Optimality of Public Persuasion in Job Seeking

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Abstract

We study an information design problem in which a school advisor strategically discloses information to promote her student in a job market with n potential employers. The advisor can send different signals to different employers (i.e., *private persuasion*) or broadcast the same signal to all employers (i.e., *public persuasion*). After receiving the signals, employers can communicate with each other (either simultaneously or sequentially, using either cheap talk or some degree of commitment) to resolve more uncertainty about the candidate in their self-interest. We show that as long as the candidate can accept at most one offer and has a known preference among the employers, public persuasion is optimal, regardless of how employers communicate. The optimal public persuasion can be derived from a first-best relaxation problem that only imposes the employers' participation constraints. We then focus on a specific case in which the candidate's characteristics can be summarized as a one-dimensional variable, and all of the receivers' utility functions are linear in this variable. We derive the optimal mechanism in a closed form for the two-receiver case. In the general case, a convex optimization problem with n decision variables and constraints can be efficiently solved to obtain an optimal mechanism. We provide structural properties and new understandings of the optimal mechanism from the dual perspective.

Subject classifications: Bayesian persuasion, public information, multiple receivers, Lagrangian dual

1 Introduction

In this paper, we study a Bayesian persuasion problem faced by a school advisor who promotes her student in a job market with n potential employers (e.g., schools with open junior faculty positions). The student has a known preference among the employers and can accept at most one offer. The advisor holds private information about the student's characteristics relevant to the employers' hiring requirements (e.g., research potential, teaching experience, and communication skills, etc.). The advisor can commit to an information disclosure mechanism that strategically discloses the candidate's characteristics (e.g., through targeted recommendation letters) to the employers to maximize the candidate's expected payoff. Notably, the advisor can use either a *public* persuasion mechanism to share the same information with all employers or a *private* persuasion mechanism to send tailored information to different employers based on their specific hiring standards.

A key feature of our model is to consider the subsequent communication among receivers after receiving signals from the sender, which is commonly observed in practice. Specifically, employers may communicate with each other (either simultaneously or sequentially, using either cheap talk or some degree of commitment) to resolve more uncertainty about the candidate in their self-interest. Then, based on the signal received from the sender and the additional information from other receivers, each employer decides whether to extend a job offer to the candidate. It is a priori unclear what the optimal persuasion mechanism is. Moreover, the potential for subsequent communication among receivers further complicates the information design problem.

We note that the receivers in this context are both cooperators and competitors. The communication reduces uncertainty about the candidate's characteristics, which benefits each receiver. However, since the sender can accept only one offer, competition among the receivers arises. Specifically, if an employer knows that a candidate is of high quality, he may withhold this information from other employers to avoid competition, especially if the sender prefers other employers.

As our first main result, we show that, perhaps strikingly, public persuasion is always optimal regardless of the detailed communication protocol used by the receivers (Section 3).[1] Since all of the employers receive the same information under a public persuasion mechanism, subsequent communication cannot convey any payoff-related information; thus, it becomes irrelevant. Therefore, the sender eliminates any room for the receivers to communicate and infer about each other for the sender's benefit.

[1]It might be striking because the optimal persuasion mechanism differs among receivers when considering each receiver in isolation. Moreover, public persuasion is optimal even when the sender knows that receivers cannot communicate (but are aware of each other's existence), as we elaborate further in Remark 4.3.

Furthermore, we show that the optimal public persuasion mechanism can be solved from a first-best relaxation problem that imposes only employers' participation constraints. Specifically, in the first-best problem, a central planner allocates a candidate with characteristics w to employers. An employer hires the candidate when the candidate is allocated to him. The first-best problem solves the optimal randomized allocation to maximize the sender's expected payoff ensuring only a nonnegative expected utility for each employer. We show that an optimal public persuasion mechanism can be derived from an optimal solution to the first-best problem, and its expected payoff matches the first-best upper bound.

Although an optimal public persuasion can be solved from the aforementioned first-best problem, it becomes an infinite-dimensional linear program (LP) when the candidate's characteristics w are infinite, which is challenging to solve. As our second main result, we focus on the efficient computation of an optimal public persuasion mechanism by considering a specific case in which the candidate's characteristics w can be summarized as a one-dimensional variable, and all receivers' utility functions are linear in this variable (Section 4). We derive optimality conditions of an optimal mechanism and provide structural properties and useful interpretations based on the Lagrangian dual of the first-best problem, where we dualize the participation constraints (Section 4.2). In the Lagrangian, each employer i is associated with a line passing through the point (α_i, v_i) with a nonnegative slope μ_i, where α_i denotes the recruiting bar[2] of employer i, v_i denotes the payoff of employer i's offer to the candidate, and μ_i denotes the dual variable associated with employer i's participation constraint. The Lagrangian assigns a candidate with characteristics w to employer i with a positive probability only if employer i's line is above the x-axis and not dominated by other employers' lines at point w. A public persuasion mechanism is optimal if and only if all participation constraints are binding and there exists a dual variable under which the mechanism is optimal to the corresponding Lagrangian.

Based on the optimality conditions, we derive the optimal persuasion mechanism in closed form when there are two employers, where one employer has a higher hiring bar but also brings a higher payoff (Section 4.3). The main trade-off is that an offer from a more competitive employer brings a higher payoff; however, targeting this employer more aggressively is costly because it reduces the overall probability of receiving an offer. Depending on the relative desirability of the two employers and their hiring bars, the optimal persuasion mechanism carefully balances this trade-off.

We then consider the general case (Section 4.4). We first show that the first-best problem

[2] That is, the utility of hiring a candidate is nonnegative for employer i if and only if the "quality" w exceeds α_i.

can be reduced to a convex optimization problem with n decision variables and constraints, and thus can be efficiently solved. The convex problem maximizes the sender's expected payoff by optimally determining the probability that the candidate joins each employer, subject to a variant of the aforementioned participation constraint that ensures the assignment of the candidate meets the hiring bar of the top k employers for any $k \leq n$. The convex problem is analogous to the one considered in Candogan (2022) but is slightly simplified. We establish the equivalence of the first-best and convex problems from both the primal and dual perspectives, recover many of the results from Candogan (2022), and provide new understandings based on the dual of the convex problem. Given an optimal solution to the convex program, we can construct an optimal persuasion mechanism in various ways. In addition to the deterministic persuasion mechanism with a double-interval structure as illustrated in Candogan (2022), we present a randomized persuasion mechanism with a monotone structure. Specifically, under the randomized persuasion mechanism, the student's payoff first-order stochastically increases with her characteristics w. This monotone property incentivizes students to invest effort in improving their "quality" w, thus can be desirable in practice.

The rest of the paper is organized as follows. Section 1.1 reviews some related work. Section 2 formulates the problem. In Section 3, we demonstrate that public persuasion mechanisms are optimal in our setup, regardless of the communication detail of the receivers. In addition, the optimal public persuasion mechanism can be solved from a first-best relaxation problem that requires only the employers' participation constraints. Section 4 addresses the efficient computation of an optimal public persuasion mechanism when the candidate's characteristics can be summarized as a one-dimensional variable, and all receivers' utility functions are linear in this variable. We provide optimality conditions of a public persuasion mechanism based on duality in Section 4.2. Section 4.3 characterizes the optimal mechanism in closed form for the two-receiver case, and Section 4.4 examines the general case. Section 5 concludes.

1.1 Related Literature

Our paper is related to the literature on Bayesian persuasion and information design. The seminar paper Kamenica and Gentzkow (2011) examines the problem where a designer (sender) with private information try to persuade an agent (receiver) to take a sender-preferred action. Subsequent literature extends this framework to settings with multiple receivers (e.g., Alonso and Câmara 2016, Arieli and Babichenko 2019, and Section 4.1 of Kamenica 2019 for a recent review). As Kamenica (2019) highlights, *if sender can send separate signals to each receiver, and if either (a) a* *receiver's optimal action depends on what other receivers do or (b) sender's utility is not separable across receiver's actions, then the problem becomes significantly more difficult.* Our setup falls within this challenging regime.

Many existing studies have not incorporated post-signal communication among receivers as we do. Two exceptions are Galperti and Perego (2023) and Candogan et al. (2023), which consider informational spillovers among the receivers. In both works, these spillovers are pre-specified by a directed network, in which arcs represent potential informational spillovers among receivers. In contrast, our model allows for strategic communication and an arbitrary communication protocol. Galperti and Perego (2023) characterize the set of all possible equilibrium outcomes that can arise from any information structure under spillover and seeding constraints. Candogan et al. (2023) show that the optimal information design problem is generally computationally challenging under information spillovers, except for certain specific cases.

Candogan (2022) considers a general model in which the designer's payoff is an increasing step function of the induced posterior mean and solves a finite-dimensional convex optimization to obtain an optimal public persuasion mechanism. While Candogan (2022) focuses on public persuasion mechanisms, we show that these mechanisms are optimal in our setup, even when receivers can communicate with each other post-signal and regardless of their communication method. When the candidate's characteristics can be summarized as a one-dimensional variable, and all receivers' utility functions are linear in this variable, solving an optimal public persuasion mechanism in our setup aligns with the general model of Candogan (2022). In this case, we slightly simplify the convex optimization problem in Candogan (2022), recover many of the results from Candogan (2022) and provide new understandings of the optimal persuasion mechanism based on the dual problems.

Bergemann and Morris (2016) and Bergemann and Morris (2019) relate the multi-receiver persuasion problem to the game-theoretic concept of Bayes correlated equilibrium (BCE). This relationship leads to a natural linear programming (LP) formulation for obtaining an optimal persuasion mechanism. Specifically, the decision variables in the LP are joint probabilities of the state and the receivers' actions, and the constraints completely characterize the set of BCEs.[3] Our first-best relaxation problem (2) is also an LP. However, in our LP, the decision variables are marginal allocation probabilities under a mechanism. The LP imposes only participation constraints that any mechanism must satisfy, and thus, does not precisely characterize the set of equilibrium outcomes.

[3]That is, a joint distribution sustains a BCE if and only if it is feasible to the LP.

Finally, Bergemann and Morris (2019) also explore when public persuasion mechanisms are optimal (Section 4.1 there). Their model does not incorporate post-signal communications. They show that public persuasion mechanisms are optimal when receivers' actions are strategic complements, as these mechanisms induce a positive correlation in the receivers' actions. However, in our setup, the receivers' actions are not strategic complements.

Kolotilin (2018) and Dworczak and Martini (2019) also use duality to characterize optimality conditions and to interpret an optimal persuasion mechanism. However, we study different problems, formulate the optimization problem in different ways, and apply duality differently. Specifically, Kolotilin (2018) dualize a consistency constraint for the marginal distribution of the sender's state and Dworczak and Martini (2019) dualize the mean-preserving spread constraint. In contrast, we dualize the employers' participation constraints.

Ostrovsky and Schwarz (2010) and Boleslavsky and Cotton (2015) study school grading problems similar to our setting. Ostrovsky and Schwarz (2010) consider a model with a continuum of schools (senders) and employers (receivers) and study the schools' equilibrium grading policies (persuasion mechanism). Each school is assumed to use a public persuasion mechanism. Boleslavsky and Cotton (2015) consider a setup with two schools (senders) and one evaluator (receiver), where each school determines both its investment level in quality and grading policies. They investigate the two schools' equilibrium strategies.

Finally, other extensions of Bayesian persuasion have been considered in the literature, including multiple senders (Gentzkow and Kamenica 2017), privately-informed receivers (Kolotilin et al. 2017, Guo and Shmaya 2019), and dynamic models (Ely 2017), which are not included in our model. In addition, numerous works focus on various operational applications, such as incentivizing exploration (Papanastasiou et al. 2018), signaling product availability (Drakopoulos et al. 2021), signaling congestion in queueing systems (Anunrojwong et al. 2023), and informing the severity of a pandemic (De Véricourt et al. 2021); for a comprehensive review, see Candogan (2020).

1.2 Notation and Terminology

We let $\mathbb{N}$ denote the set of nonnegative integers. For any two integers $a, b \in \mathbb{N}$ with $a \leq b$, we let $[a:b] = \{a, a+1, \ldots, b-1, b\}$ denote a sequence of integers starting from a and ending with b, and we denote $[n] = [1:n]$ for any $n \in \mathbb{N}_+$. For any real number $x \in \mathbb{R}$, we let $(x)^+ \triangleq \max\{x, 0\}$ denote the maximum of x and 0.

2 Problem Formulation

We consider a school advisor (referred to as "she") who promotes her student in a job market with n potential employers (referred to as "he"; e.g., schools with open junior faculty positions) via strategic information disclosure (e.g., targeted recommendation letters). The student can accept at most one offer and has a known preference among the employers. Specifically, we denote by $v_i > 0$ the utility from the offer of employer i, and we rank employers in decreasing preference; that is, $v_i > v_j$ if $i < j$, as assumed in Assumption 2.1. If the student does not secure a job, we normalize her utility to zero.

Assumption 2.1. The utility v_i from accepting employer i's offer satisfies $0 < v_n < \cdots < v_2 < v_1$.

Let $w \in \Omega$ represent the characteristics of the student, where Ω is a general state space.[4] While the realization of w is privately observable to the school advisor, employers only possess a prior distribution $G(w)$ regarding the student's characteristics, reflecting the reputation of the advisor's students. For each employer i, let $u_i(w)$ denote the utility of hiring a student with characteristics w; the utility of not hiring is zero.

Information Disclosure Mechanism We study a Bayesian persuasion setup in which the advisor (the sender), who has commitment power, designs an information disclosure mechanism to promote her student to the n employers (the receivers). Let S_i denote the set of signals employed by the advisor to interact with employer i and $\boldsymbol{S} = \bigotimes_{i=1}^{n} S_i$ represent the set of all signals. Upon observing the characteristics w, the advisor sends a signal $s_i \in S_i$ to each employer i according to a joint distribution $f(\boldsymbol{s}|w)$, where $\boldsymbol{s} = (s_1, \cdots, s_n) \in \boldsymbol{S}$ denotes the concatenation of the sent signals. We define the information mechanism $f(\cdot|w)$ as a *public* mechanism if

1. The signals share a common signal space S, that is, $S_i = S_j = S$ for all $i, j \in [n]$; and
2. The signals $(s_i)_{i\in[n]}$ are perfectly correlated, that is, $f(\boldsymbol{s}|w) = 0$ for any signal $\boldsymbol{s} = (s_i)_{i\in[n]}$ where $s_i \neq s_j$ for some $i, j \in [n]$.

With a public mechanism, employers always receive the same signal, eliminating the need for further communication. Conversely, if $f(\cdot|w)$ allows for different signals among employers, we refer to it as a *private* information mechanism. In this case, the employers may receive different signals, leading to varied information about the student's characteristics w.

[4]For example, we may have $\Omega \subseteq \mathbb{R}^m$, where m represents the number of attributes relevant to employers' hiring standards, such as research potential, teaching experience, and communication skills.

Communication among Receivers We assume that employers may communicate with each other after receiving the signal $\boldsymbol{s}$. We do not formally model how employers will communicate. Notably, employers may or may not be able to communicate, and if they do, it could be either simultaneously or sequentially, using either cheap talk or with some degree of commitment. Any of these communication methods can be reasonable in specific scenarios. However, as we demonstrate in Section 3, the optimal persuasion mechanism will be independent of the communication details. This is because, regardless of how employers communicate, a public information disclosure mechanism will always be optimal for the sender, leaving nothing for the receivers to communicate.

However, some notations are helpful to describe the problem. Given a specific communication protocol, let C_i denote the set of information that employer i can receive from other employers and $\mathbf{C} = \bigotimes_{i=1}^n C_i$ represent the communication space. Denote the communication outcome as $\mathbf{c} = (c_1, ..., c_n) \in \mathbf{C}$, where c_i is the information employer i receives through communication. Given a signal $\boldsymbol{s}$, suppose the cumulative distribution function of $\mathbf{c}$ is $C(\mathbf{c}|\boldsymbol{s})$, and the probability density function of $\mathbf{c}$ is $c(\mathbf{c}|\boldsymbol{s}) = \frac{dC(\mathbf{c}|\boldsymbol{s})}{d\mathbf{c}}$, possibly derived from the employers' equilibrium strategies.

Sender's Problem The game proceeds as follows:

1. The advisor commits to an information disclosure mechanism $f(\cdot|w)$ and a signal space $\mathbf{S} = \bigotimes_{i=1}^n S_i$.
2. The student's characteristics w are drawn from the cumulative probability distribution $G(w)$. A signal $\mathbf{s} = (s_i)_{i\in[n]}$ is then generated according to the disclosure mechanism $f(\cdot|w)$ and sent to the employers.
3. Employers communicate with each other after receiving the signal $\boldsymbol{s}$ using $C(\cdot|\boldsymbol{s})$, which may represent an equilibrium communication strategy in a specific scenario. After communication, each employer i decides whether to extend an offer to the student based on the signal s_i and the communication outcome c_i.
4. The student accepts the offer that maximizes her payoff, which corresponds to the employer with the smallest index among those sending offers, according to Assumption 2.1.

Given a signal and communication outcome $s \in S_i$ and $c \in C_i$, we define $\mathbf{S}^i(s) = \{\boldsymbol{s} \in \mathbf{S} : s_i = s\}$ and $\mathbf{C}^i(c) = \{\mathbf{c} \in \mathbf{C} : c_i = c\}$ as the sets of possible signals and communications, respectively. Upon observing s and c, employer i understands that the signal must be in the set $\mathbf{S}^i(s)$ and the communication outcome must be in the set $\mathbf{C}^i(c)$. He updates his belief about the student's characteristics w, the signal $\boldsymbol{s}$, and the communication outcome $\mathbf{c}$ using Bayes's rule whenever possible. Specifically, let $f_i(s,c)$ denote the probability that employer i receives a signal s and communication outcome c:

$$f_i(s,c) = \int_{w\in\Omega}\int_{\boldsymbol{s}\in\mathbf{S}^i(s)}\int_{\mathbf{c}\in\mathbf{C}^i(c)} c(\mathbf{c}|\boldsymbol{s})\, f(\boldsymbol{s}|w)\, d\mathbf{c}\, d\boldsymbol{s}\, dG(w).$$

If $f_i(s,c) > 0$, the employer i's posterior belief on the tuple $(w,\boldsymbol{s},\mathbf{c})$ is defined as

$$f_i(w,\boldsymbol{s},\mathbf{c}|s,c) = \begin{cases} \frac{dG(w) f(\boldsymbol{s}|w)\, c(\mathbf{c}|\boldsymbol{s})}{f_i(s,c)}, & \text{if } \boldsymbol{s}\in\mathbf{S}^i(s) \text{ and } \mathbf{c}\in\mathbf{C}^i(c), \\ 0, & \text{otherwise.} \end{cases}$$

Denote employer i's equilibrium strategy by $\delta_i(s_i,c_i)$, representing his probability of extending an offer after receiving a signal $s\in S_i$ and communication outcome $c\in C_i$. The optimality of employer i's strategy implies that $\delta_i(s_i,c_i)$ follows the following equation:

$$\delta_i(s_i,c_i) = \begin{cases} 0, & \text{if } \mathbb{E}\left[u_i(w)\cdot \mathbb{1}[a_j^* = 0,\, \forall\, j < i] \mid s_i, c_i\right] < 0, \\ \delta\in[0,1], & \text{if } \mathbb{E}\left[u_i(w)\cdot \mathbb{1}[a_j^* = 0,\, \forall\, j < i] \mid s_i, c_i\right] = 0, \\ 1, & \text{if } \mathbb{E}\left[u_i(w)\cdot \mathbb{1}[a_j^* = 0,\, \forall\, j < i] \mid s_i, c_i\right] > 0, \end{cases}$$

where the binary variable $a_i^*\in\{0,1\}$ represents employer i's action of extending an offer in an equilibrium and satisfies $\mathbb{P}[a_i^* = 1|s_i,c_i] = \delta_i(s_i,c_i)$, and the expectation $\mathbb{E}[\cdot|s_i,c_i]$ is taken over the posterior distribution $f_i(w,\boldsymbol{s},\mathbf{c}|s_i,c_i)$. Note that the student accepts employer i's offer if and only if none of the employers $j<i$ extends an offer, which is represented by $\mathbb{1}[a_j^* = 0,\, \forall\, j<i]$.

