, retain it if agent one does not already hold a type-two job; otherwise, submit it to the shared pool.
2. Upon the arrival of a type-one job, match it locally with a type-two job if one is available; otherwise, submit it to the shared pool.

The long-run average expected payoff of agent one under the alternative policy, denoted by V^{alt}, is

$$V^{\text{alt}} = w + (1 - w)\,\frac{\lambda}{1 + \lambda + \theta} + \epsilon\, w\, \frac{\lambda - 1 - \theta}{1 + \lambda + \theta}\,.$$

Further details are provided in Appendix B.2.

Suboptimality of Full Submission Note that the difference $V^{\text{alt}} - V^{\text{full}}$ is strictly positive when $\lambda > 1 + \theta$, regardless of the perturbation parameter ϵ or the matching probability w (which depends on the number of agents N). In such cases, agent one benefits by deviating from full submission.

Example 5.1 shows that the `MV` mechanism may fail to provide sufficient incentives for the full submission of under-demanded jobs. Moreover, adjusting the allocation probabilities alone cannot resolve this issue. This motivates us to consider a more refined version of the `MV` mechanism, which we present in Section 5.3.

5.3 The `MVC` Mechanism

In this section, we introduce the `MVC` (Marginal-Value-plus-Credit) mechanism as a careful refinement of the `MV` mechanism. We demonstrate that the `MVC` mechanism incentivizes the full submission of both over-demanded and under-demanded jobs in the oblivious equilibrium. Moreover, full job submission by all agents constitutes both an OE and an approximate Nash equilibrium under the `MVC` mechanism. We first describe the `MVC` mechanism in Section 5.3.1.

5.3.1 Description of the `MVC` Mechanism

In the `MVC` mechanism, agents earn credits (or virtual jobs) by submitting under-demanded jobs, which can later be redeemed for rewards in the form of future match allocations.

Perturbed Marginal Values As before, the `MVC` mechanism implements the periodic matching policy π^{P} in the shared pool, where submitted jobs are matched at discrete time points $t = k\Delta$ for any $k \in \mathbb{N}_+$. However, when a type-j job is matched, the submitter receives a perturbed marginal value $\tilde{p}_j$, rather than the full marginal value p_j^* as in the `MV` mechanism. Specifically, fix a small positive number $\epsilon > 0$. The perturbed marginal values $\tilde{\mathbf{p}} = (\tilde{p}_j)_{j \in [J]}$ are defined as follows:

$$\tilde{p}_j = \begin{cases} p_j^* - \epsilon & \forall\, j \in \mathcal{N}_+ \\ p_j^* = 0 & \forall\, j \in \mathcal{N}_0\,. \end{cases}$$

Therefore, a tax of value ϵ is imposed on centrally matched over-demanded jobs. The collected taxes are later awarded to agents who submit under-demanded jobs, as detailed later.

Credits for Submitting Under-Demanded Jobs Although submitting an under-demanded job of type $j \in \mathcal{N}_0$ does not directly create value for the agent (as $\tilde{p}_j = p_j^* = 0$ for all $j \in \mathcal{N}_0$), it grants the agent a credit of type j, which expires at the same rate θ_j as its corresponding job. To incentivize the submission of under-demanded jobs, these credits can be redeemed for rewards from future matches in the following two ways.

First, before their expiration, credits can be used to exempt over-demanded jobs from tax. Specifically, when an agent submits an over-demanded job of type $j \in \mathcal{N}_+$, she may pair it with sufficient credits to virtually form a linking match $m \in \mathcal{M}_\ell$.[13] By doing so, the agent receives an additional reward of ϵ upon the successful matching of the over-demanded job, effectively exempting the tax and restoring the full marginal value p_j^*. Once paired, the credits are consumed and removed from the agent's balance, even if the associated over-demanded job expires before being matched.

Second, any unused credit that expires will automatically convert into a one-time lottery entry for the next matching epoch. At the end of each matching epoch, the shared pool collects all the ϵ-values taxed from matched over-demanded jobs that were submitted in isolation (that is, without paired credits), forming a lottery pool. These collected values are then randomly distributed among the lottery entries participating in that epoch. Thus, lottery entries provide a second pathway for agents to derive value from submitting under-demanded jobs. We remark that both pathways are essential to incentivize the full submission of under-demanded jobs under the `MVC` mechanism; see Remark 5.1.

Non-Monetary Implementation via Randomized Matching Allocation We note that the above mechanism can be implemented without monetary transfers through randomized matching allocation, as in the `MV` mechanism. Specifically, when a match of type $\mathcal{M}_+$ is about to be performed at the shared pool, we assign it as follows:

1. Assign the match to a participant over-demanded job $j \in \mathcal{N}_+$ with probability p_j^*/r_m if the job is paired with credits to form a virtual linking match at the time of the submission (and thus is exempt from tax), and with probability $\tilde{p}_j/r_m$ if the job is submitted in isolation.
2. Assign the match to a lottery entry at the current matching epoch, chosen uniformly at random, with probability $n\epsilon/r_m$, where n denotes the number of participant over-demanded jobs submitted in isolation. If no lottery entry exists in the current epoch, the match is discarded rather than performed, with probability $n\epsilon/r_m$.

Finally, we let the agent of the selected over-demanded job or lottery entry perform the match and collect the matching reward r_m. We refer to this non-monetary implementation as the `MVC` mechanism.

Agents' Problem Under the `MVC` mechanism, agents may submit any number of jobs they currently hold at any time, and all submissions are irrevocable. Each agent, aiming to maximize her long-run

[13] Formally, the linking match $m \in \mathcal{M}_\ell$ requires exactly one over-demanded job of type j (see Definition 5.2). The agent must pair this job with $M_{j'm}$ credits of type j' for all $j' \in \mathcal{N}_0$.

average payoff, decides when to submit jobs and, in the case of over-demanded jobs, whether and how to pair them with credits, based on her current job inventory and credit balance. This yields a Markov decision problem with a multi-dimensional state space and a large action space, which is potentially challenging to analyze. In the next section, we show via a careful coupling argument that submitting all jobs upon arrival is optimal for any agent when over-demanded jobs are matched with high probability in the shared pool.

5.3.2 Analysis of the `MVC` Mechanism

In this section, we show that under the `MVC` mechanism, full job submission by all agents constitutes an oblivious equilibrium when the number of agents exceeds a finite threshold (Theorem 5.4), and it forms an approximate Nash equilibrium in the original problem (Proposition 5.5).

In the following, let $w_{\min} \triangleq \min_{j \in \mathcal{N}_+} w_j$ denote the minimum steady-state matching probability in the shared pool among over-demanded jobs. We first show in Lemma 5.2 that the `MVC` mechanism provides sufficient incentives for submitting over-demanded jobs.

Lemma 5.2 (Sufficient Incentives for Not Retaining Over-Demanded Jobs). *There exists two positive constants, $C_8(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and $C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$, both strictly less than one and dependent only on the aggregate job arrival rates $\boldsymbol{\lambda}$ (defined in Assumption 2.1), job departure rates $\boldsymbol{\theta}$, matching rewards $\mathbf{r}$, and matching matrix $\mathbf{M}$. When $w_{\min} \geq C_8(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and $\epsilon \leq C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and the `MVC` mechanism is used, it is strictly suboptimal for any agent solving her mean-field problem to withhold over-demanded jobs for future internal matching. In other words, an over-demanded job is either submitted or internally matched immediately upon arrival.*

We specify the values of the constants $C_8(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and $C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and prove Lemma 5.2 in Appendix B.3. Intuitively, when the matching probability $w_{\min}$ is large, the `MV` mechanism provides sufficient incentives to submit over-demanded jobs according to Proposition 5.1. Since the `MVC` mechanism only slightly perturbs the expected return of submitting over-demanded jobs when ϵ is small, it likewise provides sufficient incentives for submitting over-demanded jobs.

Lemma 5.2 indicates that when the shared-pool matching probability $w_{\min}$ is large, the only internal matches an agent might perform are linking matches (defined in Definition 5.2), and the agent would perform such matches only upon the arrival of over-demanded jobs. Next, we show in Lemma 5.3 that even performing these internal matches are suboptimal because, in contrast to the `MV` mechanism, the `MVC` mechanism incentivizes full submission of under-demanded jobs.

Lemma 5.3 (Sufficient Incentives for Submitting Under-Demanded Jobs). *There exist two positive constants, $C_{10}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and $C_{11}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$, dependent only on the problem primitives $(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and independent of the number of agents N, satisfying the following conditions:*

1. $C_8(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M}) \leq C_{10}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M}) < 1$,
2. $(1 - C_{10}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})) \cdot C_{11}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M}) \leq C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$,

where $C_8(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and $C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ are constants defined in Lemma 5.2. Furthermore, if

$$w_{\min} \geq C_{10}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M}), \quad \big(1 - w_{\min}\big) \cdot C_{11}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M}) \leq \epsilon \leq C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$$

and the number of agents N exceeds a certain constant depending only on the problem primitives, then it is uniquely optimal for any agent to fully submit jobs in her mean-field problem associated with the `MVC` mechanism.

In the proof, we show that the above conditions guarantee the full submission of under-demanded jobs. Since each match requires at least two jobs (that is, $\sum_{j \in [J]} M_{jm} \geq 2$ for all $m \in [K]$), the full submission of under-demanded jobs, together with Lemma 5.2, implies the full submission of all jobs. Further details are provided in Appendix B.4.

Lemma 5.3 demonstrates that the refinement—specifically, the benefits of submitting under-demanded jobs through tax exemptions (for matching over-demanded jobs) and lottery opportunities—fix the incentive issue for under-demanded jobs under the `MV` mechanism. Intuitively, performing internal matches ensures immediate matching for newly arrived over-demanded jobs. In contrast, over-demanded jobs submitted to the shared pool may expire before being matched. However, internal matching requires withholding under-demanded jobs to await potential future matches with over-demanded jobs, inevitably resulting in wasted under-demanded jobs due to expiration. Alternatively, submitting these under-demanded jobs to the shared pool allows each to generate a positive expected reward through lotteries. In the proof, via a coupling argument, we show that the costs associated with internal matching outweigh its benefits when over-demanded jobs have a high probability of being matched in the shared pool.

According to Lemma 5.3, agents fully submit their jobs when the shared-pool matching probability for over-demanded jobs is close to one. Conversely, Corollary 3.5 and Remark 3.3 indicate that this probability indeed approaches one when all agents fully submit jobs and the number of agents increases. Consequently, full job submission by all agents constitutes an OE when the number of agents is large, which we formally state in Theorem 5.4 and prove in Appendix B.5.

Theorem 5.4 (Oblivious Equilibrium). *Let $\epsilon = (K \cdot C_6 \cdot C_{11}) \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}$, where constants C_6 and C_{11} are specified in Corollary 3.5 and Lemma 5.3, respectively, and V^{P} denotes the long-run average matching reward of the periodic matching policy π^{P} in the centralized setting. There exists an integer $N_0(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ such that for any $N \geq N_0$, full job submission by all agents constitutes an OE under the `MVC` mechanism with parameter ϵ.*

Finally, analogous to the `MV` mechanism, full job submission by all agents remains an approximate Nash equilibrium under the `MVC` mechanism, as formally stated in Proposition 5.5. The proof of Proposition 5.5 is similar to that of Theorem 4.5 and is provided in Appendix B.6.

Proposition 5.5 (Approximate Nash Equilibrium). *Let the parameter ϵ be defined as in Theorem 5.4. Under the `MVC` mechanism, full job submission by all agents constitutes an $O\left(\frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}\right)$-approximate Nash equilibrium, regardless of how agents pair credits with over-demanded jobs when submitting them.*

We conclude this section with a remark highlighting that both pathways—tax exemption and lottery entry—are essential to incentivize the full submission of under-demanded jobs under the `MVC` mechanism.

Remark 5.1 (Necessity of Both Pathways). Using only one pathway (either tax exemptions or lottery opportunities) to incentivize the submission of under-demanded jobs cannot ensure full job submission as an OE for any number of agents. We provide further details in Appendix B.7.

6 Numerical Experiments

In the previous sections, we rigorously justified that fully submitting jobs is a near-optimal best response when all other agents do the same and the number of agents is large. Specifically, full job submission constitutes an approximate Nash equilibrium under the `MV` mechanism (Section 4.3) and a stronger oblivious equilibrium in large markets under the `MVC` mechanism (Section 5.3.2). In this section, we numerically evaluate the performance of our mechanisms in practical settings and demonstrate that full job submission remains approximately an equilibrium even with a moderate number of agents. We consider a simple synthetic example in Section 6.1 and a more realistic example using kidney exchange data in Section 6.2. All experiments are implemented in Matlab on a personal computer.

6.1 A Simple Example

We first consider a simple example with N statistically identical agents having the same job arrival rates $\boldsymbol{\lambda}_i$. Specifically, there are $J = 3$ job types, with arrival rates $\boldsymbol{\lambda}_i = [7.5\ 5\ 2.5]^{\mathrm{T}}$ for every agent $i \in [N]$ and departure rates $\boldsymbol{\theta} = [1\ 1\ 1]^{\mathrm{T}}$. Thus, on average, jobs arrive five times faster than they depart. Jobs can form $K = 5$ different matching types. The matching rewards are $\mathbf{r} = [1\ 1\ 1\ 2\ 4]^{\mathrm{T}}$, and the matching matrix is given by

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 \end{bmatrix}.$$

Specifically, each of the first three matching types requires two jobs of the same type and provides a reward of 1. The fourth matching type requires one job of type one and one job of type two and leads to a reward of 2. Finally, the fifth matching type requires one job from each type and yields the highest reward of 4.

Intuitively, without job departures, it is preferable to perform the fourth and final matchings and match the remaining type-one jobs with each other. This is because each job generates a higher reward from participating in the last two matching types than from matching within its own type. Notably, the fluid relaxation (1), which disregards job expiration, has a unique optimal solution, $\bar{\mathbf{x}} = [1.25\ 0\ 0\ 2.5\ 2.5]^{\mathrm{T}}$, and this solution is non-degenerate because all basic variables are positive. In the original problem, however, job expirations introduce a trade-off: Each agent must decide whether to wait for jobs to form a more desired match with the risk that on-hand jobs may expire. Therefore, although our example setup is simple, it captures the key trade-off agents face.

Numerical Results Suppose agents two to N submit all their jobs to the shared pool, and the `MV` mechanism is used. We compare the long-run average payoff of agent one from full job submission to the optimal value of (3), which serves as an upper bound on agent one's payoff under any strategy, even when she holds complete information about the system (Lemma 4.3).[14]

Specifically, we take $\Delta = \frac{1}{2} \cdot N^{-\frac{1}{2}}$ for the periodic matching policy π^{P} based on Lemma 3.3 and number of agents N increasing linearly from 20 to 200 with a step size of 20. Figure 1(a) presents the relative suboptimal gap $\left(= \frac{\text{fluid relaxation (3)} - \text{payoff from full submission}}{\text{payoff from full submission}}\right)$ of agent one from full submission. As the number of agents increases, the sub-optimality of full submission decreases

[14]The numerical results under the `MVC` mechanism remain essentially the same when a small perturbation parameter ϵ is used.

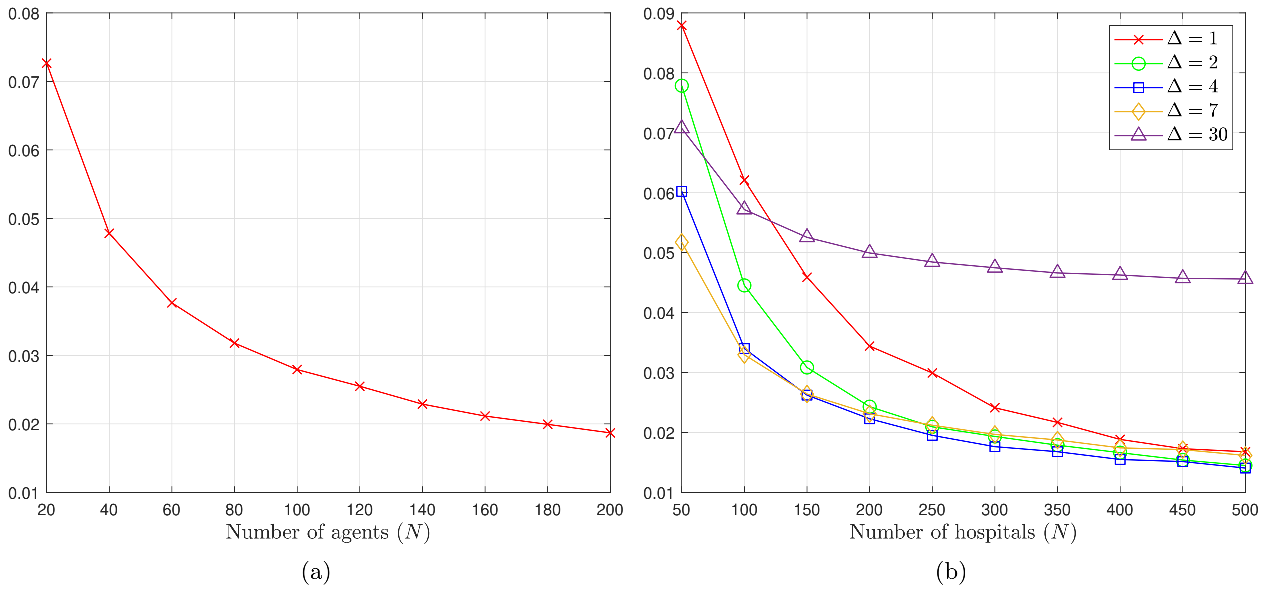

Figure 1: Relative suboptimal gap for (a) the simple synthetic example and (b) the kidney exchange example.

fast. In particular, the suboptimal gap is smaller than 5% when there are 40 agents (4.78%) and drops below 2% when there are 180 agents (1.99%). Note that the suboptimal gap we evaluate is conservative in that the fluid relaxation (3) for an agent might be a bit loose when the number of agents N is small. This is because, with fewer agents, the shared pool is less congested, and the probability of an over-demanded job being matched is lower. In contrast, in the fluid relaxation (3), an over-demanded job always secures the expected payoff from matching (which equals its marginal value) when submitted to the shared pool. Despite this, our numerical results indicate that an agent has a limited ability to strategize when all other servers submit fully and the number of agents is not very small.

