UNIVERSIDAD MAYOR DE SAN SIMÓN
FACULTAD DE CIENCIAS Y TECNOLOGÍA
CARRERA DE ELECTROMECÁNICA

TRANSFORMADAS E INTEGRALES

Apuntes de clase

Docente:
Ing. Marco Antonio Vallejo Camacho.

Índice general

Bibliografía recomendada

[1] Hwei Hsu. *Análisis de Fourier*.

[2] Serie *Schaum*. *Transformada de Laplace*.

[3] Eduardo Espinoza. *Transformada de Laplace*.

[4] Álvaro Hernando Carrasco Calvo. *Transformadas e integrales*.

Capítulo 1

Series de *Fourier*

1.1. Funciones periódicas

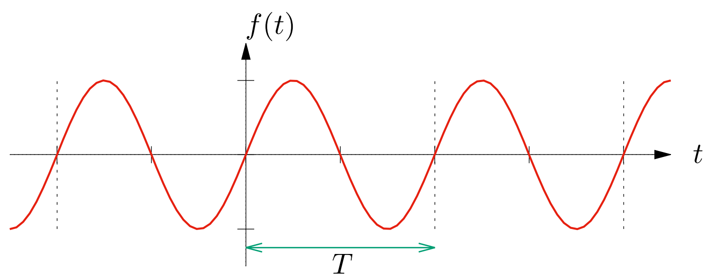


Figura 1.1: Función periódica

Una función periódica es aquella cuya gráfica se repite infinitas veces, cada cierto intervalo (**Figura 1.1**).

El menor intervalo de repetición se llama *periodo* (T).

Matemáticamente una función periódica es aquella que verifica:

$$f(t) = f(t + nT); n \in \mathbb{Z} \tag{1.1}$$

Donde T es el periodo (la menor constante que verifica la igualdad).

1.2. Propiedades de la funciones periódicas

Si: $f(t) = f(t + nT)$

Propiedad 1

$$\int_a^b f(t)\,dt = \int_{a+nT}^{b+nT} f(t)\,dt \quad n \in \mathbb{Z} \tag{1.2}$$

<u>Prueba</u>:

$$\int_a^b f(t)\,dt = \int_a^b f(t+nT)\,dt$$

Cambiando la variable:

$$\tau = t + nT$$

$$d\tau = dt$$

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_{a+nT}^{b+nT} f(\tau)\,d\tau \\ &= \int_{a+nT}^{b+nT} f(t)\,dt\end{aligned}$$

Puede verse gráficamente en la **Figura 1.2.**

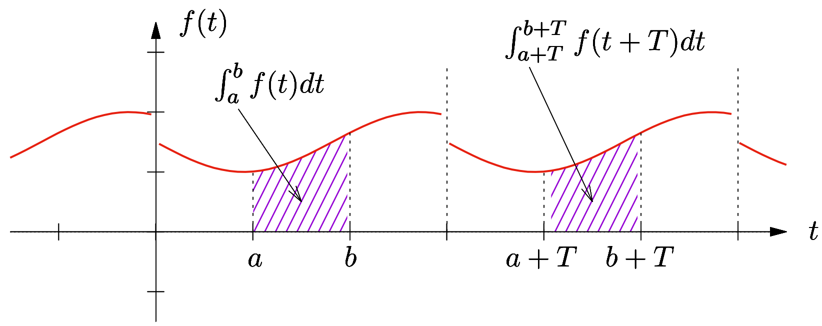


Figura 1.2: Demostración gráfica

Propiedad 2

$$\int_{a-T/2}^{a+T/2} f(t)\,dt = \int_{-T/2}^{T/2} f(t)\,dt \tag{1.3}$$

Prueba:

$$\begin{aligned}
\int_{a-T/2}^{a+T/2} f(t)\,dt &= \int_{a-T/2}^{T/2} f(t)\,dt + \int_{T/2}^{a+T/2} f(t)\,dt \\
&= \int_{a-T/2}^{T/2} f(t)\,dt + \int_{T/2-T}^{a+T/2-T} f(t)\,dt \\
&= \int_{a-T/2}^{T/2} f(t)\,dt + \int_{-T/2}^{a-T/2} f(t)\,dt \\
&= \int_{-T/2}^{T/2} f(t)\,dt
\end{aligned}$$

Puede verse gráficamente en la **Figura 1.3.**

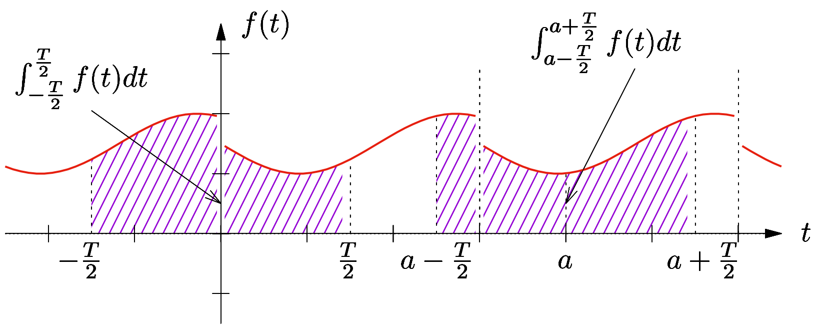


Figura 1.3: Demostración gráfica

Propiedad 3

$$\int_0^T f(t)\,dt = \int_{-T/2}^{T/2} f(t)\,dt \tag{1.4}$$

Prueba:

Si en la ecuación (1.3) $a = T/2$:

$$\begin{aligned}
\int_{-T/2}^{T/2} f(t)\,dt &= \int_{a-T/2}^{a+T/2} f(t)\,dt \\
&= \int_{T/2-T/2}^{T/2+T/2} f(t)\,dt \\
&= \int_0^T f(t)\,dt
\end{aligned}$$

Puede verse gráficamente en la **Figura 1.4.**

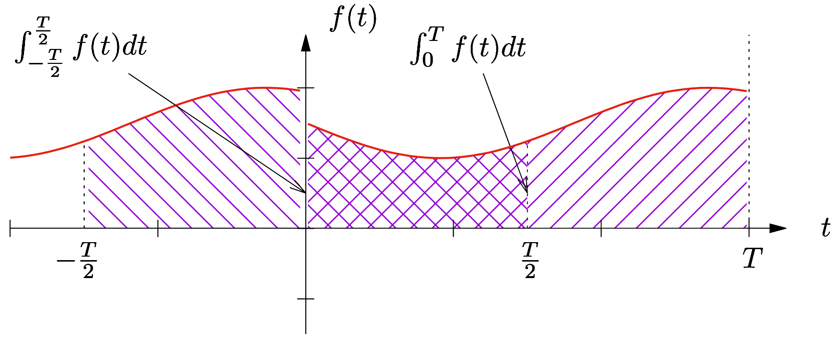


Figura 1.4: Demostración gráfica

Propiedad 4

Si $b - a = T$:

$$\int_0^T f(t)\,dt = \int_a^b f(t)\,dt \tag{1.5}$$

<u>Prueba</u>:

$$\begin{aligned}
\int_a^b f(t)\,dt &= \int_a^{a+T} f(t)\,dt \\
&= \int_a^T f(t)\,dt + \int_T^{a+T} f(t)\,dt \\
&= \int_a^T f(t)\,dt + \int_{T-T}^{a+T-T} f(t)\,dt \\
&= \int_a^T f(t)\,dt + \int_0^a f(t)\,dt \\
&= \int_0^T f(t)\,dt
\end{aligned}$$

Puede verse gráficamente en la **Figura 1.5**.

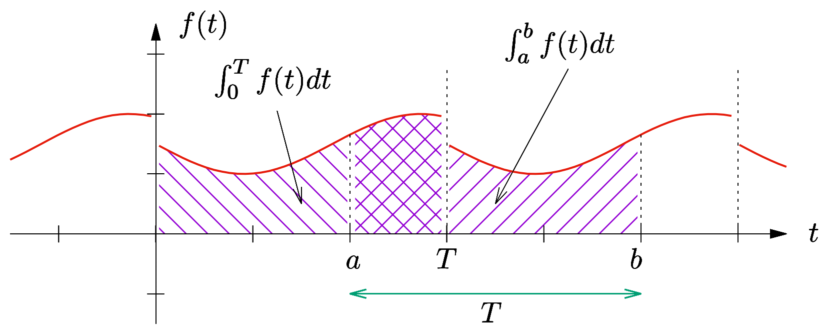


Figura 1.5: Demostración gráfica

1.2.1. Funciones seno y coseno

$$f(t) = A\ \mathrm{sen}(\omega_0 t)$$
$$f(t) = A\ \cos(\omega_0 t)$$

Donde:
A: Amplitud.
ω_0: Frecuencia angular.
$T = 2\pi/\omega_0$: Periodo.

