Geometric Transformations

Image Processing

Geometric Transformations

Geometric Transformations

- **Geometric transformations** is a spatial position change of a pixels set with coordinates (x, y) from one two-dimensional rectangular system to another one with new coordinates (x', y').
  - It is note that pixels brightness's are preserved.

Geometric Transformations

- **In Euclidean space** an image pixel corresponds to a pair of Cartesian coordinates, which are interpreted as a two-dimensional vector represented by a line from the point $(0,0)$ to the point $X_i = (x_i, y_i)$.
- **Homogeneous coordinates** are coordinates with the property that the object doesn’t change in case all coordinates are multiplied by the same nonzero number.
- The required number of homogeneous coordinates is always one more than the space dimension.

Geometric Transformations

- **For example,** in the two-dimensional space P^2 to represent the point X = (x, y) in homogeneous coordinates three coordinates $X' = (x', y', w)$ are needed:

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = w \begin{bmatrix} x \\ y \\ 1 \end{bmatrix},$$

where w – arbitrary scalar factor,

$$x = \frac{x'}{w}, y = \frac{y'}{w}.$$

Geometric Transformations

- Using triples of homogeneous coordinates and third-order matrices any linear transformation of the plane can be described.
- Geometric transformations are *matrix transformations*: $X' = TX$,
  - where T is the *coordinate transformation matrix*.
- Rewrite the equation into a form:

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}.$$

Geometric Transformations

- A point with coordinates X = (x, y) in homogeneous coordinates can be written as $X = (x', y', 1)$:

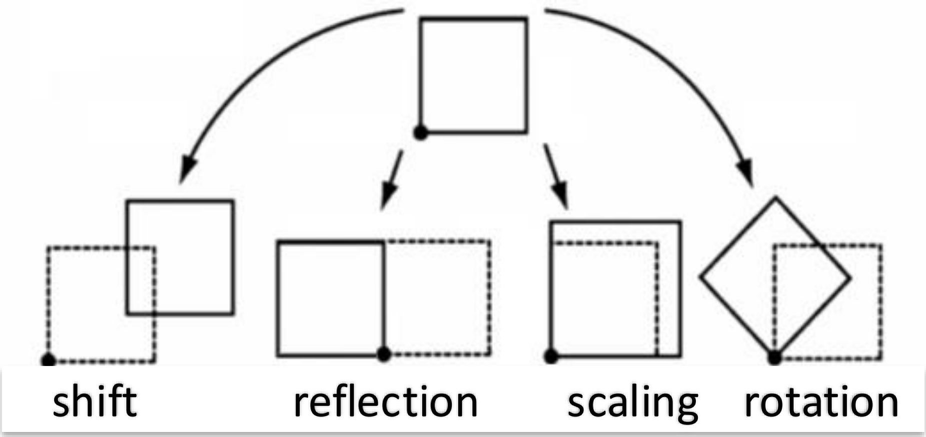
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix},$$

or in a form of equations system:

$$\begin{cases} x' = Ax + By + C \\ y' = Dx + Ey + F \end{cases}.$$

Linear Transformations

- **Euclidean mapping (transformation)** is a mapping in which the infinitely small figures shapes and the angles between the curves at the intersection points are preserved.
- *Euclidean transformations includes:*
  - Shift;
  - Reflection;
  - Uniform scaling;
  - Rotation.



Euclidean Transformations

1. **Shift**:

$$\begin{cases} x' = x + C \\ y' = y + F \end{cases}, T = \begin{bmatrix} 1 & 0 & C \\ 0 & 1 & F \\ 0 & 0 & 1 \end{bmatrix},$$

C and F – shift along axes Ox and Oy respectively.

