UNIVERSIDAD MAYOR DE SAN SIMÓN
FACULTAD DE CIENCIAS Y TECNOLOGÍA
CARRERA DE ELECTROMECÁNICA

TRANSFORMADAS E INTEGRALES

Apuntes de clase

Docente:
Ing. Marco Antonio Vallejo Camacho.

Índice general

Bibliografía recomendada

[1] Hwei Hsu. *Análisis de Fourier*.

[2] Serie *Schaum*. *Transformada de Laplace*.

[3] Eduardo Espinoza. *Transformada de Laplace*.

[4] Álvaro Hernando Carrasco Calvo. *Transformadas e integrales*.

Capítulo 1

Series de *Fourier*

1.1. Funciones periódicas

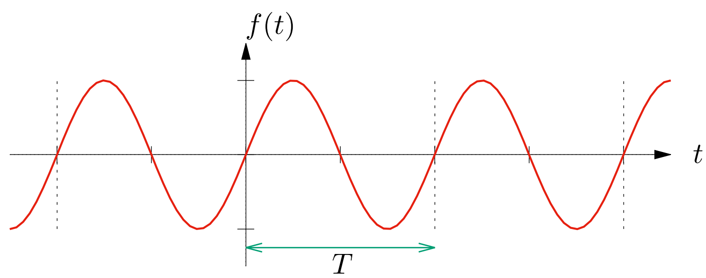


Figura 1.1: Función periódica

Una función periódica es aquella cuya gráfica se repite infinitas veces, cada cierto intervalo (**Figura 1.1**).

El menor intervalo de repetición se llama *periodo* (T).

Matemáticamente una función periódica es aquella que verifica:

$$f(t) = f(t + nT); n \in \mathbb{Z} \tag{1.1}$$

Donde T es el periodo (la menor constante que verifica la igualdad).

1.2. Propiedades de la funciones periódicas

Si: $f(t) = f(t + nT)$

Propiedad 1

$$\int_a^b f(t)\,dt = \int_{a+nT}^{b+nT} f(t)\,dt \quad n \in \mathbb{Z} \tag{1.2}$$

Prueba:

$$\int_a^b f(t)\,dt = \int_a^b f(t+nT)\,dt$$

Cambiando la variable:

$$\tau = t + nT$$

$$d\tau = dt$$

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_{a+nT}^{b+nT} f(\tau)\,d\tau \\ &= \int_{a+nT}^{b+nT} f(t)\,dt\end{aligned}$$

Puede verse gráficamente en la **Figura 1.2.**

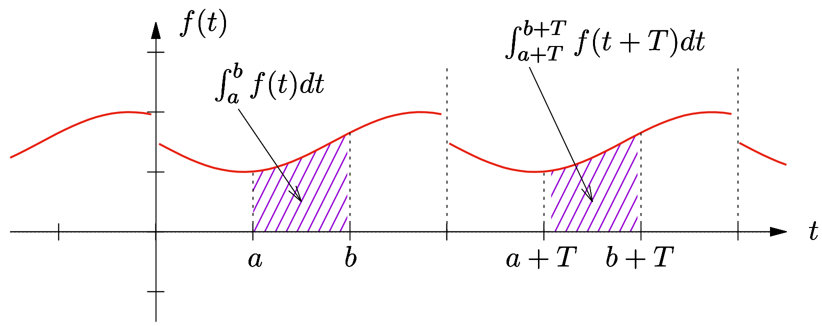


Figura 1.2: Demostración gráfica

Propiedad 2

$$\int_{a-T/2}^{a+T/2} f(t)\,dt = \int_{-T/2}^{T/2} f(t)\,dt \tag{1.3}$$

Prueba:

$$\begin{aligned}
\int_{a-T/2}^{a+T/2} f(t)\,dt &= \int_{a-T/2}^{T/2} f(t)\,dt + \int_{T/2}^{a+T/2} f(t)\,dt \\
&= \int_{a-T/2}^{T/2} f(t)\,dt + \int_{T/2-T}^{a+T/2-T} f(t)\,dt \\
&= \int_{a-T/2}^{T/2} f(t)\,dt + \int_{-T/2}^{a-T/2} f(t)\,dt \\
&= \int_{-T/2}^{T/2} f(t)\,dt
\end{aligned}$$

Puede verse gráficamente en la **Figura 1.3.**

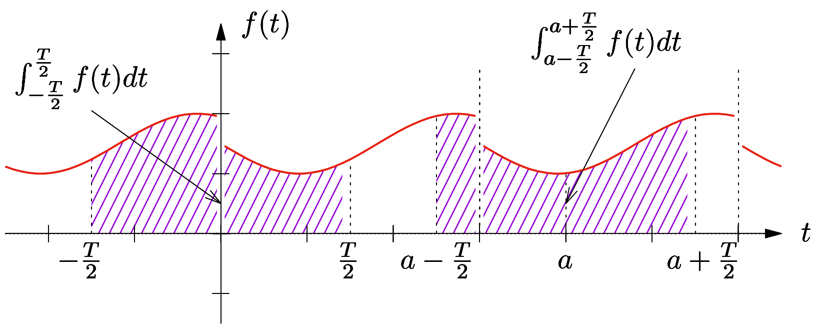


Figura 1.3: Demostración gráfica

Propiedad 3

$$\int_{0}^{T} f(t)\,dt = \int_{-T/2}^{T/2} f(t)\,dt \tag{1.4}$$

Prueba:

Si en la ecuación (1.3) $a = T/2$:

$$\begin{aligned}
\int_{-T/2}^{T/2} f(t)\,dt &= \int_{a-T/2}^{a+T/2} f(t)\,dt \\
&= \int_{T/2-T/2}^{T/2+T/2} f(t)\,dt \\
&= \int_{0}^{T} f(t)\,dt
\end{aligned}$$

Puede verse gráficamente en la **Figura 1.4**.

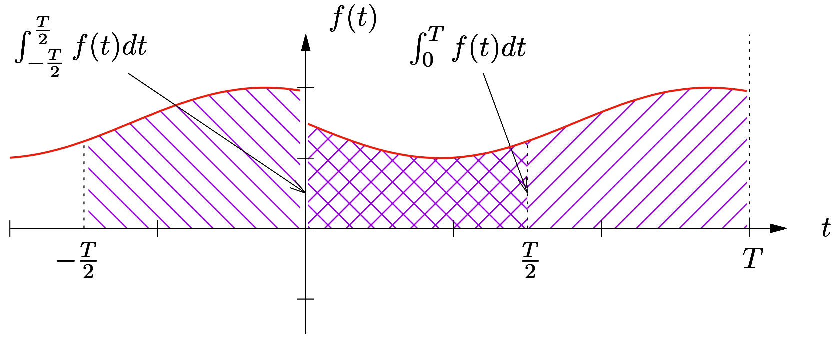


Figura 1.4: Demostración gráfica

Propiedad 4

Si $b - a = T$:

$$\int_0^T f(t)\,dt = \int_a^b f(t)\,dt \tag{1.5}$$

<u>Prueba</u>:

$$\begin{aligned}
\int_a^b f(t)\,dt &= \int_a^{a+T} f(t)\,dt \\
&= \int_a^T f(t)\,dt + \int_T^{a+T} f(t)\,dt \\
&= \int_a^T f(t)\,dt + \int_{T-T}^{a+T-T} f(t)\,dt \\
&= \int_a^T f(t)\,dt + \int_0^a f(t)\,dt \\
&= \int_0^T f(t)\,dt
\end{aligned}$$

Puede verse gráficamente en la **Figura 1.5.**

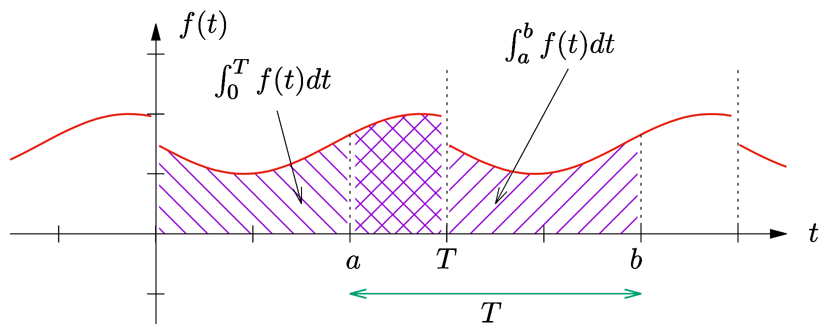


Figura 1.5: Demostración gráfica

1.2.1. Funciones seno y coseno

$$f(t) = A\ \mathrm{sen}(\omega_0 t)$$
$$f(t) = A\ \cos(\omega_0 t)$$

Donde:
A: Amplitud.
ω_0: Frecuencia angular.
$T = 2\pi/\omega_0$: Periodo.

Ejemplo: Hallar el periodo de la siguiente función:

$$f(t) = \mathrm{sen}(4t) + \mathrm{sen}(3t/2) + \mathrm{sen}(10t)$$

El periodo buscado debe contener un numero entero de veces a los 3 periodos hallados:

$$T = \begin{cases} a\,T_1; & a \in \mathbb{N} \\ b\,T_2; & b \in \mathbb{N} \\ c\,T_3; & c \in \mathbb{N} \end{cases}$$

$$a\,T_1 = b\,T_2 = c\,T_3$$
$$a\,\frac{2\pi}{4} = b\,\frac{2\pi}{3/2} = c\,\frac{2\pi}{10}$$
$$a\,\frac{\pi}{2} = b\,\frac{4\pi}{3} = c\,\frac{\pi}{5}; x30$$
$$15a = 40b = 6c$$
$$M.C.M.(15, 40, 6) = 120$$
$$\left.\begin{array}{r} 120 = 15a \to a = 8 \\ 120 = 40b \to b = 3 \\ 120 = 6c \to c = 20 \end{array}\right\} = T = 8\left(\frac{\pi}{2}\right) = 4\pi$$

Puede verse gráficamente en la **Figura 1.6**.

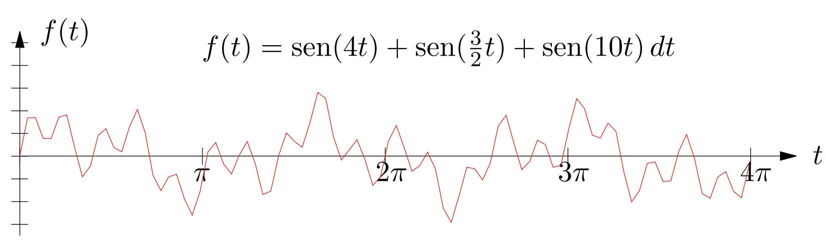


Figura 1.6: Periodo de la función

1.2.2. Propiedades ortogonales del seno y el coseno

Propiedad 1

$$\int_0^T \operatorname{sen}(n\omega_0 t)\,dt = 0 \quad n \in \mathbb{Z} \tag{1.6}$$

Prueba:

$$\begin{aligned}
\int_0^T \operatorname{sen}(n\omega_0 t)\,dt &= -\frac{\cos(n\omega_0 t)}{n\omega_0}\bigg|_0^T \\
&= -\frac{\cos(n\omega_0 T)}{n\omega_0} + \frac{\cos(0)}{n\omega_0} \\
&= -\frac{\cos(n2\pi)}{n\omega_0} + \frac{\cos(0)}{n\omega_0} \\
&= -\frac{1}{n\omega_0} + \frac{1}{n\omega_0} \\
&= 0
\end{aligned}$$

$$\int_0^T \cos(n\omega_0 t)\,dt = 0 \quad n \in \mathbb{Z} \tag{1.7}$$

Prueba:

$$\begin{aligned}
\int_0^T \cos(n\omega_0 t)\,dt &= \frac{\operatorname{sen}(n\omega_0 t)}{n\omega_0}\bigg|_0^T \\
&= \frac{\operatorname{sen}(n\omega_0 T)}{n\omega_0} - \frac{\operatorname{sen}(0)}{n\omega_0} \\
&= \frac{\operatorname{sen}(n2\pi)}{n\omega_0} - \frac{\operatorname{sen}(0)}{n\omega_0} \\
&= \frac{0}{n\omega_0} - \frac{0}{n\omega_0} \\
&= 0
\end{aligned}$$

Propiedad 2

$$\int_0^T \operatorname{sen}(m\omega_0 t)\operatorname{sen}(n\omega_0 t)\,dt = 0 \quad m, n \in \mathbb{Z} \quad m \neq n \tag{1.8}$$

Prueba:

$$\begin{aligned}
\int_0^T \operatorname{sen}(m\omega_0 t)\operatorname{sen}(n\omega_0 t)\,dt &= \int_0^T \frac{1}{2}(\cos((m-n)\omega_0 t) - \cos((m+n)\omega_0 t))dt \\
&= \frac{1}{2}\left(\int_0^T \cos((m-n)\omega_0 t)dt - \int_0^T \cos((m+n)\omega_0 t)dt\right) \\
&= \frac{1}{2}(0-0) \\
&= 0
\end{aligned}$$

$$\int_0^T \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = 0 \quad m,n \in \mathbb{Z} \quad m \neq n \tag{1.9}$$

Prueba:

$$\begin{aligned}
\int_0^T \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt &= \int_0^T \frac{1}{2}(\cos((m-n)\omega_0 t) + \cos((m+n)\omega_0 t))dt \\
&= \frac{1}{2}\left(\int_0^T \cos((m-n)\omega_0 t)dt + \int_0^T \cos((m+n)\omega_0 t)dt\right) \\
&= \frac{1}{2}(0+0) \\
&= 0
\end{aligned}$$

$$\int_0^T \operatorname{sen}(m\omega_0 t)\cos(n\omega_0 t)\,dt = 0 \quad m,n \in \mathbb{Z} \tag{1.10}$$