6.2 A Kidney Exchange Example

We next consider a more realistic multi-hospital kidney exchange example based on real data. In this example, jobs represent incompatible patient-donor pairs, and agents represent hospitals that can perform kidney exchanges. We restrict attention to bilateral exchanges.

An exchange between two patient-donor pairs is feasible if each patient is both blood-type (ABO) compatible and tissue-type compatible with the other pair's donor. ABO compatibility requires that a patient cannot receive a kidney from a donor who has a blood antigen (A or B) that the patient does not have. Figure 2 in the Appendix illustrates the ABO compatibility structure. In addition to ABO compatibility, patients must also be tissue-type compatible with donors. A commonly used measure of the difficulty a patient faces in finding a tissue-type compatible donor, among those who are ABO compatible, is the panel-reactive antibody (PRA). PRA quantifies the likelihood that a patient is tissue-type incompatible with a random donor in the population, based on the patient's antibodies. Consequently, the probability of a patient being compatible with a potential donor is $1 - \text{PRA}$. Patients with high PRA levels are more likely to have difficulty finding a tissue-type compatible donor. For further details, see Section 2 of Ashlagi and Roth (2021).

Parameter Setup For simplicity, we assume that all hospitals are statistically identical and, therefore, have the same arrival rates of patient-donor pairs. We set the total arrival rate at each hospital to be one patient-donor pair every two weeks, or approximately 26 pairs per year.

Patient-donor pairs are classified based on the donor's and the patient's blood types, and the patient's PRA score. For example, an (A–B, 65) pair consists of a donor with blood type B and a patient with blood type A whose PRA score is 65, indicating a 0.65 probability of incompatibility with an ABO-compatible donor.

To determine the frequency of different patient-donor types, we use statistics about the APKD (a major U.S. kidney exchange program) historical pool composition from 2010 to 2019, as provided in Table A.1 of Ashlagi and Roth (2021) and reproduced in Table 2 in the Appendix. Table 2 categorizes the PRA scores into 7 intervals. Consequently, we set the total number of pair types to be the 16 ABO blood type combinations multiplied by the 7 PRA intervals, resulting in 112 types. For each PRA interval, we use the midpoint as the representative PRA score. In Table 2, the second column presents the empirical distribution of the patient-donor ABO pairs, and columns three to nine present the frequencies of PRA intervals, conditional on each ABO pair. As a result, the arrival rate for each type can be calculated by multiplying the total arrival rate by the percentage of the corresponding ABO pair and the conditional PRA frequency.

Finally, following the empirical setup in Ashlagi et al. (2023), we assume an average waiting time of 360 days, or an expiration rate of 1/360 per day, for any pair type.[15] We consider each hospital's objective to be maximizing the expected number of exchanges. Therefore, two pairs can be matched only if they are ABO compatible. The expected payoff from a potential match is defined as the probability that the exchange will be successful. For a bilateral exchange between two patient-donor pairs to succeed, two independent conditions must be met: the first patient must be compatible with the second pair's donor, and the second patient must be compatible with the first pair's donor. Since these events are independent, the probability of a successful match is the

[15]See footnote 27 in Section 5 of Ashlagi et al. (2023).

product of their individual probabilities. This leads to the expected payoff from a match being: $(1 - \text{PRA of the first patient}) \times (1 - \text{PRA of the second patient})$.

Numerical Results We again assume that hospitals two to N submit all their jobs to the shared pool and that the `MV` mechanism is used, and compare the long-run average payoff of hospital one from full submission to the fluid upper bound (3). In the simulation, we vary the number of hospitals N linearly from 50 to 500 with a step size of 50, and set the matching interval length Δ in the periodic matching policy π^{P} to be $\Delta \in \{1, 2, 4, 7, 30\}$, corresponding to 1 day, 2 days, 4 days, 1 week, and 1 month, respectively. Figure 1(b) illustrates the relative suboptimal gap of hospital one from full submission under different matching interval lengths. From Figure 1(b), even with only 50 hospitals participating in the kidney exchange program, the suboptimal gap of full submission remains around 5% (specifically, 5.17% when $\Delta = 7$) assuming all other hospitals fully submit their patient-donor pairs. The suboptimal gap drops further to 2% when 250 hospitals participate (1.95% when $\Delta = 4$). It is worth noting that there are currently 256 kidney transplant centers in the U.S. (Wang and Hart 2021). When there are 250 or more hospitals, shorter matching periods (e.g., daily or every few days) outperform less frequent matching (e.g., monthly), which aligns with current practices in the U.S. (Section 2.4 of Ashlagi and Roth 2021).

7 Conclusions

Motivated by applications in multi-hospital kidney exchanges and collaboration among matchmakers in other matching markets, we studied the problem of incentivizing participation among N strategic agents, each managing a local multi-way matching problem. We focused on a limited information setting where an agent's job arrivals and actions are unobservable to others, and the designer knows only the aggregate job arrival rates and matching rewards. As our main contribution, we develop simple mechanisms based on marginal values that incentivize all agents to fully submit their jobs when the number of agents is large. As a result, the shared pool effectively operates in a centralized setting, achieving the same dynamics and performance as under centralized control. The `MV` mechanism ensures that full job submission constitutes an approximate Nash equilibrium, while the `MVC` mechanism ensures a stronger oblivious equilibrium in large markets. Numerical experiments based on synthetic examples and kidney exchange data confirm that full job submission remains approximately an equilibrium under our mechanisms, even when the number of participants is moderate. Ongoing work includes exploring near-optimal non-monetary mechanisms in more general settings, such as when agents have heterogeneous matching rewards or hold private information about those rewards.

References

Agarwal, N., I. Ashlagi, E. Azevedo, C. R. Featherstone, and Ö. Karaduman (2019). Market failure in kidney exchange. *American Economic Review 109*(11), 4026–70.

Agarwal, N., I. Ashlagi, M. Rees, and A. Roth (2021). Expanding kidney exchange. Available at `http://onepercentsteps.com/policy-briefs/expanding-kidney-exchange/`. Accessed: 2024-10-03.

Akbarpour, M., S. Li, and S. O. Gharan (2020). Thickness and information in dynamic matching markets. *Journal of Political Economy 128*(3), 783–815.

Anily, S. and M. Haviv (2010). Cooperation in service systems. *Operations Research 58*(3), 660–673.

Aouad, A. and O. Sarıtaç (2022). Dynamic stochastic matching under limited time. *Operations Research 70*(4), 2349–2383.

Arnosti, N., R. Johari, and Y. Kanoria (2021). Managing congestion in matching markets. *Manufacturing & Service Operations Management 23*(3), 620–636.

Ashlagi, I., M. Burq, C. Dutta, P. Jaillet, A. Saberi, and C. Sholley (2023). Edge-weighted online windowed matching. *Mathematics of Operations Research 48*(2), 999–1016.

Ashlagi, I., F. Fischer, I. A. Kash, and A. D. Procaccia (2015). Mix and match: A strategyproof mechanism for multi-hospital kidney exchange. *Games and Economic Behavior 91*, 284–296.

Ashlagi, I., A. Nikzad, and P. Strack (2023). Matching in dynamic imbalanced markets. *The Review of Economic Studies 90*(3), 1084–1124.

Ashlagi, I. and A. E. Roth (2014). Free riding and participation in large scale, multi-hospital kidney exchange. *Theoretical Economics 9*(3), 817–863.

Ashlagi, I. and A. E. Roth (2021). Kidney exchange: An operations perspective. *Management Science 67*(9), 5455–5478.

Aveklouris, A., L. DeValve, M. Stock, and A. Ward (2025). Matching impatient and heterogeneous demand and supply. *Operations Research 73*(3), 1637–1658.

Balseiro, S. R., O. Besbes, and G. Y. Weintraub (2015). Repeated auctions with budgets in ad exchanges: Approximations and design. *Management Science 61*(4), 864–884.

Bertsimas, D. and J. N. Tsitsiklis (1997). *Introduction to linear optimization*, Volume 6. Athena scientific Belmont, MA.

Chen, C., Y. Chen, and P. Qian (2023). Incentivizing resource pooling. *Available at SSRN 4586771*.

Cohen, M. C. and R. Zhang (2022). Competition and coopetition for two-sided platforms. *Production and Operations Management 31*(5), 1997–2014.

Farrell, J. and P. Klemperer (2007). Coordination and lock-in: Competition with switching costs and network effects. *Handbook of industrial organization 3*, 1967–2072.

Gupta, V. (2024). Greedy algorithm for multiway matching with bounded regret. *Operations Research 72*(3), 1139–1155.

Held, P. J., F. McCormick, A. Ojo, and J. P. Roberts (2016). A cost-benefit analysis of government compensation of kidney donors. *American Journal of Transplantation 16*(3), 877–885.

Hu, X., R. Caldentey, and G. Vulcano (2013). Revenue sharing in airline alliances. *Management Science 59*(5), 1177–1195.

Iyer, K., R. Johari, and M. Sundararajan (2014). Mean field equilibria of dynamic auctions with learning. *Management Science 60*(12), 2949–2970.

Karsten, F., M. Slikker, and G.-J. Van Houtum (2015). Resource pooling and cost allocation among independent service providers. *Operations Research 63*(2), 476–488.

Kerimov, S., I. Ashlagi, and I. Gurvich (2024). Dynamic matching: Characterizing and achieving constant regret. *Management Science 70*(5), 2799–2822.

Kerimov, S., I. Ashlagi, and I. Gurvich (2025). On the optimality of greedy policies in dynamic matching. *Operations Research 73*(1), 560–582.

Liu, H. and Y. Yu (2022). Incentives for shared services: multiserver queueing systems with priorities. *Manufacturing & Service Operations Management 24*(3), 1751–1759.

Mangasarian, O. L. and T.-H. Shiau (1987). Lipschitz continuity of solutions of linear inequalities, programs and complementarity problems. *SIAM Journal on Control and Optimization 25*(3), 583–595.

Netessine, S. and R. A. Shumsky (2005). Revenue management games: Horizontal and vertical competition. *Management Science 51*(5), 813–831.

Roels, G. and C. S. Tang (2017). Win-win capacity allocation contracts in coproduction and codistribution alliances. *Management science 63*(3), 861–881.

Ross, S. M. (1995). *Stochastic processes* (2 ed.). John Wiley & Sons.

Ross, S. M. and E. A. Peköz (2023). *A Second Course in Probability* (2 ed.). Cambridge University Press.

Royden, H. L. and P. Fitzpatrick (2010). *Real analysis* (4 ed.). Pearson.

Sönmez, T. and M. U. Ünver (2013). Market design for kidney exchange. In *The Handbook of Market Design*, Chapter 4. Oxford University Press.

Toulis, P. and D. C. Parkes (2015). Design and analysis of multi-hospital kidney exchange mechanisms using random graphs. *Games and Economic Behavior 91*, 360–382.

Wang, J. H. and A. Hart (2021). Global perspective on kidney transplantation: United states. *Kidney360 2*(11), 1836–1839.

Wei, Y., J. Xu, and S. H. Yu (2023). Constant regret primal-dual policy for multi-way dynamic matching. *Available at SSRN 4357216*.

Weintraub, G. Y., C. L. Benkard, and B. Van Roy (2008). Markov perfect industry dynamics with many firms. *Econometrica 76*(6), 1375–1411.

Wright, C. P., H. Groenevelt, and R. A. Shumsky (2010). Dynamic revenue management in airline alliances. *Transportation Science 44*(1), 15–37.

A Proofs in Sections 3 and 4

A.1 Proof of Lemma 3.1

Recall that

$$V^* = \max_{\pi\in\Pi} \liminf_{t\to\infty} \frac{1}{t} V^\pi(t), \qquad \text{where } V^\pi(t) = \mathbb{E}\left[\sum_{m\in[K]} r_m N_m^\pi(t)\right],$$

and Π denotes the set of all feasible policies. Note that there is no submission term in $V^\pi(t)$ because there is only one player: the central planner.

For any feasible policy $\pi \in \Pi$, the number of type-j jobs that have been matched by time t is $\sum_{m\in[K]} M_{jm} N_m^\pi(t)$, which must be less than $X_j(0) + A_j(t)$, the number of type-j jobs that have arrived by time t. Therefore, for any sample path, the total matching rewards by time t under policy π is bounded from above by the following:

$$\begin{aligned} V(t)[\mathbf{X}(0) + \mathbf{A}(t)] \triangleq \max_{\mathbf{x}\in\mathbb{N}^K} \quad & \mathbf{r}^{\mathrm{T}}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{M}\mathbf{x} \le \mathbf{X}(0) + \mathbf{A}(t). \end{aligned} \tag{5}$$

Let $\mathbf{x}[\mathbf{X}(0) + \mathbf{A}(t)]$ denote an optimal solution to (5); it holds that

$$\mathbf{M}\,\mathbb{E}\big[\mathbf{x}[\mathbf{X}(0) + \mathbf{A}(t)]\big] \le \mathbf{X}(0) + \mathbb{E}\left[\mathbf{A}(t)\right] = \mathbf{X}(0) + N\boldsymbol{\lambda} t.$$

Therefore, $\mathbb{E}\left[\mathbf{x}[\mathbf{X}(0) + \mathbf{A}(t)]\right]$ is feasible to the problem of $V^{\mathrm{F}}(\mathbf{X}(0) + N\boldsymbol{\lambda} t)$, and it follows that

$$V^\pi(t) \le \mathbb{E}\left[V(t)[\mathbf{X}(0) + \mathbf{A}(t)]\right] \le V^{\mathrm{F}}(\mathbf{X}(0) + N\boldsymbol{\lambda} t).$$

According to the definition of $V^{\mathrm{F}}(\boldsymbol{\lambda})$ and its dual problem (2), the function $V^{\mathrm{F}}(\boldsymbol{\lambda})$ is concave and piecewise linear, and thus is continuous. As a result, we have

$$\lim_{t\to\infty} \frac{1}{t} V^{\mathrm{F}}(\mathbf{X}(0) + N\boldsymbol{\lambda} t) = \lim_{t\to\infty} V^{\mathrm{F}}\left(N\boldsymbol{\lambda} + \frac{1}{t}\mathbf{X}(0)\right) = V^{\mathrm{F}}(N\boldsymbol{\lambda}) = N V^{\mathrm{F}}(\boldsymbol{\lambda}).$$

In conclusion, we have $V^* \le V^{\mathrm{F}}(N\boldsymbol{\lambda}) = N V^{\mathrm{F}}(\boldsymbol{\lambda})$.

A.2 Proof of Lemma 3.2

A.2.1 Preparations

To start, we provide some useful lemmas as preparation for the proof of Lemma 3.2. First, we bound the optimal dual variable $\mathbf{p}^*$ from above in Lemma A.1.

Lemma A.1. *Let $\mathbf{p}^* = (p_j^*)_{j\in[J]}$ be an optimal solution to (2). It follows that $p_j^* \le r_{\max}$ for all $j \in [J]$.*

Proof. We prove by contradiction. Suppose, instead, that $p_j^* > r_{\max}$ for some index $j \in [J]$. If we set $p_j^* = r_{\max}$ while keeping all other components of $\mathbf{p}^*$ unchanged, $\mathbf{p}^*$ remains feasible to (2), and this adjustment strictly improves the objective value, contradicting the optimality of $\mathbf{p}^*$. $\square$

Second, Lemma A.2 shows that the function $V^{\mathrm{F}}(\boldsymbol{\lambda})$ is Lipschitz continuous in $\boldsymbol{\lambda}$.

Lemma A.2. *For any two arrival rate vectors $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \ge 0$, we have:*

$$V^{\mathrm{F}}(\boldsymbol{\lambda}') - V^{\mathrm{F}}(\boldsymbol{\lambda}) \le r_{\max} \cdot \sum_{j\in[J]} \left(\lambda_j' - \lambda_j\right)^+.$$

Proof. By the strong duality, $V^{\mathrm{F}}(\boldsymbol{\lambda})$ equals the optimal value of (2). Let $\mathbf{p}^* = (p_j^*)_{j\in[J]}$ denote an optimal solution of (2). Therefore, we have $V^{\mathrm{F}}(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^{\mathrm{T}}\mathbf{p}^*$. Moreover, since $\mathbf{p}^*$ satisfies $\mathbf{M}^{\mathrm{T}}\mathbf{p}^* \geq \mathbf{r}$, it holds that $V^{\mathrm{F}}(\boldsymbol{\lambda}') \leq \boldsymbol{\lambda}'^{\mathrm{T}}\mathbf{p}^*$ by weak duality. As a result, we have:

$$V^{\mathrm{F}}(\boldsymbol{\lambda}') - V^{\mathrm{F}}(\boldsymbol{\lambda}) \leq \sum_{j\in[J]} p_j^*(\lambda_j' - \lambda_j) \leq \sum_{j\in[J]} p_j^*\left(\lambda_j' - \lambda_j\right)^+ \leq r_{\max}\cdot \sum_{j\in[J]}\left(\lambda_j' - \lambda_j\right)^+,$$

where the last inequality follows from Lemma A.1.

□

Finally, Lemma A.3 shows that the number of jobs that arrive within a time interval and remain till the end of the interval follows a specific Poisson distribution.

Lemma A.3. *For any time interval of length Δ, let Z_j denote the number of type-j jobs that arrive within the time interval and remain till the end of the interval. Then, Z_j follows a Poisson distribution with a mean value of $\frac{N\lambda_j}{\theta_j}(1-\exp(-\theta_j\Delta))$.*

Proof. Without loss of generality, we consider the time interval $(0,\Delta]$. We first compute the ex-ante probability $\mathbb{P}_j(\Delta)$ that a type-j job which arrived during $(0,\Delta]$ is still in the system at time Δ. Note that a type-j job will still be in the system if and only if its arrival time + waiting time $> \Delta$. Therefore, it holds that

$$\begin{aligned}
\mathbb{P}_j(\Delta) &= \mathbb{P}\left(\text{arrival time} + \text{waiting time} > \Delta\right) = \mathbb{P}\left(\text{waiting time} > \Delta - \text{arrival time}\right)\\
&= \frac{1}{\Delta}\int_0^\Delta \mathbb{P}\left(\text{waiting time} > x\right)dx = \frac{1}{\Delta}\int_0^\Delta \exp\left(-\theta_j x\right)dx\\
&= \frac{1}{\Delta\theta_j}\left(1-\exp\left(-\theta_j\Delta\right)\right),
\end{aligned}$$

where the third equality follows from the fact that the arrival time is uniformly distributed over $(0,\Delta]$, conditional on the job arriving within this time interval (see Section 2.3 of Ross 1995).

