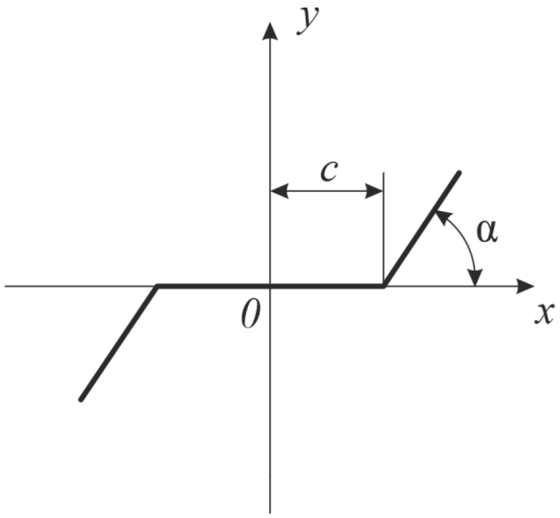
Mechanical systems dynamic with nonlinearities

Galina Demidova
Demidova@itmo.ru

Deadband

- mechanical elements in transformers



$$y = \begin{cases} 0 & |x| \le c \\ k(x-c) & x > c \\ k(x+c) & x < -c \end{cases}$$

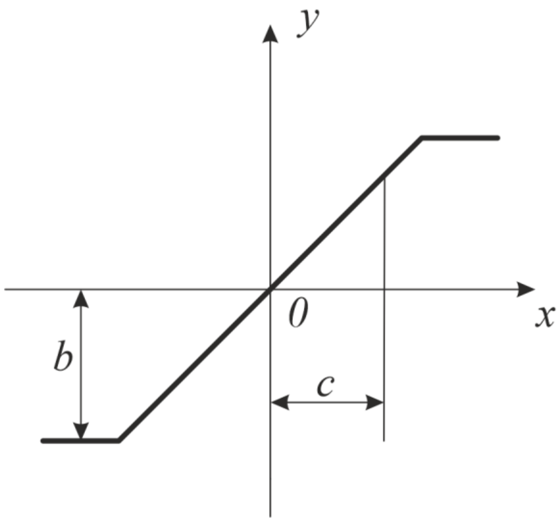
$$k = tg\alpha$$

Saturation zone

- transformers, transistors, amplifiers

$$y = \begin{cases} kx & |x| \le c \\ b & x > c \\ -b & x < -c \end{cases}$$

$$k = \frac{b}{c}$$

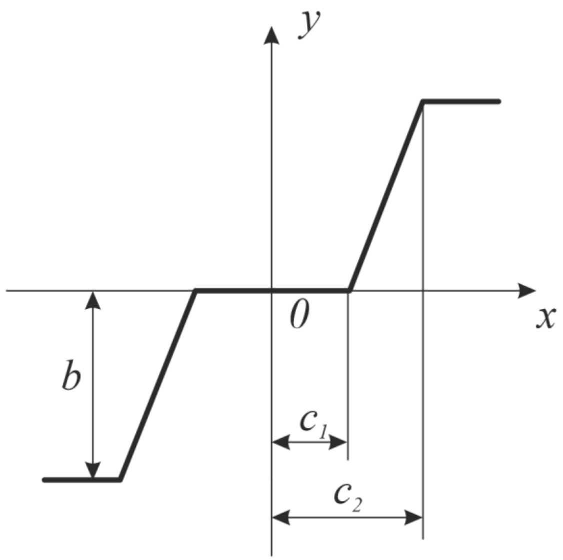


Deadband and Saturation zone

- hydrostatic and pneumatic transformer elements

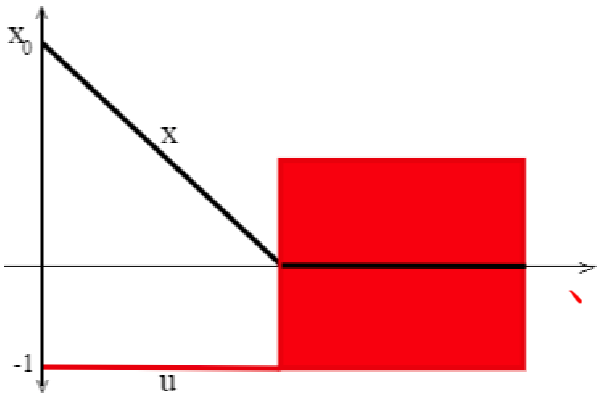
$$y = \begin{cases} -b & x \le -c_2 \\ k(x+c_1) & -c_2 < x < -c_1 \\ 0 & |x| \le c_1 \\ k(x-c_1) & c_1 < x < c_2 \\ b & x \ge c_2 \end{cases}$$

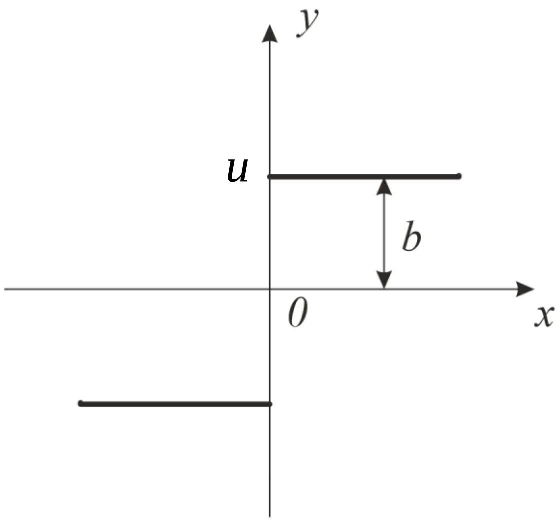
$$k = \frac{b}{c_2 - c_1}$$



Ideal relay

- discontinuous switching, delay in switching

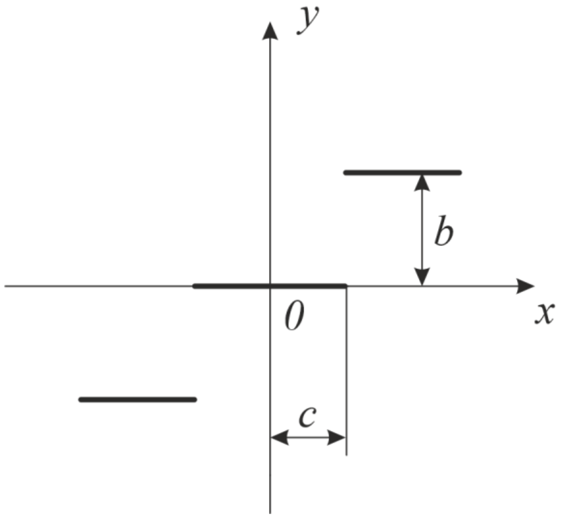




$$y = \begin{cases} b & x \geq 0 \\ -b & x < 0 \end{cases}$$

Ideal relay with deadband zone

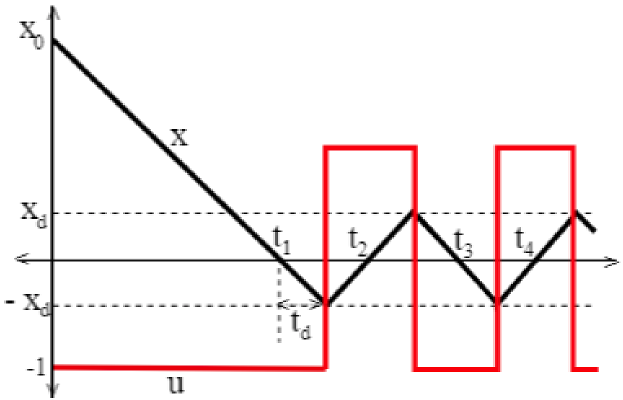
- electromagnetic devices

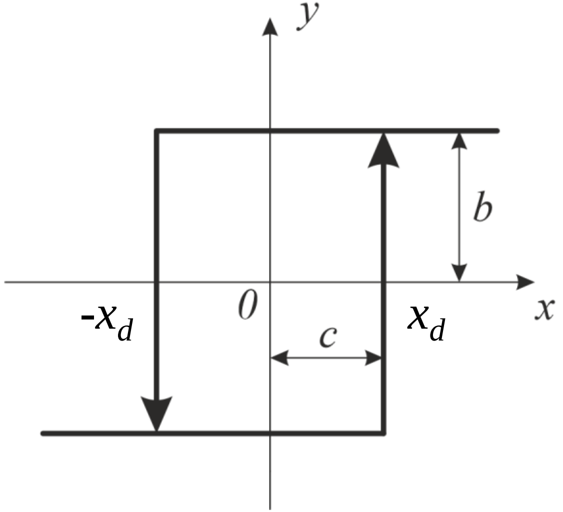


$$y = \begin{cases} 0 & |x| \le c \\ b & x > c \\ -b & x < -c \end{cases}$$

Relay with delay

- discontinuous switching, delay in switching

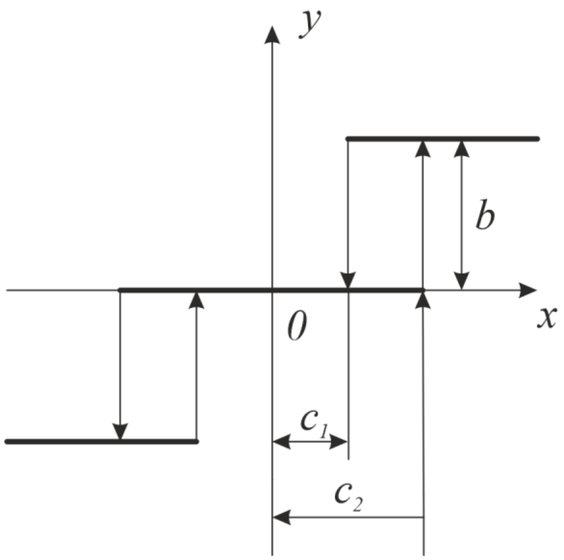




$$y = \begin{cases} -b & x \le c, \dot{x} > 0 \\ b & x > c, \dot{x} > 0 \\ b & x \ge -c, \dot{x} < 0 \\ -b & x < -c, \dot{x} < 0 \end{cases}$$

Relay with delay and deadband zone

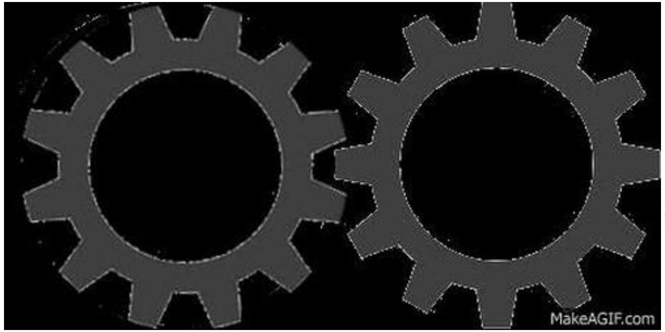
- discontinuous switching, delay in switching

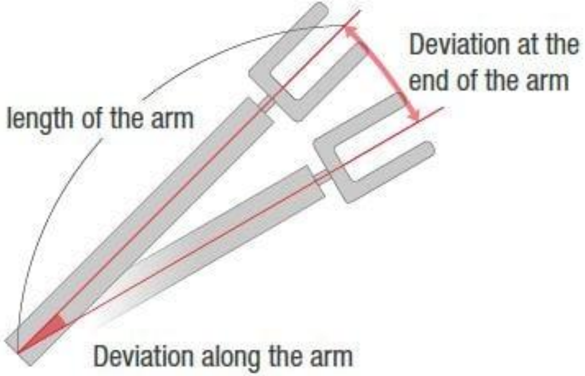


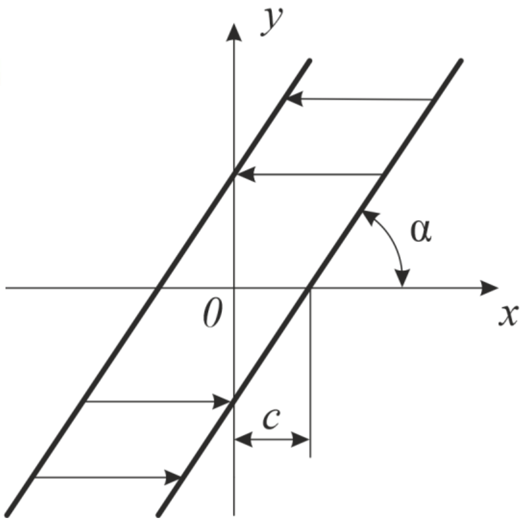
$$y = \begin{cases} -b & x \le -c_1, \dot{x} > 0 \\ 0 & -c_1 < x < c_2, \dot{x} > 0 \\ b & x \ge c_2, \dot{x} > 0 \\ b & x \ge c_1, \dot{x} < 0 \\ 0 & -c_2 < x < c_1, \dot{x} > 0 \\ -b & x \le -c_2, \dot{x} < 0 \end{cases}$$

