UNIVERSIDAD MAYOR DE SAN SIMÓN
FACULTAD DE CIENCIAS Y TECNOLOGÍA
CARRERA DE ELECTROMECÁNICA

TRANSFORMADAS E INTEGRALES

Apuntes de clase

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Índice general

Bibliografía recomendada

[1] Hwei Hsu. *Análisis de Fourier*.

[2] Serie *Schaum*. *Transformada de Laplace*.

[3] Eduardo Espinoza. *Transformada de Laplace*.

[4] Álvaro Hernando Carrasco Calvo. *Transformadas e integrales*.

Capítulo 1

Series de *Fourier*

1.1. Funciones periódicas

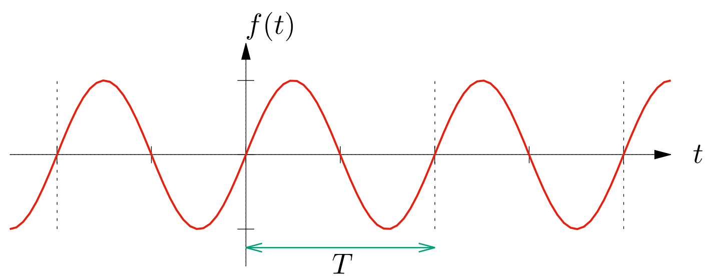


Figura 1.1: Función periódica

Una función periódica es aquella cuya gráfica se repite infinitas veces, cada cierto intervalo (**Figura 1.1**).

El menor intervalo de repetición se llama *periodo* (T).

Matemáticamente una función periódica es aquella que verifica:

$$f(t) = f(t + nT); n \in \mathbb{Z} \tag{1.1}$$

Donde T es el periodo (la menor constante que verifica la igualdad).

1.2. Propiedades de la funciones periódicas

Si $f(t) = f(t + nT)$:

Propiedad 1

$$\int_a^b f(t)\,dt = \int_{a+nT}^{b+nT} f(t)\,dt \qquad n \in \mathbb{Z} \tag{1.2}$$

Prueba:

$$\int_a^b f(t)\,dt = \int_a^b f(t+nT)\,dt$$

Cambiando la variable:

$$\tau = t + nT$$

$$d\tau = dt$$

$$\begin{aligned}\int_a^b f(t)\,dt &= \int_{a+nT}^{b+nT} f(\tau)\,d\tau \\ &= \int_{a+nT}^{b+nT} f(t)\,dt\end{aligned}$$

Puede verse gráficamente en la **Figura 1.2.**

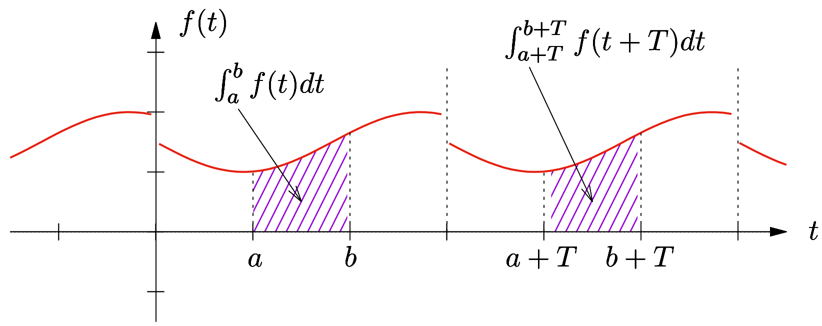


Figura 1.2: Demostración gráfica

Propiedad 2

$$\int_{a-T/2}^{a+T/2} f(t)\,dt = \int_{-T/2}^{T/2} f(t)\,dt \tag{1.3}$$

Prueba:

$$\begin{aligned}
\int_{a-T/2}^{a+T/2} f(t)\,dt &= \int_{a-T/2}^{T/2} f(t)\,dt + \int_{T/2}^{a+T/2} f(t)\,dt \\
&= \int_{a-T/2}^{T/2} f(t)\,dt + \int_{T/2-T}^{a+T/2-T} f(t)\,dt \\
&= \int_{a-T/2}^{T/2} f(t)\,dt + \int_{-T/2}^{a-T/2} f(t)\,dt \\
&= \int_{-T/2}^{T/2} f(t)\,dt
\end{aligned}$$

Puede verse gráficamente en la **Figura 1.3.**

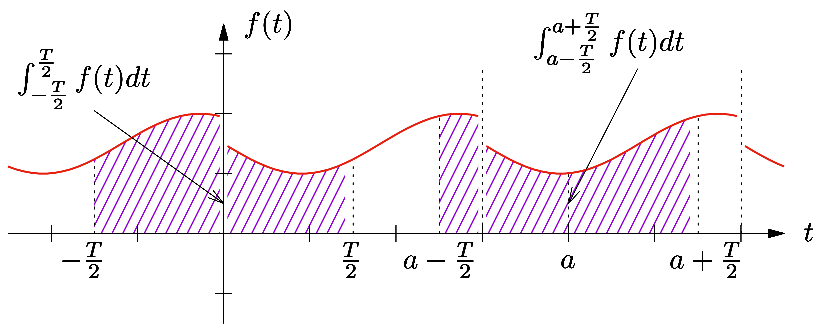


Figura 1.3: Demostración gráfica

Propiedad 3

$$\int_{0}^{T} f(t)\,dt = \int_{-T/2}^{T/2} f(t)\,dt \tag{1.4}$$

Prueba:

Si en la ecuación (1.3) $a = T/2$:

$$\begin{aligned}
\int_{-T/2}^{T/2} f(t)\,dt &= \int_{a-T/2}^{a+T/2} f(t)\,dt \\
&= \int_{T/2-T/2}^{T/2+T/2} f(t)\,dt \\
&= \int_{0}^{T} f(t)\,dt
\end{aligned}$$

Puede verse gráficamente en la **Figura 1.4**.

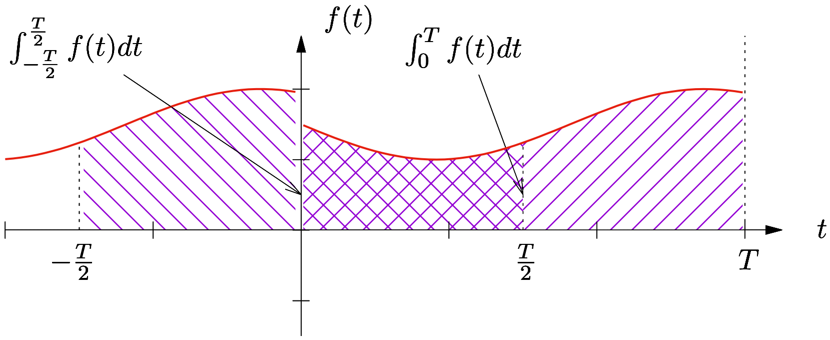


Figura 1.4: Demostración gráfica

Propiedad 4

Si $b - a = T$:

$$\int_0^T f(t)\,dt = \int_a^b f(t)\,dt \tag{1.5}$$

<u>Prueba</u>:

$$\begin{aligned}
\int_a^b f(t)\,dt &= \int_a^{a+T} f(t)\,dt \\
&= \int_a^T f(t)\,dt + \int_T^{a+T} f(t)\,dt \\
&= \int_a^T f(t)\,dt + \int_{T-T}^{a+T-T} f(t)\,dt \\
&= \int_a^T f(t)\,dt + \int_0^a f(t)\,dt \\
&= \int_0^T f(t)\,dt
\end{aligned}$$

Puede verse gráficamente en la **Figura 1.5.**

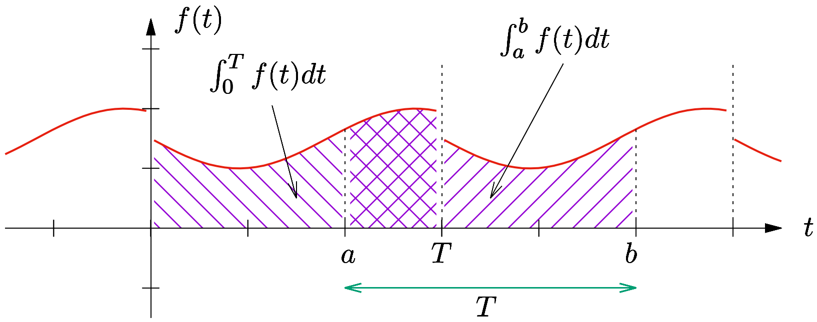


Figura 1.5: Demostración gráfica

1.2.1. Funciones seno y coseno

$$f(t) = A\ \mathrm{sen}(\omega_0\, t)$$
$$f(t) = A\ \cos(\omega_0\, t)$$

Donde:
A: Amplitud.
ω_0: Frecuencia angular.
$T = 2\pi/\omega_0$: Periodo.

Ejemplo: Hallar el periodo de la siguiente función:

$$f(t) = \mathrm{sen}(4t) + \mathrm{sen}(3t/2) + \mathrm{sen}(10t)$$

El periodo buscado debe contener un numero entero de veces a los 3 periodos hallados:

$$T = \begin{cases} a\,T_1; & a \in \mathbb{N} \\ b\,T_2; & b \in \mathbb{N} \\ c\,T_3; & c \in \mathbb{N} \end{cases}$$

$$a\,T_1 = b\,T_2 = c\,T_3$$
$$a\,\frac{2\pi}{4} = b\,\frac{2\pi}{3/2} = c\,\frac{2\pi}{10}$$
$$a\,\frac{\pi}{2} = b\,\frac{4\pi}{3} = c\,\frac{\pi}{5}; x30$$
$$15a = 40b = 6c$$
$$M.C.M.(15, 40, 6) = 120$$

$$\left.\begin{array}{r} 120 = 15a \rightarrow\ a = 8 \\ 120 = 40b \rightarrow\ b = 3 \\ 120 = 6c \rightarrow\ c = 20 \end{array}\right\} = T = 8\left(\frac{\pi}{2}\right) = 4\pi$$

Puede verse gráficamente en la **Figura 1.6.**

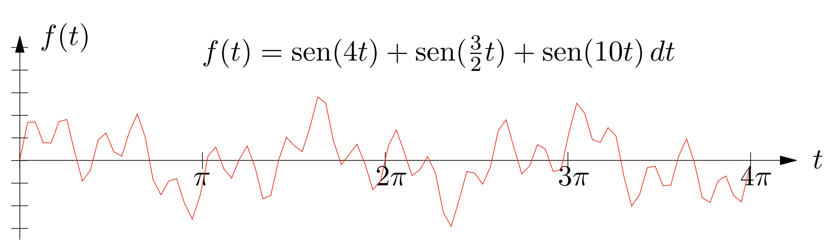


Figura 1.6: Periodo de la función

1.2.2. Propiedades ortogonales del seno y el coseno

Propiedad 1

$$\int_0^T \operatorname{sen}(n\omega_0\, t)\, dt = 0 \quad n \in \mathbb{Z} \tag{1.6}$$

Prueba:

$$\begin{aligned}
\int_0^T \operatorname{sen}(n\omega_0\, t)\, dt &= -\frac{\cos(n\omega_0\, t)}{n\omega_0}\bigg|_0^T \\
&= -\frac{\cos(n\omega_0\, T)}{n\omega_0} + \frac{\cos(0)}{n\omega_0} \\
&= -\frac{\cos(n\, 2\pi)}{n\omega_0} + \frac{\cos(0)}{n\omega_0} \\
&= -\frac{1}{n\omega_0} + \frac{1}{n\omega_0} \\
&= 0
\end{aligned}$$

$$\int_0^T \cos(n\omega_0\, t)\, dt = 0 \quad n \in \mathbb{Z} \tag{1.7}$$

Prueba:

$$\begin{aligned}
\int_0^T \cos(n\omega_0\, t)\, dt &= \frac{\operatorname{sen}(n\omega_0\, t)}{n\omega_0}\bigg|_0^T \\
&= \frac{\operatorname{sen}(n\omega_0\, T)}{n\omega_0} - \frac{\operatorname{sen}(0)}{n\omega_0} \\
&= \frac{\operatorname{sen}(n\, 2\pi)}{n\omega_0} - \frac{\operatorname{sen}(0)}{n\omega_0} \\
&= \frac{0}{n\omega_0} - \frac{0}{n\omega_0} \\
&= 0
\end{aligned}$$

Propiedad 2

$$\int_0^T \operatorname{sen}(m\omega_0\, t)\operatorname{sen}(n\omega_0\, t)\, dt = 0 \quad m, n \in \mathbb{Z} \quad m \neq n \tag{1.8}$$

Prueba:

$$\begin{aligned}\int_0^T \operatorname{sen}(m\omega_0\, t)\operatorname{sen}(n\omega_0\, t)\, dt &= \int_0^T \frac{1}{2}(\cos((m-n)\omega_0\, t) - \cos((m+n)\omega_0\, t))dt \\ &= \frac{1}{2}\left(\int_0^T \cos((m-n)\omega_0\, t)dt - \int_0^T \cos((m+n)\omega_0\, t)dt\right) \\ &= \frac{1}{2}(0-0) \\ &= 0\end{aligned}$$

$$\int_0^T \cos(m\omega_0\, t)\cos(n\omega_0\, t)\, dt = 0 \quad m, n\, \in \mathbb{Z} \quad m \neq\, n \tag{1.9}$$

Prueba:

$$\begin{aligned}\int_0^T \cos(m\omega_0\, t)\cos(n\omega_0\, t)\, dt &= \int_0^T \frac{1}{2}(\cos((m-n)\omega_0\, t) + \cos((m+n)\omega_0\, t))dt \\ &= \frac{1}{2}\left(\int_0^T \cos((m-n)\omega_0\, t)dt + \int_0^T \cos((m+n)\omega_0\, t)dt\right) \\ &= \frac{1}{2}(0+0) \\ &= 0\end{aligned}$$

$$\int_0^T \operatorname{sen}(m\omega_0\, t)\cos(n\omega_0\, t)\, dt = 0 \quad m, n\, \in \mathbb{Z} \tag{1.10}$$

Prueba:

$$\begin{aligned}\int_0^T \operatorname{sen}(m\omega_0\, t)\cos(n\omega_0\, t)\, dt &= \int_0^T \frac{1}{2}(\operatorname{sen}((m-n)\omega_0\, t) + \operatorname{sen}((m+n)\omega_0\, t))dt \\ &= \frac{1}{2}\left(\int_0^T \operatorname{sen}((m-n)\omega_0\, t)dt + \int_0^T \cos((m+n)\omega_0\, t)dt\right) \\ &= \frac{1}{2}(0+0) \\ &= 0\end{aligned}$$

Propiedad 3

$$\int_0^T \operatorname{sen}^2(n\omega_0\, t)\, dt = \frac{T}{2} \quad n\, \in \mathbb{Z} \tag{1.11}$$

Prueba:

$$
\begin{aligned}
\int_0^T \operatorname{sen}^2(n\omega_0\, t)\, dt &= \int_0^T \operatorname{sen}(n\omega_0\, t)\operatorname{sen}(n\omega_0\, t)dt \\
&= \frac{1}{2}\left(\int_0^T \cos((n-n)\omega_0\, t)dt - \int_0^T \cos((n+n)\omega_0\, t)dt\right) \\
&= \frac{1}{2}\left(\int_0^T \cos(0)dt - \int_0^T \cos((2n)\omega_0\, t)dt\right) \\
&= \frac{1}{2}\left(\int_0^T dt - \int_0^T \cos(2n\omega_0\, t)dt\right) \\
&= \frac{1}{2}(t\Big|_0^T - 0) \\
&= \frac{T}{2}
\end{aligned}
$$

$$
\int_0^T \cos^2(n\omega_0\, t)\, dt = \frac{T}{2} \qquad n \in \mathbb{Z} \tag{1.12}
$$

<u>Prueba</u>:

$$
\begin{aligned}
\int_0^T \cos^2(n\omega_0\, t)\, dt &= \int_0^T \cos(n\omega_0\, t)\cos(n\omega_0\, t)dt \\
&= \frac{1}{2}\left(\int_0^T \cos((n-n)\omega_0\, t)dt + \int_0^T \cos((n+n)\omega_0\, t)dt\right) \\
&= \frac{1}{2}\left(\int_0^T \cos(0)dt + \int_0^T \cos((2n)\omega_0\, t)dt\right) \\
&= \frac{1}{2}\left(\int_0^T dt + \int_0^T \cos(2n\omega_0\, t)dt\right) \\
&= \frac{1}{2}(t\Big|_0^T + 0) \\
&= \frac{T}{2}
\end{aligned}
$$

1.3. Series de ***Fourier***

Una función periódica que cumple ciertas condiciones puede desarrollarse mediante la serie:

$$
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0\, t) + b_n \operatorname{sen}(n\omega_0\, t)] \tag{1.13}
$$

Donde:

$\omega_0 = 2\pi/T$: Frecuencia angular de $f(t)$.

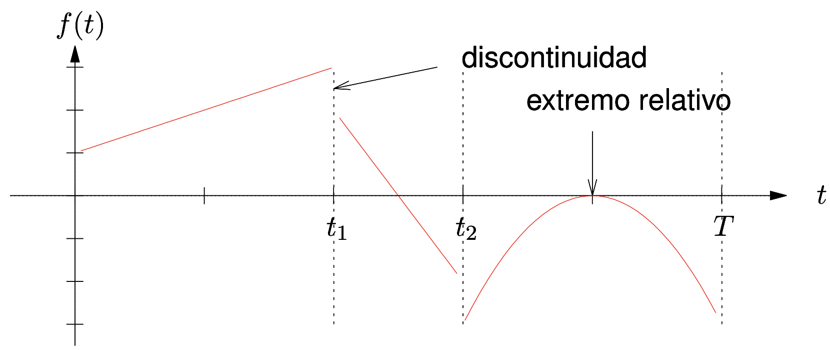
T: Periodo de $f(t)$.
$a_0; a_n; b_n$: Coeficientes de *Fourier*.
$a_0/2$: Termino constante.
$a_n \cos(n\omega_0\, t); b_n \operatorname{sen}(n\omega_0\, t)$: Armónicos, términos seno y coseno con frecuencias angulares múltiples de ω_0

$a_1 \cos(\omega_0\, t) + b_1 \operatorname{sen}(\omega_0\, t)$: Primer armónico.
$a_2 \cos(2\omega_0\, t) + b_2 \operatorname{sen}(2\omega_0\, t)$: Segundo armónico.
$a_3 \cos(3\omega_0\, t) + b_3 \operatorname{sen}(2\omega_0\, t)$: Tercer armónico.

