Optimality of Public Persuasion for Single-Good Allocation*

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Abstract

We study an information design problem in which a sender seeks to allocate an indivisible good to n receivers by strategically disclosing information. The good can be allocated to at most one receiver, and each receiver decides whether to accept based on self-interest. Relevant applications include school advisors promoting students for job placements and incubators introducing startups to potential investors. The sender can either send different signals to different receivers (i.e., *private persuasion*) or broadcast the same signal to all receivers (i.e., *public persuasion*). After receiving signals, receivers may communicate with each other in their self-interest to further reduce uncertainty about the good. We demonstrate when the sender has a known preference among the receivers, public persuasion is optimal, regardless of how receivers communicate. The optimal public persuasion can be derived from a first-best relaxation problem that imposes only the receivers' participation constraints. We then focus on a special case where the good's characteristics are captured by a one-dimensional variable, and all receivers' utility functions are linear in this variable. Using a dual approach, we derive optimality conditions for persuasion mechanisms. This leads to closed-form optimal mechanisms for the two-receiver case and an explicit characterization of the set of optimal mechanisms in the general case, thereby enhancing our understanding of their structural properties.

Subject classifications: Bayesian persuasion, public information, single-item allocation, multiple receivers, post-signal communication, Lagrangian dual

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1 Introduction

In this paper, we study a Bayesian persuasion problem in which a sender seeks to allocate an indivisible good among n receivers. The sender has a known preference over the receivers and can allocate the good to at most one receiver. Moreover, the sender holds private information about the good's characteristics, which determine the receivers' utilities. To maximize her expected payoff, the sender can commit to an information disclosure mechanism that strategically discloses these characteristics to the receivers. Notably, the sender can use either a *public* persuasion mechanism to share the same information with all receivers, or a *private* persuasion mechanism to send tailored information to different receivers based on their specific acceptance criteria. We illustrate our setting with two motivating examples.

Example 1: Job Seeking Consider a school advisor promoting a student in a job market with multiple potential employers (e.g., schools with open junior faculty positions or PhD programs at multiple universities). The advisor holds private information about the student's characteristics relevant to the employers' hiring requirements (e.g., research potential, teaching experience, and communication skills, etc.). The advisor can commit to an information disclosure mechanism that strategically discloses the candidate's characteristics (e.g., through targeted recommendation letters) to the employers to maximize the candidate's expected payoff. Each employer decides whether to extend a job offer to the student, and the student can accept at most one offer. Finally, the student has a known preference among employers. For example, candidates often prioritize employers based on factors such as alignment of interests, institutional reputation, and geographic location, which employers commonly recognize.[1]

Example 2: Startup Fundraising Consider an incubator introducing a startup to potential investors. The startup has achieved initial market traction and is seeking a significant round of venture capital (VC) funding (e.g., Series A). To attract investment, the incubator helps the startup approach potential VC investors and strategically share pertinent information about the startup's performance and product, either privately or publicly.[2] Each VC firm decides whether to extend a

[1]For example, in academic job markets, top-ranked institutions typically do not spend a lot of time worrying about whether a candidate will accept their offer, because they are generally highly attractive to any candidate. Conversely, mid-ranked institutions will factor the chance that a student will accept an offer into consideration, as they know a coveted candidate might decline their offer if a better offer comes.

[2]For example, a startup might hold private meetings with certain VCs to disclose financial information and showcase prototypes. Alternatively, it could release a public signal like a press release highlighting key metrics such as user growth or market traction that every investor sees simultaneously.

term sheet (i.e., an investment offer). Typically, the startup selects only one investor as the lead for a funding round; thus, only one offer is ultimately accepted. Furthermore, it is widely recognized that venture capital funding is not just about money. Startups often prefer lead investors who can bring additional value beyond capital—such as mentorship, prestigious VC brand recognition, specialized industry expertise, and valuable industry networks (Hsu 2004). These preferences are usually visible, either explicitly known or inferable by VC firms based on the startup's industry context and observable attributes.

Moreover, our model permits receivers to communicate among themselves after receiving signals from the sender, which is common in practice. Specifically, receivers may communicate with each other (either simultaneously or sequentially, using either cheap talk or some degree of commitment) to reduce uncertainty about the good in their self-interest.[3] Then, based on the signal received from the sender and the additional information from other receivers, each receiver decides whether to extend an offer to the sender. We note that the receivers in this context are both cooperators and competitors. The communication reduces uncertainty about the good's characteristics, which benefits each receiver. However, since the sender can accept only one offer, competition among the receivers arises. Particularly, if a receiver knows that the good is of high quality, he may withhold this information from other receivers to avoid competition, especially if the sender prefers other receivers. Therefore, the potential for subsequent communication among receivers substantially complicates the information design problem, making it unclear what an optimal persuasion mechanism is.

As our first main result, we demonstrate that public persuasion is always optimal regardless of the detailed communication protocol used by the receivers (Section 3).[4] Since all the receivers receive the same information under a public persuasion mechanism, subsequent communication cannot convey any payoff-related information and therefore becomes irrelevant. As a result, the sender eliminates any room for the receivers to communicate and infer further about the good for her own benefit.

Furthermore, we show that the optimal public persuasion mechanism can be solved from a first-best relaxation problem that imposes only receivers' participation constraints. Specifically, in

[3]For example, at academic job market conferences, colleagues from different institutions often exchange information about job market candidates during informal conversations. Analogously, the VC community is characterized by extensive formal and informal communication networks. For example, VC partners frequently co-invest in deals, attend industry events, and leverage their networks for deal sourcing and due diligence. Consequently, information and opinions about startups could be shared among VCs, sometimes selectively.

[4]This is perhaps striking because the optimal persuasion mechanism differs among receivers when considering each receiver in isolation. Moreover, public persuasion is optimal even when the sender knows that receivers cannot communicate (but are aware of each other's existence), as we elaborate further in Remark 4.3.

the first-best problem, a central planner allocates a good with characteristics w to receivers. A receiver accepts the good when it is allocated to him. The first-best relaxation problem solves the optimal randomized allocation to maximize the sender's expected payoff ensuring only a nonnegative expected utility for each receiver. We show that an optimal public persuasion mechanism can be derived from an optimal solution to the first-best relaxation problem, and its expected payoff matches the first-best upper bound.

Although an optimal public persuasion mechanism can be obtained from the aforementioned first-best relaxation problem, it becomes an infinite-dimensional linear program (LP) when the good's characteristics w are infinite, making it challenging to solve. As our second main result, we focus on efficiently computing an optimal public persuasion mechanism for the special case where the good's characteristics w can be summarized by a one-dimensional variable, and all receivers' utility functions are linear in this variable; consequently, each receiver cares only about the mean quality of the good allocated to him (Section 4). This setup has been extensively studied in the literature; see, for example, Candogan (2022), Dworczak and Martini (2019), Kleiner et al. (2021), and Arieli et al. (2023), and we compare with these works in detail in Section 1.1.

Under the linear utility assumption, we provide optimality conditions for persuasion mechanisms based on the Lagrangian dual of the first-best relaxation problem, where we dualize the participation constraints (Section 4.2). Specifically, a persuasion mechanism and a set of dual variables $\{\mu_i\}$ are optimal to the persuasion problem and Lagrangian dual, respectively, if and only if the persuasion mechanism is optimal to the Lagrangian given dual variables $\{\mu_i\}$, and the participation constraints bind for all receivers with a strictly positive dual variable (i.e., $\mu_i > 0$). We also interpret the Lagrangian dual from a geometric view.

Based on the optimality conditions, we derive optimal persuasion mechanisms in closed form when there are two receivers, where one receiver has a higher acceptance threshold but also brings a higher payoff (Section 4.3). The main trade-off is that an offer from a more competitive receiver brings a higher payoff; however, targeting this receiver more aggressively is costly because it reduces the overall probability of receiving an offer. Depending on the relative desirability of the two receivers and their acceptance thresholds, the optimal persuasion mechanism carefully balances this trade-off. For the general case with n receivers (Section 4.4), we explicitly characterize the set of all optimal mechanisms based on the geometric interpretation of the Lagrangian and the dual connection between our first-best relaxation problem and a finite-dimensional convex optimization problem analogous to the one studied in Candogan (2022). We also demonstrate various ways to construct optimal persuasion mechanisms (Appendix D).

The rest of the paper is organized as follows. Section 1.1 reviews related literature. Section 2 formulates the problem. In Section 3, we establish that public persuasion mechanisms are optimal in our setup, irrespective of how receivers communicate. Moreover, the optimal public persuasion mechanism can be obtained from a first-best relaxation problem that requires only the receivers' participation constraints. Section 4 analyzes the linear utility case. We provide optimality conditions based on duality in Section 4.2, which lead to closed-form optimal mechanisms for the two-receiver case (Section 4.3) and explicit characterizations of the set of optimal mechanisms for the general case (Section 4.4). Section 5 concludes.

1.1 Related Literature

Our work contributes to the literature on Bayesian persuasion and information design, particularly in contexts involving multiple receivers. As highlighted by Kamenica (2019), "if sender can send separate signals to each receiver, and if either (a) a receiver's optimal action depends on what other receivers do, or (b) sender's utility is not separable across receiver's actions, then the problem becomes significantly more difficult." Our setup incorporates both features. Below, we review relevant topics from the information design literature.

Persuasion with Spillovers Much of the existing literature does not incorporate post-signal communication among receivers as we do. Notable exceptions include Babichenko et al. (2022), Galperti and Perego (2023) and Candogan et al. (2023), which consider informational spillovers among receivers. In these studies, the spillover structure is pre-specified, and public persuasion is typically suboptimal. In contrast, our model allows receivers to communicate strategically in their self-interest via arbitrary communication methods. Babichenko et al. (2022) characterize when one spillover structure universally provides the sender with higher expected utility than another. Galperti and Perego (2023) characterize all possible equilibrium outcomes that can arise from an information structure given spillover and seeding constraints. Candogan et al. (2023) show that the optimal information design problem is generally computationally challenging under information spillovers, except for some specific cases.

LP Formulation Bergemann and Morris (2016) and Bergemann and Morris (2019) relate the multi-receiver persuasion problem to the game-theoretic concept of Bayes correlated equilibrium (BCE). This relationship leads to a natural LP formulation for obtaining an optimal persuasion mechanism. Specifically, the decision variables in the LP are joint probabilities of states and receivers' actions, and the constraints completely characterize the set of BCEs.[5] Our first-best relaxation problem (2) is also an LP. However, in our LP, the decision variables are marginal allocation probabilities under a mechanism. The LP imposes only participation constraints that any mechanism must satisfy, and thus, does not precisely characterize the set of equilibrium outcomes.

Optimality of Public Persuasion Bergemann and Morris (2019) also explore when public persuasion mechanisms are optimal (Section 4.1 therein). Their model does not incorporate post-signal communications. They show that public persuasion mechanisms are optimal when receivers' actions are strategic complements, as these mechanisms induce positively correlated actions. However, in our setup, receivers' actions are not strategic complements. For a binary state space without payoff externalities among receivers, Arieli and Babichenko (2019) provide necessary and sufficient conditions for the optimality of public persuasion (Theorem 3 therein). Consequently, we have identified distinct conditions—specifically, when the sender allocates a single good and has known preferences over receivers—under which public persuasion remains optimal in a strong, robust sense.

Linear Utilities In Section 4, we consider a special case in which the good's characteristics w is one-dimensional, and each receiver's utility function is linear in this variable. Consequently, each receiver cares only about the mean quality of the good allocated to him. This assumption simplifies the information design problem as the sender equivalently designs a distribution of posterior means of the good's quality. Dworczak and Martini (2019), Kleiner et al. (2021), and Arieli et al. (2023) study a general optimization problem in which the sender's utility depends on the posterior mean of the underlying state in an arbitrary way, and they design the distribution of posterior means, which must be a mean-preserving contraction of the prior distribution, to maximize the sender's expected utility. The persuasion problem studied in Section 4 corresponds to a special case of their problem, where the sender's utility function is increasing and piecewise constant in the posterior mean, as in Candogan (2022).

Candogan (2022), Kleiner et al. (2021), and Arieli et al. (2023) study the persuasion problem through an extreme points approach.[6] Their results are an optimal persuasion mechanism that has a double-interval or a more general bi-pooling structure. Our work, on the contrary, uses a dual approach to establish optimality conditions for any persuasion mechanism, and characterizes the

[5]That is, a joint distribution sustains a BCE if and only if it is feasible to the LP.

[6]Specifically, Kleiner et al. (2021) and Arieli et al. (2023) independently characterize the extreme points of the set of mean-preserving contractions of a given prior distribution. These extreme points correspond to bi-pooling distributions (termed by Arieli et al. 2023). The double-interval structure in Candogan (2022) is a special case of the bi-pooling distribution.

set of *all* optimal persuasion mechanisms.

Dworczak and Martini (2019) also use a dual approach to derive optimality conditions for their general optimization problem (presented in their Theorem 1 and Corollary 1). Specifically, they show that a distribution of posterior means sustains an optimal persuasion mechanism if and only if it, together with a price function $p(x)$, satisfies conditions (2)–(4) therein, and they interpret this price function $p(x)$ as the Walrasian equilibrium price in a persuasion economy. We observe that the envelop function $\bar{h}(w; \boldsymbol{\mu}^*) \triangleq \max\{h(w; \boldsymbol{\mu}^*), 0\}$ from our Lagrangian (defined in Section 4.2 and fully characterized in Proposition 4.6) precisely corresponds to the equilibrium price function $p(x)$ described in Dworczak and Martini (2019). Consequently, we fully characterize this equilibrium price—which is generally challenging to specify—for the special case where the sender's utility is increasing and piecewise constant in the posterior mean. Notably, our characterization of $p(x)$ relies on an alternative dual approach: we dualize the receivers' participation constraints instead of the mean-preserving contraction constraint used by Dworczak and Martini (2019). This methodological distinction may be of independent interest. We provide further discussion in Remark 4.5.

Information Design in Other Applications In addition to single-good allocation studied in this paper, numerous works have investigated efficient information disclosure in various other operational applications. For example, Papanastasiou et al. (2018) study how online platforms can strategically disclose consumer reviews to incentivize exploration of alternative options. Drakopoulos et al. (2021) examine how sellers can signal product availability to persuade customers to purchase earlier and increase revenue. Anunrojwong et al. (2023) explore how to strategically reveal queue congestion to maximize welfare. De Véricourt et al. (2021) investigate how governments can strategically inform pandemic severity to induce compliance with confinement measures. We direct interested readers to Candogan (2020) for a comprehensive review.

1.2 Notation and Terminology

We let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{N}_+$ the set of strictly positive integers. For any two integers $a, b \in \mathbb{N}$ with $a \leq b$, we let $[a:b] = \{a, a+1, \ldots, b-1, b\}$ denote a sequence of integers starting from a and ending with b, and we denote $[n] = [1:n]$ for any $n \in \mathbb{N}_+$. For any real number $x \in \mathbb{R}$, we let $(x)^+ \triangleq \max\{x, 0\}$ denote the maximum of x and 0.

2 Problem Formulation

We consider a sender (referred to as "she") who promotes a good among n potential receivers (each referred to as "he") via strategic information disclosure. The sender can allocate the good to at most one receiver and has a known preference among receivers. Specifically, we let $v_i > 0$ denote the sender's utility from allocating the good to receiver i, and we rank receivers in decreasing preference; that is, $v_i > v_j$ if $i < j$, as assumed in Assumption 2.1. If the good is not allocated, the sender's utility is normalized to zero.

Assumption 2.1. The sender's utility from allocating the good satisfies $0 < v_n < \cdots < v_2 < v_1$.

Let $w \in \Omega$ represent the characteristics of the good, where Ω is a general state space.[7] Although the sender privately observes the realization of w, receivers only possess a prior distribution $G(w)$ regarding the good's characteristics. For each receiver i, let $u_i(w)$ denote the utility from receiving a good with characteristics w; the utility of not receiving the good is zero.

Information Disclosure Mechanism We study a Bayesian persuasion setup in which the sender, who has commitment power, designs an information disclosure mechanism to promote her good to the n receivers. Let S_i denote the set of signals the sender uses to interact with receiver i and $\boldsymbol{S} = \bigotimes_{i=1}^{n} S_i$ represent the set of all signals. Upon observing the characteristics w, the sender sends a signal $s_i \in S_i$ to each receiver i according to a joint distribution $f(\boldsymbol{s}|w)$, where $\boldsymbol{s} = (s_1, \cdots, s_n) \in \boldsymbol{S}$ denotes the concatenation of the sent signals. We define the information mechanism $f(\cdot|w)$ as a *public* mechanism if

1. The signals share a common signal space S, that is, $S_i = S_j = S$ for all $i, j \in [n]$; and
2. The signals $(s_i)_{i\in[n]}$ are perfectly correlated, that is, $f(\boldsymbol{s}|w) = 0$ for any signal $\boldsymbol{s} = (s_i)_{i\in[n]}$ where $s_i \neq s_j$ for some $i, j \in [n]$.

With a public mechanism, receivers always receive the same signal, eliminating the need for further communication. Conversely, if $f(\cdot|w)$ allows for different signals among receivers, we refer to it as a *private* information mechanism. In this case, the receivers may receive different signals, leading to varied information about the good's characteristics w.

[7]For example, we may have $\Omega \subseteq \mathbb{R}^m$, where m represents the number of attributes relevant to receivers' acceptance standards, such as research potential, teaching experience, and communication skills in the student promotion example.

Communication among Receivers We assume that receivers may communicate with each other after receiving the signal $\boldsymbol{s}$. We do not formally model how receivers will communicate. Notably, receivers may or may not be able to communicate, and if they do, it could be either simultaneously or sequentially, using either cheap talk or with some degree of commitment. Any of these communication methods can be reasonable in specific scenarios. However, as we demonstrate in Section 3, the optimal persuasion mechanism will be independent of the detailed communication method. This is because, regardless of how receivers communicate, a public information disclosure mechanism will always be optimal for the sender, leaving nothing for the receivers to communicate.

However, some notations are helpful to describe the problem. Given a specific communication protocol, let C_i denote the set of information that receiver i can receive from other receivers and $\mathbf{C} = \bigotimes_{i=1}^{n} C_i$ represent the communication space. Denote the communication outcome as $\mathbf{c} = (c_1, ..., c_n) \in \mathbf{C}$, where c_i is the information receiver i receives through communication. Given a signal $\boldsymbol{s}$, suppose the cumulative distribution function of $\mathbf{c}$ is $C(\mathbf{c}|\boldsymbol{s})$, and the probability density function of $\mathbf{c}$ is $c(\mathbf{c}|\boldsymbol{s}) = \frac{dC(\mathbf{c}|\boldsymbol{s})}{d\mathbf{c}}$, possibly derived from the receivers' equilibrium strategies.

Sender's Problem The game proceeds as follows:

1. The sender commits to an information disclosure mechanism $f(\cdot|w)$ and a signal space $\mathbf{S} = \bigotimes_{i=1}^{n} S_i$.
2. The good's characteristics w are drawn from the cumulative probability distribution $G(w)$. A signal $\mathbf{s} = (s_i)_{i\in[n]}$ is then generated according to the disclosure mechanism $f(\cdot|w)$ and sent to the receivers.
3. Receivers communicate with each other after receiving the signal $\boldsymbol{s}$ using $C(\cdot|\boldsymbol{s})$, which may represent an equilibrium communication strategy in a specific scenario. After communication, each receiver i decides whether to accept the good (or extend an offer) based on the signal s_i and the communication outcome c_i.
4. The sender accepts the offer that maximizes her payoff, which corresponds to the receiver with the smallest index among those sending offers, according to Assumption 2.1.

Given a signal and communication outcome $s \in S_i$ and $c \in C_i$, we define $\mathbf{S}^i(s) = \{\boldsymbol{s} \in \mathbf{S} : s_i = s\}$ and $\mathbf{C}^i(c) = \{\mathbf{c} \in \mathbf{C} : c_i = c\}$ as the sets of possible signals and communications, respectively. Upon observing s and c, receiver i understands that the signal must be in the set $\mathbf{S}^i(s)$ and the communication outcome must be in the set $\mathbf{C}^i(c)$. He updates his belief about the good's characteristics w, the signal $\boldsymbol{s}$, and the communication outcome $\mathbf{c}$ using Bayes's rule whenever possible. Specifically, let $f_i(s,c)$ denote the probability that receiver i receives a signal s and communication outcome c:

$$f_i(s,c) = \int_{w\in\Omega}\int_{\boldsymbol{s}\in\mathbf{S}^i(s)}\int_{\mathbf{c}\in\mathbf{C}^i(c)} c(\mathbf{c}|\boldsymbol{s})\, f(\boldsymbol{s}|w)\, d\mathbf{c}\, d\boldsymbol{s}\, dG(w).$$

If $f_i(s,c) > 0$, the receiver i's posterior belief on the tuple $(w,\boldsymbol{s},\mathbf{c})$ is defined as

$$f_i(w,\boldsymbol{s},\mathbf{c}|s,c) = \begin{cases} \frac{c(\mathbf{c}|\boldsymbol{s}) f(\boldsymbol{s}|w)\, dG(w)}{f_i(s,c)}, & \text{if } \boldsymbol{s}\in\mathbf{S}^i(s) \text{ and } \mathbf{c}\in\mathbf{C}^i(c),\\ 0, & \text{otherwise.}\end{cases}$$

Denote receiver i's equilibrium strategy by $\delta_i(s,c)$, representing his probability of extending an offer after receiving a signal $s\in S_i$ and communication outcome $c\in C_i$. The optimality of receiver i's strategy implies that $\delta_i(s,c)$ follows the following equation:

$$\delta_i(s,c) = \begin{cases} 0, & \text{if } \mathbb{E}\left[u_i(w)\cdot \mathbb{1}[a_j^* = 0,\, \forall j < i] \mid s,c\right] < 0,\\ \delta\in[0,1], & \text{if } \mathbb{E}\left[u_i(w)\cdot \mathbb{1}[a_j^* = 0,\, \forall j < i] \mid s,c\right] = 0,\\ 1, & \text{if } \mathbb{E}\left[u_i(w)\cdot \mathbb{1}[a_j^* = 0,\, \forall j < i] \mid s,c\right] > 0,\end{cases}$$

where the binary variable $a_j^* \in \{0,1\}$ represents receiver j's action of extending an offer in an equilibrium and satisfies $\mathbb{P}[a_j^*=1|s_j,c_j] = \delta_j(s_j,c_j)$, and the expectation $\mathbb{E}[\cdot|s,c]$ is taken over the posterior distribution $f_i(w,\boldsymbol{s},\mathbf{c}|s,c)$. Note that the sender accepts receiver i's offer if and only if none of the receivers $j<i$ extends an offer, which is represented by $\mathbb{1}[a_j^*=0,\,\forall j<i]$.

Finally, let the random set $I(\boldsymbol{s},\mathbf{c})$ denote the receivers who extend an offer and $i(\boldsymbol{s},\mathbf{c}) \triangleq \min I(\boldsymbol{s},\mathbf{c})$ the index of the offer to accept, given the signal realization $\boldsymbol{s}\in\mathbf{S}$ and communication outcome $\mathbf{c}\in\mathbf{C}$ and under the receivers' equilibrium strategies. If $I(\boldsymbol{s},\mathbf{c}) = \varnothing$, that is, the good receives no offer, we let $i(\boldsymbol{s},\mathbf{c}) = \varnothing$ and $v_\varnothing = 0$ as the corresponding utility of the sender. The sender selects an information disclosure mechanism $f(\cdot|w)$ that maximizes her expected payoff by solving

$$V^* \triangleq \max_{f(\cdot|w)} \int_{w\in\Omega}\int_{\boldsymbol{s}\in\boldsymbol{S}}\int_{\mathbf{c}\in\mathbf{C}} \mathbb{E}_{i(\boldsymbol{s},\mathbf{c})}\left[v_{i(\boldsymbol{s},\mathbf{c})}\right]\cdot c(\mathbf{c}|\boldsymbol{s})\cdot f(\boldsymbol{s}|w)\cdot d\mathbf{c}\; d\boldsymbol{s}\; dG(w). \tag{1}$$

In (1), the expectation $\mathbb{E}_{i(\boldsymbol{s},\mathbf{c})}[\cdot]$ is taken over the possible randomness in the receivers' equilibrium offer-extending strategies when the signal and communication realizations are $\boldsymbol{s}$ and $\mathbf{c}$, respectively, and V^* denotes the expected payoff of an optimal information disclosure mechanism.

3 Optimality of Public Persuasion

In this section, we demonstrate that a public persuasion mechanism solves the sender's optimal information disclosure problem (1), regardless of how receivers communicate. We begin by introducing a relaxation of the sender's problem (1) in Section 3.1, which provides an upper bound on her optimal expected payoff V^*.