Euclidean Transformations

2. Reflection relative to the axis Ox:

$$\begin{cases} x' = x \\ y' = -y \end{cases}, T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Euclidean Transformations

3. Scaling:

$$\begin{cases} x' = \alpha x, \alpha > 0 \\ y' = \beta y, \ \beta > 0 \end{cases}, T = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

- if $\alpha < 1$ and $\beta < 1$, then the image is reduced;
- if $\alpha > 1$ and $\beta > 1$, then the image is enlarged;
- if $\alpha = \beta$, then the image scales uniformly, and the transformation is Euclidean;
- if $\alpha \neq \beta$, then the proportions will be unequal in width and height, transformation will be **affine** (not Euclidean).

Euclidean Transformations

4. Rotation on angle φ in clockwise:

$$\begin{cases} x' = x\cos\varphi - y\sin\varphi \\ y' = x\sin\varphi + y\cos\varphi \end{cases}, T = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

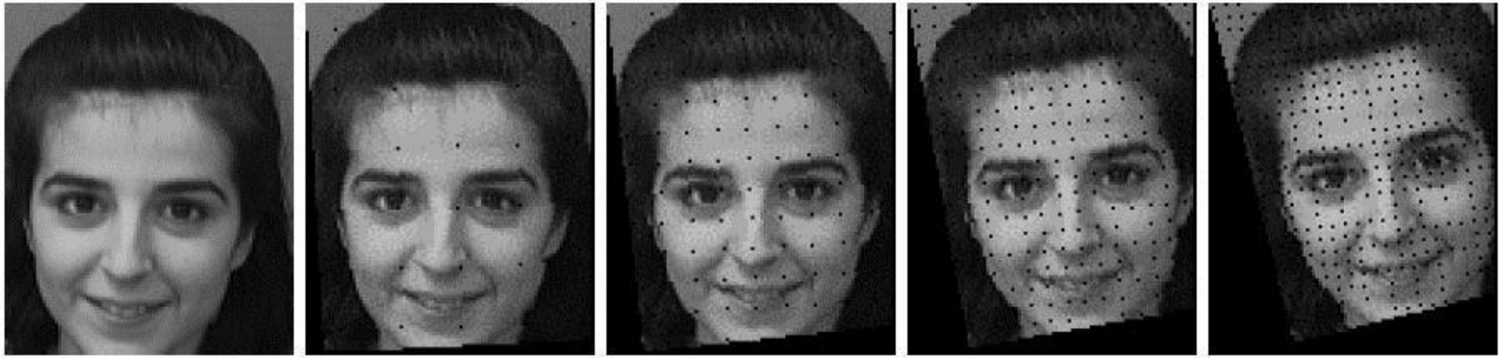
- If rotation on 90°: $\cos\varphi = 0, \sin\varphi = 1$, so:

$$\begin{cases} x' = -y \\ y' = x \end{cases}.$$

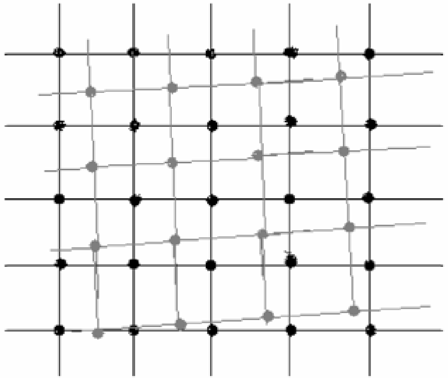
- Rotation matrix has the form:

$$T = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Pixels Uncertainty

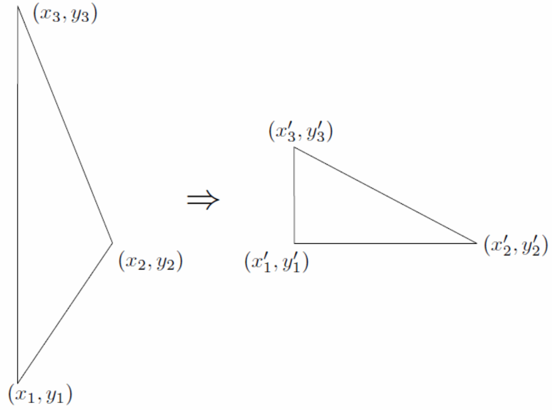


The image is rotated on 3°, 6°, 10° and 14° around the upper left corner (brightness is not defined in black pixels).