1.3.1. Condiciones de *Dirichlet*

Para que una función periódica $f(t) = f(t + nT);\ n \in \mathbb{Z}$, se desarrolle como una serie de *Fourier* debe cumplir:

- $f(t)$ debe ser continua por tramos en 1 periodo.

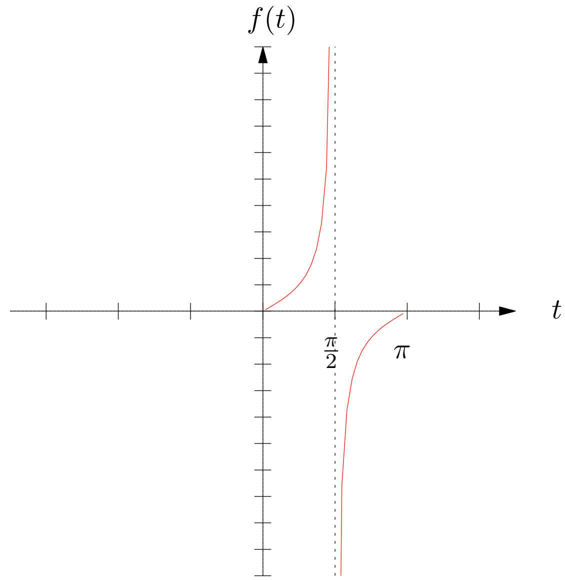


  - Debe existir un numero finito de discontinuidades (en 1 periodo).
  - Debe existir un numero finito de extremos relativos (en 1 periodo).

- La integral $\int_0^T |f(t)|\, dt < \infty$ debe ser finita.

Ejemplo:

$$f(t) = \tan(t); \quad 0 < t < \pi; \quad T = \pi$$



$$\int_0^{\pi} |\tan(t)|\, dt \to \infty$$
$$t = \frac{\pi}{2} : |\tan(t)| \to \infty$$

$\therefore$ Esta función no tiene serie de *Fourier*.

1.4. Evaluación de los coeficientes de *Fourier*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0\, t) + b_n \operatorname{sen}(n\omega_0\, t)]$$

Integrando ambas partes:

$$\begin{aligned}\int_0^T f(t)\, dt &= \int_0^T \frac{a_0}{2}\, dt + \sum_{n=1}^{\infty} \left[\int_0^T a_n \cos(n\omega_0\, t)\, dt + \int_0^T b_n \operatorname{sen}(n\omega_0\, t)\, dt\right] \\ &= \frac{a_0}{2} t \Big|_0^T \\ &= \frac{a_0}{2} T\end{aligned}$$

$$a_0 = \frac{2}{T} \int_0^T f(t)\, dt \tag{1.14}$$

Para calcular "a_n" multiplicamos por $\cos(m\omega_0\, t); m \in \mathbb{N}$ e integramos en 1 periodo.

$$\begin{aligned}\int_0^T f(t)\cos(m\omega_0\, t)\, dt &= \int_0^T \frac{a_0}{2}\cos(m\omega_0\, t)\, dt \\ &+ \sum_{n=1}^{\infty}\left[\int_0^T a_n \cos(n\omega_0\, t)\cos(m\omega_0\, t)\, dt + \int_0^T b_n \operatorname{sen}(n\omega_0\, t)\cos(m\omega_0\, t)\, dt\right] \\ &= 0 + \sum_{n=1}^{\infty}\left[\int_0^T a_n \cos(n\omega_0\, t)\cos(m\omega_0\, t)\, dt + 0\right]\end{aligned}$$

Para $n \neq m$ todos los elementos de la sumatoria serán igual a 0. Por tanto:

$$\begin{aligned}\int_0^T f(t)\cos(n\omega_0\, t)\, dt &= \int_0^T a_n \cos^2(n\omega_0\, t)\, dt \\ &= a_n \frac{T}{2}\end{aligned}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0\, t)\, dt \tag{1.15}$$

Para calcular "b_n" multiplicamos por $\operatorname{sen}(m\omega_0\, t); m \in \mathbb{N}$ e integramos en 1 periodo.

$$\begin{aligned}\int_0^T f(t)\operatorname{sen}(m\omega_0\, t)\, dt &= \int_0^T \frac{a_0}{2}\operatorname{sen}(m\omega_0\, t)\, dt \\ &+ \sum_{n=1}^{\infty}\left[\int_0^T a_n \cos(n\omega_0\, t)\operatorname{sen}(m\omega_0\, t)\, dt + \int_0^T b_n \operatorname{sen}(n\omega_0\, t)\operatorname{sen}(m\omega_0\, t)\, dt\right] \\ &= 0 + \sum_{n=1}^{\infty}\left[0 + \int_0^T b_n \operatorname{sen}(n\omega_0\, t)\operatorname{sen}(m\omega_0\, t)\, dt\right]\end{aligned}$$

Para $n \neq\ m$ todos los elementos de la sumatoria serán igual a 0. Por tanto:

$$\begin{aligned}\int_0^T f(t)\operatorname{sen}(n\omega_0\, t)\, dt &= \int_0^T b_n \operatorname{sen}^2(n\omega_0\, t)\, dt \\ &= b_n \frac{T}{2}\end{aligned}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\operatorname{sen}(n\omega_0\, t)\, dt \tag{1.16}$$

1.5. Formulas para las series de *Fourier*

$$\operatorname{sen}(\pi\, n) = 0; \quad n \in \mathbb{N}$$
$$\cos(\pi\, n) = (-1)^n; \quad n \in \mathbb{N}$$
$$\operatorname{sen}(2\pi\, n) = 0; \quad n \in \mathbb{N}$$

$$\cos(2\pi\, n) = 1; \qquad n \in \mathbb{N}$$

$$\int \operatorname{sen}(at)\, dt = -\frac{\cos(at)}{a}$$

$$\int \cos(at)\, dt = \frac{\operatorname{sen}(at)}{a}$$

$$\int t \operatorname{sen}(at)\, dt = -\frac{t}{a}\cos(at) + \frac{1}{a^2}\operatorname{sen}(at)$$

$$\int t \cos(at)\, dt = \frac{t}{a}\operatorname{sen}(at) + \frac{1}{a^2}\cos(at)$$

$$\int e^{at}\, dt = \frac{1}{a}e^{at}$$

$$\int t\, e^{at}\, dt = \frac{t}{a}\, e^{at} - \frac{1}{a^2}e^{at}$$

Capítulo 2

Análisis de formas de onda periódica

2.1. Funciones pares e impares

Una función es **par** si:

$$f(-t) = f(t) \tag{2.1}$$

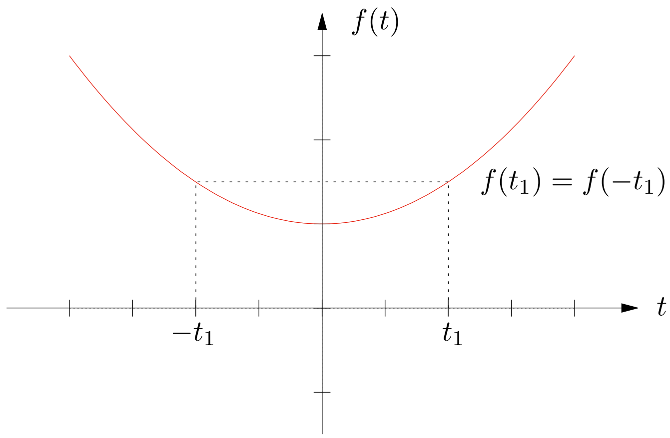


Figura 2.1: La gráfica se refleja respecto al eje central.

Ejemplo 1:

$$f(t) = t^2$$

$$f(-t) = (-t)^2 = t^2 = f(t)$$

Ejemplo 2:

$$f(t) = \cos(t)$$

$$f(t) = \cos(-t) = \cos(t) = f(t)$$

Una función es **impar** si:

$$f(-t) = -f(t) \tag{2.2}$$

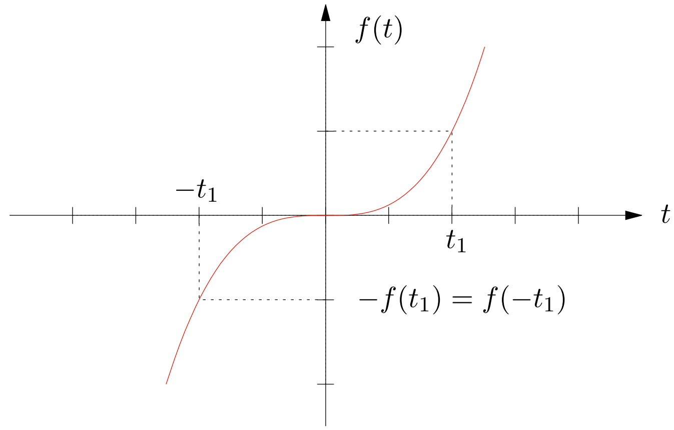


Figura 2.2: La gráfica se refleja
1ro respecto al eje central
2do respecto al eje horizontal.

Ejemplo 3:

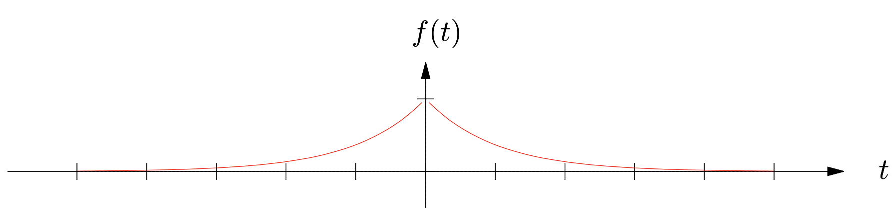
$$f(t) = t^3$$
$$f(-t) = (-t)^3 = -t^3 = -f(t)$$

Ejemplo 4:

$$f(t) = \operatorname{sen}(t)$$
$$f(t) = \operatorname{sen}(-t) = -\operatorname{sen}(t) = -f(t)$$

Ejemplo 5:

$$f(t) = \begin{cases} e^t & t < 0 \\ e^{-t} & t > 0 \end{cases}$$
$$f(-t) = \left\{\begin{matrix} e^{-t} & -t < 0 \rightarrow t > 0 \\ e^t & -t > 0 \rightarrow t < 0 \end{matrix}\right\} = f(t)$$



2.1.1. Propiedades de las funciones pares e impares

Propiedad 1

Si $f(t)$ es **par** y $g(t)$ es **par**, entonces $h(t) = f(t)g(t)$ es **par**.

Prueba:

$$\begin{cases} f(-t) = f(t) \\ g(-t) = g(t) \end{cases}$$

$$\begin{aligned} h(-t) &= f(-t)g(-t) \\ &= f(t)g(t) \\ &= h(t) \end{aligned}$$

Propiedad 2

Si $f(t)$ es **impar** y $g(t)$ es **impar**, entonces $h(t) = f(t)g(t)$ es **par.**

Prueba:

$$\begin{cases} f(-t) = -f(t) \\ g(-t) = -g(t) \end{cases}$$

Propiedad 3

Si $f(t)$ es **par** y $g(t)$ es **impar**, entonces $h(t) = f(t)g(t)$ es **impar.**

Prueba:

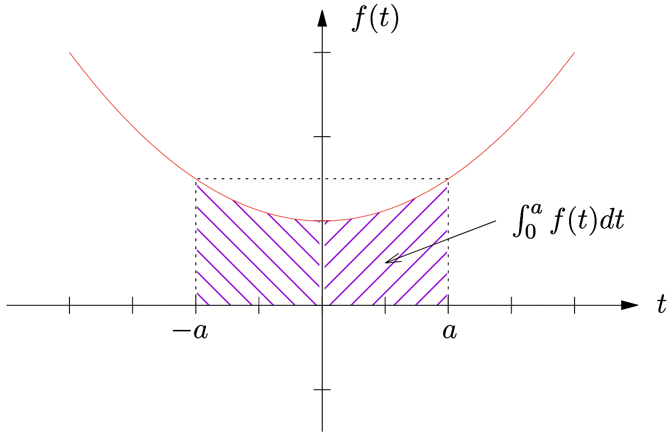
$$\begin{cases} f(-t) = f(t) \\ g(-t) = -g(t) \end{cases}$$

$$\begin{aligned} h(-t) &= f(-t)g(-t) \\ &= f(t)(-g(t)) \\ &= -f(t)g(t) \\ &= -h(t) \end{aligned}$$

Propiedad 4

Si $f(t)$ es **par**, entonces:

$$\int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt \tag{2.3}$$



Prueba:

$$\begin{aligned}
\int_{-a}^{a} f(t)\,dt &= \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt \\
&= \int_{-a}^{0} f(-t)\,dt + \int_{0}^{a} f(t)\,dt
\end{aligned}$$

$$\tau = -t$$

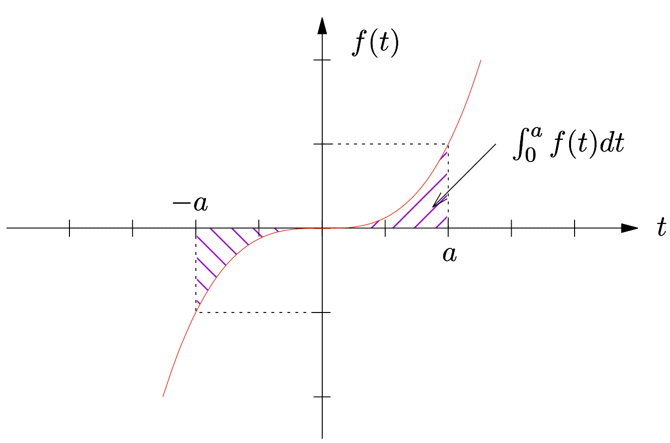
$$d\tau = -dt$$

$$\begin{aligned}
\int_{-a}^{a} f(t)\,dt &= \int_{a}^{0} f(\tau)\,(-d\tau) + \int_{0}^{a} f(t)\,dt \\
&= \int_{0}^{a} f(\tau)\,d\tau + \int_{0}^{a} f(t)\,dt \\
&= 2\int_{0}^{a} f(t)\,dt
\end{aligned}$$

Propiedad 5

Si $f(t)$ es **impar**, entonces:

$$\int_{-a}^{a} f(t)\,dt = 0 \tag{2.4}$$



Prueba:

$$\begin{aligned}
\int_{-a}^{a} f(t)\,dt &= \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt \\
&= \int_{-a}^{0} -f(-t)\,dt + \int_{0}^{a} f(t)\,dt
\end{aligned}$$

$$\tau = -t$$

$$d\tau = -dt$$

$$\begin{aligned}
\int_{-a}^{a} f(t)\,dt &= -\int_{a}^{0} f(\tau)\,(-d\tau) + \int_{0}^{a} f(t)\,dt \\
&= -\int_{0}^{a} f(\tau)\,d\tau + \int_{0}^{a} f(t)\,dt \\
&= 0
\end{aligned}$$

2.1.2. Evaluación de coeficientes de *Fourier*

Simetría par

$$\begin{aligned}
a_0 &= \frac{2}{T}\int_{0}^{T} f(t)\,dt \\
&= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt \\
&= \frac{2}{T}\left(2\int_{0}^{T/2} f(t)\,dt\right) \\
&= \frac{4}{T}\int_{0}^{T/2} f(t)\,dt
\end{aligned}$$

$$a_0 = \frac{4}{T}\int_{0}^{T/2} f(t)\,dt \tag{2.5}$$

$$\begin{aligned}
a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0\,t)\,dt \\
&= \frac{2}{T}\left(2\int_{0}^{T/2} f(t)\cos(n\omega_0\,t)\,dt\right) \\
&= \frac{4}{T}\int_{0}^{T/2} f(t)\cos(n\omega_0\,t)\,dt
\end{aligned}$$

$$a_n = \frac{4}{T}\int_{0}^{T/2} f(t)\cos(n\omega_0\,t)\,dt \tag{2.6}$$

$$\begin{aligned} b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt \\ &= 0 \end{aligned}$$

$$b_n = 0 \tag{2.7}$$

Simetría impar

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^T f(t)\,dt \\ &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt \\ &= 0 \end{aligned}$$

$$a_0 = 0 \tag{2.8}$$

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega_0\,t)\,dt \\ &= 0 \end{aligned}$$

$$a_n = 0 \tag{2.9}$$

$$\begin{aligned} b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt \\ &= \frac{2}{T}\left(2\int_0^{T/2} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt\right) \\ &= \frac{4}{T}\int_0^{T/2} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt \end{aligned}$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt \tag{2.10}$$

2.2. Simetría de media onda (S.M.O.)

$f(t)$ tiene simetría de media onda si:

$$f(t) = -f(t \pm \frac{T}{2})$$

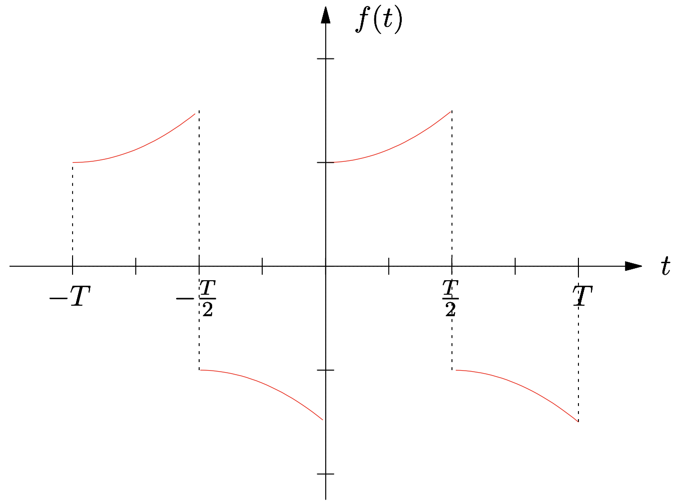


Figura 2.3: La gráfica se desplaza $1/2$ periodo
y se refleja respecto a t.

