Filter theory

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Gain and Attenuation Functions

frequency responses

magnitude $G(\omega) = |H(j\omega)| = \left| \dfrac{V_2(j\omega)}{E(j\omega)} \right|$

phase $\phi(\omega) = argument\,(H(j\omega))$

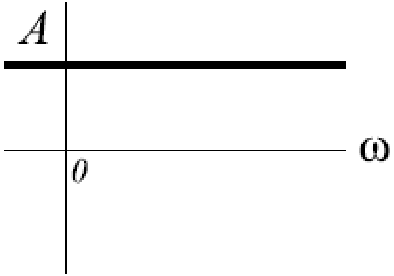
logarithmic gain $G_{dB}(\omega) = 20\log(G(\omega)) = 20\log\left(\dfrac{|V_2(j\omega)|}{|E(j\omega)|}\right)\ \text{(dB)}$

attenuation $A(\omega) = 20\log\left(\dfrac{|E(j\omega)|}{|V_2(j\omega)|}\right) = 20\log\left(\dfrac{1}{G(\omega)}\right) = -G_{dB}(\omega)\ \text{(dB)}$

Ideal Transmission

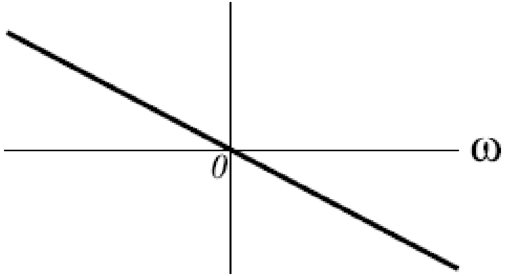
Ideal Transmission

$G(\omega)=|H(j\omega)|=A$



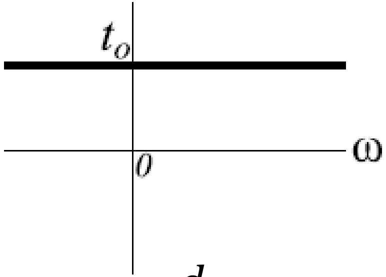
$G(\omega) = |H(j\omega)| = A$

$\varphi(\omega)=-\omega t_0$



$\varphi(\omega) = t_0\omega$

$D(\omega)$



$$D(\omega) = -\frac{d}{d\omega}\phi(\omega) = t_0$$

Real Transmission

$G(\omega) = |H(j\omega)| = f_1(\omega) \neq$const

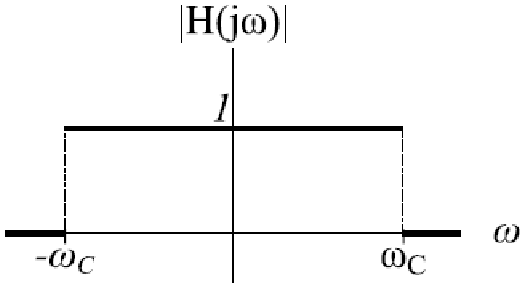
$\varphi(\omega) = f_2(\omega) \neq t_0\omega$

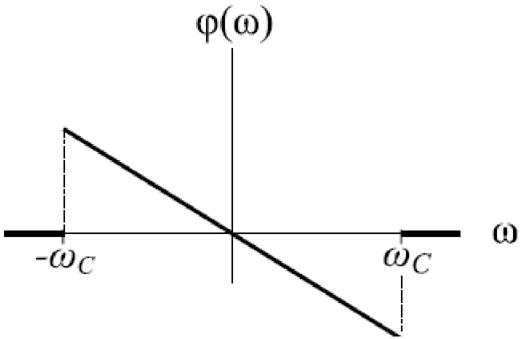
$$D(\omega) = -\frac{d}{d\omega}\phi(\omega) \neq t_0$$

Ideal Filters

ideal lowpass filter transfer function

$$H(j\omega) = \begin{cases} e^{-j\omega t_0} & for\ |\omega| \le \omega_C \\ 0 & for\ |\omega| \ge \omega_C \end{cases}$$

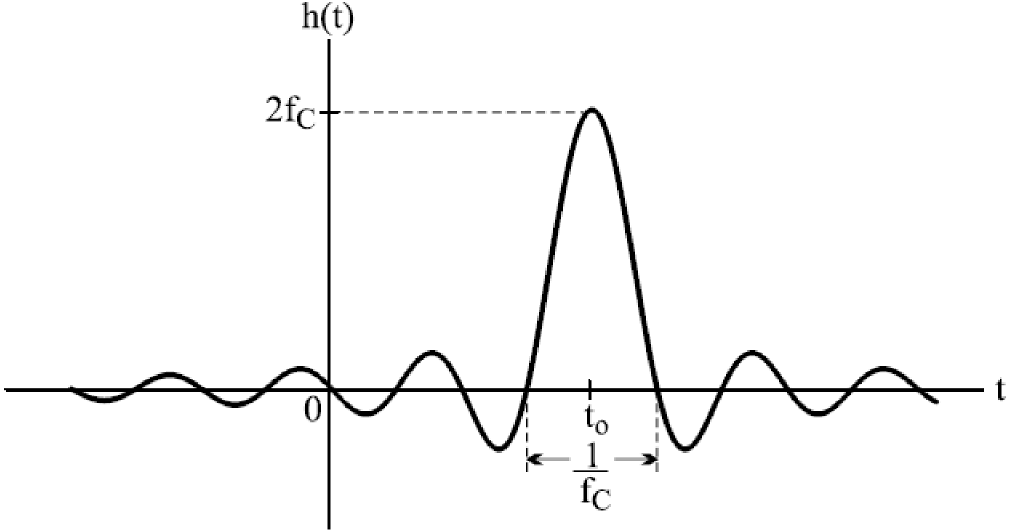




$$|H(j\omega)| = \begin{cases} 1 & for\ |\omega| \le \omega_C \\ 0 & for\ |\omega| \ge \omega_C \end{cases}$$

$$\varphi(\omega) = t_0\omega$$

Ideal Filters



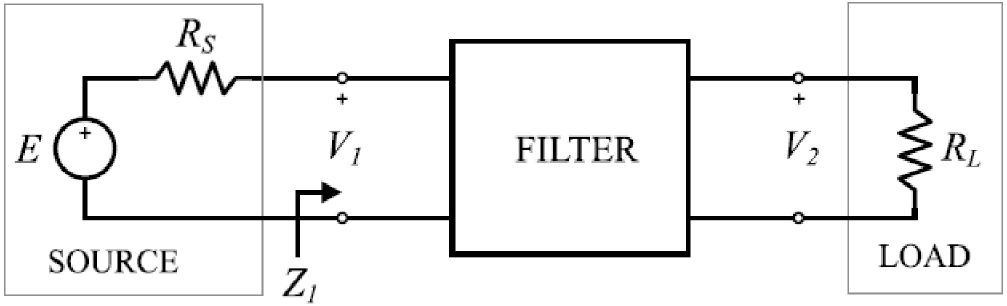
$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|e^{-j\omega t_0}e^{j\omega t}d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-t_0)}d\omega =$$

$$= \frac{\omega_C}{2\pi}\frac{sin\left(\omega_C(t-t_0)\right)}{\omega_C(t-t_0)}$$