Let $A_j(\Delta)$ denote the number of type-j jobs that have arrived by time Δ, which follows a Poisson distribution with a mean value of $\Delta N\lambda_j$. Since each of the $A_j(\Delta)$ jobs remains in the system at time Δ with a probability of $\mathbb{P}_j(\Delta)$, Z_j follows a Poisson distribution with a mean value of

$$\Delta N\lambda_j\cdot\mathbb{P}_j(\Delta) = \frac{N\lambda_j}{\theta_j}\Big(1-\exp\left(-\theta_j\Delta\right)\Big).$$

□

A.2.2 Proof of Lemma 3.2

Consider a time interval of length Δ. Let Z_j denote the number of type-j jobs that arrive within the time interval and remain till the end of the interval. According to Lemma A.3, we have that $Z_j \sim \text{Poisson}\left(\frac{N\lambda_j}{\theta_j}\left(1-\exp\left(-\theta_j\Delta\right)\right)\right)$.

Let $\mathbf{Z} = (Z_j)_{j\in[J]} \in \mathbb{R}_+^J$ be the concatenation of job arrivals, $V^{\mathrm{P}}(\Delta)$ the expected payoff of the periodic matching policy π^{P} in a period of length Δ, and $\mathbf{X}\in\mathbb{N}^J$ the number of jobs in the system by the end of the time interval and before matches are conducted. It holds that $\mathbf{X}\geq\mathbf{Z}$ for any sample path, because there could be jobs that arrived before the time interval and still remain in the system. By the definition of policy π^{P}, we solve the fluid relaxation $V^{\mathrm{F}}(\mathbf{X})$, obtain an optimal solution $\mathbf{x}^*\in\mathbb{R}_+^K$, and perform $\lfloor x_m^*\rfloor$ type-m matchings for every $m\in\mathcal{M}_+$.

We first bound the rounding error. Note that:

$$\sum_{m\in\mathcal{M}_+} r_m\lfloor x_m^*\rfloor \geq \sum_{m\in\mathcal{M}_+} r_m x_m^* - \min\left\{K,J\right\} r_{\max},$$

where the inequality follows from the fact that the fluid relaxation (1) has at most $\min\{K, J\}$ basic variables. Therefore, the following holds:

$$V^{\mathrm{P}}(\Delta) \geq V^{\mathrm{F}}(\mathbf{X}) - \min\{K, J\} \cdot r_{\max} \geq V^{\mathrm{F}}(\mathbf{Z}) - \min\{K, J\} \cdot r_{\max}\,, \tag{6}$$

where the second inequality follows from the fact that $\mathbf{X} \geq \mathbf{Z}$ for every sample path. Consequently,

$$\begin{aligned}
V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda}) - \mathbb{E}[V^{\mathrm{P}}(\Delta)] &\leq V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda}) - \mathbb{E}[V^{\mathrm{F}}(\mathbf{Z})] + \min\{K, J\} \cdot r_{\max} \\
&\leq r_{\max} \cdot \sum_{j \in [J]} \mathbb{E}\Big(\Delta N\lambda_j - Z_j\Big)^+ + \min\{K, J\} \cdot r_{\max} \\
&\leq r_{\max} \cdot \sum_{j \in [J]} \left\{\mathbb{E}\Big(\Delta N\lambda_j - \mathbb{E}[Z_j]\Big)^+ + \mathbb{E}\Big(\mathbb{E}[Z_j] - Z_j\Big)^+\right\} + \min\{K, J\} \cdot r_{\max} \\
&\leq r_{\max} \cdot \sum_{j \in [J]} \left\{\frac{1}{2}\theta_j N\lambda_j\Delta^2 + \sqrt{N\lambda_j\Delta}\right\} + \min\{K, J\} \cdot r_{\max}.
\end{aligned}$$

where the first inequality follows from (6), the second inequality from Lemma A.2, and the fourth inequality from the facts that

$$\mathbb{E}[Z_j] = N\lambda_j \int_0^{\Delta} \exp(-\theta_j x)\,dx \geq N\lambda_j \int_0^{\Delta} (1 - \theta_j x)\,dx = N\lambda_j\Delta\left(1 - \frac{1}{2}\theta_j\Delta\right), \tag{7}$$

and that

$$\begin{aligned}
\mathbb{E}\Big(\mathbb{E}[Z_j] - Z_j\Big)^+ \leq \mathbb{E}\Big|\mathbb{E}[Z_j] - Z_j\Big| \leq \left[\mathbb{E}\Big(\mathbb{E}[Z_j] - Z_j\Big)^2\right]^{\frac{1}{2}} &= \sqrt{\mathbb{V}\mathrm{ar}(Z_j)} \\
&= \sqrt{\frac{N\lambda_j}{\theta_j}\big(1 - \exp(-\theta_j\Delta)\big)} \leq \sqrt{N\lambda_j\Delta}.
\end{aligned}$$

By taking $\Delta = N^{-\frac{1}{3}}$, we have

$$\begin{aligned}
\frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{NV^{\mathrm{F}}} &= \frac{V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda}) - \mathbb{E}[V^{\mathrm{P}}(\Delta)]}{V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda})} \\
&\leq r_{\max} \cdot \frac{\sum_{j \in [J]}\left\{\sqrt{N\lambda_j\Delta} + \frac{1}{2}\theta_j N\lambda_j\Delta^2\right\} + \min\{K, J\}}{\Delta N \cdot V^{\mathrm{F}}} \\
&\leq \frac{r_{\max}}{V^{\mathrm{F}}} \cdot \sum_{j \in [J]}\left(\sqrt{\lambda_j} \cdot \frac{1}{\sqrt{\Delta N}} + \frac{\lambda_j\theta_j}{2} \cdot \Delta + \frac{1}{\Delta N}\right) \\
&\leq \frac{r_{\max}}{V^{\mathrm{F}}} \cdot \sum_{j \in [J]}\left(\sqrt{\lambda_j} + \frac{\lambda_j\theta_j}{2} + 1\right) \cdot N^{-\frac{1}{3}}.
\end{aligned}$$

Remark A.1 (Incorporating Enrollment Deferral)**.** We note that the proofs of Lemma 3.2 and Lemma 3.3 (provided in Appendix A.4) continue to hold when the enrollment deferral (Section 3.3) is incorporated. The proofs are literally the same, with the main difference being that the random variable Z_j, defined in Lemma A.3 as the number of newly arriving type-j jobs joining a specific matching epoch, now follows a Poisson distribution with a mean value of $\Delta N\lambda_j \cdot \exp(-\theta_j\Delta) \cdot \mathbb{P}_j(\Delta)$, which is greater than $(1 - \theta_j\Delta)\,\Delta N\lambda_j \cdot \mathbb{P}_j(\Delta)$.

A.3 Details of Remark 3.1

Proof of Bullet One We note that function $V^{\mathrm{F}}(\boldsymbol{\lambda})$ is piecewise linear and concave in $\boldsymbol{\lambda}$, and that $\mathbf{p}^* \in \mathbb{R}_+^J$ is an optimal dual variable of $V^{\mathrm{F}}(\boldsymbol{\lambda})$ if and only if it is a subgradient of $V^{\mathrm{F}}(\cdot)$ at the point $\boldsymbol{\lambda}$ (see Sections 5.2 and 5.3 of Bertsimas and Tsitsiklis 1997). Therefore, $\mathbf{p}^*$ is the unique optimal dual solution if and only if $\boldsymbol{\lambda}$ is not a kink of the function $V^{\mathrm{F}}(\cdot)$, and since $V^{\mathrm{F}}(\boldsymbol{\lambda})$ is piecewise linear, this condition holds if and only if $\mathbf{p}^*$ is the unique optimal dual solution in the neighborhood of $\boldsymbol{\lambda}$.

Proof of Bullet Two Let (8) be the standard form of the fluid relaxation (1) by introducing the slack variable $\mathbf{s} \in \mathbb{R}_+^J$.

$$\begin{aligned} V^{\mathrm{F}}(\boldsymbol{\lambda}) = \max_{\mathbf{x}\in\mathbb{R}_+^K,\, \mathbf{s}\in\mathbb{R}_+^J} \quad & \mathbf{r}^{\mathrm{T}}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{M}\mathbf{x} + \mathbf{s} = \boldsymbol{\lambda}. \end{aligned} \tag{8}$$

We show that $\mathbf{x}$ is a non-degenerate basic feasible solution of (1) if and only of $\mathbf{x}$ together with $\mathbf{s} = \boldsymbol{\lambda} - \mathbf{M}\mathbf{x}$ is a non-degenerate basic feasible solution of (8). For a matrix $\mathbf{M} \in \mathbb{R}^{J\times K}$ and subsets $A \subseteq [J]$ and $B \subseteq [K]$, let $\mathbf{M}_{A,B}$ denote the submatrix of $\mathbf{M}$ consisting of the rows indexed by A and the columns indexed by B. Furthermore, for a feasible solution $\mathbf{x}$ to (1), define $I \triangleq \{m \in [K] : x_m = 0\}$ and $H \triangleq \{j \in [J] : \mathbf{m}_j^{\mathrm{T}}\mathbf{x} = \lambda_j\}$, where $\mathbf{m}_j^{\mathrm{T}}$ represents the j-th row of the matching matrix $\mathbf{M}$. On one hand, $\mathbf{x}$ is a non-degenerate basic feasible solution of (1) if and only if $\mathbf{x}$ is feasible to (1) and the following conditions hold:

$$\begin{aligned} & |I| + |H| = K, \\ & \text{The sub-matrix } \mathbf{M}_{H,\,[K]\setminus I} \text{ has full rank.} \end{aligned} \tag{9}$$

On the other hand, $\mathbf{x}$ together with $\mathbf{s} = \boldsymbol{\lambda} - \mathbf{M}\mathbf{x}$ is a non-degenerate basic solution of (8) if and only if $\mathbf{x}$ is feasible to (1) and the following conditions hold:

$$\begin{aligned} & |I| + |H| = K, \\ & \text{The matrix } \left(\mathbf{M}_{[J],\,[K]\setminus I}\,, \mathbf{I}_{[J],\,[J]\setminus[H]}\right) \text{ has full rank,} \end{aligned} \tag{10}$$

where $\mathbf{I}_{[J],\,[J]\setminus[H]}$ is a sub-matrix of the $J \times J$ identity matrix. Since $\left(\mathbf{M}_{[J],\,[K]\setminus I}\,, \mathbf{I}_{[J],\,[J]\setminus[H]}\right)$ has full rank if and only if $\mathbf{M}_{H,\,[K]\setminus I}$ has full rank, (9) and (10) are equivalent.

Note that (2) is also the dual of (8). Therefore, if (1) has a non-degenerate optimal basic feasible solution, so does (8), and this implies that (2) has a unique optimal solution according to Bertsimas and Tsitsiklis (1997) (see, for example, the discussion following Example 4.6 or preceding Theorem 5.1).

A.4 Proof of Lemma 3.3

Let $\mathbf{p}^*$ be the unique optimal dual variable of $V^{\mathrm{F}}(\hat{\boldsymbol{\lambda}})$ for any $\hat{\boldsymbol{\lambda}}$ such that $||\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}||_\infty \le \delta$. We have:

$$V^{\mathrm{F}}(\hat{\boldsymbol{\lambda}}) = \hat{\boldsymbol{\lambda}}^{\mathrm{T}}\mathbf{p}^*, \ \forall\, \hat{\boldsymbol{\lambda}} \text{ such that } ||\hat{\boldsymbol{\lambda}} - \boldsymbol{\lambda}||_\infty \le \delta. \tag{11}$$

Analogous to Appendix A.2, we focus on a time interval of length Δ and define random variable Z_j to be the number of type-j jobs that arrive within the time interval and remain till the end of the interval. Z_j follows a Poisson distribution with a mean value of $\frac{N\lambda_j}{\theta_j}\left(1 - \exp\left(-\theta_j\Delta\right)\right)$ by Lemma A.3. Note that the following holds:

$$\left(1 - \theta_j\Delta\right)\Delta N\lambda_j \le \left(1 - \frac{1}{2}\theta_j\Delta\right)\Delta N\lambda_j \le \frac{N\lambda_j}{\theta_j}\left(1 - \exp\left(-\theta_j\Delta\right)\right) \le \Delta N\lambda_j\,,$$

where the second inequality follows from (7) and the third inequality follows from the fact that $e^x \geq 1 + x$ for any $x \in \mathbb{R}$. We can construct two Poisson random variables $\bar{Z}_j \sim \text{Poisson}\,(\Delta N \lambda_j)$ and $\underline{Z}_j \sim \text{Poisson}\,((1-\theta_j \Delta)\,\Delta N \lambda_j)$, such that with proper coupling,

$$\underline{Z}_j \leq Z_j \leq \bar{Z}_j$$

holds for every sample path.

A.4.1 Proof of Bullet One

Within a time interval of length Δ, the expected payoff of the periodic matching policy $V^{\mathrm{P}}(\Delta)$ is at least $V^{\mathrm{F}}(\mathbf{Z}) - \min\{K, J\} \cdot r_{\max}$ according to (6), and the payoff of the fluid relaxation is $V^{\mathrm{F}}(\Delta N \boldsymbol{\lambda})$.

Take the interval length to be $\Delta = N^{-\frac{1}{2}}$, and define the following events:

$$\begin{aligned}
&\bar{B}_j = \left\{Z_j > \lfloor \Delta N(\lambda_j + \delta) \rfloor\right\}, \forall\, j \in [J],\\
&\underline{B}_j = \left\{Z_j < \lceil \Delta N(\lambda_j - \delta) \rceil\right\}, \forall\, j \in [J],\\
&A = \left(\cup_{j \in [J]} \left(\bar{B}_j \cup \underline{B}_j\right)\right)^c \subseteq \left\{||\mathbf{Z} - \Delta N \boldsymbol{\lambda}||_\infty \leq \Delta N \delta\right\}.
\end{aligned}$$

By the concentration inequality for Poisson random variables (Lemma A.6), there exists positive constants $c_1, c_2 > 0$, such that the following hold:

$$\begin{aligned}
&\mathbb{P}\left(\bar{B}_j\right) \leq \mathbb{P}\big(\bar{Z}_j \geq \lfloor \Delta N(\lambda_j + \delta) \rfloor\big) \leq c_1 \cdot \exp\left(-c_2 \cdot N^{\frac{1}{2}}\right),\\
&\mathbb{P}\left(\underline{B}_j\right) \leq \mathbb{P}\big(\underline{Z}_j \leq \lceil \Delta N(\lambda_j - \delta) \rceil\big) \leq c_1 \cdot \exp\left(-c_2 \cdot N^{\frac{1}{2}}\right).
\end{aligned} \tag{12}$$

In the first line, the first inequality follows from the fact that $Z_j \leq \bar{Z}_j$, and the second inequality from taking $\Delta = N^{-\frac{1}{2}}$ and Lemma A.6. The second line can be justified analogously.

Using the union bound, we obtain:

$$\mathbb{P}\left(A^c\right) \leq \sum_{j \in [J]} \left(\mathbb{P}\left(\bar{B}_j\right) + \mathbb{P}\left(\underline{B}_j\right)\right) \leq 2Jc_1 \cdot \exp\left(-c_2 \cdot N^{\frac{1}{2}}\right). \tag{13}$$

In addition, we have $||\frac{\mathbf{Z}}{\Delta N} - \boldsymbol{\lambda}||_\infty \leq \delta$ conditioning on event A. Therefore, from (11) we have:

$$V^{\mathrm{F}}(\Delta N \boldsymbol{\lambda}) - V^{\mathrm{F}}(\mathbf{Z}) \leq \mathbb{1}[A] \cdot \mathbf{p}^{*\mathrm{T}} \cdot (\Delta N \boldsymbol{\lambda} - \mathbf{Z}) + \mathbb{1}[A^c] \cdot V^{\mathrm{F}}(\Delta N \boldsymbol{\lambda}).$$

Consequently,

$$\begin{aligned}
\frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{NV^{\mathrm{F}}} &= \frac{V^{\mathrm{F}}(\Delta N \boldsymbol{\lambda}) - \mathbb{E}[V^{\mathrm{P}}(\Delta)]}{V^{\mathrm{F}}(\Delta N \boldsymbol{\lambda})}\\
&\leq \frac{V^{\mathrm{F}}(\Delta N \boldsymbol{\lambda}) - \mathbb{E}\big[V^{\mathrm{F}}(\mathbf{Z})\big] + \min\{K, J\} \cdot r_{\max}}{V^{\mathrm{F}}(\Delta N \boldsymbol{\lambda})}\\
&\leq \frac{\mathbb{E}\left[\mathbb{1}[A] \cdot \mathbf{p}^{*\mathrm{T}} \cdot (\Delta N \boldsymbol{\lambda} - \mathbf{Z})\right] + \min\{K, J\} \cdot r_{\max}}{\Delta N \cdot V^{\mathrm{F}}} + \mathbb{P}\left(A^c\right),
\end{aligned} \tag{14}$$

where the first inequality follows from (6). Lemma A.4 serves as preparation to bound the first term in (14) from above.

Lemma A.4. *For any job type $j \in [J]$, the following holds:*

$$\mathbb{E}[\mathbb{1}[A] \cdot Z_j] \geq \Delta N \lambda_j \cdot (1 - \theta_j \Delta) - \Delta N \cdot \big(\lambda_j + 2J(\lambda_j + \delta)\big) \cdot c_1 \cdot \exp\left(-c_2 \cdot N^{\frac{1}{2}}\right).$$

Proof. First, note that:

$$\mathbb{E}[\mathbb{1}[A]\cdot Z_j] = \mathbb{E}[Z_j] - \mathbb{E}[\mathbb{1}[A^c]\cdot Z_j] \geq \Delta N\lambda_j \cdot \big(1-\theta_j\Delta\big) - \mathbb{E}[\mathbb{1}[A^c]\cdot Z_j],$$

where the inequality follows from $Z_j \geq \underline{Z}_j$, and hence $\mathbb{E}[Z_j] \geq \mathbb{E}\big[\underline{Z}_j\big] = \Delta N\lambda_j \cdot (1-\theta_j\Delta)$.