Ejemplo: Hallar el periodo de la siguiente función:

$$f(t) = \mathrm{sen}(4t) + \mathrm{sen}(3t/2) + \mathrm{sen}(10t)$$

El periodo buscado debe contener un numero entero de veces a los 3 periodos hallados:

$$T = \begin{cases} a\,T_1; & a \in \mathbb{N} \\ b\,T_2; & b \in \mathbb{N} \\ c\,T_3; & c \in \mathbb{N} \end{cases}$$
$$a\,T_1 = b\,T_2 = c\,T_3$$
$$a\,\frac{2\pi}{4} = b\,\frac{2\pi}{3/2} = c\,\frac{2\pi}{10}$$
$$a\,\frac{\pi}{2} = b\,\frac{4\pi}{3} = c\,\frac{\pi}{5}; x30$$
$$15a = 40b = 6c$$
$$M.C.M.(15, 40, 6) = 120$$
$$\left.\begin{aligned} 120 = 15a \rightarrow a = 8 \\ 120 = 40b \rightarrow b = 3 \\ 120 = 6c \rightarrow c = 20 \end{aligned}\right\} = T = 8\left(\frac{\pi}{2}\right) = 4\pi$$

Puede verse gráficamente en la **Figura 1.6**.

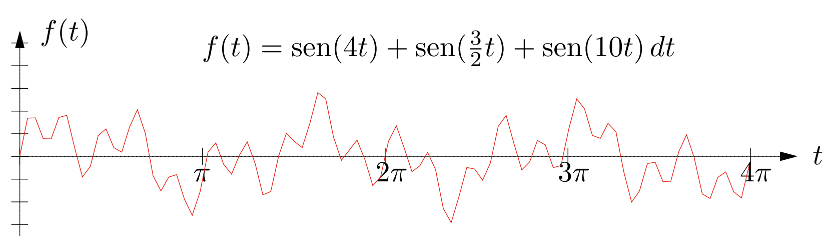


Figura 1.6: Periodo de la función

1.2.2. Propiedades ortogonales del seno y el coseno

Propiedad 1

$$\int_0^T \operatorname{sen}(n\omega_0 t)\, dt = 0 \quad n \in \mathbb{Z} \tag{1.6}$$

Prueba:

$$\begin{aligned}
\int_0^T \operatorname{sen}(n\omega_0 t)\, dt &= -\frac{\cos(n\omega_0 t)}{n\omega_0}\bigg|_0^T \\
&= -\frac{\cos(n\omega_0 T)}{n\omega_0} + \frac{\cos(0)}{n\omega_0} \\
&= -\frac{\cos(n2\pi)}{n\omega_0} + \frac{\cos(0)}{n\omega_0} \\
&= -\frac{1}{n\omega_0} + \frac{1}{n\omega_0} \\
&= 0
\end{aligned}$$

$$\int_0^T \cos(n\omega_0 t)\, dt = 0 \quad n \in \mathbb{Z} \tag{1.7}$$

Prueba:

$$\begin{aligned}
\int_0^T \cos(n\omega_0 t)\, dt &= \frac{\operatorname{sen}(n\omega_0 t)}{n\omega_0}\bigg|_0^T \\
&= \frac{\operatorname{sen}(n\omega_0 T)}{n\omega_0} - \frac{\operatorname{sen}(0)}{n\omega_0} \\
&= \frac{\operatorname{sen}(n2\pi)}{n\omega_0} - \frac{\operatorname{sen}(0)}{n\omega_0} \\
&= \frac{0}{n\omega_0} - \frac{0}{n\omega_0} \\
&= 0
\end{aligned}$$

Propiedad 2

$$\int_0^T \operatorname{sen}(m\omega_0 t) \operatorname{sen}(n\omega_0 t)\, dt = 0 \quad m, n \in \mathbb{Z} \quad m \neq n \tag{1.8}$$

Prueba:

$$\begin{aligned}\int_0^T \operatorname{sen}(m\omega_0 t)\operatorname{sen}(n\omega_0 t)\,dt &= \int_0^T \frac{1}{2}(\cos((m-n)\omega_0 t) - \cos((m+n)\omega_0 t))dt \\ &= \frac{1}{2}\left(\int_0^T \cos((m-n)\omega_0 t)dt - \int_0^T \cos((m+n)\omega_0 t)dt\right) \\ &= \frac{1}{2}(0-0) \\ &= 0\end{aligned}$$

$$\int_0^T \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt = 0 \quad m,n\in\mathbb{Z} \quad m\neq n \tag{1.9}$$

Prueba:

$$\begin{aligned}\int_0^T \cos(m\omega_0 t)\cos(n\omega_0 t)\,dt &= \int_0^T \frac{1}{2}(\cos((m-n)\omega_0 t) + \cos((m+n)\omega_0 t))dt \\ &= \frac{1}{2}\left(\int_0^T \cos((m-n)\omega_0 t)dt + \int_0^T \cos((m+n)\omega_0 t)dt\right) \\ &= \frac{1}{2}(0+0) \\ &= 0\end{aligned}$$

$$\int_0^T \operatorname{sen}(m\omega_0 t)\cos(n\omega_0 t)\,dt = 0 \quad m,n\in\mathbb{Z} \tag{1.10}$$

Prueba:

$$\begin{aligned}\int_0^T \operatorname{sen}(m\omega_0 t)\cos(n\omega_0 t)\,dt &= \int_0^T \frac{1}{2}(\operatorname{sen}((m-n)\omega_0 t) + \operatorname{sen}((m+n)\omega_0 t))dt \\ &= \frac{1}{2}\left(\int_0^T \operatorname{sen}((m-n)\omega_0 t)dt + \int_0^T \cos((m+n)\omega_0 t)dt\right) \\ &= \frac{1}{2}(0+0) \\ &= 0\end{aligned}$$

Propiedad 3

$$\int_0^T \operatorname{sen}^2(n\omega_0 t)\,dt = \frac{T}{2} \quad n\in\mathbb{Z} \tag{1.11}$$

Prueba:

$$\begin{aligned}
\int_0^T \operatorname{sen}^2(n\omega_0 t)\,dt &= \int_0^T \operatorname{sen}(n\omega_0 t)\operatorname{sen}(n\omega_0 t)dt \\
&= \frac{1}{2}\left(\int_0^T \cos((n-n)\omega_0 t)dt - \int_0^T \cos((n+n)\omega_0 t)dt\right) \\
&= \frac{1}{2}\left(\int_0^T \cos(0)dt - \int_0^T \cos((2n)\omega_0 t)dt\right) \\
&= \frac{1}{2}\left(\int_0^T dt - \int_0^T \cos(2n\omega_0 t)dt\right) \\
&= \frac{1}{2}(t\Big|_0^T - 0) \\
&= \frac{T}{2}
\end{aligned}$$

$$\int_0^T \cos^2(n\omega_0 t)\,dt = \frac{T}{2} \quad n \in \mathbb{Z} \tag{1.12}$$

Prueba:

$$\begin{aligned}
\int_0^T \cos^2(n\omega_0 t)\,dt &= \int_0^T \cos(n\omega_0 t)\cos(n\omega_0 t)dt \\
&= \frac{1}{2}\left(\int_0^T \cos((n-n)\omega_0 t)dt + \int_0^T \cos((n+n)\omega_0 t)dt\right) \\
&= \frac{1}{2}\left(\int_0^T \cos(0)dt + \int_0^T \cos((2n)\omega_0 t)dt\right) \\
&= \frac{1}{2}\left(\int_0^T dt + \int_0^T \cos(2n\omega_0 t)dt\right) \\
&= \frac{1}{2}(t\Big|_0^T + 0) \\
&= \frac{T}{2}
\end{aligned}$$

1.3. Series de *Fourier*

Una función periódica que cumple ciertas condiciones puede desarrollarse mediante la serie:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t) + b_n \operatorname{sen}(n\omega_0 t)] \tag{1.13}$$

Donde:

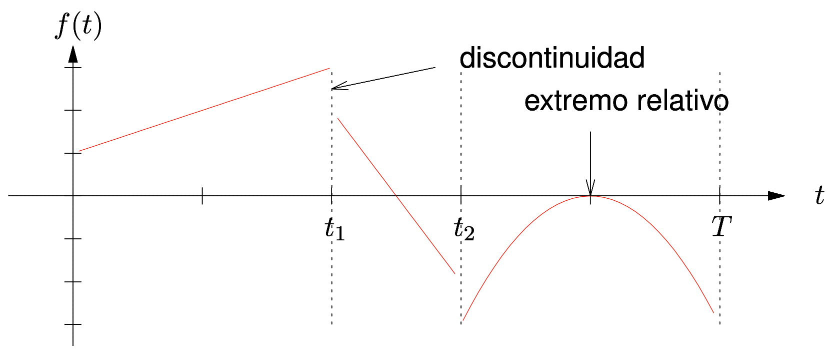
$\omega_0 = 2\pi/T$: Frecuencia angular de $f(t)$.
T: Periodo de $f(t)$.
$a_0; a_n; b_n$: Coeficientes de *Fourier*.
$a_0/2$: Termino constante.
$a_n \cos(n\omega_0 t); b_n \operatorname{sen}(n\omega_0 t)$: Armónicos, términos seno y coseno con frecuencias angulares múltiples de ω_0

$a_1 \cos(\omega_0 t) + b_1 \operatorname{sen}(\omega_0 t)$: Primer armónico.
$a_2 \cos(2\omega_0 t) + b_2 \operatorname{sen}(2\omega_0 t)$: Segundo armónico.
$a_3 \cos(3\omega_0 t) + b_3 \operatorname{sen}(2\omega_0 t)$: Tercer armónico.