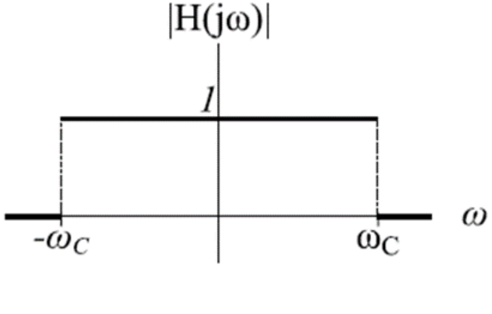
Real Electronic Filters

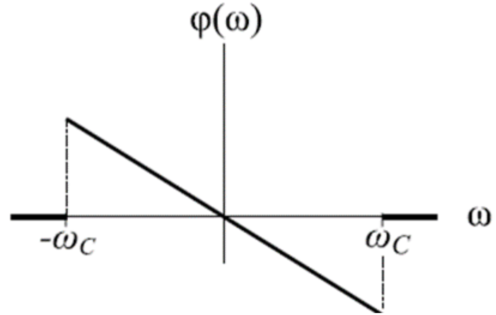


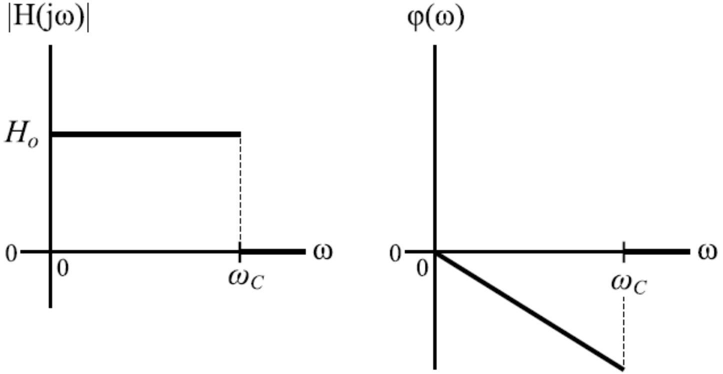
$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|e^{-j\omega t_0}e^{j\omega t}d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t-t_0)}d\omega = \frac{\omega_C}{2\pi}\frac{sin\left(\omega_C(t-t_0)\right)}{\omega_C(t-t_0)}$$

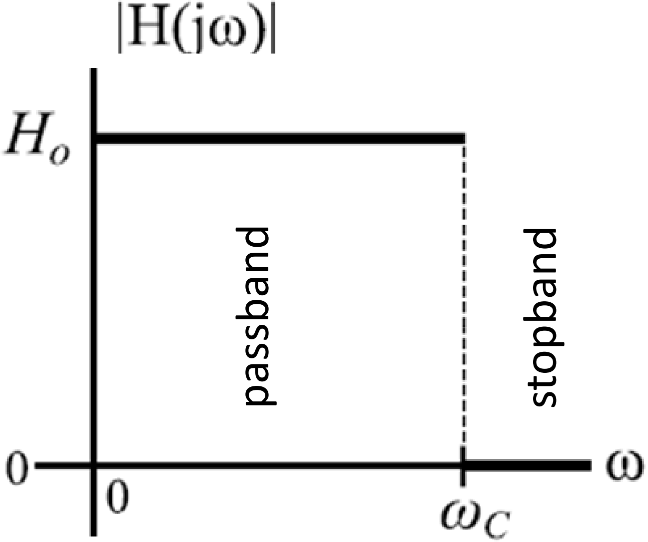
Realizable Lowpass Filters

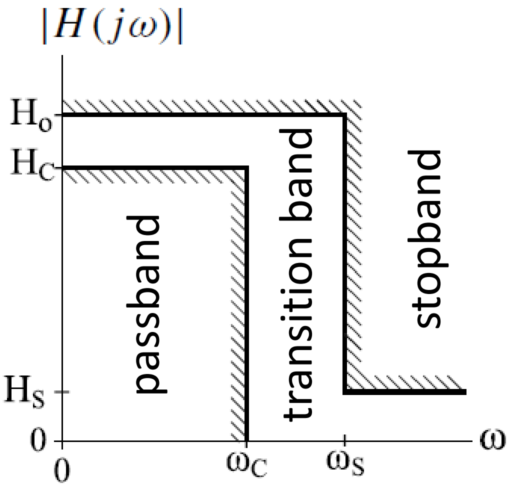






Realizable Lowpass Filters





H_C - minimum allowed gain in the passband

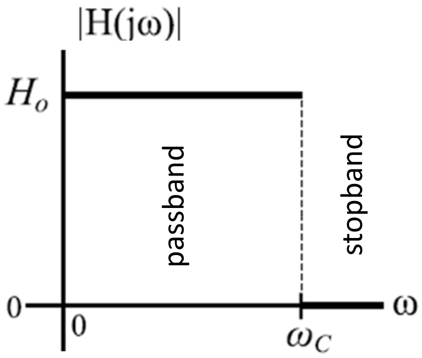
H_S - maximum allowed value of the gain in the stopband

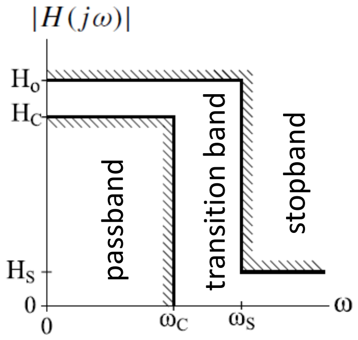
ω_C - cut-off frequency

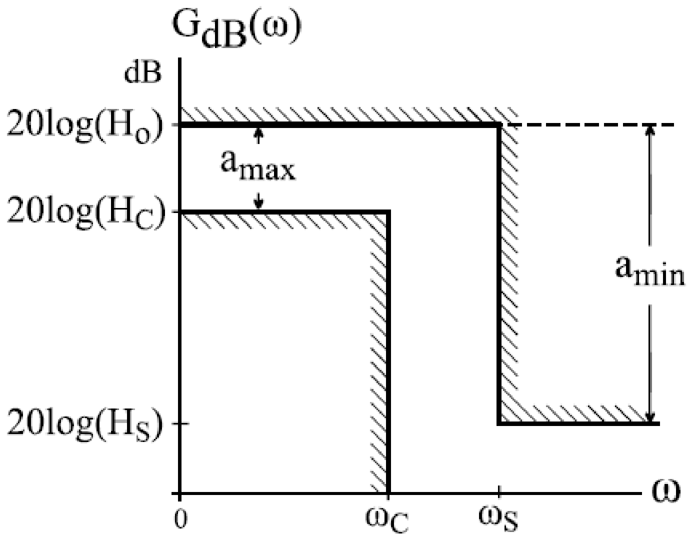
ω_S - stopband edge frequency

The five quantities H_o, H_C, H_S , ω_c *and* ω_s constitute the magnitude specifications of the realizable lowpass filter and are dictated by the requirements of the system