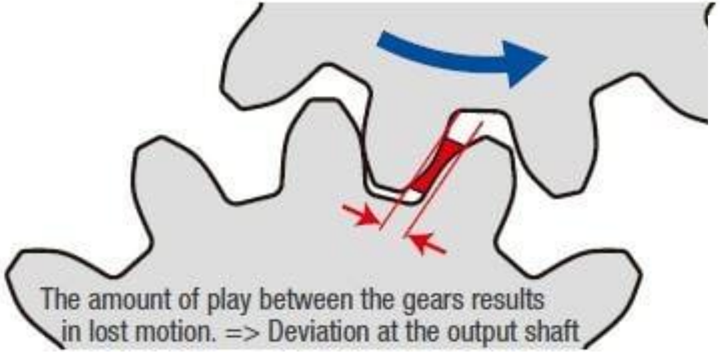
Backlash or dry friction

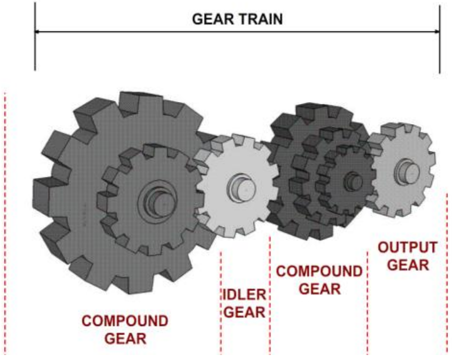
- ❑ mechanical gear









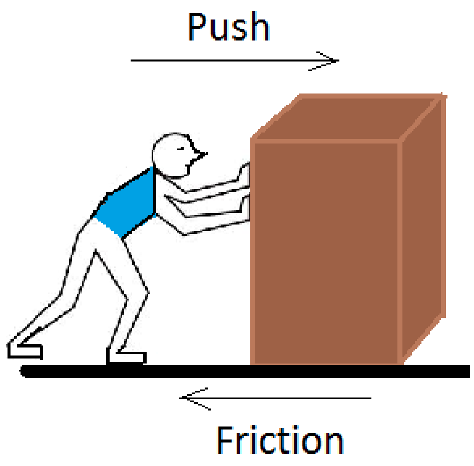


$$y = \begin{cases} k(x-c) & \dot{x} > 0 \\ k(x+c) & \dot{x} < 0 \end{cases}$$

$$k = tg\alpha$$

Friction Nonlinearity

1. Static Friction
2. Dynamic Friction
3. Limiting Friction



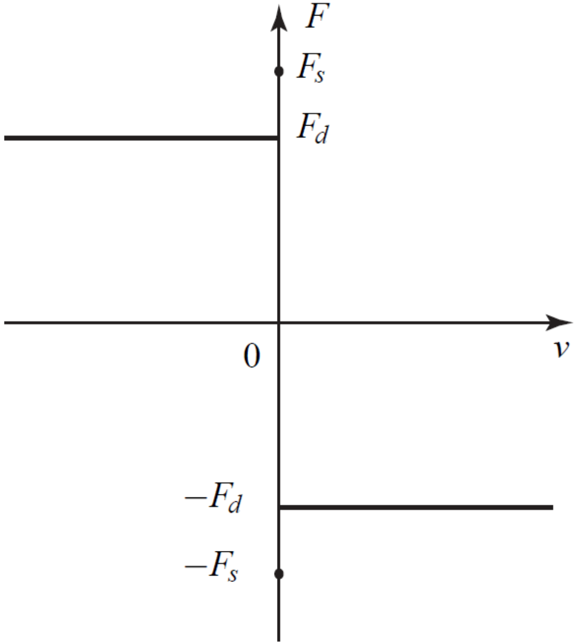
The Coulomb friction force model

$$F\begin{cases} \le \mu_s F_n & \rightarrow v = 0 \\ = -\mu_d F_n \,\mathrm{sgn}(v) & \rightarrow v \ne 0 \end{cases}$$

$$F_d = \mu_d F_n$$

$$F_s = \mu_s F_n$$

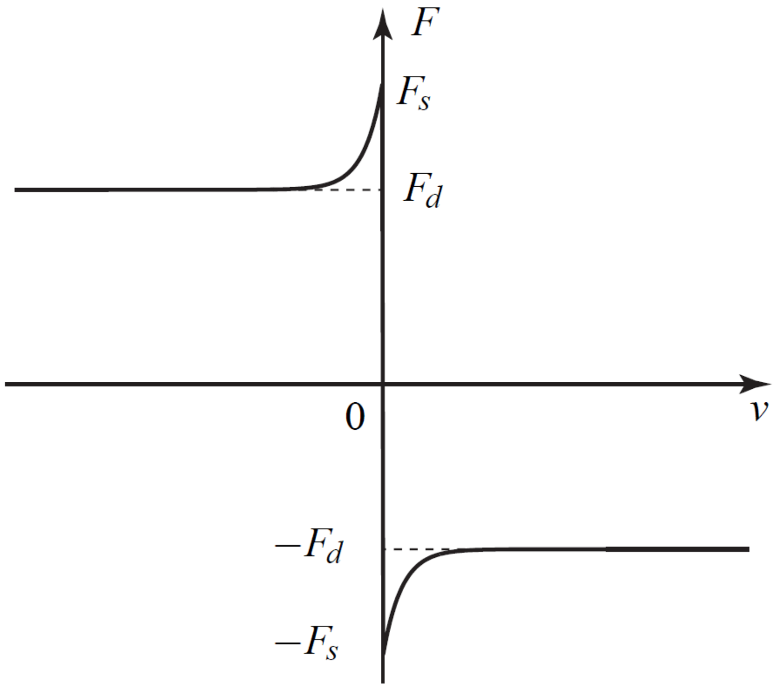
$$F = -F_d \frac{v}{|v|}$$



The Coulomb friction force model

Benson exponential friction model

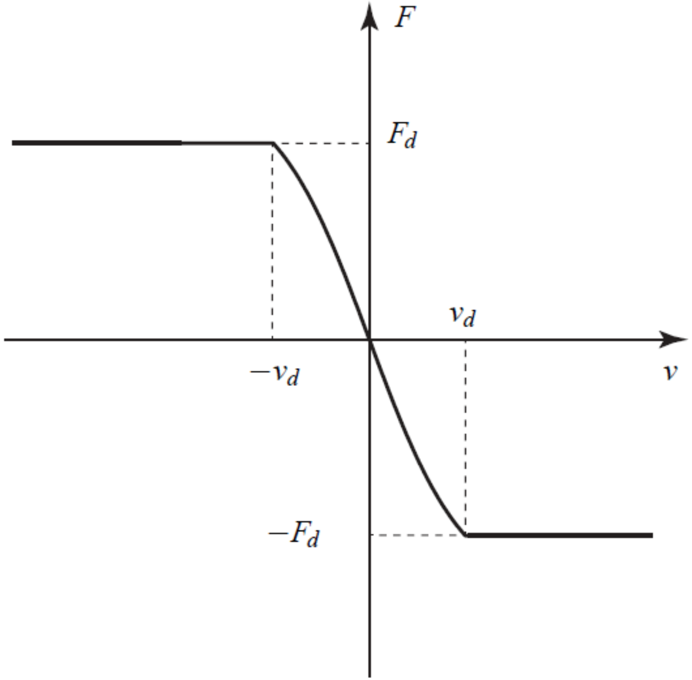
$$F = -F_d - (F_s - F_d)e^{-c|v|}\,\mathrm{sgn}(v)$$



Benson exponential friction model

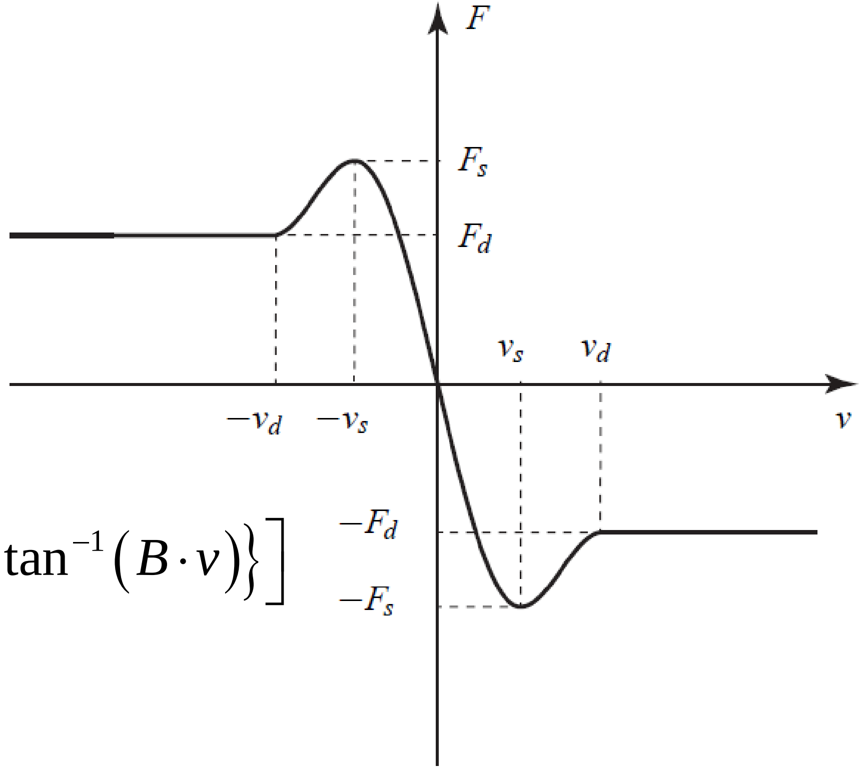
Smooth Coulomb friction model

$$F = -F_d \tanh\left({v}/{v_d} \right)$$



Smooth Coulomb friction model

Velocity-based friction model

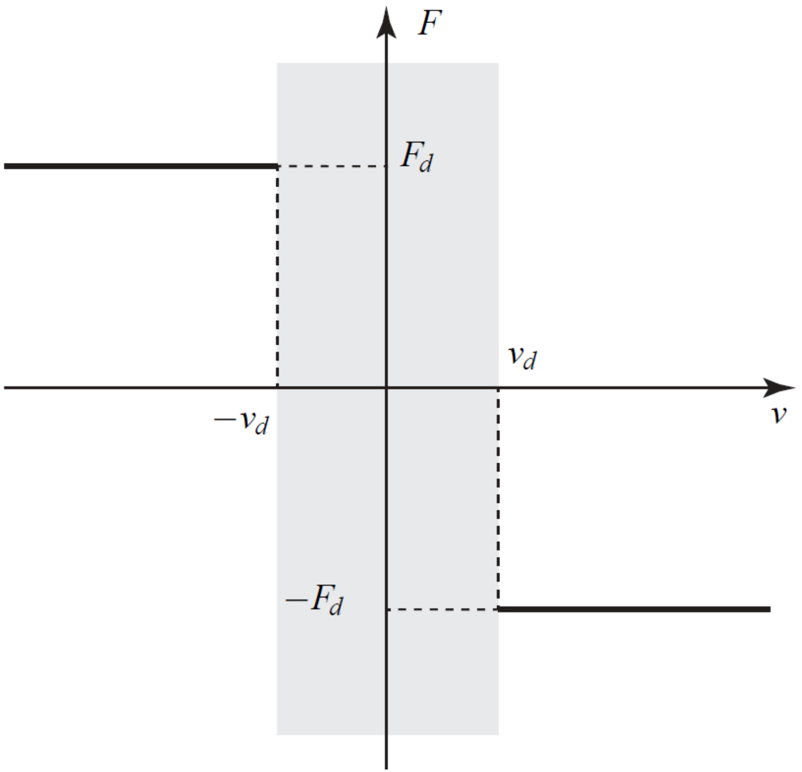


$$F = -F_s \sin\left[C \tan^{-1}(B\cdot v) - E\left\{(B\cdot v) - \tan^{-1}(B\cdot v)\right\}\right]$$

Velocity-based friction model

Karnopp friction model

$$F = \begin{cases} -\min\left(\max\left(-F_s, F_{ext}\right), F_s\right) \rightarrow |v| \le v_d \\ -F_d \operatorname{sgn}(v) \end{cases}$$

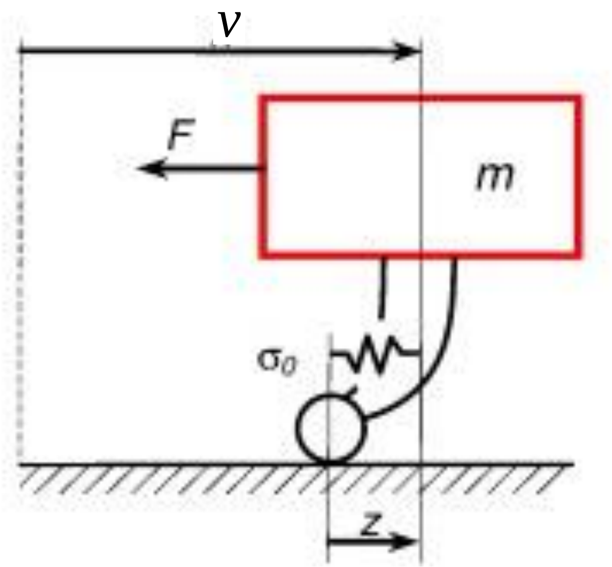


Karnopp friction model

Dahl friction model

$$F = -\sigma_0 z$$

$$\dot{z} = v \cdot \left(1 - \frac{\sigma_0 z}{F_d} \operatorname{sgn}(v)\right)^{\alpha}$$



The bristle analogy in the Dahl model

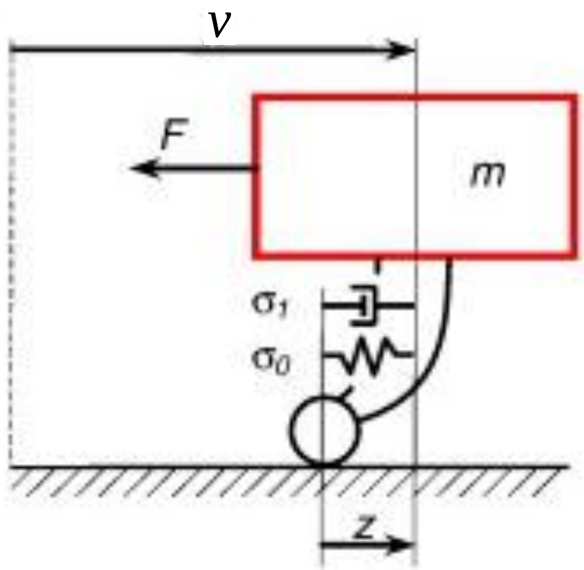
LuGre friction model

$$F = \sigma_0 z + \sigma_1 \dot{z} + \sigma_2 z$$

$$\dot{z} = v \cdot \left(1 - \frac{\sigma_0 z}{g(v)} \operatorname{sgn}(v) \right)$$

$$g(v) = F_d + (F_s - F_d) e^{-\left(v / v_{Stribeck} \right)^{\gamma}}$$

$$F = \left(F_d + (F_s - F_d) e^{-\left(v / v_{Stribeck} \right)^{\gamma}} \right) \operatorname{sgn}(v) + \sigma_2 v$$