1.3.1. Condiciones de ***Dirichlet***

Para que una función periódica $f(t) = f(t + nT); n \in \mathbb{Z}$, se desarrolle como una serie de *Fourier* debe cumplir:

- $f(t)$ debe ser continua por tramos en 1 periodo.

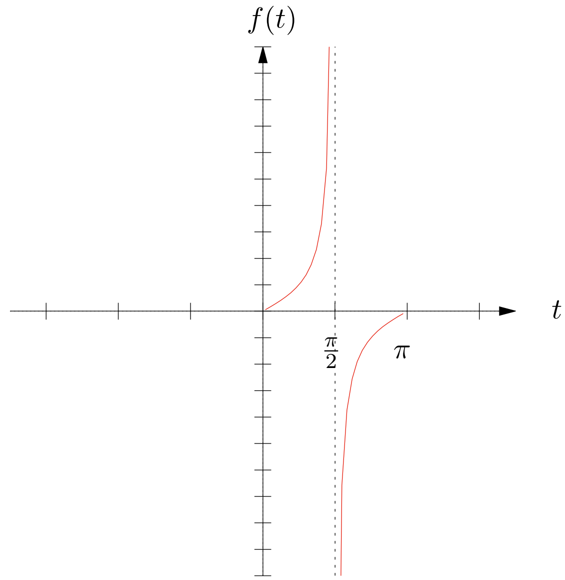


  - Debe existir un numero finito de discontinuidades (en 1 periodo).
  - Debe existir un numero finito de extremos relativos (en 1 periodo).

- La integral $\int_0^T |f(t)|\,dt < \infty$ debe ser finita.

Ejemplo:

$$f(t) = \tan(t); \quad 0 < t < \pi; \quad T = \pi$$



$$\int_0^{\pi} |\tan t| \, dt \to \infty$$
$$t = \frac{\pi}{2} : |\tan(t)| \to \infty$$

$\therefore$ Esta función no tiene serie de *Fourier*.

1.4. Evaluación de los coeficientes de *Fourier*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \operatorname{sen}(n\omega_0 t)]$$

Integrando ambas partes:

$$\begin{aligned} \int_0^T f(t)\, dt &= \int_0^T \frac{a_0}{2}\, dt + \sum_{n=1}^{\infty} \left[\int_0^T a_n \cos(n\omega_0 t)\, dt + \int_0^T b_n \operatorname{sen}(n\omega_0 t)\, dt \right] \\ &= \frac{a_0}{2} t \bigg|_0^T \\ &= \frac{a_0}{2} T \end{aligned}$$

$$a_0 = \frac{2}{T} \int_0^T f(t)\, dt \tag{1.14}$$

Para calcular "a_n" multiplicamos por $\cos(m\omega_0 t); m \in \mathbb{N}$ e integramos en 1 periodo.

$$\begin{aligned}\int_0^T f(t)\cos(m\omega_0 t)\,dt &= \int_0^T \frac{a_0}{2}\cos(m\omega_0 t)\,dt \\ &+ \sum_{n=1}^{\infty}\left[\int_0^T a_n\cos(n\omega_0 t)\cos(m\omega_0 t)\,dt + \int_0^T b_n\operatorname{sen}(n\omega_0 t)\cos(m\omega_0 t)\,dt\right] \\ &= 0 + \sum_{n=1}^{\infty}\left[\int_0^T a_n\cos(n\omega_0 t)\cos(m\omega_0 t)\,dt + 0\right]\end{aligned}$$

Para $n \neq m$ todos los elementos de la sumatoria serán igual a 0. Por tanto:

$$\begin{aligned}\int_0^T f(t)\cos(n\omega_0 t)\,dt &= \int_0^T a_n\cos^2(n\omega_0 t)\,dt \\ &= a_n\frac{T}{2}\end{aligned}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt \tag{1.15}$$

Para calcular "b_n" multiplicamos por $\operatorname{sen}(m\omega_0 t); m \in \mathbb{N}$ e integramos en 1 periodo.

$$\begin{aligned}\int_0^T f(t)\operatorname{sen}(m\omega_0 t)\,dt &= \int_0^T \frac{a_0}{2}\operatorname{sen}(m\omega_0 t)\,dt \\ &+ \sum_{n=1}^{\infty}\left[\int_0^T a_n\cos(n\omega_0 t)\operatorname{sen}(m\omega_0 t)\,dt + \int_0^T b_n\operatorname{sen}(n\omega_0 t)\operatorname{sen}(m\omega_0 t)\,dt\right] \\ &= 0 + \sum_{n=1}^{\infty}\left[0 + \int_0^T b_n\operatorname{sen}(n\omega_0 t)\operatorname{sen}(m\omega_0 t)\,dt\right]\end{aligned}$$

Para $n \neq m$ todos los elementos de la sumatoria serán igual a 0. Por tanto:

$$\begin{aligned}\int_0^T f(t)\operatorname{sen}(n\omega_0 t)\,dt &= \int_0^T b_n\operatorname{sen}^2(n\omega_0 t)\,dt \\ &= b_n\frac{T}{2}\end{aligned}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\operatorname{sen}(n\omega_0 t)\,dt \tag{1.16}$$

1.5. Formulas para las series de *Fourier*

$$\operatorname{sen}(\pi n) = 0; \quad n \in \mathbb{N}$$

$$\cos(\pi n) = (-1)^n; \quad n \in \mathbb{N}$$
$$\operatorname{sen}(2\pi n) = 0; \quad n \in \mathbb{N}$$
$$\cos(2\pi n) = 1; \quad n \in \mathbb{N}$$

$$\int \operatorname{sen}(at)\,dt = -\frac{\cos(at)}{a}$$
$$\int \cos(at)\,dt = \frac{\operatorname{sen}(at)}{a}$$

$$\int t\operatorname{sen}(at)\,dt = -\frac{t}{a}\cos(at) + \frac{1}{a^2}\operatorname{sen}(at)$$
$$\int t\cos(at)\,dt = \frac{t}{a}\operatorname{sen}(at) + \frac{1}{a^2}\cos(at)$$

$$\int e^{at}\,dt = \frac{1}{a}e^{at}$$
$$\int t\,e^{at}\,dt = \frac{t}{a}\,e^{at} - \frac{1}{a^2}e^{at}$$

Capítulo 2

Análisis de formas de onda periódica

2.1. Funciones pares e impares

Una función es **par** si:

$$f(-t) = f(t) \tag{2.1}$$

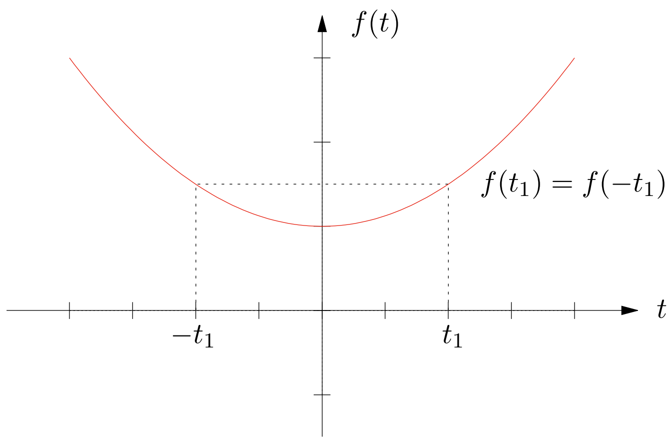


Figura 2.1: La gráfica se refleja respecto al eje central.

Ejemplo 1:

$$f(t) = t^2$$

$$f(-t) = (-t)^2 = t^2 = f(t)$$

Ejemplo 2:

$$f(t) = \cos(t)$$

$$f(t) = \cos(-t) = \cos(t) = f(t)$$

Una función es **impar** si:

$$f(-t) = -f(t) \tag{2.2}$$

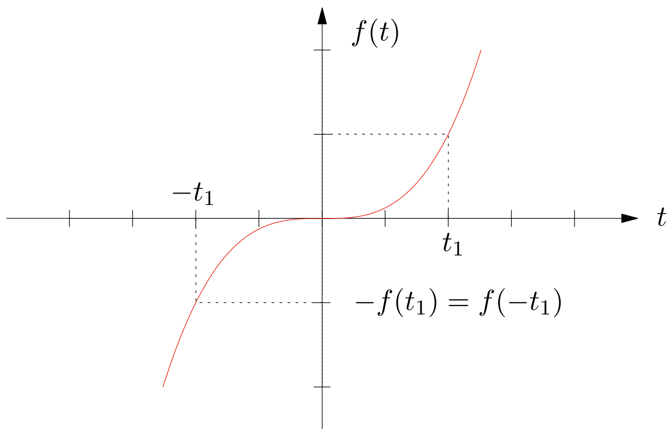


Figura 2.2: La gráfica se refleja
1ro respecto al eje central
2do respecto al eje horizontal.

