Low-Pass Filters

Image Processing

Noises Types

Impulse noise

- Mathematical model:

$$x_{i,j} = \begin{cases} d \text{ with probability } p, \\ s_{i,j} \text{ with probability } (1-p), \end{cases}$$

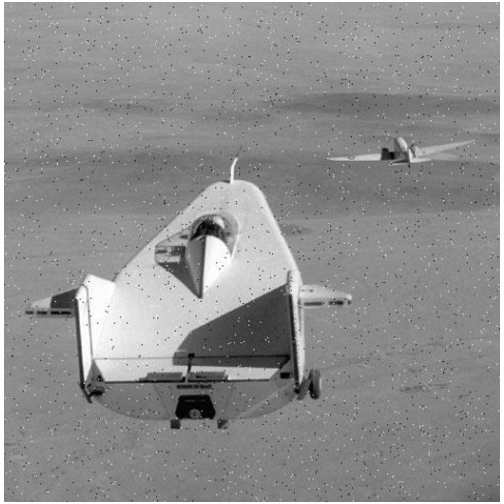
where $\{x_{i,j}\}$ – disturbed image,

$s_{i,j}$ – intensities (brightness) of source image,

p – probability noise appearing in pixel (i,j),

d – noise.

- If $d = 0$ – noise «pepper»,
- If $d = 255$ – noise «salt».



Additive noise

- Mathematical model:

$$g(x, y) = f(x, y) + \eta(x, y),$$

where $f(x, y)$ – source image,

$g(x, y)$ – noised image,

$\eta(x, y)$ – additive noise.

Multiplicative noise

- Mathematical model:

$$g(x, y) = f(x, y)\eta(x, y),$$

where $f(x, y)$ – source image,

$g(x, y)$ – noised image,

$\eta(x, y)$ – multiplicative noise.

Gaussian (normal) noise

- Mathematical model:

$$p(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(z-\mu)^2}{2\sigma^2}}$$

$p(z)$ – probability distribution density,

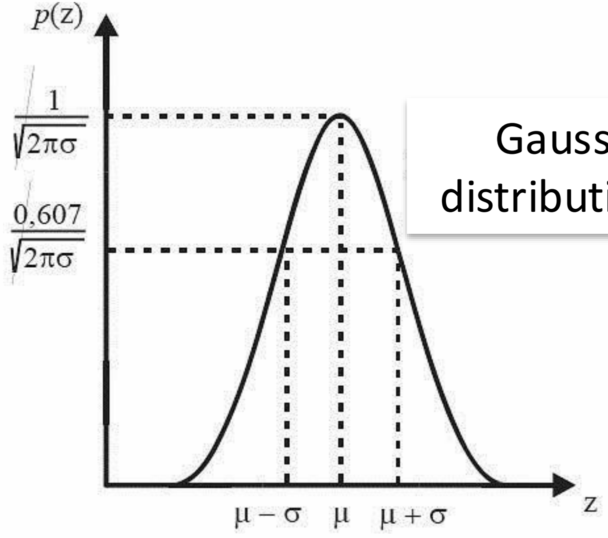
z – random variable,

μ – average value,

σ – standard deviation,

σ^2 – variance.

  - 67% z: [(μ − σ), (μ + σ)],
  - 96 % z: [(μ − 2σ), (μ + 2σ)].