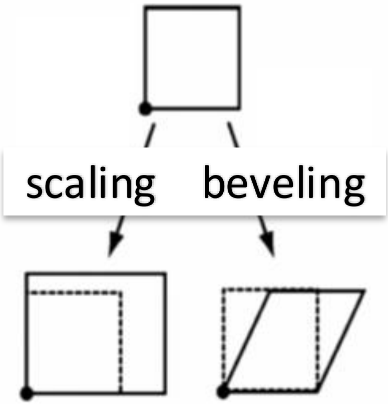
Affine Transformations

- **An affine transformation** is a transformation in which:
  - parallel lines pass into parallel lines,
  - intersecting lines pass into intersecting lines,
  - relations of the segment line's lengths lying on one straight line (or on parallel lines) are preserved,
  - relations of the figure's areas are preserved.



Affine Transformations

- Affine transformations includes Euclidean transformations, as well as **beveling** and **non-uniform scaling.**
- Any affine transformation has an *inverse affine transformation*.
- Product of the direct and inverse transformations gives a unit transformation that leaves all the points in the source place.



Affine Transformations

5. Beveling along the axis Ox:

$$\begin{cases} x' = x + sy \\ y' = y \end{cases}, T = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Affine Transformations

6. Non-uniform scaling, similar to the Euclidean mapping if $\alpha \neq \beta$:

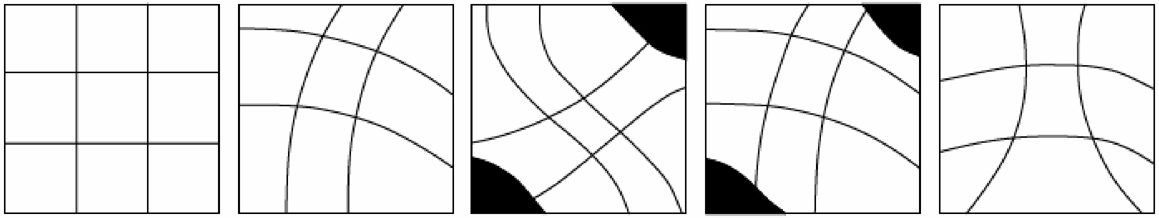
$$\begin{cases} x' = \alpha x, \alpha > 0 \\ y' = \beta y,\ \beta > 0 \end{cases}, T = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Affine Transformations

- **Arbitrary affine mapping** is a composition of consecutively performed basic transformations.
- **For example**, an alternative of the rotation matrix is the consecutive execution of three operations:
  - Beveling along the axis Ox,
  - Beveling along the axis Oy,
  - Beveling along the axis Ox.
- The rotation matrix is described as the product of three bevel matrices:

$$T = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-\tan\varphi}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \sin\varphi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{-\tan\varphi}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Nonlinear Transformations