Ejemplo 3:

$$f(t) = t^3$$
$$f(-t) = (-t)^3 = -t^3 = -f(t)$$

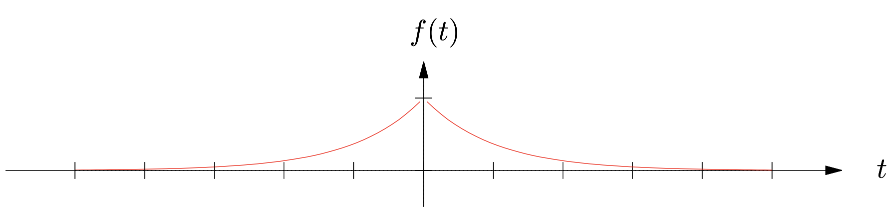
Ejemplo 4:

$$f(t) = \operatorname{sen}(t)$$
$$f(t) = \operatorname{sen}(-t) = -\operatorname{sen}(t) = -f(t)$$

Ejemplo 5:

$$f(t) = \begin{cases} e^{t} & t < 0 \\ e^{-t} & t > 0 \end{cases}$$

$$f(-t) = \left\{ \begin{matrix} e^{-t} & -t < 0 \to t > 0 \\ e^{t} & -t > 0 \to t < 0 \end{matrix} \right\} = f(t)$$



2.1.1. Propiedades de las funciones pares e impares

Propiedad 1

Si $f(t)$ es **par** y $g(t)$ es **par**, entonces $h(t) = f(t)g(t)$ es **par**.

Prueba:

$$\begin{cases} f(-t) = f(t) \\ g(-t) = g(t) \end{cases}$$

$$\begin{aligned} h(-t) &= f(-t)g(-t) \\ &= f(t)g(t) \\ &= h(t) \end{aligned}$$

Propiedad 2

Si $f(t)$ es **impar** y $g(t)$ es **impar**, entonces $h(t) = f(t)g(t)$ es **par.**

Prueba:

$$\begin{cases} f(-t) = -f(t) \\ g(-t) = -g(t) \end{cases}$$

$$\begin{aligned} h(-t) &= f(-t)g(-t) \\ &= (-f(t))(-g(t)) \\ &= f(t)g(t) \\ &= h(t) \end{aligned}$$

Propiedad 3

Si $f(t)$ es **par** y $g(t)$ es **impar**, entonces $h(t) = f(t)g(t)$ es **impar.**

Prueba:

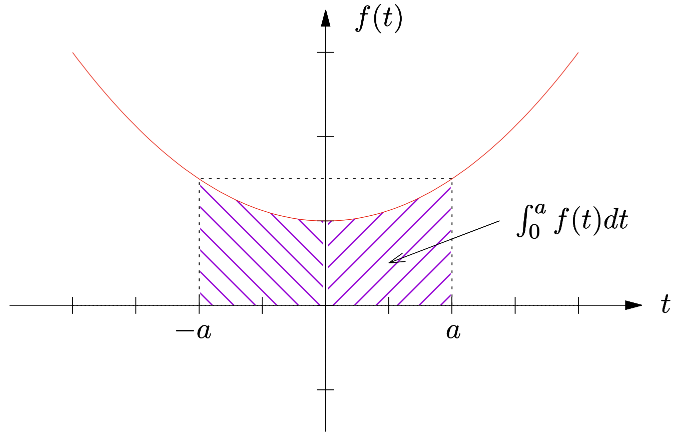
$$\begin{cases} f(-t) = f(t) \\ g(-t) = -g(t) \end{cases}$$

$$\begin{aligned} h(-t) &= f(-t)g(-t) \\ &= f(t)(-g(t)) \\ &= -f(t)g(t) \\ &= -h(t) \end{aligned}$$

Propiedad 4

Si $f(t)$ es **par**, entonces:

$$\int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt \tag{2.3}$$



Prueba:

$$\begin{aligned}\int_{-a}^{a} f(t)\,dt &= \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt\\ &= \int_{-a}^{0} f(-t)\,dt + \int_{0}^{a} f(t)\,dt\end{aligned}$$

$$\tau = -t$$

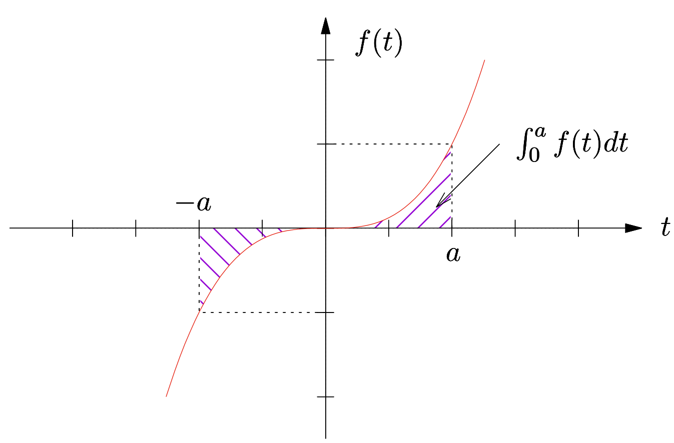
$$d\tau = -dt$$

$$\begin{aligned}\int_{-a}^{a} f(t)\,dt &= \int_{a}^{0} f(\tau)\,(-d\tau) + \int_{0}^{a} f(t)\,dt\\ &= \int_{0}^{a} f(\tau)\,d\tau + \int_{0}^{a} f(t)\,dt\\ &= 2\int_{0}^{a} f(t)\,dt\end{aligned}$$

Propiedad 5

Si $f(t)$ es **impar**, entonces:

$$\int_{-a}^{a} f(t)\,dt = 0 \qquad (2.4)$$



Prueba:

$$\begin{aligned}\int_{-a}^{a} f(t)\,dt &= \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt\\ &= \int_{-a}^{0} -f(-t)\,dt + \int_{0}^{a} f(t)\,dt\end{aligned}$$

$$\tau = -t$$

$$d\tau = -dt$$

$$\begin{aligned}\int_{-a}^{a} f(t)\,dt &= -\int_{a}^{0} f(\tau)\,(-d\tau) + \int_{0}^{a} f(t)\,dt\\ &= -\int_{0}^{a} f(\tau)\,d\tau + \int_{0}^{a} f(t)\,dt\\ &= 0\end{aligned}$$

2.1.2. Evaluación de coeficientes de *Fourier*

Simetría par

$$\begin{aligned}a_0 &= \frac{2}{T}\int_{0}^{T} f(t)\,dt\\ &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt\\ &= \frac{2}{T}\left(2\int_{0}^{T/2} f(t)\,dt\right)\\ &= \frac{4}{T}\int_{0}^{T/2} f(t)\,dt\end{aligned}$$

$$a_0 = \frac{4}{T}\int_{0}^{T/2} f(t)\,dt \tag{2.5}$$

$$\begin{aligned}a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt\\ &= \frac{2}{T}\left(2\int_{0}^{T/2} f(t)\cos(n\omega_0 t)\,dt\right)\\ &= \frac{4}{T}\int_{0}^{T/2} f(t)\cos(n\omega_0 t)\,dt\end{aligned}$$

$$a_n = \frac{4}{T}\int_{0}^{T/2} f(t)\cos(n\omega_0 t)\,dt \tag{2.6}$$

$$\begin{aligned} b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt \\ &= 0 \end{aligned}$$

$$b_n = 0 \tag{2.7}$$

Simetría impar

$$\begin{aligned} a_0 &= \frac{2}{T}\int_{0}^{T} f(t)\,dt \\ &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt \\ &= 0 \end{aligned}$$

$$a_0 = 0 \tag{2.8}$$

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0 t)\,dt \\ &= 0 \end{aligned}$$

$$a_n = 0 \tag{2.9}$$

$$\begin{aligned} b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt \\ &= \frac{2}{T}\left(2\int_{0}^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt\right) \\ &= \frac{4}{T}\int_{0}^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt \end{aligned}$$

$$b_n = \frac{4}{T}\int_{0}^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt \tag{2.10}$$

2.2. Simetría de media onda (S.M.O.)

$f(t)$ tiene simetría de media onda si:

$$f(t) = -f(t \pm \frac{T}{2})$$

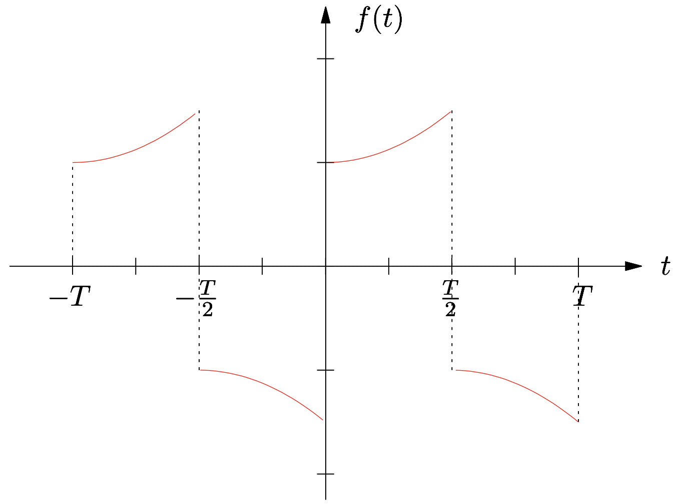


Figura 2.3: La gráfica se desplaza $1/2$ periodo
y se refleja respecto a t.