Elasto-plastic friction model

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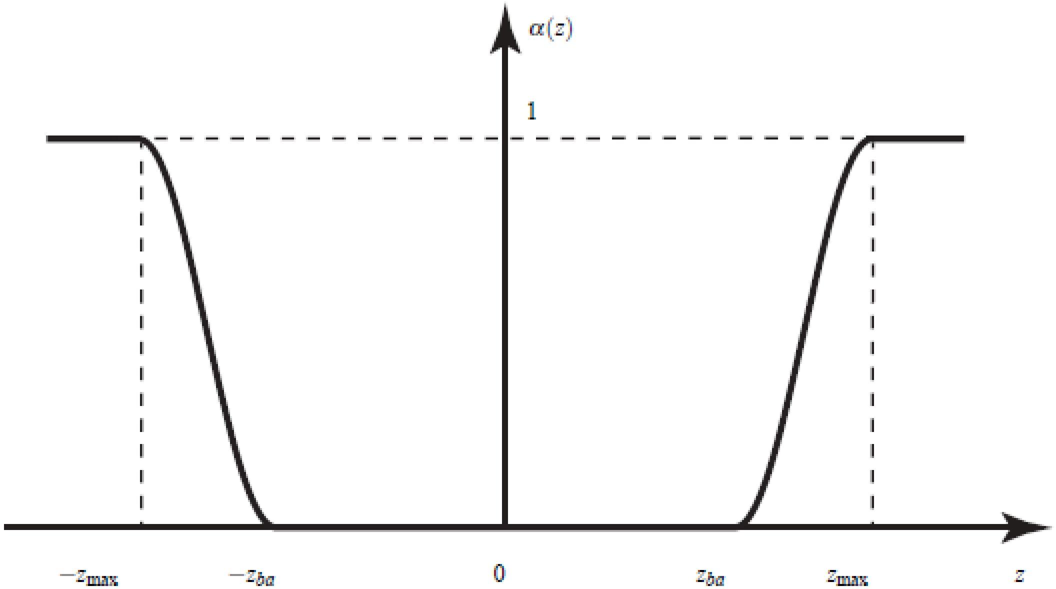
$$F = \sigma_0 z + \sigma_1 \dot{z} + \sigma_2 z$$

$$\dot{z} = v \cdot \left(1 - \alpha(z, v) \frac{\sigma z}{g(v)} \operatorname{sgn}(v) \right)$$



Elasto-plastic friction model $\alpha(z)$ parameter in case of $\operatorname{sgn}(v) = \operatorname{sgn}(\dot{z})$

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Elasto-plastic friction model

$$\alpha(z,v)=\begin{cases}0 \\ \dfrac{1}{2}\left(1+\sin\left(\pi\dfrac{z-\frac{1}{2}\left(z_{\max}+z_{ba}\right)}{z_{\max}-z_{ba}}\right)\right) \\ 1\end{cases} \rightarrow z_{ba}\le\begin{cases}|z|<z_{ba} \\ |z|<z_{\max} \\ |z|\ge z_{\max}\end{cases}$$

$$z_{\max}=g(v)/\sigma_0$$

$$\text{Otherwise,}\quad \mathit{if}\quad \operatorname{sgn}(v)\neq\operatorname{sgn}(\dot{z})\quad \mathit{then}\quad \alpha(z,v)=0$$

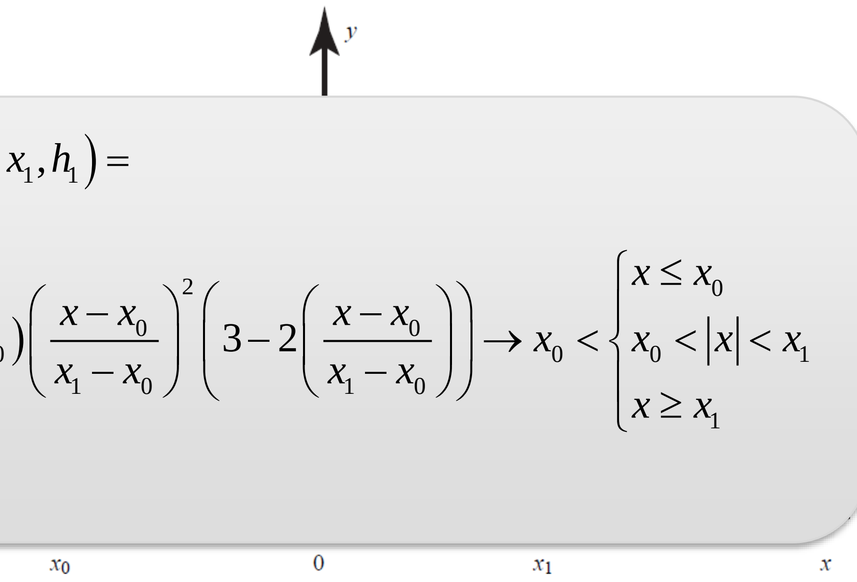
Stick-slip friction model

$$F = F_{striction} + F_{sliding}$$

$$F_{striction} = -(1-\beta) F_s \operatorname{sgn}(\Delta)$$

$$F_{sliding} = -F_d \operatorname{sgn}(v)$$

$$step(|x|, x_0, h_0, x_1, h_1) = \begin{cases} h_0 & x \le x_0 \\ h_0 + (h_1 - h_0)\left(\dfrac{x - x_0}{x_1 - x_0}\right)^2 \left(3 - 2\left(\dfrac{x - x_0}{x_1 - x_0}\right)\right) & \rightarrow x_0 < |x| < x_1 \\ h_1 & x \ge x_1 \end{cases}$$



Behaviour of the step function
$y = step(|x|, x_0, h_0, x_1, h_1)$

State	Sliding	Stiction
v	$\lvert v\rvert > v_t$	$0 \le \lvert v\rvert \le v_t$
β	1	$\mathrm{step}(\lvert v\rvert, -v_t, -1, v_t, 1)$
F_s	0	$\mathrm{step}(\lvert F_s\rvert, -\Delta_{\max}, -F_s, \Delta_{\max}, F_s)$
F_d	F_d	$\mathrm{step}(\lvert F_d\rvert, -v_t, -F_d, v_t, F_d)$
F	$F_{sliding}$	$F_{stiction} + F_{sliding}$

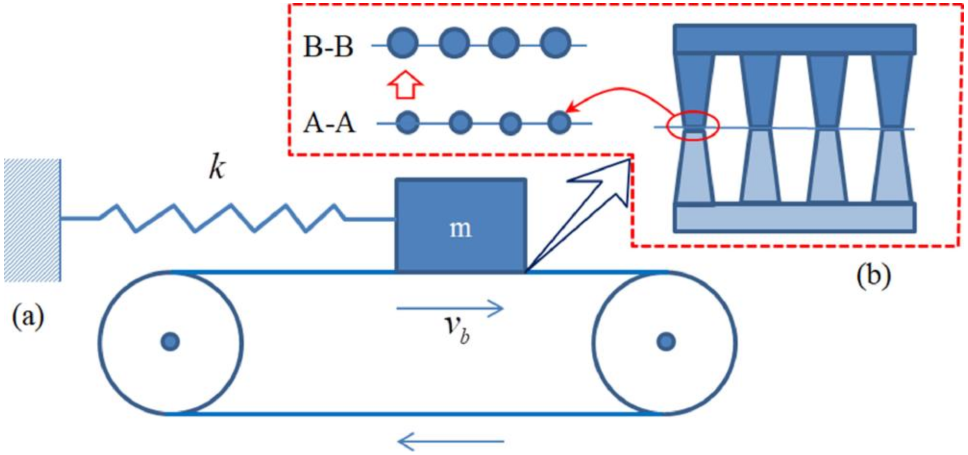
Gonthier friction model

$$F_{br} = \sigma_0 z + \sigma_1 \dot{z}$$

$$\dot{z} = s\dot{z}_{st} + (1-s)\,\dot{z}_{sl}$$

$$s = e^{-v^2 / v^2_{Stribeck}}$$

$$\begin{cases} \dot{z}_{st} = v \\ \dot{z}_{sl} = \dfrac{1}{\sigma_1} F_c - \dfrac{\sigma_0}{\sigma_1} z \end{cases}$$



Model of mass block traveling on a belt and model of contact surfaces as bristles with variation of true contact area from the beginning A-A to the end B-B of dwell-time interval.

Gonthier friction model

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$$F_c = F_d \cdot dir\left(v, v_t\right)$$

$$dir\left(v, v_t\right) = \begin{cases} \dfrac{v}{|v|} & \rightarrow |v| \geq v_t \\ \dfrac{v}{v_t}\left[\dfrac{3}{2}\dfrac{|v|}{v_t} - \dfrac{1}{2}\left(\dfrac{|v|}{v_t}\right)^3\right] & \rightarrow |v| < v_t \end{cases}$$

$$F_{\max} = F_d + \left(F_s - F_d\right) s_{dw}$$

$$\dot{s}_{dw} = \begin{cases} \dfrac{1}{\tau_{dw}}\left(s - s_{dw}\right) \rightarrow \left(s - s_{dw}\right) \geq 0 \\ \dfrac{1}{\tau_{br}}\left(s - s_{dw}\right) \rightarrow \left(s - s_{dw}\right) < 0 \end{cases}$$

$$\tau_{br} = \sigma_1 / \sigma_0$$

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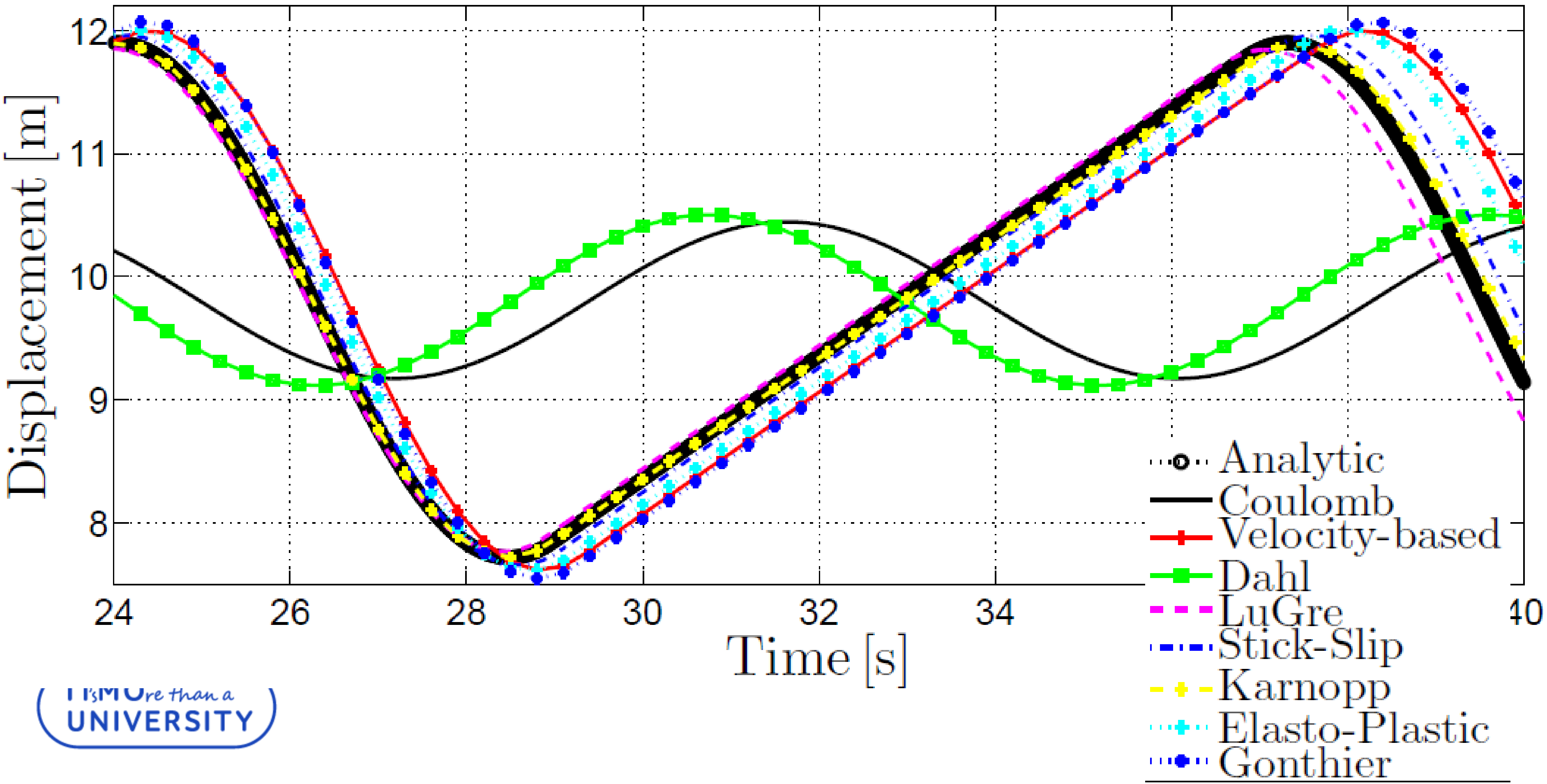
Gonthier friction model