Finally, let the random set $I(\boldsymbol{s},\mathbf{c})$ denote the employers who extend an offer and $i(\boldsymbol{s},\mathbf{c}) \triangleq \min I(\boldsymbol{s},\mathbf{c})$ the index of the offer to accept, given the signal realization $\boldsymbol{s}\in\mathbf{S}$ and communication outcome $\mathbf{c}\in\mathbf{C}$ and under the employers' equilibrium strategies. If $I(\boldsymbol{s},\mathbf{c}) = \varnothing$, that is, the student receives no offer, we let $i(\boldsymbol{s},\mathbf{c}) = \varnothing$ and $v_\varnothing = 0$ as the corresponding utility of the student. The advisor selects an information disclosure mechanism $f(\cdot|w)$ that maximizes the expected payoff of the student by solving

$$V^* \triangleq \max_{f(\cdot|w)} \int_{w\in\Omega}\int_{\boldsymbol{s}\in\boldsymbol{S}}\int_{\mathbf{c}\in\mathbf{C}} v_{i(\boldsymbol{s},\mathbf{c})}\cdot c(\mathbf{c}|\boldsymbol{s})\cdot f(\boldsymbol{s}|w)\cdot d\mathbf{c}\; d\boldsymbol{s}\; dG(w), \tag{1}$$

where V^* denotes the expected payoff of an optimal information disclosure mechanism.

3 Optimality of Public Persuasion

In this section, we illustrate that a public disclosure mechanism solves the advisor's optimal information disclosure problem (1), regardless of how employers communicate. We begin by introducing a relaxation of the designer's problem (1) in Section 3.1, which provides an upper bound on the sender's optimal expected payoff V^*.

3.1 First-Best Problem with Participation Constraints

In this section, we consider the first-best problem (2) for the sender's information design problem, where we impose only the participation constraints of the employers.

$$\begin{aligned} \bar{V} = \max_{q(i|w)\geq 0} \quad & \sum_{i=1}^{n} v_i \cdot \int_{w\in\Omega} q(i|w)\, dG(w) \\ \text{s.t.} \quad & \int_{w\in\Omega} u_i(w)\, q(i|w)\, dG(w) \geq 0,\, \forall\, i \in [n], \\ & \sum_{i\in[n]} q(i|w) \leq 1,\, \forall\, w \in \Omega. \end{aligned} \tag{2}$$

In (2), $q(i|w)$ represents the probability that a central planner allocates the candidate to employer i when her characteristics are w. An employer hires the candidate when the latter is allocated to him. The planner ensures only a nonnegative expected utility for each employer, as indicated by the first constraint in (2). We let $\bar{V}$ denote the optimal expected payoff of the central planner.

Lemma 3.1 demonstrates that (2) provides an upper bound on the sender's optimal expected payoff V^*, regardless of how employers communicate.

Lemma 3.1. *We have $\bar{V} \geq V^*$, regardless of how employers communicate.*

We prove Lemma 3.1 in Appendix A.1. Intuitively, given any disclosure mechanism $f(\cdot|w)$, let $q(i|w)$ denote the ex ante probability that employer i extends an offer and the offer gets accepted, under the employers' equilibrium strategies induced by $f(\cdot|w)$. These $\{q(i|w)\}$ are feasible to (2) and have an objective value no larger than $\bar{V}$.

3.2 Optimality of Public Persuasion

In this section, we construct a public persuasion mechanism $f^*(\cdot|w)$ from the optimal solution of (2) and show that its expected payoff attains the first-best upper bound $\bar{V}$. Therefore, the mechanism $f^*(\cdot|w)$ is optimal to (1), regardless of how receivers communicate.

Let $\{q^*(i|w)\}$ denote an optimal solution to (2). We consider a public persuasion mechanism $f^*(\cdot|w)$ with signal space $S_i = S \triangleq [n] \cup \{\varnothing\}$ for all employers $i \in [n]$. When the student's characteristics are w, the mechanism broadcasts the signal $s = i$ to all employers with probability $q^*(i|w)$ for any $i \in [n]$ and the signal $s = \varnothing$ to all employers with probability $1 - \sum_{i\in[n]} q^*(i|w)$. We can interpret the signal $s = i$ as a recommendation for only employer i to extend an offer and the signal $s = \varnothing$ as a recommendation for none of the employers to extend an offer. Lemma 3.2 shows that the public mechanism achieves the first-best upper bound $\bar{V}$.

Lemma 3.2. *Under the public persuasion mechanism $f^*(\cdot|w)$, it is an equilibrium for each employer $i \in [n]$ to extend an offer only upon receiving the signal $s = i$. Moreover, the expected payoff of the mechanism $f^*(\cdot|w)$, denoted by V^{P}, satisfies $V^{\mathrm{P}} = \bar{V}$.*

We prove Lemma 3.2 in Appendix A.2. According to Lemma 3.2, the advisor needs to focus only on public persuasion mechanisms to solve the optimal information disclosure (1). Consequently, she eliminates any communication among receivers for her own benefit, regardless of the way receivers can communicate. This holds true even if the sender knows that receivers cannot communicate but are aware of each other's existence. (we elaborate this further in Remark 4.3.) Additionally, the optimal public persuasion mechanism can be derived from (2) and achieves the first-best performance (i.e., the optimal value of (2)).

4 Simplified Optimization for One-Dimensional Linear Utility Case

When the candidate's characteristics w are infinite, the first-best problem (2) is an infinite-dimensional linear program (LP), which can be challenging to solve. In this section, we focus on a case in which the state variable w is one-dimensional, and all the receivers' utility functions are linear in w. In Section 4.2, we provide structural properties and derive the optimality condition of an optimal mechanism from the Lagrangian dual of (2), where we dualize the participation constraints. Using the optimality condition, we derive the optimal mechanism in closed form when there are two receivers in Section 4.3. For the general case (Section 4.4), problem (2) can be reduced to a convex optimization problem with n decision variables and constraints, similar to Candogan (2022), and thus can be solved efficiently. We provide a better understanding of the optimal mechanism from the Lagrangian dual of the convex optimization problem and establish the equivalence of the convex problem and problem (2) from both the primal and dual perspectives.

4.1 The Setup

To start, we formally describe the one-dimensional linear utility case. First, we assume that the candidate's characteristics w can be summarized as a one-dimensional state variable within a finite interval. Without loss of optimality, let $w \in \Omega = [0, 1]$. Moreover, we assume that w follows a continuous distribution with a strictly increasing cumulative distribution function $G(\omega)$ and a density function $g(w) > 0$ for any $w \in (0, 1)$. We summarize the above in Assumption 4.1.

Assumption 4.1. The candidate's characteristics w belong to the one-dimensional interval $\Omega = [0, 1]$ and follow a continuous distribution. Let $G(w)$ and $g(w)$ denote the cumulative distribution function and density function of w, respectively. The function $G(w)$ is strictly increasing, so its inverse, denoted by $G^{-1}(\cdot)$, exists.

Second, we assume that for each employer $i \in [n]$, the utility function for hiring a candidate with characteristics w is linear in w; that is, $u_i(w) = \theta_i \cdot \big(w - \alpha_i\big)$, where θ_i and α_i are positive constants. This assumption implies that each employer i considers only the mean value of the characteristics w among the candidates who would accept employer i's offer. Specifically, employer i will extend an offer only if this mean value exceeds his hiring threshold α_i. We state this linear utility assumption in Assumption 4.2.

Assumption 4.2. For each employer $i \in [n]$, the utility function $u_i(w)$ for a candidate with characteristic w is increasing and linear in w with a threshold value $\alpha_i > 0$; that is, $u_i(w) = \theta_i \cdot \big(w - \alpha_i\big)$, where θ_i and α_i are positive constants.

Note that since employers are ranked in decreasing preference by Assumption 2.1, there is no loss of generality to assume that the threshold values α_i also decrease in the employer index i. In other words, a more preferred employer is harder to get into. Conversely, if employer i is more preferred than j ($v_i > v_j$) but also easier to get into ($\alpha_i \leq \alpha_j$), employer j will never be targeted and can be dropped from consideration. Finally, we assume that all employers are selective, meaning their threshold values α_i are higher than the prior mean of the candidate's characteristics $\mathbb{E}_{w\sim G(w)}[w]$. We state these in Assumption 4.3.

Assumption 4.3. Let $w_0 \triangleq \mathbb{E}_{w\sim G(w)}[w]$ denote the prior mean of the candidate's characteristics w. The employers' threshold values α_i satisfy $0 < w_0 < \alpha_n < \cdots < \alpha_2 < \alpha_1 < 1$.

Based on the linear-utility Assumption 4.2, the first-best problem (2) can be written as (3):

$$
\begin{aligned}
\bar{V} = \max_{q(i|w)\geq 0} \quad & \sum_{i=1}^{n} v_i \cdot \int_0^1 q(i|w)\, g(w)\, dw \\
\text{s.t.} \quad & \int_0^1 w \cdot q(i|w)\, g(w)\, dw \geq \alpha_i \int_0^1 q(i|w)\, g(w)\, dw,\ \forall\, i \in [n], \\
& \sum_{i\in[n]} q(i|w) \leq 1,\ \forall\, w \in [0,1].
\end{aligned} \tag{3}
$$

4.1.1 Preliminary Properties of Optimal Solution of (3)

We conclude Section 4.1 by describing several properties related to an optimal solution of (3). First, for any feasible solution to (3), the probability that a candidate receives an offer is strictly less than one, as employers are selective according to Assumption 4.3. Moreover, this probability is maximized when the sender targets only the most accessible employer n. We formalize this in Proposition 4.1 and provide the proof in Appendix A.3.

Proposition 4.1. *Let $z_n \in (0,1)$ be such that $\mathbb{E}[w|w \geq z_n] = \alpha_n$, where α_n is the threshold value of employer n. For any feasible solution $\{q(i|w)\}$ of (3), we have $\sum_{i\in[n]} \int_0^1 q(i|w)dw \leq \mathbb{P}(w \geq z_n) < 1$, where the first inequality is attained when the sender targets only employer n; that is, $q(n|w) = 1$ for any $w \geq z_n$, and $q(i|w) = 0$ for any $i \neq n$ or $w < z_n$.*

Second, Assumption 4.3 implies that the participation constraints in (3) are binding with an optimal solution of (3). We state this in Proposition 4.2 and provide the proof in Appendix A.4.

Proposition 4.2. *Under Assumption 4.3, the participation constraints are binding with any optimal solution of (3).*

Finally, we show that any optimal solution exhibits a cutoff structure. Specifically, there exists a threshold value $z \in (0,1)$ such that a candidate receives an offer if and only if her characteristics w exceed z. We formalize this property in Proposition 4.3, with the proof provided in Appendix A.5.

Proposition 4.3. *Any optimal solution has a cutoff structure. That is, for any optimal solution $\{q^*(i|w)\}$ of (3), there exists a threshold value $z \in (0,1)$ such that $\sum_{i\in[n]} \int_z^1 q^*(i|w)dw = \mathbb{P}(w \geq z)$ and $\sum_{i\in[n]} \int_0^z q^*(i|w)dw = 0$.*

4.2 The Lagrangian Dual Problem

In this section, we introduce the Lagrangian dual problem of (3) that dualizes the employers' participation constraints. Specifically, denote by $\mu_i \geq 0$ the Lagrange multiplier for the participation constraint of employer $i \in [n]$. The Lagrangian relaxation, denoted by $V^{\mathrm{LR}}(\boldsymbol{\mu})$ with $\boldsymbol{\mu} = (\mu_i)_{i\in[n]} \in \mathbb{R}^n_+$, is as follows:

$$
\begin{aligned}
V^{\mathrm{LR}}(\boldsymbol{\mu}) &= \max_{\substack{q(i|w)\geq 0,\\ \sum_{i\in[n]} q(i|w)\leq 1}} \int_0^1 \sum_{i=1}^n \Big\{ v_i + \mu_i\big(w-\alpha_i\big)\Big\}\, q(i|w)\, g(w)\, dw \\
&= \int_0^1 g(w)\, dw \cdot \max_{\substack{q(i|w)\geq 0,\\ \sum_{i\in[n]} q(i|w)\leq 1}} \sum_{i=1}^n \Big\{ v_i + \mu_i\big(w-\alpha_i\big)\Big\} \cdot q(i|w).
\end{aligned} \tag{4}
$$

After relaxing the participation constraints, the Lagrangian decouples over characteristics w. Specifically, define

$$
\ell_i\big(w;\mu_i\big) \triangleq v_i + \mu_i\big(w-\alpha_i\big)
$$

as the line associated with employer $i \in [n]$. This line passes through the point (α_i, v_i) and has a nonnegative slope $\mu_i \geq 0$. In addition, let

$$
h\big(w;\boldsymbol{\mu}\big) \triangleq \max_{i\in[n]} \ell_i\big(w;\mu_i\big) = \max_{i\in[n]} \Big\{ v_i + \mu_i\big(w-\alpha_i\big)\Big\}
$$

denote the maximum of the n lines and $\bar{h}(w;\boldsymbol{\mu}) \triangleq \max\big\{h(w;\boldsymbol{\mu}), 0\big\}$. Both functions $h(w;\boldsymbol{\mu})$ and $\bar{h}(w;\boldsymbol{\mu})$ are convex, increasing (since $\mu_i \geq 0$), and piecewise linear in w. Let $\mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu})$ denote the set of optimal solutions $\{q(i|w)\}$ to $V^{\mathrm{LR}}(\boldsymbol{\mu})$; according to (4), it can be expressed as follows:

$$
\begin{aligned}
\mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu}) = \Bigg\{ q(i|w) \geq 0 : \sum_{i\in[n]} q(i|w) \leq 1,\ \forall\, w \in [0,1],\ \sum_{i\in[n]} q(i|w) = 1 \text{ if } h(w;\boldsymbol{\mu}) > 0, \text{ and} \\
q(i|w) > 0 \text{ only if } \ \ell_i(w;\mu_i) = \bar{h}(w;\boldsymbol{\mu}) \Bigg\}.
\end{aligned} \tag{5}
$$

That is, an optimal solution of $V^{\mathrm{LR}}(\boldsymbol{\mu})$ promotes a candidate with characteristics w to employer i with a positive probability only if employer i's line $\ell_i(w;\mu_i)$ is above the x-axis and not dominated by other employers' lines $\{\ell_i(w;\mu_i)\}_{j\neq i}$ at the point w. Finally, from (4) we have

$$
V^{\mathrm{LR}}(\boldsymbol{\mu}) = \int_0^1 \bar{h}\big(w;\boldsymbol{\mu}\big)\, g(w)\, dw.
$$

Since every feasible policy to (3) is feasible to (4) and attains an objective that is no smaller, $\bar{V} \leq V^{\mathrm{LR}}(\boldsymbol{\mu})$ for any $\boldsymbol{\mu} \in \mathbb{R}^n_+$. We formally state this weak-duality property in Lemma 4.4.

Lemma 4.4 (Weak Duality). *We have $\bar{V} \leq V^{\mathrm{LR}}(\boldsymbol{\mu})$ for any dual variable $\boldsymbol{\mu} \in \mathbb{R}^n_+$.*

4.2.1 The Optimal Lagrangian Dual

Since the Lagrangian $V^{\mathrm{LR}}(\boldsymbol{\mu})$ is a convex function of $\boldsymbol{\mu}$ by (4), we can solve a convex optimization problem

$$V^{\mathrm{LR}} \triangleq \min_{\boldsymbol{\mu}\in\mathbb{R}_+^n} V^{\mathrm{LR}}(\boldsymbol{\mu}) \geq \bar{V} \tag{6}$$

to obtain the tightest Lagrangian relaxation bound V^{LR}. Let $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]} \in \operatorname{argmin}_{\boldsymbol{\mu}\in\mathbb{R}_+^n} V^{\mathrm{LR}}(\boldsymbol{\mu})$ denote an optimal Lagrangian dual variable; it can be solved efficiently according to Remark 4.1.

Remark 4.1 (Solving $\boldsymbol{\mu}^*$). From Danskin's theorem (Proposition 4.5.1 in Bertsekas et al. 2003) and the fact that a convex combination of two optimal solutions of (4) is also optimal to (4), the sub-differential (i.e., set of sub-gradients) of $V^{\mathrm{LR}}(\boldsymbol{\mu})$ at any $\boldsymbol{\mu} \in \mathbb{R}_+^n$, denoted by $\partial V^{\mathrm{LR}}(\boldsymbol{\mu})$, can be expressed as

$$\partial V^{\mathrm{LR}}(\boldsymbol{\mu}) = \left\{ (g_i)_{i\in[n]} \text{ with } g_i \triangleq \int_0^1 (w-\alpha_i)\, q(i|w)\, g(w)\, dw : \{q(i|w)\} \in \mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu}) \right\}.$$

Since $V^{\mathrm{LR}}(\boldsymbol{\mu})$ and its sub-gradients can be efficiently computed, we can apply sub-gradient-based methods (e.g., the sub-gradient method or the cutting-plane method) to solve the convex program (6) and determine an optimal Lagrangian dual variable $\boldsymbol{\mu}^*$ efficiently.

Furthermore, Lemma 4.5 shows that strong duality holds, which follows standard strong duality for general convex optimization in a vector space.

Lemma 4.5 (Strong Duality). *Problem (3) and its Lagrangian relaxation (4) have the following relationship.*

1. *Strong duality holds, and there exists an optimal dual variable $\boldsymbol{\mu}^* \in \mathbb{R}_+^n$; that is, $\bar{V} = V^{\mathrm{LR}} = V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$.*
2. *$\boldsymbol{\mu} \in \mathbb{R}_+^n$ is an optimal dual variable and $\{q(i|w)\}$ is an optimal solution of (3) if and only if (1) $\{q(i|w)\} \in \mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu})$, and (2) all participation constraints in (3) are binding with $\{q(i|w)\}$.*

We prove Lemma 4.5 in Appendix A.6. Lemma 4.5 implies multiple structural properties of any optimal solution $\{q^*(i|w)\}$ and optimal dual variable $\boldsymbol{\mu}^*$ of (3). First, we show that if an employer i is considered by the sender—that is, $q_i^* \triangleq \int_0^1 q^*(i|w)dw > 0$—then the point (α_i, v_i) lies on the envelope function $h(w;\boldsymbol{\mu}^*)$ and within the interior of the line segment associated with employer i. We state this in Proposition 4.6 and provide its proof in Appendix A.7.

Proposition 4.6. *Let $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]}$ be an optimal dual variable of (3). Suppose there exists an optimal solution $\{q^*(i|w)\}$ to (3) such that $\int_0^1 q^*(i|w)dw > 0$ (i.e., employer i is considered by the sender). Then, there exist constants $\underline{b}, \bar{b} \in [0,1]$ satisfying $0 < \underline{b} < \alpha_1 < \bar{b} \leq 1$ such that $\bar{h}(w;\boldsymbol{\mu}^*) = \ell_i(w;\mu_i^*)$ (i.e., employer i's line is above the x-axis and other employers' lines) if and only if $w \in [\underline{b}, \bar{b}]$.*

Second, in Proposition 4.7, we show that the optimal dual variables $\{\mu_i^*\}$ are strictly positive and decreasing (after removing disregarded employers). Additionally, a candidate receives an offer with equal probability under any optimal solution of (3). We prove Proposition 4.7 in Appendix A.8.

