Optimality of Public Persuasion in Job Seeking

Chen Chen, Xuyuanda Qi

New York University Shanghai
cc8029@nyu.edu, xq622@nyu.edu

July 20, 2024

Abstract

We study an information design problem in which a school advisor strategically discloses information to promote her student in a job market with n potential employers. The advisor can send different signals to different employers (i.e., *private persuasion*) or broadcast the same signal to all employers (i.e., *public persuasion*). After receiving the signals, the employers can communicate with each other to reduce uncertainty about the candidate in their self-interest. We demonstrate that as long as the candidate can accept at most one offer and has a known preference among the employers, public persuasion is optimal, regardless of how employers communicate. The optimal public persuasion can be derived from a first-best relaxation problem that only imposes the employers' participation constraints. We then focus on a specific case in which the candidate's characteristics can be summarized as a one-dimensional variable, and all of the receivers' utility functions are linear in this variable. We derive the optimal mechanism in a closed form for the two-receiver case. In the general case, a convex optimization problem with n decision variables and constraints can be efficiently solved to obtain an optimal mechanism. We provide structural properties and a better understanding of the optimal mechanism from the dual perspective.

Subject classifications: Bayesian persuasion, public information, multiple receivers, Lagrangian dual

1 Introduction

In this paper, we study a Bayesian persuasion problem faced by a school advisor who promotes her student in a job market with n potential employers (e.g., schools with open junior faculty positions). The student has a known preference among the employers and can accept at most one offer. The advisor holds private information about the student's characteristics relevant to the employers' hiring requirements (e.g., research potential, teaching experience, and communication skills, etc.). The advisor can commit to an information disclosure mechanism that strategically discloses the candidate's characteristics (e.g., through targeted recommendation letters) to the employers to maximize the candidate's expected payoff. Notably, the advisor can use either a *public* persuasion mechanism to share the same information with all employers or a *private* persuasion mechanism to send tailored information to different employers based on their specific hiring standards.

A key feature of our model is the consideration of the subsequent communication among receivers after receiving signals from the sender, which is common in practice. Specifically, employers may communicate with each other (either simultaneously or sequentially, using either cheap talk or some degree of commitment) to reduce uncertainty about the candidate in their self-interest. Then, based on the signal received from the sender and the additional information from other receivers, each employer decides whether to extend a job offer to the candidate. We note that the receivers in this context are both cooperators and competitors. The communication reduces uncertainty about the candidate's characteristics, which benefits each receiver. However, since the sender can accept only one offer, competition among the receivers arises. Particularly, if an employer knows that a candidate is of high quality, he may withhold this information from other employers to avoid competition, especially if the sender prefers other employers. Therefore, the potential for subsequent communication among receivers substantially complicates the information design problem, making it unclear what an optimal persuasion mechanism is.

As our first main result, we demonstrate that public persuasion is always optimal regardless of the detailed communication protocol used by the receivers (Section 3).[1] Since all the employers receive the same information under a public persuasion mechanism, subsequent communication cannot convey any payoff-related information and therefore becomes irrelevant. As a result, the sender eliminates any room for the receivers to communicate and infer further about the candidate for her own benefit.

[1]This is perhaps striking because the optimal persuasion mechanism differs among receivers when considering each receiver in isolation. Moreover, public persuasion is optimal even when the sender knows that receivers cannot communicate (but are aware of each other's existence), as we elaborate further in Remark 4.3.

Furthermore, we show that the optimal public persuasion mechanism can be solved from a first-best relaxation problem that imposes only employers' participation constraints. Specifically, in the first-best problem, a central planner allocates a candidate with characteristics w to employers. An employer hires the candidate when the candidate is allocated to him. The first-best problem solves the optimal randomized allocation to maximize the sender's expected payoff ensuring only a nonnegative expected utility for each employer. We show that an optimal public persuasion mechanism can be derived from an optimal solution to the first-best problem, and its expected payoff matches the first-best upper bound.

Although an optimal public persuasion mechanism can be solved from the aforementioned first-best relaxation problem, it becomes an infinite-dimensional linear program (LP) when the candidate's characteristics w are infinite, which is challenging to solve. As our second main result, we then focus on the efficient computation of an optimal public persuasion mechanism for a specific case in which the candidate's characteristics w can be summarized as a one-dimensional variable, and all of the receivers' utility functions are linear in this variable (Section 4). We derive optimality conditions of an optimal mechanism and provide structural properties and useful interpretations based on the Lagrangian dual of the first-best relaxation problem, where we dualize the participation constraints (Section 4.2). In the Lagrangian, each employer i is associated with a line passing through the point (α_i, v_i) with a nonnegative slope μ_i, where α_i represents the hiring bar[2] of employer i, v_i the payoff of employer i's offer to the candidate, and μ_i the dual variable associated with employer i's participation constraint. The Lagrangian assigns a candidate with characteristics w to employer i with a positive probability only if employer i's line is above the x-axis and the other employers' lines at point w. Furthermore, a public persuasion mechanism is optimal if and only if all the receivers' participation constraints are binding and there exists a dual variable under which the mechanism is optimal to the corresponding Lagrangian.

Based on the optimality conditions, we derive the optimal persuasion mechanism in closed form when there are two employers, where one employer has a higher hiring bar but also brings a higher payoff (Section 4.3). The main trade-off is that an offer from a more competitive employer brings a higher payoff; however, targeting this employer more aggressively is costly because it reduces the overall probability of receiving an offer. Depending on the relative desirability of the two employers and their hiring bars, the optimal persuasion mechanism carefully balances this trade-off.

We then consider the general case with n employers (Section 4.4). We first show that the

[2]That is, the utility of hiring a candidate is nonnegative for employer i if and only if the "quality" w exceeds α_i.

first-best relaxation problem can be reduced to a convex optimization problem with n decision variables and constraints, and thus can be efficiently solved. The convex problem maximizes the sender's expected payoff by optimally determining the ex-ante probability that the candidate joins each employer, subject to a variant of the aforementioned participation constraint that ensures the assignment of the candidate meets the hiring bars of the top k employers for any $k \leq n$. This convex problem is analogous to the one considered in Candogan (2022) but is slightly simplified. We establish the equivalence of the first-best relaxation problem and the convex problem from both the primal and dual perspectives, recover many of the results from Candogan (2022), and provide new understandings based on the dual of the convex problem. Given an optimal solution to the convex program, we can construct an optimal persuasion mechanism in various ways. In addition to the deterministic persuasion mechanism with a double-interval structure as illustrated in Candogan (2022), we present a randomized persuasion mechanism with a monotone structure. Specifically, under the randomized persuasion mechanism, the student's payoff first-order stochastically increases with her quality w. This monotone property ensures that students benefit from higher quality, which is desirable in practice.

The rest of the paper is organized as follows. Section 1.1 reviews some related work. Section 2 formulates the problem. In Section 3, we demonstrate that public persuasion mechanisms are optimal in our setup, regardless of the communication detail of the receivers. In addition, the optimal public persuasion mechanism can be solved from a first-best relaxation problem that requires only the employers' participation constraints. Section 4 addresses the efficient computation of an optimal public persuasion mechanism when the candidate's characteristics can be summarized as a one-dimensional variable, and all receivers' utility functions are linear in this variable. We provide optimality conditions of a public persuasion mechanism based on duality in Section 4.2. Section 4.3 characterizes the optimal mechanism in closed form for the two-receiver case, and Section 4.4 examines the general case. Section 5 concludes.

1.1 Related Literature

Our work is related to the literature on Bayesian persuasion and information design. The seminal paper Kamenica and Gentzkow (2011) examines the problem in which a designer (sender) with private information tries to persuade an agent (receiver) to take a sender-preferred action. Subsequent literature extends this framework to settings with multiple receivers (e.g., Alonso and Câmara 2016, Arieli and Babichenko 2019, and Section 4.1 of Kamenica 2019 for a recent review). As Kamenica (2019) highlights, *if sender can send separate signals to each receiver, and if either (a) a receiver's* *optimal action depends on what other receivers do or (b) sender's utility is not separable across receiver's actions, then the problem becomes significantly more difficult.* Our setup falls within this challenging regime.

Many existing works have not incorporated post-signal communication among receivers as we do. Two exceptions are Galperti and Perego (2023) and Candogan et al. (2023), which consider informational spillovers among the receivers. In both works, these spillovers are pre-specified by a directed network, in which arcs represent potential informational spillovers among the receivers. In contrast, our model allows for strategic communication and an arbitrary communication protocol. Galperti and Perego (2023) characterize the set of all possible equilibrium outcomes that can arise from an information structure under spillover and seeding constraints. Candogan et al. (2023) show that the optimal information design problem is generally computationally challenging under information spillovers, except for some specific cases.

Candogan (2022) considers a general model in which the designer's payoff is an increasing step function of the induced posterior mean and solves a finite-dimensional convex optimization to obtain an optimal public persuasion mechanism. While Candogan (2022) focuses on public persuasion mechanisms, we show that these mechanisms are optimal in our setup, even when receivers can communicate with each other post-signal and regardless of their communication method. When the candidate's characteristics can be summarized as a one-dimensional variable, and all receivers' utility functions are linear in this variable, solving an optimal public persuasion mechanism in our setup aligns with the general model of Candogan (2022). In this case, we slightly simplify the convex optimization problem in Candogan (2022), and recover many of the results from Candogan (2022) and provide new understandings of the optimal persuasion mechanism based on the dual problems.

Bergemann and Morris (2016) and Bergemann and Morris (2019) relate the multi-receiver persuasion problem to the game-theoretic concept of Bayes correlated equilibrium (BCE). This relationship leads to a natural LP formulation for obtaining an optimal persuasion mechanism. Specifically, the decision variables in the LP are joint probabilities of the state and the receivers' actions, and the constraints completely characterize the set of BCEs.[3] Our first-best relaxation problem (2) is also an LP. However, in our LP, the decision variables are marginal allocation probabilities under a mechanism. The LP imposes only participation constraints that any mechanism must satisfy, and thus, does not precisely characterize the set of equilibrium outcomes. Finally, Bergemann and

[3] That is, a joint distribution sustains a BCE if and only if it is feasible to the LP.

Morris (2019) also explore when public persuasion mechanisms are optimal (Section 4.1 there). Their model does not incorporate post-signal communications. They show that public persuasion mechanisms are optimal when receivers' actions are strategic complements, as these mechanisms induce a positive correlation in the receivers' actions. However, in our setup, the receivers' actions are not strategic complements.

Kolotilin (2018) and Dworczak and Martini (2019) also use duality to characterize optimality conditions and to interpret an optimal persuasion mechanism. However, we study different problems, formulate the optimization problem in different ways, and apply duality differently. Specifically, Kolotilin (2018) dualize a consistency constraint for the marginal distribution of the sender's state and Dworczak and Martini (2019) dualize the mean-preserving spread constraint. In contrast, we dualize the employers' participation constraints.

Ostrovsky and Schwarz (2010) and Boleslavsky and Cotton (2015) study school grading problems similar to our setting. Ostrovsky and Schwarz (2010) consider a model with a continuum of schools (senders) and employers (receivers) and study the schools' equilibrium grading policies (persuasion mechanism). Each school is assumed to use a public persuasion mechanism. Boleslavsky and Cotton (2015) consider a setup with two schools (senders) and one evaluator (receiver), where each school determines both its investment level in quality and grading policies.

Finally, other extensions of Bayesian persuasion have been considered in the literature, including multiple senders (Gentzkow and Kamenica 2017), privately-informed receivers (Kolotilin et al. 2017, Guo and Shmaya 2019), and dynamic models (Ely 2017), which are not included in our model. In addition, numerous works focus on various operational applications, such as incentivizing exploration (Papanastasiou et al. 2018), signaling product availability (Drakopoulos et al. 2021), signaling congestion in queueing systems (Anunrojwong et al. 2023), and informing the severity of a pandemic (De Véricourt et al. 2021); see Candogan (2020) for a comprehensive review.

1.2 Notation and Terminology

We let $\mathbb{N}$ denote the set of nonnegative integers. For any two integers $a, b \in \mathbb{N}$ with $a \le b$, we let $[a:b] = \{a, a+1, \ldots, b-1, b\}$ denote a sequence of integers starting from a and ending with b, and we denote $[n] = [1:n]$ for any $n \in \mathbb{N}_+$. For any real number $x \in \mathbb{R}$, we let $(x)^+ \triangleq \max\{x, 0\}$ denote the maximum of x and 0.

2 Problem Formulation

We consider a school advisor (referred to as "she") who promotes her student in a job market with n potential employers (referred to as "he"; e.g., schools with open junior faculty positions) via strategic information disclosure (e.g., targeted recommendation letters). The student can accept at most one offer and has a known preference among the employers. Specifically, we denote by $v_i > 0$ the utility from the offer of employer i, and we rank employers in decreasing preference; that is, $v_i > v_j$ if $i < j$, as assumed in Assumption 2.1. If the student does not secure a job, we normalize her utility to zero.

Assumption 2.1. The utility v_i from accepting employer i's offer satisfies $0 < v_n < \cdots < v_2 < v_1$.

Let $w \in \Omega$ represent the characteristics of the student, where Ω is a general state space.[4] While the realization of w is privately observable to the school advisor, employers only possess a prior distribution $G(w)$ regarding the student's characteristics, reflecting the reputation of the advisor's students. For each employer i, let $u_i(w)$ denote the utility of hiring a student with characteristics w; the utility of not hiring is zero.

Information Disclosure Mechanism We study a Bayesian persuasion setup in which the advisor (the sender), who has commitment power, designs an information disclosure mechanism to promote her student to the n employers (the receivers). Let S_i denote the set of signals employed by the advisor to interact with employer i and $\boldsymbol{S} = \bigotimes_{i=1}^{n} S_i$ represent the set of all signals. Upon observing the characteristics w, the advisor sends a signal $s_i \in S_i$ to each employer i according to a joint distribution $f(\boldsymbol{s}|w)$, where $\boldsymbol{s} = (s_1, \cdots, s_n) \in \boldsymbol{S}$ denotes the concatenation of the sent signals. We define the information mechanism $f(\cdot|w)$ as a *public* mechanism if

1. The signals share a common signal space S, that is, $S_i = S_j = S$ for all $i, j \in [n]$; and
2. The signals $(s_i)_{i\in[n]}$ are perfectly correlated, that is, $f(\boldsymbol{s}|w) = 0$ for any signal $\boldsymbol{s} = (s_i)_{i\in[n]}$ where $s_i \neq s_j$ for some $i, j \in [n]$.

With a public mechanism, employers always receive the same signal, eliminating the need for further communication. Conversely, if $f(\cdot|w)$ allows for different signals among employers, we refer to it as a *private* information mechanism. In this case, the employers may receive different signals, leading to varied information about the student's characteristics w.

[4] For example, we may have $\Omega \subseteq \mathbb{R}^m$, where m represents the number of attributes relevant to employers' hiring standards, such as research potential, teaching experience, and communication skills.

Communication among Receivers We assume that employers may communicate with each other after receiving the signal $\boldsymbol{s}$. We do not formally model how employers will communicate. Notably, employers may or may not be able to communicate, and if they do, it could be either simultaneously or sequentially, using either cheap talk or with some degree of commitment. Any of these communication methods can be reasonable in specific scenarios. However, as we demonstrate in Section 3, the optimal persuasion mechanism will be independent of the communication details. This is because, regardless of how employers communicate, a public information disclosure mechanism will always be optimal for the sender, leaving nothing for the receivers to communicate.

However, some notations are helpful to describe the problem. Given a specific communication protocol, let C_i denote the set of information that employer i can receive from other employers and $\mathbf{C} = \bigotimes_{i=1}^{n} C_i$ represent the communication space. Denote the communication outcome as $\mathbf{c} = (c_1, ..., c_n) \in \mathbf{C}$, where c_i is the information employer i receives through communication. Given a signal $\boldsymbol{s}$, suppose the cumulative distribution function of $\mathbf{c}$ is $C(\mathbf{c}|\boldsymbol{s})$, and the probability density function of $\mathbf{c}$ is $c(\mathbf{c}|\boldsymbol{s}) = \frac{dC(\mathbf{c}|\boldsymbol{s})}{d\mathbf{c}}$, possibly derived from the employers' equilibrium strategies.

Sender's Problem The game proceeds as follows:

1. The advisor commits to an information disclosure mechanism $f(\cdot|w)$ and a signal space $\mathbf{S} = \bigotimes_{i=1}^{n} S_i$.
2. The student's characteristics w are drawn from the cumulative probability distribution $G(w)$. A signal $\mathbf{s} = (s_i)_{i\in[n]}$ is then generated according to the disclosure mechanism $f(\cdot|w)$ and sent to the employers.
3. Employers communicate with each other after receiving the signal $\boldsymbol{s}$ using $C(\cdot|\boldsymbol{s})$, which may represent an equilibrium communication strategy in a specific scenario. After communication, each employer i decides whether to extend an offer to the student based on the signal s_i and the communication outcome c_i.
4. The student accepts the offer that maximizes her payoff, which corresponds to the employer with the smallest index among those sending offers, according to Assumption 2.1.

Given a signal and communication outcome $s \in S_i$ and $c \in C_i$, we define $\mathbf{S}^i(s) = \{\boldsymbol{s} \in \mathbf{S} : s_i = s\}$ and $\mathbf{C}^i(c) = \{\mathbf{c} \in \mathbf{C} : c_i = c\}$ as the sets of possible signals and communications, respectively. Upon observing s and c, employer i understands that the signal must be in the set $\mathbf{S}^i(s)$ and the communication outcome must be in the set $\mathbf{C}^i(c)$. He updates his belief about the student's characteristics w, the signal $\boldsymbol{s}$, and the communication outcome $\mathbf{c}$ using Bayes's rule whenever possible. Specifically, let $f_i(s,c)$ denote the probability that employer i receives a signal s and communication outcome c:

$$f_i(s,c) = \int_{w\in\Omega} \int_{\boldsymbol{s}\in\mathbf{S}^i(s)} \int_{\mathbf{c}\in\mathbf{C}^i(c)} c(\mathbf{c}|\boldsymbol{s})\, f(\boldsymbol{s}|w)\, d\mathbf{c}\, d\boldsymbol{s}\, dG(w).$$

If $f_i(s,c) > 0$, the employer i's posterior belief on the tuple $(w, \boldsymbol{s}, \mathbf{c})$ is defined as

$$f_i(w,\boldsymbol{s},\mathbf{c}|s,c) = \begin{cases} \frac{dG(w) f(\boldsymbol{s}|w)\, c(\mathbf{c}|\boldsymbol{s})}{f_i(s,c)}, & \text{if } \boldsymbol{s}\in\mathbf{S}^i(s) \text{ and } \mathbf{c}\in\mathbf{C}^i(c), \\ 0, & \text{otherwise.} \end{cases}$$

Denote employer i's equilibrium strategy by $\delta_i(s,c)$, representing his probability of extending an offer after receiving a signal $s \in S_i$ and communication outcome $c \in C_i$. The optimality of employer i's strategy implies that $\delta_i(s,c)$ follows the following equation:

$$\delta_i(s,c) = \begin{cases} 0, & \text{if } \mathbb{E}\left[u_i(w)\cdot \mathbb{1}[a_j^* = 0,\, \forall\, j < i] \mid s, c\right] < 0, \\ \delta\in[0,1], & \text{if } \mathbb{E}\left[u_i(w)\cdot \mathbb{1}[a_j^* = 0,\, \forall\, j < i] \mid s, c\right] = 0, \\ 1, & \text{if } \mathbb{E}\left[u_i(w)\cdot \mathbb{1}[a_j^* = 0,\, \forall\, j < i] \mid s, c\right] > 0, \end{cases}$$

where the binary variable $a_j^* \in \{0,1\}$ represents employer j's action of extending an offer in an equilibrium and satisfies $\mathbb{P}[a_j^* = 1|s_j, c_j] = \delta_j(s_j, c_j)$, and the expectation $\mathbb{E}[\cdot|s,c]$ is taken over the posterior distribution $f_i(w,\boldsymbol{s},\mathbf{c}|s,c)$. Note that the student accepts employer i's offer if and only if none of the employers $j < i$ extends an offer, which is represented by $\mathbb{1}[a_j^* = 0,\, \forall\, j < i]$.

Finally, let the random set $I(\boldsymbol{s},\mathbf{c})$ denote the employers who extend an offer and $i(\boldsymbol{s},\mathbf{c}) \triangleq \min I(\boldsymbol{s},\mathbf{c})$ the index of the offer to accept, given the signal realization $\boldsymbol{s}\in\mathbf{S}$ and communication outcome $\mathbf{c}\in\mathbf{C}$ and under the employers' equilibrium strategies. If $I(\boldsymbol{s},\mathbf{c}) = \varnothing$, that is, the student receives no offer, we let $i(\boldsymbol{s},\mathbf{c}) = \varnothing$ and $v_\varnothing = 0$ as the corresponding utility of the student. The advisor selects an information disclosure mechanism $f(\cdot|w)$ that maximizes the expected payoff of the student by solving

$$V^* \triangleq \max_{f(\cdot|w)} \int_{w\in\Omega} \int_{\boldsymbol{s}\in\boldsymbol{S}} \int_{\mathbf{c}\in\mathbf{C}} \mathbb{E}_{i(\boldsymbol{s},\mathbf{c})}\left[v_{i(\boldsymbol{s},\mathbf{c})}\right]\cdot c(\mathbf{c}|\boldsymbol{s})\cdot f(\boldsymbol{s}|w)\cdot d\mathbf{c}\; d\boldsymbol{s}\; dG(w). \tag{1}$$

In (1), the expectation $\mathbb{E}_{i(\boldsymbol{s},\mathbf{c})}[\cdot]$ is taken over the possible randomness in the receivers' equilibrium offer-extending strategies when the signal and communication realizations are $\boldsymbol{s}$ and $\mathbf{c}$, respectively, and V^* denotes the expected payoff of an optimal information disclosure mechanism.