We now bound the second term from above, as follows:

$$\begin{aligned}
\mathbb{E}[\mathbb{1}[A^c]\cdot Z_j] &= \mathbb{E}\big[\mathbb{1}[A^c\cap \bar{B}_j]\cdot Z_j\big] + \mathbb{E}\big[\mathbb{1}[A^c\cap \bar{B}_j^c]\cdot Z_j\big] \\
&\leq \mathbb{E}[\mathbb{1}[\bar{B}_j]\cdot Z_j] + \mathbb{P}[A^c\cap \bar{B}_j^c]\cdot \Delta N\big(\lambda_j+\delta\big) \\
&\leq \mathbb{E}\left[\mathbb{1}\left[\bar{Z}_j > \lfloor \Delta N(\lambda_j+\delta)\rfloor\right]\cdot \bar{Z}_j\right] + \mathbb{P}[A^c]\cdot \Delta N\big(\lambda_j+\delta\big) \\
&\leq \Delta N\lambda_j \cdot \mathbb{P}\left(\bar{Z}_j \geq \lfloor \Delta N(\lambda_j+\delta)\rfloor\right) + 2J\cdot \Delta N\big(\lambda_j+\delta\big)\cdot c_1\cdot \exp\left(-c_2\cdot N^{\frac{1}{2}}\right) \\
&\leq \Delta N\cdot \big(\lambda_j + 2J(\lambda_j+\delta)\big)\cdot c_1 \cdot \exp\left(-c_2\cdot N^{\frac{1}{2}}\right).
\end{aligned}$$

In the above, the first inequality follows from the facts that $\bar{B}_j \subsetneq A^c$ and that $Z_j \leq \Delta N(\lambda_j+\delta)$ conditional on event $\bar{B}_j^c$, the second inequality follows from $Z_j \leq \bar{Z}_j$, the third inequality from (13) and Lemma A.7, and the fourth inequality from (12). □

From Lemma A.4, the following holds:

$$\begin{aligned}
\mathbb{E}\big[\mathbb{1}[A]\cdot \mathbf{p}^{*\mathrm{T}}\cdot(\Delta N\boldsymbol{\lambda}-\mathbf{Z})\big] &\leq \sum_{j\in[J]} p_j^*\cdot\Big(\Delta N\lambda_j - \mathbb{E}\big[\mathbb{1}[A]\cdot Z_j\big]\Big) \\
&\leq r_{\max}\cdot \sum_{j\in[J]}\Big(\Delta N\lambda_j\cdot\theta_j\Delta + \Delta N\cdot\big(\lambda_j+2J(\lambda_j+\delta)\big)\cdot c_1\cdot\exp\Big(-c_2\cdot N^{\frac{1}{2}}\Big)\Big),
\end{aligned} \tag{15}$$

where the second inequality follows from Lemma A.4 and the fact that $p_j^* \leq r_{\max}$ for any $j\in[J]$ (Lemma A.1). Combining (13) – (15), we obtain:

$$\begin{aligned}
\frac{NV^{\mathrm{F}}-V^{\mathrm{P}}}{NV^{\mathrm{F}}} &\leq \frac{V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda}) - \mathbb{E}\big[V^{\mathrm{F}}(\mathbf{Z})\big] + \min\{K,J\}\cdot r_{\max}}{V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda})} \\
&\leq \frac{r_{\max}}{V^{\mathrm{F}}}\cdot\sum_{j\in[J]}\left(\lambda_j\theta_j\Delta + \frac{1}{\Delta N} + \big(\lambda_j+2J(\lambda_j+\delta)\big)\cdot c_1\cdot\exp\left(-c_2\cdot N^{\frac{1}{2}}\right)\right) + 2Jc_1\cdot\exp\left(-c_2\cdot N^{\frac{1}{2}}\right) \\
&= O\left(N^{-\frac{1}{2}}\right)
\end{aligned}$$

by using $\Delta = N^{-\frac{1}{2}}$.

A.4.2 Proof of Bullet Two

Assume that the matching matrix $\mathbf{M}$ is totally unimodular. Then, the optimal solution to the matching problem solved by policy π^{P} at any time point $k\Delta$ is always integral. Therefore, rounding is not needed to perform the matching, and we have the following:

$$V^{\mathrm{P}}(\Delta) = V^{\mathrm{F}}(\mathbf{X}) \geq V^{\mathrm{F}}(\mathbf{Z}), \tag{16}$$

following the same notation as in (6).

Define a constant $c \triangleq \max_{j\in[J]}\big(\frac{1}{\lambda_j g(\delta/\lambda_j)} \vee \frac{2}{\lambda_j g(-\delta/\lambda_j)}\big)$, where $g(u) \triangleq (1+u)\ln(1+u) - u \geq 0$ is defined in Lemma A.6. Note that $g(0)=0$ and $g(u)>0$ for any $u\neq 0$. Take the interval length to be $\Delta = 2c\cdot\frac{\ln N}{N}$. Moreover, assume that N is sufficiently large so that: (i) $\Delta \leq \min_{j\in[J]}\frac{\delta}{\lambda_j\theta_j}$, and

(ii) $\frac{1}{2} \cdot g(-\delta/\lambda_j) \leq (1-\theta_j\Delta) \cdot g\left(\frac{\lambda_j\theta_j\Delta-\delta}{\lambda_j(1-\theta_j\Delta)}\right)$ for all $j \in [J]$. Finally, define the following events:

$$\begin{aligned}
&\bar{B}_j = \{Z_j > \Delta N(\lambda_j+\delta)\}, \forall\, j \in [J],\\
&\underline{B}_j = \{Z_j < \Delta N(\lambda_j-\delta)\}, \forall\, j \in [J],\\
&A = \left(\cup_{j\in[J]} \left(\bar{B}_j \cup \underline{B}_j\right)\right)^c = \{||\mathbf{Z} - \Delta N\boldsymbol{\lambda}||_\infty \leq \Delta N\delta\}.
\end{aligned}$$

By the concentration inequality for Poisson random variables (Lemma A.6), we have:

$$\begin{aligned}
&\mathbb{P}\left(\bar{B}_j\right) \leq \mathbb{P}\big(\bar{Z}_j \geq \Delta N(\lambda_j+\delta)\big) \leq \exp\left(-\Delta N\lambda_j \cdot g\big(\delta/\lambda_j\big)\right) \leq \frac{1}{N^2}\,,\\
&\mathbb{P}\left(\underline{B}_j\right) \leq \mathbb{P}\big(\underline{Z}_j \leq \Delta N(\lambda_j-\delta)\big) \leq \exp\left(-\Delta N\lambda_j \cdot (1-\theta_j\Delta) \cdot g\left(\frac{\lambda_j\theta_j\Delta-\delta}{\lambda_j(1-\theta_j\Delta)}\right)\right) \leq \frac{1}{N^2}\,.
\end{aligned}$$

In the first line, the first inequality follows from the fact that $Z_j \leq \bar{Z}_j$, the second inequality from Lemma A.6, and the third inequality from the definitions of the interval length Δ and constant c. Analogously, in the second line, the first inequality follows from the fact that $\underline{Z}_j \leq Z_j$, the second inequality from Lemma A.6, and the third inequality from the definitions of the interval length Δ and constant c and the aforementioned properties of Δ given that N is sufficiently large.

Using the union bound, we obtain:

$$\mathbb{P}\left(A^c\right) \leq \sum_{j\in[J]} \left(\mathbb{P}\left(\bar{B}_j\right) + \mathbb{P}\left(\underline{B}_j\right)\right) \leq \frac{2J}{N^2}\,. \tag{17}$$

In addition, we have $||\frac{\mathbf{Z}}{\Delta N} - \boldsymbol{\lambda}||_\infty \leq \delta$ conditioning on event A. Therefore, from (11) we get:

$$V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda}) - V^{\mathrm{F}}(\mathbf{Z}) \leq \mathbb{1}[A] \cdot \mathbf{p}^{*\mathrm{T}} \cdot (\Delta N\boldsymbol{\lambda} - \mathbf{Z}) + \mathbb{1}[A^c] \cdot V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda}).$$

Consequently,

$$\begin{aligned}
\frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{NV^{\mathrm{F}}} &= \frac{V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda}) - \mathbb{E}[V^{\mathrm{P}}(\Delta)]}{V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda})}\\
&\leq \frac{V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda}) - \mathbb{E}\big[V^{\mathrm{F}}(\mathbf{Z})\big]}{V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda})}\\
&\leq \frac{\mathbb{E}\left[\mathbb{1}[A] \cdot \mathbf{p}^{*\mathrm{T}} \cdot (\Delta N\boldsymbol{\lambda} - \mathbf{Z})\right]}{\Delta N \cdot V^{\mathrm{F}}} + \mathbb{P}\left(A^c\right),
\end{aligned} \tag{18}$$

where the first inequality follows from (16). Analogous to Lemma A.4, Lemma A.5 helps bound the first term in (18) from above.

Lemma A.5. *For any job type $j \in [J]$, the following holds:*

$$\mathbb{E}[\mathbb{1}[A] \cdot Z_j] \geq \Delta N\lambda_j \cdot (1-\theta_j\Delta) - \Delta\lambda_j - 2J \cdot (\lambda_j+\delta) \cdot \frac{\Delta}{N}$$

when N is larger than a certain constant.

Proof. First, note that:

$$\mathbb{E}[\mathbb{1}[A] \cdot Z_j] = \mathbb{E}[Z_j] - \mathbb{E}[\mathbb{1}[A^c] \cdot Z_j] \geq \Delta N\lambda_j \cdot \big(1-\theta_j\Delta\big) - \mathbb{E}[\mathbb{1}[A^c] \cdot Z_j],$$

where the inequality follows from $Z_j \geq \underline{Z}_j$, and hence $\mathbb{E}[Z_j] \geq \mathbb{E}\big[\underline{Z}_j\big] = \Delta N\lambda_j \cdot (1-\theta_j\Delta)$.

We now bound the second term from above, as follows:

$$\begin{aligned}\mathbb{E}[\mathbb{1}[A^c]\cdot Z_j] &= \mathbb{E}\big[\mathbb{1}[A^c\cap \bar{B}_j]\cdot Z_j\big] + \mathbb{E}\big[\mathbb{1}[A^c\cap \bar{B}_j^c]\cdot Z_j\big] \\ &\leq \mathbb{E}[\mathbb{1}[\bar{B}_j]\cdot Z_j] + \mathbb{P}[A^c\cap \bar{B}_j^c]\cdot \Delta N\big(\lambda_j+\delta\big) \\ &\leq \mathbb{E}\left[\mathbb{1}[\bar{Z}_j > \Delta N(\lambda_j+\delta)]\cdot \bar{Z}_j\right] + \mathbb{P}[A^c]\cdot \Delta N\big(\lambda_j+\delta\big) \\ &\leq \Delta N\lambda_j\cdot \mathbb{P}\left[\bar{Z}_j \geq \lfloor \Delta N(\lambda_j+\delta)\rfloor\right] + 2J\cdot\big(\lambda_j+\delta\big)\cdot\frac{\Delta}{N} \\ &\leq \Delta\lambda_j + 2J\cdot\big(\lambda_j+\delta\big)\cdot\frac{\Delta}{N}.\end{aligned}$$

In the above, the first inequality follows from $\bar{B}_j \subseteq A^c$ and $Z_j \leq \Delta N(\lambda_j+\delta)$ conditional on event $\bar{B}_j^c$. The second inequality follows from $Z_j \leq \bar{Z}_j$. The third inequality follows from (17) and Lemma A.7. Finally, let $\tilde{\delta}$ be a positive constant satisfying $\tilde{\delta} < \delta$ and $c \geq \max_{j\in[J]} \frac{1}{2\cdot\lambda_j g(\tilde{\delta}/\lambda_j)}$. The constant $\tilde{\delta}$ exists by the definition of c. The fourth inequality follows from the facts that $\Delta N\big(\lambda_j+\tilde{\delta}\big) \leq \lfloor \Delta N(\lambda_j+\delta)\rfloor$ when N is sufficiently large (specifically, when $\Delta N \geq 1/(\delta-\tilde{\delta})$) and that

$$\mathbb{P}\left(\bar{Z}_j \geq \Delta N(\lambda_j+\tilde{\delta})\right) \leq \exp\left(-\Delta N\lambda_j\cdot g\big(\tilde{\delta}/\lambda_j\big)\right) \leq \frac{1}{N}$$

by the definition of $\tilde{\delta}$ and Lemma A.6. □

From Lemma A.5, the following holds:

$$\begin{aligned}\mathbb{E}\big[\mathbb{1}[A]\cdot \mathbf{p}^{*\mathrm{T}}\cdot(\Delta N\boldsymbol{\lambda}-\mathbf{Z})\big] &\leq \sum_{j\in[J]} p_j^*\cdot\Big(\Delta N\lambda_j - \mathbb{E}\big[\mathbb{1}[A]\cdot Z_j\big]\Big) \\ &\leq r_{\max}\cdot\sum_{j\in[J]}\left(\Delta N\lambda_j\cdot\theta_j\Delta + \Delta\lambda_j + 2J\cdot(\lambda_j+\delta)\cdot\frac{\Delta}{N}\right),\end{aligned} \tag{19}$$

where the second inequality follows from Lemma A.5 and the fact that $p_j^* \leq r_{\max}$ for any $j\in[J]$ (Lemma A.1). Combining (17) - (19), we obtain:

$$\begin{aligned}\frac{NV^{\mathrm{F}}-V^{\mathrm{P}}}{NV^{\mathrm{F}}} &\leq \frac{V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda}) - \mathbb{E}\big[V^{\mathrm{F}}(\mathbf{Z})\big]}{V^{\mathrm{F}}(\Delta N\boldsymbol{\lambda})} \\ &\leq \frac{r_{\max}}{V^{\mathrm{F}}}\cdot\sum_{j\in[J]}\left(\lambda_j\theta_j\Delta + \frac{\lambda_j}{N} + 2J\cdot(\lambda_j+\delta)\cdot\frac{1}{N^2}\right) + \frac{2J}{N^2} \\ &= O\left(\frac{\ln N}{N}\right)\end{aligned}$$

by using $\Delta = \Theta\left(\frac{\ln N}{N}\right)$.

Lemma A.6 (Concentration Inequality for Poisson Random Variables)**.** *Let $X \sim Poisson(\lambda)$ be a Poisson random variable with a mean value of $\lambda > 0$. We have*

$$\mathbb{P}(X \geq \lambda(1+u)) \leq \exp(-\lambda g(u))$$

for any $u \geq 0$, and

$$\mathbb{P}(X \leq \lambda(1-u)) \leq \exp(-\lambda g(-u))$$

for any $0 \leq u < 1$, where $g(u) = (1+u)\ln(1+u) - u$.

Proof. We first prove the first inequality. For any $t \geq 0$, we have

$$\begin{aligned}\mathbb{P}(X \geq \lambda(1+u)) &= \mathbb{P}\Big[\exp(tX) \geq \exp(t\lambda(1+u))\Big] \\ &\leq \mathbb{E}\Big[\exp(tX)\Big]\exp(-t\lambda(1+u)) \\ &= \exp\big(\lambda(e^t-1) - t\lambda(1+u)\big)\end{aligned}$$

where the inequality follows from Markov's inequality and the second equality from the moment generating function of the Poisson distribution. It turns out that the right-hand side of the second equality is minimized by setting $t = \ln(1+u)$, in which case the right-hand side simplifies to $\exp(-\lambda g(u))$. The second inequality can be proven similarly. □

Lemma A.7. *Let $X \sim$ Poisson(λ) be a Poisson random variable with a mean value of $\lambda > 0$ and $z \in \mathbb{N}$ be a nonnegative integer. The following holds:*

$$\mathbb{E}\left[X \cdot \mathbb{1}[X > z]\right] = \sum_{k=z+1}^{\infty} k \cdot \frac{\lambda^k e^{-\lambda}}{k!} = \lambda \cdot \sum_{k=z}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = \lambda \cdot \mathbb{P}(X \geq z).$$

A.5 Proof of Lemma 3.4

Let x_m^P represent the long-run average matching rate of match m under policy π^{P}, and let $\mathbf{x}^P = (x_m^P)_{m\in[K]} \in \mathbb{R}_+^K$. It follows that $\mathbf{M}\mathbf{x}^P \leq N\boldsymbol{\lambda}$ because the rate of jobs joining matches must be less than the rate of job arrivals. Therefore, $\frac{\mathbf{x}^P}{N}$ is an optimal solution to the following LP (20):

$$\begin{aligned}\max_{\mathbf{x}\in\mathbb{R}_+^K} \quad & \mathbf{r}^{\mathrm{T}}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{M}\mathbf{x} \leq \boldsymbol{\lambda}, \\ & \mathbf{r}^{\mathrm{T}}\mathbf{x} \leq \frac{\mathbf{r}^{\mathrm{T}}\mathbf{x}^P}{N},\end{aligned} \tag{20}$$

because $\frac{\mathbf{x}^P}{N}$ is feasible and obtains the highest possible objective value. On the other hand, since the optimal value of the fluid relaxation (1) is V^{F}, a solution is optimal to (1) if and only if it is optimal to the following LP (21):

$$\begin{aligned}\max_{\mathbf{x}\in\mathbb{R}_+^K} \quad & \mathbf{r}^{\mathrm{T}}\mathbf{x} \\ \text{s.t.} \quad & \mathbf{M}\mathbf{x} \leq \boldsymbol{\lambda}, \\ & \mathbf{r}^{\mathrm{T}}\mathbf{x} \leq V^{\mathrm{F}}.\end{aligned} \tag{21}$$

By Lipschitz continuity of optimal solutions of LPs with respect to constraints' right-hand side (i.e., Lemma A.8 below), there exists an optimal solution $\mathbf{x}^*$ to the fluid relaxation (1) such that

$$\left\|\frac{\mathbf{x}^P}{N} - \mathbf{x}^*\right\|_\infty \leq C_4 \left|V^{\mathrm{F}} - \frac{\mathbf{r}^{\mathrm{T}}\mathbf{x}^P}{N}\right| = C_4 \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}$$

for some constant $C_4 > 0$. The rest of Lemma 3.4 follows from the fact that policy π^{P} is asymptotically optimal (i.e., Lemmas 3.2 and 3.3 hold).