Proposition 4.7. *Let $\{q^*(i|w)\}$ be an optimal solution and $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]}$ an optimal dual variable of (3). The following hold:*

1. *$\mu_i^* > 0$ for all $i \in [n]$. Therefore, the envelope function $h(w;\boldsymbol{\mu}^*)$ is strictly increasing in w.*
2. *The cutoff value in Proposition 4.3 is unique. Specifically, $\sum_{i\in[n]} \int_0^1 q^*(i|w)dw = \mathbb{P}(w \geq z^*)$ for $z^* \in (0,1)$, where z^* is the root of $h(w;\boldsymbol{\mu}^*)$, meaning that $h(z^*;\boldsymbol{\mu}^*) = 0$.*
3. *Let $P \triangleq \{i \in [n] : q_i^* \triangleq \int_0^1 q^*(i|w)dw > 0\}$ be the set of employers the sender considers. Then, $\{\mu_i^*\}_{i\in P}$ decreases with the employer index i.*

Finally, according to Lemma 4.5 Bullet 2, if we can find an optimal solution $\{q(i|w)\}$ of $V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ that ensures that all participation constraints in (3) are binding, then $\{q(i|w)\}$ is also optimal to (3) and provides an optimal (public) persuasion mechanism. However, how to identify such a desirable $\{q(i|w)\} \in \mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ through appropriate tie-breaking remains generally unclear. Despite this, in Section 4.3, we apply Lemma 4.5 to derive the optimal persuasion mechanism for the case of two receivers.

4.3 Two-Receiver Case

In this section, we assume that there are two employers $i \in \{1,2\}$, with offer values $v_1 > v_2 > 0$ and recruiting thresholds $\alpha_1 > \alpha_2 > w_0 > 0$, and we derive the optimal public persuasion mechanism based on Bullet 2 of Lemma 4.5. According to the revelation principle, it is optimal to focus on public persuasion mechanisms with a signal space $S \triangleq \{1, 2, \varnothing\}$, where the signal $s = i$ represents a recommendation for only employer i to extend an offer, and $s = \varnothing$ represents a recommendation for none of the employers to extend an offer. Moreover, incentive compatibility (IC) constraints must hold. Specifically, conditioning on the signal $s = 1$, the posterior mean of the candidate's characteristics w is α_1; therefore, employer 1 will extend an offer, and employer 2 will be indifferent because the candidate will certainly select employer 1. Conditioning on $s = 2$, the posterior mean of w is α_2, causing only employer 2 to extend an offer.[5] Finally, conditioning on $s = \varnothing$, the posterior mean of w is smaller than α_2, so neither employer will extend an offer. Let $\mathcal{M}$ denote the set of mechanisms with signal space S that satisfy the IC constraints. Specifically, $\{q(i|w)\} \in \mathcal{M}$ if they satisfy the following constraints with $n = 2$:

$$\begin{aligned}
&\int_0^1 w \cdot q(i|w)\, g(w)\, dw = \alpha_i \int_0^1 q(i|w)\, g(w)\, dw,\ \forall\, i \in [n],\\
&\int_0^1 w \cdot q(\varnothing|w)\, g(w)\, dw < \alpha_n \int_0^1 q(\varnothing|w)\, g(w)\, dw,\\
&\sum_{i\in[n]} q(i|w) + q(\varnothing|w) = 1,\ \forall\, w \in [0,1],\\
&q(i|w) \geq 0,\ \forall\, w \in [0,1],\ i \in [n] \cup \{\varnothing\}.
\end{aligned}$$

Additionally, let $q_i(M) \triangleq \int_0^1 q(i|w)\, g(w)\, dw$ denote the probability that the candidate selects employer i given a mechanism $M = \{q(i|w)\} \in \mathcal{M}$.

4.3.1 Preparation: Extreme Mechanisms Focusing on a Single Receiver

We first consider two extreme mechanisms in which the sender prioritizes either receiver 1 or 2 as preparation for characterizing the optimal mechanism in Section 4.3.2.

First, consider the mechanism $M_2 \in \mathcal{M}$ where the sender completely targets employer 2. Specifically, let $z_2 > 0$ be such that $\mathbb{E}[w|w \geq z_2] = \alpha_2$.[6] The sender transmits the signal $s = 2$ when $w \geq z_2$. Upon receiving the signal, only employer 2 will extend an offer. Since $z_2 < \alpha_1$,[7] the sender can no longer persuade employer 1 to extend an offer once targeting employer 2. Therefore, the sender transmits the signal $s = \varnothing$ when $w < z_2$. As a result, $q_1(M_2) = 0$ and $q_2(M_2) = \mathbb{P}[w \geq z_2]$. The sender receives an offer if and only if $w \geq z_2$, which occurs with probability $\mathbb{P}[w \geq z_2]$.

Second, consider the mechanism $M_1 \in \mathcal{M}$ where the sender prioritizes employer 1. Let $\bar{z}_1 > 0$ be such that $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$. The sender transmits the signal $s = 1$ when $w \geq \bar{z}_1$. Thus, $q_1(M_1) = \mathbb{P}[w \geq \bar{z}_1]$. Then, the following two scenarios can occur depending on the value of $\bar{z}_1$ relative to α_2:

- If $\bar{z}_1 > \alpha_2$: The sender can still persuade employer 2 to extend an offer after targeting receiver

[5] According to Proposition 4.2, conditioning on the signal $s = i$, the posterior mean of the candidate's characteristics w equals α_i under any optimal public mechanism.

[6] $z_2 > 0$ because $\alpha_2 > w_0$ by Assumption 4.3.

[7] Otherwise, the posterior mean would exceed α_1, which is larger than α_2.

1. Specifically, find a real value z_1 with $0 < z_1 < \alpha_2 < \bar{z}_1$ such that $\mathbb{E}[w|z_1 \le w < \bar{z}_1] = \alpha_2$. The sender transmits the signal $s = 2$ when $z_1 \le w < \bar{z}_1$ and transmits the signal $s = \varnothing$ when $w < z_1$. Therefore, $q_2(M_1) = \mathbb{P}[z_1 \le w \le \bar{z}_1]$.

- If $\bar{z}_1 \le \alpha_2$: The sender can no longer persuade employer 2 to extend an offer once targeting employer 1. In this case, let $z_1 = \bar{z}_1$. The sender transmits the signal $s = \varnothing$ when $w < z_1$; therefore, $q_2(M_1) = 0$.

In both scenarios, the sender receives an offer if and only if $w \ge z_1$, which occurs with probability $\mathbb{P}[w \ge z_1]$.

Proposition 4.8 provides several properties regarding any mechanism in the set $\mathcal{M}$.

Proposition 4.8. *Given any public persuasion mechanism $M \in \mathcal{M}$, we have the following:*

1. $q_1(M) \le q_1(M_1) = \mathbb{P}[w \ge \bar{z}_1]$.
2. $q_2(M) \le q_2(M_2) = \mathbb{P}[w \ge z_2]$.
3. *Suppose that, under a mechanism $M \in \mathcal{M}$, the candidate receives an offer if and only if $w \ge z$ for some real value $z \in (0, 1)$. Furthermore, assume $z < \bar{z}_1$ if $\alpha_2 < \bar{z}_1$. Then, $z \in [z_2, z_1]$.*[8]
4. *Conversely, for any value $z \in [z_2, z_1]$, there exists a mechanism $M \in \mathcal{M}$ such that the candidate receives an offer if and only if $w \ge z$. Moreover, for any such mechanism M, we have $q_1(M) = \mathbb{P}[w \ge z] \cdot \frac{\mathbb{E}[w|w \ge z] - \alpha_2}{\alpha_1 - \alpha_2} \ge 0$ and $q_2(M) = \mathbb{P}[w \ge z] \cdot \frac{\alpha_1 - \mathbb{E}[w|w \ge z]}{\alpha_1 - \alpha_2} \ge 0$.*

We prove Proposition 4.8 in Appendix A.9. Intuitively, the probability of joining employer 1 is highest when the sender primarily targets employer 1 (using mechanism M_1). However, this also lowers the probability of receiving any offer (which is $\mathbb{P}[w \ge z_1]$) among the "reasonable" mechanisms depicted in Bullet 3 because employer 1 is more challenging to get into. Conversely, the probability of receiving an offer is the highest when the sender exclusively targets the less competitive employer 2 (using mechanism M_2), which is $\mathbb{P}[w \ge z_2]$. Furthermore, Bullet 4 shows that any acceptance probability between these two extremes can be sustained by a mechanism that carefully balances the two employers. As we will show in Section 4.3.2, the optimal mechanism in the two-receiver case can be any of the mechanisms in Bullet 4, depending on the desirability (v_1) and hiring bar (α_1) of employer 1 relative to those of employer 2.

[8] Note that any optimal persuasion mechanism has such a cutoff structure according to Proposition 4.3. Moreover, when $\alpha_2 < \bar{z}_1$, since the sender can still persuade employer 2 to extend an offer after targeting receiver 1, there is no loss of optimality to assume that $z < \bar{z}_1$.

We conclude this section with a remark that interprets the two probabilities $q_1(M)$ and $q_2(M)$ in Bullet 4 of Proposition 4.8.

Remark 4.2 (Interpreting Bullet 4 of Proposition 4.8)**.** To understand the two probabilities $q_1(M)$ and $q_2(M)$, note that for any mechanism $M \in \mathcal{M}$ with a cutoff structure and a threshold value of z, the probabilities $q_1 \triangleq q_1(M)$ and $q_2 \triangleq q_2(M)$ must satisfy the following two linear equations:

$$\begin{aligned} q_1 + q_2 &= \mathbb{P}[w \geq z], \\ \alpha_1 q_1 + \alpha_2 q_2 &= (q_1 + q_2) \cdot \mathbb{E}[w | w \geq z], \end{aligned} \tag{7}$$

where the first equation follows from the fact that the candidate receives an offer (from either employer 1 or 2) if and only if $w \geq z$, and the second equation follows from the fact that the IC constraints are binding (i.e., $\mathbb{E}[w|s = i] = \alpha_i$) and the law of total expectation. These two equations uniquely determine the values of q_1 and q_2, as stated in Bullet 4 of Proposition 4.8. Conversely, consider a mechanism M that sends the signal $s = \varnothing$ if and only if $w < z$, and suppose the values of q_1 and q_2 satisfy (7). If $\mathbb{P}[s = 1] = q_1$ and $\mathbb{E}[w|s_1] = \alpha_1$, then it follows that $\mathbb{P}[s = 2] = q_2$ and $\mathbb{E}[w|s_2] = \alpha_2$, and vice versa.

4.3.2 Optimal Mechanism with Two Receivers

In this section, we characterize the optimal persuasion mechanism with two receivers. Intuitively, there is a trade-off: An offer from employer 1 brings a higher payoff, but targeting employer 1 more aggressively reduces the overall probability of receiving an offer.

Notably, according to Proposition 4.8 Bullet 3, the range of characteristics $w \in [z, 1]$ for which the candidate receives an offer satisfies $z \in [z_2, z_1]$ for any reasonable mechanism $M \in \mathcal{M}$. This introduces two lines, one of which (denoted by η_1) passes through the points $(z_1, 0)$ and (α_2, v_2), and the other (denoted by η_2) passes through the points $(z_2, 0)$ and (α_2, v_2), as illustrated in Figure 1. These two lines partition the value of $v_1 \in [v_2, \infty)$ into three regions.

Lemma 4.9 illustrates the optimal mechanism for the two-receiver case. It demonstrates that when the value of v_1 is sufficiently large (particularly, above line η_1), prioritizing employer 1 is optimal. Conversely, when v_1 is sufficiently small (particularly, below line η_2), completely targeting employer 2 is optimal. Finally, if v_1 falls between the two lines, the optimal mechanism requires a non-trivial trade-off between the two employers.

Lemma 4.9. *Under Assumptions 4.1 – 4.3, the optimal public persuasion mechanism with two receivers is given as follows:*

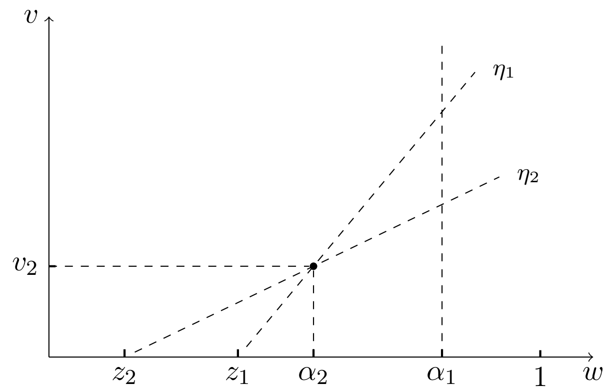

Figure 1: Partition of the value of v_1 in Lemma 4.9.

1. *If* $v_1 \le v_2 \cdot \left(\frac{\alpha_1 - z_2}{\alpha_2 - z_2}\right)$ *(i.e., the point* (α_1, v_1) *lies below line* η_2*), mechanism* M_2 *(fully targeting employer* 2*) is optimal;*
2. *If* $v_1 \ge v_2 \cdot \left(\frac{\alpha_1 - z_1}{\alpha_2 - z_1}\right)$ *(i.e., the point* (α_1, v_1) *lies above line* η_1*), mechanism* M_1 *(prioritizing employer* 1*) is optimal;*
3. *Otherwise, any mechanism* $M \in \mathcal{M}$ *satisfying Bullet 4 of Proposition 4.8 with a cutoff value* z^**, where* $z^* \triangleq \alpha_2 - v_2 \cdot \left(\frac{\alpha_1 - \alpha_2}{v_1 - v_2}\right) \in [z_2, z_1]$ *represents the* x*-intercept of the line passing through the points* (α_2, v_2) *and* (α_1, v_1)*, is optimal. In other words, the mechanism* $M \in \mathcal{M}$ *satisfies the following:*

   (a) It sends the signal $s = \varnothing$ *with a probability of one if* $w < z^*$ *and a probability of zero if* $w \ge z^*$*;*

   (b) IC constraints hold (required by definition of $\mathcal{M}$*), that is,* $\mathbb{E}[w|s = i] = \alpha_i$ *for* $i \in \{1, 2\}$*;*

   (c) $q_1(M) = q_1^* \triangleq \mathbb{P}[w \ge z^*] \cdot \frac{\mathbb{E}[w|w \ge z^*] - \alpha_2}{\alpha_1 - \alpha_2}$ *and* $q_2(M) = q_2^* \triangleq \mathbb{P}[w \ge z^*] \cdot \frac{\alpha_1 - \mathbb{E}[w|w \ge z^*]}{\alpha_1 - \alpha_2}$ *(as indicated by Bullet 4 of Proposition 4.8).*

We prove Lemma 4.9 in Appendix A.10. In the proof, we identify a set of dual variables $\boldsymbol{\mu} \in \mathbb{R}_+^n$, which, along with the proposed mechanism, satisfies Lemma 4.5 Bullet 2. This indicates that the mechanism is optimal to (3), and $\boldsymbol{\mu}$ is an optimal dual variable.

When there is a non-trivial trade-off between the two employers (Case 3 of Lemma 4.9), the optimal Lagrangian dual variable is $\boldsymbol{\mu}^* = \left(\mu_1^*, \mu_2^*\right)$ with $\mu_1^* = \mu_2^* = \frac{v_1 - v_2}{\alpha_1 - \alpha_2} > 0$ that equal the slope of the line passing through the points (α_2, v_2) and (α_1, v_1). This implies that the two employers' lines $\ell_1\left(w; \mu_1^*\right)$ and $\ell_2\left(w; \mu_2^*\right)$ completely overlap and coincide with this line (as visualized in Figure 3(b)). Therefore, according to (5), any allocation $\{q(i|w)\}$ with $q(1|w) + q(2|w) = 1$ for $w \ge z^*$ and

$q(1|w) = q(2|w) = 0$ for $w < z^*$ is optimal to the Lagrangian $V^{\text{LR}}(\boldsymbol{\mu}^*)$. As long as we appropriately divide probability one between $q(1|w)$ and $q(2|w)$ for any $w \geq z^*$, ensuring that both employers' IC constraints are binding, then it follows that the candidate joins each employer i with a probability of q_i^* according to Proposition 4.8 Bullet 4, and that the mechanism $\{q(i|w)\}$ is optimal to $\bar{V}$ according to Lemma 4.9.

We note that the (unique) aggregate allocation probabilities $\{q_i^*\}$, together with the binding IC constraints, characterize an optimal mechanism. Conversely, there are various ways to construct a set of probabilities $\{q(i|w)\}$ that satisfies Bullet 3 of Lemma 4.9 and is thus optimal to (3), as we show in the proof of Bullet 4 of Proposition 4.8. Below, we showcase two simple approaches to construct an optimal mechanism as follows and illustrate them in Example 4.1.

- (*Randomized Mechanism with a Monotone Structure*) Let $q(1|w) = q_1^*/\mathbb{P}[w \geq \bar{z}_1] \leq 1$ for $w \geq \bar{z}_1$ and $q(1|w) = 0$ otherwise, recalling that $\bar{z}_1 > 0$ is defined such that $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$. Additionally, let $q(2|w) = 1 - q(1|w)$ for $w \geq z^*$ and $q(2|w) = 0$ otherwise. This corresponds to a randomized persuasion mechanism that satisfies Bullet 3 of Lemma 4.9 and is thus optimal to (3). Note that the candidate's expected payoff, $v(w) \triangleq \sum_i v_i q(i|w)$, is increasing in w by constriction, which can be desirable in practice.[9]

- (*Deterministic Mechanism with a Double-Interval Structure*) Identify an interval $\left[\underline{b}, \bar{b}\right] \subseteq \left[\bar{z}_1, 1\right]$ such that $\mathbb{P}\left[\underline{b} \leq w \leq \bar{b}\right] = q_1^*$ and $\mathbb{E}\left[w|\underline{b} \leq w \leq \bar{b}\right] = \alpha_1$.[10] Let $q(1|w) = 1$ for $w \in \left[\underline{b}, \bar{b}\right]$, $q(2|w) = 1$ for $w \in \left[z^*, \underline{b}\right) \cup \left(\bar{b}, 1\right]$, and $q(\varnothing|w) = 1$ for $w < z^*$. This corresponds to the deterministic persuasion mechanism described in Candogan (2022). The mechanism satisfies Bullet 3 of Lemma 4.9 and is thus optimal to (3). Additionally, the mechanism exhibits a double-interval structure as each signal is associated with at most two intervals. The candidate's expected payoff is not monotone in w with this mechanism.

Example 4.1. Suppose $w \sim \text{Unif}[0, 1]$ follows a uniform distribution with support $[0, 1]$, the candidate's payoffs from the two employers' offers are $v_1 = 2$ and $v_2 = 1$, and the employers' threshold values are $\alpha_1 = 0.9$ and $\alpha_2 = 0.7$. Given these values, we have $\bar{z}_1 = 0.8$, $z_1 = 0.6$, $z^* = 0.5$, and $z_2 = 0.4$. The optimal dual variables are $\mu_1^* = \mu_2^* = 5$. Figure 2(a) illustrates the lines $\ell_1(w; \mu_1^*)$ and $\ell_2(w; \mu_2^*)$ of the two employers, which fully overlap. Additionally, we have $z^* = 0.5$, $q_1^* = 1/8$, and $q_2^* = 3/8$. There are multiple ways to construct an optimal public persuasion mechanism that

[9] An increasing payoff function $v(w)$ incentivizes students to invest effort in improving their "quality" w.

[10] Since $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$ by definition of $\bar{z}_1$ and $\mathbb{P}[w \geq \bar{z}_1] \geq q_1^*$ by Proposition 4.8 Bullet 1, such an interval exists.