3 Optimality of Public Persuasion

In this section, we illustrate that a public persuasion mechanism solves the advisor's optimal information disclosure problem (1), regardless of how employers communicate. We begin by introducing a relaxation of the designer's problem (1) in Section 3.1, which provides an upper bound on the sender's optimal expected payoff V^*.

3.1 First-Best Problem with Participation Constraints

In this section, we consider the first-best relaxation problem (2) for the sender's information design problem, where we impose only the participation constraints of the employers.

$$\begin{aligned} \bar{V} = \max_{q(i|w)\geq 0} \quad & \sum_{i=1}^{n} v_i \cdot \int_{w\in\Omega} q(i|w)\, dG(w) \\ \text{s.t.} \quad & \int_{w\in\Omega} u_i(w)\, q(i|w)\, dG(w) \geq 0,\ \forall\, i \in [n], \\ & \sum_{i\in[n]} q(i|w) \leq 1,\ \forall\, w \in \Omega. \end{aligned} \tag{2}$$

In (2), a central planner allocates the candidate with characteristics w to employer i with a probability of $q(i|w)$, and requires the employer to hire the candidate when the latter is allocated to him. The chosen $q(i|w)$ ensures a nonnegative expected utility for each employer, as indicated by the first constraint in (2). This reflects the fact that each employer should be at least break-even in expectation from hiring. In addition, any candidate is allocated to at most one employer, as indicated by the second constraint in (2). This reflects the fact that the candidate can accept at most one offer. The central planner chooses $q(i|w)$ satisfying these two constraints to maximize the candidate's expected payoff, and the optimal value is denoted by $\bar{V}$.

Lemma 3.1 demonstrates that (2) provides an upper bound on the sender's optimal expected payoff V^*, regardless of how employers communicate.

Lemma 3.1. *We have $\bar{V} \geq V^*$, regardless of how employers communicate.*

We prove Lemma 3.1 in Appendix A.1. Intuitively, given any disclosure mechanism $f(\cdot|w)$, let $q(i|w)$ denote the ex-ante probability that the candidate joins employer i when her characteristics are w, under the employers' equilibrium strategies induced by $f(\cdot|w)$. These $\{q(i|w)\}$ are feasible to (2) and have an objective value no larger than $\bar{V}$.

3.2 Optimality of Public Persuasion

In this section, we construct a public persuasion mechanism $f^*(\cdot|w)$ from the optimal solution of (2) and show that its expected payoff attains the first-best upper bound $\bar{V}$. Therefore, the mechanism $f^*(\cdot|w)$ is optimal to (1), and this optimality does not depend on the communication protocol between receivers.

Let $\{q^*(i|w)\}$ denote an optimal solution to (2). We consider a public persuasion mechanism $f^*(\cdot|w)$ with signal space $S_i = S \triangleq [n] \cup \{\varnothing\}$ for all employers $i \in [n]$. When the candidate's characteristics are w, the mechanism broadcasts the signal $s = i$ to all employers with probability $q^*(i|w)$ for any $i \in [n]$ and the signal $s = \varnothing$ to all employers with probability $1 - \sum_{i\in[n]} q^*(i|w)$. We can interpret the signal $s = i$ as a recommendation for only employer i to extend an offer and the signal $s = \varnothing$ as a recommendation for none of the employers to extend an offer. Lemma 3.2 shows that this persuasion mechanism achieves the first-best upper bound $\bar{V}$.

Lemma 3.2. *Under the public persuasion mechanism $f^*(\cdot|w)$, it is an equilibrium for each employer $i \in [n]$ to extend an offer only upon receiving the signal $s = i$. Moreover, the expected payoff of the mechanism $f^*(\cdot|w)$, denoted by V^{P}, satisfies $V^{\mathrm{P}} = \bar{V}$.*

We prove Lemma 3.2 in Appendix A.2. To understand the equilibrium in Lemma 3.2, suppose that the school advisor recommends the candidate to employer i. Employer i is willing to extend an offer because (i) his offer will be accepted with certainty given that no other employer will extend an offer, and (ii) he can break even from his offer in expectation, as indicated by the first constraint in (2). Any employer $j > i$ cannot benefit from extending an offer because the candidate will accept the more attractive offer from employer i. Any employer $j < i$ is unwilling to extend an offer because (i) his offer, if extended, will be accepted with certainty given that no better offer will be extended, and (ii) $\{q^*(i|w)\}$ being an optimal solution of (2) implies that employer j cannot break even from his offer in expectation—otherwise, the central planner in (2) can strictly improve the candidate's payoff by allocating the candidate to employer j instead of employer i without violating any constraint in (2).

Since the mechanism $f^*(\cdot|w)$ achieves the first-best upper bound $\bar{V}$, Lemma 3.1 implies that the first-best upper bound is tight (i.e., $\bar{V} = V^*$) and that $f^*(\cdot|w)$ is an optimal persuasion mechanism, independent of how employers can communicate post-signal. Since the school advisor sends the same information to all employers with mechanism $f^*(\cdot|w)$, communication becomes irrelevant. Therefore, the sender eliminates any communication among the receivers for her own benefit, regardless of the way receivers can communicate. This holds true even when the sender knows that the receivers cannot communicate but are aware of each other's existence, as we elaborate further in Remark 4.3.

4 Simplified Optimization for One-Dimensional Linear Utility Case

According to Lemma 3.2, the school advisor needs to focus only on public persuasion mechanisms to solve the optimal persuasion problem (1). Moreover, the optimal public persuasion mechanism can be derived from (2), and it achieves the first-best performance (i.e., the optimal value of (2)). However, when the candidate's characteristics w are infinite, the first-best problem (2) is an infinite-dimensional LP, which can be challenging to solve. In this section, we focus on the case where the state variable w is one-dimensional, and all of the receivers' utility functions are linear in w. we provide structural properties and derive the optimality condition of an optimal mechanism from the Lagrangian dual of (2), where we dualize the participation constraints. Using the optimality condition, we derive the optimal mechanism in closed form when there are two employers in Section 4.3. For the general case with n employers (Section 4.4), problem (2) can be reduced to a convex optimization problem with n decision variables and constraints, similar to Candogan (2022), and thus can be solved efficiently. We establish the equivalence of the convex problem and problem (2) from both the primal and dual viewpoints and provide a better understanding of the optimal mechanism based on the Lagrangian dual of the convex optimization problem.

4.1 The Setup

To start, we formally describe the one-dimensional linear utility case. First, we assume that the candidate's characteristics w can be summarized as a one-dimensional state variable within a finite interval. Without loss of optimality, let $w \in \Omega = [0, 1]$. Moreover, we assume that w follows a continuous distribution with a strictly increasing cumulative distribution function $G(\omega)$ and a density function $g(w) > 0$ for any $w \in (0, 1)$. We summarize the above in Assumption 4.1.

Assumption 4.1. The candidate's characteristics w belong to the one-dimensional interval $\Omega = [0, 1]$ and follow a continuous distribution. Let $G(w)$ and $g(w)$ denote the cumulative distribution function and density function of w, respectively. The function $G(w)$ is strictly increasing, so its inverse, denoted by $G^{-1}(\cdot)$, exists.

Second, we assume that for each employer $i \in [n]$, the utility function for hiring a candidate with characteristics w is linear in w; that is, $u_i(w) = \kappa_i \cdot \big(w - \alpha_i\big)$, where κ_i and α_i are positive constants. This assumption implies that each employer i considers only the mean value of the characteristics w among the candidates who would accept employer i's offer. Specifically, employer i will extend an offer only if this mean value exceeds his hiring threshold α_i. We state this linear utility assumption in Assumption 4.2.

Assumption 4.2. For each employer $i \in [n]$, the utility function $u_i(w)$ for a candidate with characteristic w is increasing and linear in w with a threshold value $\alpha_i > 0$; that is, $u_i(w) = \kappa_i \cdot \big(w - \alpha_i\big)$, where κ_i and α_i are positive constants.

Note that since employers are ranked in decreasing preference by Assumption 2.1, there is no loss of generality to assume that the threshold values α_i also decrease in the employer index i. In other words, a more preferred employer is harder to get into. Conversely, if employer i is more preferred than j ($v_i > v_j$) but also easier to get into ($\alpha_i \leq \alpha_j$), employer j will never be targeted and can be dropped from consideration. Finally, we assume that all employers are selective, meaning their threshold values α_i are higher than the prior mean of the candidate's characteristics $\mathbb{E}_{w\sim G(w)}[w]$. We state these in Assumption 4.3.

Assumption 4.3. Let $w_0 \triangleq \mathbb{E}_{w\sim G(w)}[w]$ denote the prior mean of the candidate's characteristics w. The employers' threshold values α_i satisfy $0 < w_0 < \alpha_n < \cdots < \alpha_2 < \alpha_1 < 1$.

Based on the linear-utility Assumption 4.2, the first-best problem (2) can be written as (3):

$$
\begin{aligned}
\bar{V} = \max_{q(i|w)\geq 0} \quad & \sum_{i=1}^{n} v_i \cdot \int_0^1 q(i|w)\, g(w)\, dw \\
\text{s.t.} \quad & \int_0^1 w \cdot q(i|w)\, g(w)\, dw \geq \alpha_i \int_0^1 q(i|w)\, g(w)\, dw,\ \forall\, i \in [n], \\
& \sum_{i\in[n]} q(i|w) \leq 1,\ \forall\, w \in [0,1].
\end{aligned} \tag{3}
$$

4.1.1 Preliminary Properties of Optimal Solution of (3)

We conclude Section 4.1 by describing several properties related to an optimal solution of (3). First, for any feasible solution to (3), the probability that a candidate receives an offer (prior to the realization of w) is strictly less than one, as employers are selective according to Assumption 4.3. Moreover, this probability is maximized when the sender targets only the most accessible employer n. We formalize this in Proposition 4.1 and provide the proof in Appendix A.3.

Proposition 4.1. *Let $z_n \in (0,1)$ be such that $\mathbb{E}[w|w \geq z_n] = \alpha_n$, where α_n is the threshold value of employer n. For any feasible solution $\{q(i|w)\}$ of (3), we have $\sum_{i\in[n]} \int_0^1 q(i|w)g(w)dw \leq \mathbb{P}(w \geq$*

$z_n) < 1$*, where the first inequality is attained when the sender targets only employer* n*; that is,* $q(n|w) = 1$ *for any* $w \geq z_n$*, and* $q(i|w) = 0$ *for any* $i \neq n$ *or* $w < z_n$*.*

Second, Assumption 4.3 implies that the participation constraints in (3) are binding with any optimal solution of (3). We state this in Proposition 4.2 and provide the proof in Appendix A.4.

Proposition 4.2. *Under Assumption 4.3, the participation constraints are binding with any optimal solution of (3).*

Finally, we show that any optimal solution exhibits a cutoff structure. Specifically, there exists a threshold value $z \in (0, 1)$ such that a candidate receives an offer if and only if her characteristics w exceeds z. We formalize this property in Proposition 4.3, with the proof provided in Appendix A.5.

Proposition 4.3. *Any optimal solution has a cutoff structure. That is, for any optimal solution* $\{q^*(i|w)\}$ *of (3), there exists a threshold value* $z \in (0, 1)$ *such that* $\sum_{i\in[n]} \int_z^1 q^*(i|w)g(w)dw = \mathbb{P}(w \geq z)$ *and* $\sum_{i\in[n]} \int_0^z q^*(i|w)g(w)dw = 0$*.*

4.2 The Lagrangian Dual Problem

In this section, we introduce the Lagrangian dual problem of (3) that dualizes the employers' participation constraints. We then interpret the Lagrangian problem from both geometric and economic views and derive the optimality conditions of an optimal persuasion mechanism.

Specifically, denote by $\mu_i \geq 0$ the Lagrange multiplier for the participation constraint of employer $i \in [n]$. The Lagrangian relaxation, denoted by $V^{\mathrm{LR}}(\boldsymbol{\mu})$ with $\boldsymbol{\mu} = (\mu_i)_{i\in[n]} \in \mathbb{R}_+^n$, is as follows:

$$
\begin{aligned}
V^{\mathrm{LR}}(\boldsymbol{\mu}) &= \max_{\substack{q(i|w)\geq 0,\\ \sum_{i\in[n]} q(i|w)\leq 1}} \int_0^1 \sum_{i=1}^n \Big\{v_i + \mu_i\big(w - \alpha_i\big)\Big\}\, q(i|w)\, g(w)\, dw \\
&= \int_0^1 \left(\max_{\substack{q(i|w)\geq 0,\\ \sum_{i\in[n]} q(i|w)\leq 1}} \sum_{i=1}^n \Big\{v_i + \mu_i\big(w - \alpha_i\big)\Big\} \cdot q(i|w) \right) \cdot g(w)\, dw.
\end{aligned} \tag{4}
$$

After dualizing the participation constraints, the Lagrangian decouples over characteristics w. Specifically, define

$$\ell_i\big(w; \mu_i\big) \triangleq v_i + \mu_i\big(w - \alpha_i\big)$$

as a line associated with employer $i \in [n]$. This line passes through the point (α_i, v_i) and has a nonnegative slope $\mu_i \geq 0$. In addition, let

$$h(w;\boldsymbol{\mu}) \triangleq \max_{i\in[n]} \ell_i(w;\mu_i) = \max_{i\in[n]} \Big\{ v_i + \mu_i (w - \alpha_i) \Big\}$$

denote the maximum of the n lines and $\bar{h}(w;\boldsymbol{\mu}) \triangleq \max\big\{h(w;\boldsymbol{\mu}), 0\big\}$. Both the functions $h(w;\boldsymbol{\mu})$ and $\bar{h}(w;\boldsymbol{\mu})$ are convex, increasing (since $\mu_i \geq 0$), and piecewise linear in w. Finally, let $\mathbf{Q}^{\text{LR}}(\boldsymbol{\mu})$ denote the set of optimal solutions $\{q(i|w)\}$ to $V^{\text{LR}}(\boldsymbol{\mu})$. According to (4), the set $\mathbf{Q}^{\text{LR}}(\boldsymbol{\mu})$ can be expressed as follows:

$$\begin{aligned} \mathbf{Q}^{\text{LR}}(\boldsymbol{\mu}) = \Bigg\{ q(i|w) : q(i|w) \geq 0 \ \text{ and } \sum_{i\in[n]} q(i|w) \leq 1, \, \forall\, w \in [0,1], \\ \sum_{i\in[n]} q(i|w) = 1 \ \text{ if } \ h(w;\boldsymbol{\mu}) > 0, \\ q(i|w) > 0 \ \text{ only if } \ \ell_i(w;\mu_i) = \bar{h}(w;\boldsymbol{\mu}) \Bigg\}. \end{aligned} \tag{5}$$

That is, an optimal solution of $V^{\text{LR}}(\boldsymbol{\mu})$ allocates a candidate with characteristics w to employer i with a positive probability only if employer i's line $\ell_i(w;\mu_i)$ is above the x-axis and not dominated by the other employers' lines $\{\ell_i(w;\mu_i)\}_{j\neq i}$ at point w.

To add an economic interpretation, note that the Lagrangian multiplier μ_i measures the extent to which employer i's participation constraint is tight. The formation of employer i's line $\ell_i(w;\mu_i)$ indicates that the school advisor's payoff from allocating a candidate with quality w to employer i has two component in the Lagrangian. The first component, v_i, is the direct payoff from letting employer i hire the candidate. The second component, $\mu_i(w - \alpha_i)$, is the indirect payoff from the impact of this hire on employer i's participation constraint. Specifically, μ_i represents the significance of this indirect effect.

If $w > \alpha_i$, by allocating the candidate to employer i, the school advisor relieves the participation constraint of employer i and can potentially place more under-qualified candidates to employer i, who might otherwise be unemployed. Conversely, if $w < \alpha_i$, by allocating the candidate to employer i, the school advisor tightens the participation constraint of employer i and can place fewer under-qualified students to employer i. Combining both direct and indirect payoffs, the school advisor allocates the candidate with quality w to the employer i with the highest positive payoff—that is, the largest value of $\ell_i(w;\mu_i)$ for all $i \in [n]$, if this highest payoff is positive. Otherwise, the advisor does not allocate this candidate to any employer to secure a payoff of zero.

Finally, from (4) we have

$$V^{\mathrm{LR}}(\boldsymbol{\mu}) = \int_0^1 \bar{h}\big(w; \boldsymbol{\mu}\big)\, g(w)\, dw.$$

Since every feasible policy to (3) is feasible to (4) and attains an objective that is no smaller, $\bar{V} \le V^{\mathrm{LR}}(\boldsymbol{\mu})$ for any $\boldsymbol{\mu} \in \mathbb{R}_+^n$. We formally state this weak-duality property in Lemma 4.4.

Lemma 4.4 (Weak Duality). *We have $\bar{V} \le V^{\mathrm{LR}}(\boldsymbol{\mu})$ for any dual variable $\boldsymbol{\mu} \in \mathbb{R}_+^n$.*

4.2.1 The Optimal Lagrangian Dual

Since the Lagrangian $V^{\mathrm{LR}}(\boldsymbol{\mu})$ is a convex function of $\boldsymbol{\mu}$ from (4), we can solve a convex optimization problem

$$V^{\mathrm{LR}} \triangleq \min_{\boldsymbol{\mu} \in \mathbb{R}_+^n} V^{\mathrm{LR}}(\boldsymbol{\mu}) \ge \bar{V} \tag{6}$$

to obtain the tightest Lagrangian relaxation bound V^{LR}. Let $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]} \in \operatorname{argmin}_{\boldsymbol{\mu}\in\mathbb{R}_+^n} V^{\mathrm{LR}}(\boldsymbol{\mu})$ denote an optimal Lagrangian dual variable; it can be solved efficiently according to Remark 4.1.

Remark 4.1 (Computing $\boldsymbol{\mu}^*$). From Danskin's theorem (Proposition 4.5.1 in Bertsekas et al. 2003) and the fact that a convex combination of any two optimal solutions of (4) is also optimal to (4), the sub-differential (i.e., set of sub-gradients) of $V^{\mathrm{LR}}(\boldsymbol{\mu})$ at any $\boldsymbol{\mu} \in \mathbb{R}_+^n$, denoted by $\partial V^{\mathrm{LR}}(\boldsymbol{\mu})$, can be expressed as

$$\partial V^{\mathrm{LR}}(\boldsymbol{\mu}) = \left\{ (g_i)_{i\in[n]} \text{ with } g_i \triangleq \int_0^1 (w - \alpha_i)\, q(i|w)\, g(w)\, dw : \{q(i|w)\} \in \mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu}) \right\}.$$

Since $V^{\mathrm{LR}}(\boldsymbol{\mu})$ and its sub-gradients can be efficiently computed, we can apply sub-gradient-based methods (e.g., the sub-gradient method or cutting-plane method) to solve the convex program (6) and determine an optimal Lagrangian dual variable $\boldsymbol{\mu}^*$ efficiently.

Furthermore, Lemma 4.5 demonstrates that strong duality holds, which follows standard strong duality for convex optimization in a vector space.

Lemma 4.5 (Strong Duality). *Problem (3) and its Lagrangian relaxation (4) have the following relationship.*

1. *Strong duality holds, and there exists an optimal dual variable $\boldsymbol{\mu}^* \in \mathbb{R}_+^n$; that is, $\bar{V} = V^{\mathrm{LR}} = V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$.*

2. *$\boldsymbol{\mu} \in \mathbb{R}^n_+$ is an optimal dual variable and $\{q(i|w)\}$ is an optimal solution of (3) if and only if (1) $\{q(i|w)\} \in \mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu})$, and (2) all participation constraints in (3) are binding with $\{q(i|w)\}$.*

We prove Lemma 4.5 in Appendix A.6. Lemma 4.5 implies multiple structural properties of an optimal solution $\{q^*(i|w)\}$ and an optimal dual variable $\boldsymbol{\mu}^*$ of (3), which we provide below. We first show that if an employer $i \in [n]$ is considered by the sender—that is, $q_i^* \triangleq \int_0^1 q^*(i|w)dw > 0$—then the point (α_i, v_i) lies on the envelope function $h(w; \boldsymbol{\mu}^*)$ and is within the interior of the line segment associated with employer i. We state this in Proposition 4.6 and provide its proof in Appendix A.7.

Proposition 4.6. *Let $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]}$ be an optimal dual variable of (3). Suppose that there exists an optimal solution $\{q^*(i|w)\}$ to (3) such that $\int_0^1 q^*(i|w)dw > 0$ (i.e., employer i is considered by the sender). Then, there exist constants $\underline{b}, \bar{b} \in [0,1]$ satisfying $0 < \underline{b} < \alpha_1 < \bar{b} \leq 1$ such that $\bar{h}(w; \boldsymbol{\mu}^*) = \ell_i(w; \mu_i^*)$ (i.e., employer i's line is above the x-axis and the other employers' lines) if and only if $w \in [\underline{b}, \bar{b}]$.*

Second, we show in Proposition 4.7 that the optimal dual variables $\{\mu_i^*\}$ are strictly positive and decreasing (after removing disregarded employers). Additionally, the probability that a candidate receives an offer is the same under any optimal solution of (3). We prove Proposition 4.7 in Appendix A.8.