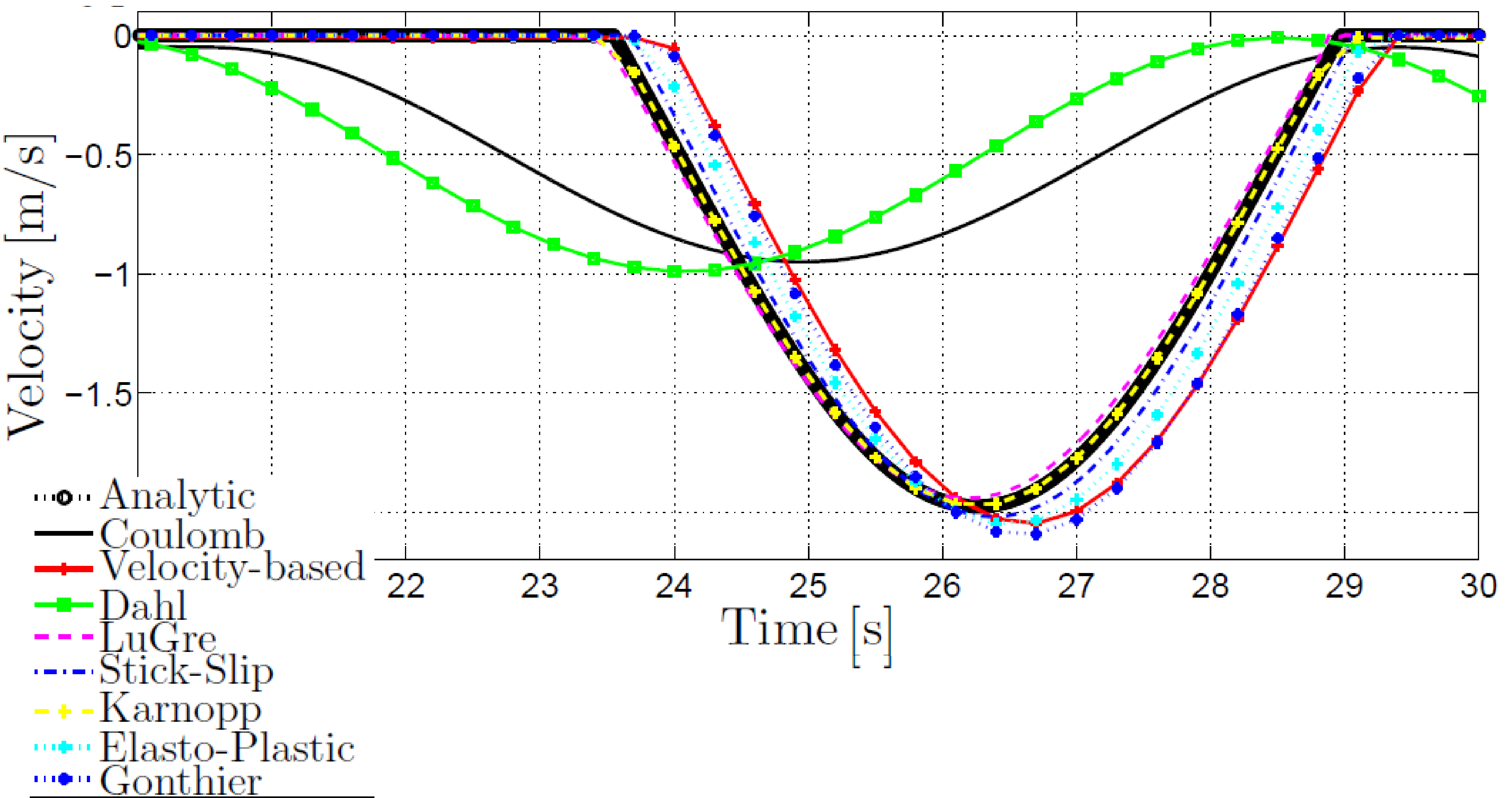
$$F_{\max} = -\mathrm{sat}\left(F_{br}, F_{\max}\right) - \sigma_2 v$$

$$\mathrm{sat}\left(F_{br}, F_{\max}\right) = \begin{cases} F_{br} & \rightarrow \left|F_{br}\right| \le F_{\max} \\ \dfrac{F_{br}}{\left|F_{br}\right|} F_{\max} & \rightarrow \left|F_{br}\right| > F_{\max} \end{cases}$$

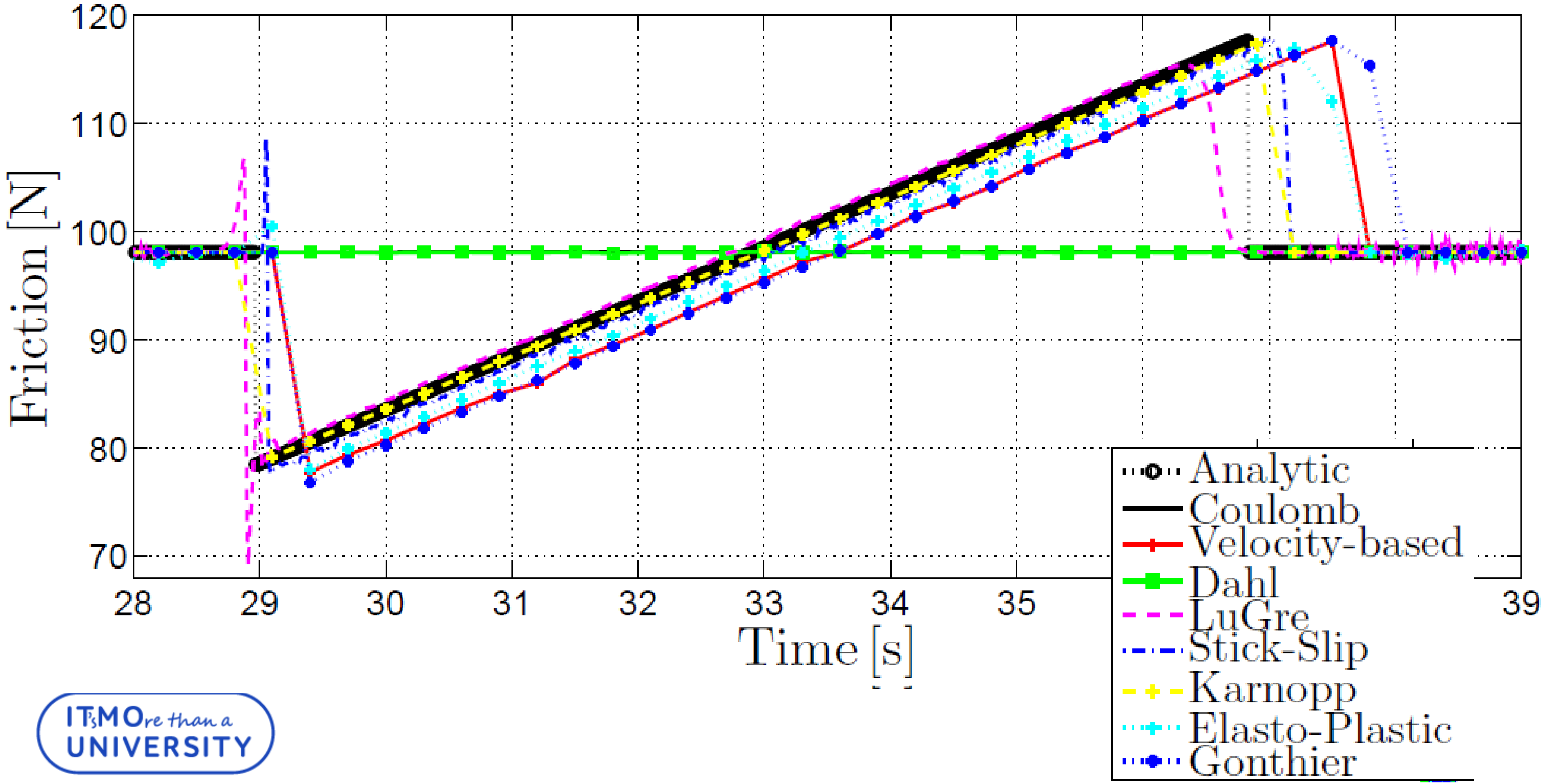
Summary of model parameters and state variables

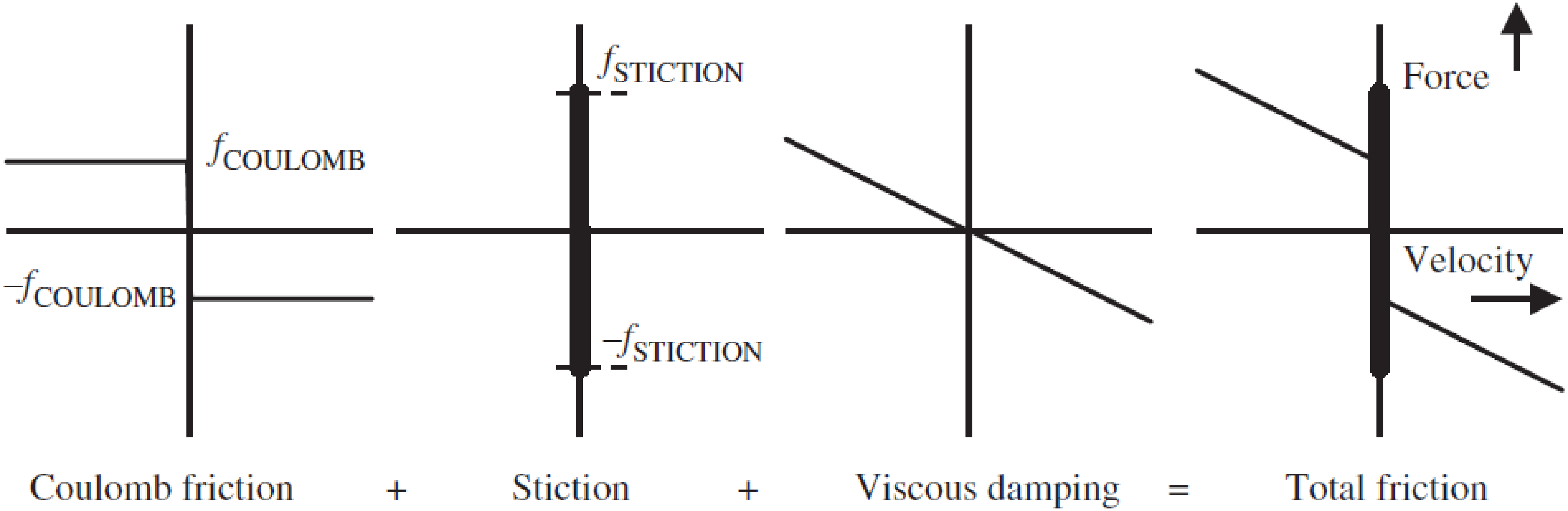
Model	Parameters	State variable
Coulomb	F_d, v_d	
Velocity based	$F_d,\ F_s,\ v_d,\ v_s$	
Karnopp	$F_d,\ F_s,\ v_d$	
Stick-slip	$F_d,\ F_s,\ v_t,\ \Delta_{max}$	
Dahl	$F_d,\ \sigma_0,\ \alpha$	z
LuGre	$F_d,\ F_s,\ \sigma_0,\ \sigma_1,\ \sigma_2,\ v_{Stribeck},\ \gamma$	z
Elasto-plastic	$F_d,\ F_s,\ \sigma_0,\ \sigma_1,\ \sigma_2,\ v_{Stribeck},\ \gamma,\ z_{ba}$	z
Gonthier	$F_d,\ F_s,\ \sigma_0,\ \sigma_1,\ \sigma_2,\ v_{Stribeck},\ v_t,\ \tau_{dw}$	$z,\ s_{dw}$


12
11
10
9
8
Displacement [m]
24
26
28
30
32
34
40
Time [s]
Analytic
Coulomb
Velocity-based
Dahl
LuGre
Stick-Slip
Karnopp
Elasto-Plastic
Gonthier

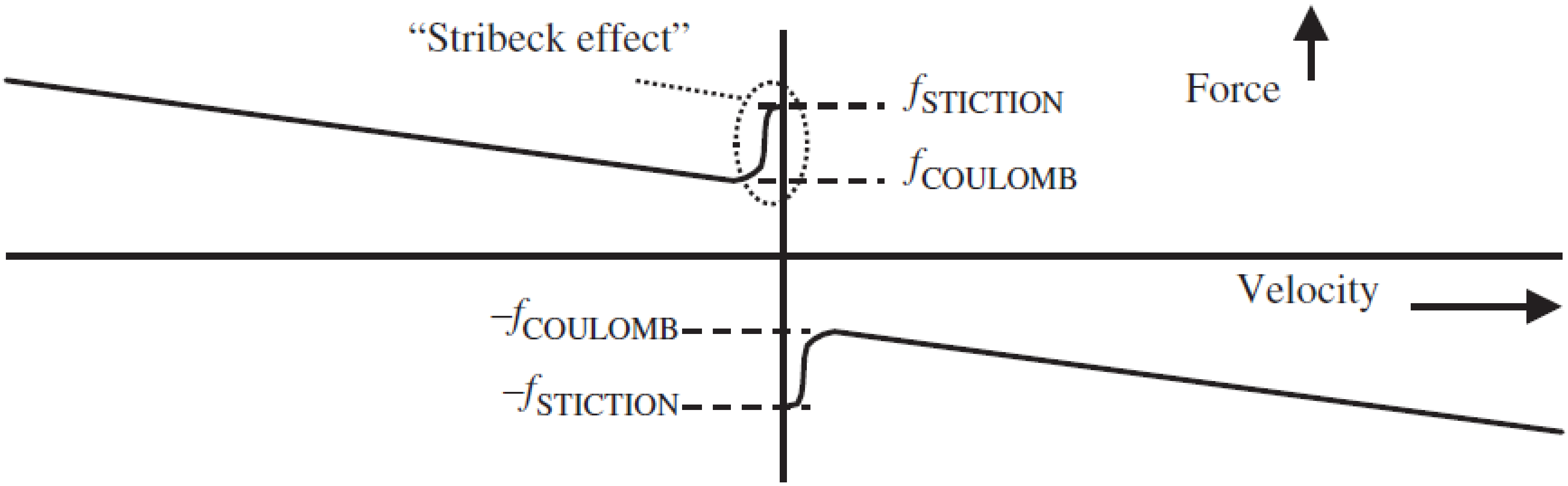


Velocity [m/s]
0
−0.5
−1
−1.5
22
23
24
25
26
27
28
29
30
Time [s]
Analytic
Coulomb
Velocity-based
Dahl
LuGre
Stick-Slip
Karnopp
Elasto-Plastic
Gonthier