2.2.1. Evaluación de coeficientes de *Fourier*

$$
\begin{aligned}
a_0 &= \frac{2}{T}\int_0^T f(t)\,dt \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\,dt + \int_{T/2}^{T} f(t)\,dt\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\,dt - \int_{t=T/2}^{t=T} f(t-\frac{T}{2})\,dt\right]
\end{aligned}
$$

$$
\tau = t - \frac{T}{2}
$$

$$
d\tau = dt
$$

$$
\begin{aligned}
a_0 &= \frac{2}{T}\left[\int_0^{T/2} f(t)\,dt - \int_{\tau=T/2-T/2}^{\tau=T-T/2} f(\tau)\,d\tau\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\,dt - \int_0^{T/2} f(\tau)\,d\tau\right]
\end{aligned}
$$

$$
a_0 = 0 \tag{2.11}
$$

$$
\begin{aligned}
a_n &= \frac{2}{T}\int_0^T f(t)\cos(n\omega_0\,t)\,dt \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0\,t)\,dt + \int_{T/2}^{T} f(t)\cos(n\omega_0\,t)\,dt\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0\,t)\,dt - \int_{T/2}^{T} f(t-\frac{T}{2})\cos(n\omega_0\,t)\,dt\right]
\end{aligned}
$$

$$
\begin{aligned}
\tau &= t - \frac{T}{2} \\
d\tau &= dt
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0\, t)\,dt - \int_{\tau=T/2-T/2}^{\tau=T-T/2} f(\tau)\cos(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0\, t)\,dt - \int_0^{T/2} f(\tau)\cos(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right]
\end{aligned}
$$

$$
\begin{aligned}
n\omega_0(\tau+T/2) &= n\omega_0\tau + n\omega_0\frac{T}{2} \\
&= n\omega_0\tau + n\frac{2\pi}{T}\frac{T}{2} \\
&= n\omega_0\tau + n\pi
\end{aligned}
$$

$$
\begin{aligned}
\cos(n\omega_0\tau + n\pi) &= \cos(n\omega_0\tau)\cos(n\pi) - \operatorname{sen}(n\pi)\operatorname{sen}(n\omega_0\tau) \\
&= \cos(n\omega_0\tau)\cos(n\pi)
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0\, t)\,dt - \int_0^{T/2} f(\tau)\cos(n\omega_0\tau)\cos(n\pi)\,d\tau\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\cos(n\omega_0\, t)\,dt - \cos(n\pi)\int_0^{T/2} f(\tau)\cos(n\omega_0\tau)\,d\tau\right] \\
&= \frac{2}{T}(1-\cos(\pi\, n))\left(\int_0^{T/2} f(t)\cos(n\omega_0\, t)\,dt\right)
\end{aligned}
$$

$$
\cos(\pi\, n) = \begin{cases} 1 & n:\text{par} \\ -1 & n:\text{impar} \end{cases}
$$

$$
\begin{cases} n:\text{par} & a_n = 0 \\ n:\text{impar} & a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0\, t)\,dt \end{cases} \tag{2.12}
$$

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_0^{T} f(t)\operatorname{sen}(n\omega_0\, t)\,dt \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0\, t)\,dt + \int_{T/2}^{T} f(t)\operatorname{sen}(n\omega_0\, t)\,dt\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0\, t)\,dt - \int_{T/2}^{T} f(t-\frac{T}{2})\operatorname{sen}(n\omega_0\, t)\,dt\right]
\end{aligned}
$$

$$
\begin{aligned}
\tau &= t - \frac{T}{2} \\
d\tau &= dt
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0\, t)\,dt - \int_{\tau=T/2-T/2}^{\tau=T-T/2} f(\tau)\operatorname{sen}(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right] \\
&= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0\, t)\,dt - \int_0^{T/2} f(\tau)\operatorname{sen}(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right]
\end{aligned}
$$

$$\begin{aligned} n\omega_0(\tau + T/2) &= n\omega_0\tau + n\omega_0\frac{T}{2} \\ &= n\omega_0\tau + n\frac{2\pi}{T}\frac{T}{2} \\ &= n\omega_0\tau + n\pi \end{aligned}$$

$$\begin{aligned} \operatorname{sen}(n\omega_0\tau + n\pi) &= \operatorname{sen}(n\omega_0\tau)\cos(n\pi) + \operatorname{sen}(n\pi)\cos(n\omega_0\tau) \\ &= \operatorname{sen}(n\omega_0\tau)\cos(n\pi) \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0\, t)\, dt - \int_0^{T/2} f(\tau)\operatorname{sen}(n\omega_0\tau)\cos(n\pi)\, d\tau\right] \\ &= \frac{2}{T}\left[\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0\, t)\, dt - \cos(n\pi)\int_0^{T/2} f(\tau)\operatorname{sen}(n\omega_0\tau)\, d\tau\right] \\ &= \frac{2}{T}(1-\cos(\pi\, n))\left(\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0\, t)\, dt\right) \end{aligned}$$

$$\cos(\pi\, n) = \begin{cases} 1 & n : \text{par} \\ -1 & n : \text{impar} \end{cases}$$

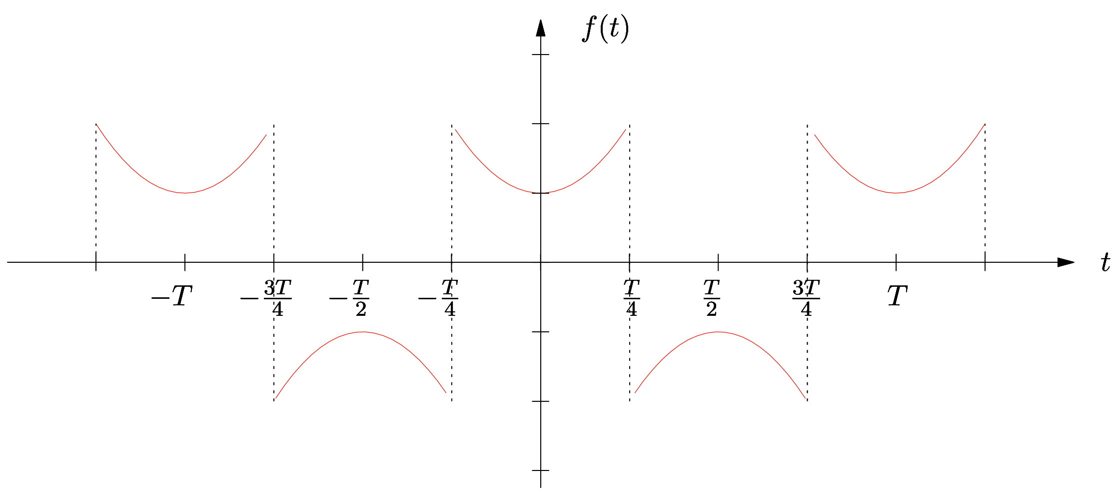
$$\begin{cases} n : \text{par} & b_n = 0 \\ n : \text{impar} & b_n = \frac{4}{T}\int_0^{T/2} f(t)\operatorname{sen}(n\omega_0\, t)\, dt \end{cases} \tag{2.13}$$

2.3. Simetría de cuarto de onda (S.C.O.)

2.3.1. Simetría de cuarto de onda par

Una función $f(t)$ tiene simetría de cuarto de onda **par** cuando:

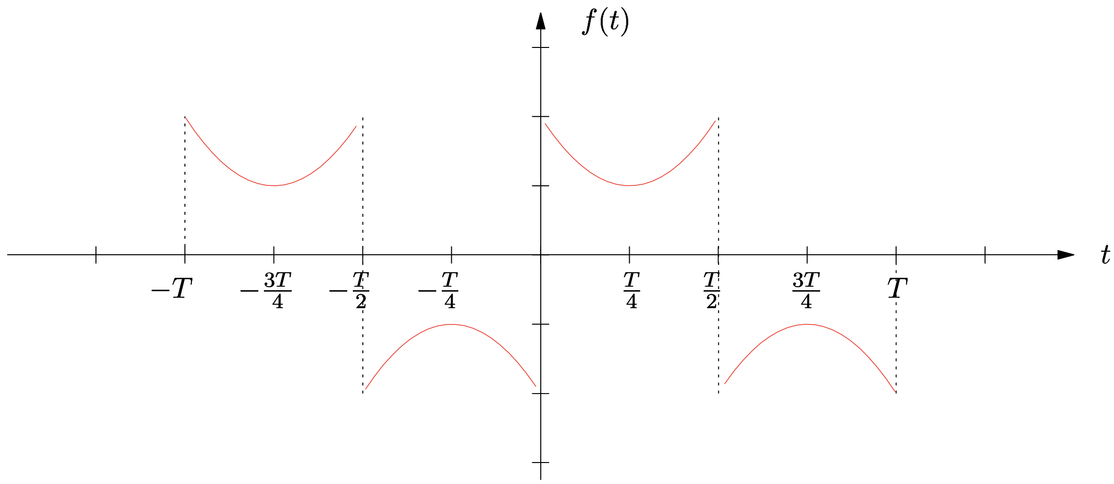
- $f(t)$ es **par**.
- $f(t)$ tiene simetría de media onda.



2.3.2. Simetría de cuarto de onda impar

Una función $f(t)$ tiene simetría de cuarto de onda **impar** cuando:

- $f(t)$ es **impar.**
- $f(t)$ tiene simetría de media onda.



2.3.3. Evaluación de coeficientes de *Fourier*

Simetría de cuarto de onda par

Como la función $f(t)$ tiene simetría de media onda:

$$a_0 = 0 \tag{2.14}$$

Como la función $f(t)$ es una función par:

$$b_n = 0 \tag{2.15}$$

Como la función $f(t)$ tiene simetría de media onda: $a_n = 0$ cuando n es par.

Para n impar:

$$\begin{aligned}
a_n &= \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0\, t)\, dt \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0\, t)\, dt + \int_{T/4}^{T/2} f(t)\cos(n\omega_0\, t)\, dt\right] \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0\, t)\, dt - \int_{T/4}^{T/2} f(t-\frac{T}{2})\cos(n\omega_0\, t)\, dt\right]
\end{aligned}$$

$$\tau = t - \frac{T}{2}$$

$$
\begin{aligned}
d\tau &= dt \\
a_n &= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0\, t)\, dt - \int_{\tau=T/4-T/2}^{\tau=T/2-T/2} f(\tau)\cos(n\omega_0(\tau+\frac{T}{2}))\, d\tau\right] \\
&= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0\, t)\, dt - \int_{-T/4}^{0} f(\tau)\cos(n\omega_0(\tau+\frac{T}{2}))\, d\tau\right]
\end{aligned}
$$

$$
\begin{aligned}
n\omega_0(\tau+T/2) &= n\omega_0\tau + n\omega_0\frac{T}{2} \\
&= n\omega_0\tau + n\frac{2\pi}{T}\frac{T}{2} \\
&= n\omega_0\tau + n\pi
\end{aligned}
$$

$$
\begin{aligned}
\cos(n\omega_0\tau + n\pi) &= \cos(n\omega_0\tau)\cos(n\pi) - \operatorname{sen}(n\pi)\operatorname{sen}(n\omega_0\tau) \\
&= \cos(n\omega_0\tau)\cos(n\pi) \\
&= -\cos(n\omega_0\tau)
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{4}{T}\left[\int_0^{T/4} f(t)\cos(n\omega_0\, t)\, dt + \int_{-T/4}^{0} f(\tau)\cos(n\omega_0\tau)\, d\tau\right] \\
&= \frac{4}{T}\left[\int_{-T/4}^{T/4} f(t)\cos(n\omega_0\, t)\, dt\right] \\
&= \frac{4}{T}\left(2\int_0^{T/4} f(t)\cos(n\omega_0\, t)\, dt\right) \\
&= \frac{8}{T}\int_0^{T/4} f(t)\cos(n\omega_0\, t)\, dt
\end{aligned}
$$

$$
\begin{cases} n : \text{par} & a_n = 0 \\ n : \text{impar} & a_n = \frac{8}{T}\int_0^{T/4} f(t)\cos(n\omega_0\, t)\, dt \end{cases} \tag{2.16}
$$

Simetría de cuarto de onda impar

Como la función $f(t)$ tiene simetría de media onda:

$$a_0 = 0 \tag{2.17}$$

Como la función $f(t)$ es una función impar:

$$a_n = 0 \tag{2.18}$$

Como la función $f(t)$ tiene simetría de media onda: $b_n = 0$ cuando n es par.

Para n impar:

$$\begin{aligned} b_n &= \frac{4}{T}\int_0^{T/2} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt \\ &= \frac{4}{T}\left[\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt + \int_{T/4}^{T/2} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt\right] \\ &= \frac{4}{T}\left[\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt - \int_{T/4}^{T/2} f(t-\frac{T}{2})\,\mathrm{sen}(n\omega_0\,t)\,dt\right] \end{aligned}$$

$$\tau = t - \frac{T}{2}$$
$$d\tau = dt$$

$$\begin{aligned} b_n &= \frac{4}{T}\left[\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt - \int_{\tau=T/4-T/2}^{\tau=T/2-T/2} f(\tau)\,\mathrm{sen}(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right] \\ &= \frac{4}{T}\left[\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt - \int_{-T/4}^{0} f(\tau)\,\mathrm{sen}(n\omega_0(\tau+\frac{T}{2}))\,d\tau\right] \end{aligned}$$

$$\begin{aligned} n\omega_0(\tau+T/2) &= n\omega_0\tau + n\omega_0\frac{T}{2} \\ &= n\omega_0\tau + n\frac{2\pi}{T}\frac{T}{2} \\ &= n\omega_0\tau + n\pi \end{aligned}$$

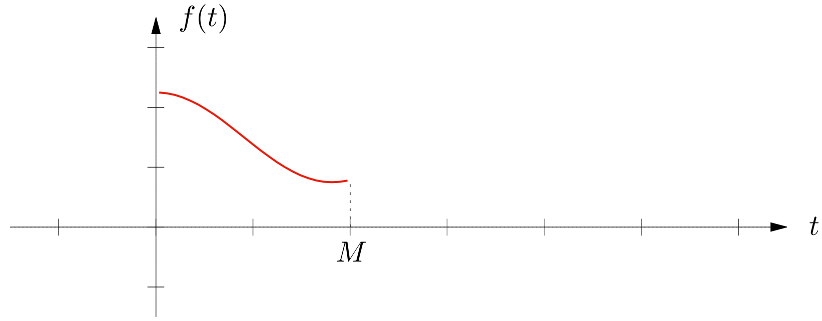
$$\begin{aligned} \mathrm{sen}(n\omega_0\tau + n\pi) &= \mathrm{sen}(n\omega_0\tau)\cos(n\pi) + \mathrm{sen}(n\pi)\cos(n\omega_0\tau) \\ &= \mathrm{sen}(n\omega_0\tau)\cos(n\pi) \\ &= -\,\mathrm{sen}(n\omega_0\tau) \end{aligned}$$

$$\begin{aligned} b_n &= \frac{4}{T}\left[\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt + \int_{-T/4}^{0} f(\tau)\,\mathrm{sen}(n\omega_0\tau)\,d\tau\right] \\ &= \frac{4}{T}\left[\int_{-T/4}^{T/4} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt\right] \\ &= \frac{4}{T}\left(2\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt\right) \\ &= \frac{8}{T}\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt \end{aligned}$$

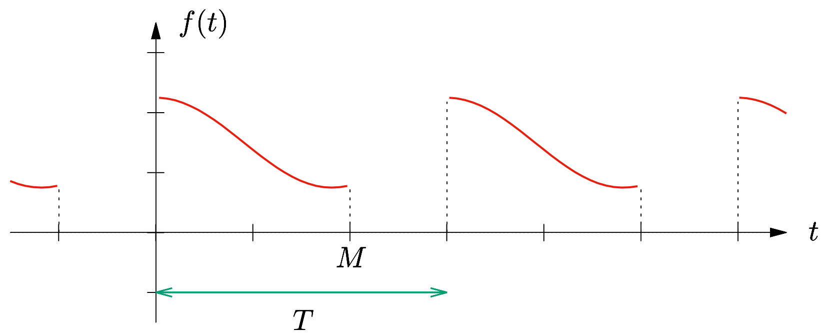
$$\begin{cases} n:\text{par} & b_n = 0 \\ n:\text{impar} & b_n = \frac{8}{T}\int_0^{T/4} f(t)\,\mathrm{sen}(n\omega_0\,t)\,dt \end{cases} \tag{2.19}$$

2.4. Expansión periódica de funciones definidas en intervalos finitos

Sea $f(t)$ una función no periódica:



$f(t)$ se convierte en periódica al repetirla un intervalo $T \geq M$.

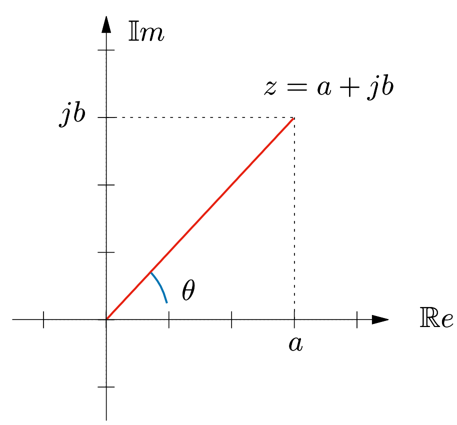


$f(t)$ puede expandirse periódicamente asignando alguna simetría conocida.