Proposition 4.7. *Let $\{q^*(i|w)\}$ be an optimal solution and $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]}$ an optimal dual variable of (3). The following hold:*

1. *$\mu_i^* > 0$ for all $i \in [n]$. Therefore, the envelope function $h(w; \boldsymbol{\mu}^*)$ is strictly increasing in w.*
2. *The cutoff value in Proposition 4.3 is unique. Specifically, $\sum_{i\in[n]} \int_0^1 q^*(i|w)dw = \mathbb{P}(w \geq z^*)$, where $z^* \in (0,1)$ is the root of $h(w; \boldsymbol{\mu}^*)$, meaning that $h(z^*; \boldsymbol{\mu}^*) = 0$.*
3. *Let $P \triangleq \{i \in [n] : q_i^* \triangleq \int_0^1 q^*(i|w)dw > 0\}$ be the set of employers the sender considers. Then, $\{\mu_i^*\}_{i\in P}$ decreases with the employer index i.*

Finally, Bullet 2 of Lemma 4.5 provides optimality conditions of a persuasion mechanism. Specifically, if we can find an optimal solution $\{q(i|w)\}$ of $V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ that ensures that all participation constraints in (3) are binding, then $\{q(i|w)\}$ is also optimal to (3) and provides an optimal (public) persuasion mechanism. However, how to identify such a desirable $\{q(i|w)\} \in \mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ through appropriate tie-breaking remains generally unclear. Despite this, in Section 4.3, we apply Lemma 4.5 to derive the optimal persuasion mechanism for the case of two receivers.

4.3 Two-Receiver Case

In this section, we assume that there are two employers $i \in \{1, 2\}$, with offer values $v_1 > v_2 > 0$ and recruiting thresholds $\alpha_1 > \alpha_2 > w_0 > 0$, and we derive the optimal public persuasion mechanism based on Bullet 2 of Lemma 4.5.

According to the revelation principle, it is optimal to focus on public persuasion mechanisms with a signal space $S \triangleq \{1, 2, \varnothing\}$ that satisfy incentive compatibility (IC) constraints. Specifically, the signal $s = i$ represents a recommendation for only employer i to extend an offer, and $s = \varnothing$ represents a recommendation for neither of the employers to extend an offer. Moreover, the IC constraints require that, conditioning on the signal $s = i$, only employer i is willing to extend an offer; and if the signal $s = \varnothing$ is generated, neither employer is willing to extend an offer. Note that Bullet 2 of Lemma 4.5 requires that all employers' participation constraints be binding with any optimal mechanism. Given the participation constraints, the incentive compatibility constraints with $s = i$ hold automatically:

1. Conditioning on $s = 1$, the posterior mean of the candidate's characteristics w is α_1; therefore, employer 1 will extend an offer, and employer 2 has no incentive to do so because the candidate will certainly select employer 1.
2. Conditioning on $s = 2$, the posterior mean of w is α_2, causing only employer 2 to extend an offer. Employer 1 strictly prefers not to make an offer as it would be loss-making.

As a result, given the binding participation constraints, the IC constraints require only that, conditioning on $s = \varnothing$, the posterior mean of w is smaller than α_2, so neither employer will extend an offer. Let $\mathcal{M}$ denote the set of mechanisms with signal space S that satisfy the IC constraints. Specifically, $\{q(i|w)\} \in \mathcal{M}$ if they satisfy the following constraints with $n = 2$:

$$
\begin{aligned}
&\int_0^1 w \cdot q(i|w)\, g(w)\, dw = \alpha_i \int_0^1 q(i|w)\, g(w)\, dw,\ \forall\, i \in [n],\\
&\int_0^1 w \cdot q(\varnothing|w)\, g(w)\, dw < \alpha_n \int_0^1 q(\varnothing|w)\, g(w)\, dw,\\
&\sum_{i\in[n]} q(i|w) + q(\varnothing|w) = 1,\ \forall\, w \in [0, 1],\\
&q(i|w) \geq 0,\ \forall\, w \in [0, 1],\ i \in [n] \cup \{\varnothing\}.
\end{aligned}
$$

Additionally, let $q_i(M) \triangleq \int_0^1 q(i|w)\, g(w)\, dw$ denote the probability that the candidate is allocated to employer i given a mechanism $M = \{q(i|w)\} \in \mathcal{M}$.

4.3.1 Preparation: Extreme Mechanisms Focusing on a Single Receiver

We first consider two extreme mechanisms in which the sender prioritizes either receiver 1 or 2 as preparation for characterizing the optimal mechanism in Section 4.3.2.

First, consider the mechanism $M_2 \in \mathcal{M}$ where the sender completely targets employer 2. Specifically, let $z_2 > 0$ be such that $\mathbb{E}[w|w \geq z_2] = \alpha_2$.[5] The sender transmits the signal $s = 2$ when $w \geq z_2$. Upon receiving the signal, only employer 2 will extend an offer. Since $z_2 < \alpha_1$,[6] the sender can no longer persuade employer 1 to extend an offer to any subset of candidates in the remaining pool $w \in [0, z_2)$ after targeting employer 2. Therefore, the sender can only transmit the signal $s = \varnothing$ when $w < z_2$. As a result, $q_1(M_2) = 0$ and $q_2(M_2) = \mathbb{P}[w \geq z_2]$. The sender receives an offer if and only if $w \geq z_2$, which occurs with probability $\mathbb{P}[w \geq z_2]$.

Second, consider the mechanism $M_1 \in \mathcal{M}$ where the sender prioritizes employer 1 and recommends to employer 2 only if candidates left by employer 1 are qualified. Let $\bar{z}_1 > 0$ be such that $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$. The sender transmits the signal $s = 1$ when $w \geq \bar{z}_1$. Thus, $q_1(M_1) = \mathbb{P}[w \geq \bar{z}_1]$. Then, the following two scenarios can occur depending on the value of $\bar{z}_1$ relative to α_2:

- If $\bar{z}_1 > \alpha_2$: The sender can still persuade employer 2 to extend an offer to some candidates in the remaining pool after targeting receiver 1. Specifically, find a real value z_1 with $0 < z_1 < \alpha_2 < \bar{z}_1$ such that $\mathbb{E}[w|z_1 \leq w < \bar{z}_1] = \alpha_2$. The sender transmits the signal $s = 2$ when $z_1 \leq w < \bar{z}_1$ and transmits the signal $s = \varnothing$ when $w < z_1$. Therefore, $q_2(M_1) = \mathbb{P}[z_1 \leq w \leq \bar{z}_1]$.

- If $\bar{z}_1 \leq \alpha_2$: The sender can no longer persuade employer 2 to extend an offer to the remaining pool once targeting employer 1. In this case, let $z_1 = \bar{z}_1$. The sender transmits the signal $s = \varnothing$ when $w < z_1$; therefore, $q_2(M_1) = 0$.

In both scenarios of M_1, the sender receives an offer if and only if $w \geq z_1$, which occurs with probability $\mathbb{P}[w \geq z_1]$.

Utilizing the knowledge of M_1 and M_2, Proposition 4.8 provides several properties regarding any mechanism in the set $\mathcal{M}$.

Proposition 4.8. *Given any public persuasion mechanism $M \in \mathcal{M}$, we have the following:*

1. $q_1(M) \leq q_1(M_1) = \mathbb{P}[w \geq \bar{z}_1]$.
2. $q_2(M) \leq q_1(M) + q_2(M) \leq q_2(M_2) = \mathbb{P}[w \geq z_2]$.

[5] We have $z_2 > 0$ because $\alpha_2 > w_0$ by Assumption 4.3.

[6] Otherwise, the posterior mean would exceed α_1, which is larger than α_2.

3. *For any mechanism $M \in \mathcal{M}$ with a cutoff $z \in (0,1)$ such that the candidate receives an offer if and only if $w \geq z$, one of the following must be true:*

   (a) $z \in [z_2, z_1]$, *or*

   (b) $z_1 < \alpha_2 < \bar{z}_1 = z$.

   In addition, case (b) is suboptimal and can disregarded in future discussions.[7]

4. *Conversely, for any value $z \in [z_2, z_1]$, there exists a mechanism $M \in \mathcal{M}$ such that the candidate receives an offer if and only if $w \geq z$. Moreover, for any such mechanism M, we have $q_1(M) = \mathbb{P}[w \geq z] \cdot \frac{\mathbb{E}[w|w \geq z] - \alpha_2}{\alpha_1 - \alpha_2} \geq 0$ and $q_2(M) = \mathbb{P}[w \geq z] \cdot \frac{\alpha_1 - \mathbb{E}[w|w \geq z]}{\alpha_1 - \alpha_2} \geq 0$.*

We prove Proposition 4.8 in Appendix A.9. Intuitively, the probability of joining employer 1 is highest when the sender primarily targets employer 1 (using mechanism M_1). However, this also lowers the probability of receiving any offer (which is $\mathbb{P}[w \geq z_1]$) among the "reasonable" mechanisms depicted in Bullet 3 because employer 1 is more challenging to get into. Conversely, the probability of receiving an offer is the highest when the sender exclusively targets the less competitive employer 2 (using mechanism M_2), which is $\mathbb{P}[w \geq z_2]$. Furthermore, Bullet 4 shows that any acceptance probability between these two extremes can be sustained by a mechanism that carefully balances the two employers. As we will show in Section 4.3.2, the optimal mechanism in the two-receiver case can be any of the mechanisms in Bullet 4, depending on the desirability (v_1) and hiring bar (α_1) of employer 1 relative to those of employer 2.

We conclude this section with a remark that interprets the two probabilities $q_1(M)$ and $q_2(M)$ in Bullet 4 of Proposition 4.8.

Remark 4.2 (Interpreting Bullet 4 of Proposition 4.8). To understand the two probabilities $q_1(M)$ and $q_2(M)$, note that for any mechanism $M \in \mathcal{M}$ with a cutoff structure and a threshold value of z, the probabilities $q_1 \triangleq q_1(M)$ and $q_2 \triangleq q_2(M)$ must satisfy the following two linear equations:

$$\begin{aligned} q_1 + q_2 &= \mathbb{P}[w \geq z], \\ \alpha_1 q_1 + \alpha_2 q_2 &= (q_1 + q_2) \cdot \mathbb{E}[w|w \geq z], \end{aligned} \tag{7}$$

where the first equation follows from the fact that the candidate receives an offer (from either employer 1 or 2) if and only if $w \geq z$, and the second equation follows from the fact that the

[7] Note that any optimal persuasion mechanism has cutoff structure according to Proposition 4.3. Moreover, when $\alpha_2 < \bar{z}_1$, since the sender can still persuade employer 2 to extend an offer to some candidates in the remaining pool after targeting receiver 1, there is no loss of optimality to assume that $z < \bar{z}_1$.

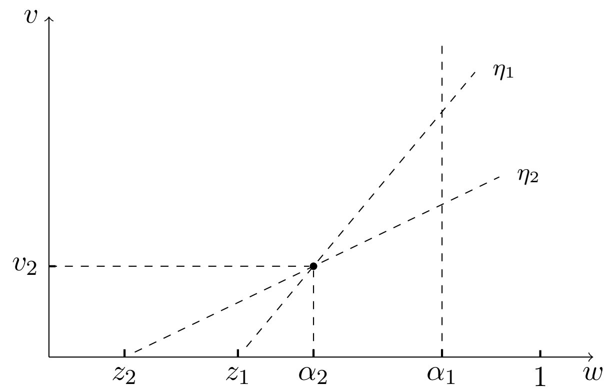

Figure 1: Partition of the value of v_1 in Lemma 4.9.

participation constraints are binding (i.e., $\mathbb{E}[w|s = i] = \alpha_i$) and the law of total expectation. These two equations uniquely determine the values of q_1 and q_2, as stated in Bullet 4 of Proposition 4.8. Conversely, consider a mechanism M that sends the signal $s = \varnothing$ if and only if $w < z$, and suppose that the values of q_1 and q_2 satisfy (7). According to (7), if $\mathbb{P}[s = 1] = q_1$ and $\mathbb{E}[w|s_1] = \alpha_1$, then it follows that $\mathbb{P}[s = 2] = q_2$ and $\mathbb{E}[w|s_2] = \alpha_2$, and vice versa.

4.3.2 Optimal Mechanism with Two Receivers

In this section, we characterize the optimal persuasion mechanism with two receivers. Intuitively, there is a trade-off: An offer from employer 1 brings a higher payoff, but targeting employer 1 more aggressively reduces the overall probability of receiving an offer.

Notably, according to Bullet 3 of Proposition 4.8, the cutoff z, such that the candidate with characteristics $w \in [z, 1]$ receives an offer, satisfies $z \in [z_2, z_1]$ for any reasonable mechanism $M \in \mathcal{M}$. This brings two lines, one of which (denoted by η_1) passes through the points $(z_1, 0)$ and (α_2, v_2), and the other (denoted by η_2) passes through the points $(z_2, 0)$ and (α_2, v_2), as illustrated in Figure 1. These two lines partition the value of $v_1 \in [v_2, \infty)$ into three regions, which determine the form of the optimal persuasion mechanism.

Lemma 4.9 illustrates the optimal persuasion mechanism for the two-receiver case. Specifically, if the value of v_1 is sufficiently large (particularly, above line η_1), prioritizing employer 1 is optimal. Alternatively, if v_1 is sufficiently small (particularly, below line η_2), completely targeting employer 2 is optimal. Finally, if v_1 falls between the two lines, the optimal mechanism requires a non-trivial trade-off between the two employers.

Lemma 4.9. *Under Assumptions 4.1 – 4.3, the optimal public persuasion mechanism with two* *receivers is given as follows:*

1. *If* $v_1 \leq v_2 \cdot \left(\frac{\alpha_1 - z_2}{\alpha_2 - z_2}\right)$ *(i.e., the point* (α_1, v_1) *lies below line* η_2*), mechanism* M_2 *(fully targeting employer* 2*) is optimal;*

2. *If* $v_1 \geq v_2 \cdot \left(\frac{\alpha_1 - z_1}{\alpha_2 - z_1}\right)$ *(i.e., the point* (α_1, v_1) *lies above line* η_1*), mechanism* M_1 *(prioritizing employer* 1*) is optimal;*

3. *Otherwise, any mechanism* $M \in \mathcal{M}$ *satisfying Bullet 4 of Proposition 4.8 with a cutoff value* z^**, where* $z^* \triangleq \alpha_2 - v_2 \cdot \left(\frac{\alpha_1 - \alpha_2}{v_1 - v_2}\right) \in [z_2, z_1]$ *represents the x-intercept of the line passing through the points* (α_2, v_2) *and* (α_1, v_1)*, is optimal. In other words, the mechanism* $M \in \mathcal{M}$ *satisfies the following:*

   (a) It sends the signal $s = \varnothing$ *with a probability of one if* $w < z^*$ *and a probability of zero if* $w \geq z^*$*;*

   (b) Participation constraints are binding (required by definition of $\mathcal{M}$*), that is,* $\mathbb{E}[w|s = i] = \alpha_i$ *for* $i \in \{1, 2\}$*;*

   (c) $q_1(M) = q_1^* \triangleq \mathbb{P}[w \geq z^*] \cdot \frac{\mathbb{E}[w|w \geq z^*] - \alpha_2}{\alpha_1 - \alpha_2}$ *and* $q_2(M) = q_2^* \triangleq \mathbb{P}[w \geq z^*] \cdot \frac{\alpha_1 - \mathbb{E}[w|w \geq z^*]}{\alpha_1 - \alpha_2}$ *(as indicated by Bullet 4 of Proposition 4.8).*

We prove Lemma 4.9 in Appendix A.10. In the proof, we identify a set of dual variables $\boldsymbol{\mu} \in \mathbb{R}_+^n$, which, together with the proposed mechanism, satisfies Bullet 2 of Lemma 4.5. This indicates that the mechanism is optimal to (3), and $\boldsymbol{\mu}$ is an optimal dual variable.

In Case 3 of Lemma 4.9, the trade-off between two employers is non-trivial. In this case, the optimal Lagrangian dual variable is $\boldsymbol{\mu}^* = \left(\mu_1^*, \mu_2^*\right)$ with $\mu_1^* = \mu_2^* = \frac{v_1 - v_2}{\alpha_1 - \alpha_2} > 0$. This value equals the slope of the line passing through the points (α_2, v_2) and (α_1, v_1). Consequently, the lines $\ell_1\left(w; \mu_1^*\right)$ and $\ell_2\left(w; \mu_2^*\right)$ of the two employers completely overlap and coincide with this line (as visualized in Figure 3(b)). As a result, according to (5), any allocation $\{q(i|w)\}$ with $q(1|w) + q(2|w) = 1$ for $w \geq z^*$ and $q(1|w) = q(2|w) = 0$ for $w < z^*$ is optimal to the Lagrangian $V^{\text{LR}}(\boldsymbol{\mu}^*)$. As long as we appropriately allocate the probability of one between $q(1|w)$ and $q(2|w)$ for any $w \geq z^*$, ensuring that the participation constraints are binding for both employers, it follows that the candidate joins each employer i with a probability of q_i^* according to Bullet 4 of Proposition 4.8, and that the mechanism $\{q(i|w)\}$ is optimal to (3) according to Lemma 4.9.

We note that although the aggregate allocation probabilities $\{q_i^*\}$ have the same value under any optimal persuasion mechanism according to Bullet 4 of Proposition 4.8, there are various ways to construct a set of probabilities $\{q(i|w)\}$ that satisfies Bullet 3 of Lemma 4.9 and is thus optimal to (3) (as we demonstrate in the proof of Bullet 4 of Proposition 4.8). Below, we present two simple approaches to construct an optimal mechanism and illustrate them in Example 4.1.

- (*Randomized Mechanism with a Monotone Structure*) Let $q(1|w) = q_1^* \,/\, \mathbb{P}[w \geq \bar{z}_1] \leq 1$ for $w \geq \bar{z}_1$ and $q(1|w) = 0$ otherwise, recalling that $\bar{z}_1 > 0$ is defined such that $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$. Additionally, let $q(2|w) = 1 - q(1|w)$ for $w \geq z^*$ and $q(2|w) = 0$ otherwise. This corresponds to a randomized persuasion mechanism that satisfies Bullet 3 of Lemma 4.9 and is, therefore, optimal to (3). Note that the candidate's expected payoff, $v(w) \triangleq \sum_i v_i q(i|w)$, is increasing in w by construction, which can be desirable in practice.[8]

- (*Deterministic Mechanism with a Double-Interval Structure*) Identify an interval $\left[\underline{b}, \bar{b}\right] \subseteq [\bar{z}_1, 1]$ such that $\mathbb{P}\left[\underline{b} \leq w \leq \bar{b}\right] = q_1^*$ and $\mathbb{E}\left[w|\underline{b} \leq w \leq \bar{b}\right] = \alpha_1$.[9] Let $q(1|w) = 1$ for $w \in \left[\underline{b}, \bar{b}\right]$, $q(2|w) = 1$ for $w \in [z^*, \underline{b}) \cup \left(\bar{b}, 1\right]$, and $q(\varnothing|w) = 1$ for $w < z^*$. This corresponds to the deterministic persuasion mechanism described in Candogan (2022). The mechanism satisfies Bullet 3 of Lemma 4.9 and is, therefore, optimal to (3). Additionally, the mechanism exhibits a double-interval structure, with each signal associated with at most two intervals. The candidate's expected payoff is not monotone in w with this mechanism.

Example 4.1. Suppose $w \sim \text{Unif}[0, 1]$ follows a uniform distribution with support $[0, 1]$, the candidate's payoffs from the two employers' offers are $v_1 = 2$ and $v_2 = 1$, and the employers' threshold values are $\alpha_1 = 0.9$ and $\alpha_2 = 0.7$. Given these values, we have $\bar{z}_1 = 0.8$, $z_1 = 0.6$, $z^* = 0.5$, and $z_2 = 0.4$. The optimal dual variables are $\mu_1^* = \mu_2^* = 5$. Figure 2(a) illustrates the lines $\ell_1(w; \mu_1^*)$ and $\ell_2(w; \mu_2^*)$ of the two employers, which fully overlap. Additionally, we have $z^* = 0.5$, $q_1^* = 1/8$, and $q_2^* = 3/8$. There are multiple ways to construct an optimal persuasion mechanism that satisfies Bullet 3 of Lemma 4.9. The previously described random persuasion mechanism is illustrated in Figure 2(b). The previously described deterministic persuasion mechanism is illustrated in Figure 2(c). We can also construct a second deterministic persuasion mechanism with a double-interval structure for this problem instance, where the signal $s = 1$ is associated with two intervals, as illustrated in Figure 2(d).

We conclude this section with a remark demonstrating that when receivers cannot communicate but are aware of each other, the vanilla private persuasion mechanism is suboptimal.

[8] An increasing payoff function $v(w)$ prevents a student from strategically degrading her "quality" w for a better offer.

[9] Since $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$ by definition and $\mathbb{P}[w \geq \bar{z}_1] \geq q_1^*$ by Bullet 1 of Proposition 4.8, such an interval exists.