Realizable Lowpass Filters







$$G_0 = 20log(H_0) \quad H_0 = 10^{\frac{G_0}{20}}$$

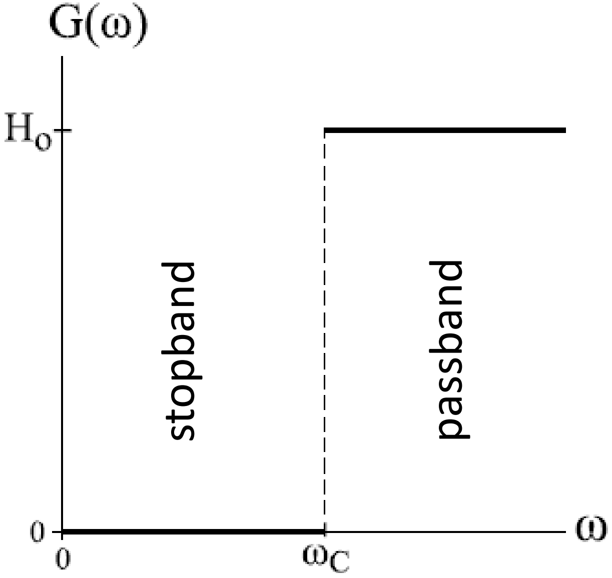
$$G_C = 20log(H_C) \quad H_C = 10^{\frac{G_C}{20}}$$

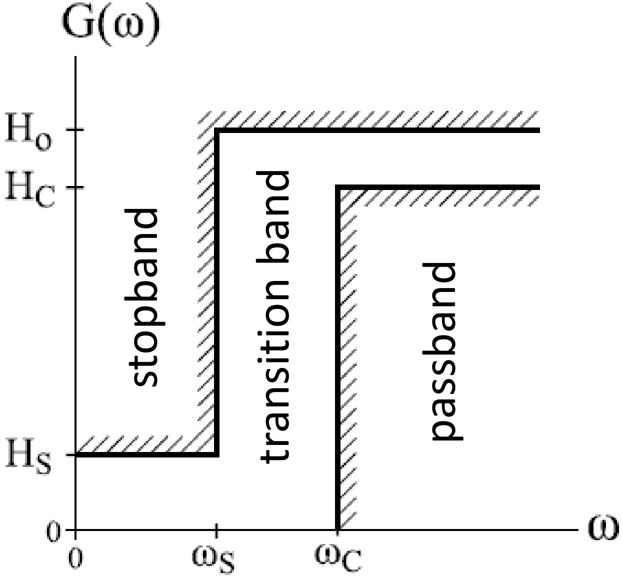
$$G_S = 20log(H_S) \quad H_S = 10^{\frac{G_S}{20}}$$

$$\alpha_{max} = 20log\left(\frac{H_0}{H_C}\right) \quad \frac{H_0}{H_C} = 10^{\frac{\alpha_{max}}{20}}$$

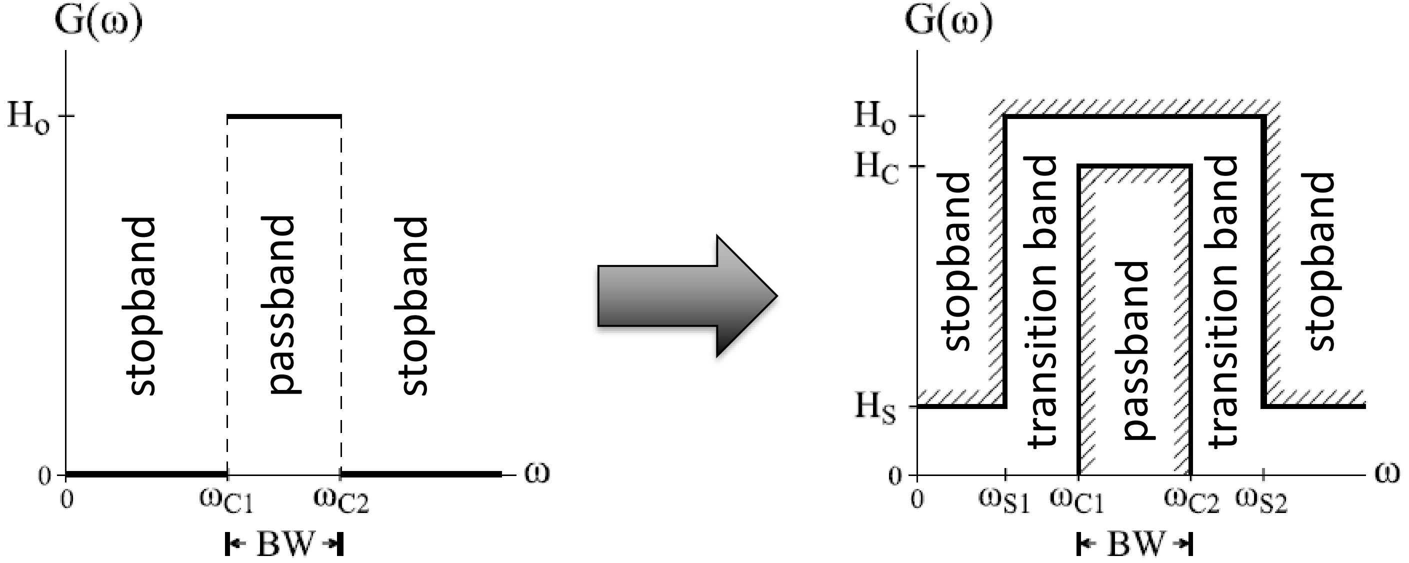
$$\alpha_{min} = 20log\left(\frac{H_0}{H_S}\right) \quad \frac{H_0}{H_S} = 10^{\frac{\alpha_{min}}{20}}$$