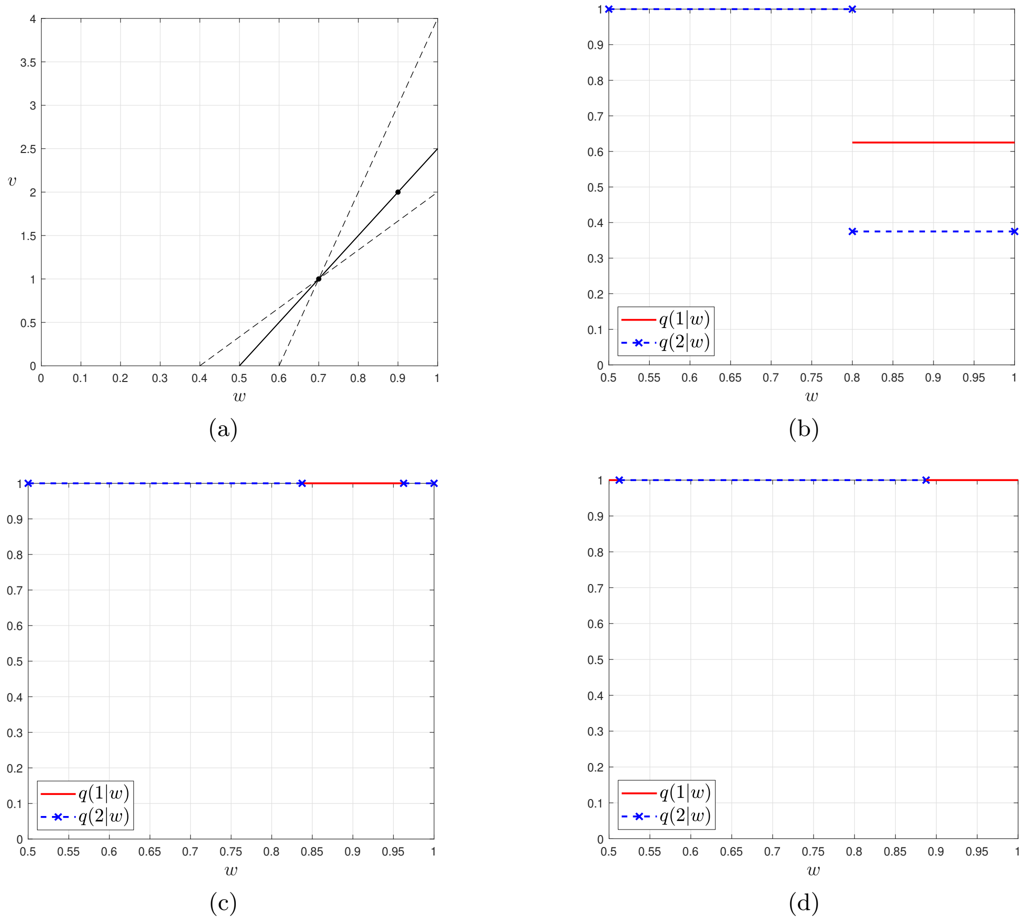

Figure 2: (a) Illustration of the two employers' lines $\ell_1(w; \mu_1^*)$ and $\ell_2(w; \mu_2^*)$ (solid), which pass through the two points $(0.7, 1)$ and $(0.9, 2)$ and fully overlap, and the two lines η_1 and η_2 in Lemma 4.9 (dashed). (b) A random optimal persuasion mechanism with $q(1|w) = 5/8$ for $w \in [0.8, 1]$, $q(2|w) = 3/8$ for $w \in [0.8, 1]$ and $q(2|w) = 1$ for $w \in [0.5, 0.8]$. (c) A deterministic optimal persuasion mechanism with $q(1|w) = 1$ for $w \in [0.8375, 0.9625]$ (centered around 0.9 and of length 1/8) and $q(2|w) = 1$ for $w \in [0.5, 0.8375] \cup [0.9625, 1]$. (d) A deterministic optimal persuasion mechanism with $q(2|w) = 1$ for $w \in [0.5125, 0.8875]$ (centered around 0.7 and of length 3/8) and $q(1|w) = 1$ for $w \in [0.5, 0.5125] \cup [0.8875, 1]$.

satisfies Lemma 4.9 Bullet 3. The previously described random persuasion mechanism is illustrated in Figure 2(b). The previously described deterministic persuasion mechanism is illustrated in Figure 2(c). We can also construct a second deterministic persuasion mechanism with a double-interval structure for this problem instance, where the signal $s = 1$ is associated with two intervals, as illustrated in Figure 2(d).

Remark 4.3 (Optimality of Public Persuasion Absent Communication Channel)**.** Finally, we remark that even if the sender knows the receivers have no communication channel and thus cannot communicate, a public persuasion mechanism remains optimal by Lemma 3.2. Conversely, treating receivers in isolation and sending separate signals to each employer using their respective optimal persuasion mechanisms is suboptimal. This involves sending the signal $s = 1$ to employer 1 when $w \geq \bar{z}_1$ and the signal $s = 2$ to employer 2 when $w \geq z_2$. Despite the lack of communication, employer 2, aware of the presence of a more preferred employer 1, will never extend an offer upon receiving the signal $s = 2$. This is because the candidate will select employer 1 with a positive probability, leading employer 2 to a negative expected utility from the candidates who accept his offer. Notably, only candidates with characteristics $w \in [z_2, \bar{z}_1)$ will select employer 2, whose expected quality is $\mathbb{E}[w|z_2 \leq w < \bar{z}_1] < \mathbb{E}[w|w \geq z_2] = \alpha_2$ and is thus smaller than α_2.

4.4 General Case

In this section, we solve (3) in the general case by reducing it to a convex optimization problem (8) with n decision variables and constraints, which can be solved efficiently. Problem (8) is analogous to problem (OPT) in Candogan (2022) but with n fewer decision variables and constraints. We also build the connection between the optimal dual variables of (3) and (8) and derive an optimal persuasion mechanism from the optimal solution of (8).

$$
\begin{aligned}
V^{\mathrm{CR}} = \max_{q_i \geq 0} \quad & \sum_{i=1}^{n} v_i q_i \\
\text{s.t.} \quad & \sum_{i \leq k} \alpha_i q_i \leq \sum_{i \leq k} q_i \cdot \mathbb{E}\left[w \,\middle|\, G(w) \geq 1 - \sum_{i \leq k} q_i \right] = \int_{1-\sum_{i \leq k} q_i}^{1} G^{-1}(x)\, dx, \; \forall\, k \in [n], \\
& \sum_{i \in [n]} q_i \leq 1.
\end{aligned} \tag{8}
$$

We first interpret (8). The decision variables q_i represent the aggregate probabilities that the candidate joins employer $i \in [n]$; specifically, q_i corresponds to $\int_0^1 q(i|w)g(w)dw$ in (3). The first constraint reflects the participation constraints for the employers. Only a limited portion of qualified candidates meet the employers' recruitment standards. This constraint requires that candidates within the top $\sum_{i \leq k} q_i$ quantile are sufficient to meet the recruiting bars (α_i) of the top k employers, given that each employer $i \in [k]$ would recruit a proportion q_i of candidates. This is a necessary condition to sustain the participation of the first k employers. The equation in this constraint follows from the fact that the random variable $G(w)$ follows a uniform distribution on $[0, 1]$. Finally, we remark that (8) is a convex optimization problem. To see this, note that $h(x) \triangleq \int_{1-x}^{1} G^{-1}(s)\, ds$ is a concave function because its derivative, $h'(x) = G^{-1}(1 - x)$, decreases in x. Therefore, the right-hand side of the first constraint is a concave function of $\{q_i\}$ because it is the composition of $h(\cdot)$ with an affine mapping.

Given a feasible solution $\{q(i|w)\}$ to (3), the set $\{q_i\}$ with $q_i = \int_0^1 q(i|w)g(w)dw$ is feasible to (8) and attains the same objective value. Therefore, (8) is a relaxation of (3). Conversely, analogous to the two-receiver case (Section 4.3), the aggregate allocation probabilities $\{q_i\}$, along with the binding IC constraints, characterize an optimal mechanism.[11] Notably, given an optimal solution $\{q_i^*\}$ to (8), we can construct a public persuasion mechanism that obtains the optimal value V^{CR}. Therefore, the relaxation (8) is tight. We state this in Proposition 4.10 and provide the proof in Appendix A.11.

Proposition 4.10. *The optimal values of (3) and (8) are equal; that is, $\bar{V} = V^{\mathrm{CR}}$. Furthermore, let $\{q^*(i|w)\}$ be an optimal solution to (3). Then, $\{q_i^*\}$, where $q_i^* = \int_0^1 q^*(i|w)g(w)dw$, is an optimal solution to (8). Conversely, if $\{q_i^*\}$ is an optimal solution to (8), then there exists an optimal solution $\{q^*(i|w)\}$ to (3) such that $q_i^* = \int_0^1 q^*(i|w)g(w)dw$.*

Let $\{q_i^*\}$ be an optimal solution to (8). In the following, we assume that $q_i^* > 0$ for all $i \in [n]$. Otherwise, we can disregard those employers with $q_i^* = 0$ with no loss of optimality.

4.4.1 Connection between the Optimal Dual Variables

In this section, we establish the connection between the optimal dual variables for the participation constraints in (3) and (8). Specifically, we show that an optimal dual variable for one problem can be converted to an optimal dual variable for the other. This provides an additional perspective on the equivalence of (3) and (8).

Let $\mathbf{q} = (q_i)_{i\in[n]} \in \mathbb{R}^n$ be a vector of allocation probabilities for the n employers, and let $L(\mathbf{q}, \boldsymbol{\lambda})$ be the Lagrangian function in which we dualize the participation constraints in (8) with a dual variable $\boldsymbol{\lambda} = (\lambda_k)_{k\in[n]} \in \mathbb{R}_+^n$; that is:

$$L(\mathbf{q}, \boldsymbol{\lambda}) = \sum_{i=1}^n v_i q_i + \sum_{k\in[n]} \lambda_k \cdot \left(\int_{1-\sum_{i\le k} q_i}^1 G^{-1}(x)\, dx - \sum_{i\le k} \alpha_i q_i \right).$$

Let $\mathbf{q}^* = (q_i^*)_{i\in[n]} \in \mathbb{R}_+^n$ denote an optimal solution to (8) and $\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]} \in \mathbb{R}_+^n$ an optimal dual variable for the participation constraints. By the KKT conditions, $\mathbf{q}^*$ also solves the following Lagrangian problem:

$$\mathbf{q}^* \in \underset{\mathbf{q}\in\mathbb{R}_+^n,\ \sum_{i\in[n]} q_i \le 1}{\operatorname{argmax}} L(\mathbf{q}, \boldsymbol{\lambda}^*)\,.$$

[11] Nevertheless, there are various ways to construct an optimal mechanism $\{q(i|w)\}$.

Since $q_i^* > 0$ for any $i \in [n]$ by assumption and $\sum_{i\in[n]} q_i^* < 1$ by Proposition 4.1, the first-order optimality condition yields

$$\frac{\partial L}{\partial q_i}(\mathbf{q}^*, \boldsymbol{\lambda}^*) = v_i + \sum_{k\geq i} \lambda_k^* \cdot \left(G^{-1}\left(1 - \sum_{j\leq k} q_j^*\right) - \alpha_i\right) = 0. \tag{9}$$

Proposition 4.11 establishes the connection between the optimal dual variables of (3) and (8), demonstrating that each can be derived from the other.

Proposition 4.11. *Suppose there exists an optimal solution $\{q_i^*\}$ to (8) such that $q_i^* > 0$ for any $i \in [n]$ (i.e., no employer is disregarded). Then, the optimal Lagrangian dual variables for the participation constraints in (3) and (8), denoted by $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]} \in \mathbb{R}_+^n$ and $\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]} \in \mathbb{R}_+^n$, respectively, are unique and satisfy $\mu_i^* = \sum_{k\geq i} \lambda_k^*$ for all $i \in [n]$.*

We prove Proposition 4.11 in Appendix A.12 by comparing the optimality conditions (9) of (8) with the optimality conditions of (3) in Section 4.2. Since the dual variables λ_k^* are nonnegative, Proposition 4.11 indicates that the dual variables $\{\mu_i^*\}$ for (3) are decreasing. This aligns with Proposition 4.7 Bullet 3 and intuitively follows from the convexity of the envelope function $h(w; \boldsymbol{\mu}^*)$ in w.

4.4.2 Structural Properties of Optimal Mechanism

Let $\{q_i^*\}$ denote an optimal solution to (8) and assume that $q_i^* > 0$ for any $i \in [n]$. Additionally, let

$$T \triangleq \left\{k \in [n] : \lambda_k^* > 0\right\}$$

denote the set of positive entries of the optimal dual variable $\boldsymbol{\lambda}^*$ for (8). Due to the complementary slackness property, the participation constraint in (8) is binding with the top k employers if $k \in T$; that is,

$$\sum_{i\leq k} \alpha_i q_i^* = \mathbb{E}\left[w \cdot \mathbb{1}\left[w \geq G^{-1}\left(1 - \sum_{i\leq k} q_i^*\right)\right]\right]. \tag{10}$$

We note that $\lambda_n^* = \mu_n^* > 0$ by Bullet 1 of Proposition 4.7 and Proposition 4.11. Therefore, $n \in T$, and the set T is nonempty.

Now suppose $T = \{1 \leq t_1 < t_2 < \cdots < t_m = n\}$ contains m employers. These m employers partition the n employers into m groups $\{T_i\}_{i\in[m]}$, with $T_1 = [t_1]$ and $T_i = [t_{i-1} + 1 : t_i]$ for any $i \in [2 : m]$. Therefore, $\bigcup_{i\in[m]} T_i = [n]$ and $T_i \cap T_j = \varnothing$ for any $i \neq j$. Moreover, each group T_i contains exactly one element from T, which is the largest element in T_i.

If a set T_i contains more than one employer (i.e., $t_{i-1}+1 < t_i$), then for any $k \in [t_{i-1}+1 : t_i-1]$, we have the following:

$$\mu_k^* = \sum_{j \geq k} \lambda_j^* = \mu_{t_i}^* = \frac{v_k - v_{t_i}}{\alpha_k - \alpha_{t_i}}, \tag{11}$$

where the first equation follows from Proposition 4.11, the second equation from the fact that $\lambda_j^* = 0$ for any $j \in [t_{i-1}+1 : t_i - 1]$, and the third equation is derived by subtracting both sides of (9) with $i = t_i$ from both sides of the same equation with $i = k$. Therefore, the optimal dual variables of employers in the same set T_i are all equal and equal to $\mu_{t_i}^*$. Furthermore, (11) implies that the points $\{(v_j, \alpha_j)\}_{j \in T_i}$ lie on a line, and the employers' lines $\ell_j(w; \mu_j^*)$ for any $j \in T_i$ completely overlap and coincide with this line.

In Lemma A.1, we completely characterize the envelope function $h(w; \boldsymbol{\mu}^*)$. Specifically, define $z_i \triangleq G^{-1}\big(1-\sum_{j=1}^{t_i} q_j^*\big)$ for any $i \in [m]$ and let $z_0 = 0$. Every group T_i of employers is associated with an interval of state variable $w \in I_i \triangleq [z_i, z_{i-1}]$. Lemma A.1 shows that the convex and piecewise linear envelope function $h(w; \boldsymbol{\mu}^*)$ coincides with line $\ell_j(w; \mu_j)$ on the interval $w \in [z_i, z_{i-1}]$ for any $j \in T_i$. Additionally, the function $h(w; \boldsymbol{\mu}^*)$ intersects the x-axis at $w = z_m$. We formally state this in Lemma 4.12, with the proof provided in Appendix A.12.3.

Lemma 4.12. *Let $\{q_i^*\}$ be an optimal solution to (8), and assume that $q_i^* > 0$ for all $i \in [n]$ (i.e., we drop ignorable employers i with $q_i^* = 0$). Let $\boldsymbol{\mu}^* = (\mu_i^*)_{i \in [n]}$ denote the optimal Lagrangian dual variable for the participation constraints in (3). The following hold:*

1. *For any $i \in [m]$ and employer $j \in T_i$, we have $\alpha_j \in (z_i, z_{i-1})$.*
2. *For any group T_i, the lines $\ell_j(w; \mu_j)$ for $j \in T_i$ fully overlap and pass though the points (α_j, v_j) for any $j \in T_i$.*
3. *For any $i \in [m]$ and employer $j \in T_i$, $h(w; \boldsymbol{\mu}) = \ell_j(w; \mu_j)$ for any $w \in \big[z_i, z_{i-1}\big]$ and $h(w; \boldsymbol{\mu}) > \ell_j(w; \mu_j)$ otherwise.*
4. *$h(w; \boldsymbol{\mu})$ is nonnegative if and only if $w \geq z_m$.*

From (5) and Lemma 4.12, an allocation $\{q(i|w)\}$ is optimal to $V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ if and only if $\{q(i|w)\}$ are nonnegative, no larger than one, and satisfy the following:

$$\begin{aligned} &\sum_{j \in T_i} q(j|w) = 1, \, \forall \, w \in (z_i, z_{i-1}), \, i \in [m], \\ &\sum_{j \in [n]} q(j|w) = 0, \, \forall \, w < z_m. \end{aligned} \tag{12}$$

In other words, a solution of $V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ allocates the interval $I_i = [z_i, z_{i-1}]$ among the employers in the subset T_i for any $i \in [m]$.

Moreover, for any group T_i and employer $k \in T_i$, contracting both sides of the first constraint in (8) from both sides of (10) with $k = t_{i-1}$ and noting that the first constraint in (8) is binding with $k = t_i$ yields the following:

$$\begin{aligned} \sum_{j\in[t_{i-1}+1:k]} \alpha_j q_j^* &\le \mathbb{E}\Big[w \cdot \mathbb{1}\Big[G^{-1}\Big(1 - \sum_{j\le k} q_j^*\Big) \le w < z_{i-1}\Big]\Big], \forall\, k \in [t_{i-1}+1 : t_i - 1], \\ \sum_{j\in T_i} \alpha_j q_j^* &= \mathbb{E}\Big[w \cdot \mathbb{1}\Big[z_i \le w < z_{i-1}\Big]\Big]. \end{aligned} \tag{13}$$

In addition, note that $\mathbb{P}[z_i \le w < z_{i-1}] = \sum_{j\in T_i} q_j^*$. Analogous to the proof of Proposition 4.10, we can allocate the state $w \in [z_i, z_{i-1}]$ to the employers in the subset T_i, possibly randomly, so that each employer i is allocated with an aggregate size of $q_i^* > 0$, and the average quality of the allocation to employer i is α_i (i.e., the participation constraint is tight). In other words, there exists an optimal solution $\{q^*(i|w)\}$ to $V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$, which satisfies

$$\begin{aligned} &\int_{w\in I_i} q^*(j|w)\, g(w)\, dw = q_j^*, \forall\, j \in T_i,\ i \in [m], \\ &\int_{w\in I_i} w \cdot q^*(j|w)\, g(w)\, dw = \alpha_j \int_{w\in I_i} q^*(j|w)\, g(w)\, dw, \forall\, j \in T_i,\ i \in [m]. \end{aligned} \tag{14}$$

From (12), (14), and Lemma 4.5 Bullet 2, such a $\{q^*(i|w)\}$ is an optimal to (3). We summarize the above in Lemma 4.13.

Lemma 4.13 (Optimality Condition). *An allocation probability $\{q(j|w)\}$ is optimal to (3) if and only if it allocates only to employers in the set T_i for any $w \in I_i$ (i.e., (12) holds) and all participation constraints in (3) are binding with $\{q(j|w)\}$. Moreover, let $\{q_j^*\}$ denote an optimal solution to (8). We can construct an optimal solution $\{q^*(j|w)\}$ to (3) such that the candidate joins each employer with probability q_j^* (i.e., (14) holds).*

From Lemma 4.13, once we solve an optimal solution $\{q_i^*\}$ and an optimal dual variable $\boldsymbol{\lambda}^*$ of (8) and obtain the partition $\{T_i\}_{i\in[m]}$, the design problem decouples over groups. Specifically, for each group T_i, the optimal mechanism allocates the state variable $w \in I_i$ to the employers in group T_i in a way that ensures that the participation constraints are binding. When the group contains only one employer, we simply allocate the entire interval to the employer. When it contains multiple employers, the allocation needs to be conducted more carefully. Analogous to the two-receiver case (Section 4.3.2), there are multiple ways to construct an optimal mechanism. Specifically, based on

(13), we can construct an optimal solution iteratively. Given that we have allocated a size q_j^* of candidates from interval I_i with a mean quality of α_j to each employer j of the first k employers in group T_i, we can also allocate a size q_j^* of candidates from the remaining candidates in interval I_i with a mean quality of α_j to employer j, where j is the $(k+1)$-th candidate in group T_i. Repeat this process until we reach the last employer of group T_i, which is employer t_i. The remaining candidates, with a size of $q_{t_i}^*$ and a mean quality of α_{t_i}, can then be allocated to the last employer.

In Section 4.4.3, we present a method to determine an allocation at each iteration step to obtain an optimal solution $\{q^*(j|w)\}$ to (3) with a monotone structure.

