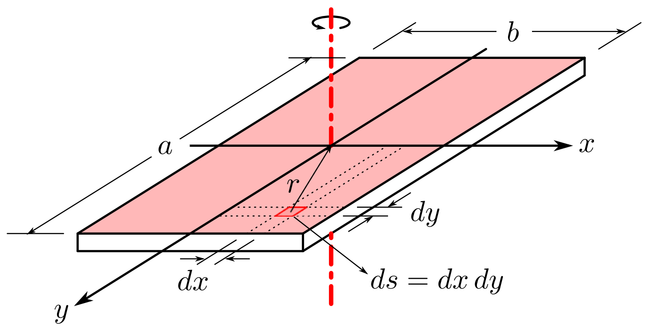
Tarea #20

Calcular el momento de inercia de una placa rectangular uniforme respecto a un eje principal que pasa por su centro de masa (cm) y es perpendicular al plano que forma.



Solución:

Dada la ecuación del momento de inercia:

$$I = \int_M r^2\, dm \tag{1}$$

Siendo r la hipotenusa de x y y, por tanto:

$$r^2 = x^2 + y^2 \tag{2}$$

Asumiendo la distribución homogénea de la masa:

$$\sigma = \frac{dm}{ds}$$

Por tanto:

$$dm = \sigma\, ds = \sigma\, dx\, dy \tag{3}$$

Reemplazando (3) en (1):

$$I = \int_0^S r^2\,\sigma\, ds = \int_{-a/2}^{a/2}\int_{-b/2}^{b/2}(x^2+y^2)\,\sigma\, dx\, dy = \sigma\int_{-a/2}^{a/2}\left(\int_{-b/2}^{b/2}(x^2+y^2)\, dx\right) dy$$

$$I = \sigma\int_{-a/2}^{a/2}\left(\int_{-b/2}^{b/2}x^2 dx + \int_{-b/2}^{b/2}y^2 dx\right) dy = \sigma\int_{-a/2}^{a/2}\left(\frac{x^3}{3}\Bigg|_{-b/2}^{b/2} + y^2 x\Bigg|_{-b/2}^{b/2}\right) dy$$

$$I = \sigma\int_{-a/2}^{a/2}\left(\frac{(\frac{b}{2})^3}{3} - \frac{(-\frac{b}{2})^3}{3} + y^2\left(\frac{b}{2}\right) - y^2\left(-\frac{b}{2}\right)\right) dy = \sigma\int_{-a/2}^{a/2}\left(\frac{b^3}{12} + b\,y^2\right) dy$$

$$I = \sigma\int_{-a/2}^{a/2}\frac{b^3}{12}\, dy + \int_{-a/2}^{a/2} b\,y^2\, dy = \sigma\left(\frac{b^3}{12}\, y\Bigg|_{-a/2}^{a/2} + b\,\frac{y^3}{3}\Bigg|_{-a/2}^{a/2}\right)$$

$$I = \sigma\left(\frac{b^3}{12}\left(\frac{a}{2}+\frac{a}{2}\right) + b\left(\frac{(\frac{a}{2})^3}{3} - \frac{(-\frac{a}{2})^3}{3}\right)\right) = \sigma\left(\frac{a\,b^3}{12} + \frac{a^3\,b}{12}\right)$$

$$I = \frac{\sigma}{12}(a\,b^3 + a^3\,b) \tag{4}$$

A partir de la ecuación (3) sabemos que:

$$M = \sigma\,s = \sigma\,a\,b$$

Despejando σ y reemplazando en la ecuación (4), obtenemos:

$$I = \frac{1}{12}\left(\frac{M}{a\,b}\right)(a\,b^3 + a^3\,b) = \frac{M}{12}\left(\frac{a\,b^3}{a\,b} + \frac{a^3\,b}{a\,b}\right)$$

Resultando finalmente:

$$I = \frac{M}{12}(a^2 + b^2) \tag{5}$$