Lemma A.8 (Theorem 2.4 of Mangasarian and Shiau 1987)**.** *Consider the following LP:*

$$P(\mathbf{b}) = \max\left\{\mathbf{r}^{\mathrm{T}}\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\right\}.$$

There exists a constant κ such that, for any vectors $\mathbf{b}, \hat{\mathbf{b}} \in \mathbb{R}_+^n$ and any optimal solution $\mathbf{x}$ to $P(\mathbf{b})$, there exists an optimal solution $\hat{\mathbf{x}}$ to $P(\hat{\mathbf{b}})$ such that $\|\mathbf{x} - \hat{\mathbf{x}}\|_\infty \leq \kappa\|\mathbf{b} - \hat{\mathbf{b}}\|_\infty$.

A.6 Proof of Corollary 3.5

Let $N_m(t)$ denote the number of type-m matches completed by time t; we have $A_{jm}(t) = M_{jm}N_m(t)$. Therefore, the following holds:

$$\mathbb{E}\left[\lim_{t\to\infty}\frac{A_{jm}(t)}{A_j(t)}\right] = \mathbb{E}\left[\lim_{t\to\infty}\frac{A_{jm}(t)}{t}\frac{t}{A_j(t)}\right] = \frac{M_{jm}}{N\lambda_j}\cdot\mathbb{E}\left[\lim_{t\to\infty}\frac{N_m(t)}{t}\right] = \frac{M_{jm}}{N\lambda_j}\cdot\lim_{t\to\infty}\mathbb{E}\left[\frac{N_m(t)}{t}\right].$$

Specifically, the second equality follows from $A_{jm}(t) = M_{jm}N_m(t)$ and the fact that $\lim_{t\to\infty} A_j(t)/t = N\lambda_j$ almost surely by the strong law for renewal processes (Proposition 6.2 in Ross and Peköz 2023). The third equality follows from the generalized dominated convergence theorem (Theorem 19 in Royden and Fitzpatrick 2010) because $N_m(t) \le A_{j'}(t)$ and $\lim_{t\to\infty}\mathbb{E}\left[\frac{A_{j'}(t)}{t}\right] = \mathbb{E}\left[\lim_{t\to\infty}\frac{A_{j'}(t)}{t}\right] = N\lambda_{j'} < \infty$, where j' denotes a job type required by match m.

The above result, together with Lemma 3.4, yields Corollary 3.5 by setting $C_6 \triangleq C_4\cdot\max_{j\in[J],m\in[K]}\frac{M_{jm}}{\lambda_j}$.

A.7 Proof of Lemma 4.3

Let x_m denote the long-run average rate at which agent i performs match m locally, and s_j the long-run average rate at which type-j jobs submitted by agent i are successfully matched at the shared pool. The vectors $\mathbf{x} = (x_m)$ and $\mathbf{s} = (s_j)$ are feasible to (3) because, for each job type, the combined rates of jobs used for local matching and jobs matched at the shared pool can not exceed the job arrival rate. Moreover, since every type-j job matched at the shared pool yields agent i an expected payoff of p_j^*, irrespective of the match type, the agent's long-run average payoff is $\mathbf{r}^{\mathrm{T}}\mathbf{x} + \mathbf{p}^{*\mathrm{T}}\mathbf{s}$, which is no larger than the optimal value of (3).

A.8 Proof of Lemma 4.4

Suppose all agents fully submit their jobs to the shared pool. Without loss of generality, assume that neither the shared pool nor any agent holds any job at time zero. Let $N_m(t)$ denote the number of type-m matches formed by the shared pool by time t, $x_m^P = \lim_{t\to\infty}\mathbb{E}[N_m(t)]/t$ the long-run average matching rate of match m at the shared pool, and $S_{ijm}(t)$ the number of type-j jobs submitted by agent i that participate in match m by time t. We have the following equation:

$$\mathbb{E}\big[S_{ijm}(t)\big] = \frac{\lambda_{ij}}{N\lambda_j}\cdot M_{jm}\cdot\mathbb{E}\big[N_m(t)\big], \tag{22}$$

where $\lambda_j = \sum_{i\in[N]}\lambda_{ij}/N$ denote the normalized aggregate arrival rate of type-j jobs. To interpret (22), note that a total of $M_{jm}N_m(t)$ type-j jobs participate in match m by time t, and each originates from agent i with probability $\frac{\lambda_{ij}}{N\lambda_j}$.

Let $\mathbf{x}^* = \big(x_m^*\big)_{m\in[K]}$ be an optimal solution to (1). The long-run average payoff of agent i, denoted by V_i, satisfies the following:

$$\begin{aligned} V_i &= \lim_{t\to\infty} \mathbb{E}\left[\frac{\sum_{j\in[J]} p_j^* \cdot \sum_{m\in[K]} S_{ijm}(t)}{t}\right] \\ &= \sum_{j\in[J]} p_j^* \cdot \frac{\lambda_{ij}}{\lambda_j} \sum_{m\in[K]} M_{jm} \cdot \frac{x_m^P}{N} \\ &\geq \sum_{j\in[J]} p_j^* \cdot \frac{\lambda_{ij}}{\lambda_j} \sum_{m\in[K]} M_{jm} \cdot \left(x_m^* - C_4 \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}\right) \\ &\geq \sum_{j\in[J]} p_j^* \cdot \frac{\lambda_{ij}}{\lambda_j} \sum_{m\in[K]} M_{jm} \cdot x_m^* - C_7 \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N} \\ &= \boldsymbol{\lambda}_i^{\mathrm{T}} \mathbf{p}^* - C_7 \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}. \end{aligned}$$

In the above, the first equality follows from the fact that every matched type-j job yields an expected payoff of p_j^*, the second equality follows from (22) and the definition of x_m^P, the first inequality from Lemma 3.4, and the second inequality from the definition of the constant C_7, Lemma A.1, and Assumption 2.1. Finally, the third equality follows from the complimentary slackness condition of (1), that is, $p_j^* \cdot (\mathbf{m}_j^{\mathrm{T}} \mathbf{x}^* - \lambda_j) = 0$ for all $j \in [J]$, where $\mathbf{m}_j^{\mathrm{T}}$ denotes the j-th row of matrix $\mathbf{M}$.

B Proof in Section 5

B.1 Proof of Proposition 5.1

The proof is identical to the proof of Lemma 5.2 in Appendix B.3 with $\epsilon = 0$.

B.2 More Details of Example 5.1

In this section, we derive agent one's long-run average expected payoff under the alternative policy. Under this policy, agent one possesses either zero or one type-two jobs, and the dynamics of the type-two jobs follow a birth-death process. Let μ_i denote the stationary probability of having $i \in \{0, 1\}$ type-two jobs; these probabilities satisfy the following balance equation:

$$\lambda \cdot \mu_0 = (1 + \theta) \cdot \mu_1.$$

Since $\mu_0 + \mu_1 = 1$, we have:

$$\mu_0 = \frac{1+\theta}{1+\lambda+\theta}, \quad \mu_1 = \frac{\lambda}{1+\lambda+\theta}.$$

The long-run average payoff under the alternative policy is given by:

$$\begin{aligned} V^{\mathrm{alt}} &= 1 \cdot \Big(\mu_0 \cdot w \cdot (1-\epsilon) + \mu_1\Big) + \lambda \cdot \mu_1 \cdot \epsilon \cdot \frac{w}{\lambda} \\ &= w + (1-w) \cdot \mu_1 + \epsilon \cdot w \cdot \big(\mu_1 - \mu_0\big) \\ &= w + (1-w) \cdot \frac{\lambda}{1+\lambda+\theta} + \epsilon \cdot w \cdot \frac{\lambda - 1 - \theta}{1+\lambda+\theta}. \end{aligned}$$

For the first equality, note that an arriving type-one job is either matched internally with probability μ_1, yielding a reward of one, or submitted with probability μ_0, generating an expected payoff of $w \cdot (1-\epsilon)$. Conversely, an arriving type-two job is submitted with probability μ_1, generating an expected payoff of $\epsilon \cdot w/\lambda$.

B.3 Proof of Lemma 5.2

B.3.1 Setting the Values of $C_8(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and $C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$

Let $p^*_{\min} \triangleq \min_{j\in\mathcal{N}_+} p^*_j > 0$ denote the minimum marginal value of over-demanded jobs, $M_{\max} \triangleq \max_{m\in[K]} \sum_{j\in\mathcal{N}_+} M_{jm}$ the maximum number of over-demanded jobs required in a match, and $\Delta r_{\min} \triangleq \min_{m\in\mathcal{M}_0} \left\{\sum_{j\in[J]} M_{jm}\, p^*_j - r_m\right\} > 0$ the smallest gap for suboptimal matchings. Define the constant $\nu \triangleq \min_{j\in[J]} \frac{\theta_j}{\theta_j + \sum_{\ell\in[J]} C_\ell\, \lambda_\ell} \in (0,1)$, where the constants $\{C_\ell\}_{\ell\in[J]}$ are specified in Assumption 2.1. We define $C_8(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and $C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ as follows:

$$C_8(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M}) = 1 - \frac{1}{8\, r_{\max}} \min\left\{2\Delta r_{\min},\, \nu p^*_{\min}\right\}$$
$$C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M}) = \frac{1}{8 M_{\max}} \min\left\{2\Delta r_{\min},\, \nu p^*_{\min}\right\}.$$

For simplicity, we denote $C_8(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and $C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ as C_8 and C_9, respectively. In Lemma B.1, we establish basic properties of these two constants.

Lemma B.1. *We have that $C_8 \geq 7/8$ and $C_9 \leq p^*_{\min}/8$.*

Proof. The result follows directly from the observations that $p^*_{\min} \leq r_{\max}$ by Lemma A.1, $\nu < 1$ by definition, and $M_{\max} \geq 1$ as each match must contain at least one over-demanded job due to the constraints of (2). $\square$

B.3.2 Proof of Lemma 5.2

We consider a representative agent i in the mean-field problem. Since every agent solves a Markov decision problem in the mean-field problem, it is without loss of optimality to assume that agent i takes actions only upon job arrivals or departures. Suppose agent i implements a policy π under which, in the long-run average, she retains a positive fraction $\rho > 0$ of type-j jobs, where type j is over-demanded (that is, $p^*_j > 0$), upon their arrival. We use a coupling argument to show that policy π is strictly suboptimal.

An Alternative Policy $\tilde{\pi}$ Consider an alternative policy $\tilde{\pi}$ that mimics policy π but always submits type-j jobs upon their arrival. We call these type-j jobs on which π and $\tilde{\pi}$ differ the *marked jobs*. This difference may affect decisions at subsequent time points. Specifically, when policy π matches an incoming job, if one of the matched participants is a marked job, policy $\tilde{\pi}$ cannot replicate this match. We call such matches *affected matches*. In other words, a match is affected if and only if it involves a marked job. In these cases, policy $\tilde{\pi}$ submits the incoming job and all other participant jobs at the moment when π executes an affected match. Note that for jobs not involved in affected matches, policies π and $\tilde{\pi}$ make identical decisions. Therefore, the payoff difference between the two policies arises solely from the marked jobs and affected matches.

Step One: Lack of Incentives for Forming Suboptimal Matches Let m be an affected match. We first show that $m \in \mathcal{M}_+$, that is, $\sum_{j\in\mathcal{N}_+} M_{jm}\, p^*_j = r_m$. Note that executing match m generates a reward of r_m, whereas submitting all participating jobs to the shared pool yields a payoff of at least $w_{\min} \sum_{j\in\mathcal{N}_+} M_{jm} \left(p_j^* - \epsilon\right)$. Suppose instead that $m \in \mathcal{M}_0$. Then:

$$\begin{aligned}
w_{\min} \sum_{j\in\mathcal{N}_+} M_{jm} \left(p_j^* - \epsilon\right) - r_m &= w_{\min} \sum_{j\in\mathcal{N}_+} M_{jm}\, p_j^* - \epsilon\, w_{\min} \sum_{j\in\mathcal{N}_+} M_{jm} - r_m \\
&\geq w_{\min} \left(r_m + \Delta r_{\min}\right) - \epsilon\, M_{\max} - r_m \\
&\geq w_{\min}\, \Delta r_{\min} - \epsilon\, M_{\max} - (1 - w_{\min}) \cdot r_{\max} \\
&> \Delta r_{\min}/2 - \Delta r_{\min}/4 - \Delta r_{\min}/4 \\
&= 0,
\end{aligned}$$

where the first inequality follows from the definitions of $\Delta r_{\min}$ and $M_{\max}$, and the third inequality follows from $w_{\min} \geq C_8 > \frac{1}{2}$ (second inequality due to Lemma B.1), $\epsilon \leq C_9$, and the definitions of C_8 and C_9. Therefore, if $m \in \mathcal{M}_0$, agent i is strictly better off submitting all participating jobs and would never internally execute match m.

Step Two: Comparing Payoffs from Marked Jobs and Affected Matches under Two Policies Consider a representative marked job of type $j \in \mathcal{N}_+$ and its affected match under policy π, provided that the job is matched. We compare the expected payoffs under the two policies by analyzing two scenarios:

1. With probability
$$\frac{\theta_j}{\theta_j + \sum_{\ell\in[J]} \lambda_{i\ell}} \;\geq\; \nu,$$
the marked job expires before a new job arrives under policy π. The inequality follows from Assumption 2.1 and the definition of ν. In this scenario, the job expires unused and generates zero reward under policy π. In contrast, under policy $\tilde{\pi}$, the same job yields a reward of at least $w_{\min} \cdot \left(p_j^* - \epsilon\right)$ through submission.
2. With probability at most $1 - \nu$, the job is matched under policy π and generates a reward of r_m, where m represents the corresponding affected match. The policy $\tilde{\pi}$ instead submits all participating jobs of match m and earns at least $w_{\min} \sum_{j\in\mathcal{N}_+} M_{jm} \cdot \left(p_j^* - \epsilon\right)$.

Let m denote the affected match if the marked job is matched under policy π. Note that $m \in \mathcal{M}_+$ by Step One. Let $V^{\pi}(j)$ and $V^{\tilde{\pi}}(j)$ denote the expected payoffs from a type-j marked job and its affected match under policies π and $\tilde{\pi}$, respectively. The payoff difference between the two policies satisfies:

$$\begin{aligned}
V^{\tilde{\pi}}(j) - V^{\pi}(j) &\geq \nu \cdot w_{\min} \cdot \left(p_j^* - \epsilon\right) + (1-\nu) \cdot \left(w_{\min} \sum_{j\in\mathcal{N}_+} M_{jm} \cdot \left(p_j^* - \epsilon\right) - r_m \right) \\
&\geq \nu \cdot w_{\min} \cdot \left(p_{\min}^* - \epsilon\right) - \epsilon \cdot M_{\max} - r_{\max} \cdot (1 - w_{\min}) \\
&> \nu p_{\min}^*/4 - \nu p_{\min}^*/8 - \nu p_{\min}^*/8 \\
&= 0,
\end{aligned}$$

where the second inequality follows from definitions of $p_{\min}^*$ and $M_{\max}$ and the facts that $m \in \mathcal{M}_+$ and $r_m \leq r_{\max}$. The third inequality follows from $w_{\min} \geq C_8 > \frac{1}{2}$ (the second inequality by Lemma B.1), $\epsilon \leq C_9 < p_{\min}^*/2$ (the second inequality by Lemma B.1), and the definitions of C_8 and C_9. Therefore, policy $\tilde{\pi}$ yields strictly higher rewards than π for marked jobs and affected matches, which implies that policy π is strictly suboptimal.

B.4 Proof of Lemma 5.3

B.4.1 Setting the Values of $C_{10}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and $C_{11}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$

We first define the following constants:

$$\nu_0 = \prod_{j \in \mathcal{N}_0} \exp\big(-C_j \lambda_j / \theta_j\big)$$
$$\rho_1 = \frac{\nu_0}{4} \cdot \frac{\sum_{j \in \mathcal{N}_+} \lambda_j}{\sum_{j \in \mathcal{N}_0} \lambda_j}$$
$$\rho_2 = \frac{\min_{j \in \mathcal{N}_0} \theta_j}{\max_{j \in \mathcal{N}_+} C_j \lambda_j}$$

where the constants $\{C_j\}$ are specified in Assumption 2.1. Using these parameters, we define $C_{10}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and $C_{11}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ as follows:

$$C_{11}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M}) = \frac{r_{\max}}{\rho_1\, \rho_2}$$
$$C_{10}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M}) = \max\left\{C_8(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M}), 1 - \frac{C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})}{C_{11}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})}\right\}$$

where $C_8(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ and $C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ are constants defined in Lemma 5.2. For simplicity, we denote $C_8(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$, $C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$, $C_{10}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$, and $C_{11}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$ as C_8, C_9, C_{10}, and C_{11}, respectively. Note that by the definition of C_{10} and C_{11}, we have:

$$C_8 \le C_{10} < 1$$
$$(1 - C_{10}) \cdot C_{11} \le C_9\,.$$

Therefore, there always exist values of $w_{\min}$ and ϵ satisfying $w_{\min} \ge C_{10}$ and $(1-w_{\min})\, C_{11} \le \epsilon \le C_9$.

B.4.2 Proof of Lemma 5.3

We consider a representative agent i in the mean-field problem. Since every agent solves a Markov decision problem in the mean-field problem, it is without loss of optimality to assume that agent i takes actions only upon job arrivals or departures.

Furthermore, since $w_{\min} \ge C_{10} \ge C_8$ and $\epsilon \le C_9$ by assumption, Lemma 5.2 implies that no agent retains over-demanded jobs for internal matching. In other words, all agents either submit or internally match over-demanded jobs immediately upon their arrival. Since each match requires at least two jobs (that is, $\sum_{j \in [J]} M_{jm} \ge 2$ for all $m \in [K]$), it suffices to show that agent i never retains under-demanded jobs in anticipation of future arrivals of over-demanded jobs. This is because the full submission of under-demanded jobs, together with Lemma 5.2, ensures the full submission of over-demanded jobs, and hence of all jobs.