4.4.3 Constructing an Optimal Mechanism with a Monotone Structure

In this section, we construct an optimal persuasion mechanism $\{q^*(j|w)\}$ iteratively that further satisfies a monotone property. Specifically, for any $w \geq w'$, the distribution $q^*(\cdot|w)$ will first-order stochastically dominate the distribution $q^*(\cdot|w')$.[12] Therefore, a student with a higher quality w is more likely to be in a better place, incentivizing a student to invest effort in improving her quality.

The monotone property automatically holds for two qualities w and w' from different intervals. Suppose $w \in I_i$ and $w' \in I_j$ with $i < j$. Since $\max T_i < \min T_j$, a student with quality w receives a better job for sure. This implies the first-order stochastically dominance. Therefore, we only need to ensure the monotone property for qualities within an interval I_i.

Algorithm 1 characterizes a way to construct an optimal solution $\{q(j|w)\}$ for $j \in T_i$ iteratively. The distribution $q(\cdot|w)$ from Algorithm 1 is first-order stochastically increasing in w; additionally, $q(j|w)$ is piecewise constant on $w \in I_i$ for any $j \in T_i$.

Algorithm 1: Optimal Persuasion Mechanism with a Monotone Structure

Input: Let $\{q_i^*\}$ be an optimal solution to (8).

Initialization: Take $b_{t_{i-1}} = z_{i-1}$ and $q_{\leq t_{i-1}}(w) = 0$ for any $w \in I_i$.

for $k \in T_i = [t_{i-1}+1 : t_i - 1]$ **do**

  Identify two values $b_k \in \left[z_i, b_{k-1}\right]$ and $\rho_k \in [0,1]$ to ensure that (14) holds for employer k with $q(k|w) = \rho_k \cdot (1 - q_{\leq k-1}(w))$ for $w \in [b_k, z_{i-1}]$ and $q(k|w) = 0$ for $w < b_k$;

  Let $q_{\leq k}(w) = q_{\leq k-1}(w) + q(k|w)$ for any $w \in I_i$.

end

Let $q(t_i|w) = 1 - q_{\leq t_i - 1}(w)$ for any $w \in I_i$ and take $b_{t_i} = z_i$.

In Algorithm 1, $q_{\leq k}(w) = \sum_{j \leq k} q(j|w)$ represents the probability that a candidate with quality w receives an offer from one of the top k employers. Note that for any $k \leq t_{i-1}$, $q_{\leq k}(w) = 0$ for

[12] More precisely, we have $\sum_{k \leq i} q^*(k|w) \geq \sum_{k \leq i} q^*(k|w')$ for any $i \in [n]$. Note that an offer from a lower-indexed employer provides a higher payoff to the candidate by Assumption 2.1.

$w \in I_i$. In each iteration, we allocate a ratio ρ_k of the remaining candidates whose quality is at least b_k to employer k. The values of ρ_k and b_k are selected so that candidates select employer k with a probability of q_k^*, and the mean quality of these candidates are α_k (i.e., (14) holds for employer k).

We note that the decreasing sequence $\{b_k\}_{k \in T_i}$ partitions the interval I_i into subintervals $I_{ik} \triangleq [b_k, b_{k-1}]$ for $k \in T_i$. Moreover, the probabilities $q(j|w)$ equal a constant $q_j(k)$ on each subinterval $w \in I_{ik}$, where the values of $q_j(k)$ satisfy the following:

$$
\begin{aligned}
&q(j|w) = q_j(k) = 0, \forall\, k \geq j+1,\, w \in I_{ik}\,, \\
&q(j|w) = q_j(k) = \rho_j \cdot \Big(1 - q_{\leq j-1}(w)\Big) = \rho_j \cdot \prod_{\ell=k}^{j-1} \Big(1 - \rho_\ell\Big), \forall\, k \leq j,\, w \in I_{ik}\,.
\end{aligned}
$$

Proposition 4.14 demonstrates that the values of $\{\rho_k\}$ and $\{b_k\}$ in Algorithm 1 exist, and the allocation $\{q(j|w)\}$ returned by Algorithm 1 is optimal to (3) and satisfies the first-order stochastic increasing property.

Proposition 4.14. *The allocation $\{q(j|w)\}$ returned by Algorithm 1 is optimal to (3) and satisfies the first-order stochastic increasing property.*

We prove Proposition 4.14 and show that the values of $\{\rho_k\}$ and $\{b_k\}$ can be easily identified in Appendix A.13. Note that when a subset T_i contains two employers, the allocation $\{q(j|w)\}$ returned by Algorithm 1 concurs with the randomized mechanism with a monotone structure described in Section 4.3.2 for the two-receiver case, on the interval I_i.

Finally, we present two useful properties of the optimal solutions of (8) as established in Candogan (2022). Specifically, there exists an optimal solution $\{q_k^*\}$ such that each subset T_i contains at most two employers with a positive probability of q_k^*. Furthermore, the optimal solution of (8) is unique if no three points of $\{(\alpha_i, v_i)\}_{i \in [n]}$ are collinear. We state these two properties in Proposition 4.15.

Proposition 4.15. *The optimal solutions of (8) satisfy the following two properties.*

1. *(Lemma 4 of Candogan 2022) Let $\{\lambda_k^*\}$ denote an optimal dual variable associated with the participation constraints in (8) and $\{T_i\}$ denote the corresponding partition of the n employers as described in Section 4.4.2. There exists an optimal solution $\{q_k^*\}$ to (8) such that $|T_i \cap P| \leq 2$ for any i, where $P \triangleq \{k \in [n] : q_k^* > 0\}$ denote the set of positive entries of $\{q_k^*\}$. In other words, each set T_i contains at most two employers with a positive probability of q_k^*.*

2. *(Appendix D of Candogan 2022) Problem (8) has a unique optimal solution* $\{q_k^*\}$ *if no three points of* $\{(\alpha_i, v_i)\}_{i\in[n]}$ *are collinear.*

We prove Proposition 4.15 in Appendix A.14 based on our previous dual analysis. Note that Bullet 2 of Proposition 4.15 becomes intuitive. Since no three points of $\{(\alpha_i, v_i)\}_{i\in[n]}$ are collinear, any group T_i contains at most two employers with a positive probability of q_k^* according to Bullet 2 of Lemma 4.12. Additionally, the values of the two probabilities are uniquely determined by two linear equations analogous to (7). We provide more details in Appendix A.14.

Proposition 4.15 implies that the information design problem in the general case can be decomposed into design problems with two receivers for each group T_i. By applying the deterministic mechanism with a double-interval structure, as described in Section 4.3.2, to each group T_i, we obtain the deterministic persuasion mechanism detailed in Candogan (2022).

5 Conclusions

We have considered a Bayesian persuasion problem in which a school advisor (the sender) strategically discloses information to persuade n employers (the receivers) to extend offers. We demonstrate that as long as receivers take binary actions (extending an offer or not), and the sender has a known preference among receivers and can accept only one offer, public persuasion is optimal in a broad sense—that is, it is so regardless of the communication detail of receivers. Thus, the sender eliminates any room for the receivers to communicate and infer about each other for her self-benefit. Additionally, the optimal public persuasion mechanism can be derived from the first-best relaxation problem that imposes only participation constraints. We are hopeful that such a strong result can be extended to a more general setting, which could be an interesting direction for future research.

We next investigate a specific setting in which the state variable is one-dimensional and the receivers' utility functions are linear (thus, a receiver considers only the posterior mean). We focus on efficient computation of the optimal (public) persuasion mechanism. We provide the optimal mechanism in closed form for the two-receiver case based on the optimality condition derived from the dual of the first-best problem. For the general case, although the optimal mechanism can be derived from a convex optimization analogous to that of Candogan (2022), we derive many of the results based on the Lagrangian dual of the problem and aim to offer new insights and a better understanding of the optimal mechanism from a dual perspective.

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A Proofs

A.1 Proof of Lemma 3.1

Fix any information disclosure mechanism $f(\cdot|w)$. For any $i \in [n]$, let

$$\begin{aligned} q(i|w) &= \mathbb{P}\left[a_i^* = 1 \text{ and } a_j^* = 0, \forall\, j < i \mid w\right] \\ &= \int_{\boldsymbol{s}} \int_{\mathbf{c}} \delta_i(s_i, c_i) \Pi_{j<i} \left(1 - \delta_j(s_j, c_j)\right) c(\mathbf{c}|\boldsymbol{s}) f(\boldsymbol{s}|w) d\mathbf{c} d\boldsymbol{s}. \end{aligned}$$

denote the probability that employer i extends an offer and the candidate accepts it under the employers' equilibrium strategies when the candidate's characteristics are w. The random binary variable $a_i^* \in \{0, 1\}$ represents employer i's action of extending an offer in the equilibrium of the game induced by the mechanism $f(\cdot|w)$. Note that the candidate will accept employer i's offer if and only if none of the employers $j < i$ extends an offer.

We first prove that the participation constraint in (2) holds; that is,

$$\int_{w \in \Omega} u_i(w)\, q(i|w)\, dG(w) = \mathbb{E}\big[u_i(w) \cdot \mathbb{1}[a_i^* = 1 \text{ and } a_j^* = 0, \forall\, j < i]\big] \geq 0.$$

To see this, note that

$$\mathbb{E}\big[u_i(w) \cdot \mathbb{1}[a_i^* = 1 \text{ and } a_j^* = 0, \forall\, j < i] \mid c_i, s_i\big] = \mathbb{1}[a_i^* = 1] \cdot \mathbb{E}\big[u_i(w) \cdot \mathbb{1}[a_j^* = 0, \forall\, j < i] \mid c_i, s_i\big] \geq 0,$$

where the equation follows from the fact that the action a_i^* is measurable with respect to the signal-communication-information pair (c_i, s_i), and the inequality follows from the optimality of the employer's equilibrium strategy—that is, employer i extends an offer only if doing so provides nonnegative utility to him. Taking expectation over (c_i, s_i) on both sides of the above inequality yields the desired result.

For the second constraint, note that for any $w \in \Omega$, we have

$$\sum_{i \in [n]} q(i|w) = \sum_{i \in [n]} \mathbb{P}\left[a_i^* = 1 \text{ for some } i \in [n] \mid w\right] \leq 1.$$

Finally, the expected payoff of the mechanism $f(\cdot|w)$ can be expressed as

$$\sum_{i=1}^{n} v_i \cdot \int_{w \in \Omega} q(i|w) \cdot dG(w),$$

which is the objective function of (2). Since $\{q(i|w)\}$ is feasible to (2) given any mechanism $f(\cdot|w)$, we have $V^* \leq \bar{V}$.

A.2 Proof of Lemma 3.2

Let $\{q^*(i|w)\}$ denote an optimal solution to (2). We first show that for any two employers j and k with $j < k$, we have

$$\int_{w \in \Omega} u_j(w)\, q(k|w)\, dG(w) < 0. \tag{15}$$

We prove this by contradiction. Assume that there exists j and k with $j < k$ such that

$$\int_{w\in\Omega} u_j(w)\, q(k|w) \cdot dG(w)\, 0.$$

Consider the new allocation rule $\tilde{q}(i|w)$ defined as:

$$\tilde{q}(i|w) = \begin{cases} q^*(j|w) + q^*(k|w) & \text{if } i = j, \\ 0 & \text{if } i = k, \\ q^*(i|w) & \text{if } i \notin \{j, k\}. \end{cases}$$

$\{\tilde{q}(i|w)\}$ is feasible to (2), and because $v_j > v_k$, $\{\tilde{q}(i|w)\}$ achieves a strictly larger objective value than $\{q^*(i|w)\}$. This contradicts the fact that $\{q^*(i|w)\}$ is optimal to (2). Thus, our assumption fails.

Since a public mechanism leaves nothing for receivers to communicate, employers make decisions based only on the public signal. We now show that it is an equilibrium for each employer $i \in [n]$ to extend an offer only upon receiving the signal $s = i$. To do so, suppose all employers other than employer i follow this strategy; we verify that it is optimal for employer i to do the same.

First, suppose employer i receives the signal $s = i$. The expected payoff for extending an offer is nonnegative because

$$\int_{w\in\Omega} u_i(w)\, dG\big(w|s = i\big) = \frac{1}{\int_w q^*(i|w)\, dG(w)} \int_{w\in\Omega} u_i(w)\, q^*(i|w)\, dG(w) \geq 0,$$

where $dG\big(w|s = i\big) = \frac{q^*(i|w)\, dG(w)}{\int_w q^*(i|w)\, dG(w)}$ denotes the posterior belief of w given $s = i$, and the inequality follows from the participation constraint in (2). Therefore, it is optimal for the employer i to extend an offer.

Second, suppose employer i receives the signal $s = k$ with $k > i$. The expected payoff for extending an offer is negative because

$$\int_{w\in\Omega} u_i(w)\, dG\big(w|s = k\big) = \frac{1}{\int_w q^*(k|w)\, dG(w)} \int_{w\in\Omega} u_i(w)\, q^*(k|w)\, dG(w) < 0,$$

where the inequality follows from (15). Therefore, employer i will not extend an offer.

Finally, suppose employer i receives the signal $s = j$ with $j < i$. Since the candidate will never accept employer i's offer (because employer j will extend an offer), employer i is indifferent between extending an offer or not.

Note that the expected payoff for the school advisor is $\bar{V}$ under this equilibrium. Therefore, the public mechanism $f^*(\cdot|w)$ is optimal to (1).

A.3 Proof of Proposition 4.1

Since the threshold value α_i is strictly decreasing in the employer index i by Assumption 4.3, the probability of receiving an offer, expressed as $\sum_{i\in[n]} \int_0^1 q(i|w)dw$, is maximized when the sender targets only employer n with the lowest threshold value α_n; that is, $q(i|w) = 0$ for any $i \neq n$ and $w \in [0, 1]$. This is because, given any feasible solution $\{q(i|w)\}$ of (3), we can construct a new solution $\{\tilde{q}(i|w)\}$ with $\tilde{q}(n|w) = \sum_{i\in[n]} q(i|w)$ and $\tilde{q}(i|w) = 0$ for any $i < n$, which is feasible to (3) and attains the same probability of receiving an offer.

On the other hand, if the sender targets only employer n, the probability is maximized with

$q(n|w) = 1$ for any $w \geq z_n$ and $q(n|w) = 0$ otherwise, resulting in a probability of $\mathbb{P}(w \geq z_n)$.

Finally, we note that $z_n > 0$ because $w_0 < \alpha_n$ by Assumption 4.3. Therefore, $\mathbb{P}(w \geq z_n) < 1$.

A.4 Proof of Proposition 4.2

Let $\{q(i|w)\}$ be a feasible solution to (3) and suppose that the participation constraint for an employer j holds with strict inequality; that is,

$$\int_0^1 w \cdot q(j|w)\, g(w)\, dw > \alpha_i \int_0^1 q(j|w)\, g(w).$$

Since $\sum_{i\in[n]} \int_0^1 q(i|w)\, g(w)\, dw < 1$ by Proposition 4.1, we can allocate more mass to employer j so that $\int_0^1 q(j|w)\, g(w)\, dw$ strictly increases, $q(i|w)$ remains unchanged for any $i \neq j$ and $w \in [0,1]$, and the participation constraint for employer j still holds. This increases the objective value and implies that $\{q(i|w)\}$ is suboptimal.

A.5 Proof of Proposition 4.3

Let $\{q(i|w)\}$ be a feasible solution of (3), and define $z \triangleq \sup\left\{z \in [0,1] : \sum_{i\in[n]} \int_0^z q(i|w)\, dw = 0\right\}$ as the (essential) lower bound of the support of $\{q(i|w)\}$. If $\sum_{i\in[n]} \int_z^1 q(i|w)dw < \mathbb{P}(w \geq z)$, there exists a point $\tilde{z} \in (z, 1)$ such that

$$\sum_{i\in[n]} \int_z^{\tilde{z}} q(i|w)\, dw = \sum_{i\in[n]} \int_{\tilde{z}}^1 (1 - q(i|w))\, dw > 0.$$

We can create a new feasible solution $\{\tilde{q}(i|w)\}$ from $\{q(i|w)\}$ by transporting the mass of $\{q(i|w)\}$ below $\tilde{z}$ to fill the "unoccupied" area above $\tilde{z}$; therefore, $\sum_{i\in[n]} \int_{\tilde{z}}^1 \tilde{q}(i|w)dw = \mathbb{P}(w \geq \tilde{z})$ and $\sum_{i\in[n]} \int_0^{\tilde{z}} \tilde{q}(i|w)dw = 0$. The two feasible solutions $\{\tilde{q}(i|w)\}$ and $\{q(i|w)\}$ have the same objective value because, by transporting, $\int_0^1 q(i|w)dw = \int_0^1 \tilde{q}(i|w)dw$ for any $i \in [n]$.

On the other hand, since $\{q(i|w)\}$ satisfies the participation constraints and we have shifted a positive mass of $\{q(i|w)\}$ from below $\tilde{z}$ to above $\tilde{z}$, the participation constraint for some employer $i \in [n]$ must hold with strict inequality with $\{\tilde{q}(i|w)\}$. According to Proposition 4.2, $\{\tilde{q}(i|w)\}$, and thus $\{q(i|w)\}$, must be suboptimal.

A.6 Proof of Lemma 4.5

Since the thresholds α_i are smaller than one by Assumption 4.3, it is straightforward to create a feasible solution to (3) where all participation constraints in (3) are satisfied with strict inequality. Therefore, strong duality holds according to Theorem 1 in Section 8.6 of Luenberger (1997).

Once strong duality is established, Bullet 2 follows from the optimality condition (Proposition 6.1.5 in Bertsekas 2016) and Proposition 4.2, which states that the participation constraints are binding with any optimal solution of (3).

A.7 Proof of Proposition 4.6

Let $\{q^*(i|w)\}$ be an optimal solution to (3) such that $\int_0^1 q^*(i|w)dw > 0$. Since $\{q^*(i|w)\}$ is also optimal to $V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ by Lemma 4.5 Bullet 2, the line of employer i, $\ell_i(w; \mu_i^*)$, is a component of the envelope function $\bar{h}(w;\boldsymbol{\mu}^*)$. This implies that there exist constants $\underline{b},\bar{b}\in[0,1]$ such that $\bar{h}(w;\boldsymbol{\mu}^*)=\ell_i(w;\mu_i^*)$ for $w\in[\underline{b},\bar{b}]$ and $\bar{h}(w;\boldsymbol{\mu}^*)>\ell_i(w;\mu_i^*)$ otherwise. We now show that $0<\underline{b}<\alpha_1<\bar{b}\le 1$.

First, note that $q^*(i|w)=0$ for any $w\in[0,\underline{b})\cup(\underline{b},1]$ because line $\ell_i(w;\mu_i^*)$ is strictly below $\bar{h}(w;\boldsymbol{\mu}^*)$ in this region. This implies that $\alpha_1\in(b_1,b_2)$ because otherwise, the participation constraint of employer i cannot be binding, which contradicts Lemma 4.5 Bullet 2.

We next prove $\underline{b}>0$ by contradiction. If $\underline{b}=0$, then for any $w>0$, we have

$$h\big(w;\boldsymbol{\mu}^*\big)\ge\ell_i\big(w;\mu_i^*\big)>\ell_i\big(0;\mu_i^*\big)=\bar{h}\big(0;\boldsymbol{\mu}^*\big)\ge 0,$$

where the first inequality follows from the definition of the envelope function $h(w;\boldsymbol{\mu}^*)$ and the second inequality from the fact that $\mu_i^*>0$ by Proposition 4.7 Bullet 1. Therefore, $\sum_{i\in[n]}q^*(i|w)=1$ for any $w>0$ according to (5), which contradicts Proposition 4.3.

A.8 Proof of Proposition 4.7

Proof of Bullet 1 We prove $\mu_i^*>0$ for any $i\in[n]$ by contradiction. If $\mu_i^*=0$ for some $i\in[n]$, then for any $w\in[0,1]$, we have $h(w;\boldsymbol{\mu}^*)\ge\ell_i(w;\mu_i^*)=v_i>0$. Consequently, $\sum_{i\in[n]}q^*(i|w)=1$ for any $w\in[0,1]$ according to (5), which contradicts Proposition 4.3.