2.2.1. Evaluación de coeficientes de *Fourier*

$$\begin{aligned}
a_0 &= \frac{2}{T}\int_0^T f(t)\,dt \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\,dt + \int_{T/2}^{T} f(t)\,dt\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\,dt - \int_{t=T/2}^{t=T} f(t-\frac{T}{2})\,dt\right] \\
&\tau = t - \frac{T}{2} \\
&d\tau = dt \\
a_0 &= \frac{2}{T}\left[\int_0^{T/2} f(t)\,dt - \int_{\tau=T/2-T/2}^{\tau=T-T/2} f(\tau)\,d\tau\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\,dt - \int_0^{T/2} f(\tau)\,d\tau\right] \\
a_0 &= 0
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt + \int_{T/2}^{T} f(t)\cos(n\omega_0 t)\,dt\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt - \int_{T/2}^{T} f(t-\frac{T}{2})\cos(n\omega_0 t)\,dt\right]
\end{aligned}$$

$$
\begin{aligned}
\tau &= t - \frac{T}{2} \\
d\tau &= dt
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt - \int_{\tau=T/2-T/2}^{\tau=T-T/2} f(\tau)\cos(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt - \int_0^{T/2} f(\tau)\cos(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right]
\end{aligned}
$$

$$
\begin{aligned}
n\omega_0(\tau+T/2) &= n\omega_0\tau + n\omega_0\frac{T}{2} \\
&= n\omega_0\tau + n\frac{2\pi}{T}\frac{T}{2} \\
&= n\omega_0\tau + n\pi
\end{aligned}
$$

$$
\begin{aligned}
\cos(n\omega_0\tau + n\pi) &= \cos(n\omega_0\tau)\cos(n\pi) - \operatorname{sen}(n\pi)\operatorname{sen}(n\omega_0\tau) \\
&= \cos(n\omega_0\tau)\cos(n\pi)
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt - \int_0^{T/2} f(\tau)\cos(n\omega_0\tau)\cos(n\pi)\,d\tau\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt - \cos(n\pi)\int_0^{T/2} f(\tau)\cos(n\omega_0\tau)\,d\tau\right] \\
&= \frac{2}{T}(1-\cos(\pi n))\left(\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt\right)
\end{aligned}
$$

$$
\cos(\pi n) = \begin{cases} 1 & n : \text{par} \\ -1 & n : \text{impar} \end{cases}
$$

$$
\begin{cases} n : \text{par} & a_n = 0 \\ n : \text{impar} & a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt \end{cases} \tag{2.12}
$$

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_0^{T} f(t)\operatorname{sen}(n\omega_0 t)\,dt \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt + \int_{T/2}^{T} f(t)\operatorname{sen}(n\omega_0 t)\,dt\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \int_{T/2}^{T} f(t-\frac{T}{2})\operatorname{sen}(n\omega_0 t)\,dt\right]
\end{aligned}
$$

$$
\begin{aligned}
\tau &= t - \frac{T}{2} \\
d\tau &= dt
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \int_{\tau=T/2-T/2}^{\tau=T-T/2} f(\tau)\operatorname{sen}(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \int_0^{T/2} f(\tau)\operatorname{sen}(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right]
\end{aligned}
$$

$$\begin{aligned} n\omega_0(\tau + T/2) &= n\omega_0\tau + n\omega_0\frac{T}{2} \\ &= n\omega_0\tau + n\frac{2\pi}{T}\frac{T}{2} \\ &= n\omega_0\tau + n\pi \end{aligned}$$

$$\begin{aligned} \operatorname{sen}(n\omega_0\tau + n\pi) &= \operatorname{sen}(n\omega_0\tau)\cos(n\pi) + \operatorname{sen}(n\pi)\cos(n\omega_0\tau) \\ &= \operatorname{sen}(n\omega_0\tau)\cos(n\pi) \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \int_0^{T/2} f(\tau)\operatorname{sen}(n\omega_0\tau)\cos(n\pi)\,d\tau\right] \\ &= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt - \cos(n\pi)\int_0^{T/2} f(\tau)\operatorname{sen}(n\omega_0\tau)\,d\tau\right] \\ &= \frac{2}{T}(1-\cos(\pi n))\left(\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt\right) \end{aligned}$$

$$\cos(\pi n) = \begin{cases} 1 & n : \text{par} \\ -1 & n : \text{impar} \end{cases}$$

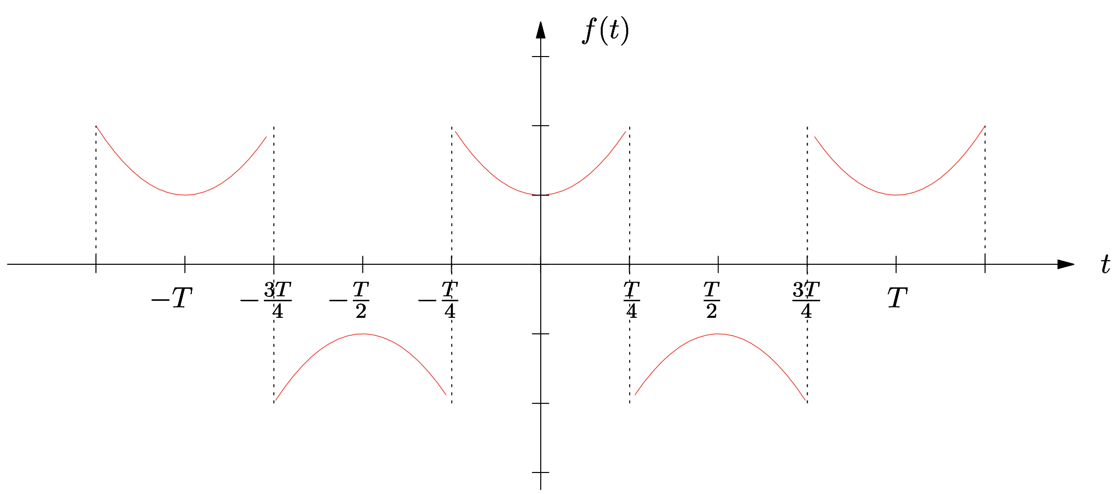
$$\begin{cases} n : \text{par} & b_n = 0 \\ n : \text{impar} & b_n = \frac{4}{T}\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0 t)\,dt \end{cases} \tag{2.13}$$

2.3. Simetría de cuarto de onda (S.C.O.)

2.3.1. Simetría de cuarto de onda par

Una función $f(t)$ tiene simetría de cuarto de onda **par** cuando:

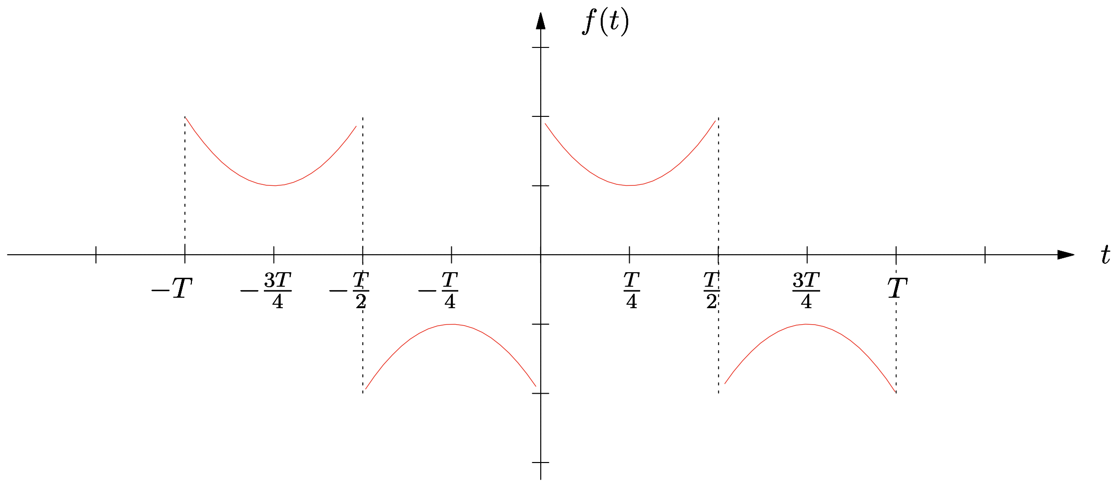
- $f(t)$ es **par**.
- $f(t)$ tiene simetría de media onda.



2.3.2. Simetría de cuarto de onda impar

Una función $f(t)$ tiene simetría de cuarto de onda **impar** cuando:

- $f(t)$ es **impar**.
- $f(t)$ tiene simetría de media onda.



2.3.3. Evaluación de coeficientes de *Fourier*

Simetría de cuarto de onda par

Como la función $f(t)$ tiene simetría de media onda:

$$a_0 = 0 \tag{2.14}$$

Como la función $f(t)$ es una función par:

$$b_n = 0 \tag{2.15}$$

Como la función $f(t)$ tiene simetría de media onda: $a_n = 0$ cuando n es par.