3.1 First-Best Problem with Participation Constraints

In this section, we introduce the first-best relaxation (2) for the sender's information design problem, where we impose only the receivers' participation constraints.

$$\begin{aligned} \bar{V} = \max_{q(i|w)\geq 0} \quad & \sum_{i=1}^{n} v_i \int_{w\in\Omega} q(i|w)\, dG(w) \\ \text{s.t.} \quad & \int_{w\in\Omega} u_i(w)\, q(i|w)\, dG(w) \geq 0,\ \forall\, i \in [n], \\ & \sum_{i\in[n]} q(i|w) \leq 1,\ \forall\, w \in \Omega. \end{aligned} \tag{2}$$

In (2), a central planner allocates the good with characteristics w to receiver i with a probability of $q(i|w)$, and requires the receiver to accept the good when the latter is allocated to him. The chosen $q(i|w)$ ensures a nonnegative expected utility for each receiver, as indicated by the first constraint in (2). This reflects the fact that each receiver should be at least break-even in expectation from accepting the good. In addition, a good is allocated to at most one receiver, as indicated by the second constraint in (2). The central planner chooses $q(i|w)$ satisfying these two constraints to maximize the sender's expected payoff, and the optimal value is denoted by $\bar{V}$.

Lemma 3.1 demonstrates that (2) provides an upper bound on the sender's optimal expected payoff V^*, regardless of how receivers communicate.

Lemma 3.1. *We have $\bar{V} \geq V^*$, regardless of how receivers communicate.*

We prove Lemma 3.1 in Appendix A.1. Intuitively, given any disclosure mechanism $f(\cdot|w)$, let $q(i|w)$ denote the ex-ante probability that the good is allocated to receiver i when its characteristics are w under the receivers' equilibrium strategies induced by $f(\cdot|w)$. These $\{q(i|w)\}$ are feasible to (2) and have an objective value no larger than $\bar{V}$.

3.2 Optimality of Public Persuasion

In this section, we construct a public persuasion mechanism $f^*(\cdot|w)$ from the optimal solution of (2) and show that its expected payoff attains the first-best upper bound $\bar{V}$. Therefore, the mechanism $f^*(\cdot|w)$ is optimal to (1), and this optimality does not depend on the communication protocol among receivers.

Let $\{q^*(i|w)\}$ denote an optimal solution to (2). We consider a public persuasion mechanism $f^*(\cdot|w)$ with signal space $S_i = S \triangleq [n] \cup \{\varnothing\}$ for all receivers $i \in [n]$. When the good's characteristics are w, the mechanism broadcasts the signal $s = i$ to all receivers with probability $q^*(i|w)$ for any $i \in [n]$ and the signal $s = \varnothing$ to all receivers with probability $1 - \sum_{i \in [n]} q^*(i|w)$. We can interpret the signal $s = i$ as a recommendation for only receiver i to extend an offer and the signal $s = \varnothing$ as a recommendation for none of the receivers to extend an offer. Theorem 3.2 shows that this persuasion mechanism achieves the first-best upper bound $\bar{V}$.

Theorem 3.2. *Under the public persuasion mechanism $f^*(\cdot|w)$, it is an equilibrium for each receiver $i \in [n]$ to extend an offer only upon receiving the signal $s = i$. Moreover, the expected payoff of the mechanism $f^*(\cdot|w)$, denoted by V^{P}, satisfies $V^{\mathrm{P}} = \bar{V}$.*

We prove Theorem 3.2 in Appendix A.2. To understand the equilibrium in Theorem 3.2, suppose the sender recommends the good to receiver i. Receiver i is willing to extend an offer because: (i) his offer will be accepted with certainty given that no other receiver will extend an offer, and (ii) he can break even from his offer in expectation, as indicated by the first constraint in (2). Any receiver $j > i$ cannot benefit from extending an offer because the sender will accept the more attractive offer from receiver i. Any receiver $j < i$ is unwilling to extend an offer because: (i) his offer, if extended, will be accepted with certainty given that no better offer will be extended, and (ii) $\{q^*(i|w)\}$ being an optimal solution of (2) implies that receiver j cannot break even from his offer in expectation—otherwise, the central planner in (2) can strictly improve the sender's payoff by allocating the good to receiver j instead of receiver i without violating any constraint in (2).

Since the mechanism $f^*(\cdot|w)$ achieves the first-best upper bound $\bar{V}$, Lemma 3.1 implies that the first-best upper bound is tight (i.e., $\bar{V} = V^*$) and that $f^*(\cdot|w)$ is an optimal persuasion mechanism, independent of how receivers can communicate post-signal. Since the sender sends the same information to all receivers with mechanism $f^*(\cdot|w)$, communication becomes irrelevant. Therefore, the sender eliminates any communication among the receivers for her own benefit, regardless of the way receivers can communicate. This holds true even when the sender knows that the receivers cannot communicate but are aware of each other's existence, as we elaborate further in Remark 4.3.

Finally, we remark that our results remain valid even when we relax the strict preference assumption (Assumption 2.1) and allow weak preferences among receivers (i.e., $v_i \geq v_j$ for any $i < j$), as detailed in Remark 3.1.

Remark 3.1 (Weak Preference). Public persuasion remains optimal and continues to achieve first-best performance even when the sender is indifferent among certain receivers (i.e., $v_i \geq v_j$ for any $i < j$). Specifically, the first-best problem (2) can still be implemented through a public persuasion mechanism in the identical way, provided the sender breaks ties according to the optimal solution of (2), and receivers anticipate this tie-breaking rule.

3.3 Discussion of Assumptions

We have demonstrated that public persuasion is broadly optimal in our setup, regardless of how receivers can communicate. The key assumption underlying this result is that the sender derives a deterministic utility v_i from allocating the good to receiver i (Assumption 2.1). In Remark 3.2, we further show that public persuasion remains optimal even when the utilities $\{v_i\}$ are uncertain and potentially correlated with the good's characteristics w, provided the sender's ordinal ranking of receivers remains fixed.

Remark 3.2 (Uncertain Preferences). Suppose the offer values $\{v_i\}$ to the sender are uncertain and potentially correlated with the good's characteristics w. Then, in general, public persuasion may no longer be optimal. However, if the sender's ordinal ranking over receivers remains fixed, public persuasion remains optimal. Specifically, in this scenario, we can adapt the first-best relaxation problem (2) and demonstrate that its optimal solution can still be implemented through a public persuasion mechanism. We provide further details in Appendix A.3.

Second, our model assumes that receivers take binary actions (i.e., accepting the good or not). In practice, a receiver may have multiple options.[8] In Remark 3.3, we demonstrate that public persuasion remains optimal even with multiple possible actions.

Remark 3.3 (Multiple Actions). Suppose each receiver can select from multiple actions regarding the good, and the sender has a known preference over these actions. Public persuasion remains optimal in this scenario. Specifically, we can again adapt the first-best relaxation problem (2) and show that its optimal solution can be implemented via a public persuasion mechanism. We provide further details in Appendix A.4.

[8] For example, in the student promotion context, an employer can extend a regular offer, provide an offer with additional benefits, or decline to make an offer.

Finally, we note in Remark 3.4 that the assumption of a single good allocation is critical. If multiple goods are available instead, public persuasion may no longer be optimal.

Remark 3.4 (Multiple Goods). Public persuasion may no longer be optimal if the sender has multiple goods to allocate. To illustrate, consider an example with one sender and two receivers. The sender has two goods to allocate, whose characteristics are perfectly correlated (i.e., $w_1 = w_2$ with probability one), and each receiver can accept at most one good. In this case, competition between receivers vanishes. Consequently, if receivers cannot communicate, the sender's persuasion problem decomposes into independent problems for each receiver. Therefore, it is optimal for the sender to send private signals to each receiver using their respective optimal persuasion strategy, rather than employing public persuasion.

4 Simplified Optimization: One-Dimensional Linear Utilities

As indicated by Theorem 3.2, the sender only needs to consider public persuasion mechanisms to solve the optimal persuasion problem (1). Moreover, the optimal public persuasion mechanism can be derived from (2) and it achieves the first-best performance (i.e., the optimal value of (2)). However, when the good's characteristics w are infinite, problem (2) is an infinite-dimensional LP, which can be challenging to solve. In this section, we focus on the special case where the state variable w is one-dimensional, and all receivers' utility functions are linear in w. We present structural properties and derive optimality conditions for a persuasion mechanism based on the Lagrangian dual of (2), where we dualize the participation constraints. These conditions lead to optimal mechanisms in closed form when there are two receivers (Section 4.3) and complete characterization of the set of all optimal persuasion mechanisms in the general case (Section 4.4).

4.1 The Setup

In this section, we formally describe the one-dimensional linear utility setup. First, we assume that the good's characteristics can be summarized by a one-dimensional state variable w within a finite interval. Without loss of generality, let $w \in \Omega = [0, 1]$. Additionally, w follows a continuous distribution with a strictly increasing cumulative distribution function $G(\omega)$ and a density function $g(w) > 0$ for all $w \in (0, 1)$. We summarize these in Assumption 4.1.

Assumption 4.1. The good's characteristics w belong to the one-dimensional interval $\Omega = [0, 1]$ and follow a continuous distribution. Let $G(w)$ and $g(w)$ denote the cumulative distribution function and density function of w, respectively. The function $G(w)$ is strictly increasing, so its inverse, denoted by $G^{-1}(\cdot)$, exists.

Second, we assume that for each receiver $i \in [n]$, his utility function $u_i(w)$ for receiving a good with characteristics w is increasing and linear in w and intersects the x-axis at $\alpha_i > 0$. Under this assumption, each receiver i cares only about the mean value of the characteristics w among the goods potentially allocated to him. In particular, receiver i accepts the good only if this mean value exceeds his threshold α_i. We state this assumption in Assumption 4.2.

Assumption 4.2. For each receiver $i \in [n]$, his utility function $u_i(w)$ for a good with characteristic w is increasing and linear in w with a threshold value $\alpha_i > 0$; that is, $u_i(\alpha_i) = 0$.

Note that since receivers are ranked in decreasing preference by Assumption 2.1, there is no loss of generality to assume that the threshold values α_i also decrease in the receiver index i.[9] This is because, if receiver i is more preferred than j ($v_i > v_j$) but also easier to get into ($\alpha_i \leq \alpha_j$), receiver j will never be targeted and can be dropped from consideration. We state this assumption in Assumption 4.3.

Assumption 4.3. The receivers' threshold values α_i satisfy $0 < \alpha_n < \cdots < \alpha_2 < \alpha_1 < 1$.

Finally, given the linear-utility Assumption 4.2, the first-best problem (2) can be written as (3):

$$
\begin{aligned}
\bar{V} = \max_{q(i|w) \geq 0} \quad & \sum_{i=1}^{n} v_i \cdot \int_0^1 q(i|w)\, g(w)\, dw \\
\text{s.t.} \quad & \int_0^1 w \cdot q(i|w)\, g(w)\, dw \geq \alpha_i \int_0^1 q(i|w)\, g(w)\, dw, \ \forall\, i \in [n], \\
& \sum_{i \in [n]} q(i|w) \leq 1, \ \forall\, w \in [0,1].
\end{aligned} \tag{3}
$$

4.2 The Lagrangian Dual Problem

In this section, we introduce the Lagrangian dual problem of (3), where we dualize the receivers' participation constraints. We then interpret the Lagrangian from a geometric view and derive the optimality condition for a persuasion mechanism.

Specifically, denote by $\mu_i \geq 0$ the Lagrange multiplier for the participation constraint of receiver

[9] In the student promotion context, this means a more preferred employer is harder to get into.

$i \in [n]$. The Lagrangian relaxation, denoted by $V^{\text{LR}}(\boldsymbol{\mu})$ with $\boldsymbol{\mu} = (\mu_i)_{i\in[n]} \in \mathbb{R}^n_+$, is as follows:

$$
\begin{aligned}
V^{\text{LR}}(\boldsymbol{\mu}) &= \max_{\substack{q(i|w)\geq 0,\\ \sum_{i\in[n]} q(i|w)\leq 1}} \int_0^1 \sum_{i=1}^n \Big\{ v_i + \mu_i\big(w-\alpha_i\big) \Big\}\, q(i|w)\, g(w)\, dw \\
&= \int_0^1 \left(\max_{\substack{q(i|w)\geq 0,\\ \sum_{i\in[n]} q(i|w)\leq 1}} \sum_{i=1}^n \Big\{ v_i + \mu_i\big(w-\alpha_i\big) \Big\} \cdot q(i|w) \right) \cdot g(w)\, dw.
\end{aligned} \tag{4}
$$

After dualizing the participation constraints, the Lagrangian decouples over characteristics w. Specifically, define

$$\ell_i\big(w;\mu_i\big) \triangleq v_i + \mu_i\big(w-\alpha_i\big)$$

as the line associated with receiver $i \in [n]$. This line passes through the point (α_i, v_i) and has a nonnegative slope $\mu_i \geq 0$. In addition, let

$$h\big(w;\boldsymbol{\mu}\big) \triangleq \max_{i\in[n]} \ell_i\big(w;\mu_i\big) = \max_{i\in[n]} \Big\{ v_i + \mu_i\big(w-\alpha_i\big) \Big\}$$

denote the upper envelope of these n lines and $\bar{h}(w;\boldsymbol{\mu}) \triangleq \max\big\{h(w;\boldsymbol{\mu}), 0\big\}$ the upper envelope of these n lines and the x-axis. Both the functions $h(w;\boldsymbol{\mu})$ and $\bar{h}(w;\boldsymbol{\mu})$ are convex, increasing (since $\mu_i \geq 0$), and piecewise linear in w. Finally, let $\mathbf{Q}^{\text{LR}}(\boldsymbol{\mu})$ denote the set of optimal solutions $\{q(i|w)\}$ to $V^{\text{LR}}(\boldsymbol{\mu})$. According to (4), the set $\mathbf{Q}^{\text{LR}}(\boldsymbol{\mu})$ can be characterized as follows:

$$
\begin{aligned}
\mathbf{Q}^{\text{LR}}(\boldsymbol{\mu}) = \Bigg\{ q(i|w) : q(i|w) \geq 0 \;\text{ and } \sum_{i\in[n]} q(i|w) \leq 1,\, \forall\, w \in [0,1], \\
\sum_{i\in[n]} q(i|w) = 1,\, \forall\, w \in [0,1] \text{ with } h\big(w;\boldsymbol{\mu}\big) > 0, \\
q(i|w) > 0 \;\text{ only if }\; \ell_i\big(w;\mu_i\big) = \bar{h}\big(w;\boldsymbol{\mu}\big),\, \forall\, i \in [n] \Bigg\}.
\end{aligned} \tag{5}
$$

That is, an optimal solution to $V^{\text{LR}}(\boldsymbol{\mu})$ allocates a good of quality w to receiver i with positive probability only if receiver i's line $\ell_i(w;\mu_i)$ lies above the x-axis and is not dominated by other receivers' lines $\big\{\ell_i(w;\mu_i)\big\}_{j\neq i}$ at point w.

To provide an economic interpretation, note that the Lagrangian multiplier μ_i quantifies how tightly receiver i's participation constraint binds. The expression for receiver i's line, $\ell_i(w;\mu_i)$, indicates that the sender's payoff from allocating a good of quality w to receiver i comprises two components in the Lagrangian:

- The first component, v_i, represents the direct payoff from allocating the good to receiver i.
- The second component, $\mu_i\,(w - \alpha_i)$, represents the indirect payoff from the allocation's impact on receiver i's participation constraint. Specifically, μ_i quantifies the magnitude of this indirect effect. If $w > \alpha_i$, allocating the good to receiver i eases receiver i's participation constraint, enabling the sender to potentially allocate more under-qualified goods to receiver i, which might otherwise be unallocated. Conversely, if $w < \alpha_i$, this allocation tightens receiver i's participation constraint, thus limiting the sender's ability to allocate under-qualified goods to receiver i in the original problem.

Combining both the direct and indirect payoffs, the sender allocates the good with quality w to the receiver with the highest positive payoff—that is, the highest value of $\ell_i(w;\mu_i)$ among all $i \in [n]$, provided this value is positive. Otherwise, the sender does not allocate this good to any receiver, securing a payoff of zero.

Finally, from (4) we have:

$$V^{\mathrm{LR}}(\boldsymbol{\mu}) = \int_0^1 \bar{h}\big(w;\boldsymbol{\mu}\big)\, g(w)\, dw.$$

Since every feasible policy to (3) is feasible to (4) and attains an objective value that is no smaller, $\bar{V} \le V^{\mathrm{LR}}(\boldsymbol{\mu})$ for any $\boldsymbol{\mu} \in \mathbb{R}^n_+$. We formally state this weak duality property in Lemma 4.1.

Lemma 4.1 (Weak Duality). *We have $\bar{V} \le V^{\mathrm{LR}}(\boldsymbol{\mu})$ for any dual variable $\boldsymbol{\mu} \in \mathbb{R}^n_+$.*

4.2.1 The Optimal Lagrangian Dual

Since the Lagrangian $V^{\mathrm{LR}}(\boldsymbol{\mu})$ is a convex function of $\boldsymbol{\mu}$ by (4), we can solve a convex optimization problem

$$V^{\mathrm{LR}} \triangleq \min_{\boldsymbol{\mu}\in\mathbb{R}^n_+} V^{\mathrm{LR}}(\boldsymbol{\mu}) \ge \bar{V} \tag{6}$$

to obtain the tightest Lagrangian relaxation bound V^{LR}. Let $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]} \in \operatorname{argmin}_{\boldsymbol{\mu}\in\mathbb{R}^n_+} V^{\mathrm{LR}}(\boldsymbol{\mu})$ denote an optimal Lagrangian dual variable, which can be efficiently computed by Remark 4.1.

Remark 4.1 (Computing $\boldsymbol{\mu}^*$). According to Danskin's theorem (Theorem 9.27 in Shapiro et al. 2021) and the fact that a convex combination of any two optimal solutions to (4) is also optimal to (4), the sub-differential (i.e., set of sub-gradients) of $V^{\mathrm{LR}}(\boldsymbol{\mu})$ at any $\boldsymbol{\mu} \in \mathbb{R}^n_+$, denoted by $\partial V^{\mathrm{LR}}(\boldsymbol{\mu})$, can be expressed as

$$\partial V^{\mathrm{LR}}(\boldsymbol{\mu}) = \left\{ (g_i)_{i\in[n]} \text{ with } g_i \triangleq \int_0^1 (w-\alpha_i)\, q(i|w)\, g(w)\, dw : \{q(i|w)\} \in \mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu}) \right\}.$$

Since both $V^{\mathrm{LR}}(\boldsymbol{\mu})$ and its sub-gradients can be efficiently computed, we can apply sub-gradient-based methods (e.g., the sub-gradient method or the cutting-plane method) to solve the convex program (6) and obtain an optimal Lagrangian dual variable $\boldsymbol{\mu}^*$ efficiently.

Furthermore, Lemma 4.2 demonstrates that strong duality holds, which follows standard strong duality for convex optimization in a vector space.

Lemma 4.2 (Strong Duality). *Problem (3) and its Lagrangian relaxation (4) satisfy the following:*

1. *Strong duality holds, and there exists an optimal dual variable $\boldsymbol{\mu}^* \in \mathbb{R}_+^n$ such that $\bar{V} = V^{\mathrm{LR}} = V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$.*
2. *$\boldsymbol{\mu} \in \mathbb{R}_+^n$ is an optimal dual variable and $\{q(i|w)\}$ is an optimal solution to (3) if and only if (1) $\{q(i|w)\} \in \mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu})$, and (2) $\{q(i|w)\}$ satisfies all participation constraints in (3), and the participation constraint for receiver i binds for all $i \in [n]$ with $\mu_i > 0$.*

We prove Lemma 4.2 in Appendix B.2. Bullet 2 of Lemma 4.2 provides optimality conditions for a persuasion mechanism. Specifically, we can find an optimal solution $\{q(i|w)\}$ to $V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ that satisfies all participation constraints in (3), with constraints binding for receivers with positive μ_i^*, then $\{q(i|w)\}$ is also optimal to (3) and yields an optimal (public) persuasion mechanism. However, how to identify such a desirable $\{q(i|w)\} \in \mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ remains generally unclear. Nevertheless, in Section 4.3, we apply Lemma 4.2 to derive the set of optimal persuasion mechanisms in closed form for the two-receiver case. For the general case (Section 4.4), we completely characterize the upper envelope function $h(w;\boldsymbol{\mu}^*)$ (Proposition 4.6), rendering the set of optimal persuasion mechanisms manifest by Lemma 4.2.

4.3 Two-Receiver Case

In this section, we consider two receivers $i \in \{1,2\}$, with offer values $v_1 > v_2 > 0$ and hiring thresholds $\alpha_1 > \alpha_2 > 0$. We derive the optimal public persuasion mechanisms based on Bullet 2 of Lemma 4.2.

Define $w_0 \triangleq \mathbb{E}_{w\sim G(w)}[w]$ as the prior mean of the good's characteristic w. We assume receiver 1 is selective, i.e., $\alpha_1 > w_0$; otherwise, the optimal mechanism is trivial since the sender can allocate the good to receiver 1 without revealing any information. Throughout this section, we also assume receiver 2 is selective, i.e., $\alpha_2 > w_0$, as formally stated in Assumption 4.4. The scenario in which receiver 2 is not selective yields similar optimal persuasion mechanisms but requires a separate discussion, which we provide in Appendix C.

Assumption 4.4. Let $w_0 \triangleq \mathbb{E}_{w\sim G(w)}[w]$ denote the prior mean of the good's characteristics w. Both receivers 1 and 2 are selective; that is, their threshold values satisfy $0 < w_0 < \alpha_2 < \alpha_1 < 1$.

Finally, for a persuasion mechanism M, we let $q_i(M)$ denote the probability that receiver i is allocated the good under mechanism M.

4.3.1 Preparation: Mechanisms Targeting a Single Receiver

We first consider two simple mechanisms, in which the sender prioritizes either receiver 1 or 2, and their interpolation, as preparation for characterizing an optimal mechanism in Section 4.3.2.

Mechanism M_1: Prioritizing Receiver 1 First, consider mechanism M_1, in which the sender prioritizes receiver 1 and recommends goods to receiver 2 only if suitable goods remain after targeting receiver 1. Specifically, define $\bar{z}_1 > 0$ such that $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$.[10] The sender sends the signal $s = 1$ if $w \geq \bar{z}_1$, resulting in allocation probability $q_1(M_1) = \mathbb{P}[w \geq \bar{z}_1]$. Then, two scenarios arise depending on the value of $\bar{z}_1$ relative to α_2:

- If $\bar{z}_1 > \alpha_2$: The sender can still persuade receiver 2 to extend an offer to some goods in the remaining pool after targeting receiver 1. Specifically, find a real value z_1 with $0 < z_1 < \alpha_2 < \bar{z}_1$ such that $\mathbb{E}[w|z_1 \leq w < \bar{z}_1] = \alpha_2$.[11] The sender sends the signal $s = 2$ if $z_1 \leq w < \bar{z}_1$, and the signal $s = \varnothing$ if $w < z_1$. Therefore, $q_2(M_1) = \mathbb{P}[z_1 \leq w < \bar{z}_1]$.

- If $\bar{z}_1 \leq \alpha_2$: The sender cannot persuade receiver 2 to accept any remaining goods after targeting receiver 1. In this case, set $z_1 = \bar{z}_1$. The sender sends the signal $s = \varnothing$ when $w < z_1$, leading to $q_2(M_1) = 0$.

In both scenarios, the sender receives an offer if and only if $w \geq z_1$, which occurs with probability $\mathbb{P}[w \geq z_1]$.

[10] We have $\bar{z}_1 > 0$ because $\alpha_1 > w_0$ by Assumption 4.4.
[11] We have $z_1 > 0$ because $\alpha_2 > w_0$ by Assumption 4.4.

Mechanism M_2: Prioritizing Receiver 2 Second, consider mechanism M_2, in which the sender completely targets receiver 2. Specifically, define $z_2 > 0$ such that $\mathbb{E}[w|w \geq z_2] = \alpha_2$. The sender sends the signal $s = 2$ whenever $w \geq z_2$. Upon receiving this signal, only receiver 2 will extend an offer. Since $z_2 < \bar{z}_1 < \alpha_1$, the sender can no longer persuade receiver 1 to extend an offer to goods in the remaining pool $[0, z_2)$ after targeting receiver 2. Therefore, the sender can only send the signal $s = \varnothing$ when $w < z_2$. As a result, $q_1(M_2) = 0$ and $q_2(M_2) = \mathbb{P}[w \geq z_2]$. The sender receives an offer if and only if $w \geq z_2$, which occurs with probability $\mathbb{P}[w \geq z_2]$.

Interpolation Between Mechanisms M_1 and M_2 In Proposition B.2 in the Appendix, we demonstrate that *every* optimal solution to (3) exhibits a cutoff structure: there exists a threshold value $z \in [0, 1]$ such that the good is allocated if and only if its characteristic w exceeds z. Clearly, any persuasion mechanism M with a cutoff value $z < z_1$ is suboptimal, because mechanism M_1 yields a higher payoff for the sender. Conversely, the cutoff value z must satisfy $z \geq z_2$; otherwise, the participation constraint of at least one receiver would be violated. We next demonstrate in Proposition 4.3 that for any $z \in [z_2, z_1]$, there exists a persuasion mechanism with cutoff point z.