Suppose that agent i implements a policy π under which she performs internal matching at a positive long-run average rate $c > 0$. We will show that policy π is strictly suboptimal. By Lemma 5.2, every internal match performed by agent i is a linking match, involving exactly one over-demanded job and one or more under-demanded jobs. Moreover, each such match is performed immediately upon the arrival of the over-demanded job.

An Alternative Policy $\tilde{\pi}$ To prove that policy π is strictly suboptimal, we consider an alternative policy $\tilde{\pi}$ that submits all jobs immediately upon their arrival but otherwise closely follows policy π. We couple the system dynamics of the two policies so that jobs in the two systems arrive and depart at the same time, even after being converted into credits by submission. This coupling is feasible because under-demanded jobs and their corresponding credits expire at identical rates.

When policy π internally executes a linking matching m involving an arriving over-demanded job and M_{jm} under-demanded jobs of type j for each $j \in \mathcal{N}_0$, policy $\tilde{\pi}$ submits the arriving over-demanded job and pairs it with M_{jm} credits of type j for each $j \in \mathcal{N}_0$. This action is feasible under the coupling described above.

Note that the difference in rewards between policies π and $\tilde{\pi}$ arises from two sources:

- Relative Benefit: For every internal linking match m performed by policy π, policy π obtains a reward of r_m. In contrast, policy $\tilde{\pi}$ submits the over-demanded job and pairs it with the corresponding credits. This yields an expected payoff of at least $w_{\min} r_m$, because the over-demanded job is matched in the shared pool with probability at least $w_{\min}$. Therefore, policy π generates higher rewards from internal matches as jobs submitted to the shared pool might expire before being matched.

- Relative Loss: Since policy π executes linking matches immediately upon the arrival of participating over-demanded jobs, some under-demanded jobs must wait for subsequent arrivals of over-demanded jobs and hence will be inevitably wasted. In contrast, these under-demanded jobs are submitted to the shared pool and become lotteries once they expire under policy $\tilde{\pi}$, each generating a positive reward.

In the following, we provide a lower bound on the loss of policy π relative to $\tilde{\pi}$ (Step Two) and an upper bound on the relative benefit (Step Three). Ultimately, we demonstrate that policy $\tilde{\pi}$ achieves a strictly larger overall payoff (Step Four).

Step One: Value of Lotteries As preparation, we first bound the value of a lottery from below. In the mean-field problem, since the shared pool remains in steady state, each lottery receives the same expected value from the collected taxes. Lemma B.2 shows that this value is at least $\rho_1 \epsilon$ when the number of agents N exceeds a constant depending only on the problem primitives. The proof is provided in Appendix B.4.3.

Lemma B.2. *In the mean-field problem, each lottery receives an expected value of at least $\rho_1 \epsilon$ from the collected taxes, provided that the number of agents N exceeds a constant depending only on the problem primitives. Here, ρ_1 is defined in Appendix B.4.1.*

Step Two: Cost of Wasted Under-Demanded Jobs under Policy π Recall that agent i internally performs linking matches at rate $c > 0$ under policy π. We apply the approach of Aouad and Sarıtaç (2022) to derive a lower bound on the rate at which under-demanded jobs are wasted while waiting to be matched. Since policy π performs an internal linking match upon the arrival of some over-demanded job (referred to as a passive job), some under-demanded job must actively wait to be matched (referred to as an active job). Following the proof of Lemma 1 in Aouad and Sarıtaç (2022), the rate at which under-demanded jobs are wasted due to waiting for future arrivals of over-demanded jobs to perform linking matches, which we denote by ρ_e, satisfies:

$$\rho_e \geq c \cdot \frac{\min_{j \in \mathcal{N}_0} \theta_j}{\max_{j \in \mathcal{N}_+} \lambda_{ij}} \geq c \cdot \frac{\min_{j \in \mathcal{N}_0} \theta_j}{\max_{j \in \mathcal{N}_+} C_j \lambda_j} = c \cdot \rho_2.$$

Consequently, under-demanded jobs expire and are wasted at a positive rate ρ_e under policy π. In contrast, these under-demanded jobs are submitted and become lotteries under policy $\tilde{\pi}$, which, by Step One, yield the agent a reward of at least

$$(c \cdot \rho_2) \cdot (\rho_1 \cdot \epsilon) = c\,\epsilon \cdot \frac{r_{\max}}{C_{11}}$$

per unit of time.

Step Three: Benefit of Internal Matching under Policy π Whenever policy π internally performs a linking match m and obtains a reward $r_m = p_j^*$—where j denotes the type of the over-demanded job in match m—policy $\tilde{\pi}$ instead submits the newly arriving over-demanded type-j job and pairs it with credits corresponding to the under-demanded jobs involved in match m, which yields an expected payoff of at least $w_{\min} \cdot p_j^*$. Therefore, the benefit of internal matching for policy π relative of $\tilde{\pi}$ is at most:

$$c \cdot (1 - w_{\min}) \cdot \max_{j \in \mathcal{N}_+} p_j^* \leq c \cdot (1 - w_{\min}) \cdot r_{\max},$$

where the inequality follows from Lemma A.1.

Step Four: Suboptimality of Policy π Since

$$c\,\epsilon \cdot \frac{r_{\max}}{C_{11}} - c \cdot (1 - w_{\min}) \cdot r_{\max} \geq 0,$$

where the inequality follows from $\epsilon \geq (1 - w_{\min}) \cdot C_{11}$, policy $\tilde{\pi}$ has a strictly higher long-run average payoff than π.

B.4.3 Proof of Lemma B.2

From Lemma 5.2, agents submit over-demanded jobs in isolation when they do not hold any under-demanded jobs or credits. We consider an alternative system in which each agent:

1. Submits all under-demanded jobs to the shared pool and never uses credits.
2. Submits all over-demanded jobs. However, a submitted over-demanded job is taxed by ϵ upon being matched only if the agent holds no credits of any job type at the time of submission.

It is straightforward to verify that, regardless of the agents' strategies in the original system, at any time t, both the number of credits and the number of lotteries in the alternative system are larger than those in the original system, and the number of the collected ϵ-values in the alternative system is smaller. Therefore, it suffices to show that each lottery in the alternative system yields an expected value of at least $\rho_1 \epsilon$, which we do now.

Let $Y(\infty)$ and $Z(\infty)$ denote, respectively, the number of collected ϵ-values and the number of lotteries in the steady state of the shared pool in the alternative system. We first establish in Lemma B.3 that both $Y(\infty)$ and $Z(\infty)$ follow Poisson distributions with mean values in the order of $\Theta(\Delta N)$. We provide the proof at the end of this section.

Lemma B.3. *In the steady state of the alternative system, the following holds:*

1. *The number of collected ϵ-values, $Y(\infty)$, follows a Poisson distribution with mean value $a\,\Delta N$, where $\frac{\nu_0}{2} \sum_{j \in \mathcal{N}_+} \lambda_j = \underline{a} \leq a \leq \bar{a} = \sum_{j \in \mathcal{N}_+} \lambda_j$.*
2. *The number of lotteries, $Z(\infty)$, follows a Poisson distribution with mean value $b\,\Delta N$, where $b = \sum_{j \in \mathcal{N}_0} \lambda_j$.*

In the mean-field problem, the shared pool remains in steady state at any matching epoch. Moreover, the taxes collected in a matching epoch are fully allocated to lotteries in that epoch whenever a lottery exists. Therefore, the expected number of ϵ-values each lottery receives is given by the following expression

$$\begin{aligned}
&\mathbb{E}\left(\frac{Y(\infty)}{Z(\infty)} \mathbb{1}[Z(\infty) \geq 1]\right) \\
\geq\, &\mathbb{E}\left(\frac{Y(\infty)}{\mathbb{E}[Z(\infty)]} \mathbb{1}[Z(\infty) \geq 1] - \frac{Y(\infty)}{\mathbb{E}[Z(\infty)]^2} \left(Z(\infty) - \mathbb{E}[Z(\infty)]\right) \mathbb{1}[Z(\infty) \geq 1]\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathbb{E}[Y(\infty)]}{\mathbb{E}[Z(\infty)]} - \mathbb{E}\left(\frac{Y(\infty)}{\mathbb{E}[Z(\infty)]}\mathbb{1}[Z(\infty)=0]\right) - \mathbb{E}\left(\frac{Y(\infty)}{\mathbb{E}[Z(\infty)]^2}\left(Z(\infty)-\mathbb{E}[Z(\infty)]\right)\right)\\
&\qquad + \mathbb{E}\left(\frac{Y(\infty)}{\mathbb{E}[Z(\infty)]^2}\left(Z(\infty)-\mathbb{E}[Z(\infty)]\right)\mathbb{1}[Z(\infty)=0]\right)\\
&= \frac{\mathbb{E}[Y(\infty)]}{\mathbb{E}[Z(\infty)]} - \frac{\mathbb{C}\mathrm{ov}[Y(\infty),Z(\infty)]}{\mathbb{E}[Z(\infty)]^2} - 2\,\frac{\mathbb{E}\big[Y(\infty)\,\mathbb{1}[Z(\infty)=0]\big]}{\mathbb{E}[Z(\infty)]}\\
&\geq \frac{a}{b} - \underbrace{\frac{\mathbb{C}\mathrm{ov}[Y(\infty),Z(\infty)]}{\mathbb{E}[Z(\infty)]^2}}_{(a)} - 2\,\underbrace{\frac{\mathbb{E}\big[Y(\infty)\,\mathbb{1}[Z(\infty)=0]\big]}{\mathbb{E}[Z(\infty)]}}_{(b)}\,,
\end{aligned}
$$

where the first inequality follows from Lemma B.4, the second equality follows from the fact that $Z(\infty)\mathbb{1}[Z(\infty)=0]=0$, and the second inequality from Lemma B.3.

We next bound the terms (a) and (b) from above using Cauchy–Schwarz inequality. For (a), we have

$$
\frac{\mathbb{C}\mathrm{ov}[Y(\infty),Z(\infty)]}{\mathbb{E}[Z(\infty)]^2} \leq \frac{\sqrt{\mathbb{V}\mathrm{ar}(Y(\infty))\mathbb{V}\mathrm{ar}(Z(\infty))}}{\mathbb{E}[Z(\infty)]^2} \leq \sqrt{\frac{\bar{a}}{b^3}\cdot\frac{1}{\Delta N}}\,,
$$

where the second inequality follows from Lemma B.3. For (b), we have

$$
\frac{\mathbb{E}\big[Y(\infty)\,\mathbb{1}[Z(\infty)=0]\big]}{\mathbb{E}[Z(\infty)]} \leq \frac{\sqrt{\mathbb{E}\left[Y(\infty)^2\right]\,\mathbb{P}\left[Z(\infty)=0\right]}}{\mathbb{E}[Z(\infty)]} \leq \frac{\sqrt{2\bar{a}}}{b}\cdot\frac{e^{-\frac{b}{2}\Delta N}}{\sqrt{\Delta N}}\,,
$$

where the second inequality also follows from Lemma B.3.

Therefore, each lottery yields an expected value of at least

$$
\begin{aligned}
&\mathbb{E}\left(\frac{Y(\infty)}{Z(\infty)}\mathbb{1}[Z(\infty)\geq 1]\right)\cdot\epsilon\\
&\geq \left(\frac{a}{b} - \sqrt{\frac{\bar{a}}{b^3}\cdot\frac{1}{\Delta N}} - \frac{2\sqrt{2\bar{a}}}{b}\cdot\frac{e^{-\frac{b}{2}\Delta N}}{\sqrt{\Delta N}}\right)\cdot\epsilon\\
&\geq \frac{a}{2b}\cdot\epsilon\\
&= \rho_1\cdot\epsilon,
\end{aligned}
$$

where the second inequality holds when N exceeds a certain constant N_0, which depends only on the problem primitives, as ΔN increases with N by the setup in Lemmas 3.2 and 3.3.

Lemma B.4. *Let $Z>0$ be a positive random variable. Then, almost surely,*

$$
\frac{1}{Z} \geq \frac{1}{\mathbb{E}[Z]} - \frac{Z-\mathbb{E}[Z]}{\mathbb{E}[Z]^2}\,.
$$

Proof of Lemma B.4. Let $f(x)=\frac{1}{x}$. Since $f(x)$ is convex for $x>0$ with derivative $f'(x)=-\frac{1}{x^2}$, it follows that

$$
f(x) \geq f(a) + f'(a)\,(x-a) \qquad \text{for all } x,a>0.
$$

By setting $x=Z$ and $a=\mathbb{E}[Z]$, we obtain the desired result. □

Proof of Lemma B.3 Bullet 1. We first show that the number of collected ϵ-values in a steady state, $Y(\infty)$, follows a Poisson distribution. To do so, let $Y(k)$ represent the number of ϵ-values collected at matching epoch $t=k\Delta$. We show that each $Y(k)$ is Poisson distributed.

Specifically, consider an over-demanded job that arrives at time $s \leq k\Delta$. Let $p(s)$ denote the probability that this job contributes an ϵ to the collected taxes at epoch $k\Delta$. In the alternative system, all agents submit jobs fully. Define

$$\rho_{ij} = \frac{\lambda_{ij}}{\sum_{j \in \mathcal{N}_+} N\lambda_j}$$

as the probability that an over-demanded job is of type $j \in \mathcal{N}_+$ and arrives at agent $i \in [N]$. Then,

$$p(s) = \sum_{i \in [N]} \sum_{j \in \mathcal{N}_+} \rho_{ij} \, \mathbb{P}\Big(A(s) \cap B(s) \cap C(s)\Big),$$

where

$$\begin{aligned}
A(s) &:= \{\text{agent } i \text{ holds no credits at time } s\},\\
B(s) &:= \{\text{the job has not expired by } k\Delta\},\\
C(s) &:= \{\text{the job is matched at epoch } k\Delta, \text{but not at earlier epochs}\}.
\end{aligned}$$

Since over-demanded jobs arrive according to a Poisson process with rate $N \sum_{j \in \mathcal{N}_+} \lambda_j$, from Proposition 2.3.2 of Ross (1995), $Y(k)$ follows a Poisson distribution with mean value

$$\mathbb{E}[Y(k)] = \left(N \sum_{j \in \mathcal{N}_+} \lambda_j\right) \frac{1}{k\Delta} \int_0^{k\Delta} p(s)\, ds.$$

Finally, since $Y(k)$ converges in distribution to $Y(\infty)$, $Y(\infty)$ also follows a Poisson distribution.

We next derive upper and lower bounds on the mean value of $Y(\infty)$. For an upper bound, note that for any $k \in \mathbb{N}_+$, the total number of ϵ-values collected in the first k matching epochs is at most

$$k\Delta \sum_{j \in \mathcal{N}_+} N\lambda_j$$

in expectation, because over-demanded jobs join the shared pool at rate $\sum_{j \in \mathcal{N}_+} N\lambda_j$, and each contributes at most ϵ to the collection. This implies that

$$\mathbb{E}[Y(\infty)] \leq \Delta N \sum_{j \in \mathcal{N}_+} \lambda_j\,.$$

We next derive a lower bound. Since all under-demanded jobs are submitted and remain unused until expiration, the number of credits of type $j \in \mathcal{N}_0$ held by agent i follows an $M/M/\infty$ queue process with job arrival rate λ_{ij} and service rate θ_j. Its stationary distribution is Poisson with mean λ_{ij}/θ_j. Therefore, when an over-demanded job arrives at agent i, the probability that agent i holds no credits of any job type is

$$\prod_{j \in \mathcal{N}_0} \exp\left(-\frac{\lambda_{ij}}{\theta_j}\right) \geq \prod_{j \in \mathcal{N}_0} \exp\left(-\frac{C_j\,\lambda_j}{\theta_j}\right) = \nu_0.$$

Each such job is matched in the shared pool with probability at least $w_{\min}$, and once matched, contributes a value of ϵ to the tax collection. Consequently, taxes are collected at a rate of at least

$$\nu_0 \cdot w_{\min} \sum_{j \in \mathcal{N}_+} N\lambda_j \geq \frac{\nu_0}{2} \sum_{j \in \mathcal{N}_+} N\lambda_j\,,$$

where the inequality follows from $w_{\min} \geq C_{10} \geq C_8 \geq 1/2$ (the last inequality is due to Lemma B.1).

This implies that

$$\mathbb{E}[Y(\infty)] \geq \Delta N \cdot \frac{\nu_0}{2} \sum_{j \in \mathcal{N}_+} \lambda_j \,. \qquad \square$$

Proof of Lemma B.3 Bullet 2. Let $Z(k)$ denote the number of lotteries at the matching epoch $t = k\Delta$. We first derive its distribution. To do so, consider a type-j credit generated s units of time before epoch $k\Delta$. Let T denote its sojourn time, which follows an exponential distribution with rate parameter θ_j. We consider two cases:

1. If $s < \Delta$, for the credit to convert into a lottery that participates in the matching epoch $k\Delta$, the credit must expire before that epoch, which occurs with probability

$$\mathbb{P}[T \leq s] = 1 - \exp\left(-\theta_j s\right) \,.$$

2. If $s \geq \Delta$, for the credit to convert into a lottery that participates in the matching epoch $k\Delta$, the credit must expire after epoch $(k-1)\Delta$ and before epoch $k\Delta$, which occurs with probability

$$\mathbb{P}[s - \Delta < T \leq s] = \exp(-\theta_j(s - \Delta)) - \exp(-\theta_j s) \,.$$

Since agents submit all under-demanded jobs to the shared pool and never use credits, type-j credits are created following a Poisson process with rate $N\lambda_j$. Let $Z_j(k)$ denote the number of lotteries at epoch $k\Delta$ that arise from expired type-j credits. According to Proposition 2.3.2 of Ross (1995), $Z_j(k)$ follows a Poisson distribution with mean value

$$\begin{aligned} E[Z_j(k)] &= N\lambda_j \cdot \left\{ \int_0^{\Delta} \Big(1 - \exp(-\theta_j s)\Big) ds + \int_{\Delta}^{k\Delta} \Big(\exp(-\theta_j(s-\Delta)) - \exp(-\theta_j s)\Big) ds \right\} \\ &= N\lambda_j \left\{ \Delta - e^{-k\theta_j\Delta} \cdot \frac{e^{\theta_j\Delta} - 1}{\theta_j} \right\} . \end{aligned}$$

Note that the lottery inventories $\{Z_j(k)\}$ are independent because they originate from jobs of different types that arrive independently. Since $Z(k) = \sum_{j \in \mathcal{N}_0} Z_j(k)$, it also follows a Poisson distribution with mean value

$$\mathbb{E}\left[Z(k)\right] = \sum_{j \in \mathcal{N}_0} N\lambda_j \left\{ \Delta - e^{-k\theta_j\Delta} \cdot \frac{e^{\theta_j\Delta} - 1}{\theta_j} \right\} .$$

Letting k increase to infinity yields the desired result. $\square$

B.5 Proof of Theorem 5.4

According to Corollary 3.5, for any over-demanded job type $j \in \mathcal{N}_+$, we have:

$$1 - w_j = \sum_{m \in [K]} \left(\frac{M_{jm} x_m^*}{\lambda_j} - q_{jm} \right) \leq K \cdot C_6 \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N} \,. \tag{23}$$

The equality follows from the fact that $\sum_{m \in [K]} M_{jm} x_m^* = \lambda_j$ for any over-demanded job type j (by complementary slackness between (1) and (2)), and that $w_j = \sum_m q_{jm}$, where the probabilities q_{jm} are defined in Corollary 3.5. The inequality follows directly from Corollary 3.5.