Para n impar:

$$\begin{aligned} a_n &= \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt \\ &= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt + \int_{T/4}^{T/2} f(t)\cos(n\omega_0 t)\,dt\right] \\ &= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt - \int_{T/4}^{T/2} f(t-\frac{T}{2})\cos(n\omega_0 t)\,dt\right] \end{aligned}$$

$$
\begin{aligned}
\tau &= t - \frac{T}{2} \\
d\tau &= dt
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt - \int_{\tau=T/4-T/2}^{\tau=T/2-T/2} f(\tau)\cos(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right] \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt - \int_{-T/4}^{0} f(\tau)\cos(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right]
\end{aligned}
$$

$$
\begin{aligned}
n\omega_0(\tau + T/2) &= n\omega_0\tau + n\omega_0\frac{T}{2} \\
&= n\omega_0\tau + n\frac{2\pi}{T}\frac{T}{2} \\
&= n\omega_0\tau + n\pi
\end{aligned}
$$

$$
\begin{aligned}
\cos(n\omega_0\tau + n\pi) &= \cos(n\omega_0\tau)\cos(n\pi) - \operatorname{sen}(n\pi)\operatorname{sen}(n\omega_0\tau) \\
&= \cos(n\omega_0\tau)\cos(n\pi) \\
&= -\cos(n\omega_0\tau)
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt + \int_{-T/4}^{0} f(\tau)\cos(n\omega_0\tau)\,d\tau\right] \\
&= \frac{4}{T}\left[\int_{-T/4}^{T/4} f(t)\cos(n\omega_0 t)\,dt\right] \\
&= \frac{4}{T}\left(2\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt\right) \\
&= \frac{8}{T}\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt
\end{aligned}
$$

$$
\begin{cases} n: \text{par} & a_n = 0 \\ n: \text{impar} & a_n = \frac{8}{T}\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt \end{cases} \tag{2.16}
$$

Simetría de cuarto de onda impar

Como la función $f(t)$ tiene simetría de media onda:

$$a_0 = 0 \tag{2.17}$$

Como la función $f(t)$ es una función impar:

$$a_n = 0 \tag{2.18}$$

Como la función $f(t)$ tiene simetría de media onda: $b_n = 0$ cuando n es par.

Para n impar:

$$
\begin{aligned}
b_n &= \frac{4}{T}\int_0^{T/2} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt + \int_{T/4}^{T/2} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt\right] \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt - \int_{T/4}^{T/2} f(t-\frac{T}{2})\,\mathrm{sen}(n\omega_0 t)\,dt\right]
\end{aligned}
$$

$$
\begin{aligned}
\tau &= t - \frac{T}{2} \\
d\tau &= dt
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{4}{T}\left[\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt - \int_{\tau=T/4-T/2}^{\tau=T/2-T/2} f(\tau)\,\mathrm{sen}(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right] \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt - \int_{-T/4}^{0} f(\tau)\,\mathrm{sen}(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right]
\end{aligned}
$$

$$
\begin{aligned}
n\omega_0(\tau + T/2) &= n\omega_0\tau + n\omega_0\frac{T}{2} \\
&= n\omega_0\tau + n\frac{2\pi}{T}\frac{T}{2} \\
&= n\omega_0\tau + n\pi
\end{aligned}
$$

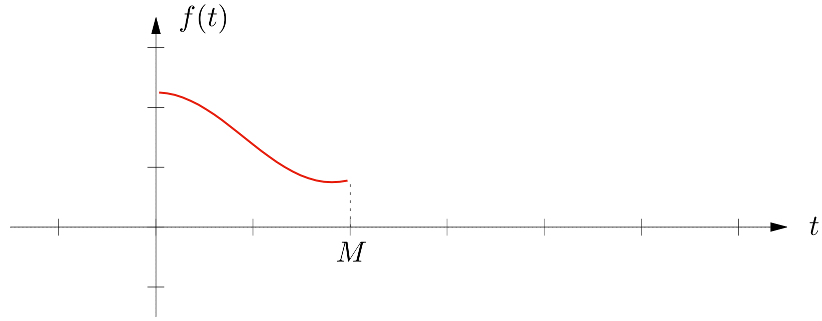
$$
\begin{aligned}
\mathrm{sen}(n\omega_0\tau + n\pi) &= \mathrm{sen}(n\omega_0\tau)\cos(n\pi) + \mathrm{sen}(n\pi)\cos(n\omega_0\tau) \\
&= \mathrm{sen}(n\omega_0\tau)\cos(n\pi) \\
&= -\,\mathrm{sen}(n\omega_0\tau)
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{4}{T}\left[\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt + \int_{-T/4}^{0} f(\tau)\,\mathrm{sen}(n\omega_0\tau)\,d\tau\right] \\
&= \frac{4}{T}\left[\int_{-T/4}^{T/4} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt\right] \\
&= \frac{4}{T}\left(2\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt\right) \\
&= \frac{8}{T}\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt
\end{aligned}
$$

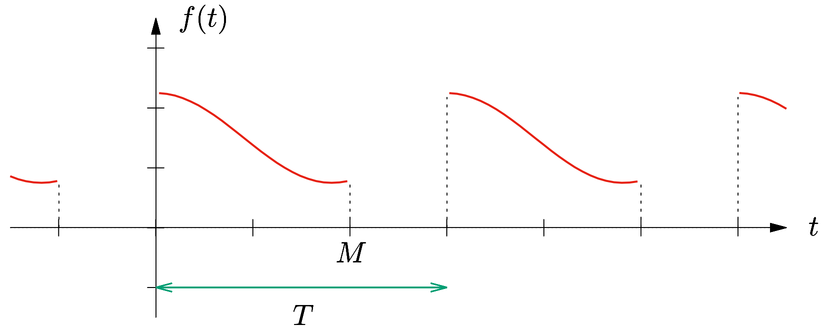
$$
\begin{cases}
n:\text{par} & b_n = 0 \\
n:\text{impar} & b_n = \frac{8}{T}\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0 t)\,dt
\end{cases}
\tag{2.19}
$$

2.4. Expansión periódica de funciones definidas en intervalos finitos

Sea $f(t)$ una función no periódica:



$f(t)$ se convierte en periódica al repetirla un intervalo $T \geq M$.

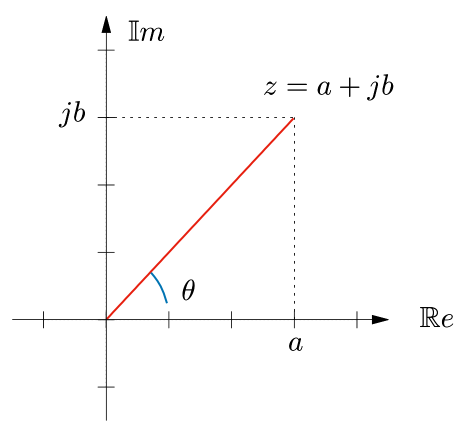


$f(t)$ puede expandirse periódicamente asignando alguna simetría conocida.

Capítulo 3

Serie compleja de *Fourier* y espectros discretos de frecuencia

3.1. Números complejos



Unidad imaginaria: $i = j = \sqrt{-1}$
Forma rectangular: $z = a + jb$
Módulo: $|z| = \sqrt{a^2 + b^2}$
Argumento: $\theta = \arctan(\frac{b}{a})$

Forma polar:

$$z = |z|\cos(\theta) + j|z|\operatorname{sen}(\theta) = |z|(\cos(\theta) + j\operatorname{sen}(\theta))$$

Formula de *Euler*:

$$e^{j\theta} = \cos(\theta) + j\operatorname{sen}(\theta) \tag{3.1}$$

Por tanto:

$$z = |z|e^{j\theta}$$

Forma exponencial o fasorial:

$$z = |z|\angle\theta$$

3.1.1. Formas complejas del seno y coseno

$$e^{j\theta} = \cos(\theta) + j\,\mathrm{sen}(\theta) \tag{3.2}$$
$$e^{-j\theta} = \cos(\theta) - j\,\mathrm{sen}(\theta) \tag{3.3}$$

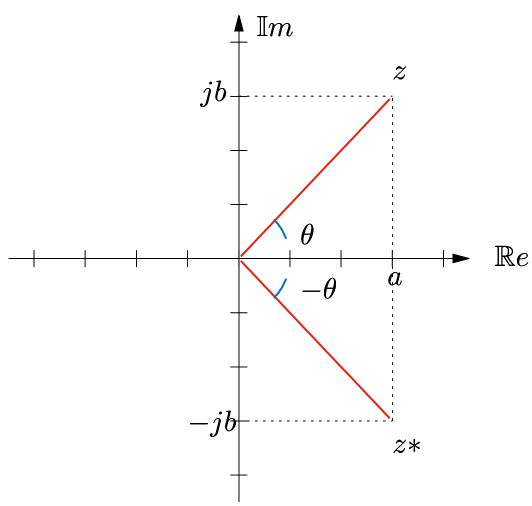
Sumando las ecuaciones 3.2 y 3.3:

$$e^{j\theta} + e^{-j\theta} = 2\cos(\theta)$$
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{3.4}$$