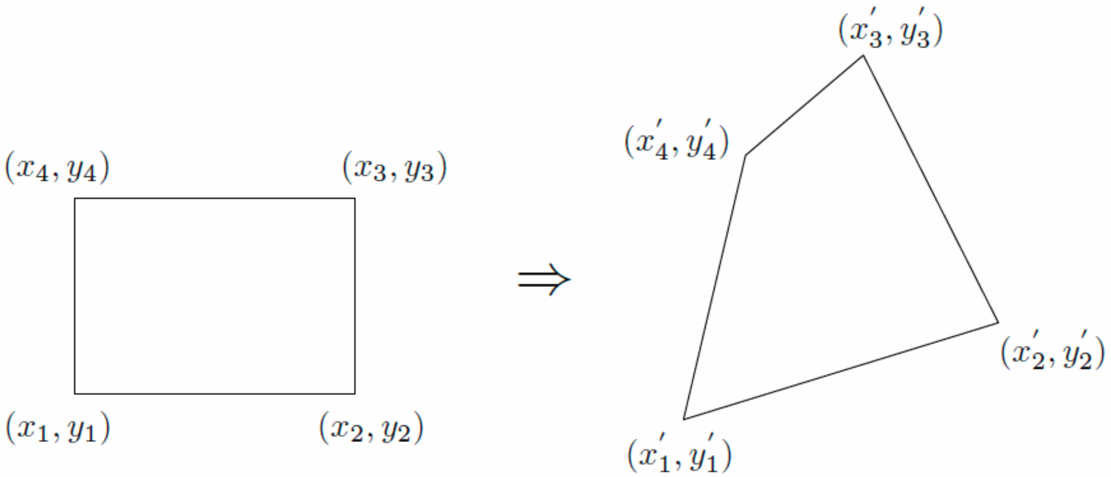


- **Projection mapping (transformation)** is a mapping in which straight lines remain straight lines, however, the geometry of the figure may be disturbed.
  - In general case, this mapping does not preserve the parallelism of lines.

Nonlinear Transformations

- A property that is preserved during **a projection transformation** is the *collinearity* of the points:
  - three points lying on one straight line (collinear) remain on the same line after the transformation.
- Projection transformation can be:
  - **parallel** (scale changes, like Euclidean transformation),
  - **projective** (figure geometry changes).

Projection Transformation



Projection projective transformation: the straight lines remained straight, but the parallel ones were mapped into intersecting.

Projective Transformation

- Transformation $P^3 \to P^2$ maps the Euclidean point of the scene $P = (x, y, z)$ (in homogeneous coordinates (x', y', z', w)) to the image point $X = (x, y)$ (in homogeneous coordinates (x', y', w)).
- To find the Cartesian coordinates of points from homogeneous coordinates, we can use the following relations:
  - $P = \left(\frac{x\prime}{w}, \frac{y\prime}{w}, \frac{z\prime}{w}\right)$ – vector $\vec{P}$ coordinates,
  - $X = \left(\frac{x\prime}{w}, \frac{y\prime}{w}\right)$ – vector $\vec{X}$ coordinates.

Projective Transformation

- Substituting into $\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$, $w = 1$ for $\vec{X}$ we'll obtain:

$$\begin{cases} x' = \dfrac{Ax + By + C}{Gx + Hy + I} \\ y' = \dfrac{Dx + Ey + F}{Gx + Hy + I} \end{cases}.$$

- Due to the normalization of coordinates on w, the projection transformation is *non-linear* in general.

Projective Transformation

- An example:

$$\begin{cases} x' = \dfrac{1{,}1x+0{,}35y}{0{,}00075x+0{,}0005y+1} \\ y' = \dfrac{0{,}2x+1{,}1y}{0{,}00075x+0{,}0005y+1} \end{cases}$$

$$\begin{cases} x' = \dfrac{1{,}1x+0{,}2y}{0{,}00075x+0{,}0005y+1} \\ y' = \dfrac{0{,}1x+0{,}9y}{0{,}00075x+0{,}0005y+1} \end{cases}$$

Polynomial Transformation

- ***A polynomial transformation*** is a transformation of the original image using polynomials.
- The coordinate transformation matrix T contains the coefficients of corresponding orders polynomials for the coordinates x and y.

Polynomial Transformation

- **For example**, in the case of a *second-order polynomial transformation*, the equations system takes the form:

$$\begin{cases} x' = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 \\ y' = b_1 + b_2x + b_3y + b_4x^2 + b_5xy + b_6y^2 \end{cases},$$

where x, y – coordinates of points in one coordinate system;

x', y' – coordinates of these points in another coordinate system;

$a_1 \dots a_6$, $b_1 \dots b_6$ – transformation coefficients.