Proof of Bullet 2 Since $\mu_i^*>0$ for any $i\in[n]$, the envelope function $h(w;\boldsymbol{\mu}^*)$ is strictly increasing. Let $z^*\in(0,1)$ be the root of $h(w;\boldsymbol{\mu}^*)$ such that $h(z^*;\boldsymbol{\mu}^*)=0$, and let $\{q^*(i|w)\}$ be an optimal solution to (3). Since $\{q^*(i|w)\}\in\mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ by Lemma 4.5, it follows that $\sum_{i\in[n]}q^*(i|w)=1$ for any $w>z^*$ and $q^*(i|w)=0$ for any $i\in[n]$ and $w<z^*$ due to (5).

Proof of Bullet 3 From Proposition 4.6, the lines of employers in the set P are components of the envelope function $\bar{h}(w;\boldsymbol{\mu}^*)$. Since $\bar{h}(w;\boldsymbol{\mu}^*)$ is convex and piecewise linear, and the slope of each component equals the dual variable μ_i^* of the corresponding employer, $\{\mu_i^*\}_{i\in P}$ are decreasing with the employer index i.

A.9 Proof of Proposition 4.8

Bullets 1 and 2 follow from Proposition 4.1 with one receiver (either employer 1 or 2).

Proof of Bullet 3 For ease of notation, we drop the dependence on the mechanism M by letting $q_1=q_1(M)$ and $q_2=q_2(M)$. If the mechanism has a cutoff structure with a threshold value of z, the following two linear equations must hold:

$$\begin{aligned}q_1+q_2&=\mathbb{P}[w\ge z],\\ \alpha_1q_1+\alpha_2q_2&=(q_1+q_2)\cdot\mathbb{E}[w|w\ge z].\end{aligned}\tag{16}$$

The first equation follows from the fact that the candidate receives an offer (from either employer 1 or 2) if and only if $w\ge z$, and the second equation follows from the law of total expectation

$$\mathbb{E}[w|w\ge z]=\frac{q_1}{q_1+q_2}\cdot\mathbb{E}[w|s=1]+\frac{q_2}{q_1+q_2}\cdot\mathbb{E}[w|s=2]$$

and the fact that $\mathbb{E}[w|s=i]=\alpha_i$ by the IC constraints for mechanisms in the set $\mathcal{M}$. The two equations in (16) determine the values of q_1 and q_2 as

$$
\begin{aligned}
q_1 &= \mathbb{P}[w \geq z] \cdot \frac{\mathbb{E}[w|w \geq z]-\alpha_2}{\alpha_1-\alpha_2}, \\
q_2 &= \mathbb{P}[w \geq z] \cdot \frac{\alpha_1-\mathbb{E}[w|w \geq z]}{\alpha_1-\alpha_2}.
\end{aligned}
\tag{17}
$$

We now validate that $z \in [z_2, z_1]$.

First, we show that $z \geq z_2$. If $z < z_2$, then $\mathbb{E}[w|w \geq z] < \mathbb{E}[w|w \geq z_2] = \alpha_2$, which implies $q_1 < 0$. Therefore, we must have $z \geq z_2$.

Next, we show that $z \leq \bar{z}_1$. If $z > \bar{z}_1$, then $\mathbb{E}[w|w \geq z] > \mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$, which implies $q_2 < 0$. Therefore, we have $z \leq \bar{z}_1$.

Finally, we show that $z \leq z_1 \leq \bar{z}_1$. If $\bar{z}_1 \leq \alpha_2$, then $z_1 = \bar{z}_1$ and we are done. Now, suppose $z_1 < \alpha_2 < \bar{z}_1$. In this case, it suffices to show that $z \notin (z_1, \bar{z}_1)$. We prove that if $z \in (z_1, \bar{z}_1)$, then $q_1 > q_1(M_1) = \mathbb{P}[w \geq \bar{z}_1]$, which contradicts Bullet 1. Specifically, let $q_1(z)$ and $q_2(z)$ denote the values of q_1 and q_2 as a function of the threshold z. From (17) we have[13]

$$
q_1(\bar{z}_1) = \mathbb{P}[w \geq \bar{z}_1], \quad q_2(\bar{z}_1) = 0
$$

and

$$
q_1(z_1) = \mathbb{P}[w \geq \bar{z}_1], \quad q_2(z_1) = \mathbb{P}[z_1 \leq w \leq \bar{z}_1].
$$

From (17), we can express $q_1(z)$ as

$$
q_1(z) = \frac{1}{\alpha_1-\alpha_2} \int_z^1 (w-\alpha_2) g(w) dw.
$$

Since the derivative is

$$
\frac{dq_1(z)}{dz} = \frac{\alpha_2 - z}{\alpha_1-\alpha_2} \cdot g(z),
$$

$q_1(z)$ is increasing in $z \in [z_1, \alpha_2]$ and decreasing in $z \in [\alpha_2, \bar{z}_1]$. Therefore, $q_1(z) > \mathbb{P}[w \geq \bar{z}_1]$ for any $z \in (z_1, \bar{z}_1)$, which contradicts Proposition 4.8 Bullet 1.

Combining the above three steps yields $z \in [z_2, z_1]$.

Proof of Bullet 4 If such a mechanism M exists, we must have $q_1 \triangleq q_1(M) = \mathbb{P}[w \geq z] \cdot \frac{\mathbb{E}[w|w \geq z]-\alpha_2}{\alpha_1-\alpha_2}$ and $q_2 \triangleq q_2(M) = \mathbb{P}[w \geq z] \cdot \frac{\alpha_1-\mathbb{E}[w|w \geq z]}{\alpha_1-\alpha_2}$ by (17). Since $z \in [z_2, z_1]$, we have $\mathbb{E}[w|w \geq z] \in [\alpha_2, \alpha_1]$, implying that $q_1 \geq 0$, $q_2 \geq 0$, and $q_2 \leq \mathbb{P}[w \geq z] \leq \mathbb{P}[w \geq z_2]$. Additionally, the proof of Bullet 3 indicates that $q_1 \leq \mathbb{P}[w \geq \bar{z}_1] \leq \mathbb{P}[w \geq z]$ when $z \in [z_2, z_1]$.

A mechanism M that meets the criteria of Bullet 4 must satisfy the following:

1. $q(1|w) + q(2|w) = 1$ for any $w \geq z$, and $q(1|w) = q(2|w) = 0$ for any $w < z$;
2. $\mathbb{P}[s=1] = q_1$, and $\mathbb{P}[s=2] = q_2$;
3. $\mathbb{E}[w|s=1] = \alpha_1$, $\mathbb{E}[w|s=2] = \alpha_2$, and $\mathbb{E}[w|s=\varnothing] < \alpha_2$.

A feasible mechanism $M \in \mathcal{M}$ can be constructed in multiple ways. For example, we can create a *deterministic* persuasion mechanism such that $q(1|w) = 1$ for $w \in T$, $q(2|w) = 1$ for $w \in [z, 1] \setminus T$,

[13] This is because $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$ and $\mathbb{E}[w|w \geq z_1] = \frac{\mathbb{P}[w \geq \bar{z}_1]}{\mathbb{P}[w \geq z_1]}\alpha_1 + \frac{\mathbb{P}[z_1 \leq w \leq \bar{z}_1]}{\mathbb{P}[w \geq z_1]}\alpha_2$.

and $q(\varnothing|w) = 1$ for $w < z$, for some subset $T \subseteq [z, 1]$. To satisfy the requirements in Bullet 4, the subset $T \subseteq [z, 1]$ must satisfy the following conditions:

1. $\mathbb{P}[w \in T] = q_1$, and $\mathbb{P}[w \in [z, 1] \setminus T] = q_2$;
2. $\mathbb{E}[w|w \in T] = \alpha_1$, $\mathbb{E}[w|w \in [z, 1] \setminus T] = \alpha_2$, and $\mathbb{E}[w|w < z] < \alpha_2$.

There are various ways to construct a feasible subset T. For instance, subset T can be an interval $[\underline{b}, \bar{b}] \subseteq [\bar{z}_1, 1]$ that includes the point α_1 and satisfies $\mathbb{P}[\underline{b} \le w \le \bar{b}] = q_1$ and $\mathbb{E}[w|\underline{b} \le w \le \bar{b}] = \alpha_1$. Such an interval $[\underline{b}, \bar{b}]$ exists because $\mathbb{E}[w|w \ge \bar{z}_1] = \alpha_1$ and $\mathbb{P}[w \ge \bar{z}_1] \ge q_1$. In addition, we have $\mathbb{P}[w \in [z, 1] \setminus T] = q_2$ and $\mathbb{E}[w|w \in [z, 1] \setminus T] = \alpha_2$ because the values of q_1 and q_2 satisfy (16). Finally, we have $\mathbb{E}[w|w < z] \le \mathbb{E}[w|w < z_1] < \alpha_2$, where the second inequality follows from the fact that $z_1 = \bar{z}_1 \le \alpha_2$ when $\bar{z}_1 \le \alpha_2$ and $\mathbb{E}[w|w < z_1] < \mathbb{E}[w|z_1 \le w < \bar{z}_1] = \alpha_2$ when $\bar{z}_1 > \alpha_2$.

A.10 Proof of Lemma 4.9

In this proof, we identify a set of dual variables $\boldsymbol{\mu} \in \mathbb{R}^n_+$, which, along with the proposed mechanism in Lemma 4.9, satisfies Lemma 4.5 Bullet 2. This indicates that the mechanism is optimal to (3), and $\boldsymbol{\mu}$ is an optimal dual variable.

Proof of Bullet 1 Suppose $v_1 \le v_2 \cdot \frac{\alpha_1 - z_2}{\alpha_2 - z_2}$, which implies that the point (α_1, v_1) lies below line t_2. We construct the employers' associated lines ℓ_1 and ℓ_2 as follows.

Let line ℓ_2 coincide with line t_2 by taking the dual variable $\mu_2 = v_2/(\alpha_2 - z_2)$. Let line ℓ_1 lie below line ℓ_2 for $w \in [z_2, 1]$. For example, this can be achieved by taking the dual variable $\mu_1 = v_1/(\alpha_1 - z_2)$. The two lines ℓ_1 and ℓ_2 are illustrated in Figure 3(a). Since line ℓ_2 dominates line ℓ_1, an optimal solution to the Lagrangian $V^{\text{LR}}(\boldsymbol{\mu})$ with $\boldsymbol{\mu} = (\mu_1, \mu_2)$ will never recommend the candidate to employer 1, regardless of the candidate's characteristics w. It is easy to verify that the mechanism M_2 and the dual variable $\boldsymbol{\mu} = (\mu_1, \mu_2)$ satisfy Lemma 4.5 Bullet 2. Therefore, mechanism M_2 is optimal to (3), and $\boldsymbol{\mu} = (\mu_1, \mu_2)$ is an optimal dual variable.

Proof of Bullet 2 Suppose $v_1 \ge v_2 \cdot \frac{\alpha_1 - z_1}{\alpha_2 - z_1}$, which implies that the point (α_1, v_1) lies above the line t_1. We construct the employers' associated lines ℓ_1 and ℓ_2 in the following two cases: when $\bar{z}_1 \le \alpha_2$ and when $\bar{z}_1 > \alpha_2$.

1. $z_1 = \bar{z}_1 \le \alpha_2$: Let line ℓ_1 be the line passing through the points $(z_1, 0)$ and (α_1, v_1) by taking the dual variable $\mu_1 = v_1/(\alpha_1 - z_1)$. Let line ℓ_2 lie below line ℓ_1 for $w \in [z_1, 1]$. For example, this can be achieved by taking $\mu_2 = v_2/(\alpha_2 - z_1)$ (in which case line ℓ_2 coincides with line t_1), because the point (α_2, v_2) lies below line ℓ_1. The lines ℓ_1 and ℓ_2 are illustrated in Figure 3(c).
2. $z_1 < \alpha_2 < \bar{z}_1$: Let line ℓ_2 coincide with line t_1 by taking the dual variable $\mu_2 = v_2/(\alpha_2 - z_1)$. Let $\mu_1 > \mu_2$ be such that line ℓ_1 intersects line ℓ_2 at $w = \bar{z}_1$.[14] After the setup, line ℓ_1 is above line ℓ_2 for $w \in [\bar{z}_1, 1]$ and line ℓ_2 is above line ℓ_1 for $w \in [z_1, \bar{z}_1]$. The lines ℓ_1 and ℓ_2 are illustrated in Figure 3(d).

It is easy to verify that the mechanism M_1 and the dual variable $\boldsymbol{\mu} = (\mu_1, \mu_2)$ satisfy Lemma 4.5 Bullet 2 in both cases. As a result, mechanism M_1 is optimal to (3), and $\boldsymbol{\mu} = (\mu_1, \mu_2)$ is an optimal dual variable.

[14] $\mu_1 > \mu_2$ because the point (α_1, v_1) lies above line t_1.

Proof of Bullet 3 Suppose the point (α_1, v_1) lies between the two lines t_1 and t_2. We define the dual variables $\mu_1 = \mu_2 = \frac{v_1 - v_2}{\alpha_1 - \alpha_2}$ so that the two lines ℓ_1 and ℓ_2 overlap and pass through (α_2, v_2) and (α_1, v_1). These lines intersect the x-axis at $w = z^* \in [z_2, z_1]$, as illustrated in Figure 3(b).

It is easy to verify that any mechanism $M \in \mathcal{M}$ feasible to Lemma 4.9 Bullet 3, together with the dual variable $\boldsymbol{\mu} = (\mu_1, \mu_2)$, satisfies Bullet 2 of Lemma 4.5. Therefore, such a mechanism M is optimal to (3), and $\boldsymbol{\mu} = (\mu_1, \mu_2)$ is an optimal dual variable. The existence of a mechanism $M \in \mathcal{M}$ meeting Bullet 3 of Lemma 4.9 is confirmed by Bullet 4 of Proposition 4.8.

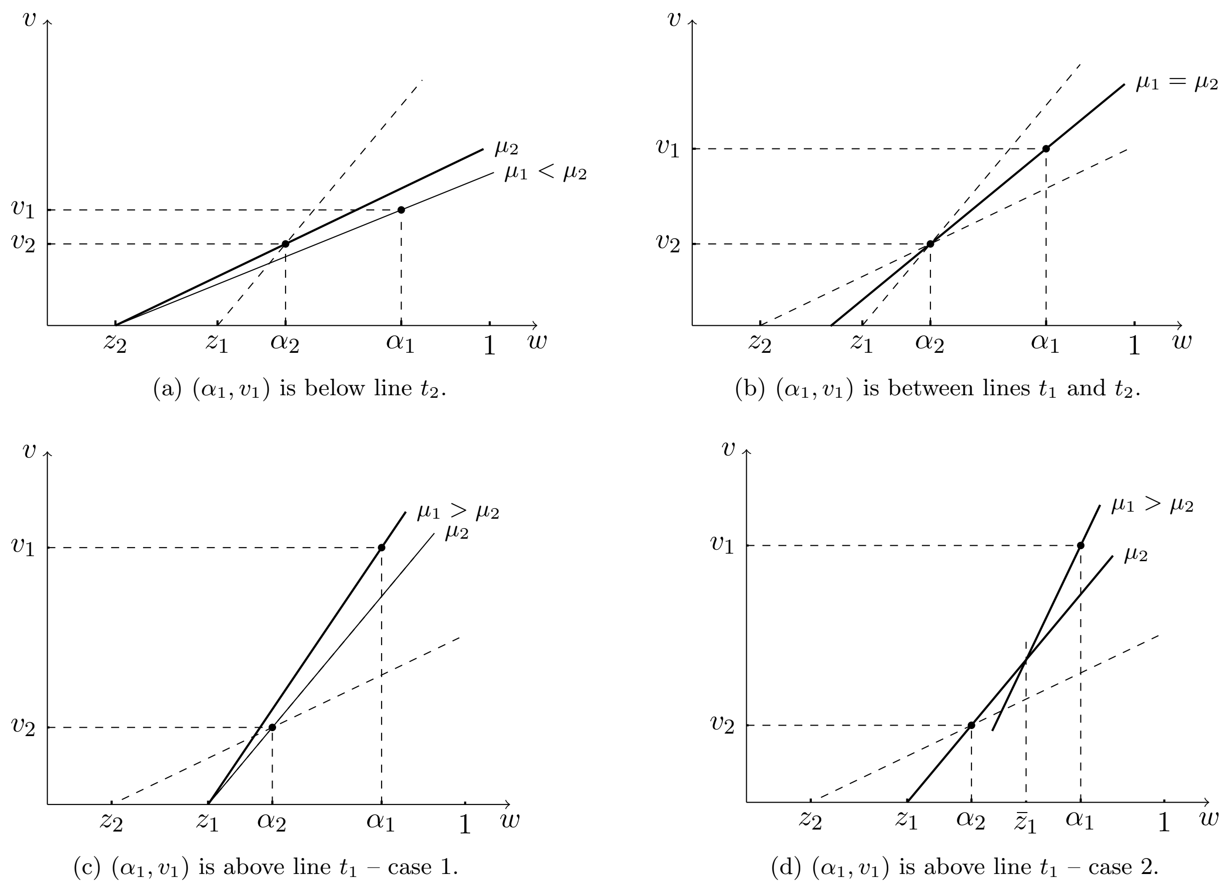

Figure 3: Visualization of two employers' associated lines.

A.11 Proof of Proposition 4.10

Step One: Proving $\bar{V} \le V^{\mathrm{CR}}$ We first prove that (8) is a relaxation of (3); therefore, $\bar{V} \le V^{\mathrm{CR}}$. Specifically, let $\{q(i|w)\}$ be a feasible solution to (3). Define $q_i = \int_0^1 q(i|w)g(w)dw$ for any $i \in [n]$. We show that $\{q_i\}$ is feasible to (8). This, together with the fact that $\{q(i|w)\}$ and $\{q_i\}$ yield the same objective value, indicates that (8) is a relaxation of (3).

To show that $\{q_i\}$ is feasible to (8), first, note that $q_i \ge 0$ for any $i \in [n]$ because $q(i|w) \ge 0$ for any $i \in [n]$ and $w \in [0, 1]$. Second,

$$\sum_{i \in [n]} q_i = \sum_{i \in [n]} \int_0^1 q(i|w)\, g(w)\, dw \le \int_0^1 g(w)\, dw = 1,$$

where the inequality follows from the fact that $\sum_{i\in[n]} q(i|w) \le 1$ for any $w \in [0,1]$.

Finally, we show $\{q_i\}$ is feasible to the first constraint in (8). To do so, let $q_{\le k}(w) = \sum_{i\le k} q(i|w)$ denote the probability that a candidate with characteristics w receives an offer from one the top k employers. Since $\{q(i|w)\}$ is a feasible solution to (3),

$$\alpha_i \int_0^1 q(i|w)\, g(w)\, dw \le \int_0^1 w \cdot q(i|w)\, g(w)\, dw.$$

Summing over $i \le k$ on both sides gives

$$\begin{aligned}
\sum_{i\le k} \alpha_i q_i \le & \int_0^1 w \cdot q_{\le k}(w)\, g(w)\, dw \\
\le & \int_{G^{-1}\left(1-\sum_{i\le k} q_i\right)}^1 w \cdot g(w)\, dw \\
= & \mathbb{E}\left[w \cdot \mathbb{1}\left[G(w) \ge 1 - \sum_{i\le k} q_i\right]\right] \\
= & \sum_{i\le k} q_i \cdot \mathbb{E}\left[w \,\middle|\, G(w) \ge 1 - \sum_{i\le k} q_i\right]
\end{aligned}$$

where the second inequality follows from the fact that $\int_0^1 q_{\le k}(w)\, g(w)\, dw = \sum_{i\le k} q_k$, and that the integration is maximized by taking $q_{\le k}(w) = 1$ for any $w \ge G^{-1}\left(1 - \sum_{i\le k} q_i\right)$ and $q_{\le k}(w) = 0$ otherwise.