Capítulo 3

Serie compleja de *Fourier* y espectros discretos de frecuencia

3.1. Números complejos



Unidad imaginaria: $i = j = \sqrt{-1}$
Forma rectangular: $z = a + jb$
Módulo: $|z| = \sqrt{a^2 + b^2}$
Argumento: $\theta = \arctan(\frac{b}{a})$

Forma polar:

$$z = |z|\cos(\theta) + j|z|\operatorname{sen}(\theta) = |z|(\cos(\theta) + j\operatorname{sen}(\theta))$$

Formula de *Euler*:

$$e^{j\theta} = \cos(\theta) + j\operatorname{sen}(\theta) \tag{3.1}$$

Por tanto:

$$z = |z|e^{j\theta}$$

Forma exponencial o fasorial:

$$z = |z|\angle\theta$$

3.1.1. Formas complejas del seno y coseno

$$e^{j\theta} = \cos(\theta) + j\,\mathrm{sen}(\theta) \tag{3.2}$$
$$e^{-j\theta} = \cos(\theta) - j\,\mathrm{sen}(\theta) \tag{3.3}$$

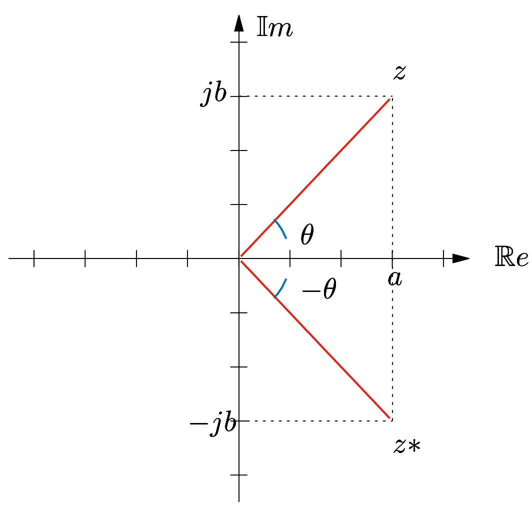
Sumando las ecuaciones 3.2 y 3.3:

$$e^{j\theta} + e^{-j\theta} = 2\cos(\theta)$$
$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{3.4}$$

Restando las ecuaciones 3.2 y 3.3:

$$e^{j\theta} - e^{-j\theta} = 2j\,\mathrm{sen}(\theta)$$
$$\mathrm{sen}(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{3.5}$$

3.1.2. Conjugado



$$z = a + jb = |z|\angle\theta$$
$$z* = a - jb = |z|\angle - \theta$$
$$(z)(z*) = |z|^2$$

3.2. Serie compleja de *Fourier*

Partiendo de la serie trigonométrica de *Fourier*:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 \, t) + b_n \operatorname{sen}(n\omega_0 \, t)]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{jn\omega_0 \, t} + e^{-jn\omega_0 \, t}}{2} \right) + b_n \left(\frac{e^{jn\omega_0 \, t} + e^{-jn\omega_0 \, t}}{2j} \right) \right]$$

$$\frac{1}{j} = -j$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \left(\frac{e^{jn\omega_0 \, t} + e^{-jn\omega_0 \, t}}{2} \right) - jb_n \left(\frac{e^{jn\omega_0 \, t} + e^{-jn\omega_0 \, t}}{2} \right) \right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2} \right) e^{jn\omega_0 \, t} + \left(\frac{a_n + jb_n}{2} \right) e^{-jn\omega_0 \, t} \right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2} \right) e^{jn\omega_0 \, t} \right] + \sum_{n=1}^{\infty} \left[\left(\frac{a_n + jb_n}{2} \right) e^{-jn\omega_0 \, t} \right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2} \right) e^{jn\omega_0 \, t} \right] + \sum_{-n=1}^{\infty} \left[\left(\frac{a_{-n} + jb_{-n}}{2} \right) e^{jn\omega_0 \, t} \right]$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{a_n - jb_n}{2} \right) e^{jn\omega_0 \, t} \right] + \sum_{n=-1}^{-\infty} \left[\left(\frac{a_n + jb_n}{2} \right) e^{jn\omega_0 \, t} \right]$$

$$f(t) = \frac{a_0}{2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{a_n - jb_n}{2} \right) e^{jn\omega_0 \, t}$$

Sean los coeficientes complejos de *Fourier*:

$$c_n = \frac{a_n - jb_n}{2}$$

$$c_0 = \frac{a_0}{2}$$

Entonces:

$$f(t) = c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n \, e^{jn\omega_0 \, t}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 \, t} \tag{3.6}$$

3.2.1. Evaluación del coeficiente complejo de *Fourier*

$$\begin{aligned}
c_n &= \frac{a_n - jb_n}{2} \\
&= \frac{1}{2}\left[\frac{2}{T}\int_0^T f(t)\cos(n\omega_0\, t)\, dt - j\frac{2}{T}\int_0^T f(t)\,\mathrm{sen}(n\omega_0\, t)\, dt\right] \\
&= \frac{1}{T}\int_0^T f(t)\left[\cos(n\omega_0\, t) - j\,\mathrm{sen}(n\omega_0\, t)\right] dt \\
&= \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0\, t}\, dt
\end{aligned}$$

$$c_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0\, t}\, dt \tag{3.7}$$

En particular:

$$c_0 = \frac{1}{T}\int_0^T f(t)\, dt \tag{3.8}$$

3.2.2. Relación entre el coeficiente complejo y los coeficientes trigonométricos

$$c_n = \frac{a_n - jbn}{2} = \frac{a_n}{2} + j\frac{-b_n}{2}$$

$$\frac{a_n}{2} = \mathbb{R}e\{c_n\}$$

$$a_n = 2\,\mathbb{R}e\{c_n\} \tag{3.9}$$

$$-\frac{b_n}{2} = \mathbb{I}m\{c_n\}$$

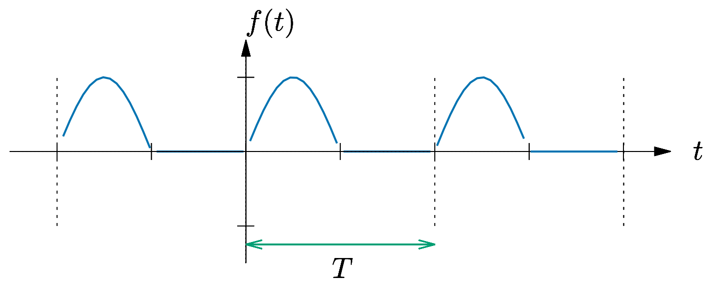
$$b_n = -2\,\mathbb{I}m\{c_n\} \tag{3.10}$$

3.3. Ondas senoidales rectificadas

3.3.1. Rectificación de media onda

$$f(t) = \begin{cases} A\,\mathrm{sen}(\omega_0\, t) & 0 < t < T/2 \\ 0 & T/2 < t < T \end{cases}$$

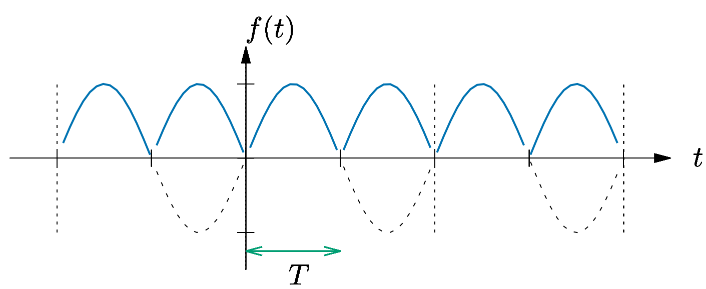
$$T = \frac{2\pi}{\omega_0}$$



El periodo de la onda rectificada es el mismo que de la onda original.

3.3.2. Rectificación de onda completa

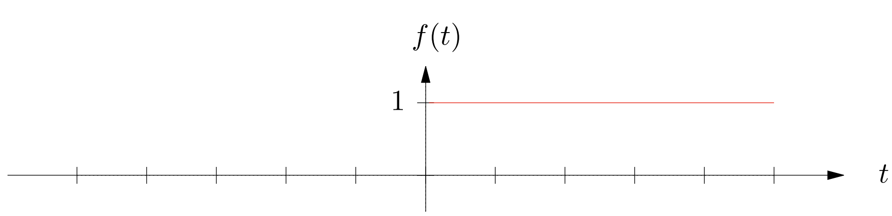
$$f(t) = A\,|\,\mathrm{sen}(\omega_0\, t)|$$



El periodo de la onda rectificada es la mitad del periodo de la onda original.

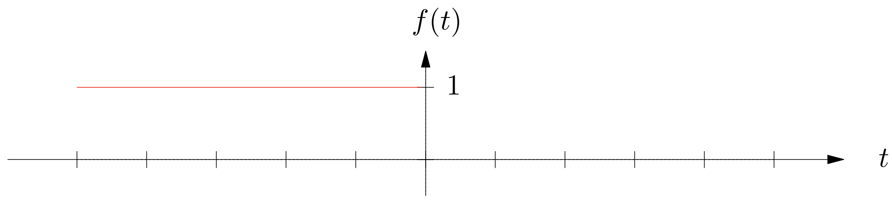
3.4. Función escalón unitario

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \tag{3.11}$$



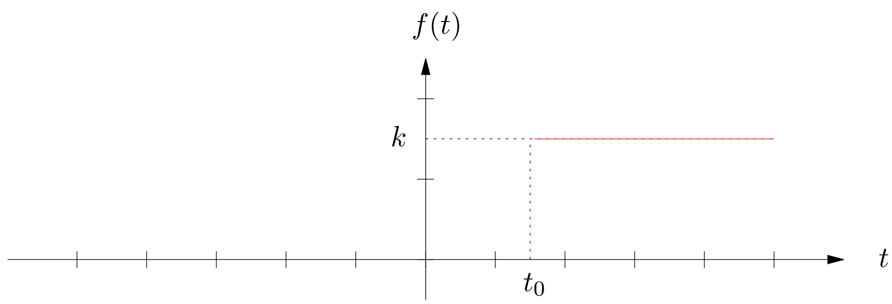
Una variante es:

$$u(-t) = \begin{cases} 1 & t < 0 \\ 0 & t > 0 \end{cases}$$

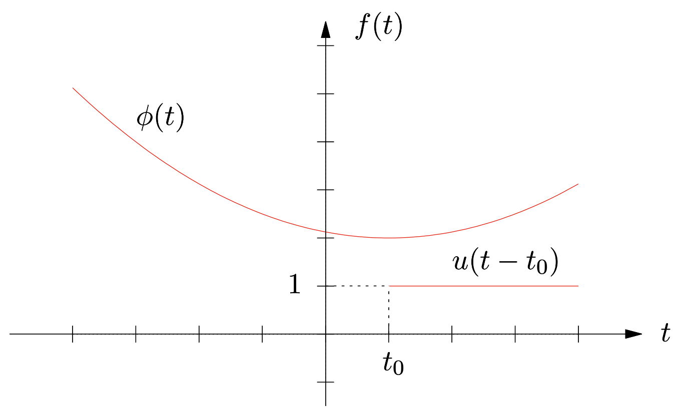


De manera general:

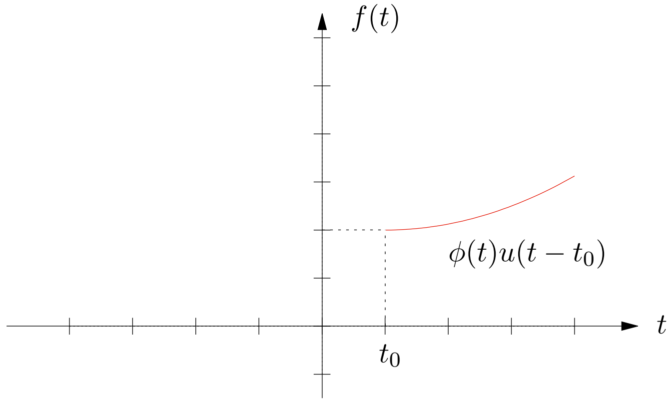
$$k\,u(t-t_0)=\begin{cases}0 & t<t_0\\ k & t>t_0\end{cases} \tag{3.12}$$



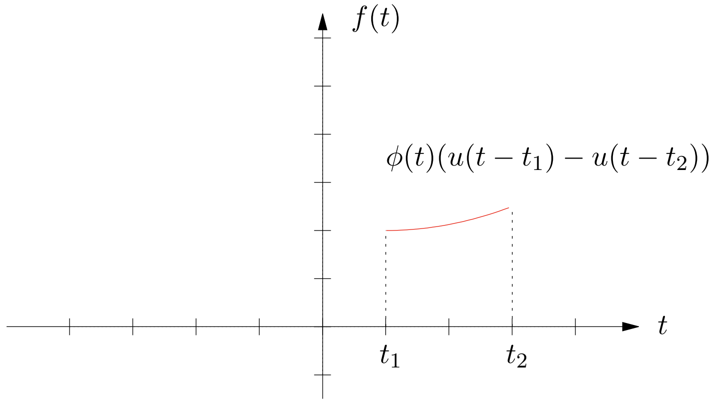
Si: $\phi(t)$ es una función de prueba:



$$\phi(t)u(t-t_0)=\begin{cases}0 & t<t_0\\ \phi(t) & t>t_0\end{cases} \tag{3.13}$$

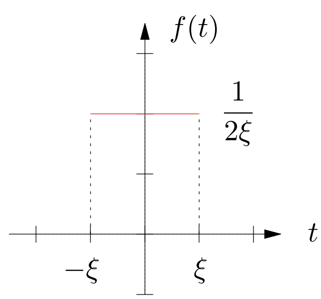


$$\phi(t)(u(t-t_1)-u(t-t_2)) = \begin{cases} 0 & t \notin [t_1, t_2] \\ \phi(t) & t \in [t_1, t_2] \end{cases}$$

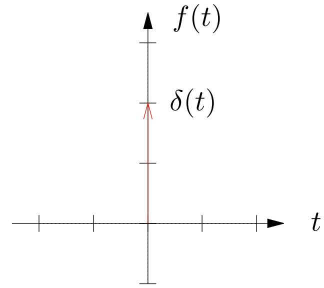


3.5. La función impulso

Pulso rectangular de área igual a 1.



Si $\xi \to 0$, entonces $\frac{1}{2\xi} \to \infty$.



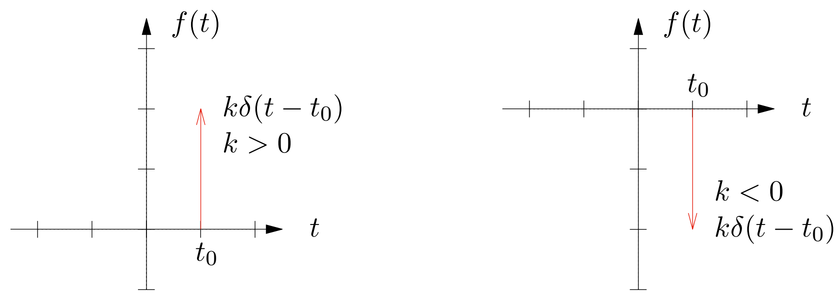
$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$

Tal que:

$$\int_{-\xi}^{\xi} \delta(t)\, dt = 1$$

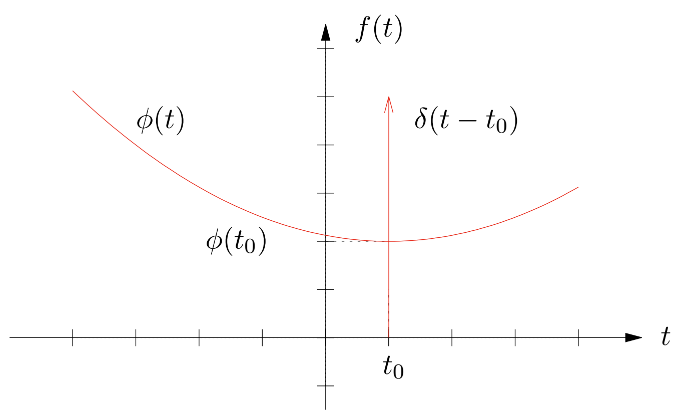
Por tanto:

$$k\delta(t) = \begin{cases} 0 & t \neq 0 \\ \pm\infty & t = 0 \end{cases} \tag{3.14}$$



Si: $\phi(t)$ es una función de prueba:

$$\phi(t)\delta(t-t_0) = \phi(t_0)\delta(t-t_0) \tag{3.15}$$



Para $t \neq 0$

$$\phi(t)\delta(t-t_0) = 0$$

Para $t = 0$

$$\phi(t)\delta(t-t_0) = \phi(t_0)\delta(t-t_0)$$

3.5.1. Propiedades de la función impulso

Propiedad 1

$$\int_a^b \delta(t-t_0)\,dt = \begin{cases} 1 & t_0 \in [a,b] \\ 0 & t_0 \notin [a,b] \end{cases}$$

En general:

$$\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1 \tag{3.16}$$

Propiedad 2

$$\int_a^b \phi(t)\,\delta(t-t_0)\,dt = \begin{cases} \phi(t_0) & t_0 \in [a,b] \\ 0 & t_0 \notin [a,b] \end{cases}$$

En general:

$$\int_{-\infty}^{\infty} \phi(t_0)\,\delta(t-t_0)\,dt = \phi(t_0) \tag{3.17}$$

<u>Prueba</u>:

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(t)\delta(t-t_0)\,dt &= \int_{-\infty}^{\infty} \phi(t_0)\delta(t-t_0)\,dt \\ &= \phi(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)\,dt \\ &= \phi(t_0) \end{aligned}$$

Propiedad 3

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(at)\,dt = \frac{1}{|a|}\int_{-\infty}^{\infty} \phi\left(\frac{t}{a}\right)\delta(t)\,dt = \frac{\phi(0)}{|a|};\, a \neq 0 \tag{3.18}$$

<u>Prueba</u>:

Realizando un cambio de variable:

$$\tau = at$$

$$d\tau = a\,dt$$

Para $a > 0$:

$$\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)\,\frac{d\tau}{a} = \frac{1}{a}\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)d\tau$$

Para $a < 0$:

$$\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)\,\frac{d\tau}{a} = -\frac{1}{a}\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)d\tau$$

Como:

$$|a| = \begin{cases} -a & a < 0 \\ a & a > 0 \end{cases}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(t)\,\delta(at)\,dt &= \frac{1}{|a|}\int_{-\infty}^{\infty} \phi\left(\frac{\tau}{a}\right)\delta(\tau)\,d\tau \\
&= \frac{1}{|a|}\phi\left(\frac{0}{a}\right) \\
&= \frac{\phi(0)}{|a|}
\end{aligned}$$

Propiedad 4

$$\delta(at) = \frac{1}{|a|}\delta(t) \tag{3.19}$$

En particular:

$$\delta(-t) = \delta(t) \tag{3.20}$$

Por tanto $\delta(t)$ es una función **par.**

<u>Prueba</u>:

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(t)\,\delta(at)\,dt &= \frac{1}{|a|}\phi(0) \\
&= \frac{1}{|a|}\int_{-\infty}^{\infty} \phi(t)\,\delta(t)\,dt
\end{aligned}$$

$$\phi(t)\,\delta(at) = \frac{1}{|a|}\phi(t)\,\delta(t)$$

$$\delta(at) = \frac{1}{|a|}\,\delta(t)$$

Para $a = -1$:

$$\delta(-t) = \frac{1}{|-1|}\,\delta(t) = \delta(t)$$

Propiedad 5

$$
\begin{aligned}
t\,\delta(t) &= 0 \\
t^n\,\delta(t) &= 0; n \in \mathbb{N}
\end{aligned}
\tag{3.21}
$$

<u>Prueba</u>:

$$\int_{-\infty}^{\infty} t^n\,\delta(t)\,dt = 0^n = 0$$

Derivando ambos miembros:

$$t^n\,\delta(t)\,dt = 0$$

3.6. Derivada de la función impulso

$$
\begin{aligned}
\delta'(t) &= \frac{d}{dt}(\delta(t)) \\
\int_{-\infty}^{\infty} \phi(t)\delta'(t)\,dt &= -\int_{-\infty}^{\infty} \phi'(t)\delta(t)\,dt = -\phi'(0)
\end{aligned}
\tag{3.22}
$$