Polynomial Transformation

- An example of second order polynomial transformation:

$$\begin{cases} x' = 0{,}1x + 0{,}9y + 0{,}002xy \\ y' = 0{,}2x + 1{,}1y + 0{,}0022xy \end{cases}$$

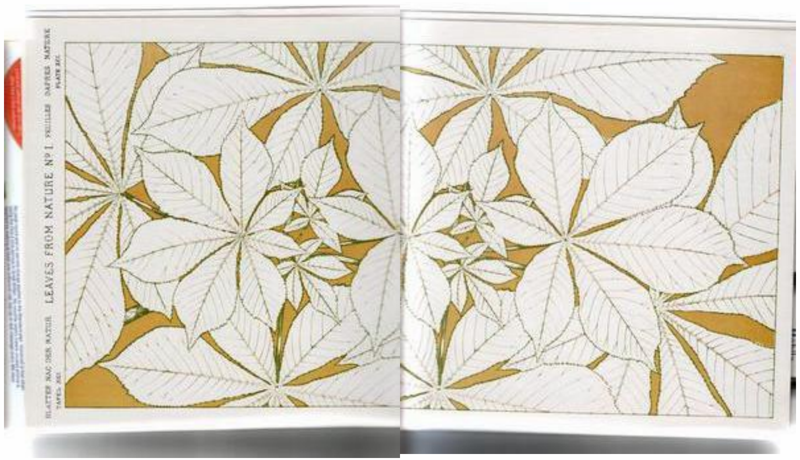
Transformations Groups

- **The Euclidean group** is a special case of the affine transformation group.
- The set of affine transformations form an **affine group**, which is a subgroup of the projective transformation group.
- These groups form the following **transformation hierarchy**:

$$\textbf{Euclidean} \subset \textbf{Affine} \subset \textbf{Projective} \subset \textbf{Nonlinear}$$

Geometric Images Correction

- **Mosaic (stitching)** is a combination of two or more images into a single image.
- Let two images were obtained by parts scanning of one large picture.
- Compulsory condition: the same objects are partially present on both images.



Geometric Images Correction

- In the general case, the images being glued may have significant differences: due to different shooting angles, camera rotation and movement of the photographed object itself, brightness changes, seasonal and daily changes, the use of a different optical system, etc.
- Consider the problem of finding a spatial transformation that allows you to determine the pixels of both images in a single coordinate system in such a way that the points corresponding to the same objects in the two images coincide.

Images Stitching

- You can use the system of the left image as the general coordinate system, then you need to find the transformation of the coordinates of all pixels of the right image (x, y) into the general coordinate system (x', y').
- To simplify the stitching problem let's assume that during the registration process there were no curvatures of straight lines, but only affine transformations.
- Affine transformations are a subset of *first-order* polynomial transformations and are described by two equations:

$$\begin{cases} x' = a_1 + a_2 x + a_3 y \\ y' = b_1 + b_2 x + b_3 y \end{cases}.$$

Images Stitching

- It’s need to find pixels that match the same objects on both images.
- Denote by
    - (x_i, y_i) are coordinates of the pixels in the right image in the coordinate system of the right image,
    - (x_i', y_i') are coordinates of the pixels in the coordinate system of the left image.

Images Stitching

- The coordinates of the same image points **are known;**
- Coefficients $a_1 \dots a_3$ and $b_1 \dots b_3$ of the first-order polynomial transformation **are unknown:**

$$\begin{cases} x' = a_1 + a_2 x + a_3 y \\ y' = b_1 + b_2 x + b_3 y \end{cases}.$$