Prueba:

$$\begin{aligned}
\int_0^T \operatorname{sen}(m\omega_0 t)\cos(n\omega_0 t)\,dt &= \int_0^T \frac{1}{2}(\operatorname{sen}((m-n)\omega_0 t) + \operatorname{sen}((m+n)\omega_0 t))dt \\
&= \frac{1}{2}\left(\int_0^T \operatorname{sen}((m-n)\omega_0 t)dt + \int_0^T \cos((m+n)\omega_0 t)dt\right) \\
&= \frac{1}{2}(0+0) \\
&= 0
\end{aligned}$$

Propiedad 3

$$\int_0^T \operatorname{sen}^2(n\omega_0 t)\,dt = \frac{T}{2} \quad n \in \mathbb{Z} \tag{1.11}$$

Prueba:

$$\begin{aligned}
\int_0^T \operatorname{sen}^2(n\omega_0 t)\,dt &= \int_0^T \operatorname{sen}(n\omega_0 t)\operatorname{sen}(n\omega_0 t)dt \\
&= \frac{1}{2}\left(\int_0^T \cos((n-n)\omega_0 t)dt - \int_0^T \cos((n+n)\omega_0 t)dt\right) \\
&= \frac{1}{2}\left(\int_0^T \cos(0)dt - \int_0^T \cos((2n)\omega_0 t)dt\right) \\
&= \frac{1}{2}\left(\int_0^T dt - \int_0^T \cos(2n\omega_0 t)dt\right) \\
&= \frac{1}{2}(t\Big|_0^T - 0) \\
&= \frac{T}{2}
\end{aligned}$$

$$\int_0^T \cos^2(n\omega_0 t)\,dt = \frac{T}{2} \quad n \in \mathbb{Z} \tag{1.12}$$

Prueba:

$$\begin{aligned}
\int_0^T \cos^2(n\omega_0 t)\,dt &= \int_0^T \cos(n\omega_0 t)\cos(n\omega_0 t)dt \\
&= \frac{1}{2}\left(\int_0^T \cos((n-n)\omega_0 t)dt + \int_0^T \cos((n+n)\omega_0 t)dt\right) \\
&= \frac{1}{2}\left(\int_0^T \cos(0)dt + \int_0^T \cos((2n)\omega_0 t)dt\right) \\
&= \frac{1}{2}\left(\int_0^T dt + \int_0^T \cos(2n\omega_0 t)dt\right) \\
&= \frac{1}{2}(t\Big|_0^T + 0) \\
&= \frac{T}{2}
\end{aligned}$$

1.3. Series de *Fourier*

Una función periódica que cumple ciertas condiciones puede desarrollarse mediante la serie:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t) + b_n \operatorname{sen}(n\omega_0 t)] \tag{1.13}$$

Donde:

$\omega_0 = 2\pi/T$: Frecuencia angular de $f(t)$.

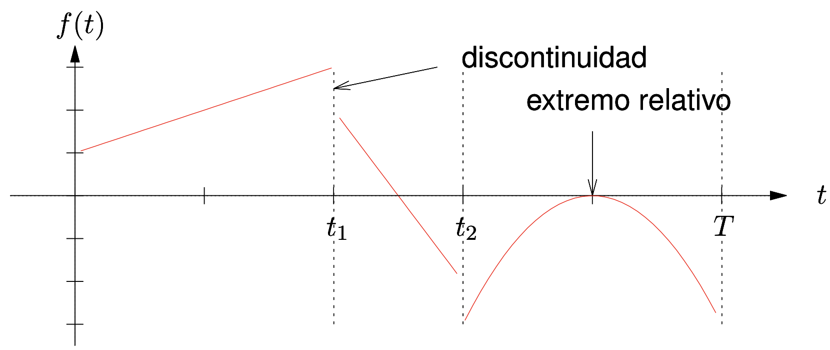
T: Periodo de $f(t)$.
$a_0; a_n; b_n$: Coeficientes de *Fourier*.
$a_0/2$: Termino constante.
$a_n \cos(n\omega_0 t); b_n \operatorname{sen}(n\omega_0 t)$: Armónicos, términos seno y coseno con frecuencias angulares múltiples de ω_0

$a_1 \cos(\omega_0 t) + b_1 \operatorname{sen}(\omega_0 t)$: Primer armónico.
$a_2 \cos(2\omega_0 t) + b_2 \operatorname{sen}(2\omega_0 t)$: Segundo armónico.
$a_3 \cos(3\omega_0 t) + b_3 \operatorname{sen}(2\omega_0 t)$: Tercer armónico.

1.3.1. Condiciones de *Dirichlet*

Para que una función periódica $f(t) = f(t + nT); n \in \mathbb{Z}$, se desarrolle como una serie de *Fourier* debe cumplir:

- $f(t)$ debe ser continua por tramos en 1 periodo.

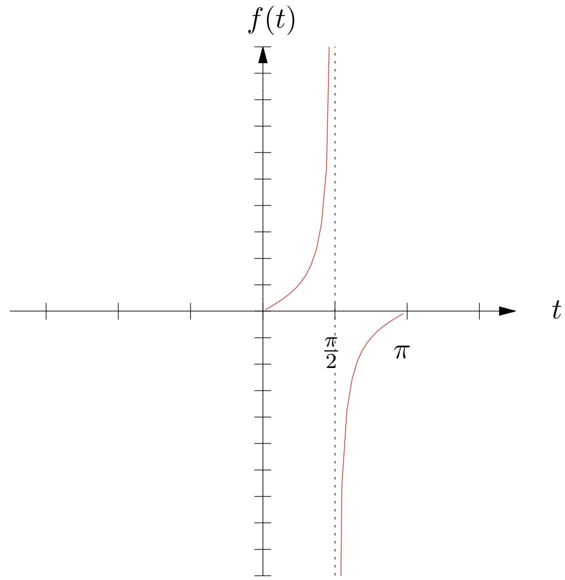


  - Debe existir un numero finito de discontinuidades (en 1 periodo).
  - Debe existir un numero finito de extremos relativos (en 1 periodo).

- La integral $\int_0^T |f(t)|\,dt < \infty$ debe ser finita.

Ejemplo:

$$f(t) = \tan(t); \quad 0 < t < \pi; \quad T = \pi$$



$$\int_0^{\pi} |\tan t|\, dt \to \infty$$

$$t = \frac{\pi}{2} : |\tan(t)| \to \infty$$

$\therefore$ Esta función no tiene serie de *Fourier*.

1.4. Evaluación de los coeficientes de *Fourier*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t) + b_n \operatorname{sen}(n\omega_0 t)]$$

Integrando ambas partes:

$$\begin{aligned}\int_0^T f(t)\, dt &= \int_0^T \frac{a_0}{2}\, dt + \sum_{n=1}^{\infty}\left[\int_0^T a_n \cos(n\omega_0 t)\, dt + \int_0^T b_n \operatorname{sen}(n\omega_0 t)\, dt\right] \\ &= \frac{a_0}{2} t \Big|_0^T \\ &= \frac{a_0}{2} T\end{aligned}$$

$$a_0 = \frac{2}{T}\int_0^T f(t)\, dt \tag{1.14}$$

Para calcular "a_n" multiplicamos por $\cos(m\omega_0 t); m \in \mathbb{N}$ e integramos en 1 periodo.

$$\begin{aligned}\int_0^T f(t)\cos(m\omega_0 t)\,dt &= \int_0^T \frac{a_0}{2}\cos(m\omega_0 t)\,dt \\ &+ \sum_{n=1}^{\infty}\left[\int_0^T a_n\cos(n\omega_0 t)\cos(m\omega_0 t)\,dt + \int_0^T b_n \operatorname{sen}(n\omega_0 t)\cos(m\omega_0 t)\,dt\right] \\ &= 0 + \sum_{n=1}^{\infty}\left[\int_0^T a_n\cos(n\omega_0 t)\cos(m\omega_0 t)\,dt + 0\right]\end{aligned}$$

Para $n \neq m$ todos los elementos de la sumatoria serán igual a 0. Por tanto:

$$\begin{aligned}\int_0^T f(t)\cos(n\omega_0 t)\,dt &= \int_0^T a_n\cos^2(n\omega_0 t)\,dt \\ &= a_n\frac{T}{2}\end{aligned}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt \tag{1.15}$$

Para calcular "b_n" multiplicamos por $\operatorname{sen}(m\omega_0 t); m \in \mathbb{N}$ e integramos en 1 periodo.

$$\begin{aligned}\int_0^T f(t)\operatorname{sen}(m\omega_0 t)\,dt &= \int_0^T \frac{a_0}{2}\operatorname{sen}(m\omega_0 t)\,dt \\ &+ \sum_{n=1}^{\infty}\left[\int_0^T a_n\cos(n\omega_0 t)\operatorname{sen}(m\omega_0 t)\,dt + \int_0^T b_n \operatorname{sen}(n\omega_0 t)\operatorname{sen}(m\omega_0 t)\,dt\right] \\ &= 0 + \sum_{n=1}^{\infty}\left[0 + \int_0^T b_n\operatorname{sen}(n\omega_0 t)\operatorname{sen}(m\omega_0 t)\,dt\right]\end{aligned}$$

Para $n \neq m$ todos los elementos de la sumatoria serán igual a 0. Por tanto:

$$\begin{aligned}\int_0^T f(t)\operatorname{sen}(n\omega_0 t)\,dt &= \int_0^T b_n\operatorname{sen}^2(n\omega_0 t)\,dt \\ &= b_n\frac{T}{2}\end{aligned}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\operatorname{sen}(n\omega_0 t)\,dt \tag{1.16}$$

1.5. Formulas para las series de *Fourier*

$$\begin{aligned}\operatorname{sen}(\pi n) = 0; &\quad n \in \mathbb{N} \\ \cos(\pi n) = (-1)^n; &\quad n \in \mathbb{N} \\ \operatorname{sen}(2\pi n) = 0; &\quad n \in \mathbb{N}\end{aligned}$$

$$\cos(2\pi n) = 1; \quad n \in \mathbb{N}$$

$$\int \operatorname{sen}(at)\,dt = -\frac{\cos(at)}{a}$$

$$\int \cos(at)\,dt = \frac{\operatorname{sen}(at)}{a}$$

$$\int t\operatorname{sen}(at)\,dt = -\frac{t}{a}\cos(at) + \frac{1}{a^2}\operatorname{sen}(at)$$

$$\int t\cos(at)\,dt = \frac{t}{a}\operatorname{sen}(at) + \frac{1}{a^2}\cos(at)$$

$$\int e^{at}\,dt = \frac{1}{a}e^{at}$$

$$\int t\,e^{at}\,dt = \frac{t}{a}\,e^{at} - \frac{1}{a^2}e^{at}$$

Capítulo 2

Análisis de formas de onda periódica

2.1. Funciones pares e impares

Una función es **par** si:

$$f(-t) = f(t) \tag{2.1}$$

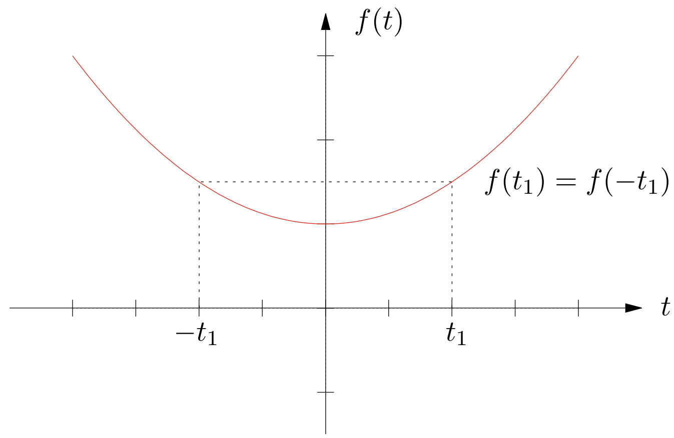


Figura 2.1: La gráfica se refleja respecto al eje central.

Ejemplo 1:

$$f(t) = t^2$$

$$f(-t) = (-t)^2 = t^2 = f(t)$$

Ejemplo 2:

$$f(t) = \cos(t)$$

$$f(t) = \cos(-t) = \cos(t) = f(t)$$

Una función es **impar** si:

$$f(-t) = -f(t) \tag{2.2}$$

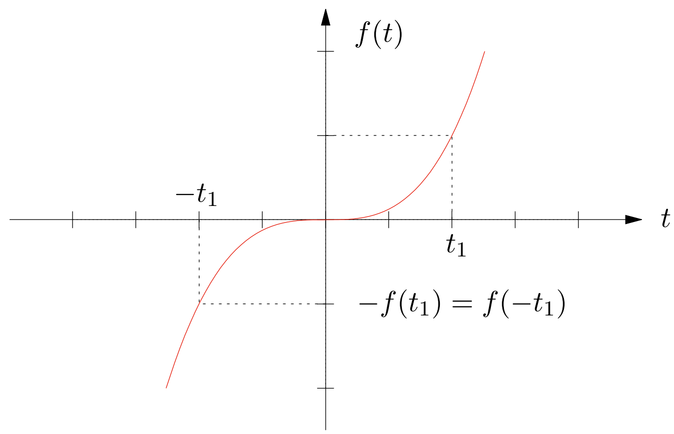


Figura 2.2: La gráfica se refleja
1ro respecto al eje central
2do respecto al eje horizontal.