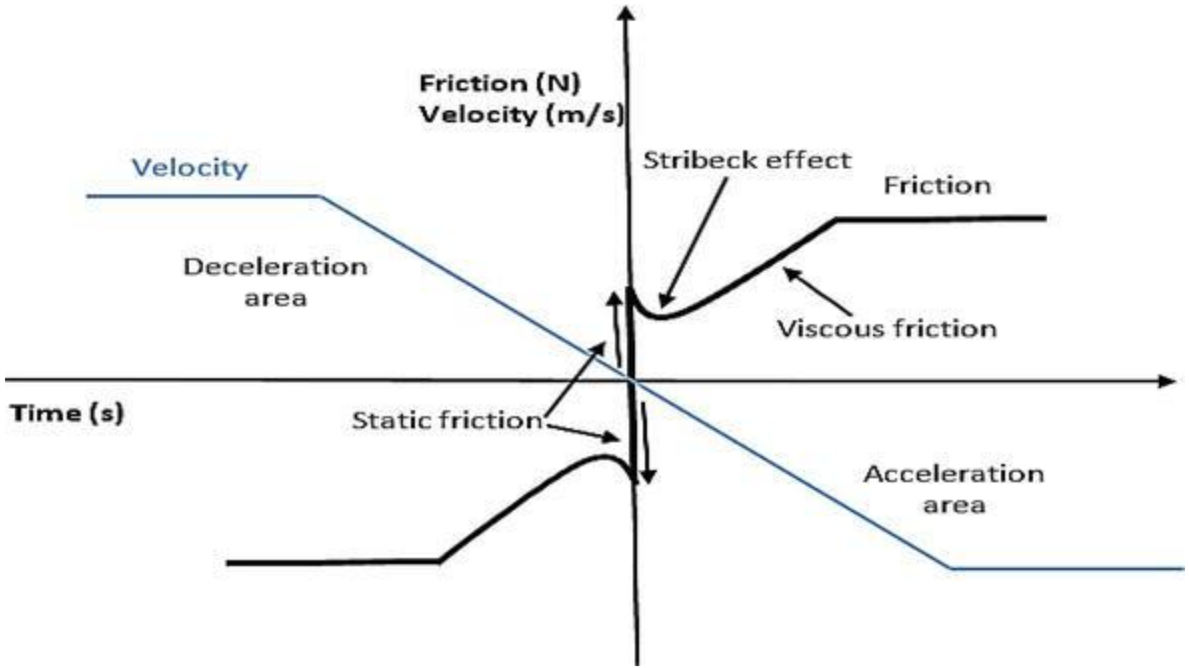

Friction [N]
Time [s]
Analytic
Coulomb
Velocity-based
Dahl
LuGre
Stick-Slip
Karnopp
Elasto-Plastic
Gonthier

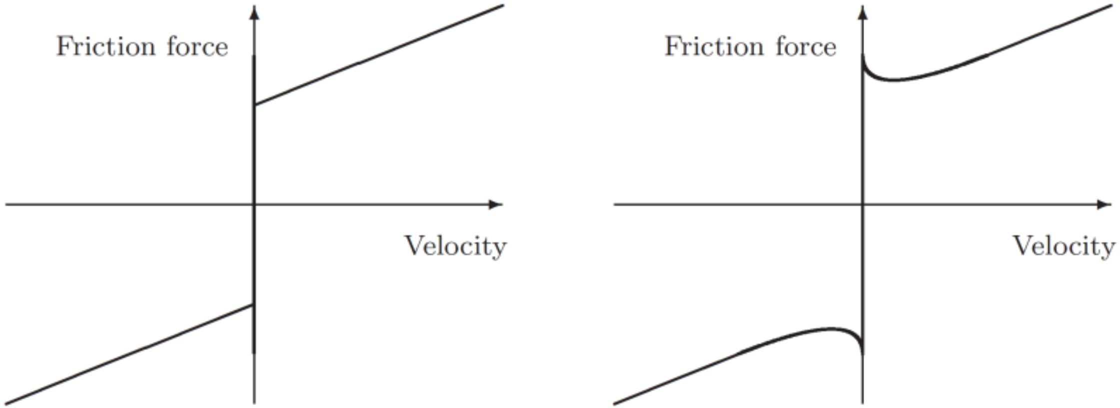


model of friction

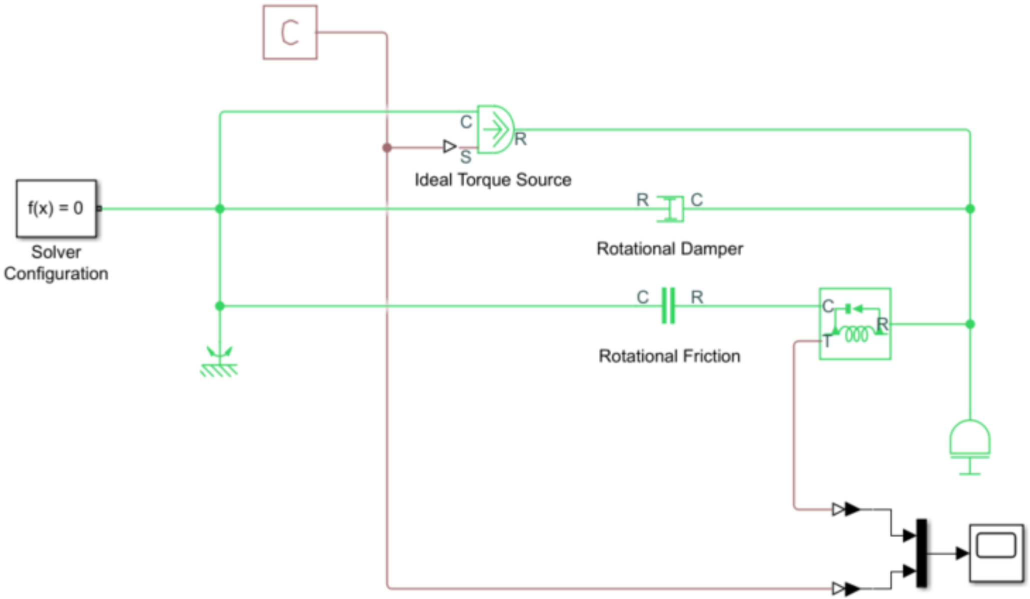


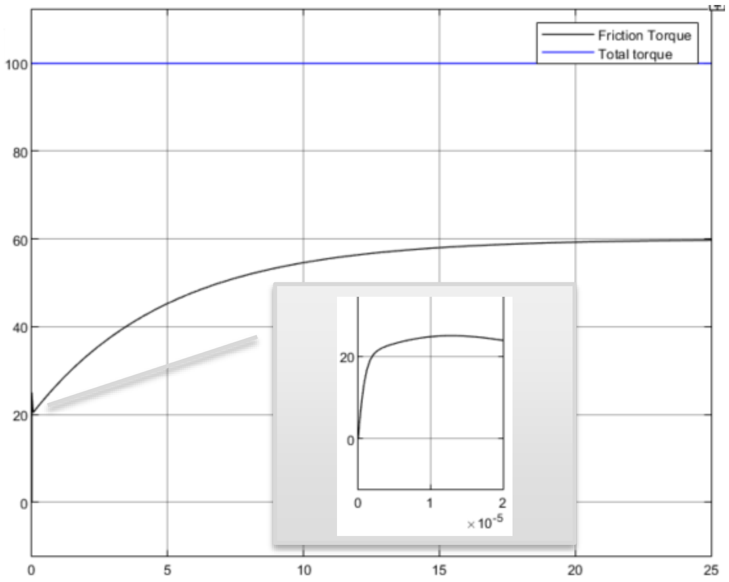
Frictional model recognizing the Stribeck effect

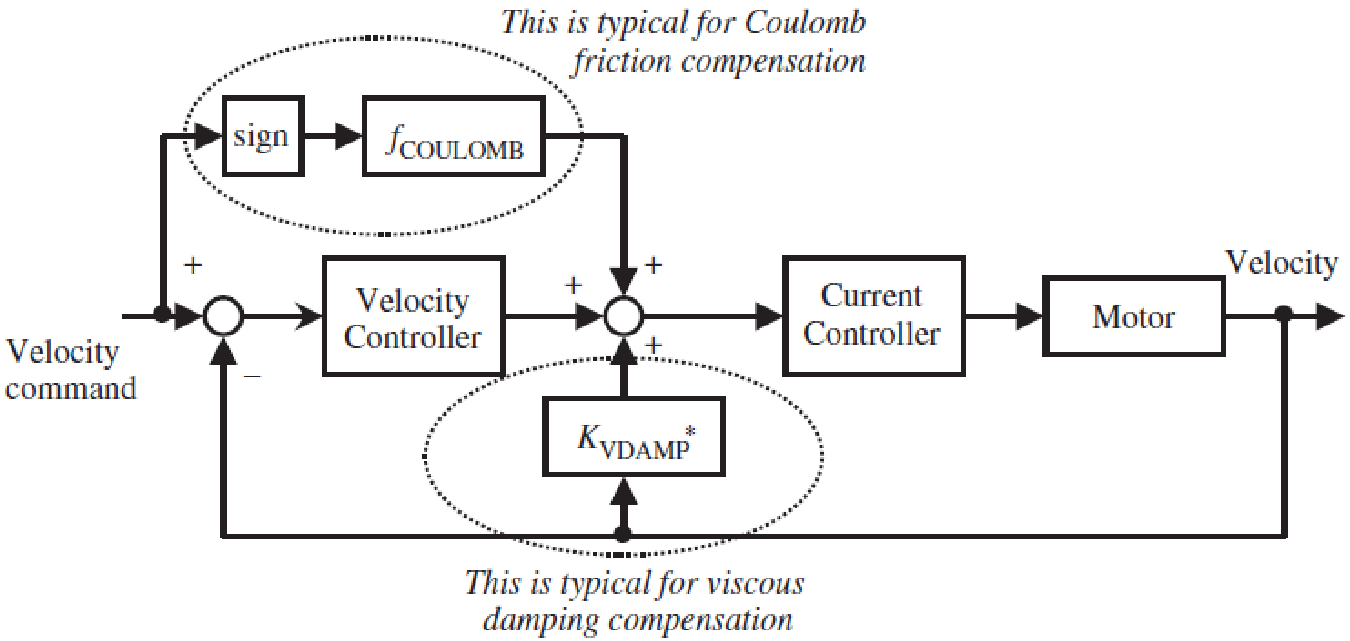

Friction (N)
Velocity (m/s)
Velocity
Stribeck effect
Friction
Deceleration area
Viscous friction
Time (s)
Static friction
Acceleration area



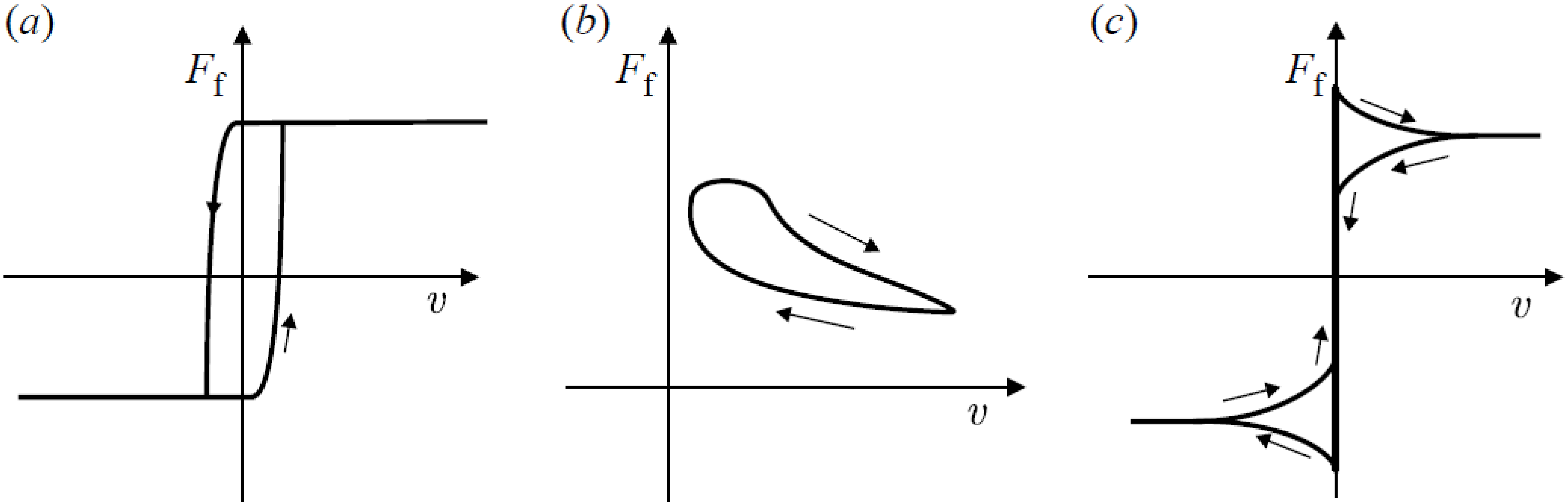
Models of friction force versus angular velocity.
(left) Static, Coulomb and viscous friction model.
(right) Negative viscous, Coulomb and viscous friction model (Stribeck).


C
C
S
R
Ideal Torque Source
f(x) = 0
Solver
Configuration
R
C
Rotational Damper
C
R
Rotational Friction
C
R
T


Friction Torque
Total torque
100
80
60
40
20
0
0
5
10
15
20
25
20
0
0
1
2
× 10^-5