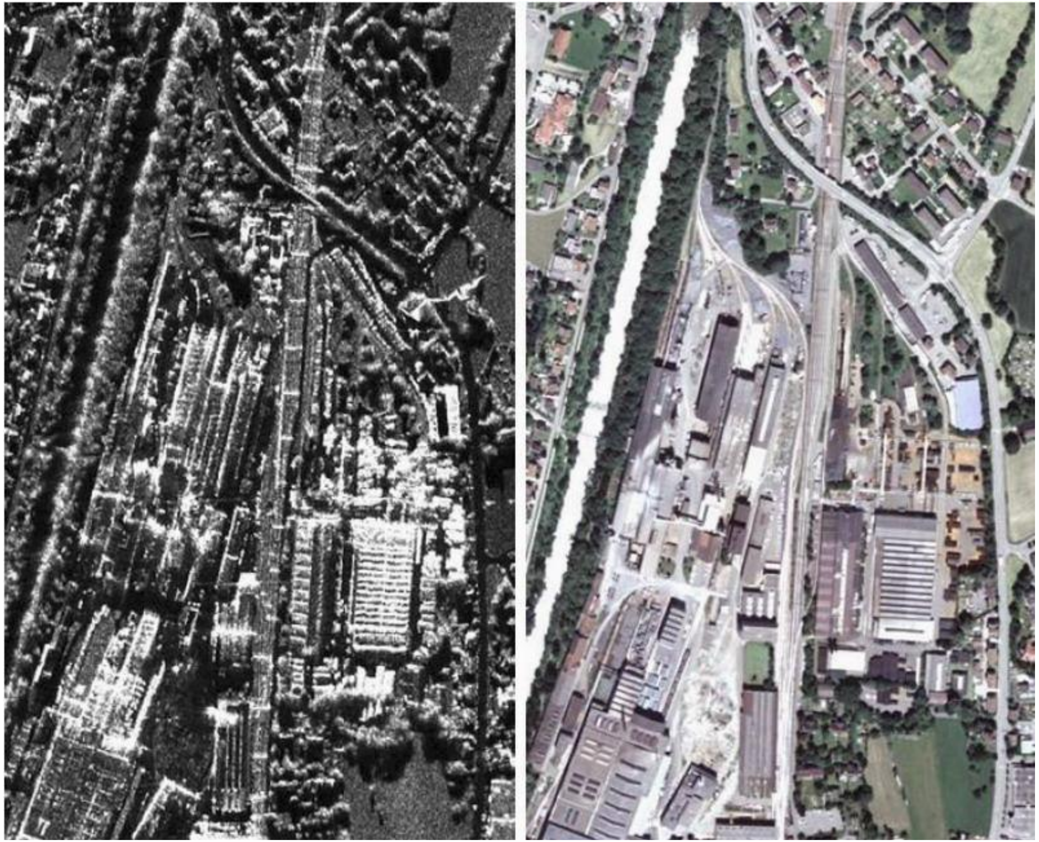


Gauss distribution

Quantization noise

- Artifacts
- Not fixed

Speckle Noise



Images Filtering

Filtering

- *Local transforms* consider brightness values in a neighborhood called a «window».
- A window is described by a matrix called a *mask* (*filter, filter core, filter kernel*).
- Matrix elements are filter *coefficients*.
- Filtering the image $f(x, y)$ with dimensions *M×N:*

$$g(x, y) = \sum_{s} \sum_{t} w(s, t) f(x + s, y + t),$$

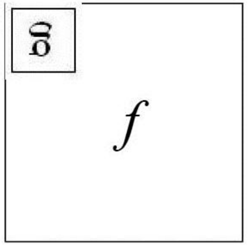
where s and t – mask w elements coordinates relative to center

(in center $s = t = 0$).

Convolution

- Filtering can be done using the *convolution operation.*
- Convolution function shows the «similarity» of one function with a reflected and shifted copy of another:

$$(f * g)(m,n) = \sum_{k,l} f(m-k, n-l)g(k,l),$$



f – image **brightness** function,

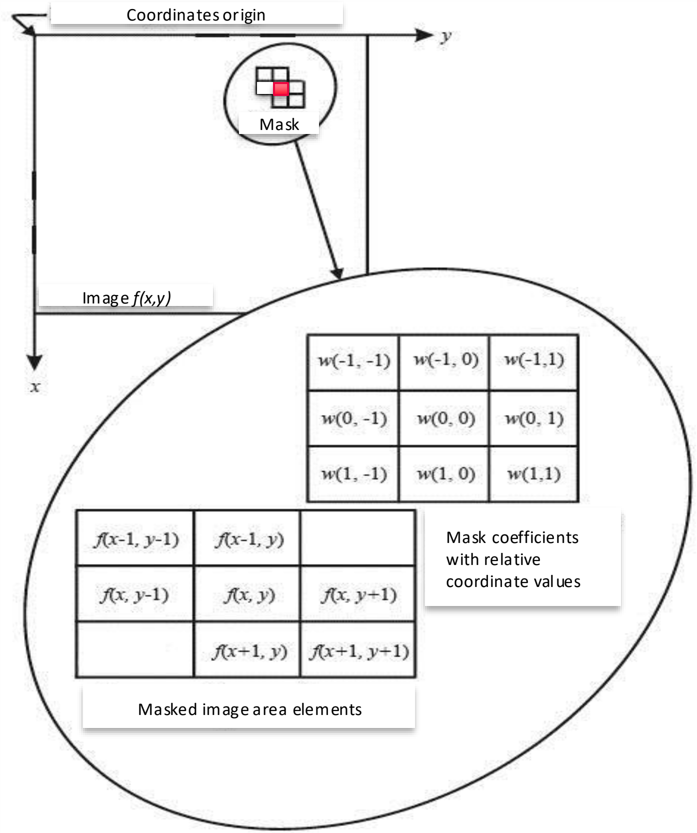
g – filter **mask**,

$(f * g)$ – **convolution** operation of image f with mask g.

Filtering

- If mask at the edge of image algorithm can:
  - restriction of the sliding window movement in the image (leave edges pixels unfiltered),
  - image expansion by adding rows and columns with zero values,
  - image expansion by adding lines and columns with values symmetrical to the edge.

Filtering



$$w(1, -1) = w(-1, 1) = 0$$

Low-Pass Filters

Low-pass Filters

- The result of low-pass filtering is image blurring.
- Low-pass filters features:
  - non-negative mask coefficients;
  - the sum of all the coefficients is equal to one.
- Examples of low-pass filter cores:

$$\frac{1}{9}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix},\frac{1}{10}\begin{bmatrix}1 & 1 & 1\\1 & 2 & 1\\1 & 1 & 1\end{bmatrix}$$

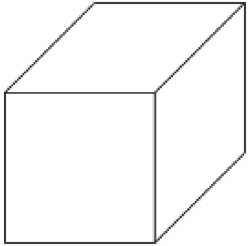
- The main group of low-pass filters is averaging (or smoothing) filters.

Arithmetic Mean Filter

(Box Filter)

- Masks with the same coefficients are used, for example:
  - for a 3x3 mask, the coefficients are 1/9,
  - for a 5x5 mask – 1/25.

$$g(x,y) = \frac{1}{M \cdot N} \sum_{i=0}^{M} \sum_{j=0}^{N} f(i,j),$$



where $g(x,y)$ – **output** pixel value,

$f(i,j)$ – **current** pixel value (mask center),

M and *N* – mask **width** and **height** correspondingly.

- The graphical representation of a 2D filter function is like a box.

Geometric Mean Filter

- Formula:

$$g(x,y) = \left[\prod_{i=0}^{M} \prod_{j=0}^{N} f(i,j)\right]^{\frac{1}{M \cdot N}},$$

where $g(x,y)$ – **output** pixel value,

$f(i,j)$ – **current** pixel value (mask center),

M and *N* – mask **width** and **height** correspondingly.

Harmonic Mean Filter

- Formula:

$$g(x,y) = \frac{M \cdot N}{\sum_{i=0}^{M} \sum_{j=0}^{N} \frac{1}{f(i,j)}},$$

where $g(x,y)$ – **output** pixel value,

$f(i,j)$ – **current** pixel value (mask center),

M and *N* – mask **width** and **height** correspondingly.

- Eliminates noise like «salt»,
- Not eliminates noise like «pepper».