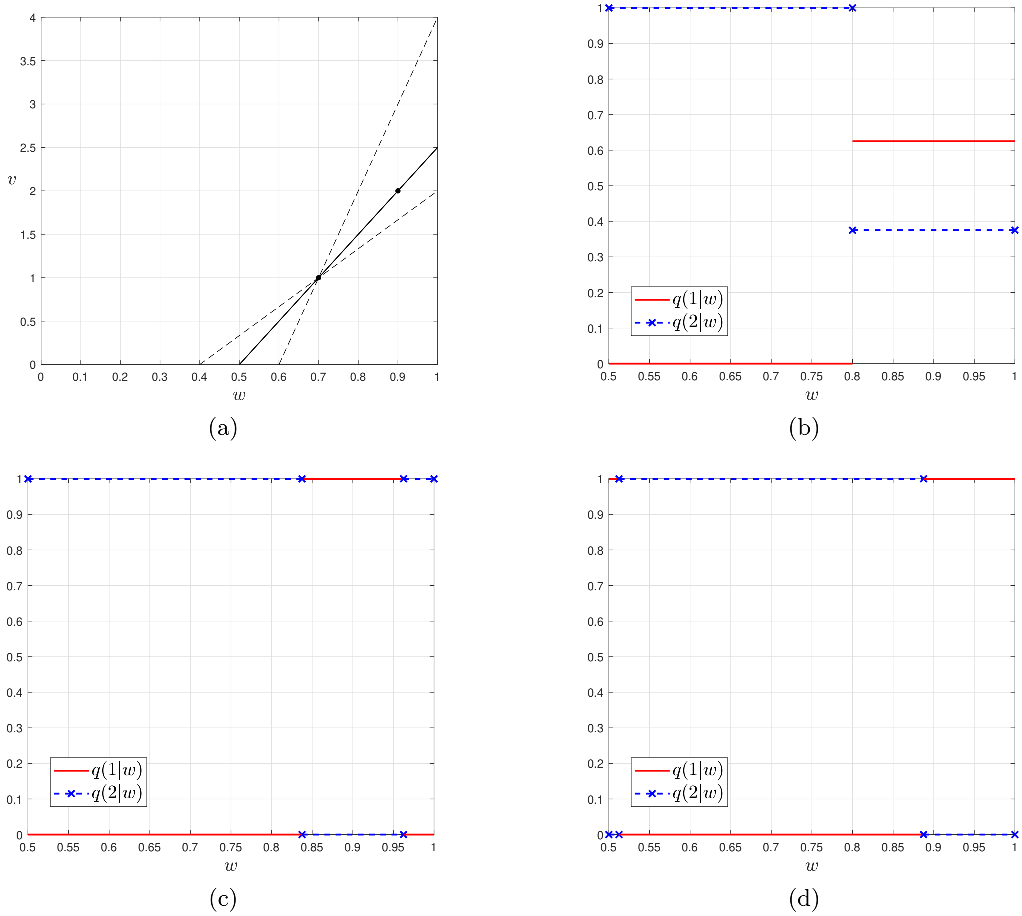

Figure 2: (a) Illustration of the two employers' lines $\ell_1(w;\mu_1^*)$ and $\ell_2(w;\mu_2^*)$ (solid), which pass through the two points $(0.7,1)$ and $(0.9,2)$ and fully overlap, and the two lines η_1 and η_2 in Lemma 4.9 (dashed). (b) A random optimal persuasion mechanism with $q(1|w)=5/8$ for $w\in[0.8,1]$, $q(2|w)=3/8$ for $w\in[0.8,1]$ and $q(2|w)=1$ for $w\in[0.5,0.8]$. (c) A deterministic optimal persuasion mechanism with $q(1|w)=1$ for $w\in[0.8375,0.9625]$ (centered around 0.9 and of length 1/8) and $q(2|w)=1$ for $w\in[0.5,0.8375]\cup[0.9625,1]$. (d) A deterministic optimal persuasion mechanism with $q(2|w)=1$ for $w\in[0.5125,0.8875]$ (centered around 0.7 and of length 3/8) and $q(1|w)=1$ for $w\in[0.5,0.5125]\cup[0.8875,1]$.

Remark 4.3 (Failure of Vanilla Private Persuasion Absent Communication)**.** When there is no communication channel between receivers, it is tempting to treat receivers in isolation and send separate signals to each employer using their respective optimal persuasion. We call this vanilla private persuasion mechanism and show this is suboptimal. First, we note that even if the sender knows the receivers have no communication channel and thus cannot communicate, a public persuasion mechanism remains optimal by Lemma 3.2. Conversely, A vanilla private persuasion mechanism involves sending the signal $s=1$ to employer 1 when $w\geq\bar{z}_1$ and the signal $s=2$ to employer 2 when $w\geq z_2$. Despite the lack of communication, employer 2, aware of the presence of a more preferred employer 1, will never extend an offer upon receiving the signal $s=2$. This is because the offer from employer 1 adversely selects candidates recommended to employer 2, leading employer 2 to a negative expected utility from the candidates who accept his offer. Notably, only candidates with characteristics $w \in [z_2, \bar{z}_1)$ will select employer 2, whose expected quality is $\mathbb{E}[w|z_2 \leq w < \bar{z}_1] < \mathbb{E}[w|w \geq z_2] = \alpha_2$, which is smaller than α_2. Given employer 2's equilibrium strategy, only candidates with quality $w \in [\bar{z}_1, 1]$ is placed, which is suboptimal according to lemma 4.9.

4.4 General Case

In this section, we show that an optimal persuasion mechanism essentially determines the expected portion of candidates hired by each employer. Specifically, we solve (3) for the general case by reducing it to a convex optimization problem (8) with n decision variables and constraints, which can be efficiently solved. Problem (8) is analogous to problem (OPT) in Candogan (2022), but with n fewer decision variables and constraints. In the following, we establish the connection between (3) and (8) from both the primal and dual viewpoints (Section 4.4.1), derive structural properties of the optimal persuasion mechanism based on the dual problem of (8) (Section 4.4.2), and construct an optimal persuasion mechanism with a monotone structure (Section 4.4.3).

$$
\begin{aligned}
V^{\mathrm{CR}} = \max_{q_i \geq 0} \quad & \sum_{i=1}^{n} v_i q_i \\
\text{s.t.} \quad & \sum_{i \leq k} \alpha_i q_i \leq \sum_{i \leq k} q_i \cdot \mathbb{E}\left[w \,\middle|\, G(w) \geq 1 - \sum_{i \leq k} q_i \right] = \int_{1-\sum_{i \leq k} q_i}^{1} G^{-1}(x)\, dx, \ \forall\, k \in [n], \\
& \sum_{i \in [n]} q_i \leq 1.
\end{aligned} \tag{8}
$$

We first interpret (8). In (8), the decision variables q_i represent the ex-ante probabilities that the candidate joins employer $i \in [n]$; specifically, q_i corresponds to $\int_0^1 q(i|w)g(w)dw$ in (3). The first constraint reflects the participation constraint for the employers. Only a limited portion of qualified candidates meet the employers' recruitment standards. This constraint requires that candidates within the top $\sum_{i \leq k} q_i$ quantile are sufficient to meet the recruiting bars (α_i) of the top k employers, given that each employer $i \in [k]$ would recruit a proportion q_i of candidates. This is a necessary condition to sustain the participation of the first k employers. The equation in this constraint follows from the fact that given any random variable w with a cumulative distribution function $G(\cdot)$, the the random variable $G(w)$ follows a uniform distribution on $[0, 1]$. Finally, we remark that

(8) is a convex optimization problem. To see this, note that $h(x) \triangleq \int_{1-x}^{1} G^{-1}(s)\,ds$ is a concave function because its derivative, $h'(x) = G^{-1}(1-x)$, decreases in x. Therefore, the right-hand side of the first constraint is a concave function of $\{q_i\}$ because it is the composition of $h(\cdot)$ with an affine mapping.

Given a feasible solution $\{q(i|w)\}$ to (3), the set $\{q_i\}$ with $q_i = \int_0^1 q(i|w)g(w)dw$ is feasible to (8) and attains the same objective value. Therefore, (8) is a relaxation of (3). Conversely, analogous to the two-receiver case (Section 4.3), the optimal aggregate allocation probabilities $\{q_i^*\}$, along with the binding participation constraints, characterize an optimal mechanism.[10] Notably, given an optimal solution $\{q_i^*\}$ to (8), we can construct a public persuasion mechanism that obtains the optimal value V^{CR}. Therefore, the relaxation (8) is tight. We state this in Proposition 4.10 and provide the proof in Appendix A.11.

Proposition 4.10. *The optimal values of (3) and (8) are equal; that is, $\bar{V} = V^{\mathrm{CR}}$. Furthermore, let $\{q^*(i|w)\}$ be an optimal solution to (3). Then, $\{q_i^*\}$, where $q_i^* = \int_0^1 q^*(i|w)g(w)dw$, is an optimal solution to (8). Conversely, if $\{q_i^*\}$ is an optimal solution to (8), then there exists an optimal solution $\{q^*(i|w)\}$ to (3) such that $q_i^* = \int_0^1 q^*(i|w)g(w)dw$.*

Let $\{q_i^*\}$ be an optimal solution to (8). In the following, we assume that $q_i^* > 0$ for all $i \in [n]$. Otherwise, we can disregard those employers with $q_i^* = 0$ with no loss of optimality.

4.4.1 Connection between the Optimal Dual Variables

In this section, we establish the equivalence of (3) and (8) from the dual point of view. Specifically, we show that an optimal dual variable for one problem can be converted to an optimal dual variable for the other.

Let $\mathbf{q} = (q_i)_{i\in[n]} \in \mathbb{R}^n$ be a vector of allocation probabilities for the n employers, and let $L(\mathbf{q}, \boldsymbol{\lambda})$ be the Lagrangian function in which we dualize the participation constraints in (8) with a dual variable $\boldsymbol{\lambda} = (\lambda_k)_{k\in[n]} \in \mathbb{R}_+^n$; that is:

$$L(\mathbf{q}, \boldsymbol{\lambda}) = \sum_{i=1}^{n} v_i q_i + \sum_{k\in[n]} \lambda_k \cdot \left(\int_{1-\sum_{i\le k} q_i}^{1} G^{-1}(x)\,dx - \sum_{i\le k} \alpha_i q_i \right).$$

Let $\mathbf{q}^* = (q_i^*)_{i\in[n]} \in \mathbb{R}_+^n$ represent an optimal solution to (8) and $\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]} \in \mathbb{R}_+^n$ an optimal dual variable for the participation constraints. By the KKT conditions, $\mathbf{q}^*$ solves the following

[10] Nevertheless, there are various ways to construct an optimal mechanism $\{q^*(i|w)\}$.

Lagrangian problem:

$$\mathbf{q}^* \in \underset{\mathbf{q}\in\mathbb{R}_+^n,\ \sum_{i\in[n]} q_i \le 1}{\operatorname{argmax}} L(\mathbf{q}, \boldsymbol{\lambda}^*)\,.$$

Since $q_i^* > 0$ for any $i \in [n]$ (i.e., we focus on non-trivial employers) and $\sum_{i\in[n]} q_i^* < 1$ by Proposition 4.1, the first-order optimality condition yields

$$\frac{\partial L}{\partial q_i}(\mathbf{q}^*, \boldsymbol{\lambda}^*) = v_i + \sum_{k\ge i} \lambda_k^* \cdot \left(G^{-1}\left(1 - \sum_{j\le k} q_j^*\right) - \alpha_i\right) = 0. \tag{9}$$

Proposition 4.11 establishes the connection between the optimal dual variables of (3) and (8), demonstrating that each can be derived from the other.

Proposition 4.11. *Suppose there exists an optimal solution $\{q_i^*\}$ to (8) such that $q_i^* > 0$ for any $i \in [n]$ (i.e., no employer is disregarded). Then, the optimal Lagrangian dual variables for the participation constraints in (3) and (8), denoted by $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]} \in \mathbb{R}_+^n$ and $\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]} \in \mathbb{R}_+^n$, respectively, are unique and satisfy $\mu_i^* = \sum_{k\ge i} \lambda_k^*$ for all $i \in [n]$.*

We prove Proposition 4.11 in Appendix A.12 by comparing the optimality condition (9) of (8) with the optimality condition of (3) in Section 4.2. Since the dual variables λ_k^* are nonnegative, Proposition 4.11 implies that the dual variables $\{\mu_i^*\}$ for (3) are decreasing. This aligns with Bullet 3 of Proposition 4.7 and intuitively follows from the fact that the envelope function $h(w; \boldsymbol{\mu}^*)$ is increasing and convex in w.

4.4.2 Structural Properties of Optimal Mechanism

In this section, we derive the structural properties of an optimal persuasion mechanism. Specifically, we demonstrate that the information design problem decouples over subsets of employers, with employers partitioned based on the positivity of the optimal dual variable $\{\lambda_k^*\}$.

Let $\{q_i^*\}$ denote an optimal solution to (8) and assume that $q_i^* > 0$ for any $i \in [n]$. Let

$$T \triangleq \left\{k \in [n] : \lambda_k^* > 0\right\}$$

denote the set of positive entries of the optimal dual variable $\boldsymbol{\lambda}^*$ for (8). Due to the complementary slackness property, the participation constraint in (8) is binding with the top k employers if $k \in T$; that is,

$$\sum_{i\le k} \alpha_i q_i^* = \mathbb{E}\left[w \cdot \mathbb{1}\left[w \ge G^{-1}\left(1 - \sum_{i\le k} q_i^*\right)\right]\right]. \tag{10}$$

We note that $\lambda_n^* = \mu_n^* > 0$ according to Bullet 1 of Proposition 4.7 and Proposition 4.11. Therefore, $n \in T$, rendering set T nonempty.

Suppose set $T = \{t_1 < t_2 < \cdots < t_m = n\}$ contains m employers. These m employers partition the n employers into m groups $\{T_i\}_{i\in[m]}$, where $T_1 = [t_1]$ and $T_i = [t_{i-1} + 1 : t_i]$ for any $i \in [2:m]$. Therefore, $\bigcup_{i\in[m]} T_i = [n]$ and $T_i \cap T_j = \varnothing$ for any $i \neq j$. Moreover, each group T_i contains exactly one element from T, which is the largest element in T_i.

If a group T_i contains more than one employer (i.e., $t_{i-1} + 1 < t_i$), then for any $k \in [t_{i-1} + 1 : t_i - 1]$, we have the following:

$$\mu_k^* = \sum_{j\geq k} \lambda_j^* = \mu_{t_i}^* = \frac{v_k - v_{t_i}}{\alpha_k - \alpha_{t_i}}\,, \tag{11}$$

where the first equation follows from Proposition 4.11, the second equation follows from the fact that $\lambda_j^* = 0$ for any $j \in [t_{i-1} + 1 : t_i - 1]$, and the third equation is derived by subtracting both sides of (9) with $i = t_i$ from both sides of the same equation with $i = k$. Therefore, the optimal dual variables for employers in the same group T_i all equal $\mu_{t_i}^*$. Furthermore, (11) implies that the points $\{(v_j, \alpha_j)\}_{j\in T_i}$ lie on a line, and the employers' lines $\ell_j(w; \mu_j^*)$ for any $j \in T_i$ completely overlap and coincide with this line.

In Lemma A.1 in the Appendix, we completely characterize the envelope function $h(w; \boldsymbol{\mu}^*)$. Let $z_i \triangleq G^{-1}\big(1 - \sum_{j=1}^{t_i} q_j^*\big)$ for any $i \in [m]$ and $z_0 = 0$. Each group T_i of employers is associated with an interval of state variable $I_i \triangleq [z_i, z_{i-1}]$. Lemma A.1 demonstrates that the envelope function $h(w; \boldsymbol{\mu}^*)$, which is convex and piecewise linear, aligns with line $\ell_j(w; \mu_j)$ on the interval $w \in [z_i, z_{i-1}]$ for any $j \in T_i$. Additionally, the function $h(w; \boldsymbol{\mu}^*)$ intersects the x-axis at $w = z_m$. We formally state the above in Lemma 4.12, with the proof provided in Appendix A.12.3.

Lemma 4.12 (Characterization of $h(w; \boldsymbol{\mu}^*)$)**.** *Let $\{q_i^*\}$ be an optimal solution to (8), and assume that $q_i^* > 0$ for all $i \in [n]$ (i.e., we drop ignorable employers i with $q_i^* = 0$). Let $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]}$ denote the optimal Lagrangian dual variable for the participation constraints in (3). The following hold:*

1. *For any $i \in [m]$ and employer $j \in T_i$, we have $\alpha_j \in \big(z_i, z_{i-1}\big)$.*
2. *For any group T_i, the lines $\ell_j(w; \mu_j^*)$ for $j \in T_i$ fully overlap, and they pass though the points (α_j, v_j) for any $j \in T_i$.*
3. *For any $i \in [m]$ and employer $j \in T_i$, $h(w; \boldsymbol{\mu}^*) = \ell_j(w; \mu_j^*)$ for any $w \in [z_i, z_{i-1}]$ and $h(w; \boldsymbol{\mu}^*) > \ell_j(w; \mu_j^*)$ for any $w \in [0, 1] \setminus [z_i, z_{i-1}]$.*

4. $h(w;\boldsymbol{\mu}^*)$ *is nonnegative if and only if* $w \geq z_m$.

According to (5) and Lemma 4.12, a set of allocation probabilities $\{q(i|w)\}$ is optimal to $V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ if and only if $\{q(i|w)\}$ are nonnegative, no larger than one, and satisfy the following:

$$\begin{aligned} &\sum_{j\in T_i} q(j|w) = 1,\ \forall\, w \in (z_i, z_{i-1}),\ i \in [m],\\ &\sum_{j\in[n]} q(j|w) = 0,\ \forall\, w < z_m. \end{aligned} \tag{12}$$

In other words, an optimal solution of $V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ allocates the interval $I_i = [z_i, z_{i-1}]$ among the employers in the subset T_i for any $i \in [m]$.

Moreover, for any group T_i and employer $k \in T_i$, subtracting both sides of the first constraint in (8) from both sides of (10) with $k = t_{i-1}$ and noting that the first constraint in (8) is binding with $k = t_i$ yields the following:

$$\begin{aligned} \sum_{j\in[t_{i-1}+1:k]} \alpha_j q_j^* &\leq \mathbb{E}\Big[w \cdot \mathbb{1}\Big[G^{-1}\Big(1 - \sum_{j\leq k} q_j^*\Big) \leq w < z_{i-1}\Big]\Big],\ \forall\, k \in [t_{i-1}+1 : t_i - 1],\\ \sum_{j\in T_i} \alpha_j q_j^* &= \mathbb{E}\Big[w \cdot \mathbb{1}\Big[z_i \leq w < z_{i-1}\Big]\Big]. \end{aligned} \tag{13}$$

In addition, note that $\mathbb{P}[z_i \leq w < z_{i-1}] = \sum_{j\in T_i} q_j^*$. Analogous to the proof of Proposition 4.10, we can allocate the state $w \in [z_i, z_{i-1}]$ to the employers in group T_i, possibly in a randomized way, so that each employer $j \in T_i$ is allocated with a size q_j^* of candidates, and the mean quality of the allocation to employer j is α_j (i.e., the participation constraint is tight). In other words, there exists an optimal solution $\{q^*(j|w)\}$ to $V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$, which satisfies

$$\begin{aligned} &\int_{w\in I_i} q^*(j|w)\, g(w)\, dw = q_j^*,\ \forall\, j \in T_i,\ i \in [m],\\ &\int_{w\in I_i} w \cdot q^*(j|w)\, g(w)\, dw = \alpha_j \int_{w\in I_i} q^*(j|w)\, g(w)\, dw,\ \forall\, j \in T_i,\ i \in [m]. \end{aligned} \tag{14}$$

From (12), (14), and Bullet 2 of Lemma 4.5, such an allocation $\{q^*(j|w)\}$ is an optimal to (3). We summarize the above in Lemma 4.13.

Lemma 4.13 (Optimality Condition). *An allocation probability* $\{q(j|w)\}$ *is optimal to (3) if and only if it allocates only to employers in the set* T_i *for any* $w \in I_i$ *(i.e., (12) holds) and all participation constraints in (3) are binding with* $\{q(j|w)\}$*. Moreover, let* $\{q_j^*\}$ *denote an optimal solution to (8). We can construct an optimal solution* $\{q^*(j|w)\}$ *to (3) such that the candidate joins each employer* *with probability q_j^* (i.e., (14) holds).*

According to Lemma 4.13, once we have solved an optimal solution $\{q_i^*\}$ and an optimal dual variable $\boldsymbol{\lambda}^*$ of (8) and obtained the partition $\{T_i\}_{i\in[m]}$ of employers, the design problem decouples over groups. Specifically, for each group T_i, the optimal mechanism allocates the state variable $w \in I_i$ to the employers in group T_i in a way that ensures that the participation constraints are binding. When the group contains only one employer, we simply allocate the entire interval I_i to the employer. When it contains multiple employers, the allocation needs to be conducted more carefully. Analogous to the two-receiver case (Section 4.3.2), there are multiple ways to construct an optimal mechanism. Specifically, based on (13), we can construct an optimal solution iteratively. Given that we have allocated a size q_j^* of candidates from interval I_i with a mean quality of α_j to each employer j within the first k employers in group T_i, we can also allocate a size $q_{j'}^*$ of candidates from the remaining candidates in interval I_i with a mean quality of $\alpha_{j'}$ to employer j', where j' denote the index of the $(k+1)$-th candidate in group T_i. Repeat this process until we reach the last employer of group T_i, which is employer t_i. The remaining candidates, with a size of $q_{t_i}^*$ and a mean quality of α_{t_i}, can then be allocated to the last employer.

In Section 4.4.3, we specify a particular allocation approach at each iteration step to obtain an optimal solution $\{q^*(j|w)\}$ to (3) with a monotone structure.

