General Theory of Graph

Jerry Chen

November 25, 2025

Abstract

In this paper, we introduce "face" and hence, generalize graph to arbitrary dimension.

Introduction

We know graph came from our intuition on points and lines. However, it's not enough to describe surfaces and higher. We need an extended theory to formulate them.

Face

We first generalize graph to include *face* and see the pattern. Clearly they should link edges and vertices, but notice that

"If face f links edge e and edge e links vertex v, f should link v."

This is the fundamental difference between 1 and higher dimension. Hence we have:

Definition. A 2-dimensional graph is set $V + E + F$ with

- $\mathrm{V} : E + F \to \mathrm{P}(V)^\times$
- $\mathrm{E} : F + V \to \mathrm{P}(E)^\times$
- $\mathrm{F} : V + E \to \mathrm{P}(F)^\times$

satisfying $\forall v \in V, \forall e \in E, \forall f \in F$,

- $v \in \mathrm{V}(e) \Leftrightarrow' e \in \mathrm{E}(v)$
- $e \in \mathrm{E}(f) \Leftrightarrow' f \in \mathrm{F}(e)$
- $f \in \mathrm{F}(v) \Leftrightarrow' v \in \mathrm{V}(f)$
- $v \in \mathrm{V}(e)$ and $e \in \mathrm{E}(f) \Rightarrow v \in \mathrm{V}(f)$
- $f \in \mathrm{F}(e)$ and $e \in \mathrm{E}(v) \Rightarrow' f \in \mathrm{F}(v)$

($'$ set is for duality, can be deduced from residual).

Graph

Have the first step, generalize to any dimension is easy:

Definition. A d-dimensional graph is set $\sum_{0\leq n\leq d} S_n$ with

- $\forall n, \mathrm{S}_n : \sum_{m\neq n} S_m \to \mathrm{P}(S_n)^\times$

satisfying $\forall s,t,u \in S_l, S_m, S_n : l < m < n \text{ or}' \; l > m > n$,

$$s \in \mathrm{S}_l(t) \text{ and } t \in \mathrm{S}_m(u) \Rightarrow s \in \mathrm{S}_l(u)$$

How to Draw

We give an alternative when faces and higher present: ignore what shapes they are (represent by dots), only consider the "link" between them (draw an arrow). The result is actually:

Definition. A "graph" is set with order $\to$:

- (Asymmetric) $\forall x \forall y, x \to y \Rightarrow y \not\to x$
- (Transitive) $\forall x \forall y \forall z, x \to y \text{ and } y \to z \Rightarrow x \to z$

Now we can describe any shape in nature.

Reference

[1] Alain Bretto "Hypergraph Theory: An Introduction"