Proposition 4.3. *For any cutoff $z \in [z_2, z_1]$, there exists a public persuasion mechanism M such that the good receives an offer if and only if $w \geq z$, and the participation constraints of both receivers bind. Moreover, under any such mechanism M, it holds that $q_1(M) = \mathbb{P}[w \geq z] \cdot \frac{\mathbb{E}[w|w \geq z] - \alpha_2}{\alpha_1 - \alpha_2} \geq 0$ and $q_2(M) = \mathbb{P}[w \geq z] \cdot \frac{\alpha_1 - \mathbb{E}[w|w \geq z]}{\alpha_1 - \alpha_2} \geq 0$.*

We prove Proposition 4.3 in Appendix B.3. Intuitively, the probability of allocating the good to receiver 1 is highest when the sender primarily targets receiver 1 (using mechanism M_1). However, this also lowers the overall probability of receiving an offer because receiver 1 has higher standards. Conversely, the probability of receiving an offer is maximized when the sender exclusively targets the less selective receiver 2 (using mechanism M_2), which is $\mathbb{P}[w \geq z_2]$. Proposition 4.3 indicates that any acceptance probability between these two extremes can be sustained by a mechanism that carefully balances the two receivers. In Section 4.3.2, we show that any such mechanism can be optimal, depending on receiver 1's desirability (v_1) and hiring bar (α_1) relative to those of receiver 2. We conclude this section with a remark that interprets the probabilities $q_1(M)$ and $q_2(M)$ in Proposition 4.3.

Remark 4.2 (Interpreting Probabilities in Proposition 4.3)**.** To interpret the probabilities $q_1(M)$ and $q_2(M)$, consider any public persuasion mechanism M characterized by a cutoff structure with threshold z. When the participation constraints of both receivers bind, the probabilities $q_1 \triangleq q_1(M)$

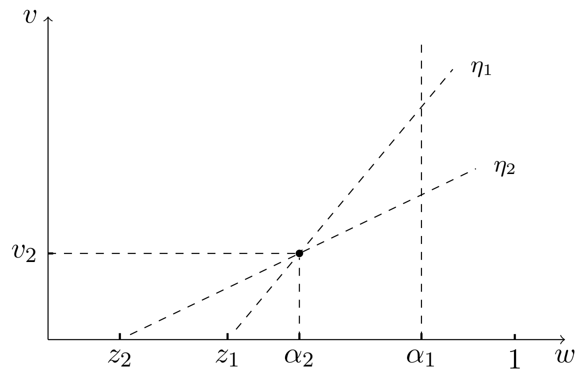

Figure 1: Visualization of the partition in Theorem 4.4.

and $q_2 \triangleq q_2(M)$ must satisfy the following two linear equations:

$$\begin{aligned} q_1 + q_2 &= \mathbb{P}[w \geq z], \\ \alpha_1 q_1 + \alpha_2 q_2 &= (q_1 + q_2) \cdot \mathbb{E}[w|w \geq z]. \end{aligned} \tag{7}$$

The first equation follows from the condition that the good receives an offer (from either receiver 1 or 2) if and only if $w \geq z$. The second equation arises from the binding participation constraints (i.e., $\mathbb{E}[w|s = i] = \alpha_i$) and the law of total expectation. These two equations uniquely determine the values of q_1 and q_2, as given in Proposition 4.3. Conversely, consider a mechanism M that sends the signal $s = \varnothing$ if and only if $w < z$. Then, if $\mathbb{P}[s = 1] = q_1$ and $\mathbb{E}[w \mid s = 1] = \alpha_1$, it follows that $\mathbb{P}[s = 2] = q_2$ and $\mathbb{E}[w \mid s = 2] = \alpha_2$, and vice versa, when q_1 and q_2 are probabilities satisfying (7).

4.3.2 Optimal Mechanisms with Two Receivers

In this section, we characterize optimal persuasion mechanisms with two receivers. Intuitively, there is a trade-off: An offer from receiver 1 yields a higher payoff, but targeting receiver 1 more aggressively reduces the overall probability of securing an offer.

Notably, by the discussion preceding Proposition 4.3, the cutoff value z—such that a good is allocated if and only if its quality $w \geq z$—satisfies $z \in [z_2, z_1]$ for any "reasonable" mechanism. This brings two lines, one of which (denoted by η_1) passes through the points $(z_1, 0)$ and (α_2, v_2), and the other (denoted by η_2) passes through the points $(z_2, 0)$ and (α_2, v_2), as illustrated in Figure 1. These two lines partition the value of $v_1 \in [v_2, \infty)$ into three regions, which determine the form of optimal persuasion mechanisms.

Theorem 4.4 characterizes the optimal persuasion mechanisms for the two-receiver case. Specifically, if the value of v_1 is sufficiently large (in particular, above line η_1), prioritizing receiver 1 is optimal. Conversely, if v_1 is sufficiently small (i.e., below line η_2), completely targeting receiver 2 is optimal. Finally, if v_1 lies between these two lines, the optimal mechanism involves a non-trivial balance between the two receivers and exhibits the structure described Proposition 4.3.

Theorem 4.4. *Under Assumptions 4.1 – 4.4, the optimal public persuasion mechanism for two receivers is characterized as follows.*

1. *If $v_1 \geq v_2 \cdot \frac{\alpha_1 - z_1}{\alpha_2 - z_1}$ (i.e., the point (α_1, v_1) lies above line η_1), mechanism M_1, which prioritizes receiver 1, is the unique optimal mechanism.*
2. *If $v_1 \leq v_2 \cdot \frac{\alpha_1 - z_2}{\alpha_2 - z_2}$ (i.e., the point (α_1, v_1) lies below line η_2), mechanism M_2, which exclusively targets receiver 2, is the unique optimal mechanism.*
3. *Otherwise, any mechanism M satisfying Proposition 4.3 with the cutoff value*

$$z^* \triangleq \alpha_2 - v_2 \cdot \frac{\alpha_1 - \alpha_2}{v_1 - v_2} \in [z_2, z_1],$$

representing the x-intercept of the line passing through points (α_2, v_2) and (α_1, v_1), is optimal. In other words, mechanism M satisfies:

   (a) *It sends signal $s = \varnothing$ with probability one if $w < z^*$, and zero otherwise.*
   (b) *Participation constraints bind; that is, $\mathbb{E}[w|s = i] = \alpha_i$ for each $i \in \{1, 2\}$.*
   (c) *The allocation probabilities are: $q_1(M) = \mathbb{P}[w \geq z^*] \cdot \frac{\mathbb{E}[w|w \geq z^*] - \alpha_2}{\alpha_1 - \alpha_2}$ and $q_2(M) = \mathbb{P}[w \geq z^*] \cdot \frac{\alpha_1 - \mathbb{E}[w|w \geq z^*]}{\alpha_1 - \alpha_2}$, as established in Proposition 4.3.*

Moreover, this completely characterizes the set of all optimal public persuasion mechanisms.

We prove Theorem 4.4 in Appendix B.4. In the proof, we identify a set of dual variables $\boldsymbol{\mu} \in \mathbb{R}^n_+$, which, together with the proposed mechanism, satisfy Bullet 2 of Lemma 4.2. This indicates that the mechanism is optimal to (3), and $\boldsymbol{\mu}$ is an optimal dual variable. Finally, in Appendix C, we characterize the optimal mechanisms when Assumption 4.4 does not hold, which have similar structures.

We note that the trade-off between the two receivers is nontrivial in Case 3 of Theorem 4.4. In this scenario, the participation constraints of both receivers bind, and the optimal Lagrangian dual variable is $\mu_1^* = \mu_2^* = \frac{v_1 - v_2}{\alpha_1 - \alpha_2} > 0$. This value corresponds to the slope of the line passing through the points (α_2, v_2) and (α_1, v_1). Consequently, the two receivers' lines, $\ell_1(w; \mu_1^*)$ and $\ell_2(w; \mu_2^*)$, completely overlap and coincide with this line (as visualized in Figure 3(b)). As a result, according to (5), any allocation $\{q(i|w)\}$ satisfying $q(1|w) + q(2|w) = 1$ for $w \geq z^*$ and $q(1|w) = q(2|w) = 0$ for $w < z^*$ is optimal to the Lagrangian $V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$. Provided we allocate the probability mass of one appropriately between $q(1|w)$ and $q(2|w)$ for all $w \geq z^*$, thereby ensuring both receivers' participation constraints bind, it follows that each receiver i is allocated the good with probability q_i^* by Proposition 4.3, and the mechanism $\{q(i|w)\}$ is optimal to (3) by Theorem 4.4.

Although the aggregate allocation probabilities $\{q_i^*\}$ are unique byProposition 4.3, there are various ways to construct a set of probabilities $\{q(i|w)\}$ that satisfy Bullet 3 of Theorem 4.4 and is thus optimal to (3). Below, we present two simple approaches to construct an optimal mechanism and illustrate them using Example 4.1.

- (*Randomized Mechanism with Monotone Structure*) Set $q(1|w) = q_1^* \,/\, \mathbb{P}[w \geq \bar{z}_1] \leq 1$ for $w \geq \bar{z}_1$ and $q(1|w) = 0$ otherwise, recalling that $\bar{z}_1 > 0$ satisfies $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$. Additionally, set $q(2|w) = 1 - q(1|w)$ for $w \geq z^*$ and $q(2|w) = 0$ otherwise. This defines a randomized persuasion mechanism, which satisfies Bullet 3 of Theorem 4.4 and is thus optimal to (3). By construction, the sender's expected payoff, $v(w) \triangleq \sum_i v_i \, q(i|w)$, increases with w, which can be desirable in practice.[12]

- (*Deterministic Mechanism with Double Interval Structure*) Select an interval $\big[\underline{b}, \bar{b}\big] \subseteq [\bar{z}_1, 1]$ such that $\mathbb{P}\big[\underline{b} \leq w \leq \bar{b}\big] = q_1^*$ and $\mathbb{E}\big[w \mid \underline{b} \leq w \leq \bar{b}\big] = \alpha_1$.[13] Set $q(1|w) = 1$ for $w \in \big[\underline{b}, \bar{b}\big]$, $q(2|w) = 1$ for $w \in [z^*, \underline{b}) \cup \big(\bar{b}, 1\big]$, and $q(\varnothing|w) = 1$ for $w < z^*$. This defines a deterministic persuasion mechanism as described in Candogan (2022). This mechanism satisfies Bullet 3 of Theorem 4.4 and is therefore optimal to (3). Moreover, it exhibits a double-interval structure, with each signal associated with at most two intervals. Notably, the sender's expected payoff under this mechanism is not monotone in w.

Example 4.1. Suppose $w \sim \mathrm{Unif}[0, 1]$ follows a uniform distribution with support $[0, 1]$, the sender's payoffs from the two receivers are $v_1 = 2$ and $v_2 = 1$, and the receivers' threshold values are $\alpha_1 = 0.9$ and $\alpha_2 = 0.7$. Given these parameters, we have $\bar{z}_1 = 0.8$, $z_1 = 0.6$, $z^* = 0.5$, and $z_2 = 0.4$. The optimal dual variables are $\mu_1^* = \mu_2^* = 5$. Figure 2(a) illustrates the two receivers' lines $\ell_1(w; \mu_1^*)$ and $\ell_2(w; \mu_2^*)$, which completely overlap. Additionally, we have $q_1^* = 1/8$ and $q_2^* = 3/8$. There are various ways to construct an optimal persuasion mechanism satisfying Bullet 3 of Theorem 4.4.

[12] For example, in the student promotion context, an increasing expected payoff prevents students from strategically degrading their quality w for better positions.

[13] Such an interval exists since $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$ and $\mathbb{P}[w \geq \bar{z}_1] \geq q_1^*$ (see lemma B.4 in the Appendix).

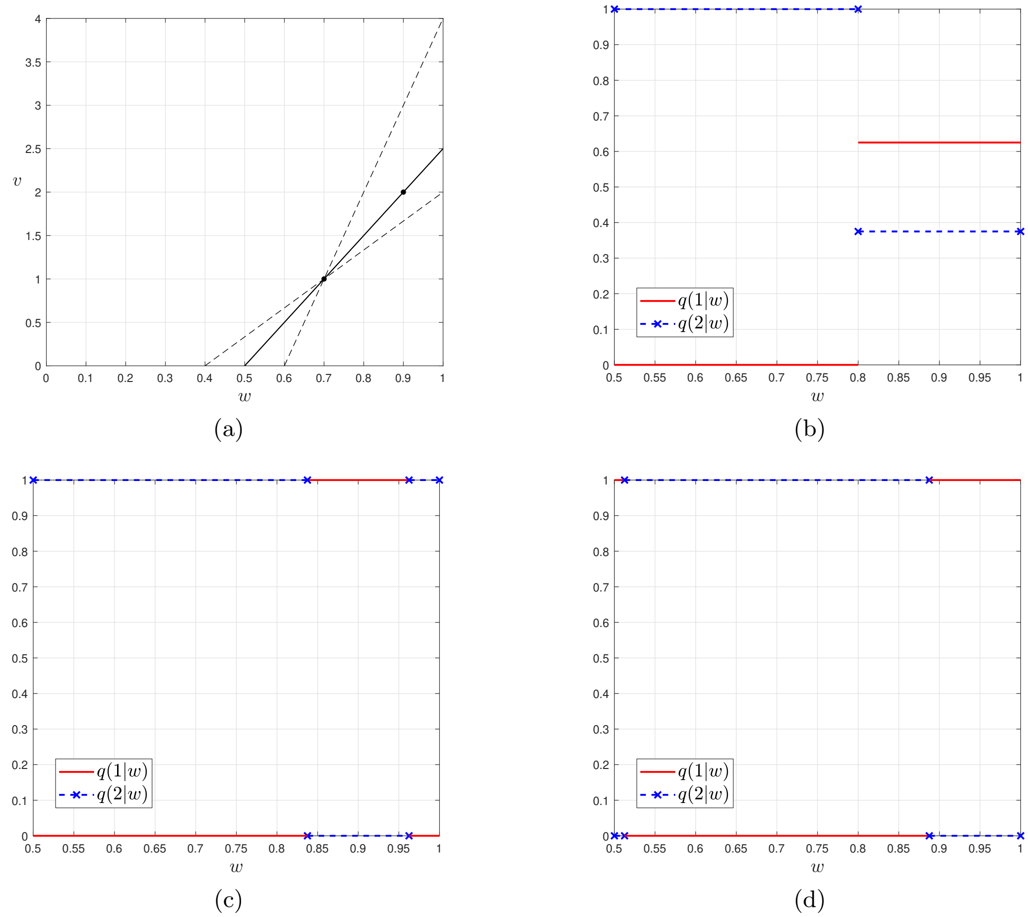

Figure 2: (a) Illustration of the two receivers' lines, $\ell_1(w;\mu_1^*)$ and $\ell_2(w;\mu_2^*)$ (solid), which overlap completely and pass through points $(0.7, 1)$ and $(0.9, 2)$, along with lines η_1 and η_2 from Theorem 4.4 (dashed). (b) A randomized optimal persuasion mechanism, where $q(1|w) = 5/8$ for $w \in [0.8, 1]$, $q(2|w) = 3/8$ for $w \in [0.8, 1]$, and $q(2|w) = 1$ for $w \in [0.5, 0.8]$. (c) A deterministic optimal persuasion mechanism, where $q(1|w) = 1$ for $w \in [0.8375, 0.9625]$ (centered around 0.9 and of length 1/8) and $q(2|w) = 1$ for $w \in [0.5, 0.8375] \cup [0.9625, 1]$. (d) A deterministic optimal persuasion mechanism, where $q(2|w) = 1$ for $w \in [0.5125, 0.8875]$ (centered around 0.7 and of length 3/8) and $q(1|w) = 1$ for $w \in [0.5, 0.5125] \cup [0.8875, 1]$.

The previously described randomized persuasion mechanism is illustrated in Figure 2(b). The previously described deterministic persuasion mechanism is illustrated in Figure 2(c). Moreover, another deterministic persuasion mechanism with a double-interval structure can be constructed for this instance, where signal $s = 1$ is associated with two intervals, as illustrated in Figure 2(d).

We conclude this section with a remark showing that when receivers cannot communicate but are aware of each other, the vanilla private persuasion mechanism is suboptimal.

Remark 4.3 (Failure of Vanilla Private Persuasion Absent Communication). When there is no communication channel between receivers, it is tempting to treat them in isolation and send separate signals to each receiver using their respective optimal persuasion strategies. We call this vanilla private persuasion mechanism and show that it is suboptimal. First, we note that even when receivers cannot communicate, a public persuasion mechanism remains optimal by Theorem 3.2. Conversely, a vanilla private persuasion mechanism sends signal $s = 1$ to receiver 1 when $w \geq \bar{z}_1$ and signal $s = 2$ to receiver 2 when $w \geq z_2$. Despite the lack of communication, receiver 2, aware of the presence of a more preferred receiver 1, will never extend an offer upon receiving signal $s = 2$. This is because the offer from receiver 1 adversely selects the goods recommended to receiver 2, resulting in negative expected utility for receiver 2. Notably, only goods with quality $w \in [z_2, \bar{z}_1)$ would eventually be allocated to receiver 2, whose expected quality satisfies $\mathbb{E}[w|z_2 \leq w < \bar{z}_1] < \mathbb{E}[w|w \geq z_2] = \alpha_2$, falling below receiver 2's acceptance threshold α_2. Given receiver 2's equilibrium strategy, only goods with quality $w \in [\bar{z}_1, 1]$ end up being allocated, rendering the vanilla private persuasion suboptimal according to Theorem 4.4.

4.4 General Case

In this section, we examine the general case and characterize the upper envelope function $h(w; \boldsymbol{\mu}^*)$ defined in Section 4.2. Then, the set of optimal persuasion mechanisms becomes clear by Bullet 2 of Lemma 4.2. To accomplish this, we first introduce a convex optimization problem (8) in Section 4.4.1, which is equivalent to the first-best problem (3), and establish their connection through duality.

4.4.1 The Convex Optimization Formulation

In this section, we introduce a convex optimization problem (8) with n decision variables and constraints and establish its equivalence to (3). Problem (8) is analogous to problem (OPT) in Candogan (2022), albeit with n fewer decision variables and constraints.[14]

$$
\begin{aligned}
V^{\mathrm{CR}} = \max_{q_i \geq 0} \quad & \sum_{i=1}^{n} v_i q_i \\
\text{s.t.} \quad & \sum_{i \leq k} \alpha_i q_i \leq \sum_{i \leq k} q_i \cdot \mathbb{E}\left[w \,\middle|\, G(w) \geq 1 - \sum_{i \leq k} q_i \right] = \int_{1-\sum_{i\leq k} q_i}^{1} G^{-1}(x)\, dx, \; \forall\, k \in [n], \\
& \sum_{i \in [n]} q_i \leq 1.
\end{aligned} \tag{8}
$$

In (8), the decision variables q_i represent the ex-ante probabilities that the good is allocated to

[14]Note that in problem (OPT) of Candogan (2022), given the optimal values $\{p_k^*\}$, it is optimal to set $z_k^* = b_k p_k^*$ there.

receiver $i \in [n]$; specifically, q_i corresponds to $\int_0^1 q(i|w)g(w)dw$ in (3). The first constraint captures the receivers' participation constraint. Only a limited portion of qualified goods meet the receivers' acceptance standards. This constraint requires that goods within the top $\sum_{i\le k} q_i$ quantile are sufficient to meet the acceptance thresholds (α_i) of the top k receivers, given that each receiver $i \in [k]$ recruits a proportion q_i of goods. This condition is necessary to maintain the participation of the first k receivers. The equality in this constraint relies on the fact that for any random variable w with cumulative distribution function $G(\cdot)$, the random variable $G(w)$ follows a uniform distribution on $[0, 1]$. Finally, we note that (8) is a convex optimization problem. To see this, define $h(x) \triangleq \int_{1-x}^1 G^{-1}(s)\,ds$. This function is concave because its derivative, $h'(x) = G^{-1}(1-x)$, is decreasing in x. Consequently, the right-hand side of the first constraint is concave in $\{q_i\}$, because it is the composition of $h(\cdot)$ with an affine mapping.

Given a feasible solution $\{q(i|w)\}$ to (3), the set $\{q_i\}$ with $q_i = \int_0^1 q(i|w)g(w)dw$ is feasible to (8) and attains the same objective value. Therefore, (8) is a relaxation of (3). Conversely, analogous to the two-receiver case (Section 4.3), the optimal aggregate allocation probabilities $\{q_i^*\}$, along with the binding participation constraints for receivers with a positive dual variable $\mu_i^* > 0$, characterize an optimal mechanism.[15] Specifically, given an optimal solution $\{q_i^*\}$ to (8), we can construct a persuasion mechanism that obtains the optimal value V^{CR}. Therefore, the relaxation (8) is tight. We state the above in Proposition 4.5 and provide its proof in Appendix B.5.

Proposition 4.5 (Primal Equivalence). *The optimal values of (3) and (8) are equal; that is, $\bar{V} = V^{\mathrm{CR}}$. Furthermore, let $\{q^*(i|w)\}$ be an optimal solution to (3). Then, $\{q_i^*\}$, where $q_i^* = \int_0^1 q^*(i|w)g(w)dw$, is an optimal solution to (8). Conversely, if $\{q_i^*\}$ is an optimal solution to (8), then there exists an optimal solution $\{q^*(i|w)\}$ to (3) such that $q_i^* = \int_0^1 q^*(i|w)g(w)dw$.*

4.4.2 Characterization of the Upper Envelope Function $h(w; \boldsymbol{\mu}^*)$

In this section, we characterize the upper envelope function $h(w; \boldsymbol{\mu}^*)$ introduced in Section 4.2. Let $\{q_i^*\}$ be an optimal solution to (8). We assume $q_i^* > 0$ for all $i \in [n]$; this does not lose generality because receivers with $q_i^* = 0$ can be disregarded from consideration. We formalize this assumption in Assumption 4.5 and maintain it throughout this section.

Assumption 4.5. There exists an optimal solution $\{q_i^*\}$ to (8) such that $q_i^* > 0$ for all $i \in [n]$.

We first introduce several parameters needed to characterize the envelope function $h(w; \boldsymbol{\mu}^*)$. Let

[15] In Proposition B.3 in the Appendix, we show that among receivers allocated the good with positive probability, participation constraints bind for all except the least preferred receiver.

$\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]} \in \mathbb{R}_+^n$ denote an optimal dual variable associated with the participation constraints in (8), and let γ^* denote an optimal dual variable for the constraint $\sum_{i\in[n]} q_i \leq 1$ in (8). Define the set:

$$T \triangleq \left\{k \in [n] : \lambda_k^* > 0\right\}$$

as the indices corresponding to positive entries in the optimal dual variable $\boldsymbol{\lambda}^*$. We assume $\lambda_n^* > 0$. The case where $\lambda_n^* = 0$ is discussed in Remark 4.4. In that scenario, we have $\lambda_{n-1}^* > 0$ (and hence, $n-1 \in T$), and the characterization of $h(w; \boldsymbol{\mu}^*)$ remains essentially unchanged.[16]

Suppose $T = \{t_1 < t_2 < \cdots < t_m = n\}$ consists of m receivers. These receivers partition all n receivers into m groups $\{T_i\}_{i\in[m]}$, with $T_1 = [t_1]$ and $T_i = [t_{i-1}+1 : t_i]$ for all $i \in [2:m]$. Define $z_i \triangleq G^{-1}\left(1 - \sum_{j=1}^{t_i} q_j^*\right)$ for each $i \in [m]$, and set $z_0 = 1$. Each receiver group T_i is associated with a state interval $I_i \triangleq [z_i, z_{i-1}]$. In Proposition 4.6, we establish the relationship between the optimal dual variables for problems (3) and (8), and characterize the upper envelope function $h(w; \boldsymbol{\mu}^*)$.