Restando las ecuaciones 3.2 y 3.3:

$$e^{j\theta} - e^{-j\theta} = 2j\,\mathrm{sen}(\theta)$$
$$\mathrm{sen}(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{3.5}$$

3.1.2. Conjugado



$$z = a + jb = |z|\angle\theta$$
$$z* = a - jb = |z|\angle - \theta$$
$$(z)(z*) = |z|^2$$

3.2. Serie compleja de *Fourier*

Partiendo de la serie trigonométrica de *Fourier*:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \operatorname{sen}(n\omega_0 t)]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) + b_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2j}\right)\right]$$

$$\frac{1}{j} = -j$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right) - jb_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2}\right)\right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t} + \left(\frac{a_n + jb_n}{2}\right) e^{-jn\omega_0 t}\right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t}\right] + \sum_{n=1}^{\infty} \left[\left(\frac{a_n + jb_n}{2}\right) e^{-jn\omega_0 t}\right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t}\right] + \sum_{-n=1}^{\infty} \left[\left(\frac{a_{-n} + jb_{-n}}{2}\right) e^{jn\omega_0 t}\right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t}\right] + \sum_{n=-1}^{-\infty} \left[\left(\frac{a_n + jb_n}{2}\right) e^{jn\omega_0 t}\right]$$

$$f(t) = \frac{a_0}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{a_n - jb_n}{2}\right) e^{jn\omega_0 t}$$

Sean los coeficientes complejos de *Fourier*:

$$c_n = \frac{a_n - jb_n}{2}$$

$$c_0 = \frac{a_0}{2}$$

Entonces:

$$f(t) = c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n e^{jn\omega_0 t}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \tag{3.6}$$

3.2.1. Evaluación del coeficiente complejo de *Fourier*

$$
\begin{aligned}
c_n &= \frac{a_n - jb_n}{2} \\
&= \frac{1}{2}\left[\frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt - j\frac{2}{T}\int_0^T f(t)\,\mathrm{sen}(n\omega_0 t)\,dt\right] \\
&= \frac{1}{T}\int_0^T f(t)\,[\cos(n\omega_0 t) - j\,\mathrm{sen}(n\omega_0 t)]\,dt \\
&= \frac{1}{T}\int_0^T f(t)\,e^{-jn\omega_0 t}\,dt
\end{aligned}
$$

$$c_n = \frac{1}{T}\int_0^T f(t)\,e^{-jn\omega_0 t}\,dt \tag{3.7}$$

En particular:

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt \tag{3.8}$$

3.2.2. Relación entre el coeficiente complejo y los coeficientes trigonométricos

$$c_n = \frac{a_n - jbn}{2} = \frac{a_n}{2} + j\frac{-b_n}{2}$$

$$
\begin{aligned}
\frac{a_n}{2} &= \mathbb{R}e\{c_n\} \\
a_n &= 2\,\mathbb{R}e\{c_n\}
\end{aligned} \tag{3.9}
$$

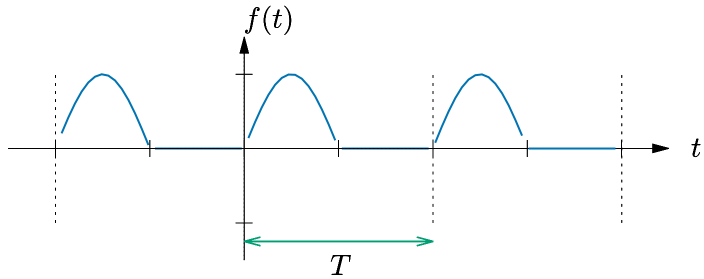
$$
\begin{aligned}
-\frac{b_n}{2} &= \mathbb{I}m\{c_n\} \\
b_n &= -2\,\mathbb{I}m\{c_n\}
\end{aligned} \tag{3.10}
$$

3.3. Ondas senoidales rectificadas

3.3.1. Rectificación de media onda

$$f(t) = \begin{cases} A\,\mathrm{sen}(\omega_0 t) & 0 < t < T/2 \\ 0 & T/2 < t < T \end{cases}$$

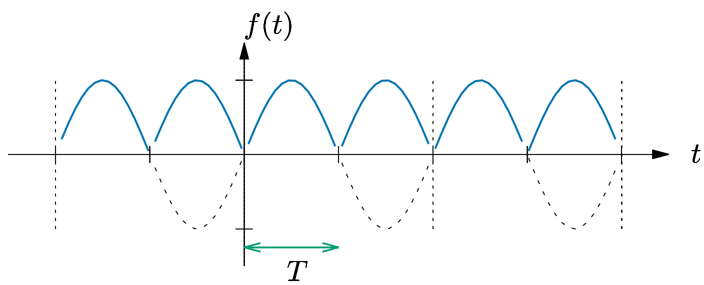
$$T = \frac{2\pi}{\omega_0}$$



El periodo de la onda rectificada es el mismo que de la onda original.

3.3.2. Rectificación de onda completa

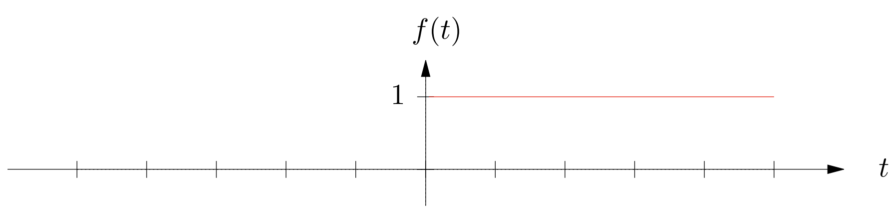
$$f(t) = A\,|\,\mathrm{sen}(\omega_0 t)|$$



El periodo de la onda rectificada es la mitad del periodo de la onda original.

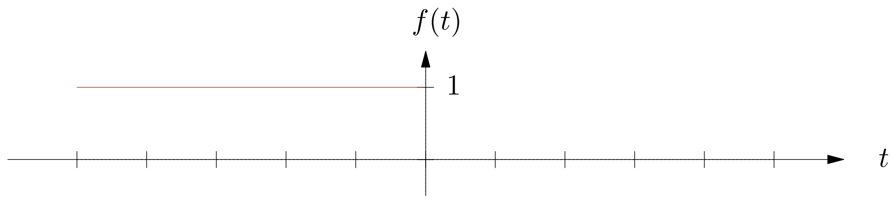
3.4. Función escalón unitario

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



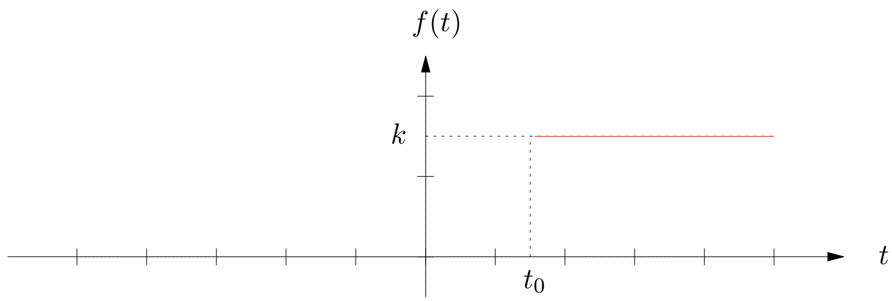
Una variante es:

$$u(-t) = \begin{cases} 1 & t < 0 \\ 0 & t > 0 \end{cases}$$

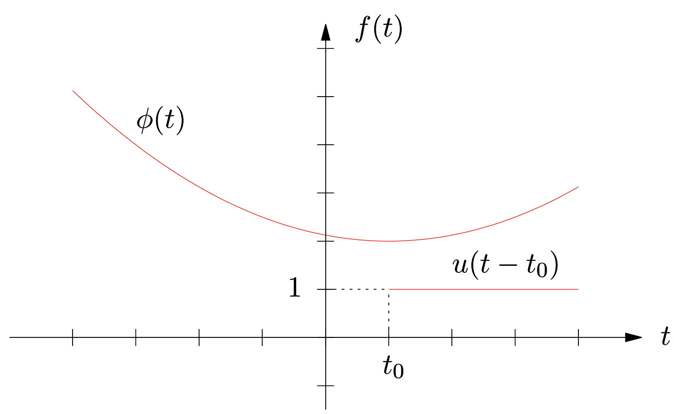


De manera general:

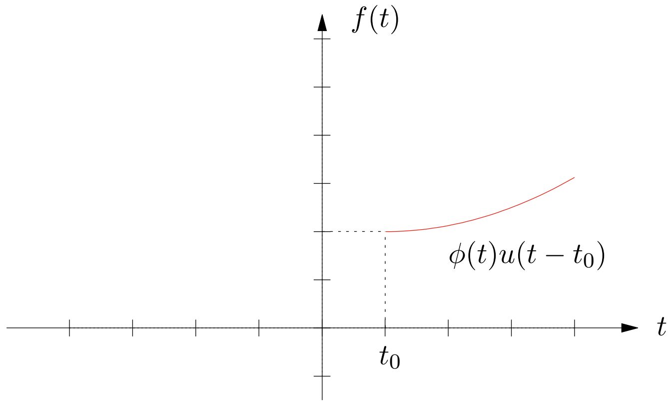
$$k\,u(t-t_0)=\begin{cases}0 & t<t_0\\ k & t>t_0\end{cases}$$