4.4.3 Constructing an Optimal Mechanism with a Monotone Structure

In this section, we construct an optimal persuasion mechanism $\{q^*(j|w)\}$ iteratively that additionally satisfies a monotone property, as defined in Definition 4.1. Specifically, for any $w \geq w'$, the distribution $q^*(\cdot|w)$ will first-order stochastically dominate the distribution $q^*(\cdot|w')$. Therefore, a student with a higher quality w is more likely to be in a better place, which is desirable in practice.

Definition 4.1 (Monotone Structure)**.** An optimal persuasion mechanism $\{q^*(j|w)\}$ satisfies a monotone property if, for any $w \geq w'$, the distribution $q^*(\cdot|w)$ first-order stochastically dominates the distribution $q^*(\cdot|w')$; in other words, we have $\sum_{k\leq i} q^*(k|w) \geq \sum_{k\leq i} q^*(k|w')$ for any $i \in [n]$.[11]

The monotone property automatically holds for two qualities w and w' from different intervals. Suppose $w \in I_i$ and $w' \in I_j$ with $i < j$. Since $\max T_i < \min T_j$, a candidate with quality w receives a better job for sure, which implies first-order stochastic dominance. Therefore, we only need to ensure the monotone structure for qualities within the same interval.

[11]Note that an offer from a lower-indexed employer provides a higher payoff to the candidate by Assumption 2.1.

Algorithm 1 presents a way to construct an optimal mechanism $\{q(j|w)\}$ for $j \in T_i$ iteratively. The distribution $q(\cdot|w)$ from Algorithm 1 is first-order stochastically increasing in w; additionally, $q(j|w)$ is piecewise constant on $w \in I_i$ for any $j \in T_i$.

Algorithm 1: Optimal Persuasion Mechanism with a Monotone Structure

Input: Let $\{q_i^*\}$ be an optimal solution to (8).

Initialization: Take $b_{t_{i-1}} = z_{i-1}$ and $q_{\leq t_{i-1}}(w) = 0$ for any $w \in I_i$.

for $k \in T_i = [t_{i-1}+1 : t_i - 1]$ **do**

  Identify two values $b_k \in \big[z_i, b_{k-1}\big]$ and $\rho_k \in [0,1]$ to ensure that (14) holds for employer k with $q(k|w) = \rho_k \cdot (1 - q_{\leq k-1}(w))$ for $w \in [b_k, z_{i-1}]$ and $q(k|w) = 0$ for $w < b_k$;

  Let $q_{\leq k}(w) = q_{\leq k-1}(w) + q(k|w)$ for any $w \in I_i$.

end

Let $q(t_i|w) = 1 - q_{\leq t_i - 1}(w)$ for any $w \in I_i$ and take $b_{t_i} = z_i$.

In Algorithm 1, $q_{\leq k}(w) = \sum_{j \leq k} q(j|w)$ represents the probability that a candidate with quality w receives an offer from one of the top k employers. Note that for any $k \leq t_{i-1}$, $q_{\leq k}(w) = 0$ for $w \in I_i$. In each iteration, we allocate a ratio ρ_k of the remaining candidates whose quality is at least b_k to employer k. The values of ρ_k and b_k are selected so the candidate joins employer k with a probability of q_k^*, and the mean quality of the candidate conditional on she joining employer k is α_k (i.e., (14) holds for employer k).

We note that the constructed sequence $\{b_k\}_{k \in T_i}$ is decreasing and partitions the interval I_i into subintervals $I_{ik} \triangleq [b_k, b_{k-1}]$ for $k \in T_i$. Additionally, the probabilities $q(j|w)$ equal a constant $q_j(k)$ on each subinterval $w \in I_{ik}$, where the values of $q_j(k)$ are specified as follows:

$$
\begin{aligned}
&q(j|w) = q_j(k) = 0, \, \forall\, k \geq j+1, \, w \in I_{ik}\,, \\
&q(j|w) = q_j(k) = \rho_j \cdot \Big(1 - q_{\leq j-1}(w)\Big) = \rho_j \cdot \prod_{\ell=k}^{j-1} \Big(1 - \rho_\ell\Big), \, \forall\, k \in [t_{i-1}+1 : j], \, w \in I_{ik}\,.
\end{aligned}
$$

Proposition 4.14 demonstrates that the values of $\{\rho_k\}$ and $\{b_k\}$ in Algorithm 1 exist, and the allocation $\{q(j|w)\}$ returned by Algorithm 1 is optimal to (3) and satisfies the first-order stochastic increasing property.

Proposition 4.14. *The allocation $\{q(j|w)\}$ returned by Algorithm 1 is optimal to (3) and satisfies the first-order stochastic increasing property.*

We prove Proposition 4.14 and demonstrate that the values of $\{\rho_k\}$ and $\{b_k\}$ can be easily identified in Appendix A.13. Note that when a group T_i contains two employers, the allocation $\{q(j|w)\}$ returned by Algorithm 1 concurs with the randomized mechanism with a monotone structure described in Section 4.3.2 for the two-receiver case.

Finally, we present two useful properties of the optimal solutions of (8) as established in Candogan (2022). Specifically, there exists an optimal solution $\{q_k^*\}$ such that each group T_i contains at most two employers with a positive probability of q_k^*. Additionally, the optimal solution of (8) is unique if no three points of $\{(\alpha_i, v_i)\}_{i\in[n]}$ are collinear. We state these two properties in Proposition 4.15.

Proposition 4.15. *The optimal solution of (8) satisfies the following two properties.*

1. *(Lemma 4 of Candogan 2022) Let $\{\lambda_k^*\}$ denote an optimal dual variable associated with the participation constraints in (8) and $\{T_i\}$ denote the corresponding partition of the n employers as described in Section 4.4.2. There exists an optimal solution $\{q_k^*\}$ to (8) such that $|T_i \cap P| \leq 2$ for any i, where $P \triangleq \{k \in [n] : q_k^* > 0\}$ denote the set of positive entries of $\{q_k^*\}$. In other words, each set T_i contains at most two employers with a positive probability of q_k^*.*
2. *(Appendix D of Candogan 2022) Problem (8) has a unique optimal solution $\{q_k^*\}$ if no three points of $\{(\alpha_i, v_i)\}_{i\in[n]}$ are collinear.*

We prove Proposition 4.15 in Appendix A.14 based on our previous dual analysis, which significantly simplifies the proof and renders both properties intuitive. For the first property, suppose a group T_i contains more than two employers with a positive probability of q_k^*. Since all points $\{(\alpha_j, v_j)\}_{j\in T_i}$ are collinear according to Bullet 2 of Lemma 4.12, we can reallocate the probabilities of two employers in T_i that are apart to an employer in between until we drain the probability of one of the two employers, without changing the objective value. For the second property, since no three points of $\{(\alpha_i, v_i)\}_{i\in[n]}$ are collinear, any group T_i contains at most two employers with a positive probability according to Bullet 2 of Lemma 4.12. Moreover, the values of these two probabilities are uniquely determined by two linear equations analogous to (7). We provide more details in Appendix A.14.

Proposition 4.15 implies that the information design problem in the general case can be reduced to design problems with two receivers, one for each group T_i. By applying the deterministic mechanism with a double-interval structure, as described in Section 4.3.2, to each group T_i, we obtain the deterministic persuasion mechanism described in Candogan (2022).

5 Conclusions

We have considered a Bayesian persuasion problem in which a school advisor (the sender) strategically discloses information to persuade n employers (the receivers) to extend offers. We demonstrate that as long as receivers take binary actions (extending an offer or not), and the sender has a known preference among the receivers and can accept only one offer, public persuasion is optimal in a broad sense—it is so independent of the communication details of the receivers. As a result, the sender eliminates any room for the receivers to communicate to infer further about the candidate, in her self-interest. Moreover, the optimal public persuasion mechanism can be derived from the first-best relaxation problem that imposes only participation constraints. We are hopeful that such a strong result can be extended to more general settings, which could be an interesting direction for future research.

We next investigate a specific setting where the state variable is one-dimensional, and the receivers' utility functions are linear functions (therefore, a receiver cares only about the posterior mean of the candidate's quality). We focus on efficient computation of the optimal (public) persuasion mechanism. We provide the optimal mechanism in closed form for the two-receiver case based on the optimality condition derived from the dual of the first-best relaxation problem. For the general case, although the optimal mechanism can be derived from a convex optimization analogous to that of Candogan (2022), we establish many of the results and provide new insights and a better understanding of the optimal mechanism based on the Lagrangian dual of the problem.

References

Alonso, R. and Câmara, O. (2016), 'Persuading voters', *American Economic Review* **106**(11), 3590–3605.

Anunrojwong, J., Iyer, K. and Manshadi, V. (2023), 'Information design for congested social services: Optimal need-based persuasion', *Management Science* **69**(7), 3778–3796.

Arieli, I. and Babichenko, Y. (2019), 'Private bayesian persuasion', *Journal of Economic Theory* **182**, 185–217.

Bergemann, D. and Morris, S. (2016), 'Bayes correlated equilibrium and the comparison of information structures in games', *Theoretical Economics* **11**(2), 487–522.

Bergemann, D. and Morris, S. (2019), 'Information design: A unified perspective', *Journal of Economic Literature* **57**(1), 44–95.

Bertsekas, D., Nedic, A. and Ozdaglar, A. (2003), *Convex analysis and optimization*, Vol. 1, Athena Scientific.

Bertsekas, D. P. (2016), *Nonlinear programming*, 3 edn, Athena Scientific.

Boleslavsky, R. and Cotton, C. (2015), 'Grading standards and education quality', *American Economic Journal: Microeconomics* **7**(2), 248–279.

Candogan, O. (2020), Information design in operations, *in* 'Pushing the Boundaries: Frontiers in Impactful OR/OM Research', INFORMS, pp. 176–201.

Candogan, O. (2022), 'Persuasion in networks: Public signals and cores', *Operations Research* **70**(4), 2264–2298.

Candogan, O., Guo, Y. and Xu, H. (2023), 'On information design with spillovers', *Available at SSRN 3537289* .

De Véricourt, F., Gurkan, H. and Wang, S. (2021), 'Informing the public about a pandemic', *Management Science* **67**(10), 6350–6357.

Drakopoulos, K., Jain, S. and Randhawa, R. (2021), 'Persuading customers to buy early: The value of personalized information provisioning', *Management Science* **67**(2), 828–853.

Dworczak, P. and Martini, G. (2019), 'The simple economics of optimal persuasion', *Journal of Political Economy* **127**(5), 1993–2048.

Ely, J. C. (2017), 'Beeps', *American Economic Review* **107**(1), 31–53.

Galperti, S. and Perego, J. (2023), 'Games with information constraints: seeds and spillovers', *Available at SSRN 3340090* .

Gentzkow, M. and Kamenica, E. (2017), 'Bayesian persuasion with multiple senders and rich signal spaces', *Games and Economic Behavior* **104**, 411–429.

Guo, Y. and Shmaya, E. (2019), 'The interval structure of optimal disclosure', *Econometrica* **87**(2), 653–675.

Kamenica, E. (2019), 'Bayesian persuasion and information design', *Annual Review of Economics* **11**, 249–272.

Kamenica, E. and Gentzkow, M. (2011), 'Bayesian persuasion', *American Economic Review* **101**(6), 2590–2615.

Kolotilin, A. (2018), 'Optimal information disclosure: A linear programming approach', *Theoretical Economics* **13**(2), 607–635.

Kolotilin, A., Mylovanov, T., Zapechelnyuk, A. and Li, M. (2017), 'Persuasion of a privately informed receiver', *Econometrica* **85**(6), 1949–1964.

Luenberger, D. G. (1997), *Optimization by vector space methods*, John Wiley & Sons.

Ostrovsky, M. and Schwarz, M. (2010), 'Information disclosure and unraveling in matching markets', *American Economic Journal: Microeconomics* **2**(2), 34–63.

Papanastasiou, Y., Bimpikis, K. and Savva, N. (2018), 'Crowdsourcing exploration', *Management Science* **64**(4), 1727–1746.

A Proofs

A.1 Proof of Lemma 3.1

Fix any information disclosure mechanism $f(\cdot|w)$. For any $i \in [n]$, let

$$\begin{aligned} q(i|w) &= \mathbb{P}\left[a_i^* = 1 \text{ and } a_j^* = 0, \forall\, j < i \mid w\right] \\ &= \int_{\boldsymbol{s}} \int_{\mathbf{c}} \delta_i(s_i, c_i) \Pi_{j<i} \left(1 - \delta_j(s_j, c_j)\right) c(\mathbf{c}|\boldsymbol{s}) f(\boldsymbol{s}|w) d\mathbf{c} d\boldsymbol{s}. \end{aligned}$$

denote the probability that employer i extends an offer and the candidate accepts it under the employers' equilibrium strategies when the candidate's characteristics are w. The random binary variable $a_i^* \in \{0, 1\}$ represents employer i's action of extending an offer in the equilibrium of the game induced by the mechanism $f(\cdot|w)$. Note that the candidate will accept employer i's offer if and only if none of the employers $j < i$ extends an offer.

We first prove that the participation constraint in (2) holds; that is,

$$\int_{w \in \Omega} u_i(w)\, q(i|w)\, dG(w) = \mathbb{E}\big[u_i(w) \cdot \mathbb{1}[a_i^* = 1 \text{ and } a_j^* = 0, \forall\, j < i]\big] \geq 0.$$

To see this, note that

$$\begin{aligned} &\mathbb{E}\big[u_i(w) \cdot \mathbb{1}[a_i^* = 1 \text{ and } a_j^* = 0, \forall\, j < i] \mid c_i, s_i\big] \\ =&\mathbb{E}\big[\mathbb{1}[a_i^* = 1] \mid c_i, s_i\big] \cdot \mathbb{E}\big[u_i(w) \cdot \mathbb{1}[a_j^* = 0, \forall\, j < i] \mid c_i, s_i\big] \geq 0, \end{aligned}$$

where the equation follows from the fact that the action a_i^* is independent of a_j^* and w conditional on the signal-communication-information pair (c_i, s_i), and the inequality follows from the optimality of the employer's equilibrium strategy—that is, employer i extends an offer only if doing so provides nonnegative utility to him. Taking expectation over (c_i, s_i) on both sides of the above inequality yields the desired result.

For the second constraint, note that for any $w \in \Omega$, we have

$$\sum_{i \in [n]} q(i|w) = \sum_{i \in [n]} \mathbb{P}\left[a_i^* = 1 \text{ for some } i \in [n] \mid w\right] \leq 1.$$

Finally, the expected payoff of the mechanism $f(\cdot|w)$ can be expressed as

$$\sum_{i=1}^{n} v_i \cdot \int_{w \in \Omega} q(i|w) \cdot dG(w),$$

which is the objective function of (2). Since $\{q(i|w)\}$ is feasible to (2) given any mechanism $f(\cdot|w)$, we have $V^* \leq \bar{V}$.

A.2 Proof of Lemma 3.2

Let $\{q^*(i|w)\}$ denote an optimal solution to (2). We first show that for any two employers j and k with $j < k$, we have

$$\int_{w \in \Omega} u_j(w)\, q^*(k|w)\, dG(w) < 0. \tag{15}$$

We prove this by contradiction. Assume that there exists j and k with $j < k$ such that

$$\int_{w\in\Omega} u_j(w)\, q^*(k|w)\, dG(w) \geq 0.$$

Consider the new allocation rule $\tilde{q}(i|w)$ defined as:

$$\tilde{q}(i|w) = \begin{cases} q^*(j|w) + q^*(k|w) & \text{if } i = j, \\ 0 & \text{if } i = k, \\ q^*(i|w) & \text{if } i \notin \{j,k\}. \end{cases}$$

$\{\tilde{q}(i|w)\}$ is feasible to (2), and because $v_j > v_k$, $\{\tilde{q}(i|w)\}$ achieves a strictly larger objective value than $\{q^*(i|w)\}$. This contradicts the fact that $\{q^*(i|w)\}$ is optimal to (2). Thus, our assumption fails.

Since a public persuasion mechanism leaves no payoff-related information for the receivers to communicate, there exists an equilibrium where employers make decisions based only on the public signal and ignore potential communication among themselves. We now show that it is an equilibrium for each employer $i \in [n]$ to extend an offer only upon receiving the signal $s = i$. To do so, suppose all employers other than employer i follow this strategy; we verify that it is optimal for employer i to do the same.

First, suppose employer i receives the signal $s = i$. The expected payoff for extending an offer is nonnegative because

$$\int_{w\in\Omega} u_i(w)\, dG\big(w|s=i\big) = \frac{1}{\int_w q^*(i|w)\, dG(w)} \int_{w\in\Omega} u_i(w)\, q^*(i|w)\, dG(w) \geq 0,$$

where $dG\big(w|s=i\big) = \frac{q^*(i|w)\, dG(w)}{\int_w q^*(i|w)\, dG(w)}$ denotes the posterior belief of w given $s = i$, and the inequality follows from the participation constraint in (2). Therefore, it is optimal for the employer i to extend an offer.

Second, suppose employer i receives the signal $s = k$ with $k > i$. The expected payoff for extending an offer is negative because

$$\int_{w\in\Omega} u_i(w)\, dG\big(w|s=k\big) = \frac{1}{\int_w q^*(k|w)\, dG(w)} \int_{w\in\Omega} u_i(w)\, q^*(k|w)\, dG(w) < 0,$$

where the inequality follows from (15). Therefore, employer i will not extend an offer.

Finally, suppose employer i receives the signal $s = j$ with $j < i$. Since the candidate will never accept employer i's offer (because employer j will extend an offer), employer i is indifferent between extending an offer or not.

Note that the expected payoff for the school advisor is $\bar{V}$ under this equilibrium. Therefore, the public mechanism $f^*(\cdot|w)$ is optimal to (1).

A.3 Proof of Proposition 4.1

Since the threshold value α_i is strictly decreasing in the employer index i by Assumption 4.3, the probability of receiving an offer, expressed as $\sum_{i\in[n]} \int_0^1 q(i|w)g(w)dw$, is maximized when the sender targets only employer n with the lowest threshold value α_n; that is, $q(i|w) = 0$ for any $i \neq n$ and $w \in [0,1]$. This is because, given any feasible solution $\{q(i|w)\}$ of (3), we can construct a new solution $\{\tilde{q}(i|w)\}$ with $\tilde{q}(n|w) = \sum_{i\in[n]} q(i|w)$ and $\tilde{q}(i|w) = 0$ for any $i < n$, which is feasible to

(3) and attains the same probability of receiving an offer. On the other hand, if $\{q(i|w)\}$ assigns a strictly positive probability to any employer $i < n$, the participation constraint of employer n is not binding under the new construction $\{\tilde{q}(i|w)\}$. Given Assumption 4.3, we can allocate more mass to employer n without violating its participation constraint.

On the other hand, if the sender targets only employer n, the probability is maximized with $q(n|w) = 1$ for any $w \geq z_n$ and $q(n|w) = 0$ otherwise, resulting in a probability of $\mathbb{P}(w \geq z_n)$.

Finally, we note that $z_n > 0$ because $w_0 < \alpha_n$ by Assumption 4.3. Therefore, $\mathbb{P}(w \geq z_n) < 1$.

A.4 Proof of Proposition 4.2

Let $\{q(i|w)\}$ be a feasible solution to (3) and suppose that the participation constraint for an employer j holds with strict inequality; that is,

$$\int_0^1 w \cdot q(j|w)\, g(w)\, dw > \alpha_j \int_0^1 q(j|w)\, g(w).$$

Since $\sum_{i\in[n]} \int_0^1 q(i|w)\, g(w)\, dw < 1$ by Proposition 4.1, we can allocate more mass to employer j so that $\int_0^1 q(j|w)\, g(w)\, dw$ strictly increases, $q(i|w)$ remains unchanged for any $i \neq j$ and $w \in [0, 1]$, and the participation constraint for employer j still holds. This increases the objective value and implies that $\{q(i|w)\}$ is suboptimal.

A.5 Proof of Proposition 4.3

Let $\{q(i|w)\}$ be a feasible solution of (3), and define $z \triangleq \sup\left\{z \in [0, 1] : \sum_{i\in[n]} \int_0^z q(i|w)\, dw = 0\right\}$ as the (essential) lower bound on the support of $\{q(i|w)\}$. If $\sum_{i\in[n]} \int_z^1 q(i|w)dw < \mathbb{P}(w \geq z)$, there exists a point $\tilde{z} \in (z, 1)$ such that

$$\sum_{i\in[n]} \int_z^{\tilde{z}} q(i|w)\, dw = \sum_{i\in[n]} \int_{\tilde{z}}^1 (1 - q(i|w))\, dw > 0.$$

We can create a new feasible solution $\{\tilde{q}(i|w)\}$ from $\{q(i|w)\}$ by transporting the mass of $\{q(i|w)\}$ below $\tilde{z}$ to fill the "unoccupied" area above $\tilde{z}$; therefore, $\sum_{i\in[n]} \int_{\tilde{z}}^1 \tilde{q}(i|w)dw = \mathbb{P}(w \geq \tilde{z})$ and $\sum_{i\in[n]} \int_0^{\tilde{z}} \tilde{q}(i|w)dw = 0$. The two feasible solutions $\{\tilde{q}(i|w)\}$ and $\{q(i|w)\}$ have the same objective value because, by transporting, $\int_0^1 q(i|w)dw = \int_0^1 \tilde{q}(i|w)dw$ for any $i \in [n]$.

On the other hand, since $\{q(i|w)\}$ satisfies the participation constraints and we have shifted a positive mass of $\{q(i|w)\}$ from below $\tilde{z}$ to above $\tilde{z}$, the participation constraint for some employer $i \in [n]$ must hold with strict inequality with $\{\tilde{q}(i|w)\}$. According to Proposition 4.2, $\{\tilde{q}(i|w)\}$, and thus $\{q(i|w)\}$, must be suboptimal.