Proposition 4.6 (Characterization of Envelope Function $h(w; \boldsymbol{\mu}^*)$)**.** *Suppose Assumption 4.5 holds. Let $\boldsymbol{\lambda}^* \in \mathbb{R}_+^n$ be an optimal dual variable associated with the participation constraints in (8). Define $\boldsymbol{\mu} = (\mu_i)_{i\in[n]}$ by setting $\mu_i = \sum_{k\geq i} \lambda_k^*$ for each $i \in [n]$. Then, $\boldsymbol{\mu}$ is an optimal dual variable for the participation constraints in (3); that is, $V^{\mathrm{LR}}(\boldsymbol{\mu}) = \bar{V}$. Moreover, suppose $\lambda_n^* > 0$. The receivers' lines $\ell_j(w; \mu_j)$ and the upper envelope function $h(w; \boldsymbol{\mu}) \triangleq \max_{j\in[n]} \ell_j(w; \mu_j)$ are characterized as follows:*

1. *For each group T_i and receiver $j \in T_i$, the threshold satisfies $\alpha_j \in (z_i, z_{i-1})$. Moreover, within each group T_i, the lines $\ell_j(w; \mu_j)$ coincide for all $j \in T_i$ and pass through the points (α_j, v_j) for all $j \in T_i$.*
2. *For any two receivers $j \in T_i$ and $k \in T_{i+1}$ from adjacent groups (where $i \leq m-1$), their lines $\ell_j(w; \mu_j)$ and $\ell_k(w; \mu_k)$ intersect at $w = z_i$. Additionally, for every receiver $j \in T_m$ (the last group), line $\ell_j(w; \mu_j)$ intersects the x-axis at $w = z_m > 0$ if $\sum_{j\in[n]} q_j^* < 1$, and intersects the y-axis at point $\gamma^* \in [0, v_n]$ if $\sum_{j\in[n]} q_j^* = 1$ (which implies $z_m = 0$).*
3. *The envelope function $h(w; \boldsymbol{\mu})$ satisfies $h(w; \boldsymbol{\mu}) = \ell_j(w; \mu_j)$ for all group T_i, receiver $j \in T_i$, and state $w \in [z_i, z_{i-1}]$. Moreover, $h(w; \boldsymbol{\mu}) = \ell_j(w; \mu_j)$ for all $j \in T_m$ and $w \in [0, z_m]$.*

We prove Proposition 4.6 in Appendix B.6 by comparing the optimality conditions of (3) and (8). Proposition 4.6 demonstrates that the optimal dual variables for the participation constraints

[16] When $\lambda_n^* = 0$, the participation constraint for receiver n does not bind. The persuasion problem simplifies to the one involving only the first $n-1$ receivers, with any unallocated goods assigned to receiver n.

in (8) correspond to differences between the optimal dual variables for the participation constraints in (3). Consequently, one set of dual solutions can be directly derived from the other. Additionally, the envelope function $h(w; \boldsymbol{\mu}^*)$, constructed in Proposition 4.6, is linear on each interval $[z_i, z_{i-1}]$ and is positive if and only if $w \geq z_m$.

We conclude this section with two remarks: the first addresses the case where $\lambda_n^* = 0$, and the second discusses the connection to Dworczak and Martini (2019).

Remark 4.4 (The case of $\lambda_n^* = 0$)**.** Suppose Assumption 4.5 holds and $\lambda_n^* = 0$. Then, the following properties hold: (i) $\lambda_{n-1}^* > 0$, implying $n - 1 \in T$, (ii) $\sum_{j \in [n]} q_j^* = 1$, and (iii) $\gamma^* = v_n$. Suppose the set $T = \{t_1 < t_2 < \cdots < t_m = n-1\}$ consists of m receivers. These receivers partition all n receivers into $m+1$ groups $\{T_i\}$, where $T_1 = [t_1]$, $T_i = [t_{i-1}+1 : t_i]$ for all $i \in [2:m]$, and $T_{m+1} = \{n\}$. Define endpoints $z_i \triangleq G^{-1}\left(1 - \sum_{j=1}^{t_i} q_j^*\right)$ for each $i \in [m]$, and set $z_0 = 1$ and $z_{m+1} = 0$. The dual variable $\boldsymbol{\mu}$ constructed in Proposition 4.6 remains optimal for (3). Moreover, the characterization of $h(w; \boldsymbol{\mu}^*)$ is identical to that in Proposition 4.6, with an additional feature that $h(w; \boldsymbol{\mu}^*) = \gamma^* = v_n > 0$ is constant (horizontal) on the interval $[0, z_m]$. We provide further details in Appendix B.6.5.

Remark 4.5 (Connection to Dworczak and Martini 2019)**.** We observe that our envelop function $\bar{h}(w; \boldsymbol{\mu}^*) \triangleq \max\{h(w; \boldsymbol{\mu}^*), 0\}$ precisely corresponds to the equilibrium price function $p(x)$ in the optimality condition of Dworczak and Martini (2019) (see Theorem 1 therein). Consequently, we fully characterize their equilibrium price—which is generally challenging to specify explicitly—for the case where the sender's utility is increasing and piecewise constant in the posterior mean of the underlying state. Notably, we derive the equilibrium price $p(x)$ using an alternative dual approach: we dualize the receivers' participation constraints instead of the mean-preserving contraction constraint used by Dworczak and Martini (2019). This methodological distinction may be of independent interest.

4.4.3 Characterization of Optimal Persuasion Mechanisms

In this section, we characterize the set of optimal persuasion mechanisms. Once the upper envelope function $h(w; \boldsymbol{\mu}^*)$ associated with an optimal dual variable $\boldsymbol{\mu}^*$ has been characterized (Proposition 4.6 and Remark 4.4), the set of optimal persuasion mechanisms can be directly derived from Bullet 2 of Lemma 4.2. Specifically, since $h(w; \boldsymbol{\mu}^*)$ is positive if and only if $w \geq z_m$ and coincides with the lines $\ell_j(w; \mu_j^*)$ for all $j \in T_i$ on every interval $[z_i, z_{i-1}]$, (5) implies that allocation probabilities $\{q(i|w)\}$ is optimal to $V^{\text{LR}}(\boldsymbol{\mu}^*)$ if and only if they satisfy:

$$
\begin{aligned}
&\sum_{j \in T_i} q(j|w) = 1, \, \forall\, w \in (z_i, z_{i-1}), \, i \in [m], \\
&\sum_{j \in [n]} q(j|w) = 0, \, \forall\, w < z_m.
\end{aligned}
\tag{9}
$$

In other words, an optimal solution to $V^{\text{LR}}(\boldsymbol{\mu}^*)$ allocates goods with quality $w \in I_i$ exclusively to receivers in group T_i for each $i \in [m]$.

Moreover, for any group T_i and receiver $k \in T_i$, subtracting both sides of the first constraint in (8) from both sides of the same constraint with $k = t_{i-1}$, and noting that this constraint is binding for $k = t_{i-1}$ and $k = t_i$, yields the following:

$$
\begin{aligned}
\sum_{j \in [t_{i-1}+1:k]} \alpha_j q_j^* &\le \mathbb{E}\Big[w \cdot \mathbb{1}\Big[G^{-1}\Big(1 - \sum_{j \le k} q_j^*\Big) \le w < z_{i-1}\Big]\Big], \, \forall\, k \in [t_{i-1}+1 : t_i - 1], \\
\sum_{j \in T_i} \alpha_j q_j^* &= \mathbb{E}\Big[w \cdot \mathbb{1}\Big[z_i \le w < z_{i-1}\Big]\Big].
\end{aligned}
\tag{10}
$$

Additionally, we have $\mathbb{P}[z_i \le w < z_{i-1}] = \sum_{j \in T_i} q_j^*$. Analogous to the proof of Proposition 4.5, we can allocate states $w \in [z_i, z_{i-1}]$ among receivers in group T_i (possibly in a randomized manner), such that each receiver $j \in T_i$ receives an aggregate allocation of size $q_j^* > 0$, and the posterior mean for each receiver j equals α_j (i.e., the participation constraint is tight).[17] In other words, there exists an optimal solution $\{q^*(j|w)\}$ to $V^{\text{LR}}(\boldsymbol{\mu}^*)$ satisfying:

$$
\begin{aligned}
&\int_{w \in I_i} q^*(j|w) g(w) dw = q_j^*, \, \forall\, j \in T_i, i \in [m], \\
&\int_{w \in I_i} w \cdot q^*(j|w) g(w) dw = \alpha_j \int_{w \in I_i} q^*(j|w) g(w) dw, \, \forall\, j \in T_i, i \in [m].
\end{aligned}
\tag{11}
$$

By (9), (11), and Bullet 2 of Lemma 4.2, this allocation $\{q^*(j|w)\}$ is optimal to (3). We summarize these results in Theorem 4.7.

Theorem 4.7 (Optimality Condition). *An allocation probability $\{q(j|w)\}$ is optimal to (3) if and only if it allocates only to receivers in the set T_i for all $w \in I_i$ (i.e., (9) holds) and all participation constraints in (3) are binding under $\{q(j|w)\}$.*[18] *Moreover, let $\{q_j^*\}$ be an optimal solution to (8).*

[17] This applies when $\lambda_n^* > 0$, as assumed in Proposition 4.6, in which case all receivers' participation constraints bind (see also Proposition B.3 in the Appendix). If $\lambda_n^* = 0$, receiver n's participation constraint might not bind, and goods with quality $w \in [0, z_m)$ are allocated to receiver n, following the notation in Remark 4.5.

[18] This applies when $\lambda_n^* > 0$. If $\lambda_n^* = 0$, then the participation constraints for all receivers except receiver n bind, and goods with quality $w \in [0, z_m)$ are allocated to receiver n, following the notation in Remark 4.5.

Then we can construct an optimal solution $\{q^*(j|w)\}$ to (3) such that the good is allocated to each receiver j with a probability of q_j^*.

According to Theorem 4.7, once we have identified an optimal solution $\{q_i^*\}$ and optimal dual variable $\boldsymbol{\lambda}^*$ to (8), and obtained the corresponding partition $\{T_i\}_{i\in[m]}$ of receivers, the persuasion problem decouples over groups. Within each subset T_i, the sender manages trade-offs among receivers, analogous to Case 3 in Theorem 4.4 for the two-receiver case. Conversely, between subsets T_i and T_j with $i < j$, the sender prioritizes receivers in T_i over those in T_j, reflecting the priority structure observed in mechanism M_1, which is optimal in Case 1 of Theorem 4.4.

Within each group T_i, the optimal mechanism allocates states $w \in I_i$ exclusively to receivers in this group, in a way that ensures that all participation constraints bind. If a group contains only one receiver, we simply allocate the entire interval I_i to the receiver. However, when a group contains multiple receivers, the allocation must be executed more carefully. Analogous to the two-receiver case (Section 4.3.2), there are multiple ways to construct an optimal mechanism. Specifically, based on (10), we can iteratively build an optimal solution. Suppose we have already allocated a size q_ℓ^* of goods from interval I_i with a mean quality α_ℓ to each of the first k receivers in group T_i. Then, we can also allocate a size q_j^* of goods, with mean quality α_j, from the remaining goods in interval I_i to the next receiver j, where j denotes the $(k+1)$-th receiver in group T_i. Repeat this procedure until we reach the final receiver in the group, receiver t_i. The remaining quantity $q_{t_i}^*$, with mean quality α_{t_i}, is then allocated to receiver t_i.

In Appendix D, we specify a particular allocation approach at each iteration step to obtain an optimal solution $\{q^*(j|w)\}$ to (3) that has a monotone structure. Furthermore, we demonstrate that a deterministic persuasion mechanism exhibiting a double-interval structure, as described in Candogan (2022), can be easily derived using results derived from our dual analysis.

5 Conclusions

We have studied a Bayesian persuasion problem in which a sender seeks to allocate an indivisible good to n receivers by strategically disclosing information. We demonstrate that as long as the sender has a known preference over the receivers and can allocate a good to at most one receiver, public persuasion is optimal regardless of how receivers can communicate. Moreover, the optimal public persuasion mechanism can be derived from the first-best relaxation problem that imposes only participation constraints. Extending such a strong result to more general settings, or obtaining results that shed light on the sub-optimality gap when public persuasion is not optimal, could be interesting future directions.

We next investigate a specific setting in which the state variable is one-dimensional, and the receivers' utility functions are linear (therefore, a receiver cares only about the good's mean quality). We focus on efficient computation of the optimal (public) persuasion mechanism. We derive optimality conditions for persuasion mechanisms based on a dual approach, which lead to optimal mechanisms in closed form for the two-receiver case and an explicit characterization of all optimal persuasion mechanisms for the general case. This dual-based analysis enriches our understanding towards structural perperties of optimal persuasion mechanisms.

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A Proofs and Additional Details for Section 3

A.1 Proof of Lemma 3.1

Fix any information disclosure mechanism $f(\cdot|w)$. For any $i \in [n]$, let

$$\begin{aligned} q(i|w) &= \mathbb{P}\left[a_i^* = 1 \text{ and } a_j^* = 0,\, \forall\, j < i \mid w\right] \\ &= \int_{\boldsymbol{s}} \int_{\mathbf{c}} \delta_i(s_i, c_i) \Pi_{j<i} \left(1 - \delta_j(s_j, c_j)\right) c(\mathbf{c}|\boldsymbol{s}) f(\boldsymbol{s}|w) d\mathbf{c} d\boldsymbol{s} \end{aligned}$$

denote the probability that receiver i extends an offer and the good is allocated to him under the receivers' equilibrium strategies when the good's characteristics are w. The random binary variable $a_i^* \in \{0, 1\}$ represents receiver i's action of extending an offer in the equilibrium of the game induced by the mechanism $f(\cdot|w)$. Note that the good will be allocated to receiver i if and only if none of the receivers $j < i$ extends an offer.

We first prove that the participation constraint in (2) holds; that is,

$$\int_{w \in \Omega} u_i(w)\, q(i|w)\, dG(w) = \mathbb{E}\big[u_i(w) \cdot \mathbb{1}[a_i^* = 1 \text{ and } a_j^* = 0,\, \forall\, j < i]\big] \geq 0.$$

To see this, note that

$$\begin{aligned} &\mathbb{E}\big[u_i(w) \cdot \mathbb{1}[a_i^* = 1 \text{ and } a_j^* = 0,\, \forall\, j < i] \mid c_i, s_i\big] \\ =&\mathbb{E}\big[\mathbb{1}[a_i^* = 1] \mid c_i, s_i\big] \cdot \mathbb{E}\big[u_i(w) \cdot \mathbb{1}[a_j^* = 0,\, \forall\, j < i] \mid c_i, s_i\big] \geq 0, \end{aligned}$$

where the equality follows from the fact that the action a_i^* is independent of a_j^* and w conditional on the signal-communication-information pair (c_i, s_i), and the inequality follows from the optimality of the receiver's equilibrium strategy—that is, receiver i extends an offer only if doing so provides nonnegative expected utility to him. Taking expectation over (c_i, s_i) on both sides of the above inequality yields the desired result.

For the second constraint, note that for any $w \in \Omega$, we have

$$\sum_{i \in [n]} q(i|w) = \sum_{i \in [n]} \mathbb{P}\left[a_i^* = 1 \text{ for some } i \in [n] \mid w\right] \leq 1.$$

Finally, the expected payoff of the mechanism $f(\cdot|w)$ can be expressed as

$$\sum_{i=1}^{n} v_i \int_{w \in \Omega} q(i|w)\, dG(w),$$

which is the objective function of (2). Since $\{q(i|w)\}$ is feasible to (2) given any mechanism $f(\cdot|w)$, we have $V^* \leq \bar{V}$.

A.2 Proof of Theorem 3.2

Let $\{q^*(i|w)\}$ denote an optimal solution to (2). We first show that for any two receivers j and k with $j < k$, we have

$$\int_{w \in \Omega} u_j(w)\, q^*(k|w)\, dG(w) < 0. \tag{12}$$

We prove this by contradiction. Assume that there exists j and k with $j < k$ such that

$$\int_{w\in\Omega} u_j(w)\, q^*(k|w)\, dG(w) \geq 0.$$

Consider the new allocation rule $\tilde{q}(i|w)$ defined as:

$$\tilde{q}(i|w) = \begin{cases} q^*(j|w) + q^*(k|w) & \text{if } i = j, \\ 0 & \text{if } i = k, \\ q^*(i|w) & \text{if } i \notin \{j,k\}. \end{cases}$$

$\{\tilde{q}(i|w)\}$ is feasible to (2), and because $v_j > v_k$, $\{\tilde{q}(i|w)\}$ achieves a strictly larger objective value than $\{q^*(i|w)\}$. This contradicts the fact that $\{q^*(i|w)\}$ is optimal to (2). Thus, our assumption fails.

Since a public persuasion mechanism leaves no payoff-related information for the receivers to communicate, receivers make decisions based only on the public signal and ignore potential communication among themselves. We now show that it is an equilibrium for each receiver $i \in [n]$ to extend an offer only upon receiving the signal $s = i$. To do so, suppose all receivers other than receiver i follow this strategy; we verify that it is optimal for receiver i to do the same.

First, suppose receiver i receives the signal $s = i$. The expected payoff for extending an offer is nonnegative because

$$\int_{w\in\Omega} u_i(w)\, dG\big(w|s=i\big) = \frac{1}{\int_w q^*(i|w)\, dG(w)} \int_{w\in\Omega} u_i(w)\, q^*(i|w)\, dG(w) \geq 0,$$

where $dG\big(w|s=i\big) = \frac{q^*(i|w)\, dG(w)}{\int_w q^*(i|w)\, dG(w)}$ denotes the posterior belief of w given $s = i$, and the inequality follows from the participation constraint in (2). Therefore, it is optimal for the receiver i to extend an offer.

Second, suppose receiver i receives the signal $s = k$ with $k > i$. The expected payoff for extending an offer is negative because

$$\int_{w\in\Omega} u_i(w)\, dG\big(w|s=k\big) = \frac{1}{\int_w q^*(k|w)\, dG(w)} \int_{w\in\Omega} u_i(w)\, q^*(k|w)\, dG(w) < 0,$$

where the inequality follows from (12). Therefore, receiver i will not extend an offer.

Finally, suppose receiver i receives the signal $s = j$ with $j < i$. Since the sender will never accept receiver i's offer (because receiver j will extend an offer), receiver i is indifferent between extending an offer or not.

Note that the expected payoff for the sender is $\bar{V}$ under this equilibrium. Therefore, the public mechanism $f^*(\cdot|w)$ is optimal to (1).

A.3 Details of Remark 3.2

In our base model, we assume that the sender obtains a deterministic utility v_i from allocating the good to each receiver $i \in [n]$. In this section, we consider a more general setting in which the offer values $\{v_i\}$ are uncertain and potentially correlated with the good's characteristics w.

The General Model We define a general state space $\Phi = \Omega \times \Theta$. As before, Ω represents the space of the good's characteristics. Each receiver $i \in [n]$ derives utility $u_i(w)$ from receiving a good with characteristics $w \in \Omega$, and zero from not receiving the good. Meanwhile, Θ represents the sender's preference space. Specifically, a sender with preference $\theta \in \Theta$ obtains utility $v_i(\theta)$ from allocating the good to receiver $i \in [n]$, and zero utility if the good remains unallocated.

The sender privately observes both the realization of the good's characteristics w and the preference type θ, whereas receivers only have a prior joint distribution over the underlying state $(w, \theta) \in \Phi$. We let $G(w)$ denote the marginal cumulative distribution of the good's characteristics w, and $H(\theta|w)$ denote the cumulative distribution of the sender's preference type θ conditional on w. The sender can commit to an information disclosure mechanism to reveal the underlying state $(w, \theta) \in \Phi$ to the receivers.

A.3.1 Positive Result: Fixed Ordinal Ranking

In this section, we demonstrate that public persuasion remains optimal when the sender's ordinal ranking over receivers is fixed; we formally state this assumption in Assumption A.1.

Assumption A.1. We have $v_i(\theta) > v_j(\theta)$ for any receivers $i < j$ and realization $\theta \in \Theta$.

We first rewrite the first-best relaxation problem (2) for the general model by replacing v_i with $v_i(\theta)$ and $q(i|w)$ with $q(i|w, \theta)$, and by taking expectations over both the good's characteristics and the sender's preferences, as given by (13).

$$
\begin{aligned}
\bar{V} = \max_{q(i|w,\theta) \geq 0} \quad & \sum_{i=1}^{n} \int_{w \in \Omega} \int_{\theta \in \Theta} v_i(\theta)\, q(i|w, \theta)\, dH(\theta|w)\, dG(w) \\
\text{s.t.} \quad & \int_{w \in \Omega} \int_{\theta \in \Theta} u_i(w)\, q(i|w, \theta)\, dH(\theta|w)\, dG(w) \geq 0,\ \forall\, i \in [n], \\
& \sum_{i \in [n]} q(i|w, \theta) \leq 1,\ \forall\, w \in \Omega,\ \theta \in \Theta.
\end{aligned} \tag{13}
$$

Problem (13) admits an interpretation analogous to that of (2). In (13), a central planner allocates a good with characteristics w to receiver i with probability $q(i|w, \theta)$, given her preference type is θ. The planner aims to maximize the sender's expected payoff while ensuring a nonnegative expected utility for each receiver.

Analogous to Lemma 3.1, (13) provides an upper bound on the sender's optimal expected payoff V^* in the general model, regardless of how receivers communicate, as we formally state in Lemma A.1.

Lemma A.1 (Extension of Lemma 3.1). *Problem (13) provides an upper bound on the sender's optimal expected payoff V^* in the general model, regardless of how receivers communicate.*

The proof mimics the proof of Lemma 3.1 presented in Appendix A.1; hence, we omit the details. Intuitively, for any disclosure mechanism $f(\cdot|w, \theta)$, define $q(i|w, \theta)$ as the ex-ante probability of allocating the good to receiver i when the good's characteristics are w, the sender's preference type is θ, and all receivers follow the equilibrium strategies induced by $f(\cdot|w, \theta)$. Analogous to the proof of Lemma 3.1, these probabilities $\{q(i|w)\}$ are feasible to (2) and yield an objective value no larger than $\bar{V}$.

Next, we demonstrate that an optimal solution to (13) can be implemented via a public persuasion mechanism, whose expected payoff attains the upper bound $\bar{V}$. Therefore, public persuasion remains optimal. Specifically, let $\{q^*(i|w, \theta)\}$ denote an optimal solution to (13). Consider a public persuasion mechanism $f^*(\cdot|w, \theta)$ with signal space $S_i = S \triangleq [n] \cup \{\varnothing\}$ for all receivers $i \in [n]$. When the good's characteristics are w and the sender's preference type is θ, the mechanism broadcasts the signal $s = i$ to all receivers with probability $q^*(i|w,\theta)$ for any $i \in [n]$ and the signal $s = \varnothing$ to all receivers with probability $1 - \sum_{i\in[n]} q^*(i|w,\theta)$. We can interpret the signal $s = i$ as a recommendation for only receiver i to extend an offer and signal $s = \varnothing$ as a recommendation for none of the receivers to extend an offer. Theorem A.2 shows that this persuasion mechanism achieves the first-best upper bound $\bar{V}$.

Theorem A.2 (Extension of Theorem 3.2). *Under the public persuasion mechanism $f^*(\cdot|w,\theta)$, it is an equilibrium for each receiver $i \in [n]$ to extend an offer if and only if he receives the signal $s = i$. Moreover, the expected payoff of the mechanism $f^*(\cdot|w,\theta)$ attains the upper-bound value $\bar{V}$.*

Proof. The proof is analogous to the proof of Theorem 3.2 in Appendix A.2. Let $\{q^*(i|w,\theta)\}$ denote an optimal solution to (13). Following the same argument in Appendix A.2, for any two receivers j and k with $j < k$, we have:

$$\int_{w\in\Omega}\int_{\theta\in\Theta} u_j(w)\, q^*(k|w,\theta)\, dH(\theta|w)\, dG(w) < 0\,. \tag{14}$$

Otherwise, the new allocation rule $\{\tilde{q}(i|w,\theta)\}$, defined as

$$\tilde{q}(i|w,\theta) = \begin{cases} q^*(j|w,\theta) + q^*(k|w,\theta) & \text{if } i = j,\\ 0 & \text{if } i = k,\\ q^*(i|w,\theta) & \text{if } i \notin \{j,k\}. \end{cases}$$

is feasible to (13) and yields a strictly higher objective value than $\{q^*(i|w,\theta)\}$ by Assumption A.1.

Since a public persuasion mechanism leaves no payoff-related information for the receivers to communicate, receivers make decisions based only on the public signal. We now show that it is an equilibrium for each receiver $i \in [n]$ to extend an offer only upon receiving the signal $s = i$. To do so, suppose all receivers other than receiver i follow this strategy; we verify that it is optimal for receiver i to do the same.

First, suppose receiver i receives the signal $s = i$. The expected payoff for extending an offer is nonnegative by the participation constraint in (13). Therefore, it is optimal for receiver i to extend an offer.

Second, suppose receiver i receives signal $s = k$ with $k > i$. The expected payoff for extending an offer is negative by (14). Therefore, receiver i will not extend an offer.

Third, suppose receiver i receives signal $s = j$ with $j < i$. Since the sender will never accept receiver i's offer (because receiver j will extend an offer, and the sender always prefers receiver j's offer by Assumption A.1), receiver i is indifferent between extending an offer or not.

Finally, we note that the expected payoff for the sender is $\bar{V}$ under this equilibrium. □

To conclude this section, we note that Assumption A.1 provides a sufficient condition for public persuasion to be optimal. Generally speaking, as long as no receiver can cherry-pick goods intended for other receivers to obtain positive expected utility, public persuasion remains optimal. Extending these conditions to more general settings is an interesting direction for future research.

A.3.2 Negative Result: General Case

In this section, we show that public persuasion may no longer be optimal in the general model when the offer values $\{v_i\}$ correlate arbitrarily with the good's characteristics w. We illustrate this through the following simple example.

