Tarea #14

1. Ejercicio 1

Demostrar que una solución de la ecuación:

$$x'' + \omega^2 x = 0 \tag{1}$$

es:

$$x = A \cdot cos(\omega t) \tag{2}$$

Si las condiciones iniciales son: $x = A$, $x' = 0$ para $t = 0$.

Solución:

$$x'' + \omega^2 x = 0$$
$$x'' = -\omega^2 x$$
$$x' \frac{dx'}{dx} = -\omega^2 x$$
$$x' dx' = -\omega^2 x dx$$

Considerando las condiciones iniciales: $x = A$, $x' = 0$ en $t = 0$:

$$\int_0^{x'} x' dx' = -\omega^2 \int_A^x x dx$$
$$\left.\frac{x'^2}{2}\right|_0^{x'} = -\omega^2 \left.\frac{x^2}{2}\right|_A^x$$
$$\frac{x'^2}{2} = -\omega^2 \left(\frac{x^2}{2} - \frac{A^2}{2}\right)$$
$$x'^2 = -\omega^2 (A^2 - x^2)$$
$$x' = \omega\sqrt{A^2 - x^2}$$
$$\frac{dx}{dt} = \omega\sqrt{A^2 - x^2}$$
$$\omega dt = \frac{dx}{\sqrt{A^2 - x^2}}$$
$$\omega \int_0^t dt = \int_A^x \frac{dx}{\sqrt{A^2 - x^2}}$$

Realizando el siguiente cambio de variable:

$$x = A \cdot cos(\theta)$$
$$dx = -A \cdot sen(\theta) d\theta$$

Obtenemos:

$$\omega t = \int \frac{-Asen(\theta)d\theta}{\sqrt{A^2 - (A \cdot cos(\theta))^2}}$$
$$\omega t = \int \frac{-sen(\theta)d\theta}{\sqrt{1 - cos^2(\theta)}}$$
$$\omega t = \int -d\theta$$
$$\omega t = -\theta$$

Sabiendo que $cos(\theta) = cos(-\theta)$:

$$\omega t = arccos\left(\frac{x}{A}\right)\Bigg|_A^x$$
$$\omega t = arccos\left(\frac{x}{A}\right) - arccos\left(\frac{A}{A}\right)$$
$$\omega t = arccos\left(\frac{x}{A}\right)$$
$$cos(\omega t) = \frac{x}{A}$$
$$x = A \cdot cos(\omega t) \tag{3}$$

2. Ejercicio 2

Encontrar una solución a la ecuación:

$$x'' + \omega^2 x = 0 \tag{4}$$

es:

$$x = A \cdot cos(\omega t) \tag{5}$$

Con condiciones iniciales: $x = 0$, $x' = x'_0$ para $t = 0$.

Solución:

$$x'' + \omega^2 x = 0$$
$$x'' = -\omega^2 x$$
$$x' \frac{dx'}{dx} = -\omega^2 x$$
$$x' dx' = -\omega^2 x dx$$

Considerando las condiciones iniciales: $x = 0$, $x' = x'_0$ en $t = 0$:

$$\int_{x'_0}^{x'} x' dx' = -\omega^2 \int_0^x x dx$$

$$\left. \frac{x'^2}{2} \right|_{x'_0}^{x'} = -\omega^2 \left. \frac{x^2}{2} \right|_0^x$$

$$\frac{x'^2}{2} - \frac{x_0'^2}{2} = -\omega^2 \frac{x^2}{2}$$

$$x'^2 - x_0'^2 = -\omega^2 x^2$$

$$x' = \sqrt{x_0'^2 - \omega^2 x^2}$$

$$\frac{dx}{dt} = \sqrt{x_0'^2 - \omega^2 x^2}$$

$$dt = \frac{dx}{\sqrt{x_0'^2 - \omega^2 x^2}}$$

$$\int_0^t dt = \int_0^x \frac{dx}{\sqrt{x_0'^2 - \omega^2 x^2}}$$

$$t = \int_0^x \frac{dx}{\sqrt{x_0'^2 \left(1 - \frac{\omega^2 x^2}{x_0'^2}\right)}}$$

$$t = \frac{1}{x'_0} \int_0^x \frac{dx}{\sqrt{1 - \frac{\omega^2 x^2}{x_0'^2}}}$$

Realizando el siguiente cambio de variable:

$$\frac{\omega x}{x'_0} = sen(\theta)$$

$$\frac{\omega dx}{x'_0} = cos(\theta) d\theta$$

Obtenemos:

$$t = \frac{1}{x'_0} \int \frac{x'_0 cos(\theta) d\theta}{\omega \sqrt{1 - sen^2(\theta)}}$$

$$t = \frac{1}{\omega} \int \frac{cos(\theta) d\theta}{\sqrt{1 - sen^2(\theta)}}$$

$$t = \frac{1}{\omega} \int d\theta$$

$$t = \frac{1}{\omega} \theta$$

$$t = \frac{1}{\omega} \cdot arcsen\left(\frac{\omega x}{x'_0}\right)\Bigg|_0^x$$

$$t = \frac{1}{\omega}\left(arcsen\left(\frac{\omega x}{x_0'}\right) - arcsen\left(\frac{\omega \cdot 0}{x_0'}\right)\right)$$

$$\omega t = arcsen\left(\frac{\omega x}{x_0'}\right)$$

$$sen(\omega t) = \frac{\omega x}{x_0'}$$

$$x = \frac{x_0'}{\omega} \cdot sen(\omega t) \tag{6}$$