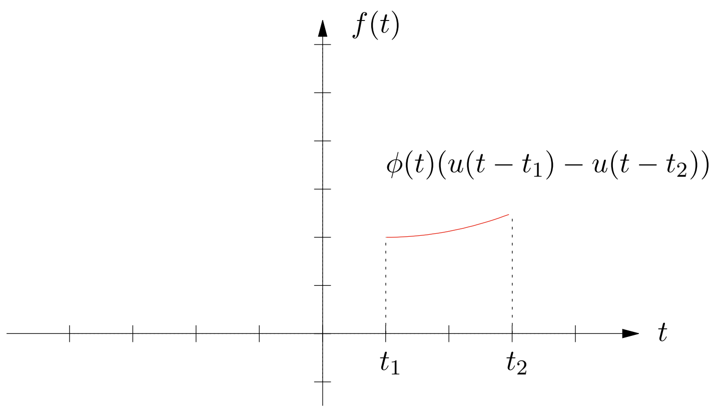
Si: $\phi(t)$ es una función de prueba:



$$\phi(t)u(t-t_0)=\begin{cases}0 & t<t_0\\ \phi(t) & t>t_0\end{cases}$$

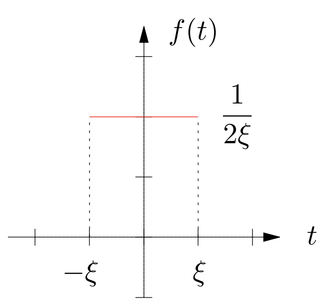


$$\phi(t)(u(t-t_1)-u(t-t_2)) = \begin{cases} 0 & t \notin [t_1, t_2] \\ \phi(t) & t \in [t_1, t_2] \end{cases}$$

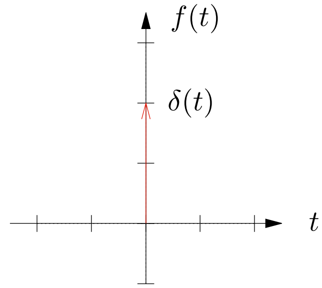


3.5. La función impulso

Pulso rectangular de área igual a 1.



Si $\xi \to 0$, entonces $\frac{1}{2\xi} \to \infty$.



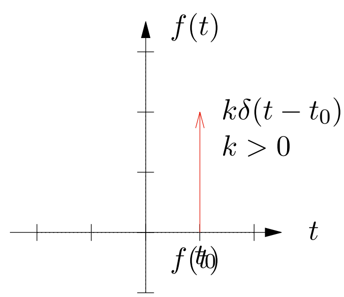
$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

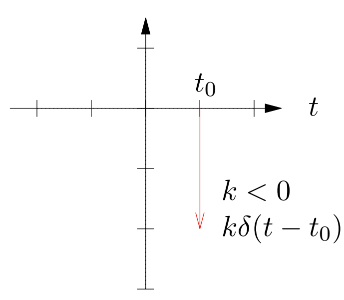
Tal que:

$$\int_{-\xi}^{\xi} \delta(t)\, dt = 1$$

Por tanto:

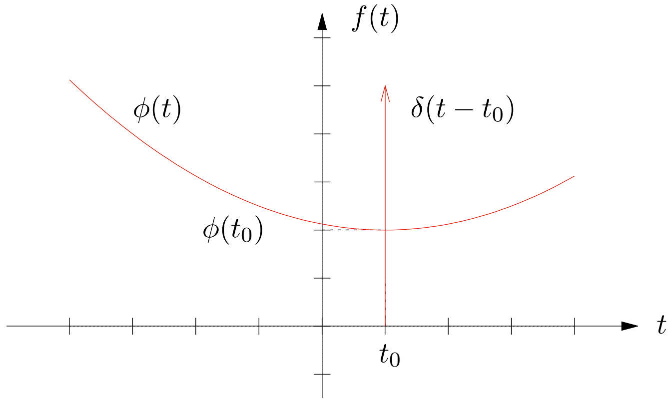
$$k\delta(t) = \begin{cases} 0 & t \neq 0 \\ \pm\infty & t = 0 \end{cases}$$





Si: $\phi(t)$ es una función de prueba:

$$\phi(t)\delta(t-t_0) = \phi(t_0)\delta(t-t_0)$$



Para $t \neq 0$

$$\phi(t)\delta(t-t_0) = 0$$

Para $t = 0$

$$\phi(t)\delta(t-t_0) = \phi(t_0)\delta(t-t_0)$$

3.5.1. Propiedades de la función impulso

Propiedad 1

$$\int_a^b \delta(t-t_0)\,dt = \begin{cases} 1 & t_0 \in [a,b] \\ 0 & t_0 \notin [a,b] \end{cases}$$

En general:

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$$

Propiedad 2

$$\int_a^b \phi(t)\,\delta(t-t_0)\,dt = \begin{cases} \phi(t_0) & t_0 \in [a,b] \\ 0 & t_0 \notin [a,b] \end{cases}$$

En general:

$$\int_{-\infty}^{\infty} \phi(t_0)\,\delta(t-t_0)\,dt = \phi(t_0)$$

Prueba:

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(t)\delta(t-t_0)\,dt &= \int_{-\infty}^{\infty} \phi(t_0)\delta(t-t_0)\,dt \\ &= \phi(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)\,dt \\ &= \phi(t_0)\end{aligned}$$

Propiedad 3

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(at)\,dt = \frac{1}{|a|}\int_{-\infty}^{\infty} \phi\left(\frac{t}{a}\right)\delta(t)\,dt = \frac{\phi(0)}{|a|}; a \neq 0$$

Prueba:

Realizando un cambio de variable:

$$\tau = at$$

$$d\tau = a\,dt$$

Para $a > 0$:

$$\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)\,\frac{d\tau}{a} = \frac{1}{a}\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)d\tau$$

Para $a < 0$:

$$\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)\,\frac{d\tau}{a} = -\frac{1}{a}\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)d\tau$$

Como:

$$|a| = \begin{cases} -a & a < 0 \\ a & a > 0 \end{cases}$$

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(t)\,\delta(at)\,dt &= \frac{1}{|a|}\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)\,d\tau \\ &= \frac{1}{|a|}\phi\left(\frac{0}{a}\right) \\ &= \frac{\phi(0)}{|a|}\end{aligned}$$

Propiedad 4

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

En particular:

$$\delta(-t) = \delta(t)$$

Por tanto $\delta(t)$ es una función **par**.

Prueba:

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(t)\,\delta(at)\,dt &= \frac{1}{|a|}\phi(0) \\ &= \frac{1}{|a|}\int_{-\infty}^{\infty} \phi(t)\,\delta(t)\,dt\end{aligned}$$

$$\phi(t)\,\delta(at) = \frac{1}{|a|}\phi(t)\,\delta(t)$$

$$\delta(at) = \frac{1}{|a|}\,\delta(t)$$

Para $a = -1$:

$$\delta(-t) = \frac{1}{|-1|}\,\delta(t) = \delta(t)$$

Propiedad 5

$$t\,\delta(t) = 0$$

$$t^n\,\delta(t) = 0; n \in \mathbb{N}$$

Prueba:

$$\int_{-\infty}^{\infty} t^n\,\delta(t)\,dt = 0^n = 0$$

Derivando ambos miembros:

$$t^n\,\delta(t)\,dt = 0$$

3.6. Derivada de la función impulso

$$\delta'(t) = \frac{d}{dt}(\delta(t))$$

$$\int_{-\infty}^{\infty} \phi(t)\delta'(t)\, dt = -\int_{-\infty}^{\infty} \phi'(t)\delta(t)\, dt = -\phi'(0)$$

Prueba:

Realizando la integración por partes:

$$\begin{aligned} u &= \phi(t) \\ du &= \phi'(t)\, dt \\ dv &= \delta'(t-t_0)\, dt \\ v &= \delta(t-t_0) \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(t)\delta'(t-t_0)\, dt &= (\phi(t)\delta(t-t_0)\Big|_{-\infty}^{\infty}) - \int_{-\infty}^{\infty} \delta(t-t_0)\phi'(t)\, dt \\ &= 0 - \int_{-\infty}^{\infty} \delta(t-t_0)\phi'(t)\, dt \\ &= -\phi'(t_0) \end{aligned}$$

Derivadas de orden superior

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(t)\delta''(t)\, dt &= \int_{-\infty}^{\infty} \phi(t)(\delta'(t))'\, dt \\ &= -\int_{-\infty}^{\infty} \phi'(t)\delta'(t)\, dt \\ &= \int_{-\infty}^{\infty} \phi''(t)\delta(t)\, dt \\ &= \phi''(0) \end{aligned}$$

De igual manera:

$$\int_{-\infty}^{\infty} \phi(t)\delta'''(t-t_0)\, dt = -\phi'''(t_0)$$

En general:

$$\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(t-t_0)\, dt = (-1)^n\phi^{(n)}(t_0)$$

3.7. Derivada de la función escalón unitario