Example A.1. Consider an example with two receivers, each with linear utilities, as in Section 4. The good's characteristic w is one-dimensional and uniformly distributed over $[0, 1]$. Suppose the receivers have acceptance thresholds $\alpha_1 = 0.9$ and $\alpha_2 = 0.7$. We correlate the offer values and the good's quality w as follows: when w exceeds α_1, the sender marginally prefers receiver 1 over receiver 2, with offer values $v_1 = 2$ and $v_2 = 1$. However, when w falls below α_1, the sender strongly prefer receiver 2, with $v_1 = 2$ and $v_2 = 100$.

Private Persuasion Attains First-Best Value Since v_2 is significantly large when the good's quality w is below α_1, the optimal solution to the first-best relaxation (13) exclusively targets receiver 2, i.e., allocating all goods with $w \in [0.4, 1]$ to receiver 2 and none to receiver 1. This solution can be implemented through private persuasion if the receivers cannot communicate. Specifically, the sender provides no information to receiver 1 and informs receiver 2 whether the good's quality exceeds 0.4. Consequently, receiver 1 refrains from making offers, while receiver 2 extends an offer only when $w \geq 0.4$.[19]

Failure of Public Persuasion We next show that public persuasion cannot achieve the first-best outcome. Under public persuasion, receiver 1 observes the goods recommended to receiver 2 (because signals are public) and can selectively target these goods with competing offers. Consequently, goods with quality $w \in [0.9, 1]$ are allocated to receiver 1, as the sender prefers receiver 1 whenever $w \geq \alpha_1$. This selection effectively removes the highest-quality goods from receiver 2's pool, resulting in negative utility for receiver 2 and making the first-best outcome unsustainable in equilibrium.

A.4 Details of Remark 3.3

In our base model, we assume that receivers take binary actions (i.e., either accepting or rejecting the good). In practice, a receiver may have multiple options. For example, in the student promotion example, an employer might extend a regular offer, provide an offer with additional benefits, or an offer for an alternative position, or decline to make an offer. In this section, we generalize our setting to allow receivers to choose among multiple possible actions regarding the good, and we demonstrate that all our results from Section 3 continue to hold.

A.4.1 A Model with Multiple Actions

In this section, we extend our base model from Section 2 by allowing each receiver to select from multiple actions. Specifically, for each receiver $i \in [n]$, let $\bar{A}_i \triangleq A_i \cup \{\varnothing\}$ denote his action set, where A_i represents multiple acceptance options available to receiver i and $\varnothing$ denotes the action of rejection. For each acceptance action $a_i \in A_i$, let $u_i(w, a_i)$ denote receiver i's utility when he offers acceptance option a_i and a good with characteristics w is allocated to him. The utility of not receiving the good is zero; hence, $u_i(w, \varnothing) = 0$ for all $i \in [n]$ and $w \in \Omega$.

The sender allocates an indivisible good and obtains a deterministic utility $v(a_i)$ if it is accepted through option a_i. We assume these offer values $\{v(a_i)\}_{a_i \in A_i, i \in [n]}$ are distinct and positive in Assumption A.2. The sender's utility is zero if the good is not allocated.

Assumption A.2. The sender receives deterministic utility $v(a_i)$ when allocating the good to receiver i through acceptance option a_i. The offer values $\{v(a_i)\}_{a_i \in A_i, i \in [n]}$ are distinct and positive.

[19] If receiver 1 were to extend an offer, given receiver 2's strategy, only goods with quality $w \in [0, 0.4) \cup [0.9, 1]$ would ultimately be allocated to receiver 1, whose expected quality falls below receiver 1's acceptance threshold α_1.

Finally, since the offer values $\{v(a_i)\}$ are distinct, we can sort the acceptance options $\bigcup_{i\in[n]} A_i$ in descending order according to their offer values, and partition them into groups $\{B_k\}$ such that:

- Each group B_k consists exclusively of acceptance actions available to one receiver (i.e., $B_k \subseteq A_i$ for some $i \in [n]$);
- Any two adjacent groups B_{k-1} and B_k correspond to different receivers;
- For any two acceptance options $a_i \in B_{k-1}$ and $a_j \in B_k$ from adjacent groups, we have $v(a_i) > v(a_j)$; that is, the sender strictly prefers options in group B_{k-1} over those in group B_k.

For each group B_k, we define $i(B_k) \in [n]$ as the receiver to whom these actions belong (i.e., $B_k \subseteq A_{i(B_k)}$).

A.4.2 The Relaxation Problem

In Appendix A.4.3, we demonstrate that public persuasion remains optimal when receivers can select from multiple actions. To do so, we first reformulate the first-best relaxation problem (2) for the multiple-action setting by incorporating additional incentive compatibility constraints for each receiver, as presented in (15).

$$
\begin{aligned}
\bar{V} = \max_{q(a_i|w)\geq 0} \quad & \sum_{i\in[n]} \sum_{a_i\in A_i} \int_{w\in\Omega} v(a_i)\, q(a_i|w)\, dG(w) \\
\text{s.t.} \quad & \int_{w\in\Omega} u_i(w,a_i)\, q(a_i|w)\, dG(w) \geq 0,\ \forall\, a_i \in A_i,\ i\in[n], \\
& \int_{w\in\Omega} u_i(w,a_i)\, q(a_i|w)\, dG(w) \geq \int_{w\in\Omega} u_i(w,\tilde{a}_i)\, q(a_i|w)\, dG(w),\ \forall\, a_i, \tilde{a}_i \in B_k,\ k\in\mathbb{N}_+, \\
& \sum_{i\in[n]} \sum_{a_i\in A_i} q(a_i|w) \leq 1,\ \forall\, w\in\Omega.
\end{aligned} \tag{15}
$$

Analogous to (2), in (15), a planner allocates a good with characteristics w to receiver i under acceptance option a_i with probability $q(a_i|w)$. The planner maximizes the sender's expected payoff while ensuring nonnegative expected utility for each action and satisfying a new incentive compatibility constraints for each receiver, as captured by the second constraint in (15). To interpret this constraint, suppose that $a_i \in B_k$. When the sender recommends action a_i to receiver i, receiver i understands if accepting with option a_i ensures allocation of the good to him, any other acceptance option $\tilde{a}_i \in B_k$ would also guarantee him the allocation. As a result, receiver i will select the action within B_k that provides him with the highest expected utility based on his posterior belief about the good's characteristics. Therefore, the second constraint in (15) represents the incentive compatibility condition that receiver i follows the sender's recommendation, which is without loss of optimality due to the revelation principle.

Analogous to Lemma 3.1, problem (15) is a relaxation to the sender's information design problem, and thus it provides an upper bound on the sender's optimal expected payoff V^* in the multiple-action setting, regardless of how receivers communicate, which we formally state in Lemma A.3.

Lemma A.3 (Extension of Lemma 3.1). *Problem (15) provides an upper bound on the sender's optimal expected payoff V^* in the multiple-action setting, regardless of how receivers communicate.*

The proof closely follows that of Lemma 3.1 in Appendix A.1; hence, we omit the details here. Intuitively, for any disclosure mechanism $f(\cdot|w)$, define $q(a_i|w)$ as the ex-ante probability that the good is allocated to receiver i under acceptance option a_i, when the good's characteristics are w and all receivers follow the equilibrium strategies induced by $f(\cdot|w)$. We can show that these probabilities $\{q(a_i|w)\}$ are feasible to (15) and yield an objective value no larger than $\bar{V}$.

A.4.3 Public Persuasion Remains Optimal

In this section, we demonstrate that an optimal solution to (15) can be implemented through a public persuasion mechanism, whose expected payoff attains the upper bound value $\bar{V}$. Therefore, public persuasion remains optimal.

Specifically, let $\{q^*(a|w)\}_{a\in\bigcup_{i\in[n]}A_i}$ denote an optimal solution to (15). Consider a public persuasion mechanism $f^*(\cdot|w)$ with signal space $S_i = S \triangleq \bigcup_{i\in[n]} A_i \cup \{\varnothing\}$ for all receivers $i \in [n]$. When the good's characteristics are w, the mechanism broadcasts the signal $s = a$ to all receivers with probability $q^*(a|w)$ for any $a \in \bigcup_{i\in[n]} A_i$ and the signal $s = \varnothing$ to all receivers with probability $1 - \sum_{i\in[n]}\sum_{a\in A_i} q^*(a|w)$. We can interpret signal $s = a$ as a recommendation to accept the good under acceptance option a and signal $s = \varnothing$ as a recommendation for none of the receivers to extend any acceptance offer. Let $v(\varnothing) = 0$ denote the sender's utility when the good is unallocated. For any recommendation $a \in \bigcup_{i\in[n]} A_i \cup \{\varnothing\}$, let:

$$a_j(a) \triangleq \arg\max\{\tilde{a} \in A_j \cup \{\varnothing\} : v(\tilde{a}) \le v(a)\}$$

represent the strongest competing offer receiver j can extend without exceeding the sender's utility for option a.[20] Note that $a_j(\varnothing) = \varnothing$ for all $j \in [n]$. Additionally, when $a = a_i \in A_i$ and each receiver j extends offer $a_j(a_i)$, only receiver i's offer a_i is accepted. Nevertheless, the competing offers from other receivers help sustain an equilibrium under the public persuasion mechanism $f^*(\cdot|w)$, whose performance achieves the first-best upper bound $\bar{V}$, as we demonstrate in Theorem A.4.

Theorem A.4 (Extension of Theorem 3.2)**.** *Under the public persuasion mechanism $f^*(\cdot|w)$, it is an equilibrium for each receiver $j \in [n]$ to extend an offer $a_j(a)$ when he receives the signal $s = a$. Moreover, the expected payoff of the mechanism $f^*(\cdot|w)$ attains the upper bound value $\bar{V}$.*

Proof. The proof is analogous to that of Theorem 3.2 provided in Appendix A.2. Let $\{q^*(a|w)\}$ denote an optimal solution to (15). Analogous to the proof in Appendix A.2, for any two acceptance options $a \in B_k$ and $\tilde{a} \in B_\ell$ from different groups $k < \ell$, we have:

$$\int_{w\in\Omega} u_{i(B_k)}(w, a)\; q^*(\tilde{a}|w)\, dG(w) < 0\,. \tag{16}$$

Otherwise, we could reallocate the probability mass $\int_{w\in\Omega} q^*(\tilde{a}|w)\, dG(w)$ to actions within group B_k to obtain a new allocation that is feasible to (15). Since the sender prefers any option in B_k over option $\tilde{a}$, the new allocation brings a strictly higher objective value than $\{q^*(a|w)\}$, contradicting the optimality of $\{q^*(a|w)\}$. Using the same argument, we can also show that for any receiver i and any action $a \in A_i$, we have:

$$\int_{w\in\Omega} u_i(w, a)\, q^*(\varnothing|w)\, dG(w) < 0\,, \tag{17}$$

[20] The functions $a_j(a)$ are well-defined since the offer values are distinct and positive, and $v(\varnothing) = 0$.

where $q^*(\varnothing|w) \triangleq 1 - \sum_{i\in[n]} \sum_{a\in A_i} q^*(a|w)$ represents the probability of receiving signal $s = \varnothing$.

Since a public persuasion mechanism leaves no payoff-related information for the receivers to communicate, receivers make decisions based only on the public signal. We now show that it is an equilibrium for each receiver $i \in [n]$ to extend an offer $a_j(a)$ upon receiving the signal $s = a$. To do so, suppose all receivers except receiver i follow this strategy; we verify that it is optimal for receiver i to adopt the same strategy.

First, suppose receiver i receives the signal $s = \varnothing$. By (17), extending any acceptance offer $a_i \in A_i$ yields a negative expected payoff. Therefore, it is optimal for receiver i to select $a_i(\varnothing) = \varnothing$, i.e., reject the good.

Second, suppose receiver i receives the signal $s = a$ with $a = a_j \in A_j$ for some $j \neq i$; that is, the sender recommends another receiver j to extend an offer a_j. If receiver i extends a stronger offer a_i with $v(a_i) > v(a_j)$, the good will be allocated to him with certainty given other receivers' strategies. However, doing so only brings him a negative expected payoff by (16). Therefore, receiver i is better off not receiving the good and is indifferent about extending the weaker offer $a_i(a)$, as this offer will not be accepted given receiver j's strategy.

Finally, suppose receiver i receives the signal $s = a$ with $a = a_i \in A_i$; that is, the sender recommends that receiver i extend the acceptance offer a_i. We now show that following the sender's recommendation yields the highest expected payoff for receiver i. To see this, assume that $a_i \in B_k$ for some subset B_k. The argument proceeds as follows:

- First, the participation constraints in (15) ensure that action a_i yields a nonnegative expected payoff to receiver i. Moreover, the incentive compatibility constraints in (15) guarantee that action a_i provides receiver i with a strictly higher expected payoff than any other action in the set B_k.
- Second, if receiver i extends a stronger offer $\tilde{a} \in B_\ell$ for some $\ell < k$, he will receive the good with certainty. However, according to (16), this action yields a negative expected payoff. Therefore, receiver i is better off not choosing any action $\tilde{a} \in B_\ell$ for $\ell < k$.
- Finally, receiver i has no incentive to take any weaker action $\tilde{a} \in B_\ell$ with $\ell > k$: given other receivers extending slightly weaker competing offers $a_j(a)$, doing so would cause receiver i to lose the good (because at least one of those competing offers will be more attractive to the sender than any offer in B_ℓ).

Consequently, the optimal strategy for receiver i is to follow the recommended action $a_i(a) = a_i$.

Therefore, it constitutes an equilibrium under the public persuasion $f^*(\cdot|w)$ when each receiver $j \in [n]$ extends an offer $a_j(a)$ upon receiving the signal $s = a$. Moreover, the sender's expected payoff under this equilibrium is $\bar{V}$, as $\{q^*(a|w)\}$ precisely correspond to the equilibrium allocation probabilities. □

B Proofs and Additional Details for Section 4

B.1 Preliminary Properties of Optimal Persuasion Mechanisms

In this section, we describe several properties of an optimal solution to (3). First, Proposition B.1 shows that for any feasible solution to (3), the probability of allocating a good (prior to observing w) is maximized when the sender exclusively targets the most accessible receiver n.

Proposition B.1. *Define* $z_n \triangleq \min\{z \geq 0 : \mathbb{E}[w|w \geq z] \geq \alpha_n\}$*, where* α_n *is receiver* n*'s threshold value. For any feasible solution* $\{q(i|w)\}$ *of (3), we have* $\sum_{i\in[n]} \int_0^1 q(i|w)g(w)dw \leq \mathbb{P}(w \geq z_n)$*, with equality attained when the sender exclusively targets receiver* n*; that is,* $q(n|w) = 1$ *for any* $w \geq z_n$*, and* $q(i|w) = 0$ *for any* $i \neq n$ *or* $w < z_n$*.*

Proof. Since the threshold value α_i is strictly decreasing in the receiver index i by Assumption 4.3, the probability of receiving an offer, given by $\sum_{i\in[n]} \int_0^1 q(i|w)g(w)dw$, is maximized when the sender targets only receiver n, who has the lowest threshold α_n. That is, $q(i|w) = 0$ for all $i \neq n$ and $w \in [0, 1]$.

To see why, given any feasible solution $\{q(i|w)\}$ to (3), we can construct a new solution $\{\tilde{q}(i|w)\}$ by setting $\tilde{q}(n|w) = \sum_{i\in[n]} q(i|w)$ and $\tilde{q}(i|w) = 0$ for all $i < n$. Note that $\{\tilde{q}(i|w)\}$ is feasible to (3) and achieves the same acceptance probability. Moreover, if the original solution $\{q(i|w)\}$ assigns a positive probability to any receiver $i < n$, the participation constraint of receiver n will be loose under the new solution $\tilde{q}(i|w)$, which allows for further allocation of probability mass to receiver n without violating his participation constraint.

On the other hand, if the sender targets only receiver n, the acceptance probability is maximized with $q(n|w) = 1$ for all $w \geq z_n$ and $q(n|w) = 0$ otherwise, resulting in an acceptance probability of $\mathbb{P}(w \geq z_n)$. □

Second, Proposition B.2 shows that any optimal solution exhibits a cutoff structure. Specifically, there exists a threshold value $z \in [0, 1]$ such that a good is allocated if and only if its characteristics w exceeds z.

Proposition B.2. *Any optimal solution has a cutoff structure. That is, for any optimal solution* $\{q^*(i|w)\}$ *to (3), there exists a threshold value* $z \in [0, 1]$ *such that* $\sum_{i\in[n]} \int_z^1 q^*(i|w)g(w)dw = \mathbb{P}(w \geq z)$ *and* $\sum_{i\in[n]} \int_0^z q^*(i|w)g(w)dw = 0$*.*

Proof. Let $\{q(i|w)\}$ be a feasible solution of (3), and define $z \triangleq \sup\{z \in [0, 1] : \sum_{i\in[n]} \int_0^z q(i|w)\, dw = 0\}$ as the lower bound on the support of $\{q(i|w)\}$. If $\sum_{i\in[n]} \int_z^1 q(i|w)dw < \mathbb{P}(w \geq z)$, there exists a point $\tilde{z} \in (z, 1)$ satisfying:

$$\sum_{i\in[n]} \int_z^{\tilde{z}} q(i|w)\, dw = \sum_{i\in[n]} \int_{\tilde{z}}^1 (1 - q(i|w))\, dw > 0.$$

We can create a new feasible solution $\{\tilde{q}(i|w)\}$ from $\{q(i|w)\}$ by transporting the mass of $\{q(i|w)\}$ from below $\tilde{z}$ to fill the "unoccupied" region above $\tilde{z}$; therefore, $\sum_{i\in[n]} \int_{\tilde{z}}^1 \tilde{q}(i|w)dw = \mathbb{P}(w \geq \tilde{z})$ and $\sum_{i\in[n]} \int_0^{\tilde{z}} \tilde{q}(i|w)dw = 0$. The two feasible solutions $\{\tilde{q}(i|w)\}$ and $\{q(i|w)\}$ have the same objective value because, by transporting, $\int_0^1 q(i|w)dw = \int_0^1 \tilde{q}(i|w)dw$ for any $i \in [n]$.

On the other hand, since $\{q(i|w)\}$ satisfies the participation constraints and we have shifted a positive mass of $\{q(i|w)\}$ from below $\tilde{z}$ to above $\tilde{z}$, the participation constraint for some receiver $i \in [n]$ must hold with strict inequality with $\{\tilde{q}(i|w)\}$. Given that $\tilde{z} > z \geq 0$, we can allocate some unallocated mass $w \in [0, \tilde{z})$ to this receiver without violating his participation constraint, thereby strictly increasing the sender's payoff. □

Finally, let $\{q^*(i|w)\}$ be an optimal solution to (3), and let $q_i^* \triangleq \int_0^1 q^*(i|w)g(w)dw$ denote the ex-ante probability that receiver i obtains the good. Without loss of generality, assume $q_i^* > 0$ for all $i \in [n]$, as receivers with $q_i^* = 0$ can be disregarded from consideration. Proposition B.3 shows that, for any optimal solution $\{q^*(i|w)\}$ to (3), the participation constraints for the first $n - 1$

receivers always bind. Receiver n's participation constraint need not bind in general, but it must bind when $\sum_{i\in[n]} q_i^* < 1$.

Proposition B.3. *Let $\{q^*(i|w)\}$ be an optimal solution to (3). Define $q_i^* \triangleq \int_0^1 q^*(i|w)g(w)\,dw$ and assume $q_i^* > 0$ for all $i \in [n]$. The participation constraints for the first $n-1$ receivers always bind. Additionally, receiver n's participation constraint binds if $\sum_{i\in[n]} q_i^* < 1$.*

Proof. We first assume $\sum_{i\in[n]} q_i^* < 1$ and demonstrate that all receivers' participation constraints bind. Suppose instead that receiver j's participation constraint holds with strict inequality:

$$\int_0^1 w \cdot q^*(j|w)\,g(w)\,dw > \alpha_j \int_0^1 q^*(j|w)\,g(w).$$

Then, since there is unallocated probability mass (as $\sum_{i\in[n]} q_i^* < 1$), we can allocate some of this mass to receiver j until his participation constraint binds, thereby strictly increasing the sender's expected payoff. This contradicts optimality.

We next assume $\sum_{i\in[n]} q_i^* = 1$ and show that the participation constraints of the first $n-1$ receivers must bind. If not, suppose the participation constraint for receiver $j \le n-1$ holds with strict inequality. Then, reallocating mass from receiver n to receiver j would strictly increase the sender's expected payoff, contradicting optimality. □

B.2 Proof of Lemma 4.2

Since the thresholds α_i are smaller than one by Assumption 4.3, it is straightforward to create a feasible solution to (3) where all participation constraints in (3) are satisfied with strict inequality. Therefore, strong duality holds and an optimal dual variable $\boldsymbol{\mu}^*$ exists according to Theorem 1 in Section 8.6 of Luenberger (1997). Once strong duality is established, Bullet 2 follows from the optimality condition (see Proposition 6.1.5 in Bertsekas 2016).

B.3 Proof of Proposition 4.3

For ease of notation, we drop the dependence on the mechanism M by letting $q_1 = q_1(M)$ and $q_2 = q_2(M)$. If the mechanism has a cutoff structure with a threshold z, and the participation constraints for both receivers bind, the following two linear equations must hold:

$$\begin{aligned} q_1 + q_2 &= \mathbb{P}[w \ge z], \\ \alpha_1 q_1 + \alpha_2 q_2 &= (q_1 + q_2) \cdot \mathbb{E}[w|w \ge z]. \end{aligned} \tag{18}$$

The first equation follows from the fact that the good is allocated (to either receiver 1 or 2) if and only if $w \ge z$, and the second equation follows from the cutoff structure, the law of total expectation:

$$\mathbb{E}[w|w \ge z] = \frac{q_1}{q_1+q_2} \cdot \mathbb{E}[w|s=1] + \frac{q_2}{q_1+q_2} \cdot \mathbb{E}[w|s=2],$$

and the fact that $\mathbb{E}[w|s=i] = \alpha_i$ by the binding participation constraints. The two equations in (18) determine the values of q_1 and q_2 as

$$\begin{aligned} q_1 &= \mathbb{P}[w \ge z] \cdot \frac{\mathbb{E}[w|w \ge z] - \alpha_2}{\alpha_1 - \alpha_2}, \\ q_2 &= \mathbb{P}[w \ge z] \cdot \frac{\alpha_1 - \mathbb{E}[w|w \ge z]}{\alpha_1 - \alpha_2}. \end{aligned} \tag{19}$$

Note that we have $\mathbb{E}[w|w \geq z] \in [\alpha_2, \alpha_1]$ when $z \in [z_2, z_1]$. Hence, $q_1, q_2 \in [0, 1]$ are well-defined probabilities.

We now construct public persuasion mechanisms M satisfying Proposition 4.3. Such mechanisms must fulfill the following conditions:

1. $q(1|w) + q(2|w) = 1$ for all $w \geq z$, and $q(1|w) = q(2|w) = 0$ for all $w < z$;
2. $\mathbb{P}[s = 1] = q_1$, and $\mathbb{P}[s = 2] = q_2$;
3. $\mathbb{E}[w|s = 1] = \alpha_1$, $\mathbb{E}[w|s = 2] = \alpha_2$, and $\mathbb{E}[w|s = \varnothing] < \alpha_2$.

A feasible mechanism M can be constructed in multiple ways. For example, we can construct a deterministic persuasion mechanism by setting: $q(1|w) = 1$ for $w \in T$, $q(2|w) = 1$ for $w \in [z, 1] \setminus T$, and $q(\varnothing|w) = 1$ for $w < z$, for some subset $T \subseteq [z, 1]$. To satisfy Proposition 4.3, the subset T must meet these conditions:

1. $\mathbb{P}[w \in T] = q_1$ and $\mathbb{P}[w \in [z, 1] \setminus T] = q_2$;
2. $\mathbb{E}[w|w \in T] = \alpha_1$, $\mathbb{E}[w|w \in [z, 1] \setminus T] = \alpha_2$, and $\mathbb{E}[w|w < z] < \alpha_2$.

There are, again, various ways to construct such a subset T. For instance, T can be chosen as an interval $[\underline{b}, \bar{b}] \subseteq [\bar{z}_1, 1]$ that contains α_1 and satisfies

$$\mathbb{P}[\underline{b} \leq w \leq \bar{b}] = q_1 \quad \text{and} \quad \mathbb{E}[w \mid \underline{b} \leq w \leq \bar{b}] = \alpha_1.$$

The existence of such an interval $[\underline{b}, \bar{b}]$ is guaranteed since $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$ and $\mathbb{P}[w \geq \bar{z}_1] \geq q_1$ (see Lemma B.4). Furthermore, conditions

$$\mathbb{P}[w \in [z, 1] \setminus T] = q_2 \quad \text{and} \quad \mathbb{E}[w \mid w \in [z, 1] \setminus T] = \alpha_2$$

hold by (18). Finally, we verify that $\mathbb{E}[w|w < z] \leq \mathbb{E}[w|w < z_1] < \alpha_2$, where the second inequality follows from two scenarios: (i) if $\bar{z}_1 \leq \alpha_2$, then $z_1 = \bar{z}_1 \leq \alpha_2$; (ii) if $\bar{z}_1 > \alpha_2$, then $\mathbb{E}[w \mid w < z_1] < \mathbb{E}[w \mid z_1 \leq w < \bar{z}_1] = \alpha_2$.