- To calculate the unknown coefficients, a minimum quantity of common points $t_{min} = 3$ is required.
- Quantity can be calculated for the transformation of the *n*-th order as follows:

$$t_{min} = \frac{\big((n+1)(n+2)\big)}{2}.$$

Images Stitching

- **Known coordinates** of the common points before and after transformation $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x'_1, y'_1), (x'_2, y'_2), (x'_3, y'_3) substitute into the equations system of first-order polynomial transformation and obtain three pairs of equations with unknowns (a_i, b_i):

$$\begin{cases} x'_1 = a_1 + a_2 x_1 + a_3 y_1 \\ y'_1 = b_1 + b_2 x_1 + b_3 y_1 \end{cases}, \begin{cases} x'_2 = a_1 + a_2 x_2 + a_3 y_2 \\ y'_2 = b_1 + b_2 x_2 + b_3 y_2 \end{cases}, \begin{cases} x'_3 = a_1 + a_2 x_3 + a_3 y_3 \\ y'_3 = b_1 + b_2 x_3 + b_3 y_3 \end{cases}.$$

- In a matrix form:

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Images Stitching

- To calculate the coefficients (a_i, b_i), each part of the matrix equation must be multiplied by the inverse matrix on the left.
- For example, for a_i:

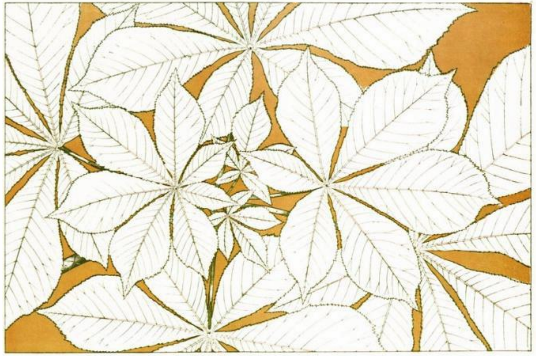
$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

- In matrix form the coefficients a_i and b_i are calculated by the formulas:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{bmatrix}.$$

Images Stitching

- Substituting the obtained coefficients into the equations system for the first order polynomial transformation and recalculating the coordinates of all pixels, we obtain:



- Higher-order transformations can be used to correct more complex types of distortion, for example, stitching images of highlands taken from a plane.

Images Stitching

- In the case of a *second-order polynomial transformation*, the equations system is given in the form:

$$\begin{cases} x' = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 \\ y' = b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 xy + b_6 y^2 \end{cases},$$

where (x, y) – coordinates of points in one coordinate system **(known);**

(x', y') – coordinates of these points in another coordinate system **(known);**

$(a_1 \dots a_6)$, $(b_1 \dots b_6)$ – coefficients **(unknown).**

Images Stitching

- The minimum required quantity of corresponding points pairs before and after the transformation for this case is 6:

$$(x_1, y_1) \dots (x_6, y_6) \text{ and } (x'_1, y'_6) \dots (x'_1, y'_6).$$

- In a matrix form:

$$\begin{bmatrix} x'_1 \\ \dots \\ x'_6 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 \end{bmatrix} \begin{bmatrix} a_1 \\ \dots \\ a_6 \end{bmatrix},$$

$$\begin{bmatrix} y'_1 \\ \dots \\ y'_6 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 \end{bmatrix} \begin{bmatrix} b_1 \\ \dots \\ b_6 \end{bmatrix}.$$

Images Stitching

- Multiplying to inverse matrices obtain the equations for finding the coefficients in matrix form:

$$\begin{bmatrix} a_1 \\ \dots \\ a_6 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 \end{bmatrix}^{-1} \begin{bmatrix} x_1' \\ \dots \\ x_6' \end{bmatrix},$$

$$\begin{bmatrix} b_1 \\ \dots \\ b_6 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_6 & y_6 & x_6^2 & x_6 y_6 & y_6^2 \end{bmatrix}^{-1} \begin{bmatrix} y_1' \\ \dots \\ y_6' \end{bmatrix}.$$

Images Stitching

- Example:

Projective Distortion

Correction

- Projective transformation is described by:

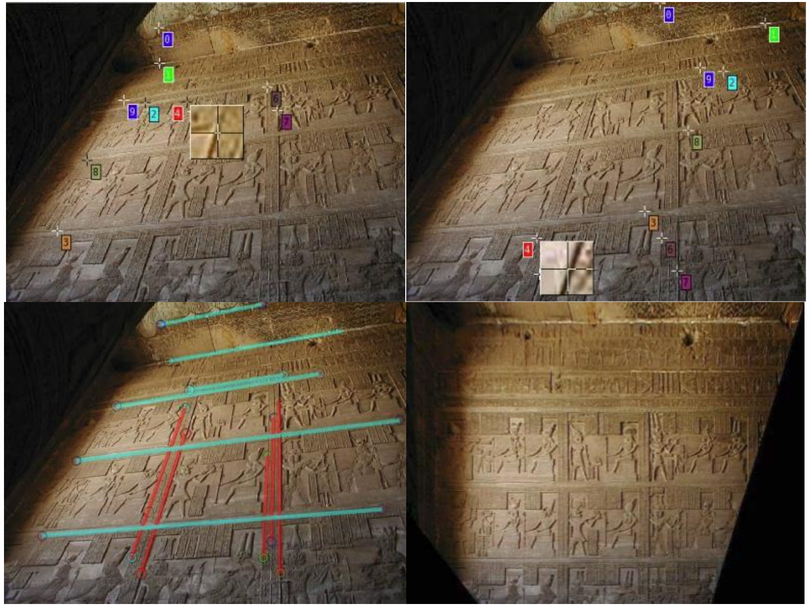
$$\begin{cases} x' = \dfrac{Ax + By + C}{Gx + Hy + I} \\ y' = \dfrac{Dx + Ey + F}{Gx + Hy + I} \end{cases}.$$

- To calculate 8 unknown coefficients (A ... H, I = 1), 8 points (4 pairs) are minimally required.
- By solving a system of linear equations, unknown transformation parameters (A ... H) are calculated.

Projective Distortion

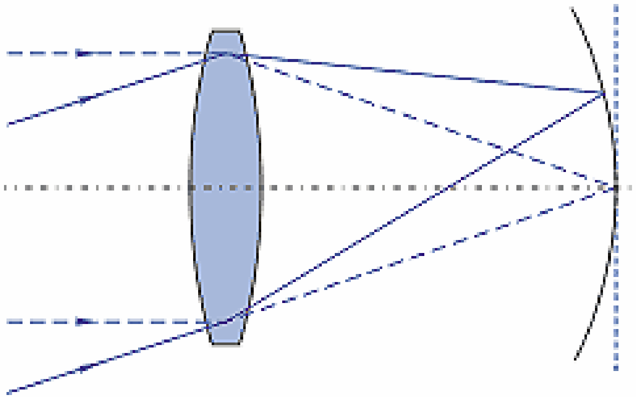
Correction

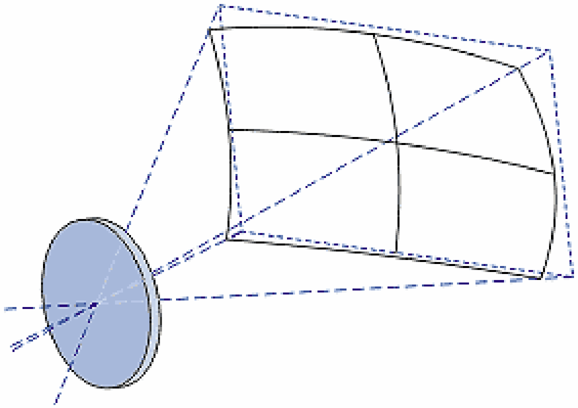
- To search for unknown parameters, it is necessary to mark the ends of the segment lines, which should be vertical and horizontal in the corrected image.