Realizable Highpass (HP) Filters





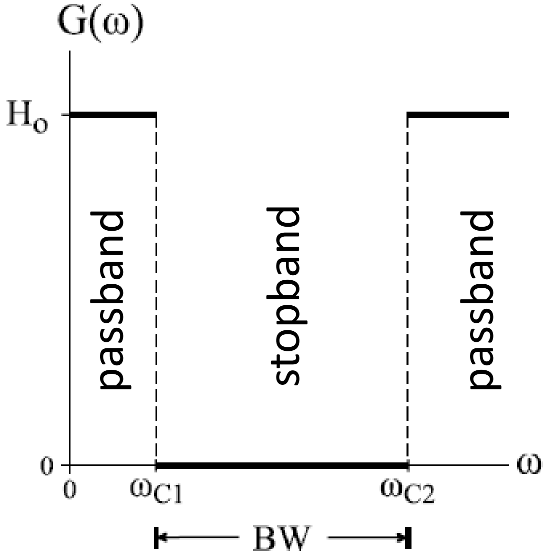
Realizable Bandpass (BP) Filters

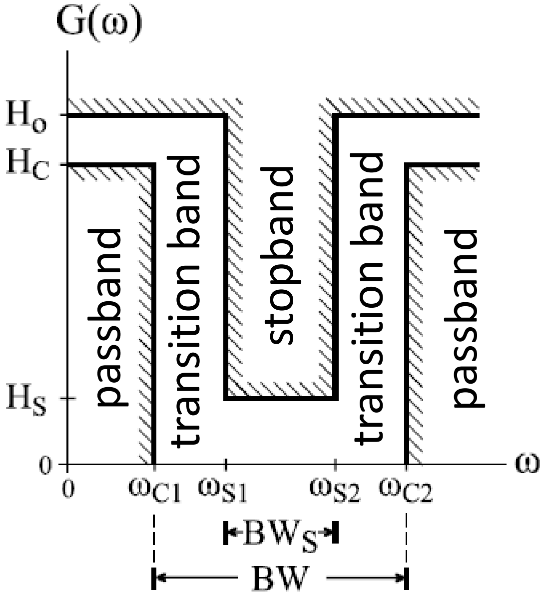


Bandwidth $BW = \Delta f = \omega_{C2} - \omega_{C1}$

center frequency $\omega_0 = \sqrt{\omega_{C2} * \omega_{C1}}$

Realizable Band-Reject (BR) Filters



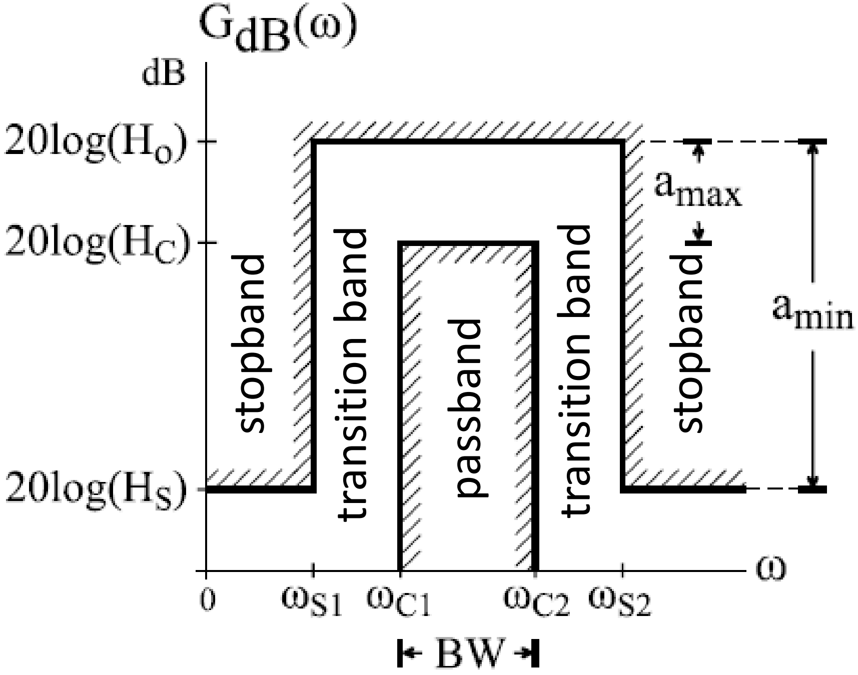


BandWidth $BW = \omega_{C2} - \omega_{C1}$

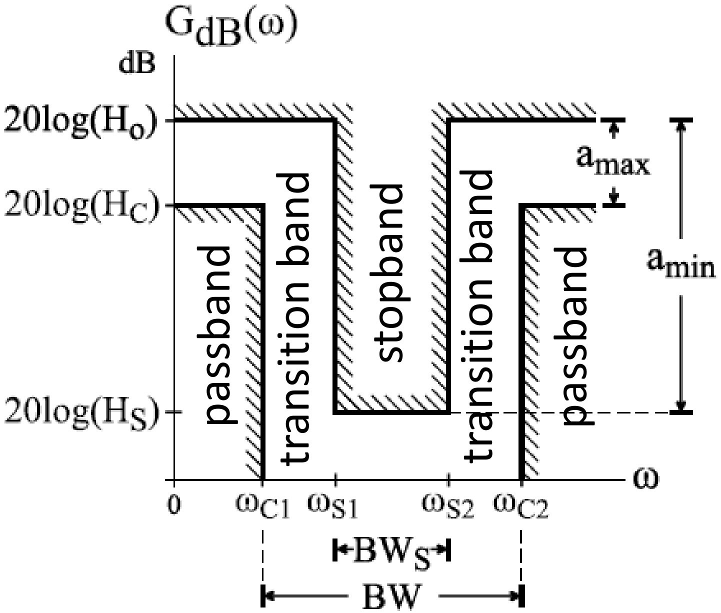
Stopband BandWidth $BWS = \omega_{S2} - \omega_{S1}$

Center Frequency $\omega_0 = \sqrt{\omega_{S2} * \omega_{S1}}$

Logarithmic Gain Bandpass And Band-reject Filter



Bandpass filter



Band-reject filter

Classification of analogue electronic filters

Classification filter by input signal, their internal signals and the output signals

i. Continuous-time (CT)

$f(t)$ at any t

ii. Sampled-data

$$f_s(t) = \sum_{n=-\infty}^{+\infty} f(nT) * \delta(t - nT)$$

$f(t)$ - continuous time signal
T - sampling period
$f_s(t) = 0$ for $nT \leq t < (n+1)T$