Counterharmonic Mean Filter

- Formula:

$$g(x,y) = \frac{\sum_{i=0}^{M}\sum_{j=0}^{N} f(i,j)^{Q+1}}{M \cdot N \cdot \sum_{i=0}^{M}\sum_{j=0}^{N} f(i,j)^{Q}},$$

where Q – filter order:

- If $Q > 0$ filter eliminates noise of type «pepper»,
- If $Q < 0$ filter eliminates noise of type «salt».
- If $Q = 0$ filter is **arithmetic,**
- If $Q = -1$ filter is **harmonic.**

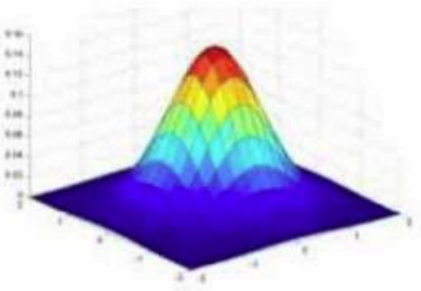
Gauss Filter

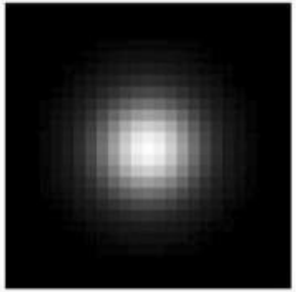
- Formula:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, G_\sigma = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}},$$

where μ – center point coordinate,

σ – variance, described width of «bell».





0.003	0.013	0.022	0.013	0.003
0.013	0.059	0.097	0.059	0.013
0.022	0.097	0.159	0.097	0.022
0.013	0.059	0.097	0.059	0.013
0.003	0.013	0.022	0.013	0.003

5 x 5, σ = 1

Properties of Gauss Filter

- Separable:

$$G_\sigma = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-y^2}{2\sigma^2}}$$

  - It allows to reduce the number of calculations from $(2r+1)^2$ up to $2(2r+1)$ on each pixel (in *r* times).
- Convolution performed twice with a mask with a radius filter *r*, gives the same result as once with a radius mask $r\sqrt{2}$.

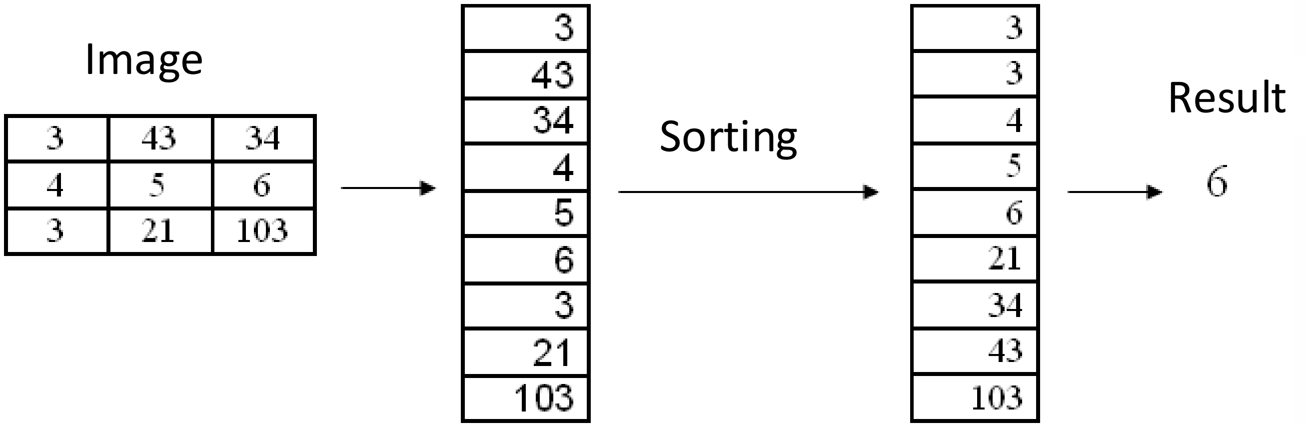
Properties of Gauss Filter

- The greater σ, the more blurred the image when applying the filter.
- The radius of the filter *r* is chosen equal to 3σ.
- The size of the mask is $2r + 1$, so it is described by a matrix of size $(6\sigma + 1) * (6\sigma + 1)$.