From (23), for any number of agents N, we have:

$$1 - w_{\min} = \max_{j \in \mathcal{N}_+} \left\{1 - w_j\right\} \leq K \cdot C_6 \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}.$$

Hence, there exists an integer $N_0 \in \mathbb{N}$ such that, for all $N \geq N_0$:

1. $w_{\min} \geq C_{10}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$,
2. $(1 - w_{\min})\, C_{11}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M}) \leq \epsilon = K \cdot C_6 \cdot C_{11}(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M}) \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N} \leq C_9(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$.
3. N is greater than the constant specified in Lemma 5.3.

Consequently, by Lemma 5.3, full job submission by all agents constitutes an OE under the `MVC` mechanism with parameter ϵ.

B.6 Proof of Proposition 5.5

Step One Suppose that all agents fully submit their jobs to the shared pool. Since every over-demanded job of type j yields a reward of at least $(p_j^* - \epsilon)$ if matched, by the same proof of Lemma 4.4, the long-run average payoff of an agent i from full submission is at least:

$$\boldsymbol{\lambda}_i^{\mathrm{T}} \mathbf{p}^* - \epsilon \cdot \sum_{j \in \mathcal{N}_+} \lambda_{ij} - C_7 \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N} \geq \boldsymbol{\lambda}_i^{\mathrm{T}} \mathbf{p}^* - \left(K \cdot C_6 \cdot C_{11} \cdot \sum_{j \in \mathcal{N}_+} C_j \lambda_j + C_7 \right) \cdot \frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}$$

where C_7 is a positive constant specified in Lemma 4.4, and the term $\frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}$ diminishes to zero at a rate characterized by Lemmas 3.2 and 3.3.

Step Two Suppose that agents two to N fully submit their jobs to the shared pool and, whenever jobs are over-demanded, pair them with credits in an arbitrary way. We first show in Lemma B.5 that, the long-run average reward agent one can obtain from tax collection is at most a constant multiple of ϵ regardless of the strategy she uses.

Lemma B.5. *Suppose that:*

1. *Agents two to N submit all their jobs to the shared pool and, whenever jobs are over-demanded, pair them with credits in an arbitrary way;*
2. *The number of agents N is larger than some constant to ensure $\Delta \sum_{j \in \mathcal{N}_+} C_j \lambda_j \leq 1$, where Δ is the matching interval length used by the periodic matching policy π^{P} in the shared pool, as specified in Lemmas 3.2 and 3.3.*

Then, the long-run average reward agent one can obtain from tax collection is at most $c\,\epsilon$, regardless of the strategy agent one uses, where c is a positive constant depending only on the problem primitives $(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$.

Recall that the shared pool performs matching at every time epoch $t = k\Delta$, with $k \in \mathbb{N}_+$. In the proof of Lemma B.5, we show that the number of lotteries associated with agent one at time $t = k\Delta$ is only a fraction $\Theta\big(\frac{1}{N}\big)$ of the total number of lotteries at that time when k is large. This ensures that agent one obtains only a fraction $\Theta\big(\frac{1}{N}\big)$ of the collected taxes, and thus establishes Lemma B.5. We provide further details in Appendix B.6.1.

Since submitting an over-demanded job of type j yields a reward of at most p_j^*, from Lemma B.5, the long-run average payoff of agent one is at most the optimal value of the following optimization problem:

$$\begin{aligned}\max_{\mathbf{x}\in\mathbb{R}_+^K,\,\mathbf{s}\in\mathbb{R}_+^J} \quad & \mathbf{r}^{\mathrm{T}}\mathbf{x} + \mathbf{p}^{*\mathrm{T}}\mathbf{s} + c\,\epsilon \\ \text{s.t.} \quad & \mathbf{M}\mathbf{x} + \mathbf{s} \leq \boldsymbol{\lambda}_i\end{aligned}$$

The optimal value is (c.f. (3)):

$$\boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{p}^* + c\,\epsilon = \boldsymbol{\lambda}_i^{\mathrm{T}}\mathbf{p}^* + c\cdot K\cdot C_6\cdot C_{11}\cdot\frac{NV^{\mathrm{F}} - V^{\mathrm{P}}}{N}\,.$$

Wrap-Up From the above two steps, the benefit of unilaterally deviating from full submission is only a negligible term of order $O\left(\frac{NV^{\mathrm{F}}-V^{\mathrm{P}}}{N}\right)$. Consequently, full job submission by all agents constitutes an approximate Nash equilibrium under the `MVC` mechanism, regardless of how agents pair credits with over-demanded jobs.

B.6.1 Proof of Lemma B.5

Without loss of generality, assume that at time zero, neither the shared pool nor any agent holds jobs or credits. In addition, assume that agents two to N submit every job to the shared pool.

Recall that the shared pool performs matching at every epoch $t = k\Delta$, with $k \in \mathbb{N}_+$. Fix such an epoch $t = k\Delta$. In the following, we first bound the number of lotteries held by agent one at that epoch from above (Step One), and the number of lotteries held by agent two to N at that epoch from below (Step Two). We then show that the number of lotteries associated with agent one at time $t = k\Delta$ is only a fraction $\Theta\left(\frac{1}{N}\right)$ of the total lotteries at that time when k is large. Consequently, agent one obtains only a fraction $\Theta\left(\frac{1}{N}\right)$ of the collected taxes, thereby establishing the desired result.

Step One: Bounding Agents One's Lotteries from Above For any integer $k \in \mathbb{N}_+$, let $Z_1(k)$ denote the number of lotteries held by agent one that participate in the matching epoch $k\Delta$. Since, for any under-demanded job type $j \in \mathcal{N}_0$, the sojourn time of type-j jobs and their associated credits follow the same exponential distribution, $Z_1(k)$ attains its maximum (in the sample path sense) when agent one submits all under-demanded jobs to the shared pool and never pairs the resulting credits with over-demanded jobs. In this case, Lemma B.6 demonstrates that $Z_1(k)$ follows a Poisson distribution with mean value

$$\mathbb{E}[Z_1(k)] \leq \sum_{j\in\mathcal{N}_0} C_j\lambda_j\Delta.$$

Lemma B.6. *Suppose agent one submits all under-demanded jobs to the shared pool and never pairs credits with over-demanded jobs. This strategy maximizes the number of lotteries $Z_1(k)$ agent one holds at any matching epoch $k\Delta$. Furthermore, under this strategy, $Z_1(k)$ follows a Poisson distribution with mean value*

$$\mathbb{E}\left[Z_1(k)\right] = \sum_{j\in\mathcal{N}_0}\lambda_{1j}\left\{\Delta - e^{-k\theta_j\Delta}\cdot\frac{e^{\theta_j\Delta}-1}{\theta_j}\right\} \leq \sum_{j\in\mathcal{N}_0} C_j\lambda_j\Delta\,.$$

Proof of Lemma B.6. Consider a type-j credit generated by agent one s units of time prior to the matching epoch $k\Delta$. Let T denote its sojourn time, which follows an exponential distribution with rate parameter θ_j. We consider two cases:

1. If $s < \Delta$, for the credit to convert into a lottery that participates in the matching epoch $k\Delta$,

the credit must expire before that epoch, which occurs with probability

$$\mathbb{P}[T \leq s] = 1 - \exp\left(-\theta_j s\right) .$$

2. If $s \geq \Delta$, for the credit to convert into a lottery that participates in the matching epoch $k\Delta$, the credit must expire after epoch $(k-1)\Delta$ and before epoch $k\Delta$, which occurs with probability
$$\mathbb{P}[s - \Delta < T \leq s] = \exp(-\theta_j(s-\Delta)) - \exp(-\theta_j s) .$$

Since agent one submits all under-demanded jobs to the shared pool and never uses credits, type-j credits are created following a Poisson process with rate λ_{1j}. Let $Z_{1j}(k)$ denote the number of lotteries at epoch $k\Delta$ that arise from expired type-j credits of agent one. By Proposition 2.3.2 of Ross (1995), $Z_{1j}(k)$ follows a Poisson distribution with mean value

$$\begin{aligned} E[Z_{1j}(k)] &= \lambda_{1j} \cdot \left\{ \int_0^{\Delta} \Big(1 - \exp(-\theta_j s)\Big) ds + \int_{\Delta}^{k\Delta} \Big(\exp(-\theta_j(s-\Delta)) - \exp(-\theta_j s)\Big) ds \right\} \\ &= \lambda_{1j} \left\{ \Delta - e^{-k\theta_j \Delta} \cdot \frac{e^{\theta_j \Delta} - 1}{\theta_j} \right\} . \end{aligned}$$

Note that $Z_1(k) = \sum_{j \in \mathcal{N}_0} Z_{1j}(k)$. Moreover, these lottery inventories $\{Z_{1j}(k)\}$ are independent because they originate from jobs of different types that arrive independently. Therefore, $Z_1(k)$ also follows a Poisson distribution with mean value

$$\mathbb{E}\left[Z_1(k)\right] = \sum_{j \in \mathcal{N}_0} \lambda_{1j} \left\{ \Delta - e^{-k\theta_j \Delta} \cdot \frac{e^{\theta_j \Delta} - 1}{\theta_j} \right\} \leq \sum_{j \in \mathcal{N}_0} C_j \lambda_j \Delta . \qquad \square$$

Step Two: Bounding Agents Two to N's Lotteries from Below For any integer $k \in \mathbb{N}_+$, let $Z_{-1}(k)$ denote the number of lotteries held by agent two to N that enter the matching epoch $k\Delta$. Lemma B.7 demonstrates that when k is large, $Z_{-1}(k)$ is sample-path-wise no smaller than a Poisson random variable $\tilde{Z}_{-1}(k)$ with mean value $\mathbb{E}[\tilde{Z}_{-1}(k)] = \Delta \cdot \Theta(N)$.

Lemma B.7. *There exists a constant $c_1(N)$, which depends on the number of agents N and problem primitives $(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$, such that for any $k \geq c_1(N)$, regardless of how agents two to N pair credits with over-demanded jobs, the number of lotteries $Z_{-1}(k)$ held by agents two to N is, sample-path-wise, no smaller than a Poisson random variable $\tilde{Z}_{-1}(k)$ with mean value*

$$\mathbb{E}[\tilde{Z}_{-1}(k)] \geq c_2 \, \Delta \, N,$$

where c_2 is a positive constant depending only on the problem primitives $(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$.

Proof of Lemma B.7. Define
$$\phi_i \triangleq \sum_{j \in \mathcal{N}_+} \lambda_{ij} \leq \sum_{j \in \mathcal{N}_+} C_j \lambda_j$$
as the aggregate arrival rate of over-demanded jobs at agent i. Consider a type-j credit generated by agent i s units of time before the matching epoch $k\Delta$, and let T denote its sojourn time, which follows an exponential distribution with rate parameter θ_j. We consider two cases:

1. If $s < \Delta$, for the credit to convert into a lottery that enters the matching epoch $k\Delta$, it must expire before that epoch and cannot be paired with an over-demanded job prior to expiration. A sufficient condition is that the credit expires before the epoch $k\Delta$ and no over-demanded

jobs arrive at agent i before its expiration, which occurs with probability

$$\exp(-\phi_i s) \cdot \mathbb{P}[T \leq s] = \exp(-\phi_i s) \cdot (1 - \exp(-\theta_j s)) \ .$$

2. If $s \geq \Delta$, for the credit to convert into a lottery that enters the matching epoch $k\Delta$, it must expire after epoch $(k-1)\Delta$ and before epoch $k\Delta$, and it must not be paired with an over-demanded job before expiration. A sufficient condition is that the credit expires after epoch $(k-1)\Delta$ and before epoch $k\Delta$, and that no over-demanded jobs arrive at agent i before its expiration. This occurs with probability

$$\exp(-\phi_i s) \cdot \mathbb{P}[s - \Delta < T \leq s] = \exp(-\phi_i s) \cdot (\exp(-\theta_j (s-\Delta)) - \exp(-\theta_j s)) \ .$$

We consider an alternative system in which, for any agent $i \in [2:N]$, each of her credits perishes without converting into a lottery if an over-demanded job arrives before the credit's expiration. Let $\tilde{Z}_{-1}(k)$ denote the number of lotteries held by agent two to N in this alternative system that participate in epoch $k\Delta$. Clearly, $\tilde{Z}_{-1}(k)$ is no larger than $Z_{-1}(k)$, regardless of how agents two to N use credits. We now show that $\tilde{Z}_{-1}(k)$ satisfies the property stated in Lemma B.7.

Recall that agents two through N submit all jobs upon arrival. For a credit generated by these agents, define

$$\rho_{ij} = \frac{\lambda_{ij}}{\sum_{j \in \mathcal{N}_0} \sum_{i=2}^{N} \lambda_{ij}}$$

as the probability that it is of type $j \in \mathcal{N}_0$ and generated by agent $i \in [2:N]$. By Proposition 2.3.2 of Ross (1995), $\tilde{Z}_{-1}(k)$ follows a Poisson distribution with mean value

$$\begin{aligned}
E\left[\tilde{Z}_{-1}(k)\right] =& \left(\sum_{j \in \mathcal{N}_0} \sum_{i=2}^{N} \lambda_{ij}\right) \cdot \left\{ \int_0^{\Delta} \sum_{j \in \mathcal{N}_0} \sum_{i=2}^{N} \rho_{ij} \cdot \exp(-\phi_i s) \Big(1 - \exp(-\theta_j s)\Big) \, ds \right. \\
& \left. + \int_{\Delta}^{k\Delta} \sum_{j \in \mathcal{N}_0} \sum_{i=2}^{N} \rho_{ij} \cdot \exp(-\phi_i s) \Big(\exp(-\theta_j (s-\Delta)) - \exp(-\theta_j s)\Big) \, ds \right\} \\
=& \sum_{j \in \mathcal{N}_0} \sum_{i=2}^{N} \lambda_{ij} \left\{ \int_0^{\Delta} \exp(-\phi_i s) \Big(1 - \exp(-\theta_j s)\Big) ds \right. \\
& \left. + \int_{\Delta}^{k\Delta} \exp(-\phi_i s) \Big(\exp(-\theta_j (s-\Delta)) - \exp(-\theta_j s)\Big) ds \right\} \\
=& \sum_{j \in \mathcal{N}_0} \sum_{i=2}^{N} \lambda_{ij} \cdot \frac{\theta_j - e^{-\Delta \phi_i} \theta_j - e^{-k\Delta(\theta_j + \phi_i)} \left(-1 + e^{\Delta \theta_j}\right) \phi_i}{\phi_i (\theta_j + \phi_i)} \\
\geq& \sum_{j \in \mathcal{N}_0} \sum_{i=2}^{N} \lambda_{ij} \cdot \frac{\frac{\Delta \theta_j}{2} - e^{-k\Delta(\theta_j + \phi_i)} \left(-1 + e^{\Delta \theta_j}\right)}{\theta_j + \phi_i}
\end{aligned}$$

where the inequality follows from the facts that $1 - e^{-x} \geq x/2$ for all $x \in [0,1]$ and that $\Delta \phi_i \leq \Delta \sum_{j \in \mathcal{N}_+} C_j \lambda_j \leq 1$ (second inequality due to assumption in Lemma B.5).