<u>Prueba</u>:

Realizando la integración por partes:

$$u = \phi(t)$$

$$
\begin{aligned}
du &= \phi'(t)\,dt \\
dv &= \delta'(t-t_0)\,dt \\
v &= \delta(t-t_0)
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} \phi(t)\delta'(t-t_0)\,dt &= (\phi(t)\delta(t-t_0)\Big|_{-\infty}^{\infty}) - \int_{-\infty}^{\infty} \delta(t-t_0)\phi'(t)\,dt \\
&= 0 - \int_{-\infty}^{\infty} \delta(t-t_0)\phi'(t)\,dt \\
&= -\phi'(t_0)
\end{aligned}
$$

Derivadas de orden superior

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(t)\delta''(t)\,dt &= \int_{-\infty}^{\infty} \phi(t)(\delta'(t))'\,dt \\
&= -\int_{-\infty}^{\infty} \phi'(t)\delta'(t)\,dt \\
&= \int_{-\infty}^{\infty} \phi''(t)\delta(t)\,dt \\
&= \phi''(0)
\end{aligned}$$

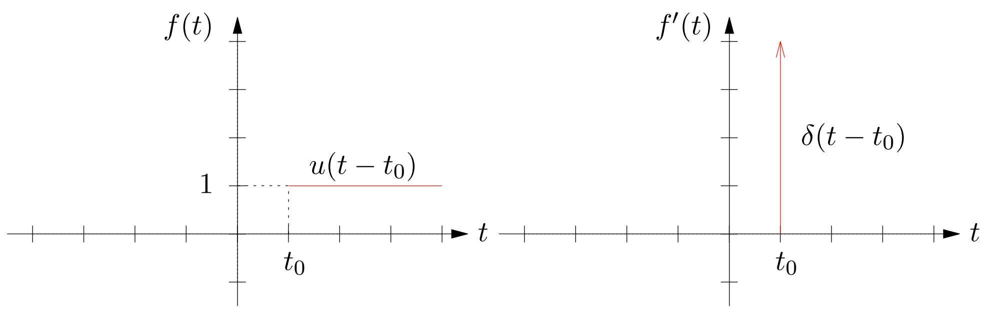
De igual manera:

$$\int_{-\infty}^{\infty} \phi(t)\delta'''(t-t_0)\,dt = -\phi'''(t_0)$$

En general:

$$\int_{-\infty}^{\infty} \phi(t)\delta^{(n)}(t-t_0)\,dt = (-1)^n\phi^{(n)}(t_0) \tag{3.23}$$

3.7. Derivada de la función escalón unitario



$$u'(t-t_0) = \delta(t-t_0) \tag{3.24}$$

<u>Prueba</u>:

$$\int_{-\infty}^{\infty} u'(t)\phi(t)\,dt$$

Realizando la integración por partes:

$$u = \phi(t)$$

$$du = \phi'(t)\,dt$$

$$dv = u'(t)\,dt$$

$$
\begin{aligned}
v &= u(t) \\
\int_{-\infty}^{\infty} \phi(t)u'(t)\,dt &= (\phi(t)u(t)\Big|_{-\infty}^{\infty}) - \int_{-\infty}^{\infty} u(t)\phi'(t)\,dt \\
&= -\int_{-\infty}^{\infty} u(t)\phi'(t)\,dt \\
&= -\int_{0}^{\infty} 1\,\phi'(t)\,dt \\
&= -\phi(t)\Big|_{0}^{\infty} \\
&= -\phi(\infty) + \phi(0)
\end{aligned}
$$

Asumiendo que $\phi(\pm\infty) = 0$:

$$\int_{-\infty}^{\infty} \phi(t)u'(t)\,dt = \phi(0)$$

Sabiendo que:

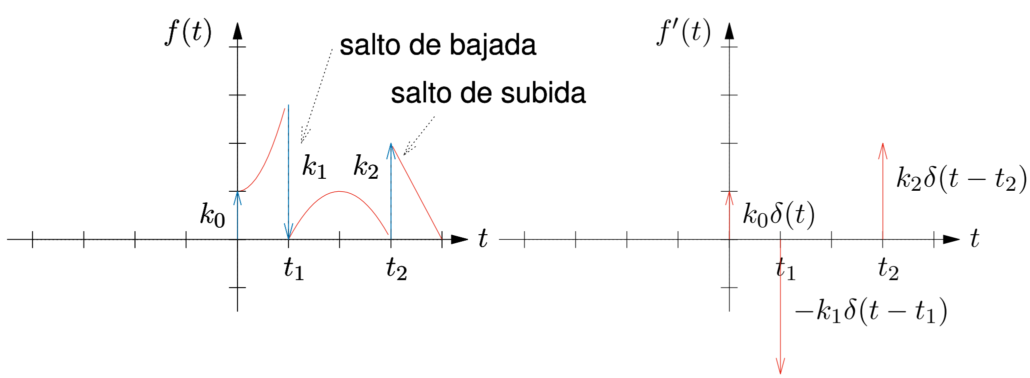
$$\int_{-\infty}^{\infty} \phi(t)\delta(t)\,dt = \phi(0)$$

Por tanto:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \phi(t)u'(t)\,dt &= \int_{-\infty}^{\infty} \phi(t)\delta(t)\,dt \\
\phi(t)u'(t) &= \phi(t)\delta(t) \\
u'(t) &= \delta(t)
\end{aligned}
$$

3.8. Derivada de una función con discontinuidades de salto

Las derivadas de los saltos de subida y bajada, van a originar impulsos hacia arriba y hacia abajo, respectivamente.



3.9. Series de *Fourier* por el método de diferenciación

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 \, t} \qquad c_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 \, t}\, dt$$

$$f'(t) = \sum_{n=-\infty}^{\infty} jn\omega_0 \, c_n \, e^{jn\omega_0 \, t} \qquad \gamma_n' = \frac{1}{T}\int_0^T f'(t)\, e^{-jn\omega_0 \, t}\, dt \tag{3.25}$$

$$f''(t) = \sum_{n=-\infty}^{\infty} (jn\omega_0)^2 \, c_n \, e^{jn\omega_0 \, t} \qquad \gamma_n'' = \frac{1}{T}\int_0^T f''(t)\, e^{-jn\omega_0 \, t}\, dt \tag{3.26}$$

$$f^{(k)}(t) = \sum_{n=-\infty}^{\infty} (jn\omega_0)^k \, c_n \, e^{jn\omega_0 \, t} \qquad \gamma_n^{(k)} = \frac{1}{T}\int_0^T f^{(k)}(t)\, e^{-jn\omega_0 \, t}\, dt \tag{3.27}$$

- Se deriva $f(t)$ hasta anularla y calcular para cada derivada: $\gamma_n^{(k)}$
- Las derivadas de $f(t)$ solo van a tomar en cuenta los impulsos obtenidos de los saltos previos.
- El coeficiente complejo c_n se obtendrá de la forma:

$$c_n = c_n' + c_n'' + \cdots + c_n^{(k)} \tag{3.28}$$

Donde:

$$c_n' = \frac{\gamma_n'}{jn\omega_0} \tag{3.29}$$

$$c_n'' = \frac{\gamma_n''}{(jn\omega_0)^2} \tag{3.30}$$

$$c_n^{(k)} = \frac{\gamma_n^{(k)}}{(jn\omega_0)^k} \tag{3.31}$$

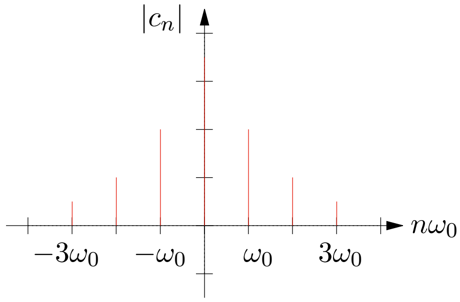
3.10. Espectros de frecuencia discreta

Los espectros de frecuencia serán gráficas discretas de modulo y argumento del coeficiente complejo de *Fourier* en función de múltiplos de la frecuencia: ω_0.

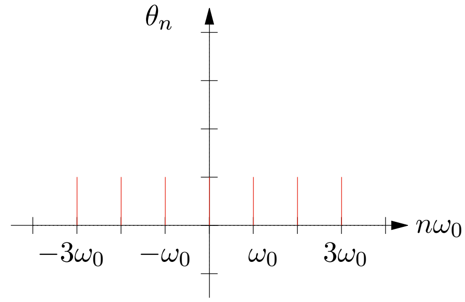
$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 \, t}$$

$$c_n = A_n + jB_n$$

$$\left.\begin{aligned} |c_n| &= \sqrt{A_n^2 + B_n^2} \\ \theta_n &= \arctan\left(\frac{B_n}{A_n}\right) \end{aligned}\right\} \text{funciones discretas de } n\omega_0 \quad n \in \mathbb{Z}$$



Espectro de amplitud o módulo



Espectro de fase o argumento

3.11. Teorema de la multiplicación

Dadas dos funciones periódicas con el mismo periodo T: $f_1(t)$ y $f_2(t)$.

Donde: $c_1 n$ y $c_2 n$, son los respectivos coeficientes complejos de *Fourier*.

$$\frac{1}{T}\int_0^T f_1(t) f_2(t)\,dt = \sum_{n=-\infty}^{\infty} c_1(n)\,c_2(-n) = \sum_{n=-\infty}^{\infty} c_1(-n)\,c_2(n) \tag{3.32}$$

<u>Prueba</u>:

$$\begin{aligned}
\frac{1}{T}\int_0^T f_1(t) f_2(t)\,dt &= \frac{1}{T}\int_0^T \left[\sum_{n=-\infty}^{\infty} c_1(n)\,e^{jn\omega_0\,t}\right] f_2(t)\,dt \\
&= \sum_{n=-\infty}^{\infty} c_1(n)\,\frac{1}{T}\int_0^T f_2(t)\,e^{jn\omega_0\,t}\,dt \\
&= \sum_{n=-\infty}^{\infty} c_1(n) c_2(-n)
\end{aligned}$$

3.12. Teorema de *Parseval*

Sea: $f_1(t) = f_2(t) = f(t)$, con coeficientes de *Fourier*: $c_1(n) = c_2(n) = c_n$:

Partiendo del teorema de multiplicación:

$$\begin{aligned}\frac{1}{T}\int_0^T f^2(t)\,dt &= \sum_{n=-\infty}^{\infty} c_{(n)}\,c_{(-n)}\\ &= \sum_{n=-\infty}^{\infty} c_{(n)}\,c^*_{(n)}\\ &= \sum_{n=-\infty}^{\infty} |c_n|^2\\ &= c_0^2 + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} |c_n|^2\end{aligned}$$

$$c_n = \frac{a_n - jb_n}{2}$$

$$|c_n|^2 = \frac{a_n^2 + b_n^2}{4}$$

$$\begin{aligned}\frac{1}{T}\int_0^T f^2(t)\,dt &= c_0^2 + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{a_n^2 + b_n^2}{4}\\ &= c_0^2 + \frac{1}{4}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) + \frac{1}{4}\sum_{n=-1}^{-\infty}(a_n^2 + b_n^2)\end{aligned}$$

Cambiando n por $-n$:

$$\frac{1}{T}\int_0^T f^2(t)\,dt = c_0^2 + \frac{1}{4}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) + \frac{1}{4}\sum_{-n=-1}^{-\infty}(a_{(-n)}^2 + b_{(-n)}^2)$$

Sabiendo que:

$$a_n^2 = a_{(-n)}^2$$

$$(-b_n)^2 = b_n^2$$

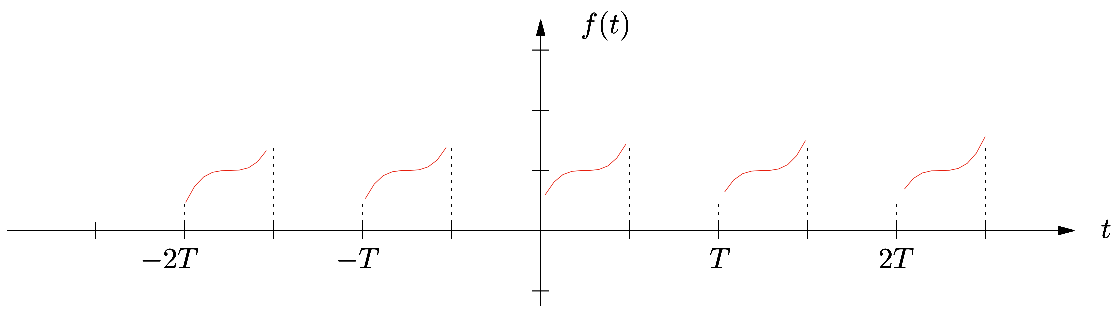
$$\begin{aligned}\frac{1}{T}\int_0^T f^2(t)\,dt &= c_0^2 + \frac{1}{4}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) + \frac{1}{4}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\\ &= c_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\end{aligned}$$

$$\frac{1}{T}\int_0^T f^2(t)\,dt = c_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{3.33}$$

Capítulo 4

Transformada de *Fourier*

4.1. Integrales de *Fourier*



Cuando $f(t)$ es periódica tiene la siguiente representación (forma compleja):

$$f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{jn\omega_0\, t}$$

Donde:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-jn\omega_0\, t}\, dt$$

Si $T \to \infty$, entonces $\omega_0 \to 0$, y la función deja de ser periódica.

$$f(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} f(\tau)\, e^{-jn\omega_0\, \tau}\, d\tau\right] e^{jn\omega_0\, t}$$

$$\frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$f(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-T/2}^{T/2} f(\tau)\, e^{-jn\omega_0\, \tau}\, d\tau\right] e^{jn\omega_0\, t}\, \omega_0$$

$$\omega_0 = d\omega$$

$$n\omega_0 = \omega$$
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega\tau}\, d\tau\right] e^{j\omega\, t}\, d\omega \tag{4.1}$$

Se define como transformada de *Fourier*:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}\, dt \tag{4.2}$$

Se define como transformada inversa de *Fourier*:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega\, t}\, d\omega \tag{4.3}$$

4.2. Transformada de *Fourier*

Dada una función $f(t)$ se define la transformada de *Fourier*:

$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}\, dt \tag{4.4}$$

Donde: $f(t)$ es una función no periódica.

La transformada de *Fourier* convierte una función del dominio del tiempo (t) al dominio de la frecuencia (ω) la cual será una variable continua.

Para $f(t) \in \mathbb{R}$:

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}\, dt \\
&= \int_{-\infty}^{\infty} f(t)[\cos(\omega\, t) - j\,\mathrm{sen}(\omega\, t)]\, dt \\
&= \int_{-\infty}^{\infty} f(t)\cos(\omega\, t)\, dt - j\int_{-\infty}^{\infty} f(t)\,\mathrm{sen}(\omega\, t)\, dt \\
&= R(\omega) + jX(\omega)
\end{aligned}$$

$$R(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega\, t)\, dt$$

$$X(\omega) = -\int_{-\infty}^{\infty} f(t)\,\mathrm{sen}(\omega\, t)\, dt$$

La parte real $R(\omega)$ es una función par:

$$R(-\omega) = R(\omega)$$

La parte imaginaria $X(\omega)$ es una función impar:

$$X(-\omega) = -X(\omega)$$

Por tanto:

Si $f(t)$ es par, entonces $X(\omega) = 0$.

Si $f(t)$ es impar, entonces $R(\omega) = 0$.

4.3. Espectros continuos de frecuencia

$$F(\omega) = R(\omega) + jX(\omega)$$

Modulo:

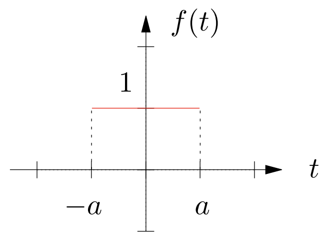
$$|F(\omega)| = \sqrt{R^2(\omega) + X^2(\omega)}$$

Argumento:

$$\theta(\omega) = \arctan\left(\frac{X(\omega)}{R(\omega)}\right)$$

Ejemplo 1: Hallar $F(\omega)$ de la función y graficar los espectros.

$$f(t) = u(t+a) - u(t-a)$$



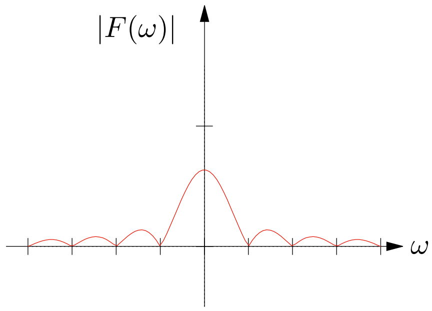
$$\begin{aligned} F(\omega) &= \int_{-a}^{a} 1\, e^{-j\omega\, t}\, dt \\ &= \left.\frac{e^{-j\omega\, t}}{-j\omega}\right|_{-a}^{a} \\ &= \frac{e^{-ja\omega} - e^{ja\omega}}{-j\omega} \\ &= \frac{-2j\,\mathrm{sen}(a\omega)}{-j\omega} \\ &= \frac{2\,\mathrm{sen}(a\omega)}{\omega} \end{aligned}$$

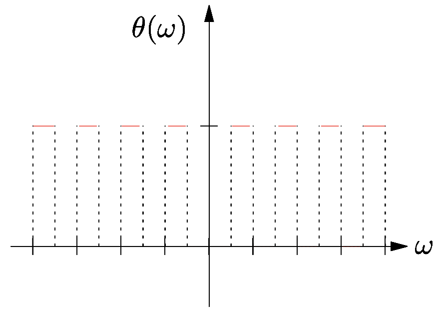
$$\mathcal{F}\{u(t+a) - u(t-a)\} = \frac{2\,\mathrm{sen}(a\omega)}{\omega} \tag{4.5}$$

$$|F(\omega)| = \left|\frac{2\,\mathrm{sen}(a\omega)}{\omega}\right|$$

Para $\omega = 0$, existe una discontinuidad:

$$\lim_{\omega \to 0}\left(\frac{2\,\mathrm{sen}(a\omega)}{\omega}\right) = 2a$$





4.4. Propiedades de la transformada de *Fourier*

4.4.1. Linealidad

$$\mathcal{F}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(\omega) + a_2 F_2(\omega) \tag{4.6}$$