A.6 Proof of Lemma 4.5

Since the thresholds α_i are smaller than one by Assumption 4.3, it is straightforward to create a feasible solution to (3) where all participation constraints in (3) are satisfied with strict inequality. Therefore, strong duality holds and an optimal dual variable $\boldsymbol{\mu}^*$ exists according to Theorem 1 in Section 8.6 of Luenberger (1997).

Once strong duality is established, Bullet 2 follows from the optimality condition (Proposition 6.1.5 in Bertsekas 2016) and Proposition 4.2, which states that the participation constraints are binding with any optimal solution of (3).

A.7 Proof of Proposition 4.6

Let $\{q^*(i|w)\}$ be an optimal solution to (3) such that $\int_0^1 q^*(i|w)dw > 0$. Since $\{q^*(i|w)\}$ is also optimal to $V^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ by Lemma 4.5 Bullet 2, the line of employer i, $\ell_i(w;\mu_i^*)$, is a component of the envelope function $\bar{h}(w;\boldsymbol{\mu}^*)$. This implies that there exist constants $\underline{b},\bar{b} \in [0,1]$ such that $\bar{h}(w;\boldsymbol{\mu}^*) = \ell_i(w;\mu_i^*)$ for $w \in [\underline{b},\bar{b}]$ and $\bar{h}(w;\boldsymbol{\mu}^*) > \ell_i(w;\mu_i^*)$ otherwise. We now show that $0 < \underline{b} < \alpha_1 < \bar{b} \leq 1$.

First, note that $q^*(i|w) = 0$ for any $w \in [0,\underline{b}) \cup (\bar{b},1]$ because line $\ell_i(w;\mu_i^*)$ is strictly below $\bar{h}(w;\boldsymbol{\mu}^*)$ in this region. This implies that $\alpha_1 \in (b_1, b_2)$ because otherwise, the participation constraint of employer i cannot be binding, which contradicts Lemma 4.5 Bullet 2.

We next prove $\underline{b} > 0$ by contradiction. If $\underline{b} = 0$, then for any $w > 0$, we have

$$h\big(w;\boldsymbol{\mu}^*\big) \geq \ell_i\big(w;\mu_i^*\big) > \ell_i\big(0;\mu_i^*\big) = \bar{h}\big(0;\boldsymbol{\mu}^*\big) \geq 0,$$

where the first inequality follows from the definition of the envelope function $h(w;\boldsymbol{\mu}^*)$ and the second inequality from the fact that $\mu_i^* > 0$ by Proposition 4.7 Bullet 1. Therefore, $\sum_{i\in[n]} q^*(i|w) = 1$ for any $w > 0$ according to (5), which contradicts Proposition 4.3.

A.8 Proof of Proposition 4.7

Proof of Bullet 1 We prove $\mu_i^* > 0$ for any $i \in [n]$ by contradiction. If $\mu_i^* = 0$ for some $i \in [n]$, then for any $w \in [0,1]$, we have $h(w;\boldsymbol{\mu}^*) \geq \ell_i(w;\mu_i^*) = v_i > 0$. Consequently, $\sum_{i\in[n]} q^*(i|w) = 1$ for any $w \in [0,1]$ according to (5), which contradicts Proposition 4.3.

Proof of Bullet 2 Since $\mu_i^* > 0$ for any $i \in [n]$, the envelope function $h(w;\boldsymbol{\mu}^*)$ is strictly increasing. Let $z^* \in (0,1)$ be the root of $h(w;\boldsymbol{\mu}^*)$ such that $h(z^*;\boldsymbol{\mu}^*) = 0$, and let $\{q^*(i|w)\}$ be an optimal solution to (3). Since $\{q^*(i|w)\} \in \mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu}^*)$ by Lemma 4.5, it follows that $\sum_{i\in[n]} q^*(i|w) = 1$ for any $w > z^*$ and $q^*(i|w) = 0$ for any $i \in [n]$ and $w < z^*$ according to (5).

Proof of Bullet 3 From Proposition 4.6, the lines of employers in the set P are components of the envelope function $\bar{h}(w;\boldsymbol{\mu}^*)$. Since $\bar{h}(w;\boldsymbol{\mu}^*)$ is convex and piecewise linear, and the slope of each component equals the dual variable μ_i^* of the corresponding employer, $\{\mu_i^*\}_{i\in P}$ are decreasing with the employer index i.

A.9 Proof of Proposition 4.8

Bullets 1 and 2 follow from Proposition 4.1.

Proof of Bullet 3 In the following, we prove that if the cutoff value z satisfies $z < \bar{z}_1$ if $\alpha_2 < \bar{z}_1$, then we have $z \in \big[z_2, z_1\big]$.

For ease of notation, we drop the dependence on the mechanism M by letting $q_1 = q_1(M)$ and $q_2 = q_2(M)$. If the mechanism has a cutoff structure with a threshold value of z, the following two linear equations must hold:

$$\begin{aligned} q_1 + q_2 &= \mathbb{P}[w \geq z], \\ \alpha_1 q_1 + \alpha_2 q_2 &= (q_1 + q_2)\cdot \mathbb{E}[w|w \geq z]. \end{aligned} \tag{16}$$

The first equation follows from the fact that the candidate receives an offer (from either employer 1 or 2) if and only if $w \geq z$, and the second equation follows from the cutoff structure, the law of total expectation

$$\mathbb{E}[w|w \geq z] = \frac{q_1}{q_1+q_2} \cdot \mathbb{E}[w|s=1] + \frac{q_2}{q_1+q_2} \cdot \mathbb{E}[w|s=2],$$

and the fact that $\mathbb{E}[w|s=i] = \alpha_i$ by the IC constraints for mechanisms in the set $\mathcal{M}$. The two equations in (16) determine the values of q_1 and q_2 as

$$\begin{aligned} q_1 &= \mathbb{P}[w \geq z] \cdot \frac{\mathbb{E}[w|w \geq z] - \alpha_2}{\alpha_1 - \alpha_2}, \\ q_2 &= \mathbb{P}[w \geq z] \cdot \frac{\alpha_1 - \mathbb{E}[w|w \geq z]}{\alpha_1 - \alpha_2}. \end{aligned} \tag{17}$$

We now validate that $z \in [z_2, z_1]$.

First, we show that $z \geq z_2$. If $z < z_2$, then $\mathbb{E}[w|w \geq z] < \mathbb{E}[w|w \geq z_2] = \alpha_2$, which implies $q_1 < 0$. Therefore, we must have $z \geq z_2$.

Next, we show that $z \leq \bar{z}_1$. If $z > \bar{z}_1$, then $\mathbb{E}[w|w \geq z] > \mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$, which implies $q_2 < 0$. Therefore, we have $z \leq \bar{z}_1$.

Finally, we show that $z \leq z_1 \leq \bar{z}_1$. If $\bar{z}_1 \leq \alpha_2$, then $z_1 = \bar{z}_1$ and we are done. Now, suppose $z_1 < \alpha_2 < \bar{z}_1$. In this case, it suffices to show that $z \notin (z_1, \bar{z}_1)$. We prove that if $z \in (z_1, \bar{z}_1)$, then $q_1 > q_1(M_1) = \mathbb{P}[w \geq \bar{z}_1]$, which contradicts Bullet 1. Specifically, let $q_1(z)$ and $q_2(z)$ denote the values of q_1 and q_2 as a function of the threshold z. From (17) we have[12]

$$q_1(\bar{z}_1) = \mathbb{P}[w \geq \bar{z}_1], \quad q_2(\bar{z}_1) = 0$$

and

$$q_1(z_1) = \mathbb{P}[w \geq \bar{z}_1], \quad q_2(z_1) = \mathbb{P}[z_1 \leq w \leq \bar{z}_1].$$

From (17), we can express $q_1(z)$ as

$$q_1(z) = \frac{1}{\alpha_1 - \alpha_2} \int_z^1 (w - \alpha_2) g(w) dw.$$

Since the derivative is

$$\frac{dq_1(z)}{dz} = \frac{\alpha_2 - z}{\alpha_1 - \alpha_2} \cdot g(z),$$

$q_1(z)$ is increasing in $z \in [z_1, \alpha_2]$ and decreasing in $z \in [\alpha_2, \bar{z}_1]$. Therefore, $q_1(z) > \mathbb{P}[w \geq \bar{z}_1]$ for any $z \in (z_1, \bar{z}_1)$, which contradicts Proposition 4.8 Bullet 1.

Combining the above three steps yields $z \in [z_2, z_1]$.

Proof of Bullet 4 If such a mechanism M exists, we must have $q_1 \triangleq q_1(M) = \mathbb{P}[w \geq z] \cdot \frac{\mathbb{E}[w|w \geq z] - \alpha_2}{\alpha_1 - \alpha_2}$ and $q_2 \triangleq q_2(M) = \mathbb{P}[w \geq z] \cdot \frac{\alpha_1 - \mathbb{E}[w|w \geq z]}{\alpha_1 - \alpha_2}$ by (17). Since $z \in [z_2, z_1]$, we have $\mathbb{E}[w|w \geq z] \in [\alpha_2, \alpha_1]$, implying that $q_1 \geq 0$, $q_2 \geq 0$, and $q_2 \leq \mathbb{P}[w \geq z] \leq \mathbb{P}[w \geq z_2]$. Additionally, the proof of Bullet 3 indicates that $q_1 \leq \mathbb{P}[w \geq \bar{z}_1] \leq \mathbb{P}[w \geq z]$ when $z \in [z_2, z_1]$.

A mechanism M that meets the criteria of Bullet 4 must satisfy the following:

1. $q(1|w) + q(2|w) = 1$ for any $w \geq z$, and $q(1|w) = q(2|w) = 0$ for any $w < z$;
2. $\mathbb{P}[s=1] = q_1$, and $\mathbb{P}[s=2] = q_2$;

[12]This is because $\mathbb{E}[w|w \geq \bar{z}_1] = \alpha_1$ and $\mathbb{E}[w|w \geq z_1] = \frac{\mathbb{P}[w \geq \bar{z}_1]}{\mathbb{P}[w \geq z_1]}\alpha_1 + \frac{\mathbb{P}[z_1 \leq w \leq \bar{z}_1]}{\mathbb{P}[w \geq z_1]}\alpha_2$.

3. $\mathbb{E}[w|s=1]=\alpha_1$, $\mathbb{E}[w|s=2]=\alpha_2$, and $\mathbb{E}[w|s=\varnothing]<\alpha_2$.

A feasible mechanism $M\in\mathcal{M}$ can be constructed in multiple ways. For example, we can create a *deterministic* persuasion mechanism such that $q(1|w)=1$ for $w\in T$, $q(2|w)=1$ for $w\in[z,1]\setminus T$, and $q(\varnothing|w)=1$ for $w<z$, for some subset $T\subseteq[z,1]$. To satisfy the requirements in Bullet 4, the subset $T\subseteq[z,1]$ must satisfy the following conditions:

1. $\mathbb{P}[w\in T]=q_1$, and $\mathbb{P}[w\in[z,1]\setminus T]=q_2$;
2. $\mathbb{E}[w|w\in T]=\alpha_1$, $\mathbb{E}[w|w\in[z,1]\setminus T]=\alpha_2$, and $\mathbb{E}[w|w<z]<\alpha_2$.

There are, again, various ways to construct a feasible set T. For instance, set T can be an interval $[\underline{b},\bar{b}]\subseteq[\bar{z}_1,1]$ that includes the point α_1 and satisfies $\mathbb{P}[\underline{b}\le w\le\bar{b}]=q_1$ and $\mathbb{E}[w|\underline{b}\le w\le\bar{b}]=\alpha_1$. A feasible interval $[\underline{b},\bar{b}]$ exists because $\mathbb{E}[w|w\ge\bar{z}_1]=\alpha_1$ and $\mathbb{P}[w\ge\bar{z}_1]\ge q_1$. In addition, we have $\mathbb{P}[w\in[z,1]\setminus T]=q_2$ and $\mathbb{E}[w|w\in[z,1]\setminus T]=\alpha_2$ because the values of q_1 and q_2 satisfy (16). Finally, we have $\mathbb{E}[w|w<z]\le\mathbb{E}[w|w<z_1]<\alpha_2$, where the second inequality follows from the fact that $z_1=\bar{z}_1\le\alpha_2$ when $\bar{z}_1\le\alpha_2$ and $\mathbb{E}[w|w<z_1]<\mathbb{E}[w|z_1\le w<\bar{z}_1]=\alpha_2$ when $\bar{z}_1>\alpha_2$.

A.10 Proof of Lemma 4.9

In this proof, we identify a set of dual variables $\boldsymbol{\mu}\in\mathbb{R}^n_+$, which, along with the proposed mechanism in Lemma 4.9, satisfies Lemma 4.5 Bullet 2. This indicates that the mechanism is optimal to (3), and $\boldsymbol{\mu}$ is an optimal dual variable.

Proof of Bullet 1 Suppose $v_1\le v_2\cdot\frac{\alpha_1-z_2}{\alpha_2-z_2}$, which implies that the point (α_1,v_1) lies below line η_2. We construct the employers' associated lines ℓ_1 and ℓ_2 as follows.

Let line ℓ_2 coincide with line η_2 by taking the dual variable $\mu_2=v_2/(\alpha_2-z_2)$. Let line ℓ_1 lie below line ℓ_2 for $w\in[z_2,1]$. For example, this can be achieved by taking the dual variable $\mu_1=v_1/(\alpha_1-z_2)$. The two lines ℓ_1 and ℓ_2 are illustrated in Figure 3(a). Since line ℓ_2 dominates line ℓ_1, an optimal solution to the Lagrangian $V^{\mathrm{LR}}(\boldsymbol{\mu})$ with $\boldsymbol{\mu}=(\mu_1,\mu_2)$ will never recommend the candidate to employer 1, regardless of the candidate's characteristics w. It is easy to verify that the mechanism M_2 and the dual variable $\boldsymbol{\mu}=(\mu_1,\mu_2)$ satisfy Lemma 4.5 Bullet 2. Therefore, mechanism M_2 is optimal to (3), and $\boldsymbol{\mu}=(\mu_1,\mu_2)$ is an optimal dual variable.

Proof of Bullet 2 Suppose $v_1\ge v_2\cdot\frac{\alpha_1-z_1}{\alpha_2-z_1}$, which implies that the point (α_1,v_1) lies above the line η_1. We construct the employers' associated lines ℓ_1 and ℓ_2 in the following two cases: when $\bar{z}_1\le\alpha_2$ and when $\bar{z}_1>\alpha_2$.

1. $z_1=\bar{z}_1\le\alpha_2$: Let line ℓ_1 be the line passing through the points $(z_1,0)$ and (α_1,v_1) by taking the dual variable $\mu_1=v_1/(\alpha_1-z_1)$. Let line ℓ_2 lie below line ℓ_1 for $w\in[z_1,1]$. For example, this can be achieved by taking $\mu_2=v_2/(\alpha_2-z_1)$ (in which case line ℓ_2 coincides with line η_1), because the point (α_2,v_2) lies below line ℓ_1. The lines ℓ_1 and ℓ_2 are illustrated in Figure 3(c).
2. $z_1<\alpha_2<\bar{z}_1$: Let line ℓ_2 coincide with line η_1 by taking the dual variable $\mu_2=v_2/(\alpha_2-z_1)$. Let line ℓ_1 be the line passing through the points $(\bar{z}_1,\frac{v_2}{\alpha_2-z_1}(\bar{z}_1-\alpha_2)+v_2)$ and (α_1,v_1) by taking the dual variable $\mu_1=\frac{\frac{v_2}{\alpha_2-z_1}(\bar{z}_1-\alpha_2)+v_2-v_1}{\bar{z}_1-\alpha_1}$. It is easy to verify that line ℓ_1 intersects line ℓ_2 at $w=\bar{z}_1$ and $\mu_1>\mu_2$.[13] After the setup, line ℓ_1 is above line ℓ_2 for $w\in[\bar{z}_1,1]$ and line ℓ_2 is above line ℓ_1 for $w\in[z_1,\bar{z}_1]$. The lines ℓ_1 and ℓ_2 are illustrated in Figure 3(d).

[13]Intuitively, $\mu_1>\mu_2$ because the point (α_1,v_1) lies above line η_1.

It is easy to verify that the mechanism M_1 and the dual variable $\boldsymbol{\mu} = (\mu_1, \mu_2)$ satisfy Lemma 4.5 Bullet 2 in both cases. As a result, mechanism M_1 is optimal to (3), and $\boldsymbol{\mu} = (\mu_1, \mu_2)$ is an optimal dual variable.

Proof of Bullet 3 Suppose $v_1 \in \left(v_2 \cdot \frac{\alpha_1 - z_2}{\alpha_2 - z_2}, v_2 \cdot \frac{\alpha_1 - z_1}{\alpha_2 - z_1}\right)$, which implies that the point (α_1, v_1) lies between the two lines η_1 and η_2. We define the dual variables $\mu_1 = \mu_2 = \frac{v_1 - v_2}{\alpha_1 - \alpha_2}$ so that the two lines ℓ_1 and ℓ_2 overlap and pass through (α_2, v_2) and (α_1, v_1). These lines intersect the x-axis at $w = z^* \in [z_2, z_1]$, as illustrated in Figure 3(b).

It is easy to verify that any mechanism $M \in \mathcal{M}$ feasible to Lemma 4.9 Bullet 3, together with the dual variable $\boldsymbol{\mu} = (\mu_1, \mu_2)$, satisfies Bullet 2 of Lemma 4.5. Therefore, such a mechanism M is optimal to (3), and $\boldsymbol{\mu} = (\mu_1, \mu_2)$ is an optimal dual variable. The existence of a mechanism $M \in \mathcal{M}$ meeting Bullet 3 of Lemma 4.9 is confirmed by Bullet 4 of Proposition 4.8.

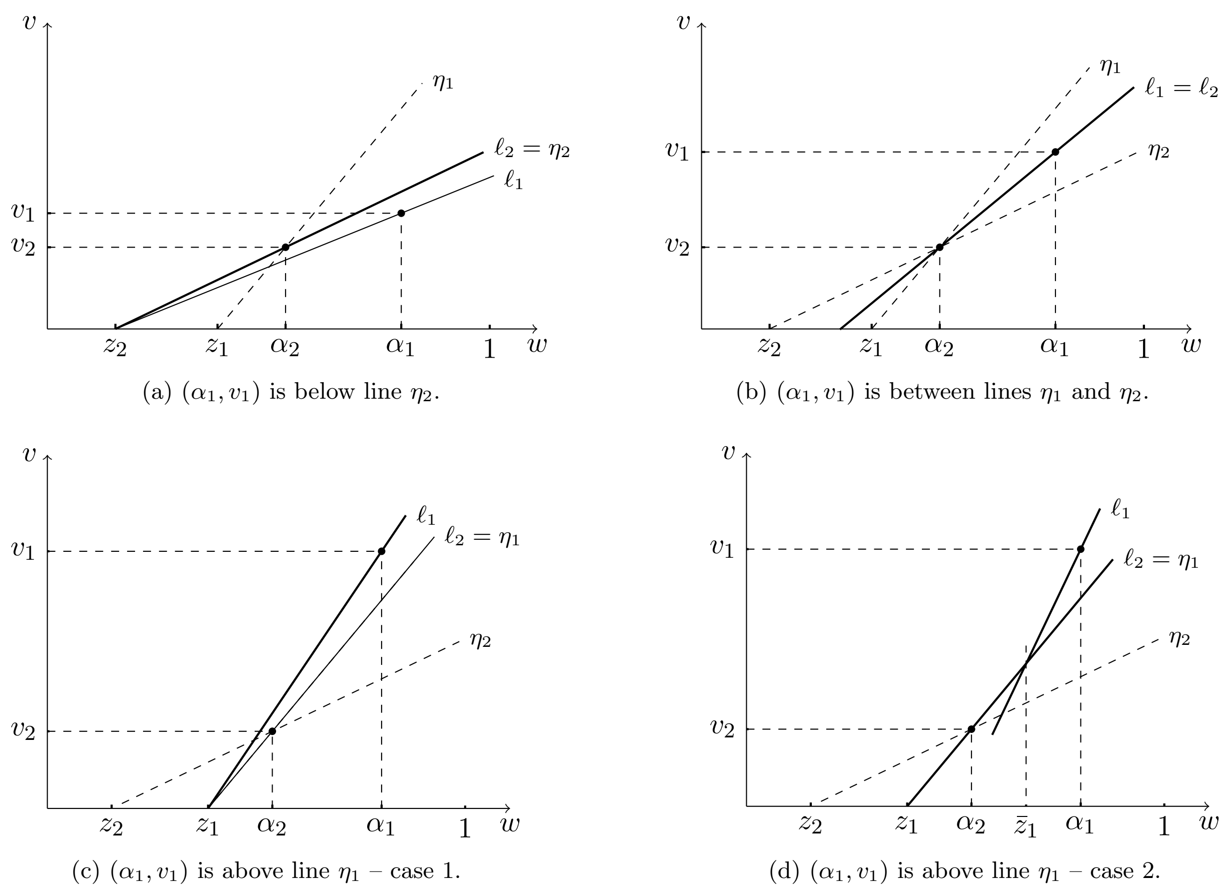

Figure 3: Visualization of two employers' associated lines.