iii. Discrete-time

$$f(nT)$$

For $-\infty < n_1 \leq n \leq n_2 < +\infty$

And $f(nT) = 0$
if $nT \leq t < (n+1)T$

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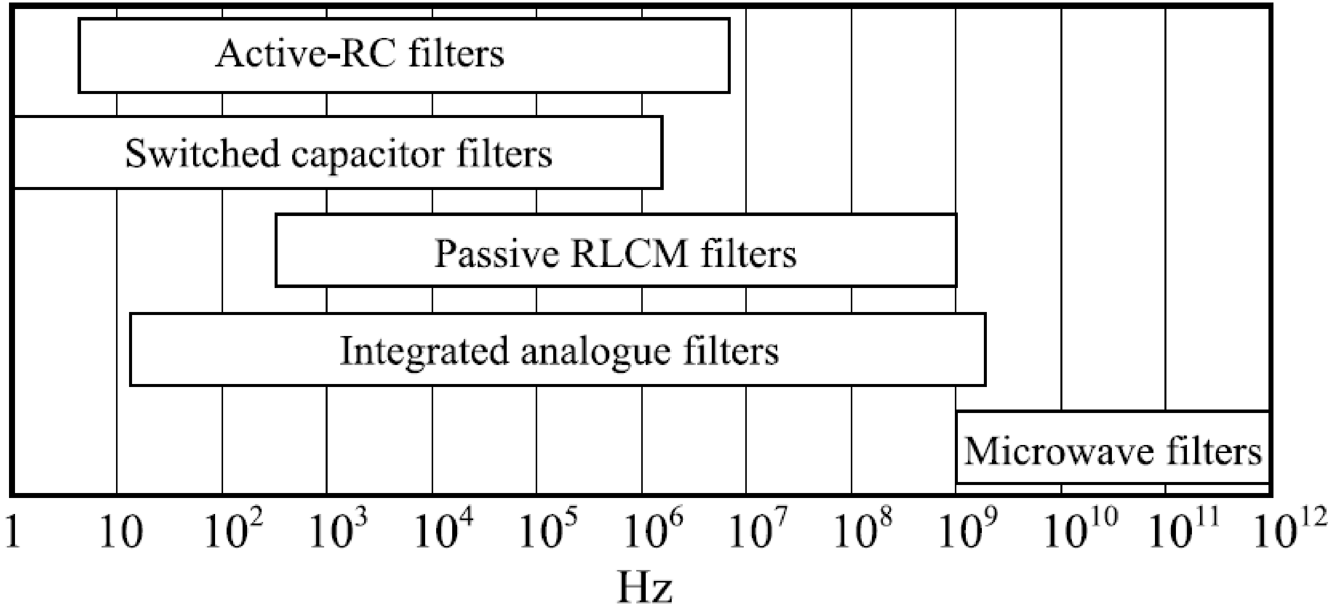
Classification of analogue electronic filters

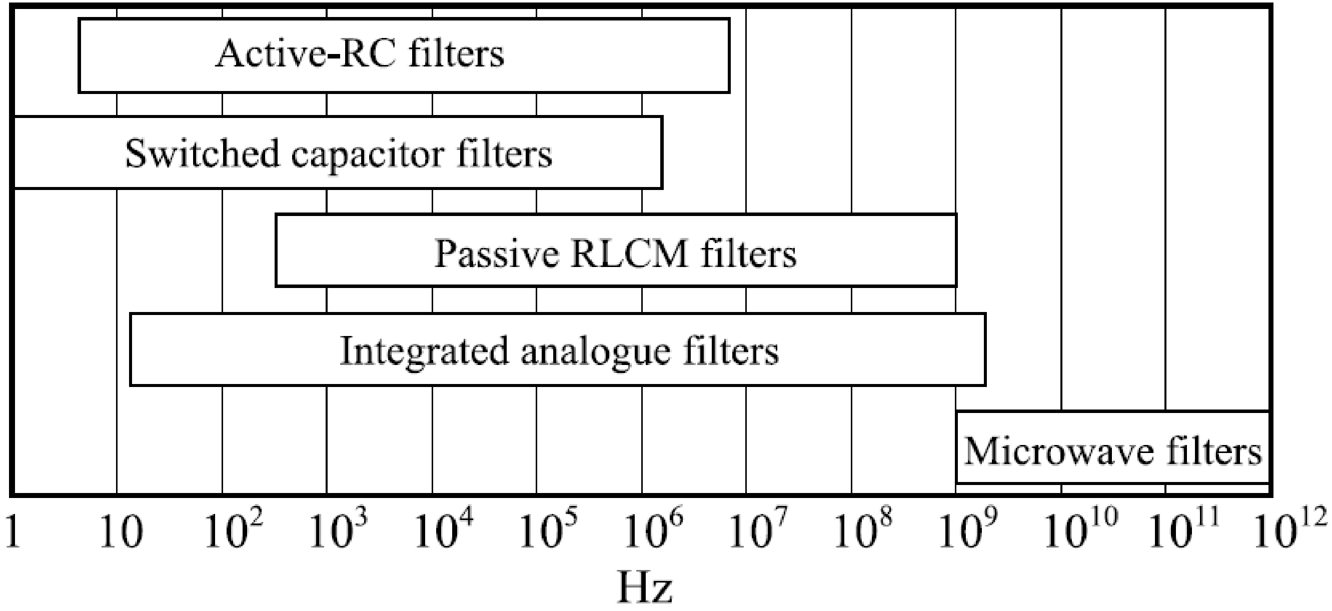
General classification of analogue electronic filters:

1. Passive RLCM filters
2. Active-RC filters
3. Integrated MOS-C filters
4. Integrated OTA-C or gm − C filters
5. Current-mode integrated filters.
6. Active switched capacitor filters designed for discrete-time signals
7. Microwave filters with distributed parameters (waveguides) and microwave filters based on microwave resonators and cavities.
8. Crystal filters
9. Mechanical and electromechanical filters

Filter frequency range

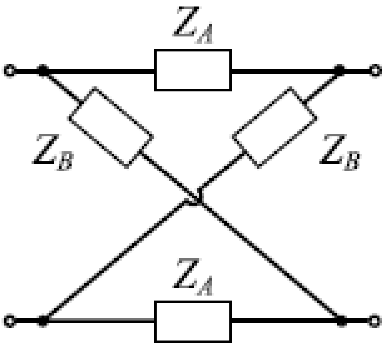


Filter frequency range

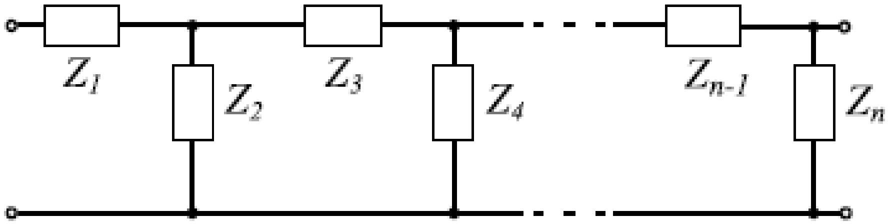


LC circuits

Lattice structure

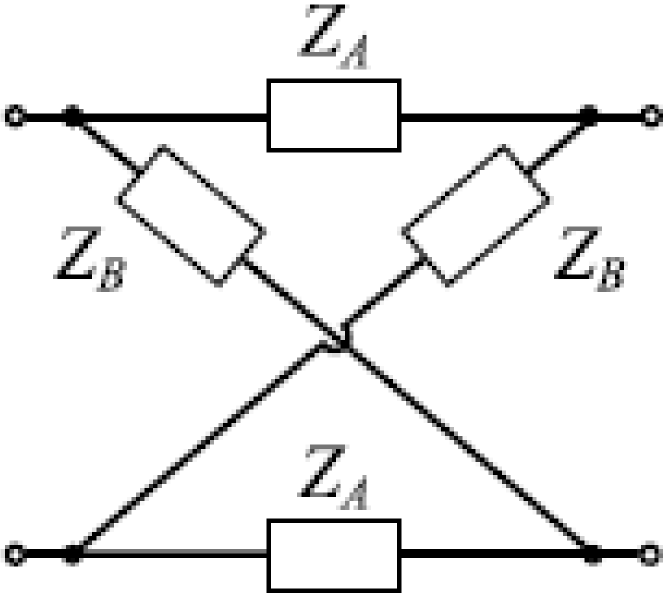


Ladder structure



Benefits LC circuits
(a) can satisfy any practical filter specifications
(b) need a minimum number of components
(c) can be designed so that they can maximize the power transferred from source to load in some frequencies in their passband.