Donde:

$$F_1(\omega) = \mathcal{F}\{f_1(t)\}$$
$$F_2(\omega) = \mathcal{F}\{f_2(t)\}$$

4.4.2. Cambio de escala

Si $\mathcal{F}\{f(t)\} = F(\omega)$:

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) \tag{4.7}$$

Prueba:

$$\mathcal{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)\, e^{-j\omega t}\, dt$$

Realizando un cambio de variable:

$$\tau = at$$
$$d\tau = a\, dt$$

Para $a > 0$:

$$\begin{aligned}\mathcal{F}\{f(at)\} &= \int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega \frac{\tau}{a}}\, \frac{d\tau}{a} \\ &= \frac{1}{a}\int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega \frac{\tau}{a}} d\tau \\ &= \frac{1}{a} F\left(\frac{\omega}{a}\right)\end{aligned}$$

Para $a < 0$:

$$\begin{aligned}
\mathcal{F}\{f(at)\} &= \int_{\infty}^{-\infty} f(\tau)\, e^{-j\omega\frac{\tau}{a}}\, \frac{d\tau}{a} \\
&= -\frac{1}{a}\int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega\frac{\tau}{a}}\, d\tau \\
&= -\frac{1}{a} F\left(\frac{\omega}{a}\right)
\end{aligned}$$

4.4.3. Desplazamiento en ω

Si $\mathcal{F}\{f(t)\} = F(\omega)$:

$$\mathcal{F}\{f(t)\, e^{jat}\} = F(\omega - a) \tag{4.8}$$

Prueba:

$$\begin{aligned}
\mathcal{F}\{f(t)\, e^{jat}\} &= \int_{-\infty}^{\infty} f(t)\, e^{jat}\, e^{-j\omega\, t}\, dt \\
&= \int_{-\infty}^{\infty} f(t)\, e^{-j(\omega - a)t}\, dt \\
&= F(\omega - a)
\end{aligned}$$

4.4.4. Desplazamiento en t

Si $\mathcal{F}\{f(t)\} = F(\omega)$:

$$\mathcal{F}\{f(t-a)\} = F(\omega)\, e^{-ja\omega} \tag{4.9}$$

Prueba:

$$\mathcal{F}\{f(t-a)\} = \int_{-\infty}^{\infty} f(t-a)\, e^{-j\omega\, t}\, dt$$

Realizando un cambio de variable:

$$\tau = t - a$$

$$d\tau = dt$$

$$\begin{aligned}
\mathcal{F}\{f(t-a)\} &= \int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega(\tau + a)}\, dt \\
&= e^{-ja\omega}\int_{-\infty}^{\infty} f(\tau)\, e^{-j\omega\tau}\, d\tau \\
&= e^{-ja\omega} F(\omega)
\end{aligned}$$

4.4.5. Simetría

Si $\mathcal{F}\{f(t)\} = F(\omega)$:

$$\mathcal{F}\{F(t)\} = 2\pi\, f(-\omega) \tag{4.10}$$

Prueba:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega\, t}\, d\omega$$

Reemplazando:

$$t \to -\omega$$
$$\omega \to\ t$$

$$\begin{aligned} f(-\omega) = &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)\, e^{jt(-\omega)}\, dt \\ &= \frac{1}{2\pi}\mathcal{F}\{F(t)\} \\ 2\pi\, f(-\omega) &= \mathcal{F}\{F(t)\} \end{aligned}$$

Ejemplo 2: Hallar $\mathcal{F}\left\{\dfrac{\operatorname{sen}(at)}{t}\right\}$ Sabiendo que:

$$\mathcal{F}\{u(t+a) - u(t-a)\} = \frac{2\operatorname{sen}(a\omega)}{\omega}$$

$$\mathcal{F}\left\{\frac{2\operatorname{sen}(at)}{t}\right\} = 2\pi(u(t+a) - u(t-a))$$

$$\mathcal{F}\left\{\frac{\operatorname{sen}(at)}{t}\right\} = \pi(u(t+a) - u(t-a)) \tag{4.11}$$

4.4.6. Multiplicación por t

Si $\mathcal{F}\{f(t)\} = F(\omega)$:

$$\mathcal{F}\{t\, f(t)\} = j\frac{dF(\omega)}{d\omega}$$

En general:

$$\mathcal{F}\{t^n\, f(t)\} = j^n\frac{d^{(n)}F(\omega)}{d\omega^n}; \quad n \in \mathbb{N} \tag{4.12}$$

Prueba:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}\, dt$$

$$\begin{aligned}
\frac{dF(\omega)}{d\omega} &= \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}(-jt)\, dt \\
&= -j \int_{-\infty}^{\infty} t\, f(t)\, e^{-j\omega\, t}\, dt \\
&= -j\, \mathcal{F}\{t\, f(t)\}
\end{aligned}$$

$$j\frac{dF(\omega)}{d\omega} = \mathcal{F}\{t\, f(t)\}$$

$$\begin{aligned}
\mathcal{F}\{t^2\, f(t)\} &= \mathcal{F}\{t\, t\, f(t)\} \\
&= j\frac{d}{d\omega}\left(\mathcal{F}\{t\, f(t)\}\right) \\
&= j\frac{d}{d\omega}\left(j\frac{dF(\omega)}{d\omega}\right) \\
&= j^2\frac{d^2F(\omega)}{d\omega^2}
\end{aligned}$$

$$\mathcal{F}\{t^n\, f(t)\} = j^n\frac{d^{(n)}\, F(\omega)}{d\omega^n}$$

Ejemplo 3: Hallar $\mathcal{F}\{t^n\, e^{-at}\, u(t)\}; \quad n \in \mathbb{N}$

Para $n = 1$:

$$\begin{aligned}
\mathcal{F}\{t\, e^{-at}\, u(t)\} &= j\frac{d}{d\omega}\left(\frac{1}{a + j\omega}\right) \\
&= (j)(-1)(a + j\omega)^{-2}(j) \\
&= \frac{1}{(a + j\omega)^2}
\end{aligned}$$

Para $n = 2$:

$$\begin{aligned}
\mathcal{F}\{t^2\, e^{-at}\, u(t)\} &= j\frac{d}{d\omega}\left(\frac{1}{(a + j\omega)^2}\right) \\
&= (j)(-2)(a + j\omega)^{-3}(j) \\
&= \frac{2}{(a + j\omega)^3}
\end{aligned}$$

Para $n = 3$:

$$\begin{aligned}
\mathcal{F}\{t^3\, e^{-at}\, u(t)\} &= j\frac{d}{d\omega}\left(\frac{2}{(a + j\omega)^3}\right) \\
&= (j)(2)(-3)(a + j\omega)^{-4}(j) \\
&= \frac{6}{(a + j\omega)^4}
\end{aligned}$$

Por tanto:

$$\mathcal{F}\{t^n\, e^{-at}\, u(t)\} = \frac{n!}{(a + j\omega)^{n+1}} \tag{4.13}$$

4.4.7. Transformada de *Fourier* de una derivada

Si $\mathcal{F}\{f(t)\} = F(\omega)$:

$$\mathcal{F}\{f'(t)\} = j\omega\, F(\omega)$$

En general:

$$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n\, F(\omega) \quad n \in \mathbb{N} \tag{4.14}$$

<u>Prueba</u>:

$$\mathcal{F}\{f'(t)\} = \int_{\infty}^{\infty} f'(t)\, e^{-j\omega\, t}\, dt$$

Realizando la integración por partes:

$$u = e^{-j\omega\, t}$$

$$du = -j\omega\, e^{-j\omega\, t}\, dt$$

$$dv = f'(t)\, dt$$

$$v = f(t)$$

$$\mathcal{F}\{f'(t)\} = \left(f(t)\, e^{-j\omega\, t} \Big|_{-\infty}^{\infty} \right) - \int_{-\infty}^{\infty} f(t)\, (-j\omega\, e^{-j\omega\, t})\, dt$$

Asumiendo $f(\pm\infty) = 0$:

$$\begin{aligned} \mathcal{F}\{f'(t)\} &= j\omega \int_{-\infty}^{\infty} f(t)\, e^{-j\omega\, t}\, dt \\ &= j\omega\, F(\omega) \end{aligned}$$

Para la segunda derivada:

$$\begin{aligned} \mathcal{F}\{f''(t)\} &= \mathcal{F}\{(f'(t))'\} \\ &= j\omega\, \mathcal{F}\{f'(t)\} \\ &= j\omega(j\omega\, F(\omega)) \\ &= (j\omega)^2\, F(\omega) \end{aligned}$$

Por tanto:

$$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n\, F(\omega)$$

4.5. Transformadas de *Fourier* especiales

4.5.1. $e^{-at}\, u(t); \quad a > 0$

$$\mathcal{F}\{e^{-at}\, u(t)\}; a > 0$$

$$
\begin{aligned}
\mathcal{F}\{e^{-at}\,u(t)\} &= \int_0^{\infty} e^{-at}\,e^{-j\omega\,t}\,dt \\
&= \int_0^{\infty} e^{-(a+j\omega)t}\,dt \\
&= \left.\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right|_0^{\infty} \\
&= \frac{0-1}{-(a+j\omega)} \\
&= \frac{1}{a+j\omega}
\end{aligned}
$$

$$
\mathcal{F}\{e^{-at}\,u(t)\} = \frac{1}{a+j\omega} \tag{4.15}
$$

4.5.2. $e^{at}\,u(-t); \quad a>0$

$$
\mathcal{F}\{e^{at}\,u(-t)\}; a>0
$$

$$
\begin{aligned}
\mathcal{F}\{e^{at}\,u(t)\} &= \int_{-\infty}^{0} e^{at}\,e^{-j\omega\,t}\,dt \\
&= \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt \\
&= \left.\frac{e^{(a-j\omega)t}}{a-j\omega}\right|_{-\infty}^{0} \\
&= \frac{1}{a-j\omega}
\end{aligned}
$$

$$
\mathcal{F}\{e^{at}\,u(-t)\} = \frac{1}{a-j\omega} \tag{4.16}
$$

4.5.3. $e^{-a|t|}$

$$
\mathcal{F}\{e^{-a|t|}\}
$$

$$
\begin{aligned}
\mathcal{F}\{e^{-a|t|}\} &= \mathcal{F}\{e^{at}u(-t) + e^{-at}u(t)\} \\
&= \frac{1}{a-j\omega} + \frac{1}{a+j\omega} \\
&= \frac{a+j\omega+a-j\omega}{(a-j\omega)(a+j\omega)} \\
&= \frac{2a}{a^2+\omega^2}
\end{aligned}
$$

$$
\mathcal{F}\{e^{-a|t|}\} = \frac{2a}{a^2+\omega^2} \tag{4.17}
$$

4.5.4. $\delta(t - t_0)$

$$\begin{aligned}
&\mathcal{F}\{\delta(t-t_0)\} \\
\mathcal{F}\{\delta(t-t_0)\} &= \int_{-\infty}^{0} \delta(t-t_0)\, e^{-j\omega\, t}\, dt \\
&= e^{-j\omega\, t_0}
\end{aligned}$$

$$\mathcal{F}\{\delta(t-t_0)\} = e^{-j\omega\, t_0} \tag{4.18}$$

En particular:

$$\mathcal{F}\{\delta(t)\} = 1$$

4.5.5. e^{jat}

Sabiendo:

$$\mathcal{F}\{\delta(t-a)\} = e^{-ja\omega}$$

Aplicando la propiedad de simetría:

$$\mathcal{F}\{e^{-jat}\} = 2\pi\delta(-\omega - a)$$

$$\begin{aligned}
\mathcal{F}\{e^{jat}\} &= 2\pi\delta(-\omega + a) \\
&= 2\pi\delta(-(\omega - a)) \\
&= 2\pi\delta(\omega - a)
\end{aligned}$$

$$\mathcal{F}\{e^{jat}\} = 2\pi\delta(\omega - a) \tag{4.19}$$

En particular:

$$\mathcal{F}\{1\} = 2\pi\delta(\omega)$$

$$\mathcal{F}\{k\} = 2\pi\, k\delta(\omega)$$

4.5.6. $\operatorname{sen}(at)$

$$\begin{aligned}
\mathcal{F}\{\operatorname{sen}(at)\} &= \mathcal{F}\left\{\frac{e^{jat} - e^{-jat}}{2j}\right\} \\
&= \frac{1}{2j}\left(2\pi\delta(\omega - a) - 2\pi\delta(\omega + a)\right) \\
&= -j\pi(\delta(\omega - a) - \delta(\omega + a))
\end{aligned}$$

$$\mathcal{F}\{\operatorname{sen}(at)\} = j\pi(\delta(\omega + a) + \delta(\omega - a)) \tag{4.20}$$

4.5.7. $\cos(at)$

$$\begin{aligned}
\mathcal{F}\{\cos(at)\} &= \mathcal{F}\left\{\frac{e^{jat}+e^{-jat}}{2}\right\} \\
&= \frac{1}{2}\left(2\pi\delta(\omega-a)+2\pi\delta(\omega+a)\right) \\
&= \pi(\delta(\omega+a)+\delta(\omega-a))
\end{aligned}$$

$$\mathcal{F}\{\cos(at)\} = \pi(\delta(\omega+a)+\delta(\omega-a)) \tag{4.21}$$

4.5.8. $u(t)$

$$u(t)+u(-t)=1$$
$$\mathcal{F}\{u(t)+u(-t)\}=\mathcal{F}\{1\}$$

Considerando:

$$\mathcal{F}\{u(t)\}=F(\omega)$$
$$\mathcal{F}\{u(-t)\}=F(-\omega)$$

Por tanto:

$$F(\omega)+F(-\omega)=2\pi\delta(\omega)$$

Se asume que la transformada de *Fourier* de la función escalón tendrá un termino impulsivo:

$$F(\omega)=\beta(\omega)+k\delta(\omega)$$
$$F(-\omega)=\beta(-\omega)+k\delta(\omega)$$
$$F(\omega)+F(-\omega)=\beta(\omega)+\beta(-\omega)+2k\delta(\omega)$$

Reemplazando:

$$\beta(\omega)+\beta(-\omega)+2k\delta(\omega)=2\pi\delta(\omega)$$

Por tanto:

$$k=\pi$$

Resultando:

$$F(\omega)=\beta(\omega)+\pi\delta(\omega)$$

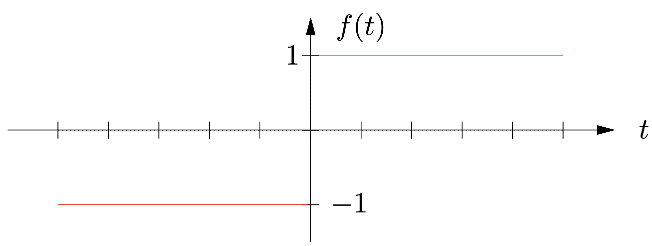
Por otro lado, se sabe que:

$$u'(t)=\delta(t)$$
$$\mathcal{F}\{u'(t)\}=\mathcal{F}\{\delta(t)\}$$
$$j\omega\mathcal{F}\{u(t)\}=1$$

$$j\omega(\beta(\omega) + \pi\delta(\omega)) = 1$$
$$j\omega\beta(\omega) + j\pi\omega\delta(\omega) = 1$$
$$j\omega\beta(\omega) = 1$$
$$\beta(\omega) = \frac{1}{j\omega}$$
$$\mathcal{F}\{u(t)\} = \frac{1}{j\omega} + \pi\delta(\omega) \tag{4.22}$$

4.6. La función signo

$$\operatorname{sgn}(t) = \begin{cases} -1 & t < 0 \\ 1 & t > 0 \end{cases} \tag{4.23}$$



Esta función puede representarse también como:

$$\operatorname{sgn}(t) = \frac{|t|}{t}$$
$$\operatorname{sgn}(t) = -1 + 2u(t)$$

A partir de la definición de la función de valor absoluto:

$$|t| = \begin{cases} -t & t < 0 \\ t & t > 0 \end{cases}$$

Es posible calcular la derivada del valor absoluto:

$$|t|' = \begin{cases} -1 & t < 0 \\ 1 & t > 0 \end{cases}$$

Por tanto:

$$|t|' = \operatorname{sgn}(t) \tag{4.24}$$

Cuya derivada es:

$$\operatorname{sgn}'(t) = 2\delta(t) \tag{4.25}$$

Calculando su transformada de *Fourier*:

$$\begin{aligned}
\mathcal{F}\{\mathrm{sgn}(t)\} &= \mathcal{F}\{-1+2u(t)\}\\
&= -2\pi\delta(\omega)+2\left(\frac{1}{j\omega}+\pi\delta(\omega)\right)\\
&= -2\pi\delta(\omega)+2\left(\frac{1}{j\omega}\right)+2\pi\delta(\omega)\\
&= \frac{2}{j\omega}
\end{aligned}$$

$$\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{2}{j\omega} \tag{4.26}$$

4.6.1. Transformada de *Fourier* de $|t|$

$$\mathcal{F}\{|t|'\} = \mathcal{F}\{\mathrm{sgn}(t)\}$$

$$j\omega\mathcal{F}\{|t|\} = \frac{2}{j\omega}$$

$$\mathcal{F}\{|t|\} = -\frac{2}{j\omega^2} \tag{4.27}$$

4.6.2. Transformada de *Fourier* de $1/t$

$$\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{2}{j\omega}$$

Por simetría:

$$\mathcal{F}\left\{\frac{2}{jt}\right\} = 2\pi\ \mathrm{sgn}(-\omega)$$

$$\mathcal{F}\left\{\frac{1}{t}\right\} = -j\pi\ \mathrm{sgn}(-\omega) \tag{4.28}$$

Calculando la segunda derivada:

$$\left(\frac{1}{t}\right)' = -\frac{1}{t^2}$$

$$\mathcal{F}\left\{-\frac{1}{t^2}\right\} = j\omega(-j\pi\ \mathrm{sgn}(\omega))$$

$$\mathcal{F}\left\{-\frac{1}{t^2}\right\} = j^2\pi\omega\ \mathrm{sgn}(\omega)$$