Ejemplo 3:

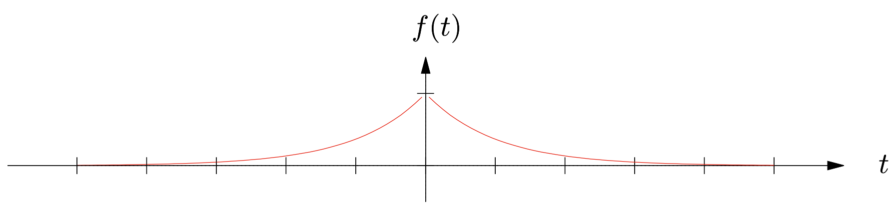
$$f(t) = t^3$$
$$f(-t) = (-t)^3 = -t^3 = -f(t)$$

Ejemplo 4:

$$f(t) = \operatorname{sen}(t)$$
$$f(t) = \operatorname{sen}(-t) = -\operatorname{sen}(t) = -f(t)$$

Ejemplo 5:

$$f(t) = \begin{cases} e^{t} & t < 0 \\ e^{-t} & t > 0 \end{cases}$$
$$f(-t) = \left\{\begin{matrix} e^{-t} & -t < 0 \to t > 0 \\ e^{t} & -t > 0 \to t < 0 \end{matrix}\right\} = f(t)$$



2.1.1. Propiedades de las funciones pares e impares

Propiedad 1

Si $f(t)$ es **par** y $g(t)$ es **par**, entonces $h(t) = f(t)g(t)$ es **par**.

Prueba:

$$\begin{cases} f(-t) = f(t) \\ g(-t) = g(t) \end{cases}$$

$$\begin{aligned} h(-t) &= f(-t)g(-t) \\ &= f(t)g(t) \\ &= h(t) \end{aligned}$$

Propiedad 2

Si $f(t)$ es **impar** y $g(t)$ es **impar**, entonces $h(t) = f(t)g(t)$ es **par.**

Prueba:

$$\begin{cases} f(-t) = -f(t) \\ g(-t) = -g(t) \end{cases}$$

Propiedad 3

Si $f(t)$ es **par** y $g(t)$ es **impar**, entonces $h(t) = f(t)g(t)$ es **impar.**

Prueba:

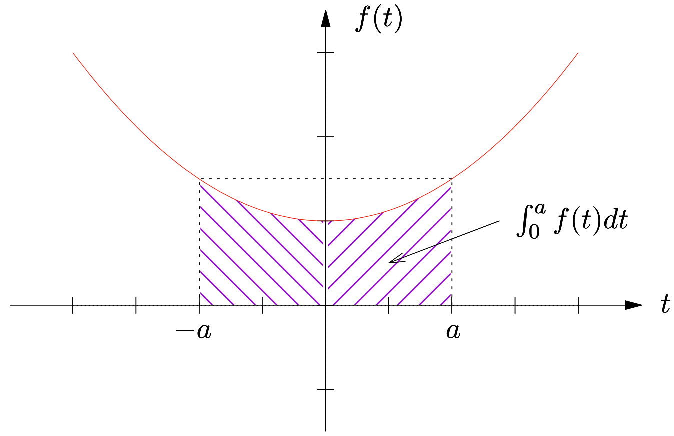
$$\begin{cases} f(-t) = f(t) \\ g(-t) = -g(t) \end{cases}$$

$$\begin{aligned} h(-t) &= f(-t)g(-t) \\ &= f(t)(-g(t)) \\ &= -f(t)g(t) \\ &= -h(t) \end{aligned}$$

Propiedad 4

Si $f(t)$ es **par**, entonces:

$$\int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt \tag{2.3}$$



Prueba:

$$
\begin{aligned}
\int_{-a}^{a} f(t)\,dt &= \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt \\
&= \int_{-a}^{0} f(-t)\,dt + \int_{0}^{a} f(t)\,dt
\end{aligned}
$$

$$\tau = -t$$

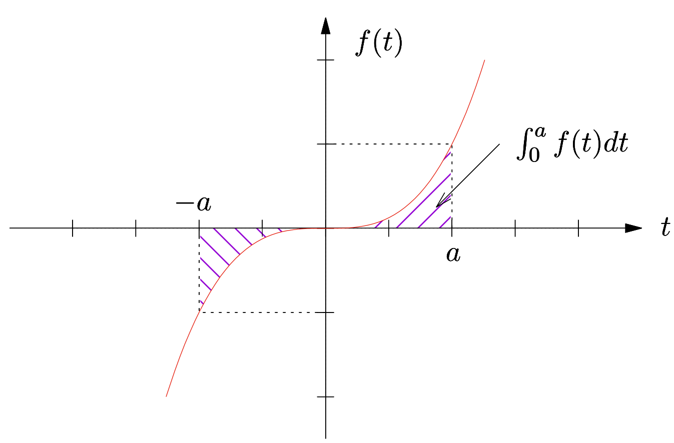
$$d\tau = -dt$$

$$
\begin{aligned}
\int_{-a}^{a} f(t)\,dt &= \int_{a}^{0} f(\tau)\,(-d\tau) + \int_{0}^{a} f(t)\,dt \\
&= \int_{0}^{a} f(\tau)\,d\tau + \int_{0}^{a} f(t)\,dt \\
&= 2\int_{0}^{a} f(t)\,dt
\end{aligned}
$$

Propiedad 5

Si $f(t)$ es **impar**, entonces:

$$\int_{-a}^{a} f(t)\,dt = 0 \tag{2.4}$$



Prueba:

$$
\begin{aligned}
\int_{-a}^{a} f(t)\,dt &= \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt \\
&= \int_{-a}^{0} -f(-t)\,dt + \int_{0}^{a} f(t)\,dt
\end{aligned}
$$

$$\tau = -t$$

$$d\tau = -dt$$

$$
\begin{aligned}
\int_{-a}^{a} f(t)\,dt &= -\int_{a}^{0} f(\tau)\,(-d\tau) + \int_{0}^{a} f(t)\,dt \\
&= -\int_{0}^{a} f(\tau)\,d\tau + \int_{0}^{a} f(t)\,dt \\
&= 0
\end{aligned}
$$

2.1.2. Evaluación de coeficientes de *Fourier*

Simetría par

$$
\begin{aligned}
a_0 &= \frac{2}{T}\int_{0}^{T} f(t)\,dt \\
&= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt \\
&= \frac{2}{T}\left(2\int_{0}^{T/2} f(t)\,dt\right) \\
&= \frac{4}{T}\int_{0}^{T/2} f(t)\,dt
\end{aligned}
$$

$$a_0 = \frac{4}{T}\int_{0}^{T/2} f(t)\,dt \tag{2.5}$$

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt \\
&= \frac{2}{T}\left(2\int_{0}^{T/2} f(t)\cos(n\omega_0 t)\,dt\right) \\
&= \frac{4}{T}\int_{0}^{T/2} f(t)\cos(n\omega_0 t)\,dt
\end{aligned}
$$

$$a_n = \frac{4}{T}\int_{0}^{T/2} f(t)\cos(n\omega_0 t)\,dt \tag{2.6}$$

$$\begin{aligned} b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt \\ &= 0 \end{aligned}$$

$$b_n = 0 \tag{2.7}$$

Simetría impar

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^T f(t)\,dt \\ &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt \\ &= 0 \end{aligned}$$

$$a_0 = 0 \tag{2.8}$$

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt \\ &= 0 \end{aligned}$$

$$a_n = 0 \tag{2.9}$$

$$\begin{aligned} b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt \\ &= \frac{2}{T}\left(2\int_0^{T/2} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt\right) \\ &= \frac{4}{T}\int_0^{T/2} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt \end{aligned}$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt \tag{2.10}$$

2.2. Simetría de media onda (S.M.O.)

$f(t)$ tiene simetría de media onda si:

$$f(t) = -f(t \pm \frac{T}{2})$$

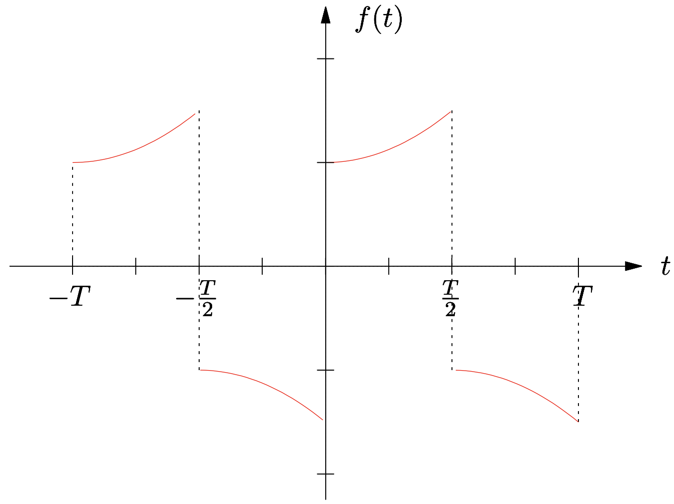


Figura 2.3: La gráfica se desplaza $1/2$ periodo
y se refleja respecto a t.

2.2.1. Evaluación de coeficientes de *Fourier*

$$
\begin{aligned}
a_0 &= \frac{2}{T}\int_0^T f(t)\,dt \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\,dt + \int_{T/2}^{T} f(t)\,dt\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\,dt - \int_{t=T/2}^{t=T} f(t-\frac{T}{2})\,dt\right]
\end{aligned}
$$

$$
\tau = t - \frac{T}{2}
$$

$$
d\tau = dt
$$

$$
\begin{aligned}
a_0 &= \frac{2}{T}\left[\int_0^{T/2} f(t)\,dt - \int_{\tau=T/2-T/2}^{\tau=T-T/2} f(\tau)\,d\tau\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\,dt - \int_0^{T/2} f(\tau)\,d\tau\right]
\end{aligned}
$$

$$
a_0 = 0 \tag{2.11}
$$

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt + \int_{T/2}^{T} f(t)\cos(n\omega_0 t)\,dt\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt - \int_{T/2}^{T} f(t-\frac{T}{2})\cos(n\omega_0 t)\,dt\right]
\end{aligned}
$$

$$\tau = t - \frac{T}{2}$$
$$d\tau = dt$$

$$\begin{aligned}
a_n &= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt - \int_{\tau=T/2-T/2}^{\tau=T-T/2} f(\tau)\cos(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt - \int_0^{T/2} f(\tau)\cos(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right]
\end{aligned}$$

$$\begin{aligned}
n\omega_0(\tau+T/2) &= n\omega_0\tau + n\omega_0\frac{T}{2} \\
&= n\omega_0\tau + n\frac{2\pi}{T}\frac{T}{2} \\
&= n\omega_0\tau + n\pi
\end{aligned}$$

$$\begin{aligned}
\cos(n\omega_0\tau + n\pi) &= \cos(n\omega_0\tau)\cos(n\pi) - \operatorname{sen}(n\pi)\operatorname{sen}(n\omega_0\tau) \\
&= \cos(n\omega_0\tau)\cos(n\pi)
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt - \int_0^{T/2} f(\tau)\cos(n\omega_0\tau)\cos(n\pi)\,d\tau\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt - \cos(n\pi)\int_0^{T/2} f(\tau)\cos(n\omega_0\tau)\,d\tau\right] \\
&= \frac{2}{T}(1-\cos(\pi n))\left(\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt\right)
\end{aligned}$$

$$\cos(\pi n) = \begin{cases} 1 & n : \text{par} \\ -1 & n : \text{impar} \end{cases}$$

$$\begin{cases} n : \text{par} & a_n = 0 \\ n : \text{impar} & a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt \end{cases} \tag{2.12}$$

$$\begin{aligned}
b_n &= \frac{2}{T}\int_0^{T} f(t)\operatorname{sen}(n\omega_0 t)\,dt \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt + \int_{T/2}^{T} f(t)\operatorname{sen}(n\omega_0 t)\,dt\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \int_{T/2}^{T} f(t-\frac{T}{2})\operatorname{sen}(n\omega_0 t)\,dt\right]
\end{aligned}$$

$$\tau = t - \frac{T}{2}$$
$$d\tau = dt$$

$$\begin{aligned}
b_n &= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \int_{\tau=T/2-T/2}^{\tau=T-T/2} f(\tau)\operatorname{sen}(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \int_0^{T/2} f(\tau)\operatorname{sen}(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right]
\end{aligned}$$

$$\begin{aligned} n\omega_0(\tau + T/2) &= n\omega_0\tau + n\omega_0\frac{T}{2} \\ &= n\omega_0\tau + n\frac{2\pi}{T}\frac{T}{2} \\ &= n\omega_0\tau + n\pi \end{aligned}$$

$$\begin{aligned} \operatorname{sen}(n\omega_0\tau + n\pi) &= \operatorname{sen}(n\omega_0\tau)\cos(n\pi) + \operatorname{sen}(n\pi)\cos(n\omega_0\tau) \\ &= \operatorname{sen}(n\omega_0\tau)\cos(n\pi) \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \int_0^{T/2} f(\tau)\operatorname{sen}(n\omega_0\tau)\cos(n\pi)\,d\tau\right] \\ &= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \cos(n\pi)\int_0^{T/2} f(\tau)\operatorname{sen}(n\omega_0\tau)\,d\tau\right] \\ &= \frac{2}{T}(1-\cos(\pi n))\left(\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt\right) \end{aligned}$$

$$\cos(\pi n) = \begin{cases} 1 & n:\text{par} \\ -1 & n:\text{impar} \end{cases}$$

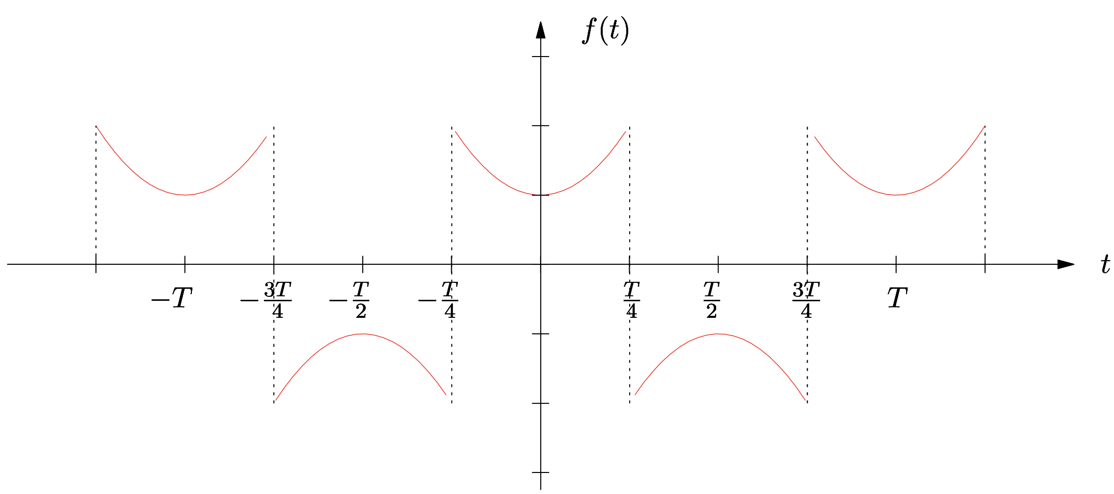
$$\begin{cases} n:\text{par} & b_n = 0 \\ n:\text{impar} & b_n = \frac{4}{T}\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt \end{cases} \tag{2.13}$$