Step Two: Proving $V^{\text{CR}} \le \bar{V}$ We next prove that $V^{\text{CR}} \le \bar{V}$. Specifically, we show that for any feasible solution $\{q_i\}$ to (8), there exists a feasible solution $\{q(i|w)\}$ to (3) with the same objective value as $\{q_i\}$, thereby implying $V^{\text{CR}} \le \bar{V}$.

Let $\{q_i\}$ be feasible to (8). Since the participation condition (i.e., the first constraint) of (8) holds for $k = 1$, we can find a portion q_1 of candidates whose mean quality just meets the threshold value α_1 of employer 1. In other words, we can find a function $q(1|w) \ge 0$ satisfying

$$\begin{aligned}
&\int_0^1 q(1|w)\, g(w)\, dw = q_1, \\
&\int_0^1 w \cdot q(1|w)\, g(w)\, dw = \alpha_1 \int_0^1 q(1|w)\, g(w)\, dw.
\end{aligned}$$

Now consider the remaining portion of candidates. Since the participation condition of (8) holds for $k = 2$, within the remaining portion of candidates, we can find a portion q_2 of candidates whose mean quality just meets the threshold value α_2 of employer 2. In other words, we can find a function $q(2|w) \ge 0$ satisfying

$$\begin{aligned}
&\int_0^1 q(2|w)\, g(w)\, dw = q_2, \\
&\int_0^1 w \cdot q(2|w)\, g(w)\, dw = \alpha_2 \int_0^1 q(2|w)\, g(w)\, dw, \\
&q(2|w) \le 1 - q(1|w),\ \forall\, w \in [0,1].
\end{aligned}$$

Repeating the process, we can find qualified portions for all employers, resulting in a set of $\{q(i|w)\}$ that is feasible to (3). Moreover, by construction, $\{q(i|w)\}$ and $\{q_i\}$ have the same objective value.

Step Three: Wrap-Up Combining the two steps, we have $\bar{V} = V^{\rm CR}$; that is, the optimal values of (3) and (8) are equal. Moreover, let $\{q^*(i|w)\}$ be an optimal solution to (3), and let $q_i^* = \int_0^1 q^*(i|w)g(w)dw$. Since $\{q_i^*\}$ is feasible to (8) and attains the same objective value as $\{q^*(i|w)\}$ by Step One, $\{q_i^*\}$ is optimal to (8). Conversely, if $\{q_i^*\}$ is an optimal solution to (8), then by Step Two, we can construct a feasible solution $\{q^*(i|w)\}$ to (3) satisfying $q_i^* = \int_0^1 q^*(i|w)g(w)dw$. This solution has an objective value $V^{\rm CR} = \bar{V}$, thus is optimal to (3).

A.12 Proof of Proposition 4.11

In the following, we first prove that the optimal Lagrangian dual variable $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]}$ for (3) is unique (Step One). We then show that if $\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]}$ is an optimal Lagrangian dual variable for (8), then $\{\mu_i\}$, with $\mu_i = \sum_{k\geq i}\lambda_k^*$, is an optimal Lagrangian dual variable for (3) (Step Two). Finally, these two steps indicate the uniqueness of the optimal Lagrangian dual variable $\boldsymbol{\lambda}^*$.

A.12.1 Step One: Uniqueness of $\boldsymbol{\mu}^*$

In this section, we prove by contradiction that the optimal Lagrangian dual variable $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]}$ of (3) is unique. Suppose, instead, that (3) has two different optimal dual variables $\boldsymbol{\mu} = (\mu_i)_{i\in[n]}$ and $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_i)_{i\in[n]}$. Let $i \triangleq \max\{j \in [n] : \mu_j \neq \tilde{\mu}_j\}$ denote the largest index where the two optimal dual variables differ. Without loss of generality, assume $\mu_i > \tilde{\mu}_i > 0$, where the second inequality follows from Proposition 4.7 Bullet 1. Let $\{q^*(i|w)\}$ be an optimal solution to (3) such that $\int_0^1 q^*(i|w)g(w)dw = q_i^* > 0$ (whose existence is validated by Proposition 4.10). According to Lemma 4.5 Bullet 2, $\{q^*(i|w)\} \in \mathbf{Q}^{\rm LR}(\boldsymbol{\mu})$ and $\{q^*(i|w)\} \in \mathbf{Q}^{\rm LR}(\tilde{\boldsymbol{\mu}})$. However, we will show that this is impossible.

First, let $\ell_j(w) \triangleq v_j + \mu_j(w - \alpha_j)$ denote the lines of employers $j \in [n]$ using the optimal dual variable $\boldsymbol{\mu} = (\mu_j)_{j\in[n]}$, and

$$\bar{h}(w) \triangleq \max_{j\in[n]}\big\{\ell_j(w)\big\}^+ = \max_{j\in[n]}\Big\{v_j + \mu_j\big(w - \alpha_j\big)\Big\}^+$$

denote the envelope function, which is convex and piecewise-linear on $w \in [0, 1]$.

Since $q_i^* > 0$, there exists constants b_1 and b_2 satisfying $0 < b_1 < \alpha_i < b_2 \leq 1$, such that $\bar{h}(w) = \ell_i(w)$ for $w \in \big[b_1, b_2\big]$ and $\bar{h}(w) > \ell_i(w)$ otherwise, according to Proposition 4.6. Since the envelope function $\bar{h}(w)$ is convex and piecewise-linear, and line $\ell_i(w)$ is dominated for any $w < b_1$, all the lines $\ell_j(w)$ with $j < i$ are also dominated for any $w < b_1$. This implies that

$$\sum_{j\leq i} q^*(j|w) = 0,\ \forall\, w < b_1. \tag{18}$$

Next, let $\tilde{\ell}_j(w) \triangleq v_j + \tilde{\mu}_j(w - \alpha_j)$ denote the lines of employers $j \in [n]$ using the optimal dual variable $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_j)_{j\in[n]}$, and

$$\tilde{h}(w) \triangleq \max_{j\in[n]}\big\{\tilde{\ell}_j(w)\big\}^+ = \max_{j\in[n]}\Big\{v_j + \tilde{\mu}_j\big(w - \alpha_j\big)\Big\}^+$$

denote the envelope function, which is convex and piecewise-linear on $w \in [0, 1]$.

Analogously, since $q_i^* > 0$, there exists constants $\tilde{b}_1$ and $\tilde{b}_2$ satisfying $0 < \tilde{b}_1 < \alpha_i < \tilde{b}_2 \le 1$, such that $\tilde{h}(w) = \tilde{\ell}_i(w)$ for $w \in [\tilde{b}_1, \tilde{b}_2]$. Since $\tilde{\mu}_j = \mu_j$ for any $j > i$, the lines $\ell_j(w)$ and $\tilde{\ell}_j(w)$ coincide for any $j > i$. Moreover, since $\tilde{\mu}_i < \mu_i$, we have $\tilde{b}_1 < b_1$, and line $\tilde{\ell}_i(w)$ is strictly above the x-axis and the lines $\{\tilde{\ell}_j(w)\}_{j>i}$ for any $w \in (\tilde{b}_1, \alpha_i)$. Therefore, according to (5), we have

$$\sum_{j \le i} q^*(j|w) = 1, \, \forall\, w \in (\tilde{b}_1, \alpha_i). \tag{19}$$

However, since $\tilde{b}_1 < b_1$, (18) and (19) cannot hold simultaneously. This implies that the optimal Lagrangian dual variable $\boldsymbol{\mu}^*$ of (3) must be unique.

A.12.2 Step Two: Connection between Optimal Dual Variables

Given an optimal Lagrangian dual variable $\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]}$ of (8), we define a dual variable $\boldsymbol{\mu} = (\mu_i)_{i\in[n]}$ with $\mu_i = \sum_{k\ge i} \lambda_k^*$ for any $i \in [n]$. In the following, we show that $V^{\mathrm{LR}}(\boldsymbol{\mu}) = \bar{V}$. Therefore, $\boldsymbol{\mu}$ is an optimal Lagrangian dual variable of (3) by the strong duality (Lemma 4.5).

Let $\{q_i^*\}$ be an optimal solution to (8), and $T \triangleq \{k \in [n] : \lambda_k^* > 0\}$ denote the set of positive elements of $\boldsymbol{\lambda}^*$. Note that if $k \in T$, then the participation constraint in (8) is binding with the top k employers due to complementary slackness; that is,

$$\sum_{i\le k} \alpha_i q_i^* = \mathbb{E}\left[w \cdot \mathbb{1}\left[w \ge G^{-1}\Big(1 - \sum_{i\le k} q_i^*\Big)\right]\right]. \tag{20}$$

The set T is nonempty because $n \in T$. To see this, let $i = n$ in (9); this gives

$$v_n + \lambda_n^* \cdot \left(G^{-1}\left(1 - \sum_{j=1}^{n} q_j^*\right) - \alpha_n\right) = 0. \tag{21}$$

Note that $G^{-1}(1 - \sum_{j=1}^n q_j^*)$ corresponds to the cutoff value z^* in Proposition 4.7 Bullet 2, which is strictly less than α_n, because the participation constraint of employer n is binding. This implies that $\lambda_n^* > 0$ (whose value is unique because the value of z^* is unique according to Proposition 4.7 Bullet 2).

Now suppose $T = \{1 \le t_1 < t_2 < \cdots < t_m = n\}$ contains m employers. These m employers partition the n employers into m groups $\{T_i\}_{i\in[m]}$ with $T_1 = [t_1]$ and $T_i = [t_{i-1}{+}1 : t_i]$ for $i \in [2{:}m]$. Therefore, we have $\bigcup_{i\in[m]} T_i = [n]$ and $T_i \cap T_j = \varnothing$ for any $i \ne j$, and each group T_i contains exactly one element from T, which is the largest element in T_i.

Define $z_i \triangleq G^{-1}\big(1 - \sum_{j=1}^{t_i} q_j^*\big)$ for any $i \in [m]$, and let $z_0 = 0$. In Lemma A.1, we provide structural properties of the employers' lines $\ell_i(w; \mu_i) = v_i + \mu_i(w - \alpha_i)$ and fully characterize the envelope functions $h(w; \boldsymbol{\mu}) \triangleq \max_{i\in[n]} \ell_i(w; \mu_i)$ and $\bar{h}(w; \boldsymbol{\mu}) \triangleq \max\{h(w; \boldsymbol{\mu}), 0\}$.

Lemma A.1. *Given an optimal Lagrangian dual variable $\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]}$ of (8), define a dual variable $\boldsymbol{\mu} = (\mu_i)_{i\in[n]}$ where $\mu_i = \sum_{k\ge i} \lambda_k^*$ for any $i \in [n]$. The following hold:*

1. *For any $i \in [m]$ and any employer $j \in T_i$, the threshold value α_j satisfies $\alpha_j \in (z_i, z_{i-1})$.*

2. *For any group T_i, the employers' lines $\ell_j(w; \mu_j)$ for any $j \in T_i$ completely overlap and pass through the points (α_j, v_j) for any $j \in T_i$.*

3. *Given any two employers $j \in T_i$ and $k \in T_{i+1}$ with $i \leq m-1$, the two lines $\ell_j(w;\mu_j)$ and $\ell_k(w;\mu_k)$ intersects at $w = z_i$. In addition, for any $j \in T_m$, line $\ell_j(w;\mu_j)$ intersects the x-axis at $w = z_m$.*

4. *$h(w;\boldsymbol{\mu}) = \ell_j(w;\mu_j)$ for any $i \in [m]$, $j \in T_i$, and $w \in \big[z_i, z_{i-1}\big]$. In addition, $h(w;\boldsymbol{\mu}) = \ell_j(w;\mu_j)$ for any $j \in T_m$ and $w \in \big[0, z_m\big]$.*

5. *$\bar{h}(w;\boldsymbol{\mu}) = h(w;\boldsymbol{\mu})$ for $w \geq z_m$ and $\bar{h}(w;\boldsymbol{\mu}) = 0$ otherwise.*

We prove Lemma A.1 in Appendix A.12.3. From (5) and Lemma A.1, an allocation $\{q(i|w)\}$ is optimal to $V^{\mathrm{LR}}(\boldsymbol{\mu})$ if and only if $\{q(i|w)\}$ are nonnegative, no larger than one, and satisfy the following:

$$\begin{aligned} &\sum_{j\in T_i} q(j|w) = 1,\ \forall\, w \in \big(z_i, z_{i-1}\big),\ i \in [m],\\ &\sum_{j\in[n]} q(j|w) = 0,\ \forall\, w < z_m. \end{aligned} \tag{22}$$

Moreover, (26) below indicates that there exists an optimal solution $\{q^*(i|w)\}$ to $V^{\mathrm{LR}}(\boldsymbol{\mu})$ such that for any $i \in [m]$ and $j \in T_i$, we have:

$$\begin{aligned} \int_{w\in I_i} q^*(j|w)\, dw &= q_j^*,\\ \int_{w\in I_i} w\cdot q^*(j|w)\, g(w)\, dw &= \alpha_j \int_{w\in I_i} q^*(j|w)\, g(w)\, dw, \end{aligned} \tag{23}$$

where $I_i \triangleq [z_i, z_{i-1}]$. We observe from (22) that for any employer $j \in T_i$, $q^*(j|w) = 0$ for any $w \notin I_i$. This observation, combined with (23), implies that for any $j \in [n]$, we have:

$$\int_0^1 q^*(j|w)\, g(w)\, dw = q_j^*, \tag{24}$$

$$\int_0^1 w\cdot q^*(j|w)\, g(w)\, dw = \alpha_j \int_0^1 q^*(j|w)\, g(w)\, dw. \tag{25}$$

Therefore,

$$\begin{aligned} V^{\mathrm{LR}}(\boldsymbol{\mu}) &= \int_0^1 \sum_{i=1}^n \Big\{v_i + \mu_i\big(w-\alpha_i\big)\Big\}\, q^*(i|w)\, g(w)\, dw\\ &= \sum_{i=1}^n v_i \cdot \int_0^1 q^*(i|w)\, g(w)\, dw\\ &= \sum_{i=1}^n v_i \cdot q_i^* = V^{\mathrm{CR}} = \bar{V}, \end{aligned}$$

where the first equation follows from the fact that $\{q^*(i|w)\}$ is optimal to $V^{\mathrm{LR}}(\boldsymbol{\mu})$, the second from (25), the third from (24), the fourth from the fact that $\{q^*\}$ is optimal to (8), and the fifth from Proposition 4.10.

A.12.3 Proof of Lemma A.1

Proof of Bullet 1 For any group T_i and element $k \in T_i$, contracting both sides of the first constraint in (8) from both sides of (20) with $k = t_{i-1}$, and noting that the first constraint in (8) is binding with $k = t_i$, yields the following:

$$\begin{aligned} \sum_{j\in[t_{i-1}+1:k]} \alpha_j q_j^* &\le \mathbb{E}\Big[w \cdot \mathbb{1}\Big[G^{-1}\Big(1 - \sum_{j\le k} q_j^*\Big) \le w < z_{i-1}\Big]\Big], \forall\, k \in [t_{i-1}+1 : t_i - 1], \\ \sum_{j\in T_i} \alpha_j q_j^* &= \mathbb{E}\Big[w \cdot \mathbb{1}\Big[z_i \le w < z_{i-1}\Big]\Big]. \end{aligned} \tag{26}$$

In addition, note that $\mathbb{P}[z_i \le w < z_{i-1}] = \sum_{j\in T_i} q_j^*$. Analogous to the proof of Proposition 4.10 (in particular, Step Two in Appendix A.11), this implies that we can allocate the state $w \in [z_i, z_{i-1}]$ to the employers in the subset T_i (possibly in a randomized way) so that each employer i is allocated with an aggregate size of $q_i^* > 0$, and the posterior mean of the allocation to employer i is α_i (i.e., the participation constraint is tight). Therefore, we must have $\alpha_j \in (z_i, z_{i-1})$ for any $j \in T_i$.

Proof of Bullet 2 If group T_i contains only one employer, there is nothing to prove. Now suppose T_i contains more than one employer (i.e., $t_{i-1} + 1 < t_i$). For any $k \in [t_{i-1} + 1 : t_i - 1]$, we have

$$\mu_k = \sum_{j\ge k} \lambda_j^* = \mu_{t_i} = \frac{v_k - v_{t_i}}{\alpha_k - \alpha_{t_i}}\,, \tag{27}$$

where the first equation follows from the definition of $\{\mu_i\}$ and the second equation follows from the fact that $\lambda_j^* = 0$ for any $j \in [t_{i-1} + 1 : t_i - 1]$. The third equation is derived by subtracting both sides of (9) with $i = t_i$ from both sides of the same equation with $i = k$. (27) implies that the points $\{(v_j, \alpha_j)\}_{j\in T_i}$ lie on a line, and the employers' lines $\ell_j(w; \mu_j)$ with any $j \in T_i$ fully overlap and coincide with this line.

Proof of Bullet 3 Based on Bullet 2, it suffices to show that (i) line $\ell_n(w; \mu_n)$ intersects the x-axis at $w = z_m$, and (ii) for any $i \in [m - 1]$, the two lines $\ell_{t_i}(w; \mu_{t_i})$ and $\ell_{t_{i+1}}(w; \mu_{t_{i+1}})$ intersect at $w = z_i$.

First, from (21) we have

$$v_n + \lambda_n^* \cdot (z_m - \alpha_n) = v_n + \mu_n \cdot (z_m - \alpha_n) = 0,$$

where the first equation follows from $\mu_n = \lambda_n^*$ by definition. Therefore, line $\ell_n(w; \mu_n)$ intersects the x-axis at $w = z_m$.

We now prove (ii) by induction. To start, note that

$$\mu_{t_i} = \sum_{j=i}^{m} \lambda_{t_j}^*, \forall i \in [m] \tag{28}$$

because $\lambda_k^* = 0$ for any $k \notin T$. We first show that (ii) holds for $i = m - 1$. Since line $\ell_n(w; \mu_n)$ passes through the point (z_{m-1}, h_{m-1}) with

$$h_{m-1} = \mu_n \cdot (z_{m-1} - z_m), \tag{29}$$

it suffices to show that line $\ell_{t_{m-1}}(w; \mu_{t_{m-1}})$ also passes through (z_{m-1}, h_{m-1}). We now verify this.

Specifically, taking $i = t_{m-1}$ in (9) yields

$$v_{t_{m-1}} + \lambda_n^* \cdot (z_m - \alpha_{t_{m-1}}) + \lambda_{t_{m-1}}^* \cdot (z_{t_{m-1}} - \alpha_{t_{m-1}}) = v_{t_{m-1}} + \mu_{t_{m-1}} \cdot \big(z_{m-1} - h_{m-1}/\mu_{t_{m-1}} - \alpha_{t_{m-1}}\big) = 0$$

where the first equation follows from (28) and (29). Therefore, it follows that

$$v_{t_{m-1}} + \mu_{t_{m-1}} \cdot (z_{m-1} - \alpha_{t_{m-1}}) = h_{m-1},$$

implying that line $\ell_{t_{m-1}}(w; \mu_{t_{m-1}})$ also passes through the point (z_{m-1}, h_{m-1}).

We now assume that (ii) holds for any $j \geq i+1$ and verify that it also holds for $j = i$. Given that (ii) holds for any $j \geq i+1$, line $\ell_{t_{i+1}}(w; \mu_{t_{i+1}})$ passes through the point (z_i, h_i) with

$$h_i = \sum_{j=i}^{m-1} \mu_{i+1} \cdot (z_i - z_{i+1}). \tag{30}$$

It suffices to show that line $\ell_{t_i}(w; \mu_{t_i})$ also passes through (z_i, h_i). To do so, take $i = t_i$ in (9); this gives:

$$v_{t_i} + \sum_{j=i}^{m} \lambda_{t_j}^* \cdot \big(z_j - \alpha_{t_i}\big) = v_{t_i} + \mu_{t_i} \cdot \Big(z_i - h_i/\mu_{t_i} - \alpha_{t_i}\Big) = 0,$$

where the first equation follows from (30) and the fact that $\lambda_{t_j}^* = \mu_{t_j} - \mu_{t_{j+1}}$ for any $j \in [m]$ (letting $\mu_{t_{m+1}} = 0$) by (28). Therefore, we have

$$v_{t_i} + \mu_{t_i} \cdot (z_i - \alpha_{t_i}) = h_i,$$

implying that line $\ell_{t_i}(w; \mu_{t_i})$ also passes through the point (z_i, h_i). Therefore, (ii) holds for $j = i$.