Lattice LC circuits



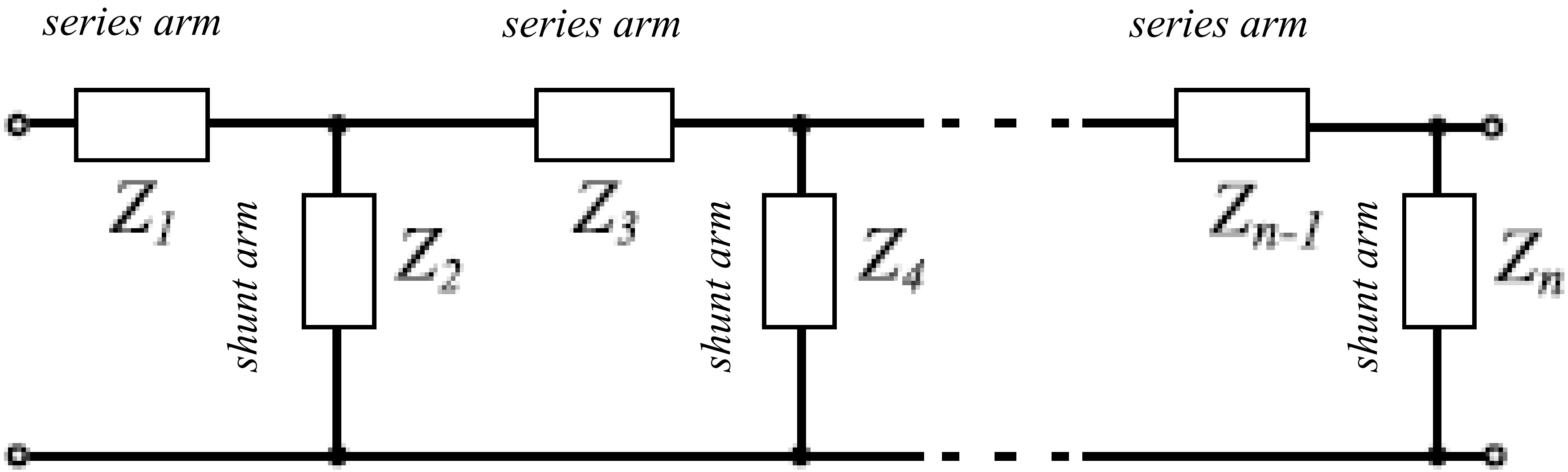
disadvantages when compared to the ladder topology:

1. for a given set of requirements, it needs more lossless elements than the corresponding ladder
2. it presents high sensitivity to component changes

To realize a transmission zero in passband, a bridge equilibrium is required:

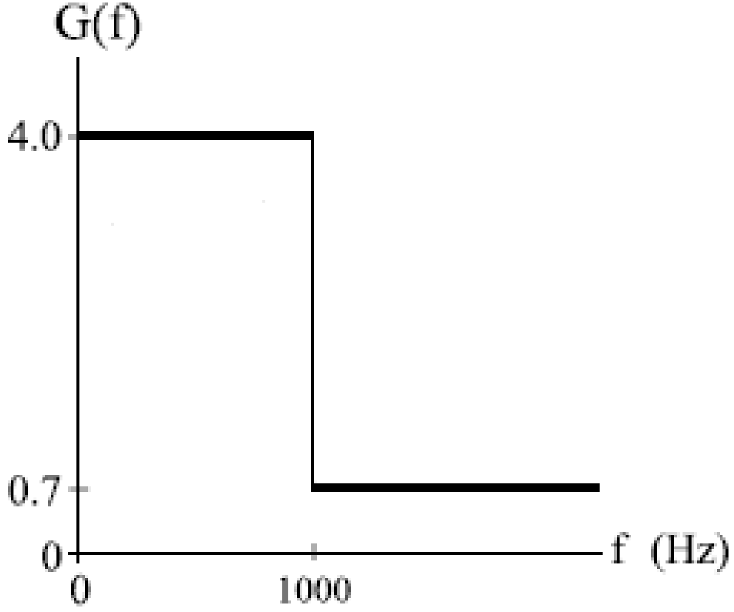
$$|Z_A(j\omega)| = |Z_B(j\omega)|$$

Ladder LC circuits



Designing a Filter. Example

We need lowpass filter with the following specifications
(a) For $0 \leq f \leq 1\ kHz$ the gain is $G(f) = 4$
(b) For $f > 1\ kHz$ the gain is $G(f) = 0.7$



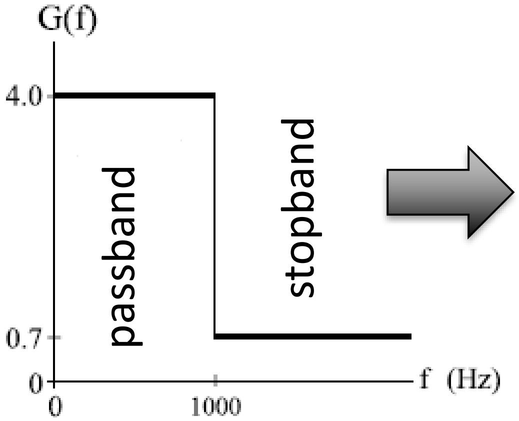
Designing a Filter. Example

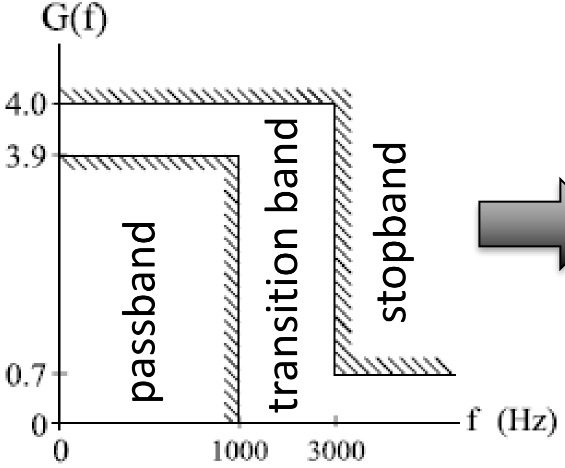
For the filter to be realizable, some tolerance should be given:

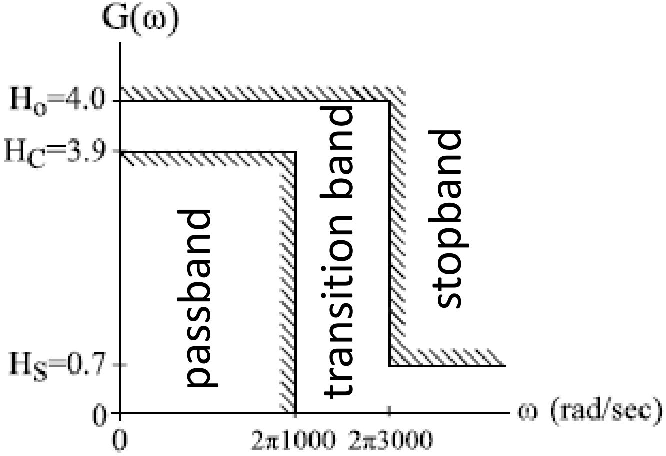
(a) For $0 \leq f \leq 1\ kHz$ the gain may vary $3.9 \leq G(f) \leq 4$

(b) For $1\ kHz \leq f \leq 3\ kHz$ the gain must be $G(f) \leq 3.9$

(c) For $f > 3\ kHz$ the gain must be $G(f) \leq 0.7$







Scaling and Normalization

let be characteristic frequency at ω_0, wish to move it to another frequency ω_X

$$\omega_0 \Longrightarrow \omega_X$$

So *scale* the frequency by $\gamma = \frac{\omega_0}{\omega_X}$

functions will be frequency-scaled by the factor γ.

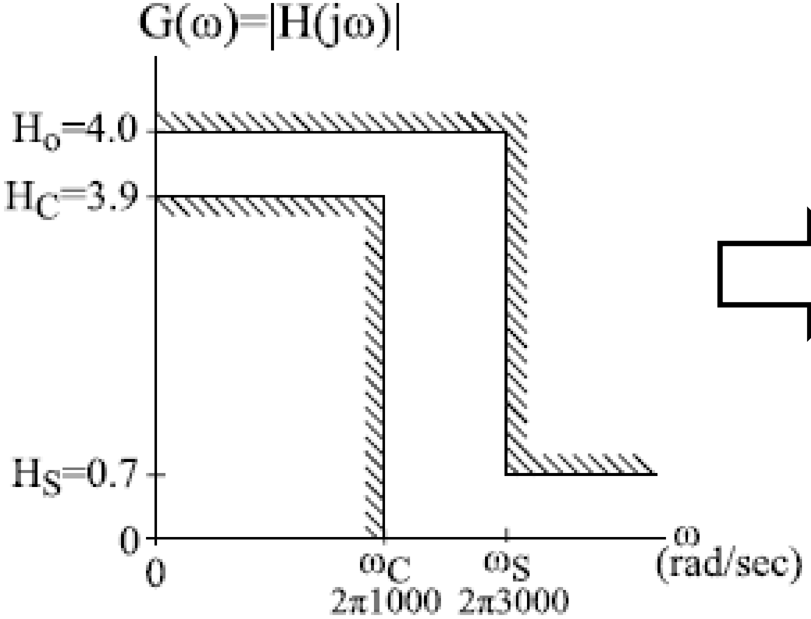
$$H(s) \Longrightarrow H(\gamma s), \mathrm{G}(\omega) \Longrightarrow G(\gamma\omega)$$

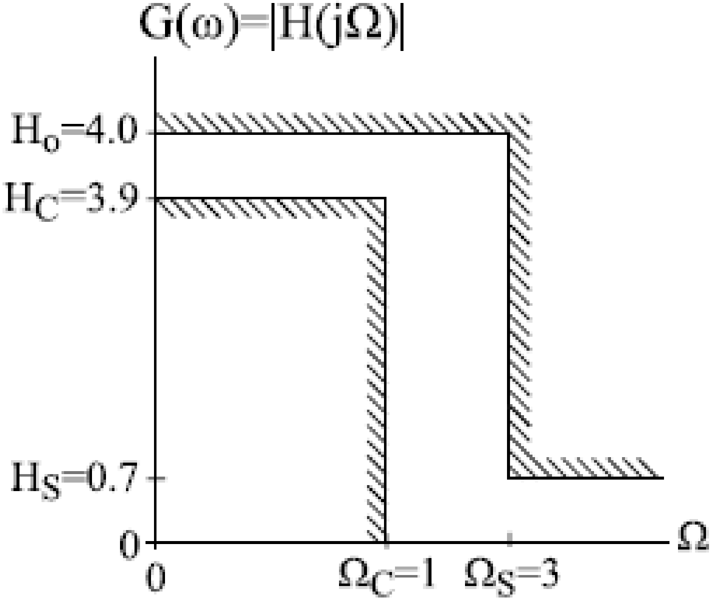
When using $\omega_X = 1$,want the characteristic frequency to move from $\omega = \omega_0$ to $\omega = 1$ this particular frequency scaling is called *frequency normalization*.

The new scaled frequency $\frac{\omega}{\omega_0} = \Omega$ is the *normalized frequency*.

A frequency-normalized filter is a filter which satisfies normalized specifications. It can be easily denormalized so that the cutoff frequency takes any desirable value, without affecting gain and attenuation characteristics.

Scaling and Normalization. Example





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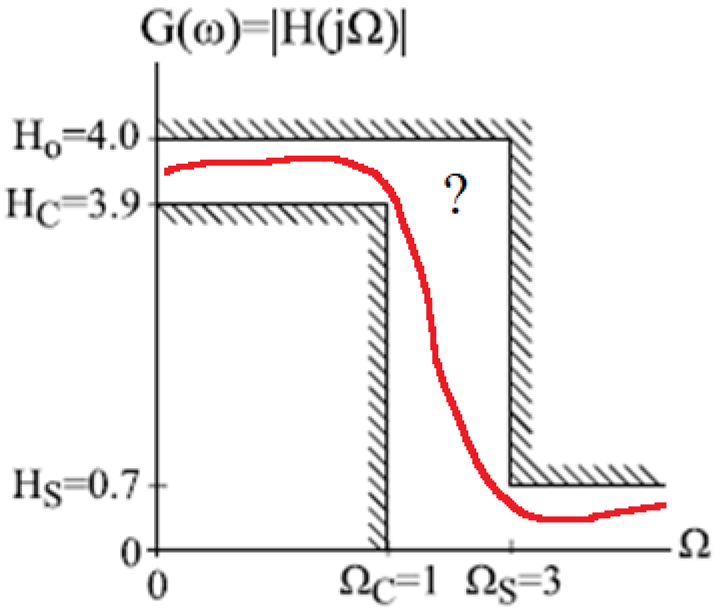
Approximation



approximation - finding the normalized gain function $G(\omega)$

and theoretically has an *infinite number of solutions*

To minimize the number of solutions, the function should be satisfy some realizability conditions