2.3. Simetría de cuarto de onda (S.C.O.)

2.3.1. Simetría de cuarto de onda par

Una función $f(t)$ tiene simetría de cuarto de onda **par** cuando:

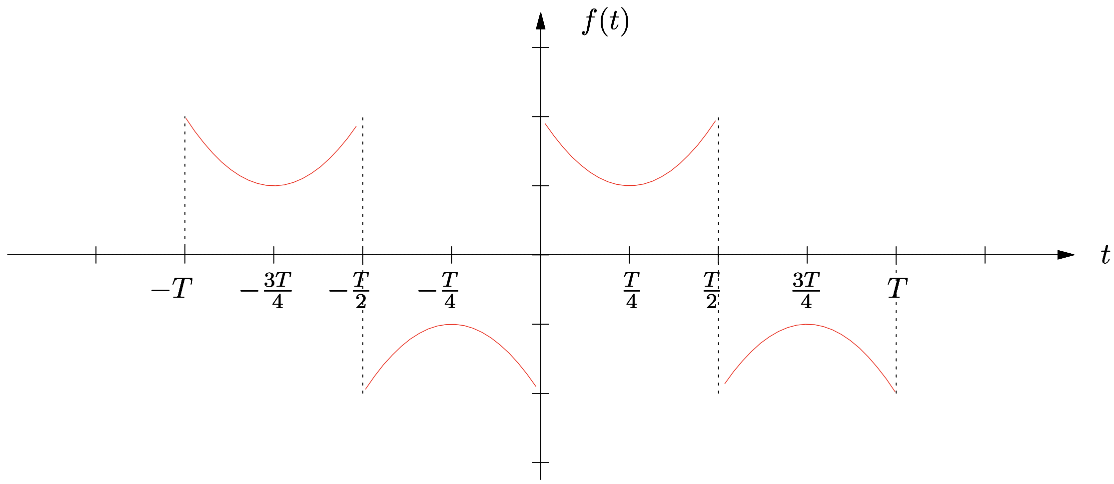
- $f(t)$ es **par**.
- $f(t)$ tiene simetría de media onda.



2.3.2. Simetría de cuarto de onda impar

Una función $f(t)$ tiene simetría de cuarto de onda **impar** cuando:

- $f(t)$ es **impar.**
- $f(t)$ tiene simetría de media onda.



2.3.3. Evaluación de coeficientes de *Fourier*

Simetría de cuarto de onda par

Como la función $f(t)$ tiene simetría de media onda:

$$a_0 = 0 \tag{2.14}$$

Como la función $f(t)$ es una función par:

$$b_n = 0 \tag{2.15}$$

Como la función $f(t)$ tiene simetría de media onda: $a_n = 0$ cuando n es par.

Para n impar:

$$\begin{aligned}
a_n &= \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt + \int_{T/4}^{T/2} f(t)\cos(n\omega_0 t)\,dt\right] \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt - \int_{T/4}^{T/2} f(t-\frac{T}{2})\cos(n\omega_0 t)\,dt\right]
\end{aligned}$$

$$\tau = t - \frac{T}{2}$$

$$\begin{aligned}
d\tau &= dt \\
a_n &= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt - \int_{\tau=T/4-T/2}^{\tau=T/2-T/2} f(\tau)\cos(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right] \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt - \int_{-T/4}^{0} f(\tau)\cos(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right]
\end{aligned}$$

$$\begin{aligned}
n\omega_0(\tau+T/2) &= n\omega_0\tau + n\omega_0\frac{T}{2} \\
&= n\omega_0\tau + n\frac{2\pi}{T}\frac{T}{2} \\
&= n\omega_0\tau + n\pi
\end{aligned}$$

$$\begin{aligned}
\cos(n\omega_0\tau + n\pi) &= \cos(n\omega_0\tau)\cos(n\pi) - \operatorname{sen}(n\pi)\operatorname{sen}(n\omega_0\tau) \\
&= \cos(n\omega_0\tau)\cos(n\pi) \\
&= -\cos(n\omega_0\tau)
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt + \int_{-T/4}^{0} f(\tau)\cos(n\omega_0\tau)\,d\tau\right] \\
&= \frac{4}{T}\left[\int_{-T/4}^{T/4} f(t)\cos(n\omega_0 t)\,dt\right] \\
&= \frac{4}{T}\left(2\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt\right) \\
&= \frac{8}{T}\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt
\end{aligned}$$

$$\begin{cases} n : \text{par} & a_n = 0 \\ n : \text{impar} & a_n = \frac{8}{T}\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt \end{cases} \tag{2.16}$$

Simetría de cuarto de onda impar

Como la función $f(t)$ tiene simetría de media onda:

$$a_0 = 0 \tag{2.17}$$

Como la función $f(t)$ es una función impar:

$$a_n = 0 \tag{2.18}$$

Como la función $f(t)$ tiene simetría de media onda: $b_n = 0$ cuando n es par.

Para n impar:

$$\begin{aligned}
b_n &= \frac{4}{T}\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\operatorname{sen}(n\omega_0 t)\,dt + \int_{T/4}^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt\right] \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \int_{T/4}^{T/2} f(t-\frac{T}{2})\operatorname{sen}(n\omega_0 t)\,dt\right]
\end{aligned}$$

$$\begin{aligned}
\tau &= t - \frac{T}{2} \\
d\tau &= dt
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{4}{T}\left[\int_0^{T/4} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \int_{\tau=T/4-T/2}^{\tau=T/2-T/2} f(\tau)\operatorname{sen}(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right] \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \int_{-T/4}^{0} f(\tau)\operatorname{sen}(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right]
\end{aligned}$$

$$\begin{aligned}
n\omega_0(\tau+T/2) &= n\omega_0\tau + n\omega_0\frac{T}{2} \\
&= n\omega_0\tau + n\frac{2\pi}{T}\frac{T}{2} \\
&= n\omega_0\tau + n\pi
\end{aligned}$$

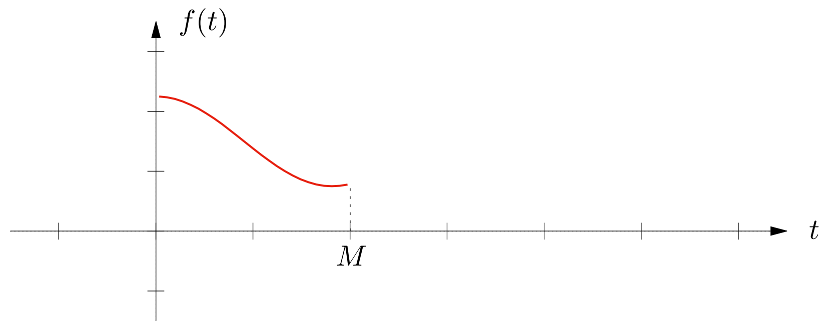
$$\begin{aligned}
\operatorname{sen}(n\omega_0\tau + n\pi) &= \operatorname{sen}(n\omega_0\tau)\cos(n\pi) + \operatorname{sen}(n\pi)\cos(n\omega_0\tau) \\
&= \operatorname{sen}(n\omega_0\tau)\cos(n\pi) \\
&= -\operatorname{sen}(n\omega_0\tau)
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{4}{T}\left[\int_0^{T/4} f(t)\operatorname{sen}(n\omega_0 t)\,dt + \int_{-T/4}^{0} f(\tau)\operatorname{sen}(n\omega_0\tau)\,d\tau\right] \\
&= \frac{4}{T}\left[\int_{-T/4}^{T/4} f(t)\operatorname{sen}(n\omega_0 t)\,dt\right] \\
&= \frac{4}{T}\left(2\int_0^{T/4} f(t)\operatorname{sen}(n\omega_0 t)\,dt\right) \\
&= \frac{8}{T}\int_0^{T/4} f(t)\operatorname{sen}(n\omega_0 t)\,dt
\end{aligned}$$

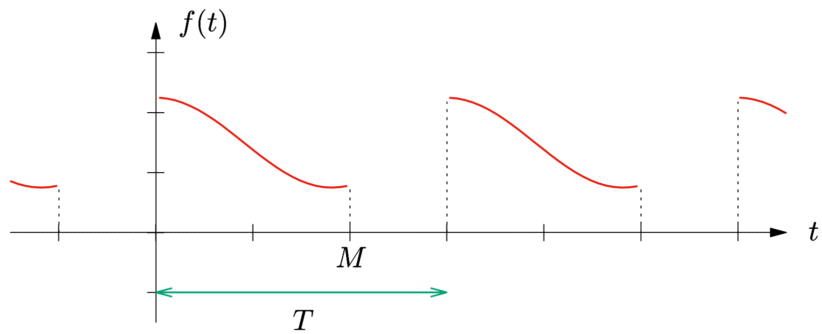
$$\begin{cases} n:\text{par} & b_n = 0 \\ n:\text{impar} & b_n = \frac{8}{T}\int_0^{T/4} f(t)\operatorname{sen}(n\omega_0 t)\,dt \end{cases} \tag{2.19}$$

2.4. Expansión periódica de funciones definidas en intervalos finitos

Sea $f(t)$ una función no periódica:



$f(t)$ se convierte en periódica al repetirla un intervalo $T \geq M$.

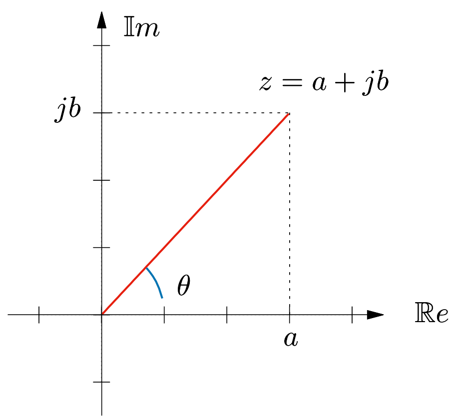


$f(t)$ puede expandirse periódicamente asignando alguna simetría conocida.

Capítulo 3

Serie compleja de *Fourier* y espectros discretos de frecuencia

3.1. Números complejos



Unidad imaginaria: $i = j = \sqrt{-1}$
Forma rectangular: $z = a + jb$
Módulo: $|z| = \sqrt{a^2 + b^2}$
Argumento: $\theta = \arctan(\frac{b}{a})$

Forma polar:

$$z = |z|\cos(\theta) + j|z|\operatorname{sen}(\theta) = |z|(\cos(\theta) + j\operatorname{sen}(\theta))$$

Formula de *Euler*:

$$e^{j\theta} = \cos(\theta) + j\operatorname{sen}(\theta) \tag{3.1}$$

Por tanto:

$$z = |z|e^{j\theta}$$

Forma exponencial o fasorial:

$$z = |z|\angle\theta$$

3.1.1. Formas complejas del seno y coseno

$$e^{j\theta} = \cos(\theta) + j\,\mathrm{sen}(\theta) \tag{3.2}$$
$$e^{-j\theta} = \cos(\theta) - j\,\mathrm{sen}(\theta) \tag{3.3}$$

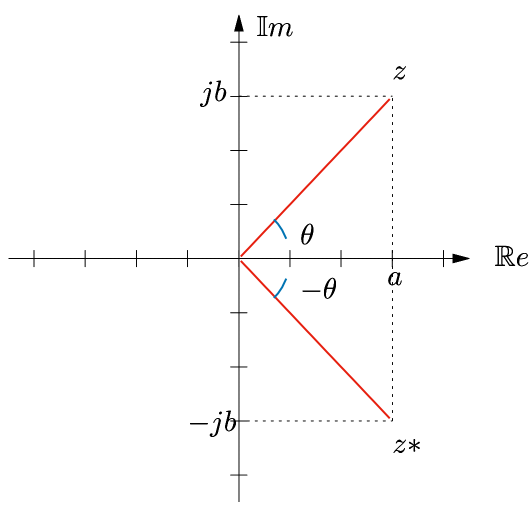
Sumando las ecuaciones 3.2 y 3.3:

$$e^{j\theta} + e^{-j\theta} = 2\cos(\theta)$$
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{3.4}$$

Restando las ecuaciones 3.2 y 3.3:

$$e^{j\theta} - e^{-j\theta} = 2j\,\mathrm{sen}(\theta)$$
$$\mathrm{sen}(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{3.5}$$

3.1.2. Conjugado



$$z = a + jb = |z|\angle\theta$$
$$z* = a - jb = |z|\angle - \theta$$
$$(z)(z*) = |z|^2$$