Calculando la derivada n-ésima:

$$\left(\frac{1}{t}\right)'' = \frac{2}{t^3}$$

$$\left(\frac{1}{t}\right)''' = -\frac{6}{t^4}$$

$$\left(\frac{1}{t}\right)^{(n)} = (-1)^k \frac{k!}{t^{k+1}}$$

$$\mathcal{F}\{f^{(k)}(t)\} = (-1)^k\, k!\, \mathcal{F}\left\{\frac{1}{t^{k+1}}\right\}$$

$$(j\omega)^k(-j\pi\ \mathrm{sgn}(\omega)) = (-1)^k\, k!\, \mathcal{F}\left\{\frac{1}{t^{k+1}}\right\}$$

$$-j^{k+1}\pi\omega^k\ \mathrm{sgn}(\omega) = (-1)^k\, k!\, \mathcal{F}\left\{\frac{1}{t^{k+1}}\right\}$$

$$-j^n\pi\omega^{n-1}\ \mathrm{sgn}(\omega) = (-1)^{n-1}\,(n-1)!\,\mathcal{F}\left\{\frac{1}{t^n}\right\}$$

$$-j^n\pi\omega^{n-1}\ \mathrm{sgn}(\omega) = (-1)^n(-1)^{-1}(n-1)!\mathcal{F}\left\{\frac{1}{t^n}\right\}$$

$$\mathcal{F}\left\{\frac{1}{t^n}\right\} = \frac{j^n\pi\omega^{n-1}\ \mathrm{sgn}(\omega)}{(-1)^n(n-1)!} \tag{4.29}$$

4.7. Tabla de transformadas de *Fourier* conocidas

	$f(t)$	$F(\omega) = \mathcal{F}\{f(t)\}$
1	$u(t+a) - u(t-a)$	$\dfrac{2\,\mathrm{sen}(a\omega)}{\omega}$
2	$\dfrac{\mathrm{sen}(at)}{t}$	$\pi[u(\omega+a) - u(\omega-a)]$
3	$e^{-at}\,u(t) \quad a>0$	$\dfrac{1}{a+j\omega}$
4	$e^{at}\,u(-t) \quad a>0$	$\dfrac{1}{a-j\omega}$
5	$e^{-a\lvert t\rvert} \quad a>0$	$\dfrac{2a}{a^2+\omega^2}$
6	$\dfrac{1}{t^2+a^2}$	$\dfrac{\pi}{a}e^{-a\lvert\omega\rvert}$
7	$\delta(t-a)$	$e^{-ja\omega}$
8	e^{jat}	$2\pi\delta(\omega-a)$
9	k	$2\pi\,k\delta(\omega)$
10	$\mathrm{sen}(at)$	$j\omega[\delta(\omega+a) - \delta(\omega-a)]$
11	$\cos(at)$	$\pi[\delta(\omega+a) + \delta(\omega-a)]$
12	$t^n\,e^{-at}\,u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}} \quad n\in\mathbb{N}$
13	$u(t)$	$\dfrac{1}{j\omega} + \pi\delta(\omega)$
14	$\mathrm{sgn}(t)$	$\dfrac{2}{j\omega}$
15	$\lvert t\rvert$	$-\dfrac{2}{\omega^2}$
16	$\dfrac{1}{t}$	$-j\pi\;\mathrm{sgn}(\omega)$
17	$\dfrac{1}{t^n}$	$\dfrac{j^n\pi\omega^{n-1}\;\mathrm{sgn}(\omega)}{(-1)^n(n-1)!}$

Capítulo 5

Transformada inversa de *Fourier*

$$\mathcal{F}\{f(t)\} = F(\omega) \rightarrow \mathcal{F}^{-1}\{F(\omega)\} = f(t) \tag{5.1}$$

Con la transformada inversa de *Fourier* se regresa del dominio de "ω" al dominio de "t".

5.1. Tabla de transformadas de *Fourier* inversas

	$F(\omega)$	$f(t) = \mathcal{F}^{-1}\{F(\omega)\}$
1	$\dfrac{1}{a + j\omega}$	$e^{-at}\,u(t) \quad a > 0$
2	$\dfrac{1}{a - j\omega}$	$e^{at}\,u(-t) \quad a > 0$
3	$\dfrac{2a}{a^2 + \omega^2}$	$e^{-a\lvert t\rvert} \quad a > 0$
4	$\frac{1}{\omega}\,\mathrm{sen}(a\omega)$	$\frac{1}{2}[u(t+a) - u(t-a)]$
5	k	$k\delta(t)$
6	$\dfrac{1}{\omega}$	$\frac{1}{2}j\ \mathrm{sgn}(t)$

5.2. Propiedades de la transformada inversa de *Fourier*

5.2.1. Linealidad

$$\mathcal{F}^{-1}\{a_1\,F_1(\omega) + a_2\,F_2(\omega)\} = a_1\,f_1(t) + a_2\,f_2(t) \tag{5.2}$$

5.2.2. Desplazamiento en t

Si $\mathcal{F}^{-1}\{F(\omega)\} = f(t)$:

$$\mathcal{F}^{-1}\{F(\omega)\, e^{-ja\omega}\} = f(t-a) \tag{5.3}$$

5.2.3. Desplazamiento en ω

Si $\mathcal{F}^{-1}\{F(\omega)\} = f(t)$:

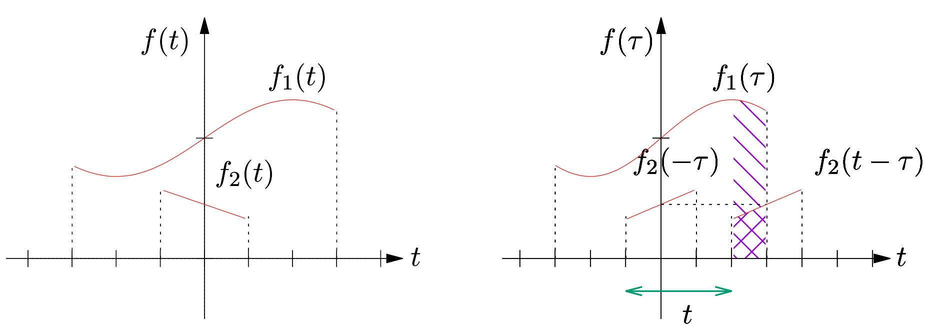
$$\mathcal{F}^{-1}\{F(\omega - a)\} = f(t)\, e^{jat} \tag{5.4}$$

5.3. Convolución

Dadas dos funciones $f_1(t)$ y $f_2(t)$ definimos la convolución mediante la siguiente integral:

$$f(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau \tag{5.5}$$

- En $f_1(t)$ se reemplaza "t" por "τ", permanece fija y no cambia.
- En $f_2(t)$ se reemplaza "t" por "$t-\tau$".La gráfica se refleja respecto al eje vertical, y luego se desplaza un "t" variable.



El intervalo de integración será el intervalo donde ambas gráficas se superponen.

5.4. Propiedades de la convolución

5.4.1. Conmutatividad

$$f_1(t) * f_2(t) = f_2(t) * f_1(t) \tag{5.6}$$

5.4.2. Asociatividad

$$f_1(t) * [f_2(t) * f_3(t)] = [f_1(t) * f_2(t)] * f_3(t) \tag{5.7}$$

5.4.3. Distributividad

$$f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t) \tag{5.8}$$

5.4.4. Función impulso

$$f_1(t) * \delta(t - t_0) = f_1(t - t_0) \tag{5.9}$$

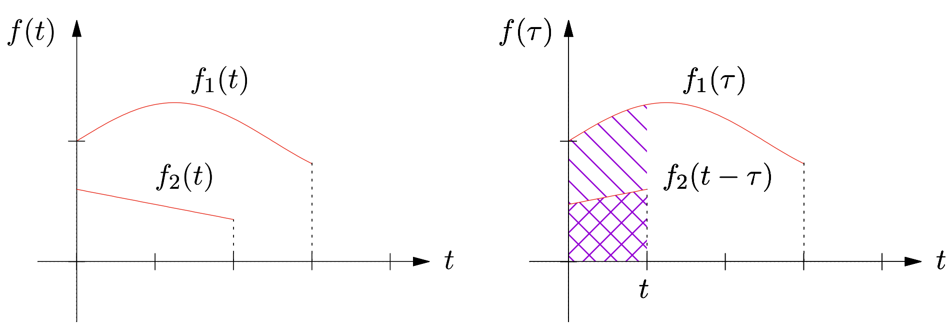
<u>Prueba</u>:

$$f_1(t) * \delta(t - t_0) = \int_{-\infty}^{\infty} f_1(t - \tau)\, \delta(\tau - t_0)\, d\tau = f_1(t - t_0)$$

5.4.5. Función escalón unitario

Si $f_1(t) = f_1(t)\, u(t)$ y $f_2(t) = f_2(t)\, u(t)$:

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau)\, f_2(t - \tau)\, d\tau \tag{5.10}$$



5.5. Transformada de *Fourier* y convolución

$$\mathcal{F}\{f_1(t) * f_2(t)\} = F_1(\omega)\, F_2(\omega) \tag{5.11}$$

Donde:

$$F_1(\omega) = \mathcal{F}\{f_1(t)\}$$

$$F_2(\omega) = \mathcal{F}\{f_2(t)\}$$

Prueba:

$$\begin{aligned}
\mathcal{F}\{f_1(t) * f_2(t)\} &= \int_{-\infty}^{\infty} \left[\int_{\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau \right] e^{-jn\omega_0 t}\, dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, e^{-jn\omega_0 t}\, d\tau\, dt \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, e^{-jn\omega_0 t}\, dt\, d\tau \\
&= \int_{-\infty}^{\infty} f_1(\tau) d\tau \int_{-\infty}^{\infty} f_2(t-\tau)\, e^{-jn\omega_0 t}\, dt
\end{aligned}$$

$$u = t - \tau \rightarrow\ t = u + \tau$$

$$du = dt$$

$$\begin{aligned}
\mathcal{F}\{f_1(t) * f_2(t)\} &= \int_{-\infty}^{\infty} f_1(\tau) d\tau \int_{-\infty}^{\infty} f_2(u)\, e^{-jn\omega_0 (u+\tau)}\, du \\
&= \int_{-\infty}^{\infty} f_1(\tau)\, e^{-jn\omega_0 \tau}\, d\tau \int_{-\infty}^{\infty} f_2(u)\, e^{-jn\omega_0 u}\, du \\
&= F_1(\omega)\, F_2(\omega)
\end{aligned}$$

5.6. Transformada inversa de *Fourier* por convolución

$$\mathcal{F}^{-1}\{F_1(\omega)\, F_2(\omega)\} = f_1(t) * f_2(t) \tag{5.12}$$

Donde:

$$f_1(t) = \mathcal{F}^{-1}\{F_1(\omega)\}$$
$$f_2(t) = \mathcal{F}^{-1}\{F_2(\omega)\}$$

5.7. Ecuaciones diferenciales ordinarias

$$\begin{aligned}
\mathcal{F}\{f'(t)\} &= j\omega\, F(\omega) \\
\mathcal{F}\{f''(t)\} &= (j\omega)^2\, F(\omega) \\
\mathcal{F}\{f^{(n)}(t)\} &= (j\omega)^n\, F(\omega)
\end{aligned}$$

Capítulo 6

Transformada de *Laplace*

Dada una función $f(t)$ definida para $t > 0$:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \tag{6.1}$$

Donde: $s = \sigma + j\omega$ es una frecuencia compleja.

La transformada de *Laplace* convierte una función del dominio de “t” al dominio de “s”.

6.1. Transformadas de funciones elementales

6.1.1. k

$$\begin{aligned}
\mathcal{L}\{k\} &= \int_0^\infty k\, e^{-st}\, dt \\
&= \left.\frac{k\, e^{-st}}{-s}\right|_0^\infty \\
&= \frac{k}{-s}(0-1)
\end{aligned}$$

$$\mathcal{L}\{k\} = \frac{k}{s} \tag{6.2}$$

6.1.2. $t^n \quad \in \mathbb{N}$

Para $n = 1$:

$$\begin{aligned}
\mathcal{L}\{t\} &= \int_0^\infty t\, e^{-st}\, dt \\
&= \left(\frac{t\, e^{-st}}{-s} \bigg|_0^\infty \right) - \left(\frac{e^{-st}}{(-s)^2} \bigg|_0^\infty \right) \\
&= \frac{-0-1}{(-s)^2} \\
&= \frac{1}{s^2}
\end{aligned}$$

Para $n = 2$:

$$\mathcal{L}\{t^2\} = \int_0^\infty t^2\, e^{-st}\, dt$$

$$u = t^2$$

$$du = 2t\, dt$$

$$dv = e^{-st}\, dt$$

$$v = \frac{e^{-st}}{-s}$$

$$\begin{aligned}
\mathcal{L}\{t^2\} &= \int_0^\infty t^2\, e^{-st}\, dt \\
&= \left(\frac{t^2\, e^{-st}}{-s} \bigg|_0^\infty \right) - \int_0^\infty \frac{e^{-st}}{-s}\, 2t\, dt \\
&= \frac{2}{s} \int_0^\infty t\, e^{-st}\, dt \\
&= \frac{2}{s} \left(\frac{1}{s^2} \right) \\
&= \frac{2}{s^3}
\end{aligned}$$

Para $n = 3$:

$$\mathcal{L}\{t^3\} = \int_0^\infty t^3\, e^{-st}\, dt$$

$$u = t^3$$

$$du = 3t^2\, dt$$

$$dv = e^{-st}\, dt$$

$$v = \frac{e^{-st}}{-s}$$

$$
\begin{aligned}
\mathcal{L}\{t^3\} &= \int_0^\infty t^3\, e^{-st}\, dt \\
&= \left(\left. \frac{t^3\, e^{-st}}{-s} \right|_0^\infty \right) - \int_0^\infty \frac{e^{-st}}{-s}\, 3t^2\, dt \\
&= \frac{3}{s} \int_0^\infty t^2\, e^{-st}\, dt \\
&= \frac{3}{s} \left(\frac{2}{s^3} \right) \\
&= \frac{6}{s^4}
\end{aligned}
$$

$$
\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \tag{6.3}
$$

6.1.3. e^{at}

$$
\begin{aligned}
\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{at}\, e^{-st}\, dt \\
&= \int_0^\infty e^{-(s-a)t}\, dt \\
&= \left(\left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^\infty \right) \\
&= \frac{0-1}{-(s-a)}
\end{aligned}
$$

$$
\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \tag{6.4}
$$

6.1.4. $\operatorname{sen}(at)$

$$
\begin{aligned}
\mathcal{L}\{\operatorname{sen}(at)\} &= \mathcal{L}\left\{ \frac{e^{jat} - e^{-jat}}{2j} \right\} \\
&= \frac{1}{2j} \left(\frac{1}{s-ja} - \frac{1}{s+ja} \right) \\
&= \frac{1}{2j} \left(\frac{s+ja-s+ja}{(s-ja)(s+ja)} \right)
\end{aligned}
$$

$$
\mathcal{L}\{\operatorname{sen}(at)\} = \frac{a}{s^2+a^2} \tag{6.5}
$$

6.1.5. $\cos(at)$

$$\begin{aligned}
\mathcal{L}\{\cos(at)\} &= \mathcal{L}\left\{\frac{e^{jat}+e^{-jat}}{2}\right\} \\
&= \frac{1}{2}\left(\frac{1}{s-ja} - \frac{1}{s+ja}\right) \\
&= \frac{1}{2}\left(\frac{s+ja+s-ja}{(s-ja)(s+ja)}\right)
\end{aligned}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2} \tag{6.6}$$

6.1.6. $\mathrm{senh}(at)$

$$\begin{aligned}
\mathcal{L}\{\mathrm{senh}(at)\} &= \mathcal{L}\left\{\frac{e^{at}-e^{-at}}{2}\right\} \\
&= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) \\
&= \frac{1}{2}\left(\frac{s+a-s+a}{(s-a)(s+a)}\right)
\end{aligned}$$

$$\mathcal{L}\{\mathrm{senh}(at)\} = \frac{a}{s^2-a^2} \tag{6.7}$$

6.1.7. $\cosh(at)$

$$\begin{aligned}
\mathcal{L}\{\cosh(at)\} &= \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\} \\
&= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) \\
&= \frac{1}{2}\left(\frac{s+a+s-a}{(s-ja)(s+ja)}\right)
\end{aligned}$$

$$\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2-a^2} \tag{6.8}$$

6.2. Propiedades de la transformada de *Laplace*

6.2.1. Linealidad

$$\mathcal{L}\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 F_1(s) + a_2 F_2(s) \tag{6.9}$$

6.2.2. Desplazamiento en s

Si $\mathcal{L}\{f(t)\} = F(s)$:

$$\mathcal{L}\{f(t)\, e^{at}\} = F(s-a) \tag{6.10}$$

6.2.3. Desplazamiento en t

Si $\mathcal{L}\{f(t)\} = F(s)$:

$$\mathcal{L}\{f(t-a)\, u(t-a)\} = F(s)\, e^{-as} \tag{6.11}$$

$f(t)$ esta definida para $t > 0$, por tanto: $f(t) = f(t)\, u(t)$.