There obviously exists a constant $c_1(N)$, depending on the number of agents N, such that for any $k \geq c_1(N)$,

$$e^{-k\Delta\theta_j} \left(-1 + e^{\Delta\theta_j}\right) \leq \frac{\Delta\theta_j}{4}, \ \forall j \in \mathcal{N}_0.$$

Therefore, when $k \geq c_1(N)$, we have

$$
\begin{aligned}
E\Big[\tilde{Z}_{-1}(k)\Big] &\geq \Delta \sum_{j \in \mathcal{N}_0} \sum_{i=2}^{N} \frac{\lambda_{ij}}{4} \cdot \frac{\theta_j}{\theta_j + \phi_i} \\
&\geq \frac{\Delta}{4} \sum_{j \in \mathcal{N}_0} \frac{\theta_j}{\theta_j + \sum_{j \in \mathcal{N}_+} C_j \lambda_j} \cdot (N - C_j)\, \lambda_j \\
&\geq c_2\, \Delta\, N
\end{aligned}
$$

for some positive constant c_2 depending only on the problem primitives $(\boldsymbol{\lambda}, \boldsymbol{\theta}, \mathbf{r}, \mathbf{M})$. The second inequality follows from the facts that $\sum_{i \in [N]} \lambda_{ij} = N\lambda_j$, $\lambda_{1j} \leq C_j \lambda_j$, and $\phi_i \leq \sum_{j \in \mathcal{N}_+} C_j \lambda_j$ for all $i \in [N]$. $\square$

Step Three: Bounding Agent One's Payoff from Collected Taxes Since the upper bound $Z_1(k)$ in Lemma B.6 depends only on the job arrivals at agent one, and the lower bound $\tilde{Z}_{-1}(k)$ in Lemma B.7 depends only on the job arrivals at agents two to N, $Z_1(k)$ and $\tilde{Z}_{-1}(k)$ are independent. Therefore, for $k \geq c_1(N)$, the fraction of rewards that agent one obtains from collected taxes is at most

$$
\begin{aligned}
\mathbb{E}\left[\frac{Z_1(k)}{Z_1(k) + \tilde{Z}_{-1}(k)} \cdot \mathbb{1}[Z_1(k) \geq 1]\right] &\leq \mathbb{E}\left[\frac{Z_1(k)}{1 + \tilde{Z}_{-1}(k)} \cdot \mathbb{1}[Z_1(k) \geq 1]\right] \\
&= \mathbb{E}\left[Z_1(k) \cdot \mathbb{1}[Z_1(k) \geq 1]\right] \cdot \mathbb{E}\left[\frac{1}{1 + \tilde{Z}_{-1}(k)}\right] \\
&\leq E[Z_1(k)] \cdot \frac{1}{E[\tilde{Z}_{-1}(k)]} \\
&\leq \frac{c_3}{N},
\end{aligned}
$$

where the equality follows from the independence of $Z_1(k)$ and $\tilde{Z}_{-1}(k)$, the second inequality from Lemma B.8, and the third inequality from Lemmas B.6 and B.7, with $c_3 \triangleq \sum_{j \in \mathcal{N}_0} C_j \lambda_j / c_2$ a constant that depends only on the problem primitives.

Note that for any $k \in \mathbb{N}_+$, the total taxes collected in the first k matching epochs are at most

$$
\epsilon \cdot k\Delta \sum_{j \in \mathcal{N}_+} N\lambda_j
$$

in expectation, because over-demanded jobs join the shared pool at a rate no larger than $\sum_{j \in \mathcal{N}_+} N\lambda_j$, and each over-demanded job contributes at most ϵ to the collection. Therefore, the long-run average payoff that agent one receives from tax collection is bounded above by

$$
\lim_{k \to \infty} \frac{1}{k\Delta}\, \epsilon \left(c_1(N)\, \Delta \sum_{j \in \mathcal{N}_+} N\lambda_j + \frac{c_3}{N} \cdot k\Delta \sum_{j \in \mathcal{N}_+} N\lambda_j \right) = \epsilon\, c_3 \sum_{j \in \mathcal{N}_+} \lambda_j\,,
$$

because agent one receives at most all the taxes in the first $c_1(N)$ matching epoches and a fraction $\frac{c_3}{N}$ of the collected taxes in the remaining epoches. We thus complete the proof by letting $c \triangleq c_3 \sum_{j \in \mathcal{N}_+} \lambda_j$.

Lemma B.8. *Let $X \sim Poisson(\lambda)$ be a Poisson random variable with mean value $\lambda > 0$. Then*

$$\mathbb{E}\left[\frac{1}{1+X}\right] = \frac{1-e^{-\lambda}}{\lambda} < \frac{1}{\lambda}.$$

Proof. Note that

$$\mathbb{E}\left[\frac{1}{1+X}\right] = \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{\lambda^k e^{-\lambda}}{k!} = \frac{e^{-\lambda}}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)!} = \frac{e^{-\lambda}}{\lambda}\left(e^{\lambda}-1\right) = \frac{1-e^{-\lambda}}{\lambda}. \qquad \square$$

B.7 Both Pathways are Essential to Incentivize Full Submission of Under-Demanded Jobs

In this section, we show that both pathways for deriving value from submitting under-demanded jobs—namely, tax exemption and lottery entry—are essential to sustain full job submission as an OE. To illustrate this, we use Example 5.1 to demonstrate that relying on only one pathway is insufficient to guarantee full submission as an OE.

B.7.1 Tax Exemption Alone Does Not Guarantee Full Submission as an OE

Consider a modification of the `MVC` mechanism that incentivizes the submission of under-demanded jobs only through tax exemption for over-demanded jobs. We show that, under this mechanism, withholding all type-two jobs strictly benefits an agent solving the mean-field problem in Example 5.1, regardless of the tax parameter $\epsilon > 0$ or the steady-state matching probability $w \in (0, 1)$ of type-one jobs in the shared pool. Therefore, full job submission by all agents cannot substitute an OE.

To illustrate this, let π^{full} denote the policy that submits all jobs while optimally pairing submitted (over-demanded) type-one jobs with type-two credits. To establish that policy π^{full} is suboptimal, we consider an alternative policy $\tilde{\pi}$, which withholds all type-two jobs while mimicking the actions of policy π^{full} for type-one jobs in the following manner: when a type-one job arrives,

1. If policy π^{full} pairs the job with a type-two credit upon submission, then policy $\tilde{\pi}$ internally matches it with a type-two job;
2. If policy π^{full} submits the job in isolation (i.e., without pairing it with a credit), then policy $\tilde{\pi}$ also submits it in isolation.

We couple the system dynamics under the two policies so that jobs in both systems arrive and depart at the same time, even when type-two jobs are converted into credits upon submission.[16] Consequently, policy $\tilde{\pi}$ can mimic the actions of policy π^{full} as described.

For type-one jobs submitted in isolation under both policies, the resulting rewards are identical. Thus, the payoff difference between the two policies arises only from type-one jobs that policy π^{full} pairs with a credit upon submission. For each such job, policy π^{full} yields an expected payoff of $w < 1$. In contrast, since policy $\tilde{\pi}$ internally matches the job, it obtains a higher payoff of 1. Therefore, policy $\tilde{\pi}$ strictly outperforms π^{full} for every type-one job that π^{full} submits with a pair. This implies that full submission is strictly suboptimal, and consequently, full job submission by all agents cannot sustain an OE when only tax exemption is applied.

B.7.2 Lottery Entry Alone Does Not Guarantee Full Submission as an OE

We now consider a modification of the `MVC` mechanism that uses only lottery entry to incentivize submission of under-demanded jobs. We show that in the corresponding mean-field problem of

[16] Recall that type-two jobs and their credits expire according to exponential clocks with identical rates.

Example 5.1, full job submission is strictly suboptimal for any tax parameter $\epsilon > 0$ and any steady-state matching probability $w \in (0, 1)$ of type-one jobs in the shared pool. Thus, full job submission by all agents cannot substitute an OE under the mechanism.

Note that without tax exemption, each submitted type-one job yields an expected payoff of $1 - \epsilon$ when matched. Meanwhile, submitted type-two jobs receive value from the redistribution of taxes collected from type-one jobs. Suppose all agents fully submit their jobs to the shared pool. Each type-two job receives an expected payoff of at most $w\epsilon/\lambda$ upon submission.[17] Since a type-two job is matched in the shared pool with probability w/λ,[18] the reward structure is equivalent if a type-two job instead receives an expected payoff of at most ϵ upon being matched.

Consequently, the payoffs from submitting type-one and type-two jobs under the mechanism are analogous to those under the $\texttt{MV}(\epsilon)$ mechanism defined in Definition 5.3. As shown in Section 5.2.1, when $\lambda > 1 + \theta$, an alternative policy that retains one type-two job for local matching strictly outperforms full job submission. Therefore, full job submission by all agents cannot constitute an OE when lottery entry is the sole incentive for submitting type-two jobs.

C Extension to Heterogeneous Matching Rewards

In this section, we consider a more general setting in which agents have heterogeneous matching rewards. Specifically, we model the dynamics of each agent as in Section 2. However, agent i generates a reward $r_{im} > 0$ from performing a type-m match, where these rewards $\{r_{im}\}$ may vary across agents. We let $\mathbf{r}_i = (r_{im})_{m \in [K]} \in \mathbb{R}_+^K$ denote the vector of matching rewards for agent i.

In the following, we first examine the centralized setting in Appendix C.1. We then show that if monetary transfers are allowed, the monetary mechanism proposed in Section 4.1 continues to make full submission a dominant strategy, thereby achieving the first-best performance, and remains approximately budget-balanced in large markets, in Appendix C.2. Finally, in Appendix C.3, we show that no non-monetary mechanism can achieve the first-best performance when matching rewards are heterogeneous.

C.1 The Centralized Setting

We first consider a centralized setting in which a central planner has full control over all agents and aims to maximize the expected long-average total matching value of the system. For each match type $m \in [K]$, let $r_m \triangleq \max_{i \in [N]} r_{im}$ denote the highest value that an agent can generate from a type-m match, and $S(m) \triangleq \{i \in [N] : r_{im} = r_m\}$ the set of agents achieving this maximum value.

When the central planner schedules a type-m match, she would assign it to an agent in $S(m)$ to perform it, so as to obtain the maximum matching reward. Accordingly, we define $\mathbf{r} = \max_{i \in [N]} \mathbf{r}_i = (r_m)_{m \in [K]} \in \mathbb{R}_+^K$ as the vector of matching rewards for the planner, where the maximization is taken componentwise.

Let V^* denote the optimal long-run average payoff achievable under centralized matching. Problem (24) provides a fluid relaxation to the centralized matching problem, which is identical to (1) with the new definition of the matching value $\mathbf{r}$, and has the same interpretation as (1).

$$
\begin{aligned}
V^{\mathrm{F}}(\boldsymbol{\lambda}) = \max_{\mathbf{x} \in \mathbb{R}_+^K} \quad & \mathbf{r}^{\mathrm{T}} \mathbf{x} \\
\text{s.t.} \quad & \mathbf{M}\mathbf{x} \le \boldsymbol{\lambda}.
\end{aligned}
\tag{24}
$$

[17] This is because type-one jobs enter the shared pool at rate N, each being matched and taxed ϵ with probability w. Meanwhile, type-two jobs enter the shared pool at rate $N\lambda$, each converting into a lottery upon expiration. The result then follows from the following two observations: (i) under the mean-field approximation, every lottery receives the same expected value from the collected taxes, because the shared pool remains in steady state under this approximation; and (ii) the rate at which taxes are allocated cannot exceed the rate at which they are collected.

[18] This is because the shared pool matches an equal number of type-one and type-two jobs, and type-one jobs are matched with probability w.

The dual problem of (24) is given by the following LP, which is identical to (2):

$$\begin{aligned} \min_{\mathbf{p} \in \mathbb{R}_+^J} \quad & \boldsymbol{\lambda}^{\mathrm{T}} \mathbf{p} \\ \text{s.t.} \quad & \mathbf{M}^{\mathrm{T}} \mathbf{p} \geq \mathbf{r}. \end{aligned} \tag{25}$$

We let $\mathbf{p}^* = (p_j^*)_{j \in [J]} \in \mathbb{R}_+^J$ denote an optimal solution to (25), and we can interpret the value of p_j^* as the marginal value of a type-j job to the centralized problem.

C.2 Mechanism with Monetary Transfer

We consider the monetary mechanism proposed in Section 4.1, which rewards an agent with a value of p_j^* whenever she submits a job of type j to the shared pool. The mechanism then performs an asymptotically optimal matching in the shared pool (e.g., by implementing the periodic matching policy π^{P} described in Section 3.1), and let agents in set $S(m)$ to execute type-m matches, charging r_m per match. In what follows, we show that the properties established in Section 4.1 continue to hold. In particular, this mechanism incentivizes all agents to fully submit their jobs and remains approximately budget-balanced.

To do so, we analyze the decision problem faced by an individual agent i under this mechanism. The fluid relaxation to the decision problem of agent i is given in (26):

$$\begin{aligned} \max_{\mathbf{x} \in \mathbb{R}_+^K,\, \mathbf{s} \in \mathbb{R}_+^J} \quad & \mathbf{r}_i^{\mathrm{T}} \mathbf{x} + \mathbf{p}^{*\mathrm{T}} \mathbf{s} \\ \text{s.t.} \quad & \mathbf{M}\mathbf{x} + \mathbf{s} \leq \boldsymbol{\lambda}_i. \end{aligned} \tag{26}$$

Note that (26) is identical to (3), except that the matching rewards $\mathbf{r}_i$ reflect agent i's individual values. The dual problem of (26) is given by (27):

$$\begin{aligned} \min_{\mathbf{p} \in \mathbb{R}_+^J} \quad & \boldsymbol{\lambda}_i^{\mathrm{T}} \mathbf{p} \\ \text{s.t.} \quad & \mathbf{M}^{\mathrm{T}} \mathbf{p} \geq \mathbf{r}_i, \\ & \mathbf{p} \geq \mathbf{p}^*. \end{aligned} \tag{27}$$

Proposition C.1 parallels Proposition 4.1 and provides optimal solutions to the primal and dual problems.

Proposition C.1. $\mathbf{x} = \mathbf{0}$ *and* $\mathbf{s} = \boldsymbol{\lambda}_i$ *is an optimal primal solution and* $\mathbf{p} = \mathbf{p}^*$ *is an optimal dual solution to (26), with the optimal value being* $\boldsymbol{\lambda}_i^{\mathrm{T}} \mathbf{p}^*$*.*

Proof. Note that $\mathbf{x} = \mathbf{0}$ and $\mathbf{s} = \boldsymbol{\lambda}_i$ is feasible to (26), and $\mathbf{p} = \mathbf{p}^*$ is feasible to (27) because $\mathbf{p} = \mathbf{p}^*$ is feasible to (2) and $\mathbf{r} = \max_{i \in [N]} \mathbf{r}_i \geq \mathbf{r}_i$. Moreover, these solutions achieve the same objective value. By weak duality, they must be optimal solutions to the primal and dual problems, respectively. □

From Proposition C.1, submitting all arriving jobs to the shared pool is optimal to an agent's fluid relaxation problem (26). Moreover, under the mechanism, the marginal value of a type-j job to an agent's problem equals the job's marginal value in the centralized problem (24), which is p_j^*. Finally, Proposition C.1 also implies that it is a dominant strategy for an agent to submit all jobs to the shared pool in the original problem, as stated in Proposition C.2.

Proposition C.2. *For any agent* $i \in [N]$*, it is a dominant strategy to submit all jobs to the shared pool upon their arrival, and this yields an expected long-run average payoff of* $\boldsymbol{\lambda}_i^{\mathrm{T}} \mathbf{p}^*$*.*

Proof. This is because full submission achieves the fluid relaxation upper bound (26). □

Finally, we remark that the monetary mechanism remains approximately budget-balanced in the large market regime, by the same reasoning as in Remark 4.1.

C.3 Mechanism without Monetary Transfer

In this section, we show that when agents have heterogeneous matching rewards, no non-monetary mechanism can achieve the first-best performance. This contrasts with the case of homogeneous matching rewards. To illustrate this, we consider a simple example with two job types and one match type, as defined in Example C.1.

Example C.1. Consider a problem instance with two job types and one matching type. Each match requires one job from each type. Jobs of both types arrive at each agent at rates 1 and expire at rates 1. There are N agents; the matching value is 1 for the first $N/2$ agents and 2 for the remaining agents.

Suppose the designer cannot use monetary transfer and can only reimburse job submissions through matching allocations. In order to incentivize the first $N/2$ agents to fully submit their jobs, the mechanism must allocate matches to each of these agents at a constant rate that compensates them more than what they could achieve individually. However, since these first $N/2$ agents generate lower matching rewards than the remaining ones, a central controller would never assign them any jobs, and they would consequently receive zero matching rewards. This implies that a non-monetary mechanism cannot achieve the first-best performance for any number of agents N.

D More on the Kidney Exchange Example in Section 6.2

Figure 2 (provided in Ashlagi and Roth 2021) illustrates the ABO compatibility structure.

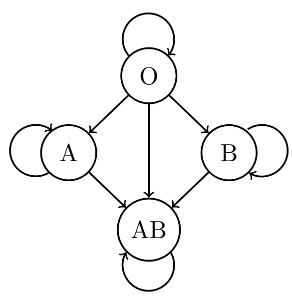

Figure 2: Illustration of ABO compatibility (Figure 1 of Ashlagi and Roth 2021). A directed arc of X to Y means that a donor with blood type X is compatible with a recipient with blood type Y.

Table 2 (reproduced from Table A.1 of Ashlagi and Roth 2021) reports statistics on the historical pool composition of APKD (a major U.S. kidney exchange program) from 2010 to 2019. The second column presents the empirical distribution of the patient-donor ABO pairs, and columns three to nine present the frequencies of PRA intervals, conditional on each ABO pair.

		Marginal frequencies (PRA intervals)						
Patient–donor ABO	% of pairs	$0-1$	$1-10$	$10-50$	$50-80$	$80-95$	$95-99$	$99-100$
AB–AB	0.2	0.0	0.0	0.0	50.0	0.0	25.0	25.0
AB–B	0.4	0.0	0.0	0.0	16.7	16.7	0.0	66.7
AB–A	0.7	0.0	8.3	0.0	8.3	50.0	0.0	33.3
AB–O	0.6	10.0	0.0	20.0	10.0	0.0	20.0	40.0
B–AB	0.9	37.5	6.2	18.8	6.2	12.5	0.0	18.8
B–B	2.4	0.0	4.9	12.2	12.2	31.7	9.8	29.3
B–A	5.8	46.5	8.1	13.1	9.1	12.1	1.0	10.1
B–O	4.2	9.9	1.4	4.2	16.9	19.7	15.5	32.4
A–AB	1.0	41.2	5.9	5.9	11.8	17.6	0.0	17.6
A–B	3.6	30.6	9.7	6.5	14.5	9.7	1.6	27.4
A–A	9.7	4.2	1.8	16.9	19.3	18.1	10.8	28.9
A–O	8.8	12.7	4.7	9.3	19.3	15.3	18.0	20.7
O–AB	2.3	46.2	10.3	23.1	5.1	12.8	0.0	2.6
O–B	9.2	47.1	10.8	14.0	7.6	8.3	4.5	7.6
O–A	29.4	49.9	10.0	12.8	8.8	6.4	3.6	8.6
O–O	20.7	4.5	2.8	13.9	17.3	23.9	16.2	21.3

Table 2: APKD pool composition (2010-2019) (Table A.1 of Ashlagi and Roth 2021).