3.2. Serie compleja de *Fourier*

Partiendo de la serie trigonométrica de *Fourier*:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \operatorname{sen}(n\omega_0 t)]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) + b_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2j}\right)\right]$$

$$\frac{1}{j} = -j$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) - jb_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right)\right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t} + \left(\frac{a_n + jb_n}{2}\right) e^{-jn\omega_0 t}\right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t}\right] + \sum_{n=1}^{\infty} \left[\left(\frac{a_n + jb_n}{2}\right) e^{-jn\omega_0 t}\right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t}\right] + \sum_{-n=1}^{\infty} \left[\left(\frac{a_{-n} + jb_{-n}}{2}\right) e^{jn\omega_0 t}\right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t}\right] + \sum_{n=-1}^{-\infty} \left[\left(\frac{a_n + jb_n}{2}\right) e^{jn\omega_0 t}\right]$$

$$f(t) = \frac{a_0}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t}$$

Sean los coeficientes complejos de *Fourier*:

$$c_n = \frac{a_n - jb_n}{2}$$

$$c_0 = \frac{a_0}{2}$$

Entonces:

$$f(t) = c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n e^{jn\omega_0 t}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{3.6}$$

3.2.1. Evaluación del coeficiente complejo de *Fourier*

$$\begin{aligned}
c_n &= \frac{a_n - jb_n}{2} \\
&= \frac{1}{2}\left[\frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt - j\frac{2}{T}\int_0^T f(t)\,\mathrm{sen}(n\omega_0 t)\,dt\right] \\
&= \frac{1}{T}\int_0^T f(t)\left[\cos(n\omega_0 t) - j\,\mathrm{sen}(n\omega_0 t)\right]dt \\
&= \frac{1}{T}\int_0^T f(t)\,e^{-jn\omega_0 t}\,dt
\end{aligned}$$

$$c_n = \frac{1}{T}\int_0^T f(t)\,e^{-jn\omega_0 t}\,dt \tag{3.7}$$

En particular:

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt \tag{3.8}$$

3.2.2. Relación entre el coeficiente complejo y los coeficientes trigonométricos

$$c_n = \frac{a_n - jbn}{2} = \frac{a_n}{2} + j\frac{-b_n}{2}$$

$$\frac{a_n}{2} = \mathbb{R}e\{c_n\}$$

$$a_n = 2\,\mathbb{R}e\{c_n\} \tag{3.9}$$

$$-\frac{b_n}{2} = \mathbb{I}m\{c_n\}$$

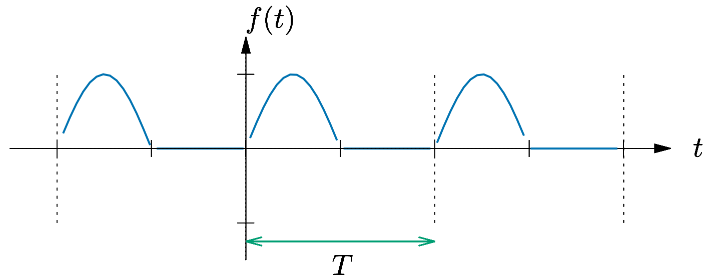
$$b_n = -2\,\mathbb{I}m\{c_n\} \tag{3.10}$$

3.3. Ondas senoidales rectificadas

3.3.1. Rectificación de media onda

$$f(t) = \begin{cases} A\,\mathrm{sen}(\omega_0 t) & 0 < t < T/2 \\ 0 & T/2 < t < T \end{cases}$$

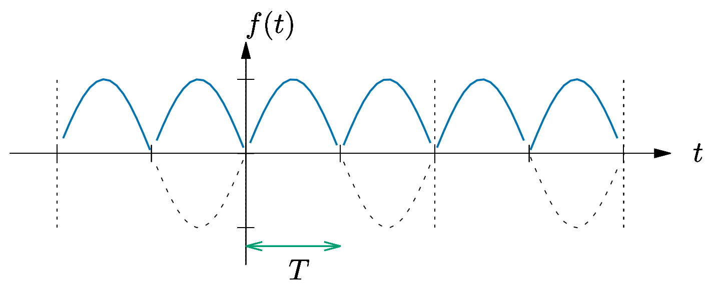
$$T = \frac{2\pi}{\omega_0}$$



El periodo de la onda rectificada es el mismo que de la onda original.

3.3.2. Rectificación de onda completa

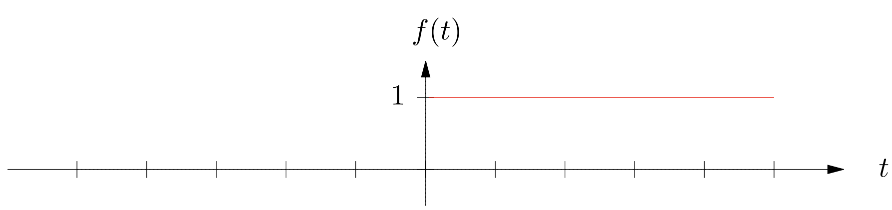
$$f(t) = A\,|\operatorname{sen}(\omega_0 t)|$$



El periodo de la onda rectificada es la mitad del periodo de la onda original.

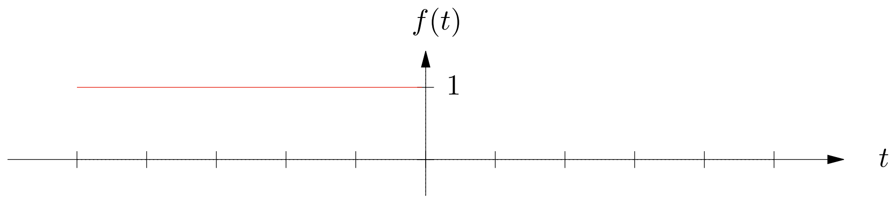
3.4. Función escalón unitario

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{3.11}$$



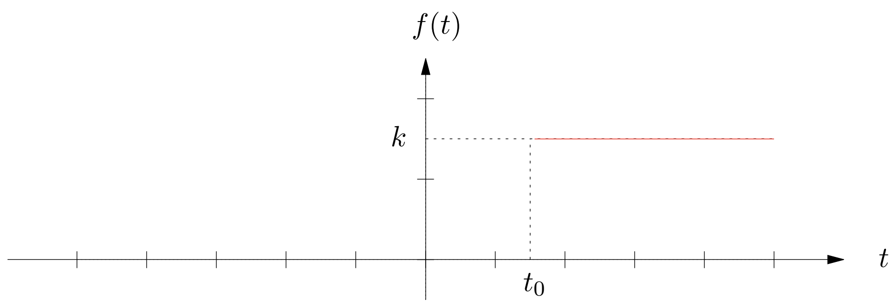
Una variante es:

$$u(-t) = \begin{cases} 1 & t < 0 \\ 0 & t > 0 \end{cases}$$

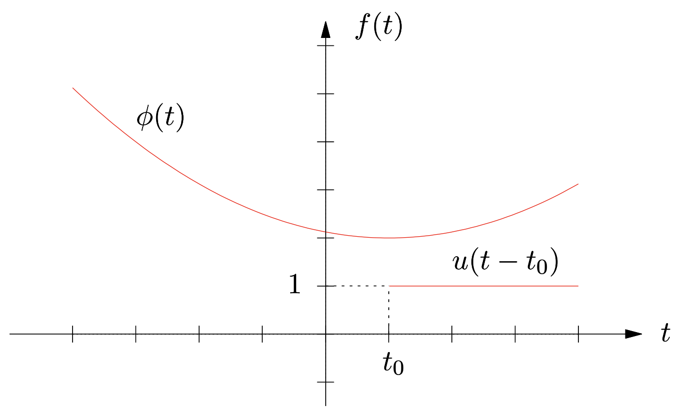


De manera general:

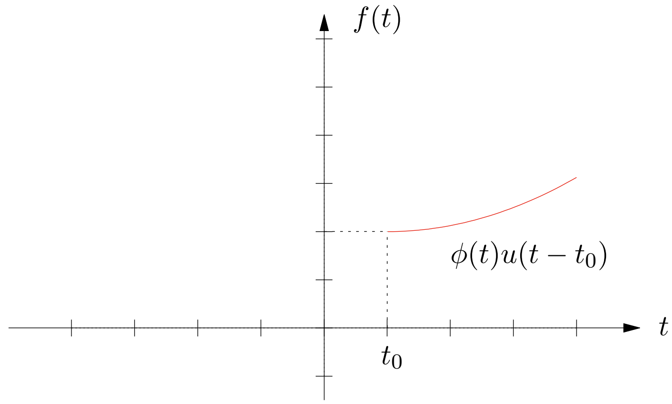
$$k\,u(t-t_0)=\begin{cases}0 & t<t_0\\ k & t>t_0\end{cases} \tag{3.12}$$



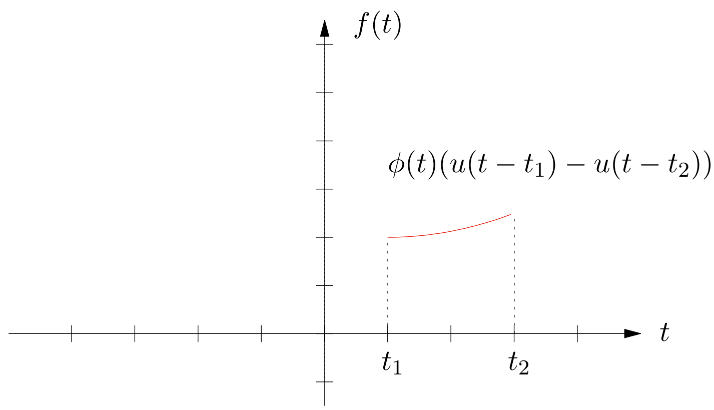
Si: $\phi(t)$ es una función de prueba:



$$\phi(t)u(t-t_0)=\begin{cases}0 & t<t_0\\ \phi(t) & t>t_0\end{cases} \tag{3.13}$$

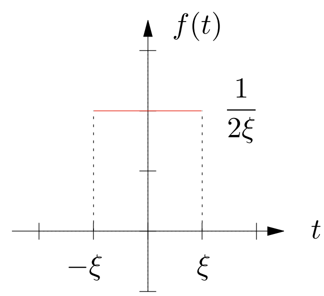


$$\phi(t)(u(t-t_1)-u(t-t_2)) = \begin{cases} 0 & t \notin [t_1, t_2] \\ \phi(t) & t \in [t_1, t_2] \end{cases}$$

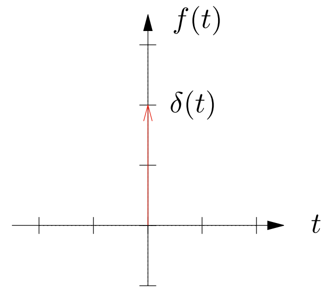


3.5. La función impulso

Pulso rectangular de área igual a 1.



Si $\xi \to 0$, entonces $\frac{1}{2\xi} \to \infty$.



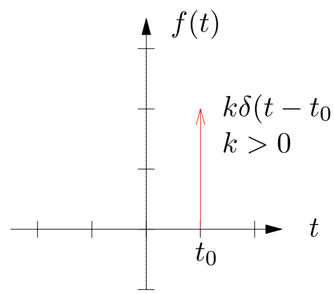
$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

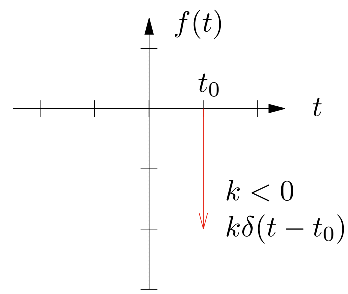
Tal que:

$$\int_{-\xi}^{\xi} \delta(t)\, dt = 1$$

Por tanto:

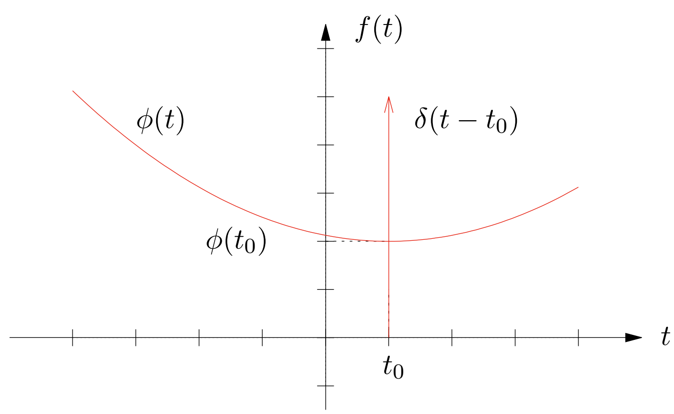
$$k\delta(t) = \begin{cases} 0 & t \neq 0 \\ \pm\infty & t = 0 \end{cases} \tag{3.14}$$





Si: $\phi(t)$ es una función de prueba:

$$\phi(t)\delta(t-t_0) = \phi(t_0)\delta(t-t_0) \tag{3.15}$$



Para $t \neq 0$

$$\phi(t)\delta(t-t_0) = 0$$

Para $t = 0$

$$\phi(t)\delta(t-t_0) = \phi(t_0)\delta(t-t_0)$$

3.5.1. Propiedades de la función impulso

Propiedad 1

$$\int_a^b \delta(t-t_0)\,dt = \begin{cases} 1 & t_0 \in [a,b] \\ 0 & t_0 \notin [a,b] \end{cases}$$

En general:

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1 \tag{3.16}$$

Propiedad 2

$$\int_a^b \phi(t)\,\delta(t-t_0)\,dt = \begin{cases} \phi(t_0) & t_0 \in [a,b] \\ 0 & t_0 \notin [a,b] \end{cases}$$

En general:

$$\int_{-\infty}^{\infty} \phi(t_0)\,\delta(t-t_0)\,dt = \phi(t_0) \tag{3.17}$$

Prueba:

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(t)\delta(t-t_0)\,dt &= \int_{-\infty}^{\infty} \phi(t_0)\delta(t-t_0)\,dt \\ &= \phi(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)\,dt \\ &= \phi(t_0) \end{aligned}$$

Propiedad 3

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(at)\,dt = \frac{1}{|a|}\int_{-\infty}^{\infty} \phi\left(\frac{t}{a}\right)\delta(t)\,dt = \frac{\phi(0)}{|a|}; a \neq 0 \tag{3.18}$$

Prueba:

Realizando un cambio de variable:

$$\tau = at$$

$$d\tau = a\,dt$$

Para $a > 0$:

$$\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right) \delta(\tau)\,\frac{d\tau}{a} = \frac{1}{a}\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right) \delta(\tau) d\tau$$

Para $a < 0$:

$$\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right) \delta(\tau)\,\frac{d\tau}{a} = -\frac{1}{a}\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right) \delta(\tau) d\tau$$

Como:

$$|a| = \begin{cases} -a & a < 0 \\ a & a > 0 \end{cases}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(t)\,\delta(at)\,dt &= \frac{1}{|a|}\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right) \delta(\tau)\,d\tau \\
&= \frac{1}{|a|}\phi\left(\frac{0}{a}\right) \\
&= \frac{\phi(0)}{|a|}
\end{aligned}$$

Propiedad 4

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{3.19}$$

En particular:

$$\delta(-t) = \delta(t) \tag{3.20}$$

Por tanto $\delta(t)$ es una función **par.**

<u>Prueba</u>:

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(t)\,\delta(at)\,dt &= \frac{1}{|a|}\phi(0) \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} \phi(t)\,\delta(t)\,dt
\end{aligned}$$

$$\phi(t)\,\delta(at) = \frac{1}{|a|}\phi(t)\,\delta(t)$$

$$\delta(at) = \frac{1}{|a|}\,\delta(t)$$

Para $a = -1$:

$$\delta(-t) = \frac{1}{|-1|}\,\delta(t) = \delta(t)$$

Propiedad 5

$$
\begin{aligned}
t\,\delta(t) &= 0 \\
t^n\,\delta(t) &= 0; n \in \mathbb{N}
\end{aligned}
\tag{3.21}
$$

Prueba:

$$\int_{-\infty}^{\infty} t^n\,\delta(t)\,dt = 0^n = 0$$

Derivando ambos miembros:

$$t^n\,\delta(t)\,dt = 0$$

3.6. Derivada de la función impulso

$$
\begin{aligned}
\delta'(t) &= \frac{d}{dt}(\delta(t)) \\
\int_{-\infty}^{\infty} \phi(t)\delta'(t)\,dt &= -\int_{-\infty}^{\infty} \phi'(t)\delta(t)\,dt = -\phi'(0)
\end{aligned}
\tag{3.22}
$$

Prueba:

Realizando la integración por partes:

$$u = \phi(t)$$

$$
\begin{aligned}
du &= \phi'(t)\,dt \\
dv &= \delta'(t-t_0)\,dt \\
v &= \delta(t-t_0)
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} \phi(t)\delta'(t-t_0)\,dt &= (\phi(t)\delta(t-t_0)\Big|_{-\infty}^{\infty}) - \int_{-\infty}^{\infty} \delta(t-t_0)\phi'(t)\,dt \\
&= 0 - \int_{-\infty}^{\infty} \delta(t-t_0)\phi'(t)\,dt \\
&= -\phi'(t_0)
\end{aligned}
$$

Derivadas de orden superior

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(t)\delta''(t)\,dt &= \int_{-\infty}^{\infty} \phi(t)(\delta'(t))'\,dt \\
&= -\int_{-\infty}^{\infty} \phi'(t)\delta'(t)\,dt \\
&= \int_{-\infty}^{\infty} \phi''(t)\delta(t)\,dt \\
&= \phi''(0)
\end{aligned}$$

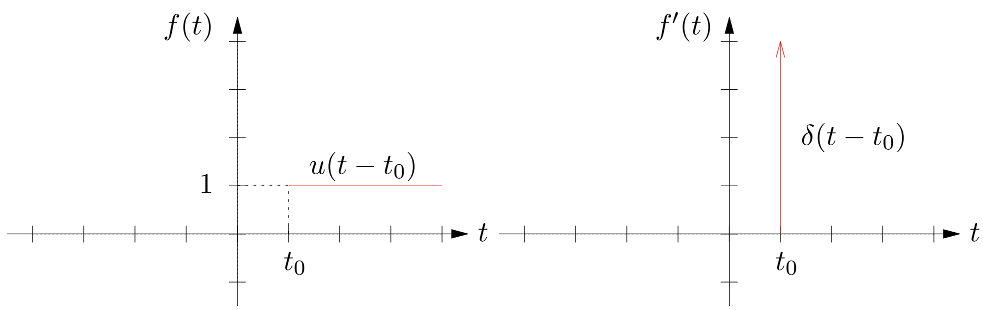
De igual manera:

$$\int_{-\infty}^{\infty} \phi(t)\delta'''(t-t_0)\,dt = -\phi'''(t_0)$$

En general:

$$\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(t-t_0)\,dt = (-1)^n\phi^{(n)}(t_0) \tag{3.23}$$

3.7. Derivada de la función escalón unitario



$$u'(t-t_0) = \delta(t-t_0) \tag{3.24}$$

<u>Prueba</u>:

$$\int_{-\infty}^{\infty} u'(t)\phi(t)\,dt$$

Realizando la integración por partes:

$$u = \phi(t)$$

$$du = \phi'(t)\,dt$$

$$dv = u'(t)\,dt$$

$$v = u(t)$$

$$\int_{-\infty}^{\infty} \phi(t)u'(t)\,dt = (\phi(t)u(t)\Big|_{-\infty}^{\infty}) - \int_{-\infty}^{\infty} u(t)\phi'(t)\,dt$$
$$= -\int_{-\infty}^{\infty} u(t)\phi'(t)\,dt$$
$$= -\int_{0}^{\infty} 1\,\phi'(t)\,dt$$
$$= -\phi(t)\Big|_{0}^{\infty}$$
$$= -\phi(\infty) + \phi(0)$$

Asumiendo que $\phi(\pm\infty) = 0$:

$$\int_{-\infty}^{\infty} \phi(t)u'(t)\,dt = \phi(0)$$

Sabiendo que:

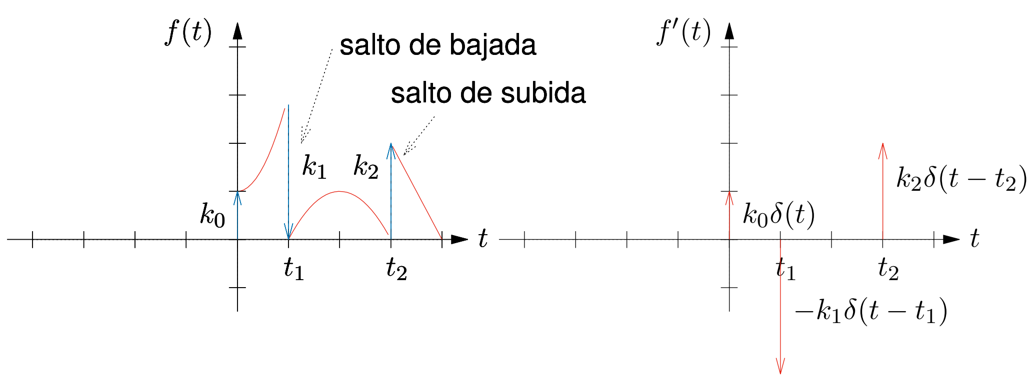
$$\int_{-\infty}^{\infty} \phi(t)\delta(t)\,dt = \phi(0)$$

Por tanto:

$$\int_{-\infty}^{\infty} \phi(t)u'(t)\,dt = \int_{-\infty}^{\infty} \phi(t)\delta(t)\,dt$$
$$\phi(t)u'(t) = \phi(t)\delta(t)$$
$$u'(t) = \delta(t)$$

3.8. Derivada de una función con discontinuidades de salto

Las derivadas de los saltos de subida y bajada, van a originar impulsos hacia arriba y hacia abajo, respectivamente.



3.9. Series de *Fourier* por el método de diferenciación

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t} \qquad c_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$$

$$f'(t) = \sum_{n=-\infty}^{\infty} jn\omega_0 c_n\, e^{jn\omega_0 t} \qquad \gamma'_n = \frac{1}{T}\int_0^T f'(t)\, e^{-jn\omega_0 t}\, dt$$

$$f''(t) = \sum_{n=-\infty}^{\infty} (jn\omega_0)^2 c_n\, e^{jn\omega_0 t} \qquad \gamma''_n = \frac{1}{T}\int_0^T f''(t)\, e^{-jn\omega_0 t}\, dt$$

$$f^{(k)}(t) = \sum_{n=-\infty}^{\infty} (jn\omega_0)^k c_n\, e^{jn\omega_0 t} \qquad \gamma^{(k)}_n = \frac{1}{T}\int_0^T f^{(k)}(t)\, e^{-jn\omega_0 t}\, dt$$

- Se deriva $f(t)$ hasta anularla y calcular para cada derivada: $\gamma_n^{(k)}$
- Las derivadas de $f(t)$ solo van a tomar en cuenta los impulsos obtenidos de los saltos previos.
- El coeficiente complejo c_n se obtendrá de la forma:

$$c_n = c'_n + c''_n + \cdots + c^{(k)}_n \tag{3.25}$$

Donde:

$$c'_n = \frac{\gamma'_n}{jn\omega_0} \tag{3.26}$$

$$c''_n = \frac{\gamma''_n}{(jn\omega_0)^2} \tag{3.27}$$

$$c^{(k)}_n = \frac{\gamma^{(k)}_n}{(jn\omega_0)^k} \tag{3.28}$$

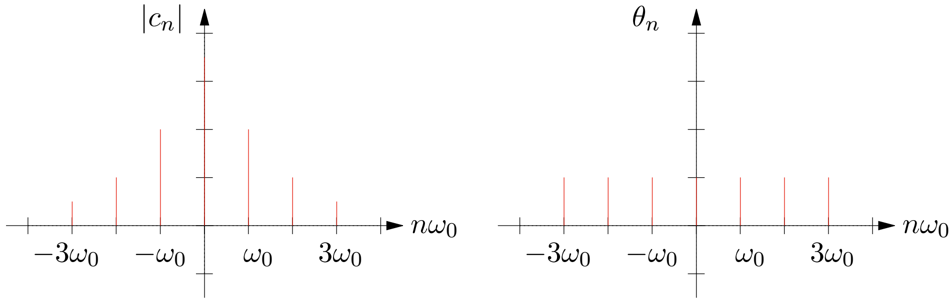
3.10. Espectros de frecuencia discreta

Los espectros de frecuencia serán gráficas discretas de modulo y argumento del coeficiente complejo de *Fourier* en función de múltiplos de la frecuencia: ω_0.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0 t}$$

$$c_n = A_n + jB_n$$

$$\left.\begin{aligned} |c_n| &= \sqrt{A_n^2 + B_n^2} \\ \theta_n &= \arctan\left(\frac{B_n}{A_n}\right) \end{aligned}\right\} \text{funciones discretas de } n\omega_0 \quad n \in \mathbb{Z}$$



Espectro de amplitud o módulo   Espectro de fase o argumento

3.11. Teorema de la multiplicación

Dadas dos funciones periódicas con el mismo periodo T: $f_1(t)$ y $f_2(t)$.

Donde: $c_1 n$ y $c_2 n$, son los respectivos coeficientes complejos de *Fourier*.

$$\frac{1}{T}\int_0^T f_1(t)f_2(t)\,dt = \sum_{n=-\infty}^{\infty} c_{1(n)}\,c_{2(-n)} = \sum_{n=-\infty}^{\infty} c_{1(-n)}\,c_{2(n)} \tag{3.29}$$

Prueba:

$$\begin{aligned}
\frac{1}{T}\int_0^T f_1(t)f_2(t)\,dt &= \frac{1}{T}\int_0^T \left[\sum_{n=-\infty}^{\infty} c_{1(n)}\,e^{jn\omega_0 t}\right] f_2(t)\,dt \\
&= \sum_{n=-\infty}^{\infty} c_{1(n)}\,\frac{1}{T}\int_0^T f_2(t)\,e^{jn\omega_0 t}\,dt \\
&= \sum_{n=-\infty}^{\infty} c_{1(n)}c_{2(-n)}
\end{aligned}$$

3.12. Teorema de *Parseval*

Sea: $f_1(t) = f_2(t) = f(t)$, con coeficientes de *Fourier*: $c_1(n) = c_2(n) = c_n$:

Partiendo del teorema de multiplicación:

$$\begin{aligned}
\frac{1}{T}\int_0^T f^2(t)\,dt &= \sum_{n=-\infty}^{\infty} c_{(n)}\,c_{(-n)} \\
&= \sum_{n=-\infty}^{\infty} c_{(n)}\,c^*_{(n)} \\
&= \sum_{n=-\infty}^{\infty} |c_n|^2 \\
&= c_0^2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |c_n|^2
\end{aligned}$$

$$\begin{aligned}
c_n &= \frac{a_n - jb_n}{2} \\
|c_n|^2 &= \frac{a_n^2 + jb_n^2}{4}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{T}\int_0^T f^2(t)\,dt &= c_0^2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a_n^2 + jb_n^2}{4} \\
&= c_0^2 + \frac{1}{4}\sum_{n=1}^{\infty}(a_n^2 + jb_n^2) + \frac{1}{4}\sum_{n=-1}^{-\infty}(a_n^2 + jb_n^2)
\end{aligned}$$

Cambiando n por $-n$:

$$\frac{1}{T}\int_0^T f^2(t)\,dt = c_0^2 + \frac{1}{4}\sum_{n=1}^{\infty}(a_n^2 + jb_n^2) + \frac{1}{4}\sum_{-n=-1}^{-\infty}(a_{(-n)}^2 + jb_{(-n)}^2)$$

Sabiendo que:

$$\begin{aligned}
a_n^2 &= a_{(-n)}^2 \\
(-b_n)^2 &= b_n^2
\end{aligned}$$

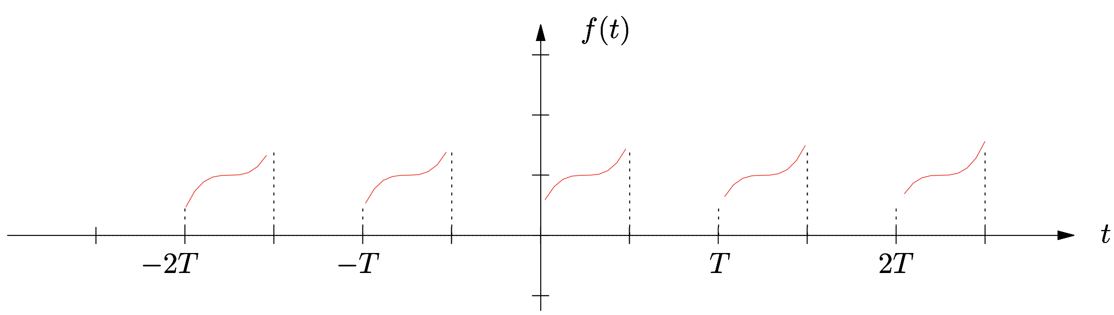
$$\begin{aligned}
\frac{1}{T}\int_0^T f^2(t)\,dt &= c_0^2 + \frac{1}{4}\sum_{n=1}^{\infty}(a_n^2 + jb_n^2) + \frac{1}{4}\sum_{n=1}^{\infty}(a_n^2 + jb_n^2) \\
&= c_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + jb_n^2)
\end{aligned}$$

$$\frac{1}{T}\int_0^T f^2(t)\,dt = c_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + jb_n^2) \tag{3.30}$$

Capítulo 4

Transformada de *Fourier*

4.1. Integrales de *Fourier*



Cuando $f(t)$ es periódica tiene la siguiente representación (forma compleja):

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 t}$$

Donde:

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, e^{-jn\omega_0 t} \, dt$$

Si $T \to \infty$, entonces $\omega_0 \to 0$, y la función deja de ser periódica.