A.11 Proof of Proposition 4.10

Step One: Proving $\bar{V} \le V^{\text{CR}}$ We first prove that (8) is a relaxation of (3); therefore, $\bar{V} \le V^{\text{CR}}$. Specifically, let $\{q(i|w)\}$ be a feasible solution to (3). Define $q_i = \int_0^1 q(i|w)g(w)dw$ for any $i \in [n]$. We show that $\{q_i\}$ is feasible to (8). This, together with the fact that $\{q(i|w)\}$ and $\{q_i\}$ yield the same objective value, indicates that (8) is a relaxation of (3).

To show that $\{q_i\}$ is feasible to (8), first, note that $q_i \geq 0$ for any $i \in [n]$ because $q(i|w) \geq 0$ for any $i \in [n]$ and $w \in [0,1]$. Second,

$$\sum_{i\in[n]} q_i = \sum_{i\in[n]} \int_0^1 q(i|w)\, g(w)\, dw \leq \int_0^1 g(w)\, dw = 1,$$

where the inequality follows from the fact that $\sum_{i\in[n]} q(i|w) \leq 1$ for any $w \in [0,1]$.

Finally, we show $\{q_i\}$ is feasible to the first constraint in (8). To do so, let $q_{\leq k}(w) = \sum_{i\leq k} q(i|w)$ denote the probability that a candidate with characteristics w receives an offer from one the top k employers. Since $\{q(i|w)\}$ is a feasible solution to (3),

$$\alpha_i \int_0^1 q(i|w)\, g(w)\, dw \leq \int_0^1 w \cdot q(i|w)\, g(w)\, dw.$$

Summing over $i \leq k$ on both sides gives

$$\begin{aligned}
\sum_{i\leq k} \alpha_i q_i \leq & \int_0^1 w \cdot q_{\leq k}(w)\, g(w)\, dw \\
\leq & \int_{G^{-1}\left(1-\sum_{i\leq k} q_i\right)}^1 w \cdot g(w)\, dw \\
=& \mathbb{E}\left[w \cdot \mathbb{1}\left[G(w) \geq 1 - \sum_{i\leq k} q_i\right]\right] \\
=& \sum_{i\leq k} q_i \cdot \mathbb{E}\left[w \middle| G(w) \geq 1 - \sum_{i\leq k} q_i\right]
\end{aligned}$$

where the second inequality follows from the fact that $\int_0^1 q_{\leq k}(w)\, g(w)\, dw = \sum_{i\leq k} q_k$, and that the integration is maximized by taking $q_{\leq k}(w) = 1$ for any $w \geq G^{-1}\left(1 - \sum_{i\leq k} q_i\right)$ and $q_{\leq k}(w) = 0$ otherwise.

Step Two: Proving $V^{\text{CR}} \leq \bar{V}$ We next prove that $V^{\text{CR}} \leq \bar{V}$. Specifically, we show that for any feasible solution $\{q_i\}$ to (8), there exists a feasible solution $\{q(i|w)\}$ to (3) with the same objective value as $\{q_i\}$, thereby implying $V^{\text{CR}} \leq \bar{V}$.

Let $\{q_i\}$ be feasible to (8). Since the participation condition (i.e., the first constraint) of (8) holds for $k = 1$, we can find a portion q_1 of candidates whose mean quality just meets the threshold value α_1 of employer 1. In other words, we can find a function $q(1|w) \geq 0$ satisfying

$$\begin{aligned}
&\int_0^1 q(1|w)\, g(w)\, dw = q_1, \\
&\int_0^1 w \cdot q(1|w)\, g(w)\, dw = \alpha_1 \int_0^1 q(1|w)\, g(w)\, dw.
\end{aligned}$$

Now consider the remaining portion of candidates. Since the participation condition of (8) holds for $k = 2$, within the remaining portion of candidates, we can find a portion q_2 of candidates whose mean quality just meets the threshold value α_2 of employer 2. In other words, we can find a function

$q(2|w) \geq 0$ satisfying

$$\int_0^1 q(2|w)\, g(w)\, dw = q_2,$$
$$\int_0^1 w \cdot q(2|w)\, g(w)\, dw = \alpha_2 \int_0^1 q(2|w)\, g(w)\, dw,$$
$$q(2|w) \leq 1 - q(1|w),\ \forall\, w \in [0, 1].$$

Repeating the process, we can find qualified portions for all employers, resulting in a set of $\{q(i|w)\}$ that is feasible to (3). Moreover, by construction, $\{q(i|w)\}$ and $\{q_i\}$ have the same objective value.

Step Three: Wrap-Up Combining the two steps, we have $\bar{V} = V^{\mathrm{CR}}$; that is, the optimal values of (3) and (8) are equal. Moreover, let $\{q^*(i|w)\}$ be an optimal solution to (3), and let $q_i^* = \int_0^1 q^*(i|w)g(w)dw$. Since $\{q_i^*\}$ is feasible to (8) and attains the same objective value as $\{q^*(i|w)\}$ by Step One, $\{q_i^*\}$ is optimal to (8). Conversely, if $\{q_i^*\}$ is an optimal solution to (8), then by Step Two, we can construct a feasible solution $\{q^*(i|w)\}$ to (3) satisfying $q_i^* = \int_0^1 q^*(i|w)g(w)dw$. This solution has an objective value $V^{\mathrm{CR}} = \bar{V}$, thus is optimal to (3).

A.12 Proof of Proposition 4.11

In the following, we first prove that the optimal Lagrangian dual variable $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]}$ for (3) is unique (Step One). We then show that if $\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]}$ is an optimal Lagrangian dual variable for (8), then $\{\mu_i\}$, with $\mu_i = \sum_{k\geq i} \lambda_k^*$, is an optimal Lagrangian dual variable for (3) (Step Two). Finally, these two steps indicate the uniqueness of the optimal Lagrangian dual variable $\boldsymbol{\lambda}^*$.

A.12.1 Step One: Uniqueness of $\boldsymbol{\mu}^*$

In this section, we prove by contradiction that the optimal Lagrangian dual variable $\boldsymbol{\mu}^* = (\mu_i^*)_{i\in[n]}$ of (3) is unique. Suppose, instead, that (3) has two different optimal dual variables $\boldsymbol{\mu} = (\mu_i)_{i\in[n]}$ and $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_i)_{i\in[n]}$. Let $i \triangleq \max\{j \in [n] : \mu_j \neq \tilde{\mu}_j\}$ denote the largest index where the two optimal dual variables differ. Without loss of generality, assume $\mu_i > \tilde{\mu}_i > 0$, where the second inequality follows from Proposition 4.7 Bullet 1.

Let $\{q^*(i|w)\}$ be an optimal solution to (3) such that $\int_0^1 q^*(i|w)g(w)dw = q_i^* > 0$ (whose existence is validated by Proposition 4.10). According to Lemma 4.5 Bullet 2, we have $\{q^*(i|w)\} \in \mathbf{Q}^{\mathrm{LR}}(\boldsymbol{\mu})$ and $\{q^*(i|w)\} \in \mathbf{Q}^{\mathrm{LR}}(\tilde{\boldsymbol{\mu}})$. However, in the following, we will show that such an optimal solution $\{q^*(i|w)\}$ does not exist.

First, let $\ell_j(w) \triangleq v_j + \mu_j(w - \alpha_j)$ denote the lines of employers $j \in [n]$ using the optimal dual variable $\boldsymbol{\mu} = (\mu_j)_{j\in[n]}$, and

$$\bar{h}(w) \triangleq \max_{j\in[n]} \big\{\ell_j(w)\big\}^+ = \max_{j\in[n]} \Big\{v_j + \mu_j\big(w - \alpha_j\big)\Big\}^+$$

denote the envelope function, which is convex and piecewise-linear on $w \in [0, 1]$.

Since $q_i^* > 0$, there exists constants b_1 and b_2 satisfying $0 < b_1 < \alpha_i < b_2 \leq 1$, such that $\bar{h}(w) = \ell_i(w)$ for $w \in \big[b_1, b_2\big]$ and $\bar{h}(w) > \ell_i(w)$ otherwise, according to Proposition 4.6. Since the envelope function $\bar{h}(w)$ is convex and piecewise-linear, and line $\ell_i(w)$ is dominated for any $w < b_1$, all the lines $\ell_j(w)$ with $j < i$ are also dominated for any $w < b_1$. This implies that

$$\sum_{j \le i} q^*(j|w) = 0, \forall\, w < b_1. \tag{18}$$

Next, let $\tilde{\ell}_j(w) \triangleq v_j + \tilde{\mu}_j(w - \alpha_j)$ denote the lines of employers $j \in [n]$ using the optimal dual variable $\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_j)_{j\in[n]}$, and

$$\tilde{h}(w) \triangleq \max_{j\in[n]} \{\tilde{\ell}_j(w)\}^+ = \max_{j\in[n]} \Big\{v_j + \tilde{\mu}_j\big(w - \alpha_j\big)\Big\}^+$$

denote the envelope function, which is convex and piecewise-linear on $w \in [0, 1]$.

Analogously, since $q_i^* > 0$, there exists constants $\tilde{b}_1$ and $\tilde{b}_2$ satisfying $0 < \tilde{b}_1 < \alpha_i < \tilde{b}_2 \le 1$, such that $\tilde{h}(w) = \tilde{\ell}_i(w)$ for $w \in \big[\tilde{b}_1, \tilde{b}_2\big]$. Since $\tilde{\mu}_j = \mu_j$ for any $j > i$, the lines $\ell_j(w)$ and $\tilde{\ell}_j(w)$ coincide for any $j > i$. Moreover, since $\tilde{\mu}_i < \mu_i$, we have $\tilde{b}_1 < b_1$, and line $\tilde{\ell}_i(w)$ is strictly above the x-axis and the lines $\{\tilde{\ell}_j(w)\}_{j>i}$ for any $w \in \big(\tilde{b}_1, \alpha_i\big)$. Therefore, according to (5), we have

$$\sum_{j \le i} q^*(j|w) = 1, \forall\, w \in \big(\tilde{b}_1, \alpha_i\big). \tag{19}$$

However, since $\tilde{b}_1 < b_1$, (18) and (19) cannot hold simultaneously. This implies that the optimal Lagrangian dual variable $\boldsymbol{\mu}^*$ of (3) must be unique.

A.12.2 Step Two: Connection between Optimal Dual Variables

Given an optimal Lagrangian dual variable $\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]}$ of (8), we define a dual variable $\boldsymbol{\mu} = (\mu_i)_{i\in[n]}$ with $\mu_i = \sum_{k\ge i} \lambda_k^*$ for any $i \in [n]$. In the following, we show that $V^{\mathrm{LR}}(\boldsymbol{\mu}) = \bar{V}$. Therefore, $\boldsymbol{\mu}$ is an optimal Lagrangian dual variable of (3) by the strong duality (Lemma 4.5).

Let $\{q_i^*\}$ be an optimal solution to (8), and $T \triangleq \{k \in [n] : \lambda_k^* > 0\}$ denote the set of positive elements of $\boldsymbol{\lambda}^*$. Note that if $k \in T$, then the participation constraint in (8) is binding with the top k employers due to complementary slackness; that is,

$$\sum_{i \le k} \alpha_i q_i^* = \mathbb{E}\Bigg[w \cdot \mathbb{1}\Bigg[w \ge G^{-1}\Big(1 - \sum_{i\le k} q_i^*\Big)\Bigg]\Bigg]. \tag{20}$$

The set T is nonempty because $n \in T$. To see this, let $i = n$ in (9); this gives

$$v_n + \lambda_n^* \cdot \left(G^{-1}\left(1 - \sum_{j=1}^{n} q_j^*\right) - \alpha_n\right) = 0. \tag{21}$$

Note that $G^{-1}(1 - \sum_{j=1}^{n} q_j^*)$ corresponds to the cutoff value z^* in Proposition 4.7 Bullet 2, which is strictly less than α_n, because the participation constraint of employer n is binding. This implies that $\lambda_n^* > 0$ (whose value is unique because the value of z^* is unique according to Proposition 4.7 Bullet 2).

Following the notation in Section 4.4.2, suppose set $T = \{t_1 < t_2 < \cdots < t_m = n\}$ contains m employers. These m employers partition the n employers into m groups $\{T_i\}_{i\in[m]}$ with $T_1 = [t_1]$ and $T_i = [t_{i-1}+1 : t_i]$ for $i \in [2:m]$, and each group T_i contains exactly one element from T, which is the largest element in T_i. In addition, let $z_i \triangleq G^{-1}\big(1 - \sum_{j=1}^{t_i} q_j^*\big)$ for any $i \in [m]$ and $z_0 = 0$, and define subinterval $I_i = [z_i, z_{i-1}]$ for any $i \in [m]$.

In Lemma A.1, we provide structural properties of the employers' lines $\ell_i(w;\mu_i) = v_i + \mu_i(w - \alpha_i)$ and fully characterize the envelope functions $h(w;\boldsymbol{\mu}) \triangleq \max_{i\in[n]} \ell_i(w;\mu_i)$ and $\bar{h}(w;\boldsymbol{\mu}) \triangleq \max\{h(w;\boldsymbol{\mu}), 0\}$.

Lemma A.1. *Given an optimal Lagrangian dual variable $\boldsymbol{\lambda}^* = (\lambda_k^*)_{k\in[n]}$ of (8), define a dual variable $\boldsymbol{\mu} = (\mu_i)_{i\in[n]}$ where $\mu_i = \sum_{k\geq i} \lambda_k^*$ for any $i \in [n]$. The following hold:*

1. *For any $i \in [m]$ and any employer $j \in T_i$, the threshold value α_j satisfies $\alpha_j \in \big(z_i, z_{i-1}\big)$.*
2. *For any group T_i, the lines $\ell_j(w;\mu_j)$ for any $j \in T_i$ completely overlap, and they pass through the points (α_j, v_j) for any $j \in T_i$.*
3. *Given any two employers $j \in T_i$ and $k \in T_{i+1}$ with $i \leq m-1$, the two lines $\ell_j(w;\mu_j)$ and $\ell_k(w;\mu_k)$ intersects at $w = z_i$. In addition, for any employer $j \in T_m$, the line $\ell_j(w;\mu_j)$ intersects the x-axis at $w = z_m$.*
4. *$h(w;\boldsymbol{\mu}) = \ell_j(w;\mu_j)$ for any $i \in [m]$, $j \in T_i$, and $w \in \big[z_i, z_{i-1}\big]$. In addition, $h(w;\boldsymbol{\mu}) = \ell_j(w;\mu_j)$ for any $j \in T_m$ and $w \in \big[0, z_m\big]$.*
5. *$\bar{h}(w;\boldsymbol{\mu}) = h(w;\boldsymbol{\mu})$ for $w \geq z_m$ and $\bar{h}(w;\boldsymbol{\mu}) = 0$ otherwise.*

We prove Lemma A.1 in Appendix A.12.3. According to (5) and Lemma A.1, a set of allocation probabilities $\{q(j|w)\}$ is optimal to $V^{\text{LR}}(\boldsymbol{\mu})$ if and only if $\{q(j|w)\}$ are nonnegative, no larger than one, and satisfy the following:

$$\begin{aligned} &\sum_{j\in T_i} q(j|w) = 1,\ \forall\, w \in \big(z_i, z_{i-1}\big),\ i \in [m], \\ &\sum_{j\in[n]} q(j|w) = 0,\ \forall\, w < z_m. \end{aligned} \tag{22}$$

Moreover, (26) below indicates that there exists an optimal solution $\{q^*(j|w)\}$ to $V^{\text{LR}}(\boldsymbol{\mu})$ such that for any $i \in [m]$ and $j \in T_i$, we have:

$$\begin{aligned} \int_{w\in I_i} q^*(j|w)\, dw &= q_j^*, \\ \int_{w\in I_i} w \cdot q^*(j|w)\, g(w)\, dw &= \alpha_j \int_{w\in I_i} q^*(j|w)\, g(w)\, dw. \end{aligned} \tag{23}$$

We observe from (22) that for any employer $j \in T_i$, $q^*(j|w) = 0$ for any $w \notin I_i$. This observation, combined with (23), implies that for any $j \in [n]$, we have:

$$\int_0^1 q^*(j|w)\, g(w)\, dw = q_j^*, \tag{24}$$

$$\int_0^1 w \cdot q^*(j|w)\, g(w)\, dw = \alpha_j \int_0^1 q^*(j|w)\, g(w)\, dw. \tag{25}$$

Therefore,

$$\begin{aligned}
V^{\mathrm{LR}}(\boldsymbol{\mu}) &= \int_0^1 \sum_{j=1}^n \Big\{ v_j + \mu_j\big(w - \alpha_j\big) \Big\}\, q^*(j|w)\, g(w)\, dw \\
&= \sum_{j=1}^n v_j \cdot \int_0^1 q^*(j|w)\, g(w)\, dw \\
&= \sum_{j=1}^n v_j \cdot q_j^* = V^{\mathrm{CR}} = \bar{V},
\end{aligned}$$

where the first equation follows from the fact that $\{q^*(j|w)\}$ is optimal to $V^{\mathrm{LR}}(\boldsymbol{\mu})$, the second from (25), the third from (24), the fourth from the fact that $\{q_j^*\}$ is optimal to (8), and the fifth from Proposition 4.10.

A.12.3 Proof of Lemma A.1

Proof of Bullet 1 For any group T_i and element $k \in T_i$, subtracting both sides of the first constraint in (8) from both sides of (20) with $k = t_{i-1}$, and noting that the first constraint in (8) is binding with $k = t_i$, yields the following:

$$\begin{aligned}
\sum_{j \in [t_{i-1}+1:k]} \alpha_j q_j^* &\le \mathbb{E}\Big[w \cdot \mathbb{1}\Big[G^{-1}\Big(1 - \sum_{j \le k} q_j^*\Big) \le w < z_{i-1}\Big]\Big], \, \forall\, k \in [t_{i-1}+1 : t_i - 1], \\
\sum_{j \in T_i} \alpha_j q_j^* &= \mathbb{E}\Big[w \cdot \mathbb{1}\Big[z_i \le w < z_{i-1}\Big]\Big].
\end{aligned} \tag{26}$$

In addition, we have $\mathbb{P}[z_i \le w < z_{i-1}] = \sum_{j \in T_i} q_j^*$. Analogous to the proof of Proposition 4.10 (in particular, Step Two in Appendix A.11), we can allocate the state $w \in [z_i, z_{i-1}]$ to employers in group T_i (possibly in a randomized way), so that each employer j is allocated with an aggregate size of $q_j^* > 0$, and the posterior mean of the allocation to employer j is α_j (i.e., the participation constraint is tight). This implies that $\alpha_j \in (z_i, z_{i-1})$ for any $j \in T_i$.

Proof of Bullet 2 If group T_i contains only one employer, there is nothing to prove. Now suppose T_i contains more than one employer (i.e., $t_{i-1} + 1 < t_i$). For any $k \in [t_{i-1} + 1 : t_i - 1]$, we have

$$\mu_k = \sum_{j \ge k} \lambda_j^* = \mu_{t_i} = \frac{v_k - v_{t_i}}{\alpha_k - \alpha_{t_i}}\,, \tag{27}$$

where the first equation follows from the definition of $\{\mu_i\}$ and the second equation follows from the fact that $\lambda_j^* = 0$ for any $j \in [t_{i-1} + 1 : t_i - 1]$. The third equation is derived by subtracting both sides of (9) with $i = t_i$ from both sides of the same equation with $i = k$. (27) implies that the points $\{(v_j, \alpha_j)\}_{j \in T_i}$ lie on a line, and the employers' lines $\ell_j(w; \mu_j)$ with any $j \in T_i$ fully overlap and coincide with this line.

Proof of Bullet 3 For ease of notation, we drop the dependence on the dual variable $\boldsymbol{\mu} = (\mu_j)_{j \in [n]}$ by letting $\ell_j(w) \triangleq \ell_j(w; \mu_j) = v_j + \mu_j(w - \alpha_j)$. Based on Bullet 2, it suffices to show that: (i) line $\ell_n(w)$ intersects the x-axis at $w = z_m$, and (ii) for any $i \in [m-1]$, the two lines $\ell_{t_i}(w)$ and $\ell_{t_{i+1}}(w)$ intersect at $w = z_i$.

First, from (21) we have

$$v_n + \lambda_n^* \cdot (z_m - \alpha_n) = v_n + \mu_n \cdot (z_m - \alpha_n) = 0,$$

where the first equation follows from $\mu_n = \lambda_n^*$ by definition. Therefore, line $\ell_n(w)$ intersects the x-axis at $w = z_m$.

We now prove (ii) by induction. To start, note that

$$\mu_{t_i} = \sum_{j=i}^{m} \lambda_{t_j}^*, \ \forall i \in [m] \tag{28}$$

because $\lambda_k^* = 0$ for any $k \notin T$. We first show that (ii) holds for $i = m-1$. Since line $\ell_n(w)$ passes through the point (z_{m-1}, h_{m-1}) with

$$h_{m-1} = \mu_n \cdot (z_{m-1} - z_m), \tag{29}$$

it suffices to show that line $\ell_{t_{m-1}}(w)$ also passes through (z_{m-1}, h_{m-1}). We now verify this. Specifically, taking $i = t_{m-1}$ in (9) yields

$$v_{t_{m-1}} + \lambda_n^* \cdot (z_m - \alpha_{t_{m-1}}) + \lambda_{t_{m-1}}^* \cdot (z_{m-1} - \alpha_{t_{m-1}}) = v_{t_{m-1}} + \mu_{t_{m-1}} \cdot \big(z_{m-1} - h_{m-1}/\mu_{t_{m-1}} - \alpha_{t_{m-1}}\big) = 0$$

where the first equation follows from (28) and (29). Therefore, it follows that

$$v_{t_{m-1}} + \mu_{t_{m-1}} \cdot (z_{m-1} - \alpha_{t_{m-1}}) = h_{m-1},$$

implying that line $\ell_{t_{m-1}}(w)$ also passes through the point (z_{m-1}, h_{m-1}).