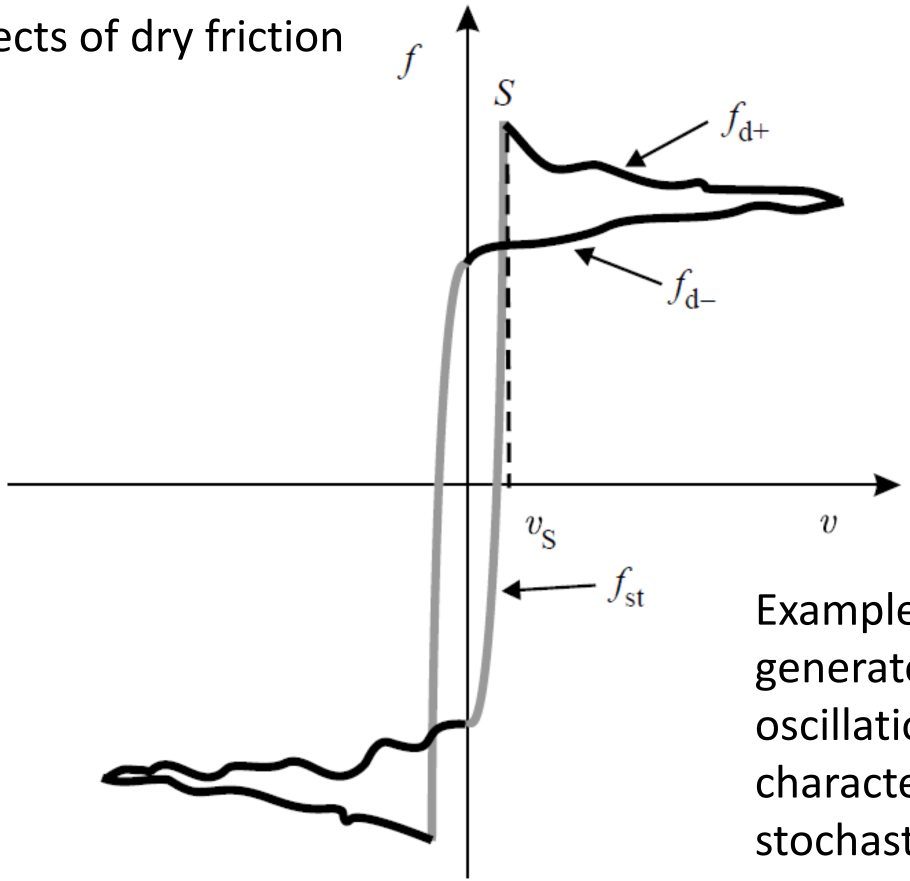


Typical compensation techniques for Coulomb friction and viscous damping

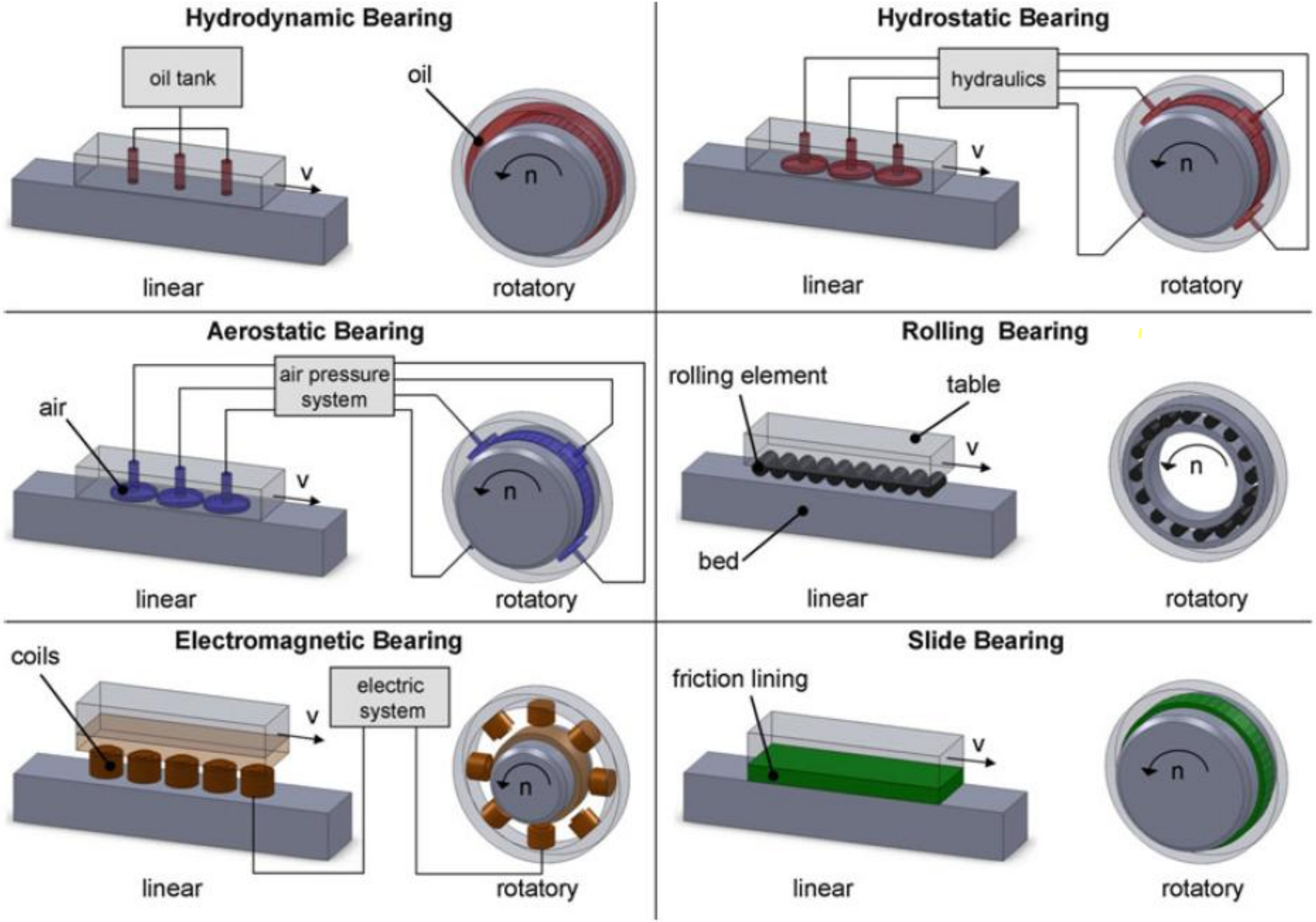


(a) Hysteretic effects of dry friction: contact compliance, (b) frictional memory and (c) non-reversible friction characteristic

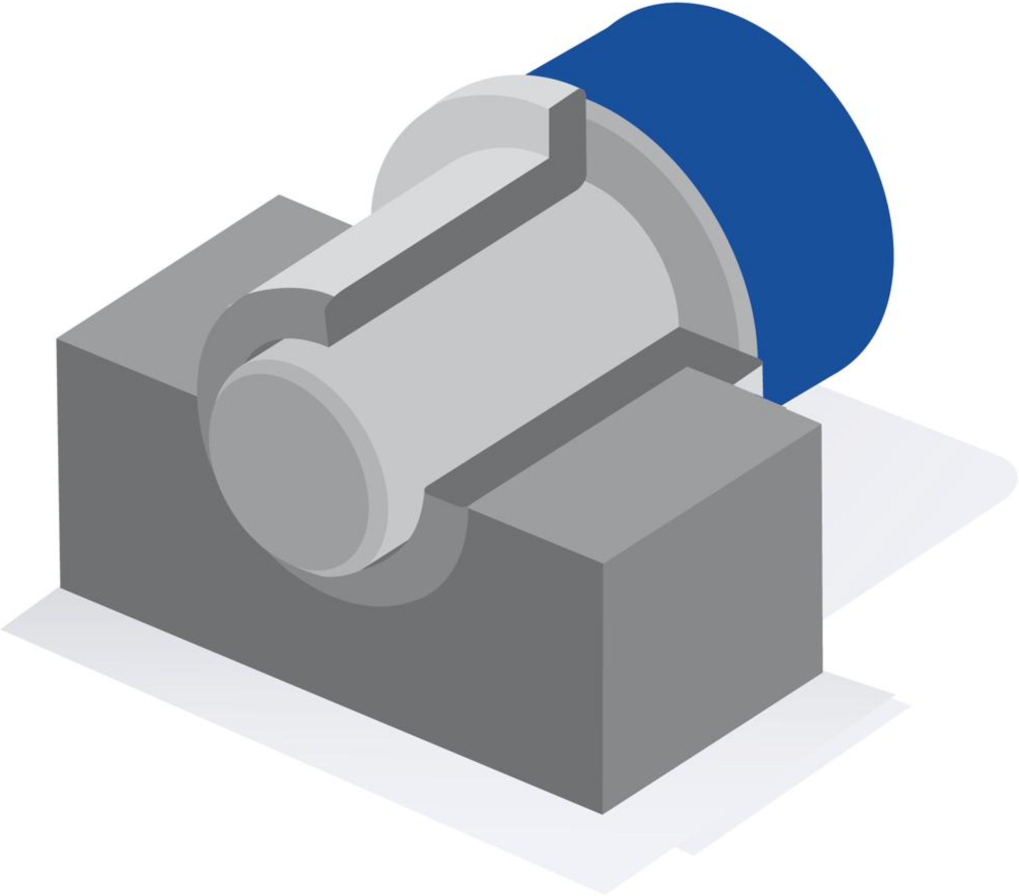
Hysteretic effects of dry friction

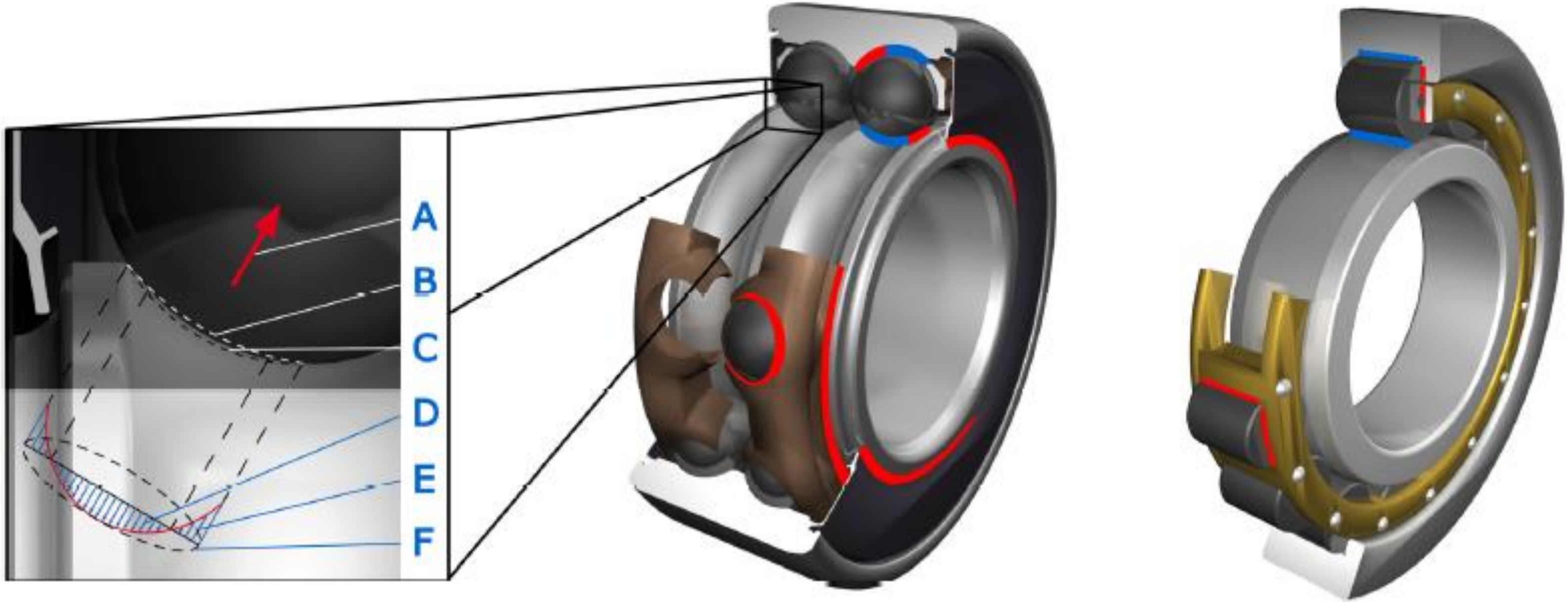


Example of dry friction characteristic generated during one cycle of oscillation. The perturbations of the characteristic's curves due to a stochastic component are visible.