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} f(\tau) \, e^{-jn\omega_0 \tau} \, d\tau \right] e^{jn\omega_0 t}$$

$$\frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f(\tau) \, e^{-jn\omega_0 \tau} \, d\tau \right] e^{jn\omega_0 t} \, \omega_0$$

$$\omega_0 = d\omega$$

$$n\omega_0 = \omega$$
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega\tau}\, d\tau\right]\, e^{j\omega t}\, d\omega \tag{4.1}$$

Se define como transformada de *Fourier*:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \tag{4.2}$$

Se define como transformada inversa de *Fourier*:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \tag{4.3}$$

4.2. Transformada de *Fourier*

Dada una función $f(t)$ se define la transformada de *Fourier*:

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \tag{4.4}$$

Donde: $f(t)$ es una función no periódica.

La transformada de *Fourier* convierte una función del dominio del tiempo (t) al dominio de la frecuencia (ω) la cual será una variable continua.

Para $f(t) \in \mathbb{R}$:

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \\ &= \int_{-\infty}^{\infty} f(t)[\cos(\omega t) - j\operatorname{sen}(\omega t)]\, dt \\ &= \int_{-\infty}^{\infty} f(t)\cos(\omega t)\, dt - j\int_{-\infty}^{\infty} f(t)\operatorname{sen}(\omega t)\, dt \\ &= R(\omega) + jX(\omega) \end{aligned}$$

$$R(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t)\, dt$$
$$X(\omega) = -\int_{-\infty}^{\infty} f(t)\operatorname{sen}(\omega t)\, dt$$

La parte real $R(\omega)$ es una función par:

$$R(-\omega) = R(\omega)$$

La parte imaginaria $X(\omega)$ es una función impar:

$$X(-\omega) = -X(\omega)$$

Por tanto:

Si $f(t)$ es par, entonces $X(\omega) = 0$.

Si $f(t)$ es impar, entonces $R(\omega) = 0$.

4.3. Espectros continuos de frecuencia

$$F(\omega) = R(\omega) + jX(\omega)$$

Modulo:

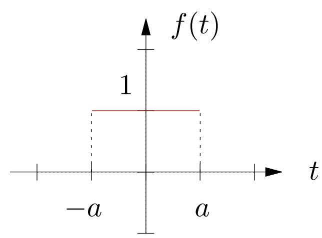
$$|F(\omega)| = \sqrt{R^2(\omega) + X^2(\omega)}$$

Argumento:

$$\theta(\omega) = \arctan\left(\frac{X(\omega)}{R(\omega)}\right)$$

Ejemplo 1: Hallar $F(\omega)$ de la función y graficar los espectros.

$$f(t) = u(t+a) - u(t-a)$$



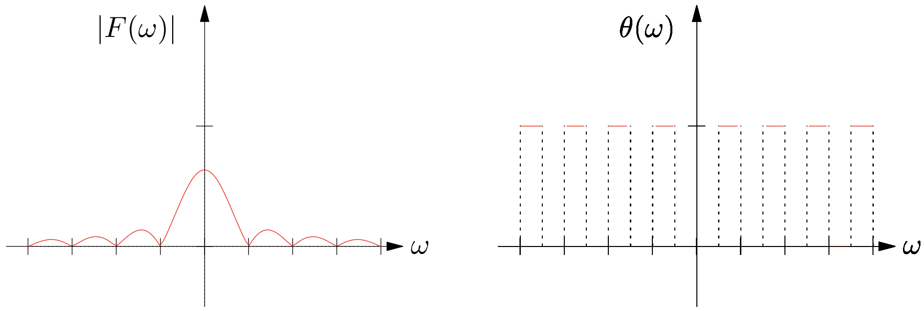
$$\begin{aligned} F(\omega) &= \int_{-a}^{a} 1\, e^{-j\omega t}\, dt \\ &= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-a}^{a} \\ &= \frac{e^{-ja\omega} - e^{ja\omega}}{-j\omega} \\ &= \frac{-2j\,\mathrm{sen}(a\omega)}{-j\omega} \\ &= \frac{2\,\mathrm{sen}(a\omega)}{\omega} \end{aligned}$$

$$\mathcal{F}\{u(t+a) - u(t-a)\} = \frac{2\,\mathrm{sen}(a\omega)}{\omega} \tag{4.5}$$

$$|F(\omega)| = |\frac{2\,\mathrm{sen}(a\omega)}{\omega}|$$

Para $\omega = 0$, existe una discontinuidad:

$$\lim_{\omega\to 0}\left(\frac{2\,\mathrm{sen}(a\omega)}{\omega}\right) = 2a$$



4.4. Propiedades de la transformada de *Fourier*

4.4.1. Linealidad

$$\mathcal{F}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(\omega) + a_2 F_2(\omega) \tag{4.6}$$

Donde:

$$F_1(\omega) = \mathcal{F}\{f_1(t)\}$$
$$F_2(\omega) = \mathcal{F}\{f_2(t)\}$$

4.4.2. Cambio de escala

Si: $\mathcal{F}\{f(t)\} = F(\omega)$:

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) \tag{4.7}$$

<u>Prueba</u>:

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)\, e^{-j\omega t}\, dt$$

Realizando un cambio de variable:

$$\tau = at$$
$$d\tau = a\, dt$$

Para $a > 0$:

$$\begin{aligned} \mathcal{F}\{f(at)\} &= \int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega \frac{\tau}{a}}\, \frac{d\tau}{a} \\ &= \frac{1}{a} \int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega \frac{\tau}{a}}\, d\tau \\ &= \frac{1}{a} F\left(\frac{\omega}{a}\right) \end{aligned}$$

Para $a < 0$:

$$
\begin{aligned}
\mathcal{F}\{f(at)\} &= \int_{\infty}^{-\infty} f(\tau)\, e^{-j\omega\frac{\tau}{a}}\, \frac{d\tau}{a} \\
&= -\frac{1}{a}\int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega\frac{\tau}{a}} d\tau \\
&= -\frac{1}{a} F\left(\frac{\omega}{a}\right)
\end{aligned}
$$

4.4.3. Desplazamiento en ω

Si: $\mathcal{F}\{f(t)\} = F(\omega)$:

$$
\mathcal{F}\{f(t)\, e^{jat}\} = F(\omega - a) \tag{4.8}
$$

Prueba:

$$
\begin{aligned}
\mathcal{F}\{f(t)\, e^{jat}\} &= \int_{-\infty}^{\infty} f(t)\, e^{jat}\, e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} f(t)\, e^{-j(\omega - a)t}\, dt \\
&= F(\omega - a)
\end{aligned}
$$

4.4.4. Desplazamiento en t

Si: $\mathcal{F}\{f(t)\} = F(\omega)$:

$$
\mathcal{F}\{f(t-a)\} = F(\omega)\, e^{-ja\omega} \tag{4.9}
$$

Prueba:

$$
\mathcal{F}\{f(t-a)\} = \int_{-\infty}^{\infty} f(t-a)\, e^{-j\omega t}\, dt
$$

Realizando un cambio de variable:

$$
\tau = t - a
$$

$$
d\tau = dt
$$

$$
\begin{aligned}
\mathcal{F}\{f(t-a)\} &= \int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega(\tau + a)}\, dt \\
&= e^{-ja\omega} \int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega\tau}\, d\tau \\
&= e^{-ja\omega} F(\omega)
\end{aligned}
$$

4.4.5. Simetría

Si: $\mathcal{F}\{f(t)\} = F(\omega)$:

$$\mathcal{F}\{F(t)\} = 2\pi f(-\omega) \tag{4.10}$$

Prueba:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

Reemplazando:

$$t \to -\omega$$

$$\omega \to t$$

$$\begin{aligned} f(-\omega) = &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)\, e^{jt(-\omega)}\, dt \\ &= \frac{1}{2\pi}\mathcal{F}\{F(t)\} \\ 2\pi f(-\omega) &= \mathcal{F}\{F(t)\} \end{aligned}$$

Ejemplo 2: Hallar $\mathcal{F}\{\dfrac{\operatorname{sen}(at)}{t}\}$ Sabiendo que:

$$\mathcal{F}\{u(t+a) - u(t-a)\} = \frac{2\operatorname{sen}(a\omega)}{\omega}$$

$$\mathcal{F}\{\frac{2\operatorname{sen}(at)}{t}\} = 2\pi(u(t+a) - u(t-a))$$

$$\mathcal{F}\{\frac{\operatorname{sen}(at)}{t}\} = \pi(u(t+a) - u(t-a)) \tag{4.11}$$

4.4.6. Multiplicación por t

Si: $\mathcal{F}\{f(t)\} = F(\omega)$:

$$\mathcal{F}\{t\, f(t)\} = j\frac{dF(\omega)}{d\omega}$$

En general:

$$\mathcal{F}\{t^n\, f(t)\} = j^n \frac{d^{(n)}F(\omega)}{d\omega^n}; \quad n \in \mathbb{N} \tag{4.12}$$

Prueba:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$
\begin{aligned}
\frac{dF(\omega)}{d\omega} &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}(-jt)\, dt \\
&= -j \int_{-\infty}^{\infty} t\, f(t)\, e^{-j\omega t}\, dt \\
&= -j\, \mathcal{F}\{t\, f(t)\}
\end{aligned}
$$

$$
\begin{aligned}
j\frac{dF(\omega)}{d\omega} &= \mathcal{F}\{t\, f(t)\} \\
\mathcal{F}\{t^2\, f(t)\} &= \mathcal{F}\{t\, t\, f(t)\} \\
&= j\frac{d}{d\omega}\left(\mathcal{F}\{t\, f(t)\}\right) \\
&= j\frac{d}{d\omega}\left(j\frac{dF(\omega)}{d\omega}\right) \\
&= j^2\frac{d^2F(\omega)}{d\omega^2} \\
\mathcal{F}\{t^n\, f(t)\} &= j^n\frac{d^nF(\omega)}{d\omega^n}
\end{aligned}
$$

Ejemplo 3: Hallar $\mathcal{F}\{t^n e^{-at}u(t)\}; \quad n \in \mathbb{N}$

Para $n = 1$:

$$
\begin{aligned}
\mathcal{F}\{t\, e^{-at}u(t)\} &= j\frac{d}{d\omega}\left(\frac{1}{a+j\omega}\right) \\
&= (j)(-1)(a+j\omega)^{-2}(j) \\
&= \frac{1}{(a+j\omega)^2}
\end{aligned}
$$

Para $n = 2$:

$$
\begin{aligned}
\mathcal{F}\{t^2\, e^{-at}u(t)\} &= j\frac{d}{d\omega}\left(\frac{1}{(a+j\omega)^2}\right) \\
&= (j)(-2)(a+j\omega)^{-3}(j) \\
&= \frac{2}{(a+j\omega)^3}
\end{aligned}
$$

Para $n = 3$:

$$
\begin{aligned}
\mathcal{F}\{t^3\, e^{-at}u(t)\} &= j\frac{d}{d\omega}\left(\frac{2}{(a+j\omega)^3}\right) \\
&= (j)(2)(-3)(a+j\omega)^{-4}(j) \\
&= \frac{6}{(a+j\omega)^4}
\end{aligned}
$$

Por tanto:

$$
\mathcal{F}\{t^n\, e^{-at}u(t)\} = \frac{n!}{(a+j\omega)^{n+1}} \tag{4.13}
$$

4.4.7. Transformada de *Fourier* de una derivada

Si: $\mathcal{F}\{f(t)\} = F(\omega)$:

$$\mathcal{F}\{f'(t)\} = j\omega F(\omega)$$

En general:

$$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega) \quad n \in \mathbb{N} \tag{4.14}$$

Prueba:

$$\mathcal{F}\{f'(t)\} = \int_{\infty}^{\infty} f'(t)\, e^{-j\omega t}\, dt$$

Realizando la integración por partes:

$$u = e^{-j\omega t}$$

$$du = -j\omega\, e^{-j\omega t}\, dt$$
$$dv = f'(t)\, dt$$
$$v = f(t)$$
$$\mathcal{F}\{f'(t)\} = \left(f(t)\, e^{-j\omega t} \Big|_{-\infty}^{\infty} \right) - \int_{-\infty}^{\infty} f(t)\, (-j\omega\, e^{-j\omega t})\, dt$$