Distortion Correction

- Distortion is an optical deformation expressed in the curvature of straight lines.
- Light rays passing through the center of the lens converge at a point farther from the lens than rays that pass through its edges.

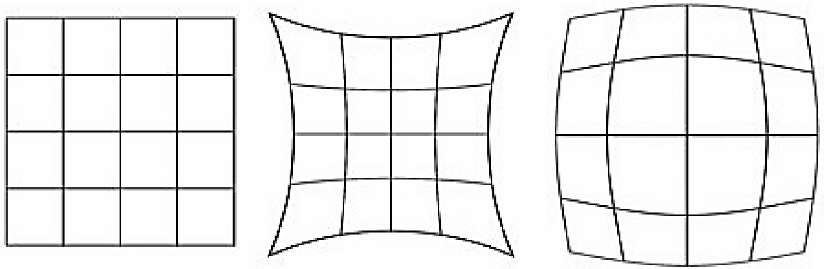




Distortion Correction

- Distortion does not violate the sharpness and brightness of the images but introduces distortion into its shape.
- Straight lines are represented by curves, except for those that lie in the same plane with the optical axis.
- **Pincushion distortion**: positive, appears with wide-angle lenses when shooting at maximum focal length.
- **Barrel-shaped distortion**: negative, appears in telephoto lenses when shooting at a minimum focal length.

Distortion Correction



Kinds of the distortion $sign(F_3) = sign(b_0)$ $sign(F_3) \neq sign(b_0)$

- $n = 1$: $\vec{R} = b_0\vec{r}$
- $n = 3$: $\vec{R} = b_0\vec{r} + F_3 r^2 \vec{r}$

b_0 – linear increasing coefficient;

F_3 – third order distortion coefficient;

r – vector length $\vec{r}$.

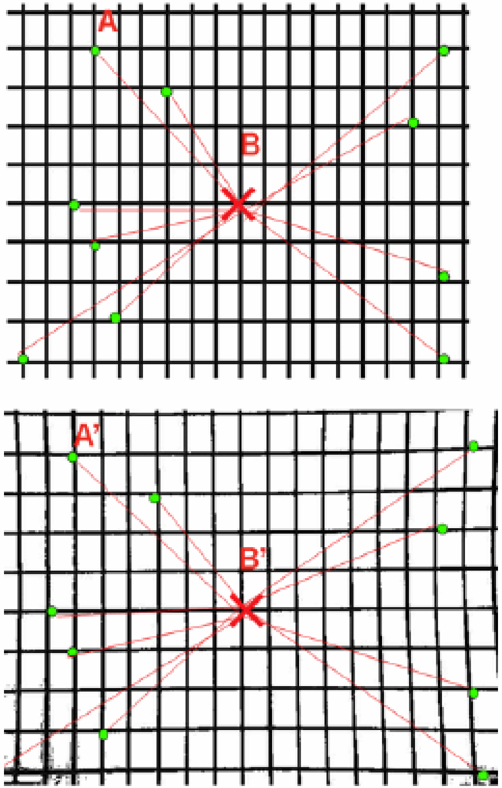
Distortion Correction

- $\vec{r} = (x, y)$ – a vector specifies the coordinates in a plane perpendicular to the optical axis;
- All rays that leave this point and pass through the optical system will fall into the image point with coordinates $\vec{R}$.
- For the same optical system, the distortion depends on the distance to the lens and on the coefficient b_0.
- Long-focus lenses have less distortion than normal lenses, and wide-angle lenses have more.
- Distortion slightly depends on the length of the reflected wave.

Distortion Correction

- To calculate the parameters of the correction transformation, we can use the image of a **regular grid** and its curved images:
  - Pairs of corresponding points are selected;
  - The vectors connecting these points with the origin are calculated;
  - The obtained parameters are substituted into the distortion equation and a system of linear equations is solved to calculate the unknown coefficients b_0 and F_3.

Distortion Correction



THANK YOU FOR YOUR TIME!

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