Nonlinear Filtering

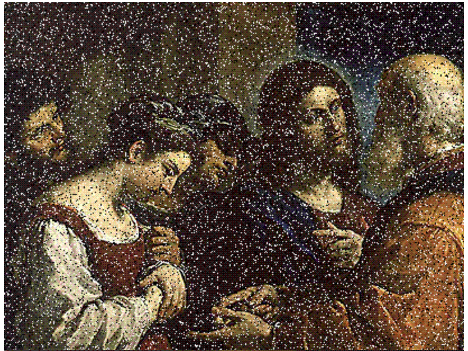
Median Filter

- The neighborhood of a mask may be arbitrary.
- Pixel brightness values in the neighborhood are sorted in ascending order.
- The result of filtering is the central pixel.



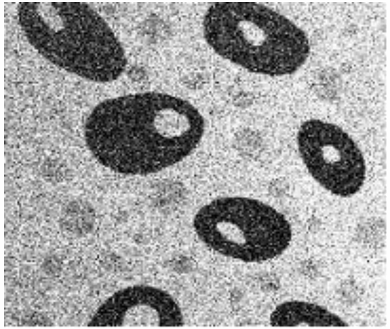
Median Filtering

Original image

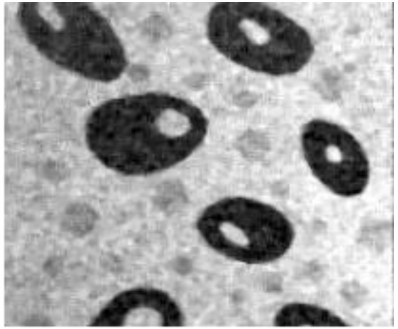


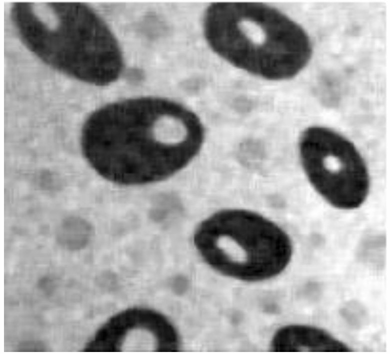
3x3 mask, applied 3 times

Original image

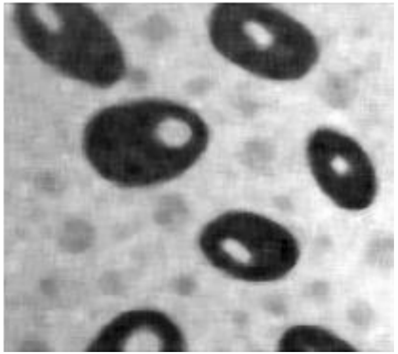


5x5 mask





7x7 mask



9x9 mask

Weighted Median Filter

- The mask uses weights (2, 3, etc.).
- The number of the median element after sorting is **(N + 1) / 2**,
  - where N is the number of brightness values in the sorting, equal to the sum of the mask weights.
  - in a sorting we should **repeat pixel intensity** by his weight factor.
- Properties:
  - inseparable;
  - nonlinear;
  - at grayscale images doesn’t introduce new brightness values;
  - qualitatively removes impulse-type noise.