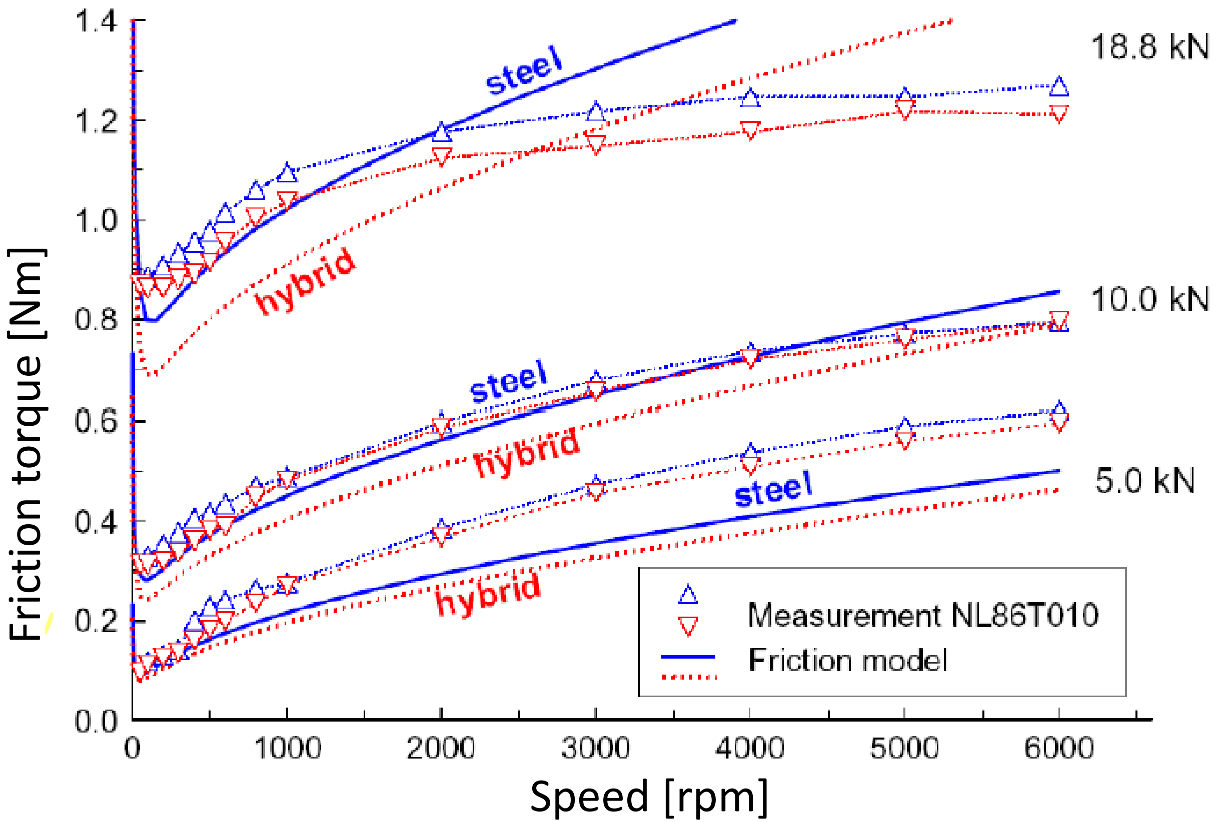
General bearing types





Rolling bearings and their sources of friction

$$F = F_{rf} + F_{sl} + F_{seal} + F_{drag}$$

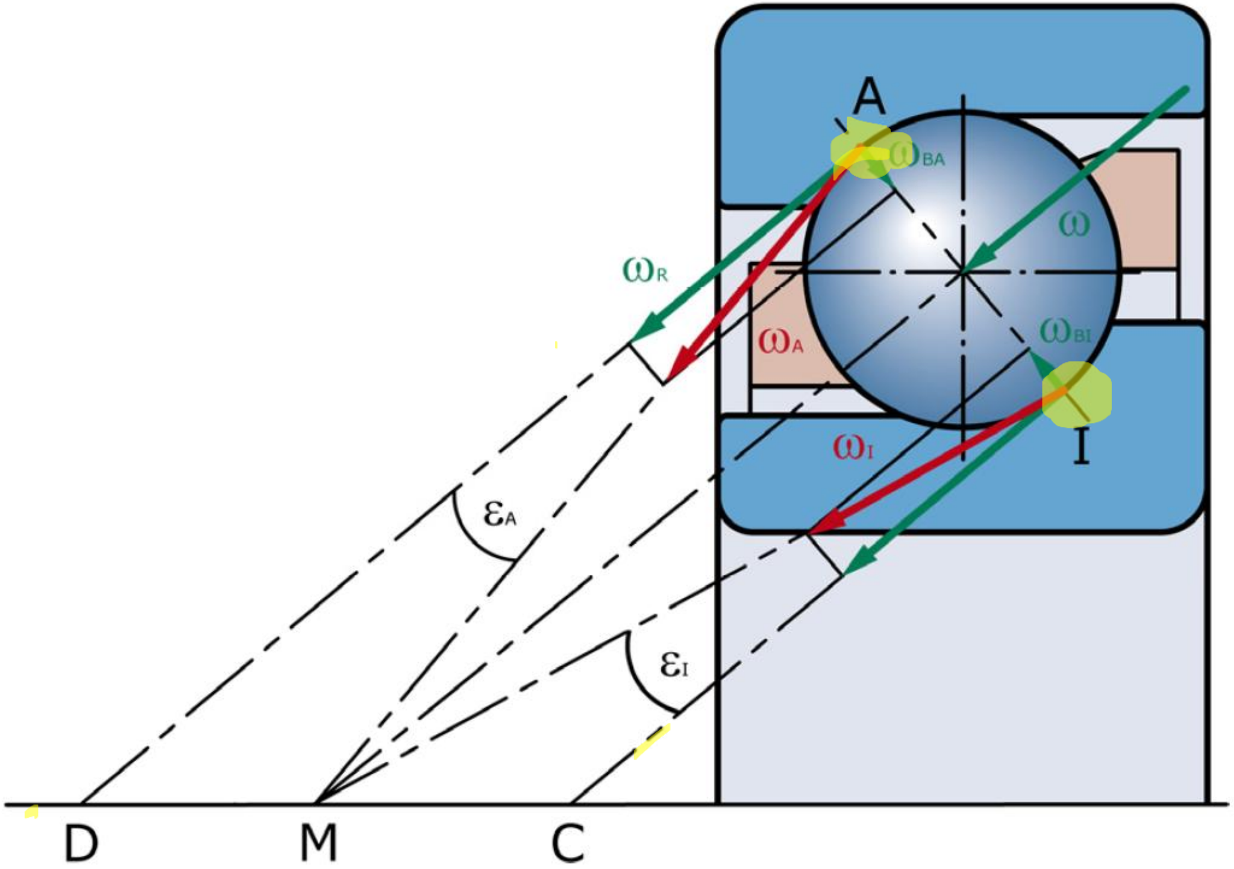


$$\omega_R = \omega_A \cos\varepsilon_A$$

$$\omega_B = \omega_A \sin\varepsilon_A$$

$$\varepsilon = \frac{\omega_B}{\omega_R}$$

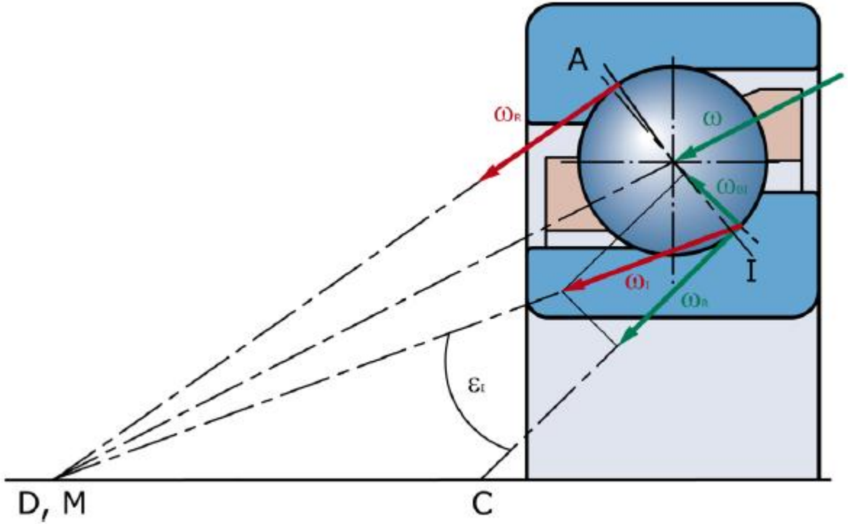
$$\tan\varepsilon = 0$$



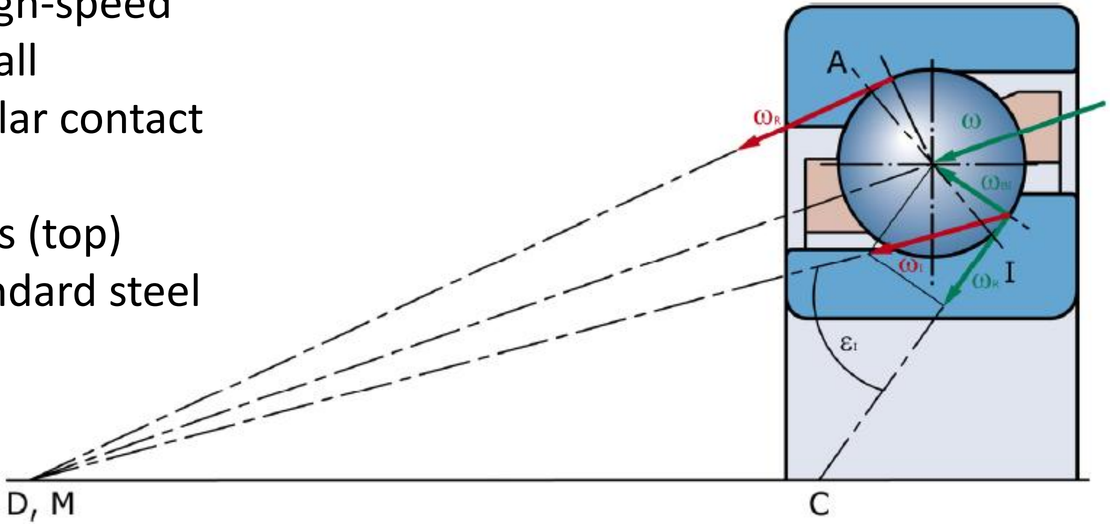
Kinematics of an angular contact ball bearing, in general

$$\varepsilon_{I\_silicon\_nitride} < \varepsilon_{I\_steel}$$

$$\varepsilon = \frac{\omega_B}{\omega_R}$$

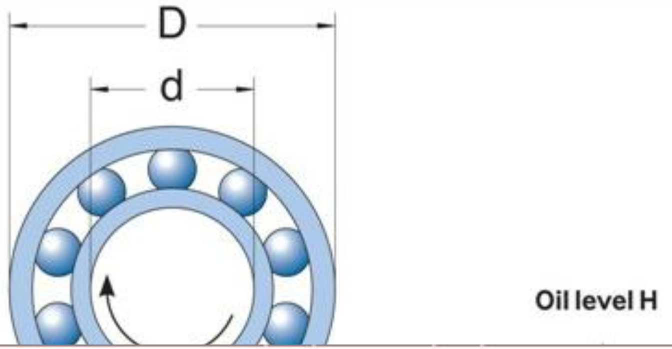


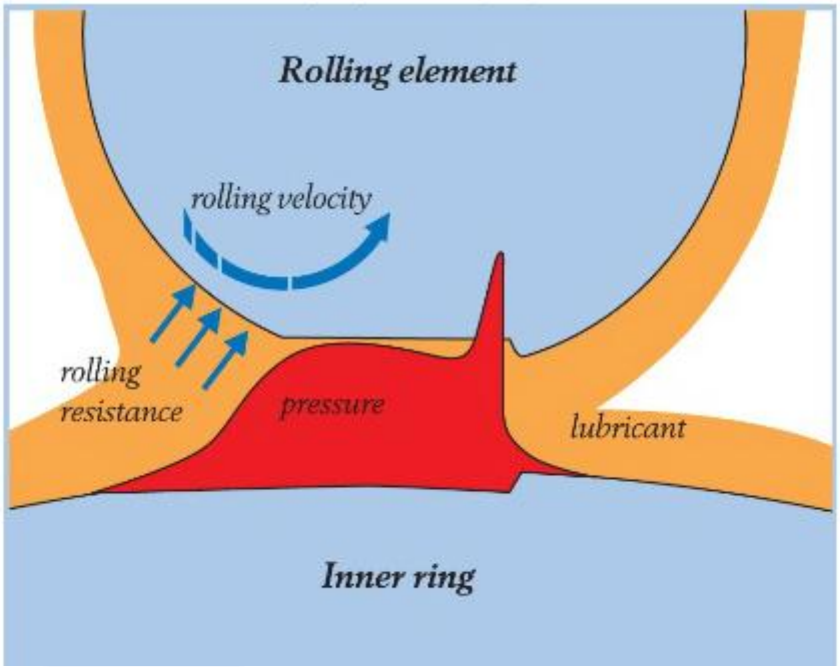
Hybrid Bearing



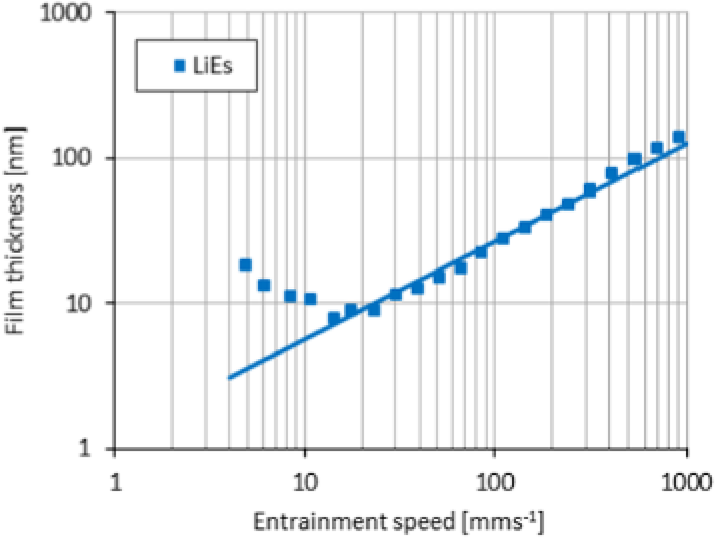
Standard Bearing

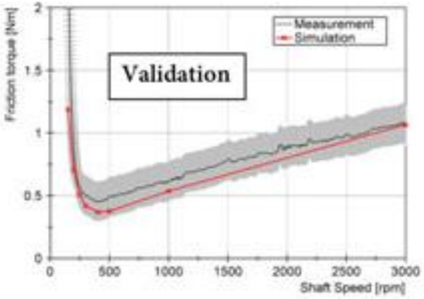
Kinematics for high-speed with outer ring ball guidance in angular contact ball bearings with ceramic balls (top) compared to standard steel balls (bottom)

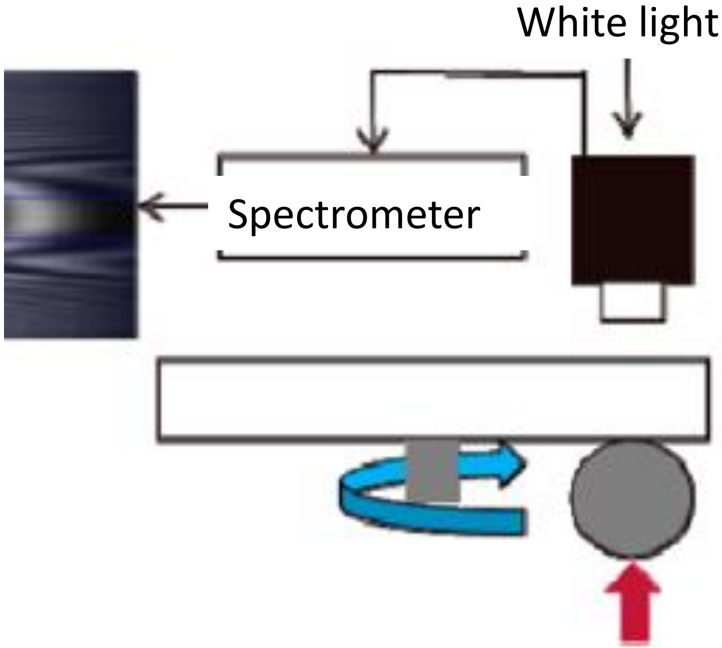

D
d
Oil level H


Rolling element
rolling velocity
rolling resistance
pressure
lubricant
Inner ring

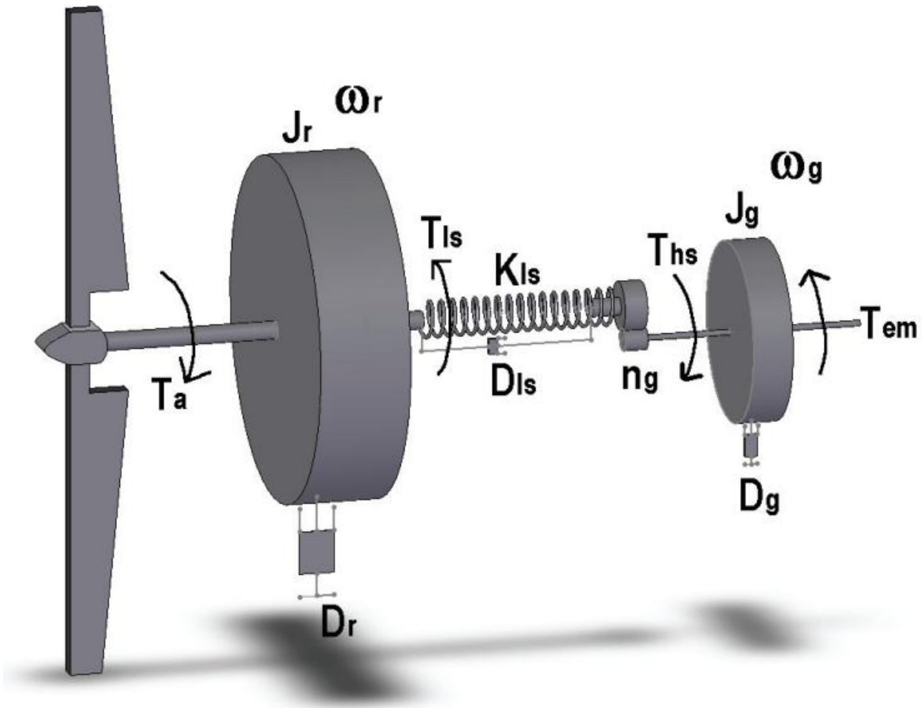

Frictional torque Nm
0.08
0.06
0.04
0.02
0
0
2 000
4 000
6 000
8 000
1 x 10⁴
1.2 x 10⁴
1.4 x 10⁴
Rotational speed, rpm
Total frictional torque
Rolling frictional torque
Sliding frictional torque
Drag frictional torque