Segunda forma:

$$\mathcal{L}\{f(t)\, u(t-a)\} = \int_a^\infty f(t)\, e^{-st}\, dt$$

$$\tau = t - a \;\rightarrow\; t = \tau + a$$

$$d\tau = dt$$

$$\begin{aligned}\mathcal{L}\{f(t)\, u(t-a)\} &= \int_0^\infty f(\tau + a)\, e^{-s(\tau+a)}\, dt \\ &= e^{-as} \int_0^\infty f(\tau + a)\, e^{-st}\, d\tau\end{aligned}$$

$$\mathcal{L}\{f(t)\, u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\} \tag{6.12}$$

6.2.4. Multiplicación por t

Si $\mathcal{L}\{f(t)\} = F(s)$:

$$\mathcal{L}\{t\, f(t)\} = -\frac{dF(s)}{ds} \tag{6.13}$$

$$\mathcal{L}\{t^n\, f(t)\} = (-1)^n \frac{d^{(n)}F(s)}{ds^n} \tag{6.14}$$

6.2.5. División por t

Si $\mathcal{L}\{f(t)\} = F(s)$:

$$\mathcal{L}\left\{\frac{1}{t}f(t)\right\} = \int_s^{\infty} F(s)\,ds \tag{6.15}$$

<u>Prueba</u>:

$$g(t) = \frac{1}{t}f(t) \;\rightarrow\; f(t) = t\,g(t)$$
$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\,g(t)\}$$
$$F(s) = -\frac{dG(s)}{ds}$$
$$\int_s^{\infty} dG(s) = \int_s^{\infty} -dF(s)\,ds$$
$$G(\infty) - G(s) = -\int_s^{\infty} F(s)\,ds$$

Asumiendo que $G(\infty) = 0$:

$$G(s) = \int_s^{\infty} F(s)\,ds$$
$$\mathcal{L}\left\{\frac{1}{t}f(t)\right\} = \int_s^{\infty} F(s)\,ds$$

Ejemplo:

$$\mathcal{L}\left\{\frac{1}{t}sen(at)\right\}$$
$$\mathcal{L}\left\{\frac{1}{t}sen(at)\right\} = \int_s^{\infty} \frac{a}{s^2+a^2}\,ds$$
$$= \arctan\left(\frac{s}{a}\right)\Big|_s^{\infty}$$
$$= \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right)$$
$$u = \frac{\pi}{2} - \arctan\left(\frac{s}{a}\right) \rightarrow \arctan\left(\frac{s}{a}\right) = \frac{\pi}{2} - u \rightarrow \tan\left(\frac{\pi}{2} - u\right) = \frac{s}{a}$$
$$\tan(u) = \frac{a}{s} \rightarrow u = \arctan\left(\frac{a}{s}\right)$$
$$\mathcal{L}\left\{\frac{1}{t}sen(at)\right\} = \arctan\left(\frac{a}{s}\right)$$

6.3. Tabla de transformadas de *Laplace*

	$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
1	k	$\frac{k}{s}$
2	t^n	$\frac{n!}{s^{n+1}} \quad n \in \mathbb{N}$
3	e^{at}	$\frac{1}{s-a}$
4	$\text{sen}(at)$	$\frac{a}{s^2+a^2}$
5	$\cos(at)$	$\frac{s}{s^2+a^2}$
6	$\text{senh}(at)$	$\frac{a}{s^2-a^2}$
7	$\cosh(at)$	$\frac{s}{s^2-a^2}$
8	$u(t-a)$	$\frac{1}{s}e^{-as}$
9	$\delta(t-a)$	e^{-at}
10	$\frac{1}{t}\,\text{sen}(at)$	$\arctan\left(\frac{a}{s}\right)$

6.4. Transformada de *Laplace* de derivadas

Si $\mathcal{L}\{f(t)\} = F(s)$, las transformadas de sus derivadas serán:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \tag{6.16}$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - f(0)s - f'(0) \tag{6.17}$$

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - f(0)s^2 - f'(0)s - f''(0) \tag{6.18}$$

En general, de una derivada n-ésima:

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - f(0)s^{n-1} - f'(0)s^{n-2} - f''(0)s^{n-3} - \cdots - f^{(n-1)}(0) \tag{6.19}$$

Donde: $f(0); f'(0); f''(0); \ldots; f^{(n-1)}(0)$ son los valores iniciales en $t = 0$ de la función y de sus derivadas.

Prueba:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt$$

$$\begin{aligned}
u &= e^{-st} \\
du &= -s\,e^{-st}\,dt \\
dv &= f'(t)\,dt \\
v &= f(t)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= (f(t)\,e^{-st}\Big|_0^\infty) - \int_0^\infty f(t)(-s\,e^{-st})\,dt \\
&= s\int_0^\infty f(t)\,e^{-st}\,dt - f(0) \\
&= sF(s) - f(0)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0) \\
&= s(sF(s) - f(0)) - f'(0) \\
&= s^2F(s) - f(0)s - f'(0)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{f'''(t)\} &= s\mathcal{L}\{f''(t)\} - f''(0) \\
&= s(s^2F(s) - f(0)s - f'(0)) - f''(0) \\
&= s^3F(s) - f(0)s^2 - f'(0)s - f''(0)
\end{aligned}$$

6.5. Transformada de *Laplace* de integrales

Si $\mathcal{L}\{f(t)\} = F(s)$:

$$\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s) \tag{6.20}$$

Prueba:

$$\begin{gathered}
g(t) = \int_0^t f(t)\,dt \rightarrow\ g'(t) = f(t) \\
\mathcal{L}\{g'(t)\} = \mathcal{L}\{f(t)\} \\
sG(s) - g(0) = F(s) \\
s\mathcal{L}\{\left\{\int_0^t f(t)\,dt\right\} = F(s) \\
\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s}F(s)
\end{gathered}$$

6.6. La función *Gamma*

$$\Gamma(n) = \int_0^\infty x^{n-1}\,e^{-x}\,dx \tag{6.21}$$

6.6.1. Propiedades de la función *Gamma*

Propiedad 1

$$\Gamma(n) = (n-1)\Gamma(n-1) \tag{6.22}$$

En general:

$$\Gamma(n) = (n-1)(n-2)(n-3)\ldots(n-r)\Gamma(n-r) \tag{6.23}$$

Prueba:

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx$$
$$u = x^{n-1}$$
$$du = (n-1)\, x^{n-2}\, dx$$
$$dv = e^{-x}\, dx$$
$$v = -e^{-x}$$
$$\Gamma(n) = (-x^{n-1}\, e^{-x}\Big|_0^\infty) - \int_0^\infty -e^{-x}(n-1)x^{n-2}\, dx$$
$$= (n-1)\int_0^\infty x^{(n-1)-1}\, e^{-x}\, dx$$
$$= (n-1)\Gamma(n-1)$$
$$\Gamma(n-1) = (n-2)\Gamma(n-2)$$
$$\Gamma(n-2) = (n-3)\Gamma(n-3)$$
$$\Gamma(n) = (n-1)(n-2)(n-3)\ldots(n-r)\Gamma(n-r)$$

Propiedad 2

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \tag{6.24}$$

Prueba:

$$\Gamma(n) = (n-1)\Gamma(n-1)$$
$$\Gamma(n+1) = n\Gamma(n)$$
$$\Gamma(n) = \frac{1}{n}\Gamma(n+1)$$

Propiedad 3

Si $n \in \mathbb{N}$:

$$\Gamma(n) = (n-1)! \tag{6.25}$$

$$\Gamma(n+1) = n! \tag{6.26}$$

Prueba:

$$\begin{aligned}
\Gamma(n) &= (n-1)(n-2)(n-3)\dots(3)(2)(1)\,\Gamma(1) \\
\Gamma(1) &= \int_0^\infty x^{1-1}\,e^{-x}\,dx \\
&= -e^{-x}\Big|_0^\infty \\
&= -(0-1) \\
&= 1
\end{aligned}$$

En particular $n = 1$:

$$\begin{aligned}
\Gamma(1) &= (1-1)! \\
\Gamma(1) &= 0! \\
1 &= 0! \qquad (6.27)
\end{aligned}$$

6.7. Evaluación de la función *Gamma*

Dada la integral:

$$\begin{aligned}
I &= \int_0^\infty e^{-x^2}\,dx \\
I &= \int_0^\infty e^{-y^2}\,dy \\
I^2 = \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dy\,dx
\end{aligned}$$

Transformando a coordenadas polares:

$$x^2 + y^2 = r^2$$

$$dx\,dy = r\,dr\,d\theta$$

Por tanto:

$$I^2 = \lim_{M\to\infty} \int_0^{\frac{\pi}{2}} \int_0^{M} e^{-r^2} r\, dr\, d\theta$$

$$u = -r^2$$

$$du = -2r\, dr$$

$$\begin{aligned}
I^2 &= \lim_{M\to\infty} \int_0^{\frac{\pi}{2}} d\theta \int_0^{M} e^{u} \left(\frac{-dt}{2}\right) \\
&= \lim_{M\to\infty} \int_0^{\frac{\pi}{2}} d\theta \left(-\frac{e^{-r^2}}{2}\Bigg|_0^M \right) \\
&= -\frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta (0-1) \\
&= \frac{1}{2}\frac{\pi}{2} \\
&= \frac{\pi}{4}
\end{aligned}$$

$$I = \int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} x^{-\frac{1}{2}} e^{-x}\, dx$$

$$x = u^2$$

$$dx = 2u\, du$$

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \int_0^{\infty} (u^2)^{-\frac{1}{2}} e^{-u^2}\, 2u\, du \\
&= 2\int_0^{\infty} e^{-u^2}\, du \\
&= 2\frac{\sqrt{\pi}}{2} \\
&= \sqrt{\pi}
\end{aligned}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{6.28}$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = \frac{\sqrt{\pi}}{-\frac{1}{2}} = -2\sqrt{\pi}$$

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi} \tag{6.29}$$

6.8. Transformada de *Laplace* con la función *Gamma*

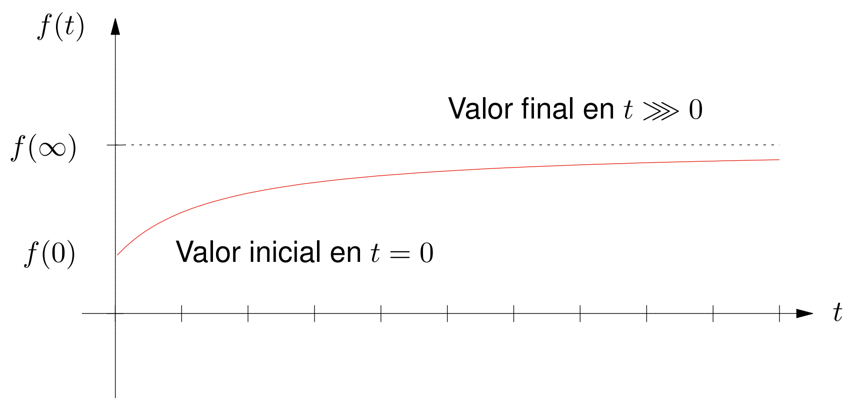
$$
\begin{aligned}
\mathcal{L}\{t^n\} &= \int_0^\infty t^n\, e^{-st}\, dt \\
u = st &\to t = \frac{u}{s} \\
du &= s\, dt \\
\mathcal{L}\{t^n\} &= \int_0^\infty \left(\frac{u}{s}\right)^n e^{-u}\, \frac{du}{s} \\
&= \frac{1}{s^{n+1}} \int_0^\infty u^n\, e^{-u}\, du \\
&= \frac{\Gamma(n+1)}{s^{n+1}}
\end{aligned}
$$

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}} \tag{6.30}$$

En particular para $n \in \mathbb{N}$:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \tag{6.31}$$

6.9. Teoremas del valor inicial y final



6.9.1. Teorema del valor inicial

Si $\mathcal{L}\{f(t)\} = F(s)$:

$$f(0) = \lim_{t \to 0} f(t) = \lim_{t \to \infty} sF(s) \tag{6.32}$$

<u>Prueba</u>:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$
$$\int_0^\infty f'(t)\, e^{-st}\, dt = sF(s) - f(0)$$
$$\lim_{s\to\infty} \int_0^\infty f'(t)\, e^{-st}\, dt = \lim_{s\to\infty} (sF(s) - f(0))$$
$$0 = \lim_{s\to\infty} (sF(s) - f(0))$$
$$f(0) = \lim_{s\to\infty} sF(s)$$

6.9.2. Teorema del valor final

Si $\mathcal{L}\{f(t)\} = F(s)$:

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{t\to 0} sF(s) \tag{6.33}$$

Prueba:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$
$$\int_0^\infty f'(t)\, e^{-st}\, dt = sF(s) - f(0)$$
$$\lim_{s\to 0} \int_0^\infty f'(t)\, e^{-st}\, dt = \lim_{s\to 0}(sF(s) - f(0))$$
$$f(t)\Big|_0^\infty = \lim_{s\to 0}(sF(s) - f(0))$$
$$f(\infty) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$
$$f(\infty) = \lim_{s\to 0} sF(s)$$

Capítulo 7

Transformada inversa de *Laplace*

Si $\mathcal{L}\{f(t)\} = F(s)$:

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \tag{7.1}$$

Definida para $t > 0$.

La transformada inversa de *Laplace* regresa del dominio de “s” al dominio de “t”.

Ejemplo:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\}$$

$$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$t^n = \mathcal{L}^{-1}\left\{\frac{\Gamma(n+1)}{s^{n+1}}\right\}$$

$$\frac{t^n}{\Gamma(n+1)} = \mathcal{L}^{-1}\left\{\frac{1}{s^{n+1}}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\Gamma(n)} \tag{7.2}$$

En particular $n \in \mathbb{N}$:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!} \tag{7.3}$$

7.1. Tabla de transformadas de *Laplace* inversas

	$F(s)$	$f(t) = \mathcal{L}^{-1}\{F(s)\}; t > 0$
1	$\dfrac{k}{s}$	k
2	$\dfrac{1}{s^n}$	$\dfrac{t^{n-1}}{\Gamma(n)}; \quad \dfrac{t^{n-1}}{(n-1)!}\, n \in \mathbb{N}$
3	$\dfrac{1}{s-a}$	e^{at}
4	$\dfrac{1}{s^2+a^2}$	$\dfrac{1}{a}\operatorname{sen}(at)$
5	$\dfrac{s}{s^2+a^2}$	$\cos(at)$
6	$\dfrac{1}{s^2-a^2}$	$\dfrac{1}{a}\operatorname{senh}(at)$
7	$\dfrac{s}{s^2-a^2}$	$\cosh(at)$
8	$\arctan\left(\dfrac{a}{s}\right)$	$\dfrac{1}{t}\operatorname{sen}(at)$
9	k	$k\delta(t)$
10	e^{-as}	$\delta(t-a)$

7.2. Propiedades de la transformada inversa de *Laplace*

7.2.1. Linealidad

$$\mathcal{L}^{-1}\{a_1\, F_1(s) + a_2\, F_2(s)\} = a_1\, f_1(t) + a_2\, f_2(t) \tag{7.4}$$

7.2.2. Desplazamiento en t

Si $\mathcal{L}^{-1}\{F(s)\} = f(t)$:

$$\mathcal{L}^{-1}\{F(s)\, e^{-as}\} = f(t-a)u(t-a) \tag{7.5}$$

7.2.3. Desplazamiento en s

Si $\mathcal{L}^{-1}\{F(s)\} = f(t)$:

$$\mathcal{L}^{-1}\{F(s-a)\} = f(t)\, e^{at} \tag{7.6}$$

7.2.4. División por s

Si $\mathcal{L}^{-1}\{F(s)\} = f(t)$:

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\,dt \tag{7.7}$$

En general:

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s^n}\right\} = \int_0^t \int_0^t \cdots \int_0^t f(t)\,dt \ldots dt\,dt \tag{7.8}$$

7.2.5. Transformada inversa de la derivada

Si $\mathcal{L}^{-1}\{F(s)\} = f(t)$:

$$\mathcal{L}^{-1}\{F'(s)\} = -t\,f(t) \tag{7.9}$$

En general:

$$\mathcal{L}^{-1}\{F^{(n)}(s)\} = (-1)^n\,t^n\,f(t) \tag{7.10}$$

7.3. Descomposición en fracciones parciales

En la mayoría de los casos se tienen funciones racionales de la siguiente forma:

$$F(s) = \frac{P(s)}{Q(s)}$$

Donde el grado del numerador debe ser menor al grado del denominador. Para hallar la transformada inversa se debe descomponer en fracciones parciales de acuerdo a los factores del denominador. Se tienen los siguientes casos:

7.3.1. Factores lineales no repetidos

$$\frac{P(s)}{(s-a_1)(s-a_2)\ldots(s-a_n)} = \frac{A_1}{s-a_1} + \frac{A_2}{s-a_2} + \cdots + \frac{A_n}{s-a_n}$$

7.3.2. Factores lineales y repetidos

$$\frac{P(s)}{(s-a)^m(s-b)^n} = \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_m}{(s-a)^m} + \frac{B_1}{(s-b)} + \frac{B_2}{(s-a)^2} + \cdots + \frac{B_n}{(s-b)^n}$$

7.3.3. Factores cuadráticos (con raíces imaginarias o complejas)

$$\frac{P(s)}{(s^2+a_1s+b_1)(s^2+a_2s+b_2)} = \frac{A_1s+B_1}{s^2+a_1s+b_1} + \frac{A_2s+B_2}{s^2+a_2s+b_2}$$

7.4. Convolución

$$\begin{aligned}
f_1(t) * f_2(t) &= \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau \\
\mathcal{L}\{f_1(t) * f_2(t)\} &= \int_0^\infty \left[\int_0^t f_1(\tau) f_2(t-\tau)\, d\tau\right] e^{-st}\, dt \\
&= \int_{t=0}^{t=\infty} \int_{\tau=0}^{\tau=t} f_1(\tau) f_2(t-\tau) e^{-st}\, d\tau\, dt \\
&= \lim_{M\to\infty} \int_{\tau=0}^{\tau=M} \int_{t=\tau}^{t=M} f_1(\tau) f_2(t-\tau) e^{-st}\, dt\, d\tau \\
&= \lim_{M\to\infty} \int_0^M f_1(\tau)\, d\tau \int_\tau^M f_2(t-\tau) e^{-st}\, dt \\
u = t - \tau &\rightarrow\ t = u + \tau \\
du &= dt \\
\mathcal{L}\{f_1(t) * f_2(t)\} &= \lim_{M\to\infty} \int_0^M f_1(\tau)\, d\tau \int_0^{M-\tau} f_2(u) e^{-s(u+\tau)}\, du \\
&= \int_0^\infty f_1(\tau) e^{-st}\, d\tau \int_0^\infty f_2(u) e^{-su}\, du \\
&= F_1(s)\, F_2(s) \\
\mathcal{L}\{f_1(t) * f_2(t)\} &= F_1(s)\, F_2(s) \qquad (7.11)
\end{aligned}$$

7.5. Transformada inversa por convolución

$$\mathcal{L}^{-1}\{F_1(s)\, F_2(s)\} = f_1(t) * f_2(t) \qquad (7.12)$$

Donde:

$$f_1(t) = \mathcal{L}^{-1}\{F_1(s)\}$$
$$f_2(t) = \mathcal{L}^{-1}\{F_2(s)\}$$

Capítulo 8

Aplicaciones de la transformada de *Laplace*

8.1. Ecuaciones diferenciales lineales

Pasos:

1. Se tiene Ecuación diferencial en el dominio del tiempo “t”.
2. Se cuenta con condiciones iniciales.
3. Se aplica la transformada de *Laplace* y se convierte en una ecuación algebraica en el dominio de “s”.
4. Se soluciona la ecuación en el dominio de “s”.
5. Se aplica la transformada inversa de *Laplace* y se lleva la solución a el dominio de “t”.

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \tag{8.1}$$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - f(0)s - f'(0) \tag{8.2}$$

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - f(0)s^2 - f'(0)s - f''(0) \tag{8.3}$$