Adaptive Median Filter

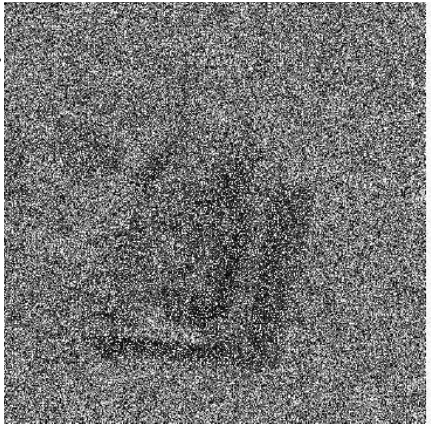
- **Idea:** increasing the size of the window S during the filtering process, depending on local statistical characteristics.
- **Key terms:**
  - $S \times S$ – window's size.
  - Z_{min} – minimum value in the window;
  - Z_{max} – maximum value in the window;
  - Z_{med} – median value in the window;
  - Z_{ij} – pixel value with coordinates (i, j);
  - S_{max} – maximum allowed window size.

Adaptive Median Filter
(Algorithm)

1. Set the initial filtering window size S and maximum S_{max}.
2. For pixel (i, j) with intensity Z_{ij}:
   - **Calculate:** $Z_{min}, Z_{max}, Z_{med}, A_1 = Z_{med} - Z_{min}, A_2 = Z_{med} - Z_{max}$.
   - If $A_1 > 0$ and $A_2 < 0$ **go to step 3.**
     - Otherwise, increase window size;
   - If current size $S \leq S_{max}$ **repeat step 2**;
     - Otherwise, **filtering result is $\boldsymbol{Z_{ij}}$.**
3. Calculate $B_1 = Z_{ij} - Z_{min}, B_2 = Z_{ij} - Z_{max}$.
   - If $B_1 > 0$ and $B_2 < 0$, **filtering result is $\boldsymbol{Z_{ij}}$.**
   - Otherwise, **filtering result is $\boldsymbol{Z_{med}}$.**
4. Change coordinates (i, j)
   - If is not end of image, **go to step 2**.
   - Otherwise, end.

Adaptive Median Filtering

- Advantages:
  - optimal removal of impulse noise;
  - smoothing other types of noise.
- Disadvantages:
  - increases the amount of computati



Noised image

Filtered image

Rank Filtering

- A rank filter of order r ($1 \le r \le N$, where N is the number of elements in the neighborhood) selects the element with number r from the resulting series and assigns its value as a result of pixel filtering.
  - If the number N is odd and $r = (N + 1) / 2$, the filter becomes median.
  - If $r = 1$, the filter selects the minimum brightness value in the window and called «min-filter».
  - If $r = N$, the filter selects the maximum brightness value in the window and called «max filter».
- The rank can be set in percent, then the choice of the minimum value corresponds to 0%, median – 50%, and maximum – 100%.

Activity Time

What is the result of
median filtering?

$$\begin{bmatrix} 13 & 11 & 3 \\ 1 & \mathbf{9} & 2 \\ 0 & 2 & 14 \end{bmatrix}$$

What is the result of
median filtering?

$$\begin{bmatrix} 13 & 11 & 3 \\ 1 & \mathbf{9} & 2 \\ 0 & 2 & 14 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 13 & 11 & 3 \\ 1 & \mathbf{3} & 2 \\ 0 & 2 & 14 \end{bmatrix}$$

What is the result of
weighted median filtering?

$$\text{Mask } M = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix},$$

$$\text{Fragment of image } Im = \begin{bmatrix} 0 & 1 & 2 & 3 & 2 \\ 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 2 \\ 2 & 3 & 1 & 0 & 1 \\ 2 & 1 & 2 & 3 & 1 \end{bmatrix}.$$

- <u>Leave the edge pixels unfiltered.</u>

What is the result of

min / max filtering?

$$\text{Fragment of image } Im = \begin{bmatrix} 0 & 1 & 2 & 3 & 2 \\ 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 2 \\ 2 & 3 & 1 & 0 & 1 \\ 2 & 1 & 2 & 3 & 1 \end{bmatrix}.$$

- Leave the edge pixels unfiltered.

THANK YOU
FOR YOUR TIME!

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