Lemma B.4. *Let $q_1(z)$ denote the probability q_1 defined in* (19)*. Then, $q_1(z) \leq q_1(z_1) = \mathbb{P}[w \geq \bar{z}_1]$ for all $z \in [z_2, z_1]$.*

Proof. Since

$$\mathbb{E}[w \,|\, w \geq z_1] = \frac{\mathbb{P}[w \geq \bar{z}_1]}{\mathbb{P}[w \geq z_1]}\alpha_1 + \frac{\mathbb{P}[z_1 \leq w \leq \bar{z}_1]}{\mathbb{P}[w \geq z_1]}\alpha_2 \,,$$

we have $q_1(z_1) = \mathbb{P}[w \geq \bar{z}_1]$ by (19). We next show that $q_1(z) \leq q_1(z_1)$ for any $z \in [z_2, z_1]$.

From (19), we can express $q_1(z)$ as:

$$q_1(z) = \frac{1}{\alpha_1 - \alpha_2} \int_z^1 (w - \alpha_2) g(w) dw.$$

which yields the derivative:

$$\frac{dq_1(z)}{dz} = \frac{\alpha_2 - z}{\alpha_1 - \alpha_2} \cdot g(z).$$

Since $z_1 \leq \alpha_2$, $q_1(z)$ is increasing on $[z_2, z_1]$. Therefore, $q_1(z) \leq q_1(z_1) = \mathbb{P}[w \geq \bar{z}_1]$ for all $z \in [z_2, z_1]$. □

B.4 Proof of Theorem 4.4

In this proof, we identify a set of dual variables $\boldsymbol{\mu} \in \mathbb{R}^n_+$, which, together with the mechanism proposed in Theorem 4.4, satisfy Bullet 2 of Lemma 4.2. This indicates that the mechanism is optimal to (3), and $\boldsymbol{\mu}$ is an optimal dual variable.

Proof of Bullet 2 Suppose $v_1 \leq v_2 \cdot \frac{\alpha_1 - z_2}{\alpha_2 - z_2}$, which implies that the point (α_1, v_1) lies below line η_2. We construct the receivers' lines ℓ_1 and ℓ_2 as follows.

Let line ℓ_2 coincide with line η_2 by taking the dual variable $\mu_2 = \frac{v_2}{\alpha_2 - z_2}$. Set line ℓ_1 to lie below line ℓ_2 for all $w \in [z_2, 1]$. For instance, this can be achieved by taking the dual variable $\mu_1 = \frac{v_1}{\alpha_1 - z_2}$. The lines ℓ_1 and ℓ_2 are illustrated in Figure 3(a). Since line ℓ_2 dominates ℓ_1, an optimal solution to the Lagrangian $V^{\text{LR}}(\boldsymbol{\mu})$ with $\boldsymbol{\mu} = (\mu_1, \mu_2)$ will never allocate the good to receiver 1, irrespective of the good's quality w. It is easy to verify that the mechanism M_2 and dual variable $\boldsymbol{\mu} = (\mu_1, \mu_2)$ satisfy Lemma 4.2 Bullet 2. Therefore, mechanism M_2 is optimal to (3), and $\boldsymbol{\mu} = (\mu_1, \mu_2)$ is an optimal dual variable. Moreover, M_2 is the unique mechanism that satisfies Lemma 4.2 Bullet 2 given $\boldsymbol{\mu} = (\mu_1, \mu_2)$.

Proof of Bullet 3 Suppose $v_1 \in \left(v_2 \cdot \frac{\alpha_1 - z_2}{\alpha_2 - z_2}, v_2 \cdot \frac{\alpha_1 - z_1}{\alpha_2 - z_1}\right)$, which implies that the point (α_1, v_1) lies between the two lines η_1 and η_2. Define dual variables $\mu_1 = \mu_2 = \frac{v_1 - v_2}{\alpha_1 - \alpha_2}$, so that the lines ℓ_1 and ℓ_2 fully overlap and pass through the points (α_2, v_2) and (α_1, v_1). These lines intersect the x-axis at $w = z^* \in [z_2, z_1]$, as illustrated in Figure 3(b).

It is easy to verify that any mechanism M feasible to Theorem 4.4 Bullet 3, together with the dual variables $\boldsymbol{\mu} = (\mu_1, \mu_2)$, satisfies Bullet 2 of Lemma 4.2. Therefore, such a mechanism M is optimal to (3), and $\boldsymbol{\mu} = (\mu_1, \mu_2)$ is an optimal dual variable. Moreover, given the optimal dual variable $\boldsymbol{\mu} = (\mu_1, \mu_2)$, a mechanism M satisfies Bullet 2 of Lemma 4.2 if and only if it meets Bullet 3 of Theorem 4.4.

Proof of Bullet 1 Suppose $v_1 \geq v_2 \cdot \frac{\alpha_1 - z_1}{\alpha_2 - z_1}$,[21] which implies that the point (α_1, v_1) lies above line η_1. We construct the receivers' lines ℓ_1 and ℓ_2 by considering two cases: (i) $\bar{z}_1 \leq \alpha_2$ and (ii) $\bar{z}_1 > \alpha_2$.

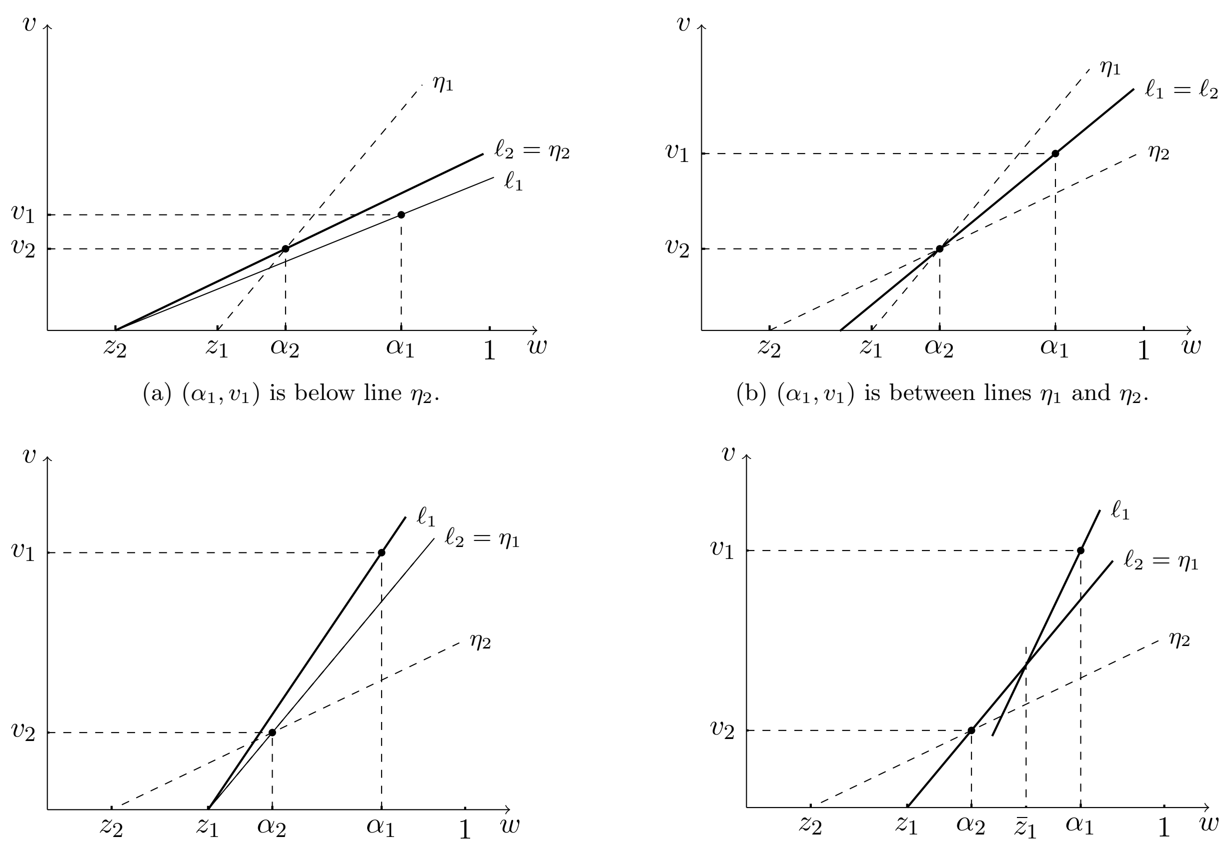
1. $z_1 = \bar{z}_1 \leq \alpha_2$: Let line ℓ_1 pass through the points $(z_1, 0)$ and (α_1, v_1) by choosing dual variable $\mu_1 = \frac{v_1}{\alpha_1 - z_1}$. Let line ℓ_2 lie below ℓ_1 for all $w \in [z_1, 1]$. For example, this can be achieved by choosing $\mu_2 = \frac{v_2}{\alpha_2 - z_1}$ – causing line ℓ_2 to coincide with line η_1 – as the point (α_2, v_2) lies below line ℓ_1. The lines ℓ_1 and ℓ_2 are illustrated in Figure 3(c).
2. $z_1 < \alpha_2 < \bar{z}_1$: Let line ℓ_2 coincide with line η_1 by choosing dual variable $\mu_2 = \frac{v_2}{\alpha_2 - z_1}$. Set line ℓ_1 to pass through the points $(\bar{z}_1, \frac{v_2}{\alpha_2 - z_1}(\bar{z}_1 - \alpha_2) + v_2)$ and (α_1, v_1) by selecting dual variable $\mu_1 = \frac{\frac{v_2}{\alpha_2 - z_1}(\bar{z}_1 - \alpha_2) + v_2 - v_1}{\bar{z}_1 - \alpha_1}$. It is easy to verify that line ℓ_1 intersects line ℓ_2 at $w = \bar{z}_1$ and that $\mu_1 > \mu_2$.[22] With this setup, line ℓ_1 lies above line ℓ_2 for $w \in [\bar{z}_1, 1]$ and line ℓ_2 lies above line ℓ_1 for $w \in [z_1, \bar{z}_1]$. The lines ℓ_1 and ℓ_2 are illustrated in Figure 3(d).

It can be easily verified that mechanism M_1 and the dual variables $\boldsymbol{\mu} = (\mu_1, \mu_2)$ satisfy Bullet 2 of Lemma 4.2 in both cases. Therefore, mechanism M_1 is optimal to problem (3), and $\boldsymbol{\mu} = (\mu_1, \mu_2)$ are optimal dual variables. Moreover, M_1 is the unique mechanism satisfying Bullet 2 of Lemma 4.2 given $\boldsymbol{\mu} = (\mu_1, \mu_2)$.

[21] If $z_1 = \bar{z}_1 \leq \alpha_2$, we set the right-hand side of the inequality to positive infinity.

[22] Intuitively, $\mu_1 > \mu_2$ because the point (α_1, v_1) lies above line η_1.

Complete Characterization of Optimal Persuasion Mechanisms Finally, we note that in all the three cases above, given the optimal dual variables $\boldsymbol{\mu} = (\mu_1, \mu_2)$ identified in each case, a mechanism M satisfies Bullet 2 of Lemma 4.2 if and only if it meets the corresponding condition described above. Therefore, by Bullet 2 of Lemma 4.2, a persuasion mechanism M is optimal to (3) if and only if it satisfies Theorem 4.4.



(a) (α_1, v_1) is below line η_2.

(b) (α_1, v_1) is between lines η_1 and η_2.

(c) (α_1, v_1) is above line η_1 – case 1.

(d) (α_1, v_1) is above line η_1 – case 2.

Figure 3: Visualization of two receivers' associated lines.

B.5 Proof of Proposition 4.5

Step One: Proving $\bar{V} \le V^{\mathrm{CR}}$ We first prove that (8) is a relaxation of (3); therefore, $\bar{V} \le V^{\mathrm{CR}}$. Specifically, let $\{q(i|w)\}$ be a feasible solution to (3). Define $q_i = \int_0^1 q(i|w)g(w)dw$ for any $i \in [n]$. We show that $\{q_i\}$ is feasible to (8). This, together with the fact that $\{q(i|w)\}$ and $\{q_i\}$ yield the same objective value, indicates that (8) is a relaxation of (3).

To show that $\{q_i\}$ is feasible to (8), first, note that $q_i \ge 0$ for any $i \in [n]$ because $q(i|w) \ge 0$ for any $i \in [n]$ and $w \in [0, 1]$. Second,

$$\sum_{i \in [n]} q_i = \sum_{i \in [n]} \int_0^1 q(i|w)\, g(w)\, dw \le \int_0^1 g(w)\, dw = 1,$$

where the inequality follows from the fact that $\sum_{i \in [n]} q(i|w) \le 1$ for any $w \in [0, 1]$.

Finally, we show $\{q_i\}$ is feasible to the first constraint in (8). To do so, let $q_{\leq k}(w) = \sum_{i \leq k} q(i|w)$ denote the probability that a good with characteristics w receives an offer from one of the top k receivers. Since $\{q(i|w)\}$ is a feasible solution to (3),

$$\alpha_i \int_0^1 q(i|w)\, g(w)\, dw \leq \int_0^1 w \cdot q(i|w)\, g(w)\, dw.$$

Summing over $i \leq k$ on both sides gives

$$\begin{aligned}
\sum_{i \leq k} \alpha_i q_i \leq & \int_0^1 w \cdot q_{\leq k}(w)\, g(w)\, dw \\
\leq & \int_{G^{-1}\left(1-\sum_{i \leq k} q_i\right)}^1 w \cdot g(w)\, dw \\
= & \mathbb{E}\left[w \cdot \mathbb{1}\left[G(w) \geq 1 - \sum_{i \leq k} q_i\right]\right] \\
= & \sum_{i \leq k} q_i \cdot \mathbb{E}\left[w \middle| G(w) \geq 1 - \sum_{i \leq k} q_i\right]
\end{aligned}$$

where the second inequality follows from the fact that $\int_0^1 q_{\leq k}(w)\, g(w)\, dw = \sum_{i \leq k} q_k$, and that the integration is maximized by taking $q_{\leq k}(w) = 1$ for all $w \geq G^{-1}\big(1 - \sum_{i \leq k} q_i\big)$ and $q_{\leq k}(w) = 0$ otherwise.

Step Two: Proving $V^{\text{CR}} \leq \bar{V}$ We next prove that $V^{\text{CR}} \leq \bar{V}$. Specifically, we show that for any feasible solution $\{q_i\}$ to (8), there exists a feasible solution $\{q(i|w)\}$ to (3) with the same objective value as $\{q_i\}$, thereby implying $V^{\text{CR}} \leq \bar{V}$.

Let $\{q_i\}$ be feasible to (8). Since the participation condition (i.e., the first constraint) of (8) holds for $k = 1$, we can find a portion q_1 of goods whose mean quality just meets the threshold value α_1 of receiver 1. In other words, we can find a function $q(1|w) \geq 0$ satisfying:

$$\begin{aligned}
&\int_0^1 q(1|w)\, g(w)\, dw = q_1, \\
&\int_0^1 w \cdot q(1|w)\, g(w)\, dw = \alpha_1 \int_0^1 q(1|w)\, g(w)\, dw.
\end{aligned}$$

Now consider the remaining portion of goods. Since the participation condition of (8) holds for $k = 2$, within the remaining portion of goods, we can find a portion q_2 of goods whose mean quality just meets the threshold value α_2 of receiver 2. In other words, we can find a function $q(2|w) \geq 0$ satisfying:

$$\begin{aligned}
&\int_0^1 q(2|w)\, g(w)\, dw = q_2, \\
&\int_0^1 w \cdot q(2|w)\, g(w)\, dw = \alpha_2 \int_0^1 q(2|w)\, g(w)\, dw, \\
&q(2|w) \leq 1 - q(1|w),\ \forall\, w \in [0, 1].
\end{aligned}$$

Repeating the process, we can find qualified portions for all receivers, resulting in a set of $\{q(i|w)\}$ that is feasible to (3). Moreover, by construction, $\{q(i|w)\}$ and $\{q_i\}$ have the same objective value.

Step Three: Wrap-Up Combining the two steps, we have $\bar{V} = V^{\text{CR}}$; that is, the optimal values of (3) and (8) are equal. Moreover, let $\{q^*(i|w)\}$ be an optimal solution to (3), and let $q_i^* = \int_0^1 q^*(i|w)g(w)dw$. Since $\{q_i^*\}$ is feasible to (8) and attains the same objective value as $\{q^*(i|w)\}$ by Step One, $\{q_i^*\}$ is optimal to (8). Conversely, if $\{q_i^*\}$ is an optimal solution to (8), then by Step Two, we can construct a feasible solution $\{q^*(i|w)\}$ to (3) satisfying $q_i^* = \int_0^1 q^*(i|w)g(w)dw$. This solution has an objective value $V^{\text{CR}} = \bar{V}$, thus is optimal to (3).

B.6 Proof of Proposition 4.6

In this section, we prove that if $\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]}$ is an optimal Lagrangian dual variable for (8), then $\{\mu_i\}$, with $\mu_i = \sum_{k\geq i} \lambda_k^*$, is an optimal Lagrangian dual variable for (3). We further characterize the receivers' lines $\ell_j(w;\mu_j)$ and the upper envelope function $h(w;\boldsymbol{\mu})$. To achieve this, we first derive the optimality conditions for (8) in Appendix B.6.1.

B.6.1 Optimality Condition for (8)

Let $\mathbf{q} = (q_i)_{i\in[n]} \in \mathbb{R}_+^n$ be a vector of allocation probabilities for the n receivers, and let $L(\mathbf{q}, \boldsymbol{\lambda}, \gamma)$ represent the Lagrangian function of (8) obtained by dualizing the participation constraints with dual variables $\boldsymbol{\lambda} = (\lambda_k)_{k\in[n]} \in \mathbb{R}_+^n$ and the constraint $\sum_{i\in[n]} q_i \leq 1$ with dual variable $\gamma \geq 0$:

$$L(\mathbf{q}, \boldsymbol{\lambda}, \gamma) = \sum_{i=1}^n v_i q_i + \sum_{k\in[n]} \lambda_k \left(\int_{1-\sum_{i\leq k} q_i}^1 G^{-1}(x)\, dx - \sum_{i\leq k} \alpha_i q_i \right) + \gamma \left(1 - \sum_{i\in[n]} q_i \right).$$

Denote by $\mathbf{q}^* = (q_i^*)_{i\in[n]} \in \mathbb{R}_+^n$ an optimal solution to (8), and $\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]} \in \mathbb{R}_+^n$ and $\gamma^* \geq 0$ optimal dual variables to (8). By the KKT conditions, the vector $\mathbf{q}^*$ solves the following Lagrangian problem:

$$\mathbf{q}^* \in \underset{\mathbf{q}\in\mathbb{R}_+^n,\ \sum_{i\in[n]} q_i \leq 1}{\operatorname{argmax}} L(\mathbf{q}, \boldsymbol{\lambda}^*, \gamma^*)\,.$$

Since $q_i^* > 0$ for all $i \in [n]$ (i.e., we consider only non-disregarded receivers), the first-order optimality conditions yield:

$$\frac{\partial L}{\partial q_i}(\mathbf{q}^*, \boldsymbol{\lambda}^*, \gamma^*) = v_i - \gamma^* + \sum_{k\geq i} \lambda_k^* \left(G^{-1}\left(1 - \sum_{j\leq k} q_j^* \right) - \alpha_i \right) = 0, \quad \forall\, i \in [n]. \tag{20}$$

Finally, Proposition B.5 presents several preliminary properties of the optimal dual variables associated with (8).

Proposition B.5. *Any optimal solution $\mathbf{q}^* \in \mathbb{R}_+^n$ and optimal dual variables $\boldsymbol{\lambda}^* \in \mathbb{R}_+^n$ and $\gamma^* \geq 0$ of (8) satisfy the following properties:*

1. *The optimal dual variable γ^* satisfies $\gamma^* \leq v_n$;*
2. *If $\sum_{i\in[n]} q_i^* < 1$, then we have $\lambda_n^* > 0$ and $\gamma^* = 0$;*
3. *If $\lambda_n^* = 0$, then we have $\sum_{i\in[n]} q_i^* = 1$ and $\gamma^* = v_n$.*

Proof. Proof of Bullet One: If $\sum_{i\in[n]} q_i^* < 1$, then we have $\gamma^* = 0$ by complementary slackness. Otherwise, suppose $\sum_{i\in[n]} q_i^* = 1$. Then, (20) with $i = n$ implies that

$$\gamma^* = v_n - \lambda_n^* \alpha_n \leq v_n,$$

where the equality follows from $G^{-1}(0) = 0$ and the inequality from $\lambda_n^* \geq 0$.

Proof of Bullet Two: If $\sum_{i\in[n]} q_i^* < 1$, complementary slackness implies $\gamma^* = 0$. Additionally, setting $i = n$ in (20) yields:

$$\lambda_n^* \left(\alpha_n - G^{-1}\left(1 - \sum_{j\in[n]} q_j^*\right)\right) = v_n > 0\,.$$

We remark that $G^{-1}\left(1 - \sum_{j\in[n]} q_j^*\right) < \alpha_n$. To see this, note that any optimal persuasion mechanism exhibits a cutoff structure, such that a good is allocated if and only if its characteristics w exceeds a threshold value $z \in [0,1]$ (see Proposition B.2), and $G^{-1}\left(1 - \sum_{j\in[n]} q_j^*\right)$ corresponds to this threshold. Therefore, we must have $G^{-1}\left(1 - \sum_{j\in[n]} q_j^*\right) < \alpha_n$, because, otherwise, receiver n's participation constraint does not bind, allowing the sender to allocate some of the remaining unassigned mass to receiver n and thereby strictly increase the sender's expected payoff.[23] Consequently, the above equality implies that $\lambda_n^* > 0$.

Proof of Bullet Three: If $\lambda_n^* = 0$, we have $\sum_{i\in[n]} q_i^* = 1$ by Bullet Two. Additionally, setting $i = n$ in (20) gives $\gamma^* = v_n$. □

In what follows, we first prove Proposition 4.6 under the assumption that $\sum_{j\in[n]} q_j^* < 1$. The proofs for the remaining cases follow a similar argument.

B.6.2 Characterization of $h(w;\boldsymbol{\mu})$ when $\sum_{j\in[n]} q_j^* < 1$

Define the dual variable $\boldsymbol{\mu} = (\mu_i)_{i\in[n]}$ by $\mu_i = \sum_{k\geq i} \lambda_k^*$ for each $i \in [n]$. In this section, we assume $\sum_{j\in[n]} q_j^* < 1$ and verify the properties of the receivers' lines $\ell_j(w;\mu_j)$ and the upper envelope function $h(w;\boldsymbol{\mu})$ in Proposition 4.6.

When $\sum_{j\in[n]} q_j^* < 1$, Bullet 2 of Proposition B.5 implies $\lambda_n^* > 0$ and $\gamma^* = 0$. Therefore, the first-order optimality condition (20) becomes

$$\frac{\partial L}{\partial q_i}(\mathbf{q}^*, \boldsymbol{\lambda}^*, \gamma^* = 0) = v_i + \sum_{k\geq i} \lambda_k^* \left(G^{-1}\left(1 - \sum_{j\leq k} q_j^*\right) - \alpha_i\right) = 0, \quad \forall\, i \in [n]. \tag{21}$$

In addition, following the notation from Section 4.4.2, let

$$T \triangleq \left\{k \in [n] : \lambda_k^* > 0\right\}$$

denote the set of indices corresponding to positive entries in the optimal dual variable $\boldsymbol{\lambda}^*$. Since $\lambda_n^* > 0$, we have $n \in T$.

Suppose $T = \{t_1 < t_2 < \cdots < t_m = n\}$ consists of m receivers. These receivers partition the set of n receivers into m groups $\{T_i\}_{i\in[m]}$, where $T_1 = [t_1]$ and $T_i = [t_{i-1} + 1 : t_i]$ for $i \in [2 : m]$.

[23] Such unallocated mass exists because $\sum_{i\in[n]} q_i^* < 1$ by assumption.

Moreover, each group T_i contains exactly one element from T, which is its largest element.

If $k \in T$, complementary slackness implies that the participation constraint in (8) is binding for the top k receivers; that is,

$$\sum_{i \le k} \alpha_i q_i^* = \mathbb{E}\Big[w \cdot \mathbb{1}\Big[w \ge G^{-1}\Big(1 - \sum_{i \le k} q_i^*\Big)\Big]\Big]. \tag{22}$$

Finally, define $z_i \triangleq G^{-1}\left(1 - \sum_{j=1}^{t_i} q_j^*\right)$ for each $i \in [m]$ and set $z_0 = 1$, and define subinterval $I_i = [z_i, z_{i-1}]$ for each $i \in [m]$.