Proof of Bullet 4 Bullet 4 follows from Bullets 2 and 3 and the fact that the dual variables $\{\mu_i\}$—which corresponds to the slopes of $\ell_i(w; \mu_i)$—are decreasing.

Proof of Bullet 5 According to Bullet 3, the function $h(w; \boldsymbol{\mu})$ is nonnegative if and only if $w \geq z_m$. Therefore, $\bar{h}(w; \boldsymbol{\mu}) = h(w; \boldsymbol{\mu})$ for $w \geq z_m$, and $\bar{h}(w; \boldsymbol{\mu}) = 0$ otherwise.

A.13 Proof of Proposition 4.14

In the following, we first prove that the values of $\{b_k\}$ and $\{\rho_k\}$ in Algorithm 1 exist and can be identified efficiently (Appendix A.13.1). We then show that the assignment probability $q(j|w)$ returned from Algorithm 1 is optimal to (3) and possesses the first-order stochastically increasing property (Appendix A.13.2).

A.13.1 Existence of $\{b_k\}$ and $\{\rho_k\}$

We prove by induction that the values of $\{\rho_k\}$ and $\{b_k\}$ in Algorithm 1 exist and can be computed efficiently.

Induction Step We first determine the values of $b_{t_{i-1}+1}$ and $\rho_{t_{i-1}+1}$. From (13), the following hold:

$$\mathbb{E}\left[w \middle| G^{-1}\Big(1 - \sum_{j \le t_{i-1}+1} q_j^*\Big) \le w \le z_{i-1}\right] \ge \alpha_{t_{i-1}+1}\,,$$
$$\mathbb{E}\Big[w \Big| z_i \le w \le z_{i-1}\Big] = \sum_{j \in T_i} \alpha_j \cdot \frac{q_j^*}{\sum_{j \in T_i} q_j^*} \le \alpha_{t_{i-1}+1}\,,$$

where the inequality in the second line follows from the fact that $\alpha_{t_{i-1}+1} \ge \alpha_j$ for any $j \in T_i$. Therefore, there exists a value $b_{t_{i-1}+1}$ satisfying that $z_i \le b_{t_{i-1}+1} \le G^{-1}\Big(1 - \sum_{j \le t_{i-1}+1} q_j^*\Big) \le z_{i-1}$ such that $\mathbb{E}\big[w \big| b_{t_{i-1}+1} \le w \le z_{i-1}\big] = \alpha_{t_{i-1}+1}$. In addition, let $\rho_{t_{i-1}+1} = q^*_{t_{i-1}+1} \,/\, \mathbb{P}[b_{t_{i-1}+1} \le w \le z_{i-1}] \le 1$. The probability $q(t_{i-1}+1|w)$ satisfies (14) by the setup of $b_{t_{i-1}+1}$ and $\rho_{t_{i-1}+1}$.

Iteration Step Let k be an integer with $k \in [t_{i-1}+2 : t_i - 1]$. Suppose that for any $j \in [t_{i-1}+1 : k-1]$, we have determined the values of ρ_j and b_j such that the probability $q(j|w)$ satisfies (14). We now identify the values of ρ_k and b_k such that the probability $q(k|w)$ also satisfies (14).

To achieve this, let $b_k = b \in \big[z_i, z_{i-1}\big]$ and $\rho_k = \rho \in \Big[\underline{\rho}_k, 1\Big]$, with $\underline{\rho}_k \triangleq \frac{q_k^*}{\sum_{\ell=k}^{t_i} q_\ell^*}$. Additionally, let $q(k|w) = \rho_k \cdot \big(1 - q_{\le k-1}(w)\big)$ for $w \in \big[b_k, z_{i-1}\big]$ and $q(k|w) = 0$ for $w \in \big[z_i, b_k\big]$. We define the following two functions:

$$F\big(b, \rho\big) \triangleq \int_{w \in I_i} q(k|w)\, g(w)\, dw\,,$$
$$Q\big(b, \rho\big) \triangleq \int_{w \in I_i} w \cdot q(k|w)\, g(w)\, dw\,.$$

The probability $q\big(k|w\big)$ satisfies (14) under the choice of $b_k = b$ and $\rho_k = \rho$ if and only if $F\big(b,\rho\big) = q_k^*$ and $Q\big(b,\rho\big) = \alpha_k q_k^*$.

Evidently, the function $F\big(b,\rho\big)$ is strictly increasing in ρ and strictly decreasing in b. Therefore, for any $\rho \in \big[\underline{\rho}_k, 1\big]$, there exists a unique value, denoted by $b(\rho)$, that satisfies $F\big(b(\rho),\rho\big) = q_k^*$. In particular, we have $b\Big(\underline{\rho}_k\Big) = z_i$. To see this, note that the following holds:

$$\begin{aligned} F\Big(z_i, \underline{\rho}_k\Big) &= \underline{\rho}_k \cdot \int_{w \in I_i} \big(1 - q_{\le k-1}(w)\big)\, g(w)\, dw \\ &= \underline{\rho}_k \cdot \left(\int_{w \in I_i} g(w)\, dw - \sum_{j=t_{i-1}+1}^{k-1} \int_{w \in I_i} q(j|w)\, g(w)\, dw \right) \\ &= q_k^*\,, \end{aligned}$$

where the first equation follows from the definition of $q\big(k|w\big)$, and the third equation follows from the facts that $\mathbb{P}[z_i \le w < z_{i-1}] = \sum_{j \in T_i} q_j^*$ and that the probability $q(j|w)$ satisfies (14) for any $j \le k-1$, and the definition of $\underline{\rho}_k$. Moreover, the function $b(\rho)$ is strictly increasing with ρ. Its inverse, denoted by $\rho(b)$, exists and is also strictly increasing.

We now define the function
$$Q(b) \triangleq Q\Big(b, \rho(b)\Big).$$

Since $F\big(b, \rho(b)\big) = q_k^*$ for any value of b, it suffices to find a value $b \in \big[z_i, b_{k-1}\big]$ satisfying $Q\big(b\big) =$

$\alpha_k q_k^*$, which we do now.

First, note that the function $Q(b)$ is increasing. This is because, as b increases, we transport the fixed proportion q_k^* of candidates to the right, which increases the mean quality of these candidates.

Second, we inspect the value of $Q(z_i)$. Specifically, the following holds:

$$\begin{aligned} Q(z_i) &= \underline{\rho}_k \cdot \int_{w \in I_i} w \cdot \big(1 - q_{\leq k-1}(w)\big)\, g(w)\, dw \\ &= \underline{\rho}_k \cdot \left(\int_{w\in I_i} w\, g(w)\, dw - \sum_{j=t_{i-1}+1}^{k-1} \int_{w \in I_i} w \cdot q(j|w)\, g(w)\, dw \right) \\ &= \frac{q_k^*}{\sum_{\ell=k}^{t_i} q_\ell^*} \cdot \left(\sum_{\ell \in T_i} \alpha_\ell q_\ell^* - \sum_{\ell=t_{i-1}+1}^{k-1} \alpha_\ell q_\ell^* \right) \\ &= \frac{\sum_{\ell=k}^{t_i} \alpha_\ell q_\ell^*}{\sum_{\ell=k}^{t_i} q_\ell^*} \cdot q_k^* \\ &\leq \alpha_k q_k^*, \end{aligned} \tag{31}$$

where the third equation follows from the second line of (13) and that probability $q(j|w)$ satisfies (14) for any $j \leq k-1$, and the inequality follows from the fact that α_ℓ decreases with index ℓ.

Finally, we derive two more inequalities. If $b(1) \geq b_{k-1}$ (in other words, the "unoccupied" area to the right of b_{k-1} and above the function $q_{\leq k-1}(w)$ is larger than q_k^*), we have

$$Q(b_{k-1}) = \alpha_{k-1} q_k^* > \alpha_k q_k^*\,, \tag{32}$$

because in this case, $q(k|w) = c \cdot q(k-1|w)$ for some constant $c > 0$ and for any $w \in I_i$.

Alternatively, suppose $b(1) \leq b_{k-1}$. Then, it follows that $q_{\leq k}(w) = 1$ for $w \in \big[b(1), z_{i-1}\big]$ and $q_{\leq k}(w) = 0$ for $w < b(1)$, which indicates that $b(1) = G^{-1}\Big(1 - \sum_{j \leq k} q_j^*\Big)$. As a result, the following hold:

$$\begin{aligned} Q\big(b(1)\big) &= \int_{w \in I_i} w \cdot q_{\leq k}(w)\, g(w)\, dw - \int_{w\in I_i} w \cdot q_{\leq k-1}(w)\, g(w)\, dw \\ &= \int_{b(1)}^{z_{i-1}} w \cdot q_{\leq k}(w)\, g(w)\, dw - \sum_{j=t_{i-1}+1}^{k-1} \int_{w \in I_i} w \cdot q(j|w)\, g(w)\, dw \\ &= \mathbb{E}\left[w \cdot \mathbb{1}\left[G^{-1}\Big(1 - \sum_{j \leq k} q_j^*\Big) \leq w \leq z_{i-1} \right]\right] - \sum_{j=t_{i-1}+1}^{k-1} \int_{w\in I_i} w \cdot q(j|w)\, g(w)\, dw \\ &\geq \sum_{j=t_{i-1}+1}^{k} \alpha_j q_j^* - \sum_{j=t_{i-1}+1}^{k-1} \alpha_j q_j^* \\ &= \alpha_k q_k^*\,, \end{aligned} \tag{33}$$

where the inequality follows from the first equation in (13).

Since $Q(b)$ is continuous and increasing in b, (31) – (33) imply that there exists a value $b_k \in \big[z_i, \min\{b_{k-1}, b(1)\}\big]$ satisfying $Q(b_k) = \alpha_k q_k^*$. Moreover, the value of b_k can be efficiently identified using binary search. Let $\rho_k = \rho(b_k)$. The probability $q\big(k|w\big)$ satisfies (14) under the choice of b_k and ρ_k.

Final Step Let $q(t_i|w) = 1 - q_{\leq t_i - 1}(w)$ for any $w \in I_i$. Since $q(j|w)$ satisfies (14) for any $j \leq t_i - 1$, the second equation in (13) and the fact that $\mathbb{P}[z_i \leq w < z_{i-1}] = \sum_{j \in T_i} q_j^*$ imply that $q(t_i|w)$ also satisfies (14).

A.13.2 Optimality and FOSD Property

Let $\{q(j|w)\}$ denote the output of Algorithm 1. $\{q(j|w)\}$ is optimal to (3) according to Lemma 4.13.

We now prove that the distribution $q(\cdot|w)$ first-order stochastically increases with w on the interval I_i. By definition, this is equivalent to proving that the cumulative distribution function $q_{\leq k}(w)$ is increasing in w for any $k \in T_i$. We prove this by induction. First, $q_{\leq t_{i-1}}(w) = 0$ for any $w \in I_i$ by definition, which serves as the induction step. Next, suppose $q_{\leq k-1}(w)$ is increasing on $w \in I_i$ for some $k \in T_i$, we show that $q_{\leq k}(w)$ is also increasing. To do so, fix two points $w, w' \in I_i$ with $w' < w$. If $w' < b_k$, we have

$$q_{\leq k}(w') = q_{\leq k-1}(w') \leq q_{\leq k-1}(w) \leq q_{\leq k}(w),$$

where the inequality follows from the fact that $q_{\leq k-1}(w)$ increases with w. Alternatively, if $w' \geq b_k$, we have

$$\begin{aligned}
q_{\leq k}(w') &= q_{\leq k-1}(w') + \rho_k \cdot \big(1 - q_{\leq k-1}(w')\big) \\
&= q_{\leq k-1}(w') + \rho_k \cdot \big(q_{\leq k-1}(w) - q_{\leq k-1}(w')\big) + \rho_k \cdot \big(1 - q_{\leq k-1}(w)\big) \\
&\leq q_{\leq k-1}(w') + q_{\leq k-1}(w) - q_{\leq k-1}(w') + \rho_k \cdot \big(1 - q_{\leq k-1}(w)\big) \\
&= q_{\leq k-1}(w) + \rho_k \cdot \big(1 - q_{\leq k-1}(w)\big) \\
&= q_{\leq k}(w),
\end{aligned}$$

where the inequality follows from the fact that $q_{\leq k-1}(w) \geq q_{\leq k-1}(w')$ and $\rho_k \geq 0$.

A.14 Proof of Proposition 4.15

A.14.1 Proof of Bullet 1

Let $\{\lambda_k^*\}$ denote an optimal dual variable for the participation constraints in (8) and $\{T_i\}$ denote the resulting partition of the n employers as described in Section 4.4.2. For a feasible solution $\{q_k\}$ to (8), let

$$T_i\big(\{q_k\}\big) \triangleq \Big|T_i \cap \{k \in [n] : q_k > 0\}\Big|$$

denote the number of employers in group T_i that have a positive probability q_k.

Let $\{q_k^*\}$ denote an optimal solution $\{q_k^*\}$ to (8). Lemma A.2 shows that if there exists a group T_i that satisfies $T_i\big(\{q_k^*\}\big) > 2$, we can find a new optimal solution $\{\tilde{q}_k\}$ to (8) that is closer to the desired one in Bullet 1.

Lemma A.2. *Let $\{q_k^*\}$ denote an optimal solution $\{q_k^*\}$ to (8). If there exists a subset T_i that satisfies $T_i\big(\{q_k^*\}\big) > 2$, we can find a new optimal solution $\{\tilde{q}_k\}$ to (8) such that (i) $\tilde{q}_k = q_k^*$ for any $k \notin T_i$, and (ii) $T_i\big(\{\tilde{q}_k\}\big) < T_i\big(\{q_k^*\}\big)$.*

Repeating the process in Lemma A.2 iteratively will eventually (in at most n steps) yields a desired optimal solution to (8) that satisfies Bullet 1.

Proof of Lemma A.2. From Lemma 4.13, there exists an optimal solution $\{q^*(j|w)\}$ to (3) such that the candidate joins each employer j with probability q_j^*. In the following, we modify $\{q^*(j|w)\}$

to create a new optimal solution $\{\tilde{q}(j|w)\}$ to (3) such that the candidate joins each employer j with probability $\tilde{q}_j$, where $\{\tilde{q}_j\}$ satisfies Lemma A.2. Then, $\{\tilde{q}_j\}$ is optimal to (8) according to Proposition 4.10.

Suppose $\{a, b, c\} \subseteq T_i(\{q_k^*\})$, where a, b, and c denote three distinct integers. Without loss of generality, assume that $1 \leq a < b < c \leq n$. Therefore, $\alpha_a > \alpha_b > \alpha_c$. We consider the following two scenarios.

Case One Suppose

$$\frac{\alpha_a q_a^* + \alpha_c q_c^*}{q_a^* + q_c^*} = \alpha_b, \tag{34}$$

that is, the mean quality of the candidates joining employers a or c is precisely α_b, the recruiting bar of employer b. Let

$$\tilde{q}(j|w) = \begin{cases} q^*(a|w) + q^*(c|w) & \text{if } j = b, \\ 0 & \text{if } j \in a, c, \\ q^*(j|w) & \text{if } j \notin \{a, b, c\}. \end{cases}$$

(34) implies that the participation constraint for employer b remains binding with $\tilde{q}(j|w)$. Therefore, $\tilde{q}(j|w)$ is optimal to (3) according to Lemma 4.13. Additionally, we have

$$\tilde{q}_j \triangleq \int_0^1 \tilde{q}(j|w)\, g(w)\, dw = \begin{cases} q_b^* + q_a^* + q_c^* & \text{if } j = b, \\ 0 & \text{if } j \in a, c, \\ q_j^* & \text{if } j \notin \{a, b, c\}. \end{cases}$$

As a result, $\{\tilde{q}_j\}$ satisfies Lemma A.2 because $\{\tilde{q}_j\}$ is optimal to (8) by Proposition 4.10 and $T_i(\{\tilde{q}_j\}) = T_i(\{q_j^*\}) - 2 < T_i(\{q_j^*\})$ by construction.

Case Two Suppose (34) does not hold. Without loss of generality, assume that $\frac{\alpha_a q_a^* + \alpha_c q_c^*}{q_a^* + q_c^*} > \alpha_b$, which translates to $q_a^* > \underline{q}_a \triangleq q_c^* \cdot \frac{\alpha_b - \alpha_c}{\alpha_a - \alpha_b}$. Let $\rho_a \triangleq \underline{q}_a / q_a^* < 1$. Note that the following holds:

$$\frac{\alpha_a \underline{q}_a + \alpha_c q_c^*}{\underline{q}_a + q_c^*} = \alpha_b. \tag{35}$$

Let

$$\tilde{q}(j|w) = \begin{cases} \rho_a \cdot q^*(a|w) + q^*(c|w) & \text{if } j = b, \\ (1 - \rho_a) \cdot q^*(a|w) & \text{if } j \in a, \\ 0 & \text{if } j \in c, \\ q^*(j|w) & \text{if } j \notin \{a, b, c\}. \end{cases}$$

(35) implies that the participation constraint for employer b remains binding with $\tilde{q}(j|w)$. Therefore, $\tilde{q}(j|w)$ is optimal to (3) according to Lemma 4.13. Additionally, we have

$$\tilde{q}_j \triangleq \int_0^1 \tilde{q}(j|w)\, g(w)\, dw = \begin{cases} q_b^* + \rho_a \cdot q_a^* + q_c^* & \text{if } j = b, \\ (1 - \rho_a) \cdot q_a^* & \text{if } j \in a, \\ 0 & \text{if } j \in c, \\ q_j^* & \text{if } j \notin \{a, b, c\}. \end{cases}$$

As a result, $\{\tilde{q}_j\}$ satisfies Lemma A.2 because $\{\tilde{q}_j\}$ is optimal to (8) by Proposition 4.10 and $T_i\big(\{\tilde{q}_j\}\big) = T_i\big(\{q_j^*\}\big) - 1 < T_i\big(\{q_j^*\}\big)$ by construction. $\square$

A.14.2 Proof of Bullet 2

In this section, we demonstrate that Bullet 2 can be derived from our established results based on the dual analysis. Assume, without loss of generality, that there exists an optimal solution $\{q_k^*\}$ to (8) such that $q_k^* > 0$ for any $k \in [n]$. We then show that the values of $\{q_k^*\}$ are unique. To see that this assumption loses no generality, let

$$P_\varnothing = \Big\{k \in [n] : q_k^* = 0 \text{ for any optimal solution} \{q_k^*\} \text{ to } (8)\Big\}$$

denote the set of employers ignored by any optimal solution to (8). We can exclude the employers in set $P_\varnothing$ without affects anything. Meanwhile, define $P = [n] \setminus P_\varnothing$. Since (8) is a convex optimization problem, the set of optimal solutions is convex. This implies that there exists an optimal solution $\{q_k^*\}$ such that $q_k^* > 0$ for any $k \in P$.

Now, let $\{\lambda_k^*\}$ denote the optimal dual variable of (8). Note that the values of $\{\lambda_k^*\}$ are unique according to Proposition 4.11. Let $\{T_i\}$ denote the partition of employers described in Section 4.4.2. Since no three points of $\{(\alpha_i, v_i)\}_{i\in[n]}$ are collinear, any set T_i contains at most two employers based on Bullet 2 of Lemma 4.12. Fix a group T_i. First, suppose $T_i = \{k\}$ contains one employer. Then, we have $q_k^* = \mathbb{P}[z_i \le w \le z_{i-1}]$, whose value is uniquely determined.

Second, suppose $T_i = \{k, j\}$ contains two employers. Then, the values of q_k^* and q_j^* must satisfy

$$\begin{aligned} q_k^* + q_j^* &= \mathbb{P}[z_i \le w \le z_{i-1}], \\ \alpha_k q_k^* + \alpha_j q_j^* &= \mathbb{E}\Big[w \cdot \mathbb{1}\big[z_i \le w \le z_{i-1}\big]\Big], \end{aligned}$$

and therefore, are uniquely determined as well.