We now assume that (ii) holds for any $j \geq i+1$ and verify that it also holds for $j = i$. Given that (ii) holds for any $j \geq i+1$, line $\ell_{t_{i+1}}(w)$ passes through the point (z_i, h_i) with

$$h_i = \sum_{j=i}^{m-1} \mu_{i+1} \cdot (z_i - z_{i+1}). \tag{30}$$

It suffices to show that line $\ell_{t_i}(w)$ also passes through (z_i, h_i). To do so, take $i = t_i$ in (9); this gives:

$$v_{t_i} + \sum_{j=i}^{m} \lambda_{t_j}^* \cdot \big(z_j - \alpha_{t_i}\big) = v_{t_i} + \mu_{t_i} \cdot \Big(z_i - h_i/\mu_{t_i} - \alpha_{t_i}\Big) = 0,$$

where the first equation follows from (30) and the fact that $\lambda_{t_j}^* = \mu_{t_j} - \mu_{t_{j+1}}$ for any $j \in [m]$ (letting $\mu_{t_{m+1}} = 0$) by (28). Therefore, we have

$$v_{t_i} + \mu_{t_i} \cdot (z_i - \alpha_{t_i}) = h_i,$$

implying that line $\ell_{t_i}(w)$ also passes through the point (z_i, h_i). Therefore, (ii) holds for $j = i$.

Proof of Bullet 4 Bullet 4 follows from Bullets 2 and 3 and the fact that the dual variables $\{\mu_i\}$, which corresponds to the slopes of the lines $\ell_i(w; \mu_i)$, are decreasing.

Proof of Bullet 5 According to Bullet 3, the function $h(w;\boldsymbol{\mu})$ is nonnegative if and only if $w \geq z_m$. Therefore, $\bar{h}(w;\boldsymbol{\mu}) = h(w;\boldsymbol{\mu})$ for $w \geq z_m$, and $\bar{h}(w;\boldsymbol{\mu}) = 0$ otherwise.

A.13 Proof of Proposition 4.14

In the following, we first prove that the values of $\{b_k\}$ and $\{\rho_k\}$ in Algorithm 1 exist and can be identified efficiently (Appendix A.13.1). We then show that the assignment probability $q(j|w)$ returned from Algorithm 1 is optimal to (3) and possesses the first-order stochastically increasing property (Appendix A.13.2).

A.13.1 Existence of $\{b_k\}$ and $\{\rho_k\}$

We prove by induction that the values of $\{\rho_k\}$ and $\{b_k\}$ in Algorithm 1 exist and can be computed efficiently.

Induction Step We first determine the values of $b_{t_{i-1}+1}$ and $\rho_{t_{i-1}+1}$. From (13), the following hold:

$$\mathbb{E}\left[w \middle| G^{-1}\Big(1 - \sum_{j \leq t_{i-1}+1} q_j^*\Big) \leq w < z_{i-1}\right] \geq \alpha_{t_{i-1}+1}\,,$$
$$\mathbb{E}\Big[w \Big| z_i \leq w < z_{i-1}\Big] = \sum_{j \in T_i} \alpha_j \cdot \frac{q_j^*}{\sum_{j \in T_i} q_j^*} \leq \alpha_{t_{i-1}+1}\,,$$

where the inequality in the second line follows from the fact that $\alpha_{t_{i-1}+1} \geq \alpha_j$ for any $j \in T_i$. Therefore, there exists a value of $b_{t_{i-1}+1}$ satisfying that $z_i \leq b_{t_{i-1}+1} \leq G^{-1}\big(1 - \sum_{j \leq t_{i-1}+1} q_j^*\big) \leq z_{i-1}$ and that $\mathbb{E}\big[w \big| b_{t_{i-1}+1} \leq w \leq z_{i-1}\big] = \alpha_{t_{i-1}+1}$. Additionally, let $\rho_{t_{i-1}+1} = q^*_{t_{i-1}+1} \,/\, \mathbb{P}[b_{t_{i-1}+1} \leq w < z_{i-1}] \leq 1$. The allocation probability $q(t_{i-1}+1|w)$ satisfies (14) by the setup of $b_{t_{i-1}+1}$ and $\rho_{t_{i-1}+1}$.

Iteration Step Let k be an integer with $k \in [t_{i-1}+2 : t_i - 1]$. Suppose that for any $j \in [t_{i-1}+1 : k-1]$, we have determined the values of ρ_j and b_j such that the probability $q(j|w)$ satisfies (14). We now identify the values of ρ_k and b_k such that the probability $q(k|w)$ also satisfies (14).

To achieve this, let $b_k = b \in [z_i, z_{i-1}]$ and $\rho_k = \rho \in [\underline{\rho}_k, 1]$, with $\underline{\rho}_k \triangleq \frac{q_k^*}{\sum_{\ell=k}^{t_i} q_\ell^*}$. Additionally, let $q(k|w) = \rho_k \cdot \big(1 - q_{\leq k-1}(w)\big)$ for $w \in [b_k, z_{i-1}]$ and $q(k|w) = 0$ for $w \in [z_i, b_k)$. We define the following two functions:

$$F(b,\rho) \triangleq \int_{w \in I_i} q(k|w)\, g(w)\, dw\,,$$
$$Q(b,\rho) \triangleq \int_{w \in I_i} w \cdot q(k|w)\, g(w)\, dw\,.$$

The allocation probability $q(k|w)$ satisfies (14) under the choice of $b_k = b$ and $\rho_k = \rho$ if and only if $F(b,\rho) = q_k^*$ and $Q(b,\rho) = \alpha_k q_k^*$.

Evidently, the function $F(b,\rho)$ is strictly increasing in ρ and strictly decreasing in b. Therefore, for any $\rho \in [\underline{\rho}_k, 1]$, there exists a unique value, denoted by $b(\rho)$, that satisfies $F\big(b(\rho),\rho\big) = q_k^*$.

Specifically, we have $b(\underline{\rho}_k) = z_i$. To see this, note that the following holds:

$$
\begin{aligned}
F\big(z_i, \underline{\rho}_k\big) &= \underline{\rho}_k \cdot \int_{w \in I_i} \big(1 - q_{\leq k-1}(w)\big)\, g(w)\, dw \\
&= \underline{\rho}_k \cdot \left(\int_{w \in I_i} g(w)\, dw - \sum_{j=t_{i-1}+1}^{k-1} \int_{w \in I_i} q(j|w)\, g(w)\, dw \right) \\
&= q_k^* \,,
\end{aligned}
$$

where the first equation follows from the definition of $q(k|w)$, and the third equation follows from the facts that $\mathbb{P}[z_i \leq w < z_{i-1}] = \sum_{j \in T_i} q_j^*$ and that the probability $q(j|w)$ satisfies (14) for any $j \leq k-1$, and the definition of $\underline{\rho}_k$. Moreover, the function $b(\rho)$ is strictly increasing with ρ. Its inverse, denoted by $\rho(b)$, exists and is also strictly increasing.

We now define the function

$$
Q(b) \triangleq Q\big(b, \rho(b)\big).
$$

Since $F\big(b, \rho(b)\big) = q_k^*$ for any value of b, it suffices to find a value $b \in [z_i, b_{k-1}]$ satisfying $Q(b) = \alpha_k q_k^*$, which we do now.

First, note that the function $Q(b)$ is increasing. This is because, as b increases, we transport a fixed mass q_k^* of candidates to the right, which increases the mean quality of these candidates.

Second, we inspect the value of $Q(z_i)$. Specifically, the following holds:

$$
\begin{aligned}
Q(z_i) &= \underline{\rho}_k \cdot \int_{w \in I_i} w \cdot \big(1 - q_{\leq k-1}(w)\big)\, g(w)\, dw \\
&= \underline{\rho}_k \cdot \left(\int_{w \in I_i} w\, g(w)\, dw - \sum_{j=t_{i-1}+1}^{k-1} \int_{w \in I_i} w \cdot q(j|w)\, g(w)\, dw \right) \\
&= \frac{q_k^*}{\sum_{\ell=k}^{t_i} q_\ell^*} \cdot \left(\sum_{\ell \in T_i} \alpha_\ell q_\ell^* - \sum_{\ell=t_{i-1}+1}^{k-1} \alpha_\ell q_\ell^* \right) \\
&= \frac{\sum_{\ell=k}^{t_i} \alpha_\ell q_\ell^*}{\sum_{\ell=k}^{t_i} q_\ell^*} \cdot q_k^* \\
&\leq \alpha_k q_k^*,
\end{aligned}
\tag{31}
$$

where the third equation follows from the second line of (13) and that probability $q(j|w)$ satisfies (14) for any $j \leq k-1$, and the inequality follows from the fact that α_ℓ decreases with index ℓ.

Finally, we derive two more inequalities. If $b(1) \geq b_{k-1}$ (in other words, the "unoccupied" area to the right of b_{k-1} and above the function $q_{\leq k-1}(w)$ is larger than q_k^*), we have

$$
Q\big(b_{k-1}\big) = \alpha_{k-1} q_k^* > \alpha_k q_k^* \,, \tag{32}
$$

because in this case, $q(k|w) = c \cdot q(k-1|w)$ for some constant $c > 0$ and any $w \in I_i$.

Alternatively, suppose $b(1) \leq b_{k-1}$. Then, it follows that $q_{\leq k}(w) = 1$ for $w \in \big[b(1), z_{i-1}\big]$ and $q_{\leq k}(w) = 0$ for $w < b(1)$, which indicates that $b(1) = G^{-1}\Big(1 - \sum_{j \leq k} q_j^*\Big)$. As a result, the following hold:

$$
\begin{aligned}
Q\big(b(1)\big) &= \int_{w\in I_i} w\cdot q_{\le k}(w)\,g(w)\,dw - \int_{w\in I_i} w\cdot q_{\le k-1}(w)\,g(w)\,dw \\
&= \int_{b(1)}^{z_{i-1}} w\cdot q_{\le k}(w)\,g(w)\,dw - \sum_{j=t_{i-1}+1}^{k-1}\int_{w\in I_i} w\cdot q(j|w)\,g(w)\,dw \\
&= \mathbb{E}\Big[w\cdot \mathbb{1}\Big[G^{-1}\Big(1-\sum_{j\le k} q_j^*\Big)\le w< z_{i-1}\Big]\Big] - \sum_{j=t_{i-1}+1}^{k-1}\int_{w\in I_i} w\cdot q(j|w)\,g(w)\,dw \\
&\ge \sum_{j=t_{i-1}+1}^{k}\alpha_j q_j^* - \sum_{j=t_{i-1}+1}^{k-1}\alpha_j q_j^* \\
&= \alpha_k q_k^*\,,
\end{aligned}
\tag{33}
$$

where the inequality follows from the first equation in (13) and the fact that probability $q(j|w)$ satisfies (14) for any $j\le k-1$.

Since the function $Q(b)$ is continuous and increasing in b, (31) – (33) imply that there exists a value $b_k\in\big[z_i,\min\{b_{k-1},b(1)\}\big]$ satisfying $Q(b_k)=\alpha_k q_k^*$. Moreover, the value of b_k can be efficiently identified using binary search. Let $\rho_k=\rho(b_k)$. The probability $q(k|w)$ satisfies (14) under the choice of b_k and ρ_k.

Final Step Let $q(t_i|w)=1-q_{\le t_i-1}(w)$ for any $w\in I_i$. Since $q(j|w)$ satisfies (14) for any $j\le t_i-1$, the second equation in (13) and the fact that $\mathbb{P}[z_i\le w<z_{i-1}]=\sum_{j\in T_i} q_j^*$ imply that $q(t_i|w)$ also satisfies (14).

A.13.2 Optimality and FOSD Property

Let $\{q(j|w)\}$ denote the output of Algorithm 1. $\{q(j|w)\}$ is optimal to (3) according to Lemma 4.13.

We now prove that the distribution $q(\cdot|w)$ first-order stochastically increases with w on the interval I_i. By definition, this is equivalent to proving that the cumulative distribution function $q_{\le k}(w)$ is increasing in w for any $k\in T_i$. We prove this by induction. First, $q_{\le t_{i-1}}(w)=0$ for any $w\in I_i$ by definition, which serves as the induction step. Next, suppose $q_{\le k-1}(w)$ is increasing on $w\in I_i$ for some $k\in T_i$, we show that $q_{\le k}(w)$ is also increasing. To do so, fix two points $w,w'\in I_i$ with $w'<w$. If $w'<b_k$, we have

$$
0=q_{\le k}(w')=q_{\le k-1}(w')\le q_{\le k-1}(w)\le q_{\le k}(w),
$$

where the inequality follows from the fact that $q_{\le k-1}(w)$ increases with w. Alternatively, if $w'\ge b_k$, we have

$$
\begin{aligned}
q_{\le k}(w') &= q_{\le k-1}(w')+\rho_k\cdot\big(1-q_{\le k-1}(w')\big) \\
&= q_{\le k-1}(w')+\rho_k\cdot\big(q_{\le k-1}(w)-q_{\le k-1}(w')\big)+\rho_k\cdot\big(1-q_{\le k-1}(w)\big) \\
&\le q_{\le k-1}(w')+q_{\le k-1}(w)-q_{\le k-1}(w')+\rho_k\cdot\big(1-q_{\le k-1}(w)\big) \\
&= q_{\le k-1}(w)+\rho_k\cdot\big(1-q_{\le k-1}(w)\big) \\
&= q_{\le k}(w),
\end{aligned}
$$

where the inequality follows from the fact that $q_{\le k-1}(w)\ge q_{\le k-1}(w')$ and $\rho_k\ge 0$.

A.14 Proof of Proposition 4.15

A.14.1 Proof of Bullet 1

Let $\{\lambda_k^*\}$ denote an optimal dual variable for the participation constraints in (8) and $\{T_i\}$ denote the resulting partition of the n employers as described in Section 4.4.2. For a feasible solution $\{q_k\}$ to (8), let

$$T_i\big(\{q_k\}\big) \triangleq \Big|T_i \cap \{k \in [n] : q_k > 0\}\Big|$$

denote the number of employers in group T_i that have a positive probability q_k.

Let $\{q_k^*\}$ denote an optimal solution to (8). Lemma A.2 shows that if there exists a group T_i that satisfies $T_i\big(\{q_k^*\}\big) > 2$, we can find a new optimal solution $\{\tilde{q}_k\}$ to (8) that is closer to the desired one in Bullet 1.

Lemma A.2. *Let $\{q_k^*\}$ denote an optimal solution $\{q_k^*\}$ to (8). If there exists a subset T_i that satisfies $T_i\big(\{q_k^*\}\big) > 2$, we can find a new optimal solution $\{\tilde{q}_k\}$ to (8) such that (i) $\tilde{q}_k = q_k^*$ for any $k \notin T_i$, and (ii) $T_i\big(\{\tilde{q}_k\}\big) < T_i\big(\{q_k^*\}\big)$.*

Repeating the process in Lemma A.2 iteratively will eventually (in at most n steps) yields a desired optimal solution to (8) that satisfies Bullet 1.

Proof of Lemma A.2. From Lemma 4.13, there exists an optimal solution $\{q^*(j|w)\}$ to (3) such that the candidate joins each employer j with probability q_j^*. Suppose $T_i\big(\{q_k^*\}\big) > 2$. In the following, we modify $\{q^*(j|w)\}$ to create a new optimal solution $\{\tilde{q}(j|w)\}$ to (3) such that the candidate joins each employer j with probability $\tilde{q}_j$, where $\{\tilde{q}_j\}$ satisfies Lemma A.2. Then, $\{\tilde{q}_j\}$ is optimal to (8) according to Proposition 4.10.

Assume $\{a, b, c\} \subseteq T_i\big(\{q_k^*\}\big)$, where a, b, and c denote three distinct integers. Without loss of generality, assume that $1 \le a < b < c \le n$. Therefore, $\alpha_a > \alpha_b > \alpha_c$. We consider the following two scenarios.

Case One Suppose

$$\frac{\alpha_a q_a^* + \alpha_c q_c^*}{q_a^* + q_c^*} = \alpha_b, \tag{34}$$

that is, the mean quality of the candidates joining employers a or c is precisely α_b, the recruiting bar of employer b. Let

$$\tilde{q}(j|w) = \begin{cases} q^*(a|w) + q^*(b|w) + q^*(c|w) & \text{if } j = b, \\ 0 & \text{if } j \in a, c, \\ q^*(j|w) & \text{if } j \notin \{a, b, c\}. \end{cases}$$

(34) implies that the participation constraint for employer b remains binding with $\tilde{q}(j|w)$. Therefore, $\tilde{q}(j|w)$ is optimal to (3) according to Lemma 4.13. Additionally, we have

$$\tilde{q}_j \triangleq \int_0^1 \tilde{q}(j|w)\, g(w)\, dw = \begin{cases} q_b^* + q_a^* + q_c^* & \text{if } j = b, \\ 0 & \text{if } j \in a, c, \\ q_j^* & \text{if } j \notin \{a, b, c\}. \end{cases}$$

As a result, $\{\tilde{q}_j\}$ satisfies Lemma A.2 because $\{\tilde{q}_j\}$ is optimal to (8) by Proposition 4.10 and $T_i\big(\{\tilde{q}_j\}\big) = T_i\big(\{q_j^*\}\big) - 2 < T_i\big(\{q_j^*\}\big)$ by construction.

Case Two Suppose (34) does not hold. Without loss of generality, assume that $\frac{\alpha_a q_a^* + \alpha_c q_c^*}{q_a^* + q_c^*} > \alpha_b$, which translates to $q_a^* > \underline{q}_a \triangleq q_c^* \cdot \frac{\alpha_b - \alpha_c}{\alpha_a - \alpha_b}$. Let $\rho_a \triangleq \underline{q}_a / q_a^* < 1$. Note that the following holds:

$$\frac{\alpha_a \underline{q}_a + \alpha_c q_c^*}{\underline{q}_a + q_c^*} = \alpha_b. \tag{35}$$

Let

$$\tilde{q}(j|w) = \begin{cases} \rho_a \cdot q^*(a|w) + q^*(b|w) + q^*(c|w) & \text{if } j = b, \\ (1 - \rho_a) \cdot q^*(a|w) & \text{if } j \in a, \\ 0 & \text{if } j \in c, \\ q^*(j|w) & \text{if } j \notin \{a, b, c\}. \end{cases}$$

(35) implies that the participation constraint for employer b remains binding with $\tilde{q}(j|w)$. Therefore, $\tilde{q}(j|w)$ is optimal to (3) according to Lemma 4.13. Additionally, we have

$$\tilde{q}_j \triangleq \int_0^1 \tilde{q}(j|w)\, g(w)\, dw = \begin{cases} q_b^* + \rho_a \cdot q_a^* + q_c^* & \text{if } j = b, \\ (1 - \rho_a) \cdot q_a^* & \text{if } j \in a, \\ 0 & \text{if } j \in c, \\ q_j^* & \text{if } j \notin \{a, b, c\}. \end{cases}$$

As a result, $\{\tilde{q}_j\}$ satisfies Lemma A.2 because $\{\tilde{q}_j\}$ is optimal to (8) by Proposition 4.10 and $T_i(\{\tilde{q}_j\}) = T_i(\{q_j^*\}) - 1 < T_i(\{q_j^*\})$ by construction. □

A.14.2 Proof of Bullet 2

We prove Bullet 2 based on our established results from the dual analysis. Assume, without loss of generality, that there exists an optimal solution $\{q_k^*\}$ to (8) such that $q_k^* > 0$ for any $k \in [n]$. We then show that the values of $\{q_k^*\}$ are unique. To see that this assumption loses no generality, let

$$P_\varnothing = \Big\{ k \in [n] : q_k^* = 0 \text{ for any optimal solution} \{q_k^*\} \text{ to (8)} \Big\}$$

denote the set of employers ignored by any optimal solution to (8). We can exclude the employers in set $P_\varnothing$ without affecting anything. Meanwhile, define $P = [n] \setminus P_\varnothing$. Since (8) is a convex optimization problem, the set of optimal solutions is convex. This implies that there exists an optimal solution $\{q_k^*\}$ such that $q_k^* > 0$ for any $k \in P$.

Now, let $\{\lambda_k^*\}$ denote the optimal dual variable of (8). Note that the values of $\{\lambda_k^*\}$ are unique according to Proposition 4.11. Let $\{T_i\}$ denote the partition of employers described in Section 4.4.2. Since no three points of $\{(\alpha_i, v_i)\}_{i \in [n]}$ are collinear, any group T_i contains at most two employers based on Bullet 2 of Lemma 4.12. Fix a group T_i. First, suppose $T_i = \{k\}$ contains one employer. Then, we have $q_k^* = \mathbb{P}[z_i \le w \le z_{i-1}]$, whose value is uniquely determined.

Second, suppose $T_i = \{k, j\}$ contains two employers. Then, the values of q_k^* and q_j^* must satisfy

$$\begin{aligned} q_k^* + q_j^* &= \mathbb{P}[z_i \le w \le z_{i-1}], \\ \alpha_k q_k^* + \alpha_j q_j^* &= \mathbb{E}\Big[w \cdot \mathbb{1}\big[z_i \le w \le z_{i-1} \big] \Big], \end{aligned}$$

and therefore, are uniquely determined as well.