We now verify the properties of the receivers' lines $\ell_j(w; \mu_j)$ and upper envelope function $h(w; \boldsymbol{\mu})$ characterized in Proposition 4.6.

Proof of Bullet 1 For any group T_i and element $k \in T_i$, subtracting both sides of the first constraint in (8) from both sides of (22) with $k = t_{i-1}$, and noting that the first constraint in (8) is binding with $k = t_i$, yields the following:

$$\begin{aligned} \sum_{j \in [t_{i-1}+1:k]} \alpha_j q_j^* &\le \mathbb{E}\Big[w \cdot \mathbb{1}\Big[G^{-1}\Big(1 - \sum_{j \le k} q_j^*\Big) \le w < z_{i-1}\Big]\Big], \forall\, k \in [t_{i-1}+1 : t_i - 1], \\ \sum_{j \in T_i} \alpha_j q_j^* &= \mathbb{E}\Big[w \cdot \mathbb{1}\Big[z_i \le w < z_{i-1}\Big]\Big]. \end{aligned} \tag{23}$$

Additionally, we have $\mathbb{P}[z_i \le w < z_{i-1}] = \sum_{j \in T_i} q_j^*$.

To verify $\alpha_j \in (z_i, z_{i-1})$ for all $j \in T_i$, fix receiver $k = t_{i-1}+1$. The first inequality in (23) and the positivity of q_k imply $\alpha_k < z_{i-1}$. Otherwise, no goods from the interval $\left[G^{-1}\left(1 - \sum_{j \le k} q_j^*\right), z_{i-1}\right)$ would meet the hiring threshold α_k, contradicting the inequality.

Furthermore, subtracting the equality in (23) from the inequality in (23) evaluated at $k = t_i - 1$ yields:

$$\alpha_{t_i} q_{t_i}^* \ge \mathbb{E}\Big[w \cdot \mathbb{1}\Big[z_i \le w < G^{-1}\Big(1 - \sum_{j \le t_i - 1} q_j^*\Big)\Big]\Big].$$

Positivity of q_{t_i} implies $z_i < \alpha_{t_i}$. Otherwise, the interval $\left[z_i, G^{-1}\left(1 - \sum_{j \le t_i - 1} q_j^*\right)\right)$ would contain only goods overqualified for receiver t_i and the inequality above cannot hold. Thus, we conclude that $\alpha_j \in (z_i, z_{i-1})$ for every $j \in T_i$.

We now show that within each group T_i, the lines $\ell_j(w; \mu_j)$ coincide for all $j \in T_i$ and pass through the points (α_j, v_j) for each $j \in T_i$. If set T_i contains only one receiver, the result trivially holds. Now suppose T_i contains multiple receivers (i.e., $t_{i-1} + 1 < t_i$). For any $k \in [t_{i-1}+1 : t_i - 1]$, we have:

$$\mu_k = \sum_{j \ge k} \lambda_j^* = \mu_{t_i} = \frac{v_k - v_{t_i}}{\alpha_k - \alpha_{t_i}} \tag{24}$$

where the first equality follows from the definition of $\{\mu_i\}$ and the second equality follows from the fact that $\lambda_j^* = 0$ for any $j \in [t_{i-1} + 1 : t_i - 1]$. The third equality is obtained by subtracting both sides of (20) with $i = t_i$ from both sides of the same equation with $i = k$. (24) implies that the points $\{(v_j, \alpha_j)\}_{j \in T_i}$ lie on a line, and the receivers' lines $\ell_j(w; \mu_j)$ for any $j \in T_i$ fully overlap and coincide with this line.

Proof of Bullet 2 For ease of notation, we suppress the explicit dependence on the dual variables $\boldsymbol{\mu} = (\mu_j)_{j\in[n]}$ and denote

$$\ell_j(w) \triangleq \ell_j(w;\mu_j) = v_j + \mu_j(w - \alpha_j).$$

Based on Bullet 1, it suffices to show that: (i) the line $\ell_n(w)$ intersects the x-axis at $w = z_m > 0$, and (ii) for each $i \in [m-1]$, the lines $\ell_{t_i}(w)$ and $\ell_{t_{i+1}}(w)$ intersect at $w = z_i$.

First, from (21) with $i = n$, we obtain:

$$v_n + \lambda_n^*(z_m - \alpha_n) = v_n + \mu_n(z_m - \alpha_n) = 0,$$

where the first equality follows from $\mu_n = \lambda_n^*$ by definition. Thus, line $\ell_n(w)$ intersects the x-axis at $w = z_m$.

We now prove (ii) by induction. To start, note that

$$\mu_{t_i} = \sum_{j=i}^{m} \lambda_{t_j}^*, \ \forall i \in [m] \tag{25}$$

because $\lambda_k^* = 0$ for all $k \notin T$. We first show that (ii) holds for $i = m-1$. Since line $\ell_n(w)$ passes through the point (z_{m-1}, h_{m-1}) with

$$h_{m-1} = \mu_n \cdot (z_{m-1} - z_m), \tag{26}$$

it suffices to show that line $\ell_{t_{m-1}}(w)$ also passes through (z_{m-1}, h_{m-1}). We now verify this. Specifically, taking $i = t_{m-1}$ in (21) yields:

$$v_{t_{m-1}} + \lambda_n^* \cdot (z_m - \alpha_{t_{m-1}}) + \lambda_{t_{m-1}}^* \cdot (z_{m-1} - \alpha_{t_{m-1}}) = v_{t_{m-1}} + \mu_{t_{m-1}} \cdot \left(z_{m-1} - h_{m-1}/\mu_{t_{m-1}} - \alpha_{t_{m-1}}\right) = 0$$

where the first equality follows from (25) and (26). Therefore, it follows that:

$$v_{t_{m-1}} + \mu_{t_{m-1}} \cdot (z_{m-1} - \alpha_{t_{m-1}}) = h_{m-1},$$

implying that line $\ell_{t_{m-1}}(w)$ also passes through the point (z_{m-1}, h_{m-1}).

We now assume that (ii) holds for all $j \geq i+1$ and verify that it also holds for $j = i$. Given that (ii) holds for any $j \geq i+1$, line $\ell_{t_{i+1}}(w)$ passes through the point (z_i, h_i), where

$$h_i = \sum_{j=i}^{m-1} \mu_{i+1} \cdot (z_i - z_{i+1}). \tag{27}$$

To complete the induction step, it suffices to show that line $\ell_{t_i}(w)$ also passes through (z_i, h_i). To do so, take $i = t_i$ in (21); this gives:

$$v_{t_i} + \sum_{j=i}^{m} \lambda_{t_j}^* \cdot \left(z_j - \alpha_{t_i}\right) = v_{t_i} + \mu_{t_i} \cdot \Big(z_i - h_i/\mu_{t_i} - \alpha_{t_i}\Big) = 0,$$

where the first equality follows from (27) and the identity $\lambda_{t_j}^* = \mu_{t_j} - \mu_{t_{j+1}}$ for each $j \in [m]$ (letting $\mu_{t_{m+1}} = 0$) by (25). Consequently, we have:

$$v_{t_i} + \mu_{t_i} \cdot (z_i - \alpha_{t_i}) = h_i,$$

implying that line $\ell_{t_i}(w)$ also passes through the point (z_i, h_i). Therefore, (ii) holds for $j = i$, completing the induction.

Proof of Bullet 3 Bullet 3 follows directly from Bullets 1 and 2, and from the fact that the dual variables $\{\mu_i\}$ – which represent the slopes of the lines $\ell_i(w; \mu_i)$ – decrease with index i due to the nonnegativity of $\{\lambda_k^*\}$. Note that by Bullet 3, the function $h(w; \boldsymbol{\mu})$ is nonnegative if and only if $w \geq z_m$. Therefore, $\bar{h}(w; \boldsymbol{\mu}) = h(w; \boldsymbol{\mu})$ for $w \geq z_m$, and $\bar{h}(w; \boldsymbol{\mu}) = 0$ otherwise.

B.6.3 Optimality of Dual Variable $\boldsymbol{\mu}$ when $\sum_{j\in[n]} q_j^* < 1$

In this section, we verify that $V^{\mathrm{LR}}(\boldsymbol{\mu}) = \bar{V}$ for the dual variable $\boldsymbol{\mu}$ defined in Proposition 4.6. Therefore, by strong duality (Lemma 4.2), $\boldsymbol{\mu}$ is an optimal Lagrangian dual variable of (3).

Using the characterization of the upper envelope function $h(w; \boldsymbol{\mu})$ (Bullet 2 in Proposition 4.6) and (5), a set of allocation probabilities $\{q(j|w)\}$ is optimal to $V^{\mathrm{LR}}(\boldsymbol{\mu})$ if and only if it satisfies:

$$\begin{aligned} &\sum_{j\in T_i} q(j|w) = 1, \; \forall\, w \in \big(z_i, z_{i-1}\big), \; i \in [m], \\ &\sum_{j\in[n]} q(j|w) = 0, \; \forall\, w < z_m. \end{aligned} \tag{28}$$

Moreover, by repeating the proof of Proposition 4.5 (specifically, Step Two in Appendix B.5), (23) implies that there exists an optimal solution $\{q^*(j|w)\}$ to $V^{\mathrm{LR}}(\boldsymbol{\mu})$ such that for any $i \in [m]$ and $j \in T_i$, we have:

$$\begin{aligned} \int_{w\in I_i} q^*(j|w)\, dw &= q_j^*, \\ \int_{w\in I_i} w \cdot q^*(j|w)\, g(w)\, dw &= \alpha_j \int_{w\in I_i} q^*(j|w)\, g(w)\, dw. \end{aligned} \tag{29}$$

We note that from (28), for any receiver $j \in T_i$, $q^*(j|w) = 0$ for any $w \notin I_i$. Together with (29), this implies that for any $j \in [n]$, we have:

$$\int_0^1 q^*(j|w)\, g(w)\, dw = q_j^*, \tag{30}$$

$$\int_0^1 w \cdot q^*(j|w)\, g(w)\, dw = \alpha_j \int_0^1 q^*(j|w)\, g(w)\, dw. \tag{31}$$

Therefore,

$$\begin{aligned} V^{\mathrm{LR}}(\boldsymbol{\mu}) &= \int_0^1 \sum_{j=1}^n \Big\{ v_j + \mu_j\big(w - \alpha_j\big) \Big\}\, q^*(j|w)\, g(w)\, dw \\ &= \sum_{j=1}^n v_j \int_0^1 q^*(j|w)\, g(w)\, dw \\ &= \sum_{j=1}^n v_j \cdot q_j^* = V^{\mathrm{CR}} = \bar{V}, \end{aligned} \tag{32}$$

where the first equality follows from the fact that $\{q^*(j|w)\}$ is optimal to $V^{\mathrm{LR}}(\boldsymbol{\mu})$, the second from

(31), the third from (30), the fourth from the optimality of $\{q_j^*\}$ to (8), and the final one from Proposition 4.5.

B.6.4 Case Two: $\lambda_n^* > 0$ and $\sum_{j\in[n]} q_j^* = 1$

When $\lambda_n^* > 0$, we have $n \in T$. Suppose $T = \{t_1 < t_2 < \cdots < t_m = n\}$ consists of m receivers. Since $\sum_{j\in[n]} q_j^* = 1$, it follows that $z_m = G^{-1}\left(1 - \sum_{j=1}^{n} q_j^*\right) = 0$.

Note that the general first-order optimality condition (20) reduces to the simpler condition (21) if we define modified offer values $v_i' = v_i - \gamma^*$ for all $i \in [n]$. Since the results in Appendix B.6.2 are derived based on (21), the properties of the receivers' lines $\ell_j(w; \mu_j)$ and the upper envelope function $h(w; \boldsymbol{\mu})$ remain valid with the modified values $\{v_i'\}$.

When transforming back from $\{v_i'\}$ to the original offer values $\{v_i\}$, the receivers' lines $\ell_j(w; \mu_j)$ and the upper envelope function $h(w; \boldsymbol{\mu})$ uniformly shift upward by γ^*. Consequently, the lines $\ell_j(w; \mu_j)$ for all $j \in T_m$ and the envelope function $h(w; \boldsymbol{\mu})$ intersect the y-axis at $\gamma^* \in [0, v_n]$.

Finally, the dual variable $\boldsymbol{\mu}$ constructed in Proposition 4.6 an optimal dual variable of (3), following the same proof in Appendix B.6.3.

B.6.5 Case Three: $\lambda_n^* = 0$

Suppose Assumption 4.5 holds and $\lambda_n^* = 0$. Then Bullet 3 of Proposition B.5 implies $\sum_{j\in[n]} q_j^* = 1$ and $\gamma^* = v_n$. In this case, Lemma B.6 shows that $\lambda_{n-1}^* > 0$, implying $n - 1 \in T$.

Lemma B.6. *Suppose Assumption 4.5 holds and $\lambda_n^* = 0$. Then, it follows that $\lambda_{n-1}^* > 0$.*

Proof. Taking $i = n - 1$ in (20) and noting that $\lambda_n^* = 0$ and $\gamma^* = v_n$, we have:

$$\lambda_{n-1}^* \left(\alpha_{n-1} - G^{-1}\left(1 - \sum_{j\le n-1} q_j^*\right)\right) = v_{n-1} - \gamma^* = v_{n-1} - v_n > 0\,.$$

Given that $\lambda_{n-1}^* \ge 0$, it follows that $\lambda_{n-1}^* > 0$ and $\alpha_{n-1} > G^{-1}\left(1 - \sum_{j\le n-1} q_j^*\right)$. ☐

Suppose $T = \{t_1 < t_2 < \cdots < t_m = n-1\}$ includes m receivers. These receivers partition the set of n receivers into $m+1$ groups $\{T_i\}$, where $T_1 = [t_1]$, $T_i = [t_{i-1}+1 : t_i]$ for all $i \in [2:m]$, and $T_{m+1} = \{n\}$. Define the endpoints $z_i \triangleq G^{-1}\left(1 - \sum_{j=1}^{t_i} q_j^*\right)$ for each $i \in [m]$, and set $z_0 = 1$ and $z_{m+1} = 0$.

Applying the same approach as in Appendix B.6.4 – that is, reducing the general first-order optimality condition (20) to (21) by modifying offer values to $v_i' = v_i - \gamma^*$ for all $i \in [n]$ – we can verify that the properties of the receivers' lines $\ell_j(w; \mu_j)$ and the envelope function $h(w; \boldsymbol{\mu})$ in Proposition 4.6 remain valid, with an additional feature that $h(w; \boldsymbol{\mu}) = \gamma^* = v_n > 0$ is constant (horizontal) on the interval $[0, z_m]$.

Moreover, the dual variable $\boldsymbol{\mu}$ constructed in Proposition 4.6 continues to be optimal for (3). The proof follows the same proof in Appendix B.6.3, with one modification: in (32), the second equality utilizes (31) and the fact that $\mu_n = \lambda_n^* = 0$.

Finally, note that in this case, we have $\mu_n = \lambda_n^* = 0$ and $\mu_i \ge \lambda_{n-1}^* > 0$ for any $i \le n-1$. Therefore, the participation constraints for the first $n-1$ receivers are binding, whereas receiver n's participation constraint may not bind. This aligns with Proposition B.3.

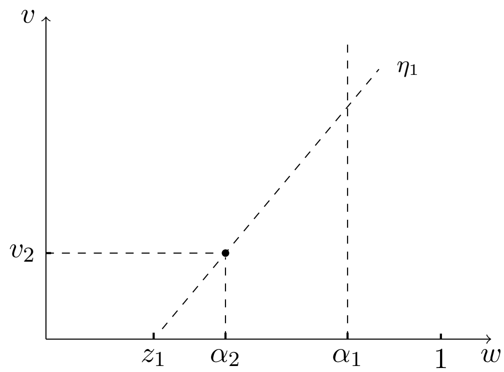


Figure 4: Visualization of the partition in Theorem C.1.

C Optimal Mechanism for Two Receivers: Other Cases

In this section, we consider two receivers $i \in \{1, 2\}$, with offer values $v_1 > v_2 > 0$ and hiring thresholds $\alpha_1 > \alpha_2 > 0$. We derive the optimal public persuasion mechanisms for scenarios not covered in Section 4.3.

We begin by specifying two trivial scenarios and add assumptions to exclude them. Let $w_0 \triangleq \mathbb{E}_{w\sim G(w)}[w]$ denote the prior mean of the good's characteristics w. First, we assume receiver 1 is selective; that is, $\alpha_1 > w_0$. Otherwise, the optimal mechanism is trivial, as the sender can allocate the good to receiver 1 without revealing any information. Next, define $\bar{z}_1 > 0$ such that $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$.[24] We further assume that $\mathbb{E}[w|w < \bar{z}_1] < \alpha_2$. Otherwise, the optimal mechanism is straightforward: allocate goods with quality $w \in [\bar{z}_1, 1]$ to receiver 1 and those with quality $w \in [0, \bar{z}_1]$ to receiver 2. Finally, we assume receiver 2 is non-selective, meaning $\alpha_2 \leq w_0$, which corresponds to the scenario not covered in Section 4.3 (see Assumption 4.4). We summarize these assumptions in Assumption C.1 and impose them throughout this section.

Assumption C.1. Let $w_0 \triangleq \mathbb{E}_{w\sim G(w)}[w]$ denote the prior mean of the good's characteristics w. Receiver 1's threshold value satisfies $\alpha_1 > w_0$. Moreover, define $\bar{z}_1 > 0$ such that $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$. Receiver 2's threshold value satisfies $\mathbb{E}[w|w < \bar{z}_1] < \alpha_2 \leq w_0$.

We now characterize the set of optimal persuasion mechanisms under Assumption C.1. These mechanisms resemble those described in Section 4.3.2, except that they either prioritize receiver 1 (Case 1 of Theorem 4.4) or balance between the two receivers as described in Proposition 4.3 (Case 3 of Theorem 4.4), but never exclusively target receiver 2 (Case 2 of Theorem 4.4).

Specifically, following the notation in Section 4.3.2, define line η_1 as passing through the points $(z_1, 0)$ and (α_2, v_2), where z_1 is the threshold value of mechanism M_1 defined in Section 4.3.1.[25] Line η_1 partitions the value of $v_1 \in [v_2, \infty)$ into two regions, as illustrated in Figure 4. If the value of v_1 is sufficiently large (in particular, above line η_1), prioritizing receiver 1 is optimal. Otherwise, the optimal mechanism balances between the two receivers as described in Proposition 4.3. We characterize these optimal persuasion mechanisms in Theorem C.1.

Theorem C.1. *Under Assumptions 4.1 – 4.3 and C.1, the optimal public persuasion mechanism for two receivers is characterized as follows.*

[24]Note that $\bar{z}_1 > 0$ follows directly from the assumption $\alpha_1 > w_0$.
[25]We have $z_1 > 0$ by Assumption C.1.

1. *If $v_1 \geq v_2 \cdot \frac{\alpha_1 - z_1}{\alpha_2 - z_1}$ (i.e., the point (α_1, v_1) lies above line η_1), then mechanism M_1, defined in Section 4.3.1, which prioritizes receiver 1, is the unique optimal mechanism.*

2. *Otherwise, define line ℓ as the line passing through points (α_2, v_2) and (α_1, v_1). let z^* be the x-intercept of line ℓ if it intersects the x-axis; otherwise, let $z^* = 0$ (in this case, line ℓ intersects the y-axis with an intercept in $[0, v_2]$). Any mechanism M satisfying Proposition 4.3 with cutoff value z^* is optimal. Furthermore, this fully characterizes the set of optimal public persuasion mechanisms.*

We prove Theorem C.1 in Appendix C.1. The proof mimics the proof of Theorem 4.4: we identify a set of dual variables $\boldsymbol{\mu} \in \mathbb{R}_+^n$, which, together with the proposed mechanism, satisfy Bullet 2 of Lemma 4.2. This indicates that the mechanism is optimal to (3), and $\boldsymbol{\mu}$ is an optimal dual variable.

C.1 Proof of Theorem C.1

In this proof, we identify a set of dual variables $\boldsymbol{\mu} \in \mathbb{R}_+^n$, which, together with the mechanism proposed in Theorem C.1, satisfy Bullet 2 of Lemma 4.2. This indicates that the mechanism is optimal to (3), and $\boldsymbol{\mu}$ is an optimal dual variable.

Proof of Bullet 2 Suppose $v_1 \leq v_2 \cdot \frac{\alpha_1 - z_1}{\alpha_2 - z_1}$,[26] which implies that the point (α_1, v_1) lies below line η_1. Define dual variables $\mu_1 = \mu_2 = \frac{v_1 - v_2}{\alpha_1 - \alpha_2} > 0$. Consequently, the two receivers' lines ℓ_1 and ℓ_2 coincide with line ℓ and pass through points (α_2, v_2) and (α_1, v_1), as illustrated in Figure 3(b) (note that line ℓ may either intersects the x-axis or intersects the y-axis below v_2 under Assumption C.1).

It is easy to verify that any mechanism M feasible to Bullet 2 of Theorem C.1, together with the dual variables $\boldsymbol{\mu} = (\mu_1, \mu_2)$ defined above, satisfies Bullet 2 of Lemma 4.2. Therefore, such a mechanism M is optimal to (3), and $\boldsymbol{\mu} = (\mu_1, \mu_2)$ is an optimal dual variable. Moreover, given the optimal dual variable $\boldsymbol{\mu} = (\mu_1, \mu_2)$, a mechanism M satisfies Bullet 2 of Lemma 4.2 if and only if it meets Bullet 2 of Theorem C.1.

Proof of Bullet 1 Suppose $v_1 \geq v_2 \cdot \frac{\alpha_1 - z_1}{\alpha_2 - z_1}$, which implies that the point (α_1, v_1) lies above line η_1. The proof is identical to the proof of Bullet 1 of Theorem 4.4, presented in Appendix B.4.

Complete Characterization of Optimal Persuasion Mechanisms Finally, we note that given the optimal dual variables $\boldsymbol{\mu} = (\mu_1, \mu_2)$ identified in both cases, a mechanism M satisfies Bullet 2 of Lemma 4.2 if and only if it satisfies Theorem C.1. Therefore, by Bullet 2 of Lemma 4.2, a persuasion mechanism M is optimal to (3) if and only if it satisfies Theorem C.1.

D Constructing Optimal Mechanisms with Specific Structure

As discussed in Section 4.4.3, the general persuasion problem with n receivers decouples over subsets of receivers $\{T_i\}$, and we can build an optimal solution in an iterative way for each subset. Moreover, there exist multiple ways to construct an optimal mechanism, with the set of optimal persuasion

[26]If $z_1 = \bar{z}_1 = \alpha_2$, we set the right-hand side of the inequality to positive infinity.

mechanisms characterized by Theorem 4.7. In this section, we propose a specific allocation approach at each iteration step to obtain an optimal solution $\{q^*(j|w)\}$ to (3) that has a monotone structure (Appendix D.1). Additionally, we show that a deterministic persuasion mechanism exhibiting a double-interval structure, as described in Candogan (2022), can be easily derived using results derived from our dual approach (Appendix D.2).

D.1 Optimal Mechanism with Monotone Structure

In this section, we construct an optimal persuasion mechanism $\{q^*(j|w)\}$ iteratively that additionally satisfies a monotone property, as defined in Definition D.1. Specifically, for any $w \geq w'$, the distribution $q^*(\cdot|w)$ will first-order stochastically dominate the distribution $q^*(\cdot|w')$. Therefore, a good with a higher quality w is more likely to be in a better place, which is desirable in practice.[27,28]

Definition D.1 (Monotone Structure)**.** An optimal persuasion mechanism $\{q^*(j|w)\}$ satisfies a monotone property if, for any $w \geq w'$, the distribution $q^*(\cdot|w)$ first-order stochastically dominates the distribution $q^*(\cdot|w')$; in other words, we have $\sum_{k\leq i} q^*(k|w) \geq \sum_{k\leq i} q^*(k|w')$ for any $i \in [n]$.[29]

The monotone property automatically holds for two qualities w and w' from different intervals. Suppose $w \in I_i$ and $w' \in I_j$ with $i < j$. Since $\max T_i < \min T_j$, a good of quality w joins a better place for sure, which implies first-order stochastic dominance. Therefore, we only need to ensure the monotone structure for qualities within the same interval.

Algorithm 1 presents a way to construct an optimal mechanism $\{q(j|w)\}$ for $j \in T_i$ iteratively. The distribution $q(\cdot|w)$ from Algorithm 1 is first-order stochastically increasing in w; additionally, $q(j|w)$ is piecewise constant on $w \in I_i$ for any $j \in T_i$.

Algorithm 1: Optimal Persuasion Mechanism with Monotone Structure

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Input: An optimal solution $\{q_i^*\}$ to (8).