Asumiendo $f(\pm\infty) = 0$:

$$\begin{aligned} \mathcal{F}\{f'(t)\} &= j\omega \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \\ &= j\omega\, F(\omega) \end{aligned}$$

Para la segunda derivada:

$$\begin{aligned} \mathcal{F}\{f''(t)\} &= \mathcal{F}\{(f'(t))'\} \\ &= j\omega\, \mathcal{F}\{f'(t)\} \\ &= j\omega(j\omega\, F(\omega)) \\ &= (j\omega)^2\, F(\omega) \end{aligned}$$

Por tanto:

$$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$$

4.5. Transformadas de *Fourier* especiales

4.5.1. $e^{-at}\, u(t); \quad a > 0$

$$\mathcal{F}\{e^{-at}\, u(t)\}; a > 0$$

$$
\begin{aligned}
\mathcal{F}\{e^{-at}\,u(t)\} &= \int_0^{\infty} e^{-at}\,e^{-j\omega t}\,dt \\
&= \int_0^{\infty} e^{-(a+j\omega)t}\,dt \\
&= \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_0^{\infty} \\
&= \frac{0-1}{-(a+j\omega)} \\
&= \frac{1}{a+j\omega}
\end{aligned}
$$

$$
\mathcal{F}\{e^{-at}\,u(t)\} = \frac{1}{a+j\omega} \tag{4.15}
$$

4.5.2. $e^{at}\,u(-t); \quad a>0$

$$
\begin{aligned}
&\mathcal{F}\{e^{at}\,u(-t)\}; a>0 \\
\mathcal{F}\{e^{at}\,u(t)\} &= \int_{-\infty}^{0} e^{at}\,e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt \\
&= \left.\frac{e^{(a-j\omega)t}}{a-j\omega}\right|_{-\infty}^{0} \\
&= \frac{1}{a-j\omega}
\end{aligned}
$$

$$
\mathcal{F}\{e^{at}\,u(-t)\} = \frac{1}{a-j\omega} \tag{4.16}
$$

4.5.3. $e^{-a|t|}$

$$
\begin{aligned}
&\mathcal{F}\{e^{-a|t|}\} \\
\mathcal{F}\{e^{-a|t|}\} &= \mathcal{F}\{e^{at}u(-t) + e^{-at}u(t)\} \\
&= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \\
&= \frac{a+j\omega+a-j\omega}{(a-j\omega)(a+j\omega)} \\
&= \frac{2a}{a^2+\omega^2}
\end{aligned}
$$

$$
\mathcal{F}\{e^{-a|t|}\} = \frac{2a}{a^2+\omega^2} \tag{4.17}
$$

4.5.4. $\delta(t-t_0)$

$$
\begin{aligned}
&\mathcal{F}\{\delta(t-t_0)\}\\
\mathcal{F}\{\delta(t-t_0)\} &= \int_{-\infty}^{0} \delta(t-t_0)\, e^{-j\omega t}\, dt\\
&= e^{-j\omega t_0}
\end{aligned}
$$

$$
\mathcal{F}\{\delta(t-t_0)\} = e^{-j\omega t_0} \tag{4.18}
$$

En particular:

$$
\mathcal{F}\{\delta(t)\} = 1
$$

4.5.5. e^{jat}

Sabiendo:

$$
\mathcal{F}\{\delta(t-a)\} = e^{-ja\omega}
$$

Aplicando la propiedad de simetría:

$$
\mathcal{F}\{e^{-jat}\} = 2\pi\delta(-\omega-a)
$$

$$
\begin{aligned}
\mathcal{F}\{e^{jat}\} &= 2\pi\delta(-\omega+a)\\
&= 2\pi\delta(-(\omega-a))\\
&= 2\pi\delta(\omega-a)
\end{aligned}
$$

$$
\mathcal{F}\{e^{jat}\} = 2\pi\delta(\omega-a) \tag{4.19}
$$

En particular:

$$
\mathcal{F}\{1\} = 2\pi\delta(\omega)
$$

$$
\mathcal{F}\{k\} = 2\pi k\delta(\omega)
$$

4.5.6. $\operatorname{sen}(at)$

$$
\begin{aligned}
\mathcal{F}\{\operatorname{sen}(at)\} &= \mathcal{F}\{\frac{e^{jat}-e^{-jat}}{2j}\}\\
&= \frac{1}{2j}\left(2\pi\delta(\omega-a) - 2\pi\delta(\omega+a)\right)\\
&= -j\pi(\delta(\omega-a) - \delta(\omega+a))
\end{aligned}
$$

$$
\mathcal{F}\{\operatorname{sen}(at)\} = j\pi(\delta(\omega+a) + \delta(\omega-a)) \tag{4.20}
$$

4.5.7. $\cos(at)$

$$\begin{aligned}\mathcal{F}\{\cos(at)\} &= \mathcal{F}\{\frac{e^{jat}+e^{-jat}}{2}\} \\ &= \frac{1}{2}\left(2\pi\delta(\omega-a)+2\pi\delta(\omega+a)\right) \\ &= \pi(\delta(\omega+a)+\delta(\omega-a))\end{aligned}$$

$$\mathcal{F}\{\cos(at)\} = \pi(\delta(\omega+a)+\delta(\omega-a)) \tag{4.21}$$

4.5.8. $u(t)$

$$u(t)+u(-t)=1$$
$$\mathcal{F}\{u(t)+u(-t)\}=\mathcal{F}\{1\}$$

Considerando:

$$\mathcal{F}\{u(t)\}=F(\omega)$$
$$\mathcal{F}\{u(-t)\}=F(-\omega)$$

Por tanto:

$$F(\omega)+F(-\omega)=2\pi\delta(\omega)$$

Se asume que la transformada de *Fourier* de la función escalón tendrá un termino impulsivo:

$$F(\omega)=\beta(\omega)+k\delta(\omega)$$
$$F(-\omega)=\beta(-\omega)+k\delta(\omega)$$
$$F(\omega)+F(-\omega)=\beta(\omega)+\beta(-\omega)+2k\delta(\omega)$$

Reemplazando:

$$\beta(\omega)+\beta(-\omega)+2k\delta(\omega)=2\pi\delta(\omega)$$

Por tanto:

$$k=\pi$$

Resultando:

$$F(\omega)=\beta(\omega)+\pi\delta(\omega)$$

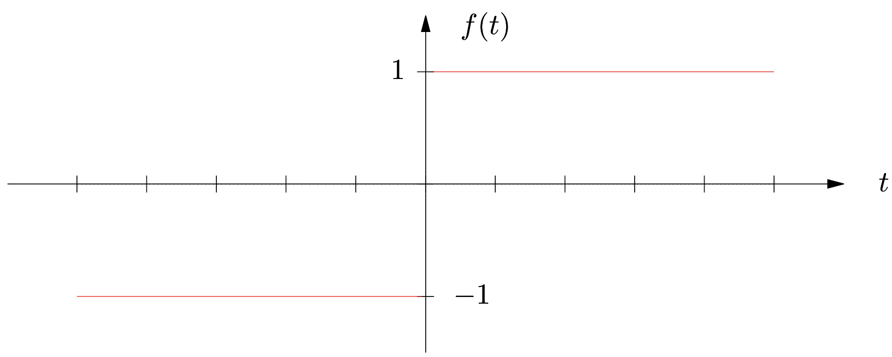
Por otro lado, se sabe que:

$$u'(t)=\delta(t)$$
$$\mathcal{F}\{u'(t)\}=\mathcal{F}\{\delta(t)\}$$
$$j\omega\mathcal{F}\{u(t)\}=1$$
$$j\omega(\beta(\omega)+\pi\delta(\omega))=1$$

$$j\omega\beta(\omega) + j\pi\omega\delta(\omega) = 1$$
$$j\omega\beta(\omega) = 1$$
$$\beta(\omega) = \frac{1}{j\omega}$$
$$\mathcal{F}\{u(t)\} = \frac{1}{j\omega} + \pi\delta(\omega) \tag{4.22}$$

4.6. La función signo

$$sgn(t) = \begin{cases} -1 & t < 0 \\ 1 & t > 0 \end{cases} \tag{4.23}$$



Esta función puede representarse también como:

$$sgn(t) = \frac{|t|}{t}$$
$$sgn(t) = -1 + 2u(t)$$

A partir de la definición de la función de valor absoluto:

$$|t| = \begin{cases} -t & t < 0 \\ t & t > 0 \end{cases}$$

Es posible calcular la derivada del valor absoluto:

$$|t|' = \begin{cases} -1 & t < 0 \\ 1 & t > 0 \end{cases}$$

Por tanto:

$$|t|' = sgn(t) \tag{4.24}$$

Cuya derivada es:

$$sgn'(t) = 2\delta(t) \tag{4.25}$$

Calculando su transformada de *Fourier*:

$$\begin{aligned}
\mathcal{F}\{sgn(t)\} &= \mathcal{F}\{-1 + 2u(t)\} \\
&= -2\pi\delta(\omega) + 2\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) \\
&= -2\pi\delta(\omega) + 2\left(\frac{1}{j\omega}\right) + 2\pi\delta(\omega) \\
&= \frac{2}{j\omega}
\end{aligned}$$

$$\mathcal{F}\{sgn(t)\} = \frac{2}{j\omega} \tag{4.26}$$

4.6.1. Transformada de *Fourier* de $|t|$

$$\begin{aligned}
\mathcal{F}\{|t|'\} &= \mathcal{F}\{sgn(t)\} \\
j\omega\mathcal{F}\{|t|\} &= \frac{2}{j\omega} \\
\mathcal{F}\{|t|\} &= -\frac{2}{j\omega^2}
\end{aligned} \tag{4.27}$$

4.6.2. Transformada de *Fourier* de $1/t$

$$\mathcal{F}\{sgn(t)\} = \frac{2}{j\omega}$$

Por simetría:

$$\begin{aligned}
\mathcal{F}\{\frac{2}{jt}\} &= 2\pi\, sgn(-\omega) \\
\mathcal{F}\{\frac{1}{t}\} &= -j\pi\, sgn(-\omega)
\end{aligned} \tag{4.28}$$

Calculando la segunda derivada:

$$\left(\frac{1}{t}\right)' = -\frac{1}{t^2}$$

$$\mathcal{F}\{-\frac{1}{t^2}\} = j\omega(-j\pi\, sgn(\omega))$$

$$\mathcal{F}\{-\frac{1}{t^2}\} = j^2\pi\omega\, sgn(\omega)$$

Calculando la derivada n-ésima:

$$\left(\frac{1}{t}\right)'' = \frac{2}{t^3}$$

$$\left(\frac{1}{t}\right)''' = -\frac{6}{t^4}$$

$$\left(\frac{1}{t}\right)^{(n)} = (-1)^k \frac{k!}{t^{k+1}}$$

$$\mathcal{F}\{f^{(k)}(t)\} = (-1)^k \, k! \, \mathcal{F}\{\frac{1}{t^{k+1}}\}$$

$$(j\omega)^k(-j\pi \, sgn(\omega)) = (-1)^k \, k! \, \mathcal{F}\{\frac{1}{t^{k+1}}\}$$

$$-j^{k+1}\pi\omega^k \, sgn(\omega) = (-1)^k \, k! \, \mathcal{F}\{\frac{1}{t^{k+1}}\}$$

$$-j^n\pi\omega^{n-1} \, sgn(\omega) = (-1)^{n-1} \, (n-1)! \, \mathcal{F}\{\frac{1}{t^n}\}$$

$$-j^n\pi\omega^{n-1} \, sgn(\omega) = (-1)^n(-1)^{-1}(n-1)!\mathcal{F}\{\frac{1}{t^n}\}$$

$$\mathcal{F}\{\frac{1}{t^n}\} = \frac{j^n\pi\omega^{n-1} \, sgn(\omega)}{(-1)^n(n-1)!} \tag{4.29}$$

4.7. Tabla de transformadas de *Fourier* conocidas

	$f(t)$	$F(\omega) = \mathcal{F}\{f(t)\}$
1	$u(t+a) - u(t-a)$	$\dfrac{2\,\mathrm{sen}(a\omega)}{\omega}$
2	$\dfrac{\mathrm{sen}(at)}{t}$	$\pi[u(\omega+a) - u(\omega-a)]$
3	$e^{-at}\,u(t) \quad a>0$	$\dfrac{1}{a+j\omega}$
4	$e^{at}\,u(-t) \quad a>0$	$\dfrac{1}{a-j\omega}$
5	$e^{-a\|t\|} \quad a>0$	$\dfrac{2a}{a^2+\omega^2}$
6	$\dfrac{1}{t^2+a^2}$	$\dfrac{\pi}{a}e^{-a\|\omega\|}$
7	$\delta(t-a)$	$e^{-ja\omega}$
8	e^{jat}	$2\pi\delta(\omega-a)$
9	k	$2\pi k\delta(\omega)$
10	$\mathrm{sen}(at)$	$j\omega[\delta(\omega+a) - \delta(\omega-a)]$
11	$\cos(at)$	$\pi[\delta(\omega+a) + \delta(\omega-a)]$
12	$t^n\,e^{-at}\,u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}} \quad n \in \mathbb{N}$
13	$u(t)$	$\dfrac{1}{j\omega} + \pi\delta(\omega)$
14	$sgn(t)$	$\dfrac{2}{j\omega}$
15	$\|t\|$	$-\dfrac{2}{\omega^2}$
16	$\dfrac{1}{t}$	$-j\pi\, sgn(\omega)$
17	$\dfrac{1}{t^n}$	$\dfrac{j^n\pi\omega^{n-1}\,sgn(\omega)}{(-1)^n(n-1)!}$