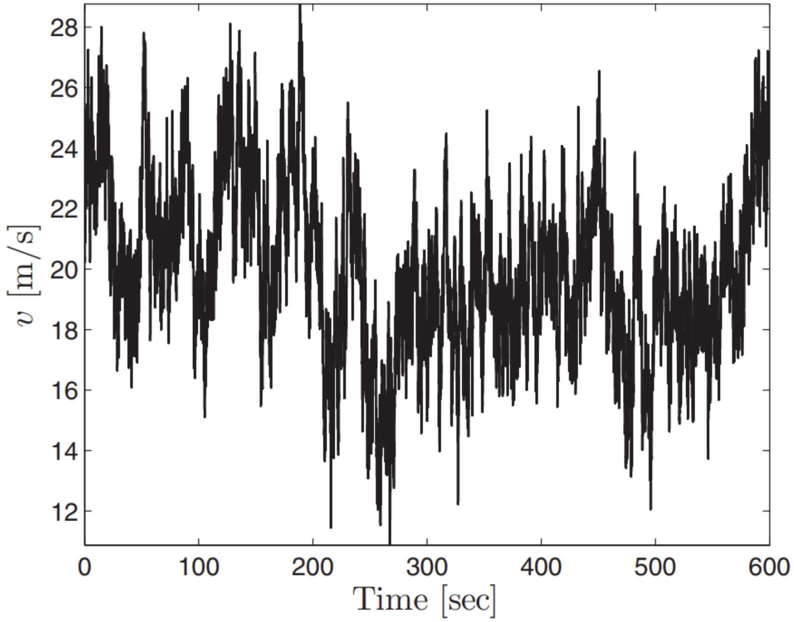




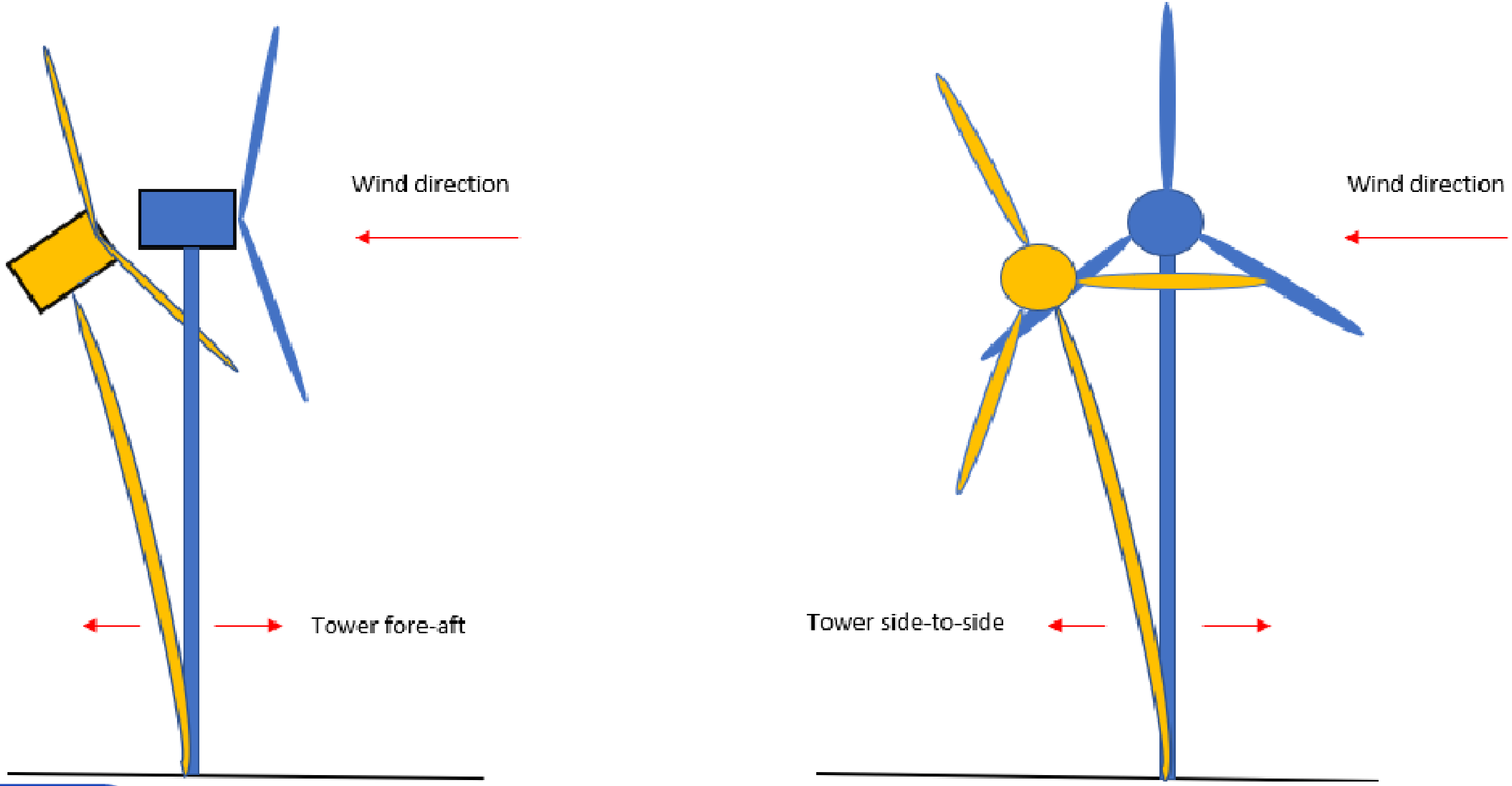
Schematic representation of film thickness



Two-mass wind turbine model



Wind speed profile of 20 m/s mean value



Fore-aft direction of wind turbine tower

Converting transfer function to state space

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \dots + a_1\frac{dy}{dt} + a_0 y = b_0 u$$

for state-space variables we choose output y (n-1) derivatives

$x_1 = y$

$x_2 = \dot{y}$

$x_3 = \ddot{y}$

$\vdots$

$x_n = \frac{d^{n-1}y}{dt^{n-1}}$

take derivatives $\Rightarrow$

$\dot{x}_1 = \dot{y} = x_2$

$\dot{x}_2 = \ddot{y} = x_3$

$\dot{x}_3 = \dddot{y} = x_4$

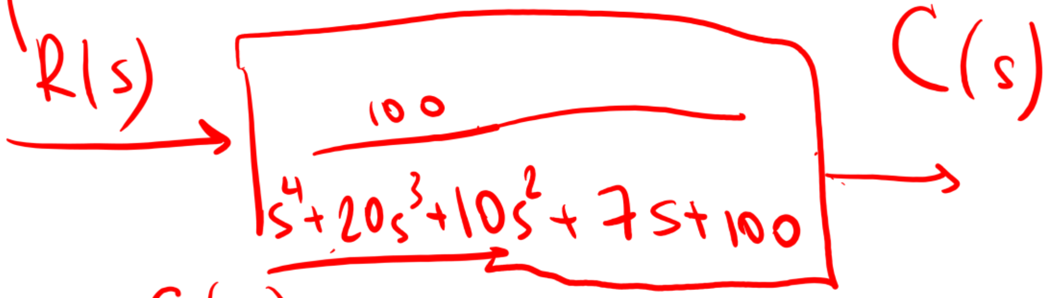
$\vdots$

$\dot{x}_n = \frac{d^n y}{dt^n}$

$$
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = x_3 \\
\dot{x}_3 = x_4 \\
\vdots \\
\dot{x}_n = -a_0 x_1 - a_1 x_2 - \dots - a_{n-1} x_n + b_0 u
\end{cases}
$$

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \vdots \\ \dot{x}_n \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots \\
-a_0 & -a_1 & -a_2 & & \cdots & -a_{n-1}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} u
$$

example



$$\frac{\text{output}}{\text{input}} \rightarrow \frac{C(s)}{R(s)} = \frac{100}{s^4+20s^3+10s^2+7s+100}$$

denominator

$$(s^4+20s^3+10s^2+7s+100)\,C(s) = 100 \cdot R(s)$$

dif. equat $\rightarrow$ take $\mathcal{L}^{-1}$

$$c^4 + 20c^3 + 10c^2 + 7c^1 + 100c = 100r$$

$x_1 = c$

$x_2 = \dot{c}$

$x_3 = \ddot{c}$

$x_4 = \dddot{c}$

fake derivatives $\Rightarrow$

$\dot{x}_1 = \dot{c} = x_2$

$\dot{x}_2 = \ddot{c} = x_3$

$\dot{x}_3 = \dddot{c} = x_4$

$\dot{x}_4 = c^4 = -20x_4 - 10x_3 - 7x_2 - 100 + 100r$

matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -100 & -7 & -10 & -20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 100 \end{bmatrix} r$$

$$Y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

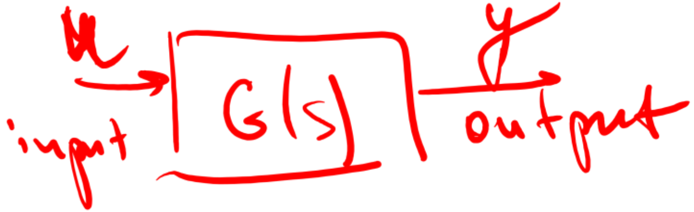
$c = x_1$

output

2x2 matrix

Converting state space to transfer function

$$\begin{cases}\dot{\bar{x}} = \bar{A}\bar{x}+\bar{B}\bar{u} & (1)\\ \bar{y}=\bar{C}\bar{x} & (2)\end{cases}$$



take Laplace eq (1) with init cond = 0

$$s\cdot\bar{X}(s)=\bar{A}\,\underbrace{\bar{X}(s)}_{\leftarrow\text{left}}+\bar{B}\,\bar{U}(s)$$

$\bar{I}$ diagonal $\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}$

$$(s\bar{I}-\bar{A})\cdot\bar{X}(s)=\bar{B}\,\bar{U}(s)$$

solve for $\bar{X}(s)$

$$\bar{X}(s) = (s\bar{I} - \bar{A})^{-1} \bar{B}\bar{U}(s) \qquad (3)$$

take Laplace eq (2)

$$\bar{Y}(s) = \bar{C}\bar{X}(s) \qquad (4)$$

transfer function $\quad G(s) = \dfrac{\bar{Y}(s)}{\bar{U}(s)} \Rightarrow$

(3) $\xrightarrow{\text{into}}$ (4) $\quad \bar{Y}(s) = \bar{C}\left[(s\bar{I} - \bar{A})^{-1} \bar{B}\bar{U}(s)\right]$

$$\bar{G}(s) \Rightarrow \bar{G}(s)\cdot\bar{u}(s) = \bar{y}(s)$$

$$\Rightarrow \frac{\bar{y}(s)}{\bar{u}(s)} = \bar{G}(s) = \underbrace{\bar{c}\,(s\bar{I}-\bar{A})^{-1}\cdot\bar{B}}_{\text{answer}}$$

example : find TF

$$\dot{x} = \underbrace{\begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u \; ; \; y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x$$

$$s\bar{I} - \bar{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & -1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$G(s) = \bar{C}\,(s\bar{I} - \bar{A})^{-1}\,\bar{B}$$

$$(s\bar{I} - \bar{A})^{-1} = \begin{bmatrix} \dfrac{s}{s^2+3s+2} & \dfrac{1}{s^2+3s+2} \\ \dfrac{-2}{s^2+3s+2} & \dfrac{s+3}{s^2+3s+2} \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{s}{s^2+3s+2} & \dfrac{1}{\cdots\cdots} \\ \dfrac{-2}{s^2\cdots\cdots} & \dfrac{s+3}{\cdots\cdots} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

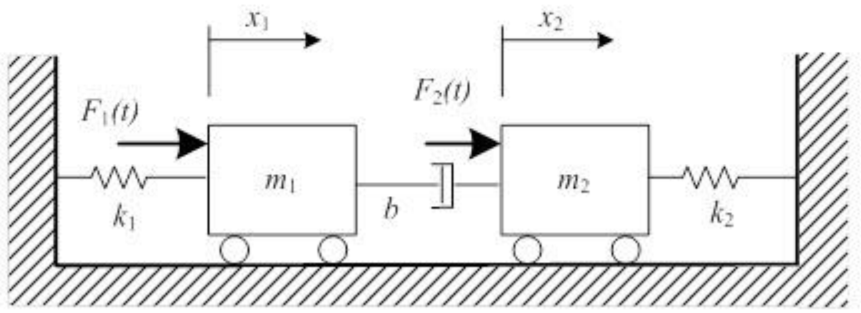
first

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \dfrac{1}{s^2+3s+2} \\ \dfrac{s+3}{s^2+3s+2} \end{bmatrix} = \frac{1}{s^2+3s+2} = \frac{1}{(s+2)(s+1)}$$

answer

Test 4

system with two masses, two springs, a damper, and two forces, as shown



Your HDU number.........

Your Name.........

1) The equations of motion

ADD your answer

2) The state-space equations (can be developed from these with the states being the original coordinates as well as their derivatives - in other words, the positions and velocities of the masses.)

ADD your answer

3) Matrices C and D. Choose the outputs to be the position of each mass. (Note that the third and fourth states are the velocity of mass 1 and mass 2, respectively.)

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} =$$ **ADD your answer**

Thank you!

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