Initialization: Set $b_{t_{i-1}} = z_{i-1}$ and initialize $q_{\leq t_{i-1}}(w) = 0$ for all $w \in I_i$.

for $k \in T_i = [t_{i-1}+1 : t_i - 1]$ **do**

  Determine values $b_k \in [z_i, b_{k-1}]$ and $\rho_k \in [0,1]$ to satisfy condition (11) for receiver k by setting:

$$q(k|w) = \begin{cases} \rho_k \cdot (1 - q_{\leq k-1}(w))\,, & \text{for all } w \in [b_k, z_{i-1}] \\ 0, & \text{for all } w < b_k \end{cases}$$

  Set $q_{\leq k}(w) = q_{\leq k-1}(w) + q(k|w)$ for all $w \in I_i$.

end

Set $q(t_i|w) = 1 - q_{\leq t_i - 1}(w)$ for all $w \in I_i$ and define $b_{t_i} = z_i$.

In Algorithm 1, $q_{\leq k}(w) = \sum_{j\leq k} q(j|w)$ represents the probability that a good of quality w receives an offer from one of the top k receivers. Note that for any $k \leq t_{i-1}$ and $w \in I_i$, we have $q_{\leq k}(w) = 0$. In each iteration, we allocate a fraction ρ_k of the remaining goods with quality at least b_k to receiver k. The values of ρ_k and b_k are selected to ensure that receiver k receives the good

[27] For instance, in the student promotion context, this monotone property prevents students from strategically degrading their "quality" w for better positions.

[28] We note that Arieli et al. (2023) also construct optimal persuasion mechanisms with a monotone structure when the distribution of posterior means exhibits a bi-pooling structure (Lemma 2 therein). Our construction is slightly more general by allowing for any distribution of posterior means sustained by an optimal persuasion mechanism.

[29] Note that an offer from a lower-indexed receiver provides a higher payoff to the sender by Assumption 2.1.

with probability q_k^*, and that the expected quality of the good, conditional on being allocated to receiver k, is exactly α_k (i.e., (11) holds for receiver k).

We note that the constructed sequence $\{b_k\}_{k \in T_i}$ decreases and partitions the interval I_i into subintervals $I_{ik} \triangleq [b_k, b_{k-1}]$ for $k \in T_i$. Additionally, the probabilities $q(j|w)$ equal a constant $q_j(k)$ on each subinterval $w \in I_{ik}$, where the values of $q_j(k)$ are specified as follows:

$$q(j|w) = q_j(k) = \begin{cases} 0 & k \geq j+1,\, w \in I_{ik} \\ \rho_j \cdot \Big(1 - q_{\leq j-1}(w)\Big) = \rho_j \cdot \prod_{\ell=k}^{j-1} \Big(1 - \rho_\ell\Big) & k \in [t_{i-1}+1 : j],\, w \in I_{ik} \end{cases}$$

Proposition D.1 demonstrates that the values of $\{\rho_k\}$ and $\{b_k\}$ in Algorithm 1 exist, and the allocation $\{q(j|w)\}$ returned by Algorithm 1 is optimal to (3) and satisfies the first-order stochastic increasing property.

Proposition D.1. *The allocation $\{q(j|w)\}$ returned by Algorithm 1 is optimal to (3) and satisfies the first-order stochastic increasing property.*

We prove Proposition D.1 and demonstrate that the values of $\{\rho_k\}$ and $\{b_k\}$ can be easily identified in Appendix D.3. Note that when a group T_i contains two receivers, the allocation $\{q(j|w)\}$ returned by Algorithm 1 concurs with the randomized mechanism with a monotone structure described in Section 4.3.2 for the two-receiver case.

D.2 Optimal Mechanism with Double-Interval Structure

In this section, we demonstrate that a deterministic persuasion mechanism with a double-interval structure, as described in Candogan (2022), can be easily derived using results from our dual analysis.

We first present two useful properties of the optimal solutions to (8) as established in Candogan (2022). Specifically, there exists an optimal solution $\{q_k^*\}$ in which each group T_i contains at most two receivers with a positive probability of q_k^*. Additionally, the optimal solution to (8) is unique if no three points of $\{(\alpha_i, v_i)\}_{i \in [n]}$ are collinear. We state these properties in Proposition D.2.

Proposition D.2. *The optimal solution of (8) satisfies the following two properties.*

1. *(Lemma 4 of Candogan 2022) Let $\{\lambda_k^*\}$ denote an optimal dual variable associated with the participation constraints in (8) and $\{T_i\}$ denote the corresponding partition of the n receivers as described in Section 4.4.2. There exists an optimal solution $\{q_k^*\}$ to (8) such that $|T_i \cap P| \leq 2$ for any i, where $P \triangleq \{k \in [n] : q_k^* > 0\}$ denote the set of positive entries of $\{q_k^*\}$. In other words, each set T_i contains at most two receivers with a positive probability of q_k^*.*
2. *(Appendix D of Candogan 2022) Problem (8) has a unique optimal solution $\{q_k^*\}$ if no three points of $\{(\alpha_i, v_i)\}_{i \in [n]}$ are collinear.*

We prove Proposition D.2 in Appendix D.4 based on results from our dual approach, which significantly simplifies the proof and makes both properties intuitive. For the first property, suppose a group T_i contains more than two receivers with positive probabilities. Since the points $\{(\alpha_j, v_j)\}_{j \in T_i}$ are collinear by Bullet 2 of Proposition 4.6, we can reallocate the probabilities of two non-adjacent receivers to an intermediate receiver until we drain the probability of one of the two original receivers, without changing the objective value. For the second property, since no three points of $\{(\alpha_i, v_i)\}_{i \in [n]}$ are collinear, Bullet 2 of Proposition 4.6 implies that any group T_i contains at most two receivers with positive probabilities. Furthermore, the values of these probabilities are uniquely determined by two linear equations analogous to (7). We provide more details in Appendix D.4.

Proposition D.2 implies that the general information design problem can be decomposed into separate design problems with two receivers, one for each group T_i. Applying the deterministic mechanism with a double-interval structure described in Section 4.3.2 to each group T_i, we obtain the deterministic persuasion mechanism described in Candogan (2022).

D.3 Proof of Proposition D.1

In this section, we first prove that the values of $\{b_k\}$ and $\{\rho_k\}$ in Algorithm 1 exist and can be identified efficiently (Appendix D.3.1). We then show that the assignment probability $q(j|w)$ returned from Algorithm 1 is optimal to (3) and possesses the first-order stochastically increasing property (Appendix D.3.2).

D.3.1 Existence of $\{b_k\}$ and $\{\rho_k\}$

We prove by induction that the values of $\{\rho_k\}$ and $\{b_k\}$ in Algorithm 1 exist and can be computed efficiently.

Induction Step We first determine the values of $b_{t_{i-1}+1}$ and $\rho_{t_{i-1}+1}$. From (10), the following hold:

$$\mathbb{E}\left[w\middle|G^{-1}\left(1-\sum_{j\le t_{i-1}+1} q_j^*\right)\le w< z_{i-1}\right]\ge \alpha_{t_{i-1}+1}\,,$$
$$\mathbb{E}\Big[w\Big|z_i\le w<z_{i-1}\Big]=\sum_{j\in T_i}\alpha_j\cdot\frac{q_j^*}{\sum_{j\in T_i}q_j^*}\le \alpha_{t_{i-1}+1}\,,$$

where the inequality in the second line follows from the fact that $\alpha_{t_{i-1}+1}\ge\alpha_j$ for any $j\in T_i$. Therefore, there exists a value of $b_{t_{i-1}+1}$ satisfying that $z_i\le b_{t_{i-1}+1}\le G^{-1}\big(1-\sum_{j\le t_{i-1}+1}q_j^*\big)\le z_{i-1}$ such that $\mathbb{E}\big[w\big|b_{t_{i-1}+1}\le w\le z_{i-1}\big]=\alpha_{t_{i-1}+1}$. Additionally, let $\rho_{t_{i-1}+1}=q^*_{t_{i-1}+1}\,/\,\mathbb{P}[b_{t_{i-1}+1}\le w< z_{i-1}]\le 1$. The allocation probability $q(t_{i-1}+1|w)$ satisfies (11) by the setup of $b_{t_{i-1}+1}$ and $\rho_{t_{i-1}+1}$.

Iteration Step Let k be an integer with $k\in[t_{i-1}+2{:}t_i-1]$. Suppose that, for all $j\in[t_{i-1}+1{:}k-1]$, we have already determined the values of ρ_j and b_j such that the probability $q(j|w)$ satisfies (11). We now identify values of ρ_k and b_k so that the probability $q(k|w)$ also satisfies (11).

To achieve this, set $b_k=b\in[z_i,z_{i-1}]$ and $\rho_k=\rho\in[\underline{\rho}_k,1]$, where

$$\underline{\rho}_k\triangleq\frac{q_k^*}{\sum_{\ell=k}^{t_i}q_\ell^*}.$$

Additionally, define the allocation probability as

$$q(k|w)=\begin{cases}\rho\cdot\big(1-q_{\le k-1}(w)\big), & w\in[b,z_{i-1}],\\ 0, & w\in[z_i,b).\end{cases}$$

Define the following two functions:

$$F(b,\rho) \triangleq \int_{w\in I_i} q(k|w)g(w)\,dw,$$
$$Q(b,\rho) \triangleq \int_{w\in I_i} w\cdot q(k|w)g(w)\,dw.$$

The allocation probability $q(k|w)$ satisfies (11) with $b_k = b$ and $\rho_k = \rho$ if and only if:

$$F(b,\rho) = q_k^* \quad \text{and} \quad Q(b,\rho) = \alpha_k q_k^*.$$

Evidently, function $F(b,\rho)$ is strictly increasing in ρ and strictly decreasing in b. Therefore, for any $\rho \in [\underline{\rho}_k, 1]$, there exists a unique constant, denoted by $b(\rho)$, that satisfies $F\big(b(\rho),\rho\big) = q_k^*$. Specifically, we have $b\big(\underline{\rho}_k\big) = z_i$, because:

$$\begin{aligned}
F\left(z_i,\underline{\rho}_k\right) &= \underline{\rho}_k \int_{w\in I_i} \big(1 - q_{\le k-1}(w)\big)\, g(w)\,dw \\
&= \underline{\rho}_k \left(\int_{w\in I_i} g(w)\,dw - \sum_{j=t_{i-1}+1}^{k-1} \int_{w\in I_i} q(j|w)\, g(w)\,dw \right) \\
&= q_k^*,
\end{aligned}$$

where the first equality follows from the definition of $q(k|w)$, and the third from the facts that $\mathbb{P}[z_i \le w < z_{i-1}] = \sum_{j\in T_i} q_j^*$ and that the probability $q(j|w)$ satisfies (11) for any $j \le k-1$, and the definition of $\underline{\rho}_k$. Moreover, the function $b(\rho)$ is strictly increasing with ρ. Thus, its inverse, denoted by $\rho(b)$, exists and is strictly increasing. Now, define the function:

$$Q(b) \triangleq Q\big(b,\rho(b)\big).$$

Since $F\big(b,\rho(b)\big) = q_k^*$ for any b, it suffices to find a value $b \in [z_i, b_{k-1}]$ satisfying $Q(b) = \alpha_k q_k^*$, which we do now.

First, note that function $Q(b)$ is increasing. This is because, as b increases, we transport a fixed mass q_k^* to higher values, which increases the mean quality of the goods allocated.

Second, we inspect the value of $Q(z_i)$. Specifically, the following holds:

$$\begin{aligned}
Q(z_i) &= \underline{\rho}_k \int_{w\in I_i} w\left(1 - q_{\le k-1}(w)\right) g(w)\,dw \\
&= \underline{\rho}_k \left(\int_{w\in I_i} w\, g(w)\,dw - \sum_{j=t_{i-1}+1}^{k-1} \int_{w\in I_i} w\cdot q(j|w)\, g(w)\,dw \right) \\
&= \frac{q_k^*}{\sum_{\ell=k}^{t_i} q_\ell^*} \left(\sum_{\ell\in T_i} \alpha_\ell q_\ell^* - \sum_{\ell=t_{i-1}+1}^{k-1} \alpha_\ell q_\ell^* \right) \\
&= \frac{\sum_{\ell=k}^{t_i} \alpha_\ell q_\ell^*}{\sum_{\ell=k}^{t_i} q_\ell^*} \cdot q_k^* \\
&\le \alpha_k q_k^*,
\end{aligned} \tag{33}$$

where the first equality uses $\rho(z_i) = \underline{\rho}_k$, the third equality follows from the second line of (10) and that probability $q(j|w)$ satisfies (11) for all $j \leq k-1$, and the inequality follows from the fact that α_ℓ decreases with index ℓ.

Finally, we derive two additional inequalities. If $b(1) \geq b_{k-1}$ (in other words, the "unoccupied" area to the right of b_{k-1} and above the function $q_{\leq k-1}(w)$ exceeds q_k^*), we have:

$$Q\big(b_{k-1}\big) = \alpha_{k-1} q_k^* > \alpha_k q_k^* \,, \tag{34}$$

because, in this case, $q(k|w) = c \cdot q(k-1|w)$ for some constant $c > 0$ and all $w \in I_i$.

Alternatively, suppose $b(1) \leq b_{k-1}$. Then, we have $q_{\leq k}(w) = 1$ for $w \in [b(1), z_{i-1}]$ and $q_{\leq k}(w) = 0$ for $w < b(1)$, which implies that $b(1) = G^{-1}\Big(1 - \sum_{j \leq k} q_j^*\Big)$. Consequently, the following holds:

$$\begin{aligned}
Q\big(b(1)\big) &= \int_{w \in I_i} w \cdot q_{\leq k}(w)\, g(w)\, dw - \int_{w \in I_i} w \cdot q_{\leq k-1}(w)\, g(w)\, dw \\
&= \int_{b(1)}^{z_{i-1}} w \cdot q_{\leq k}(w)\, g(w)\, dw - \sum_{j=t_{i-1}+1}^{k-1} \int_{w \in I_i} w \cdot q(j|w)\, g(w)\, dw \\
&= \mathbb{E}\Big[w \cdot \mathbb{1}\Big[G^{-1}\Big(1 - \sum_{j \leq k} q_j^*\Big) \leq w < z_{i-1}\Big]\Big] - \sum_{j=t_{i-1}+1}^{k-1} \int_{w \in I_i} w \cdot q(j|w)\, g(w)\, dw \\
&\geq \sum_{j=t_{i-1}+1}^{k} \alpha_j q_j^* - \sum_{j=t_{i-1}+1}^{k-1} \alpha_j q_j^* \\
&= \alpha_k\, q_k^* \,,
\end{aligned} \tag{35}$$

where the inequality follows from the first equation in (10) and the fact that the probability $q(j|w)$ satisfies (11) for all $j \leq k-1$.

Since $Q(b)$ is continuous and strictly increasing in b, inequalities (33)–(35) guarantee the existence of a value $b_k \in [z_i, \min\{b_{k-1}, b(1)\}]$ such that $Q(b_k) = \alpha_k q_k^*$. Moreover, this value b_k can be efficiently determined using binary search. Let $\rho_k = \rho(b_k)$. The resulting allocation probability $q(k|w)$ satisfies (11) with these choices of b_k and ρ_k.

Final Step Let $q(t_i|w) = 1 - q_{\leq t_i - 1}(w)$ for any $w \in I_i$. Since $q(j|w)$ satisfies (11) for any $j \leq t_i - 1$, the second equation in (10) and the fact that $\mathbb{P}[z_i \leq w < z_{i-1}] = \sum_{j \in T_i} q_j^*$ imply that $q(t_i|w)$ also satisfies (11).

D.3.2 Optimality and FOSD Property

Let $\{q(j|w)\}$ denote the output of Algorithm 1. $\{q(j|w)\}$ is optimal to (3) according to Theorem 4.7.

We now prove that the distribution $q(\cdot|w)$ first-order stochastically increases with w on the interval I_i. By definition, this is equivalent to proving that the cumulative distribution function $q_{\leq k}(w)$ is increasing in w for any $k \in T_i$. We prove this by induction. First, $q_{\leq t_{i-1}}(w) = 0$ for any $w \in I_i$ by definition, which serves as the induction step. Next, suppose $q_{\leq k-1}(w)$ is increasing on $w \in I_i$ for some $k \in T_i$, we show that $q_{\leq k}(w)$ is also increasing. To do so, fix two points $w, w' \in I_i$ with $w' < w$. If $w' < b_k$, we have:

$$0 = q_{\leq k}(w') = q_{\leq k-1}(w') \leq q_{\leq k-1}(w) \leq q_{\leq k}(w),$$

where the first inequality is because $q_{\le k-1}(w)$ increases with w. Alternatively, if $w' \ge b_k$, we have:

$$\begin{aligned}
q_{\le k}(w') &= q_{\le k-1}(w') + \rho_k \cdot \big(1 - q_{\le k-1}(w')\big) \\
&= q_{\le k-1}(w') + \rho_k \cdot \big(q_{\le k-1}(w) - q_{\le k-1}(w')\big) + \rho_k \cdot \big(1 - q_{\le k-1}(w)\big) \\
&\le q_{\le k-1}(w') + q_{\le k-1}(w) - q_{\le k-1}(w') + \rho_k \cdot \big(1 - q_{\le k-1}(w)\big) \\
&= q_{\le k-1}(w) + \rho_k \cdot \big(1 - q_{\le k-1}(w)\big) \\
&= q_{\le k}(w),
\end{aligned}$$

where the inequality follows from the fact that $q_{\le k-1}(w) \ge q_{\le k-1}(w')$ and $\rho_k \le 1$.

D.4 Proof of Proposition D.2

D.4.1 Proof of Bullet 1

Let $\{\lambda_k^*\}$ denote an optimal dual variable for the participation constraints in (8) and $\{T_i\}$ denote the resulting partition of the n receivers as described in Section 4.4.2. For a feasible solution $\{q_k\}$ to (8), let

$$T_i\big(\{q_k\}\big) \triangleq \Big|T_i \cap \{k \in [n] : q_k > 0\}\Big|$$

denote the number of receivers in group T_i that have a positive probability q_k.

Let $\{q_k^*\}$ denote an optimal solution to (8). Lemma D.3 shows that if there exists a group T_i that satisfies $T_i\big(\{q_k^*\}\big) > 2$, we can find a new optimal solution $\{\tilde{q}_k\}$ to (8) that is closer to the desired one in Bullet 1.

Lemma D.3. *Let $\{q_k^*\}$ denote an optimal solution $\{q_k^*\}$ to (8). If there exists a subset T_i that satisfies $T_i\big(\{q_k^*\}\big) > 2$, we can find a new optimal solution $\{\tilde{q}_k\}$ to (8) such that (i) $\tilde{q}_k = q_k^*$ for any $k \notin T_i$, and (ii) $T_i\big(\{\tilde{q}_k\}\big) < T_i\big(\{q_k^*\}\big)$.*

Repeating the process in Lemma D.3 iteratively will eventually (in at most n steps) yields a desired optimal solution to (8) that satisfies Bullet 1.

Proof of Lemma D.3. From Proposition 4.5, there exists an optimal solution $\{q^*(j|w)\}$ to (3) such that the good is allocated to each receiver j with probability q_j^*. Suppose $T_i\big(\{q_k^*\}\big) > 2$. In the following, we modify $\{q^*(j|w)\}$ to create a new optimal solution $\{\tilde{q}(j|w)\}$ to (3) such that the good is allocated to each receiver j with probability $\tilde{q}_j$, where $\{\tilde{q}_j\}$ satisfies Lemma D.3. Then, $\{\tilde{q}_j\}$ is optimal to (8) again according to Proposition 4.5.

Assume $\{a, b, c\} \subseteq T_i\big(\{q_k^*\}\big)$, where a, b, and c denote indices of three distinct receivers. Without loss of generality, assume that $1 \le a < b < c \le n$. Therefore, $\alpha_a > \alpha_b > \alpha_c$. We consider the following two scenarios.

Case One Suppose

$$\frac{\alpha_a q_a^* + \alpha_c q_c^*}{q_a^* + q_c^*} = \alpha_b, \tag{36}$$

that is, the mean quality of the goods allocated to receivers a or c is precisely α_b, the acceptance bar of receiver b. Let

$$\tilde{q}(j|w) = \begin{cases} q^*(a|w) + q^*(b|w) + q^*(c|w) & \text{if } j = b, \\ 0 & \text{if } j \in a, c, \\ q^*(j|w) & \text{if } j \notin \{a, b, c\}. \end{cases}$$

(36) implies that the participation constraint for receiver b remains binding with $\tilde{q}(j|w)$. Therefore, $\tilde{q}(j|w)$ is optimal to (3) according to Theorem 4.7. Additionally, we have

$$\tilde{q}_j \triangleq \int_0^1 \tilde{q}(j|w)\, g(w)\, dw = \begin{cases} q_b^* + q_a^* + q_c^* & \text{if } j = b, \\ 0 & \text{if } j \in a, c, \\ q_j^* & \text{if } j \notin \{a, b, c\}. \end{cases}$$

As a result, $\{\tilde{q}_j\}$ satisfies Lemma D.3 because $\{\tilde{q}_j\}$ is optimal to (8) by Proposition 4.5 and $T_i(\{\tilde{q}_j\}) = T_i(\{q_j^*\}) - 2 < T_i(\{q_j^*\})$ by construction.

Case Two Suppose (36) does not hold. Without loss of generality, assume that $\frac{\alpha_a q_a^* + \alpha_c q_c^*}{q_a^* + q_c^*} > \alpha_b$, which translates to $q_a^* > \underline{q}_a \triangleq q_c^* \cdot \frac{\alpha_b - \alpha_c}{\alpha_a - \alpha_b}$. Let $\rho_a \triangleq \underline{q}_a / q_a^* < 1$. Note that the following holds:

$$\frac{\alpha_a \underline{q}_a + \alpha_c q_c^*}{\underline{q}_a + q_c^*} = \alpha_b. \tag{37}$$

Let

$$\tilde{q}(j|w) = \begin{cases} \rho_a \cdot q^*(a|w) + q^*(b|w) + q^*(c|w) & \text{if } j = b, \\ (1 - \rho_a) \cdot q^*(a|w) & \text{if } j \in a, \\ 0 & \text{if } j \in c, \\ q^*(j|w) & \text{if } j \notin \{a, b, c\}. \end{cases}$$

(37) implies that the participation constraint for receiver b remains binding with $\tilde{q}(j|w)$. Therefore, $\tilde{q}(j|w)$ is optimal to (3) according to Theorem 4.7. Additionally, we have

$$\tilde{q}_j \triangleq \int_0^1 \tilde{q}(j|w)\, g(w)\, dw = \begin{cases} q_b^* + \rho_a \cdot q_a^* + q_c^* & \text{if } j = b, \\ (1 - \rho_a) \cdot q_a^* & \text{if } j \in a, \\ 0 & \text{if } j \in c, \\ q_j^* & \text{if } j \notin \{a, b, c\}. \end{cases}$$

As a result, $\{\tilde{q}_j\}$ satisfies Lemma D.3 because $\{\tilde{q}_j\}$ is optimal to (8) by Proposition 4.5 and $T_i(\{\tilde{q}_j\}) = T_i(\{q_j^*\}) - 1 < T_i(\{q_j^*\})$ by construction. □

D.4.2 Proof of Bullet 2

We prove Bullet 2 based on our established results from the dual approach. Assume, without loss of generality, that there exists an optimal solution $\{q_k^*\}$ to (8) such that $q_k^* > 0$ for any $k \in [n]$. We then show that the values of $\{q_k^*\}$ are unique. To see that this assumption loses no generality, let

$$P_\varnothing = \Big\{ k \in [n] : q_k^* = 0 \text{ for all optimal solutions } \{q_k^*\} \text{ to (8)} \Big\}$$

denote the set of receivers disregarded by all optimal solutions to (8). We can exclude the receivers in set $P_\varnothing$ without affecting anything. Meanwhile, define $P = [n] \setminus P_\varnothing$. Since (8) is a convex optimization problem, the set of optimal solutions is convex. This implies that there exists an optimal solution $\{q_k^*\}$ such that $q_k^* > 0$ for any $k \in P$.

Now, let $\{\lambda_k^*\}$ denote an optimal dual variable of (8). Let $\{T_i\}$ denote the partition of receivers described in Section 4.4.2. Since no three points of $\{(\alpha_i, v_i)\}_{i \in [n]}$ are collinear, any group T_i contains at most two receivers by Bullet 2 of Proposition 4.6. Fix a group T_i. First, suppose $T_i = \{k\}$ contains one receiver. Then, we have $q_k^* = \mathbb{P}[z_i \leq w \leq z_{i-1}]$, whose value is uniquely determined.

Second, suppose $T_i = \{k, j\}$ contains two receivers. Then, the values of q_k^* and q_j^* must satisfy

$$\begin{aligned} q_k^* + q_j^* &= \mathbb{P}[z_i \leq w \leq z_{i-1}], \\ \alpha_k q_k^* + \alpha_j q_j^* &= \mathbb{E}\Big[w \cdot \mathbb{1}\left[z_i \leq w \leq z_{i-1}\right]\Big], \end{aligned}$$

and therefore, are uniquely determined as well.