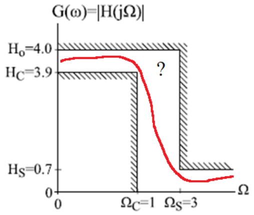
- $G(\Omega)$ – even function of *Ω*
- $G^2(\Omega)$ – even rational function

Solution that satisfies the conditions in question

$$G(\Omega) = \frac{A}{\sqrt{1 + k^2 P^2(\Omega)}}$$

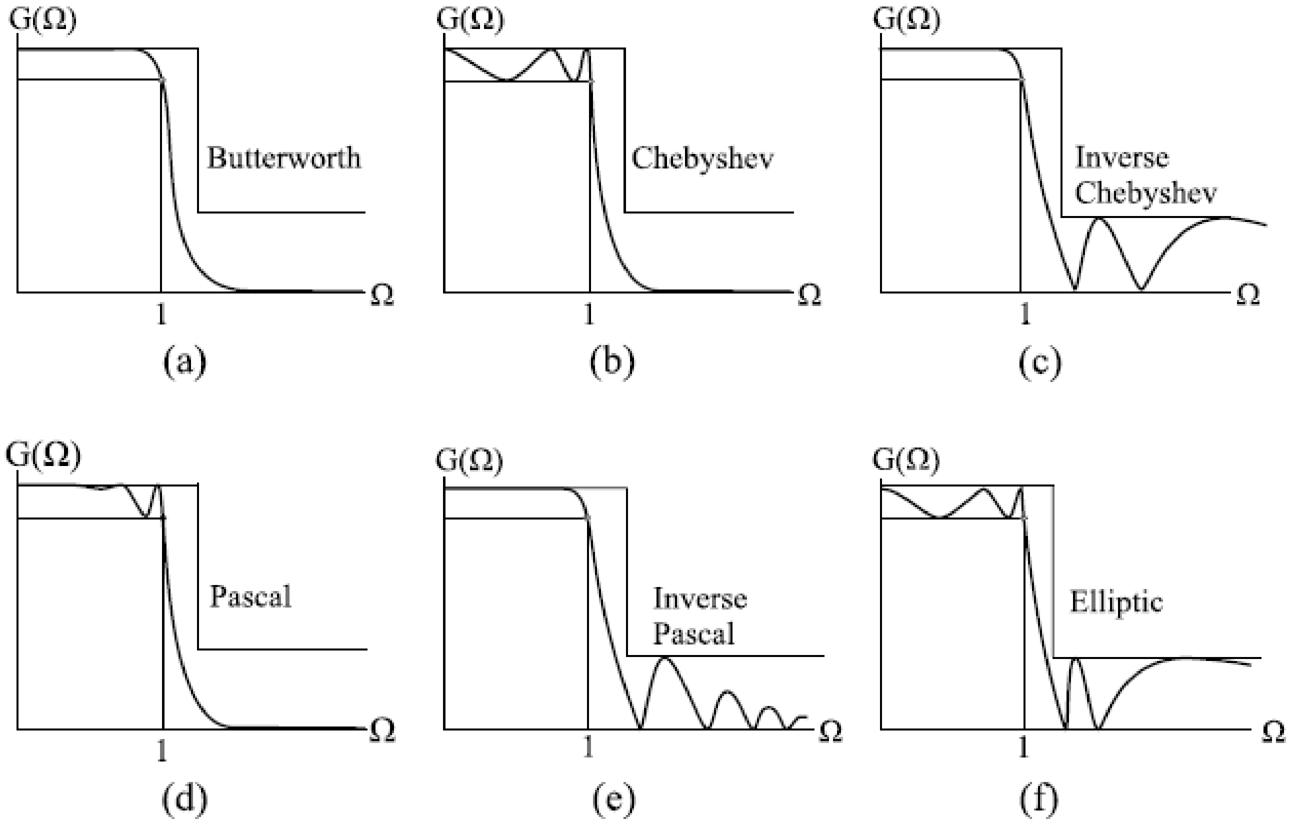
where *k* is a constant and *P(Ω)* is a polynomial or a rational function of *Ω*.

Approximation

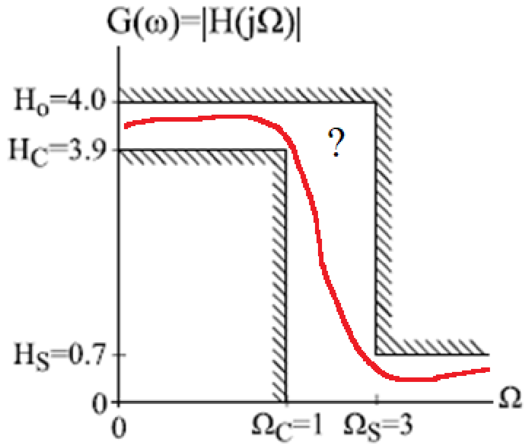


Approximation:

- Butterworth
- Chebyshev
- Inverse Chebyshev
- Bessel
- Pascal
- Inverse Pascal
- Elliptic
- And so on…



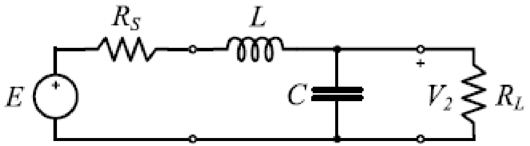
General Filter Design Procedure



design procedure:

1. Frequency is scaled using ω_C so that the normalized cutoff frequency becomes 1 and the stopband edge frequency $\Omega_S = \frac{\omega_S}{\omega_C}$
2. Using one of the known approximations, the corresponding transfer function *H(s)* is calculated, whose frequency response $|H(j\Omega)| = G(\Omega)$ satisfies the specifications.
3. The normalized filter circuit is synthesized from *H(s)*.
4. The filter is denormalized to the desired frequency and impedance level.

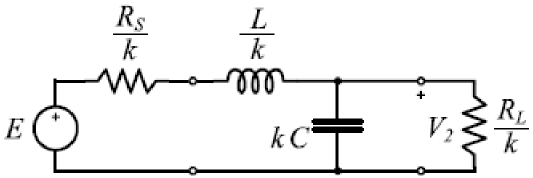
Impedance Scaling



$$H(s) = \frac{1/LC}{s^2 + (\frac{R_S}{L} + \frac{1}{R_L C})s + \frac{1+R_S/R_L}{LC}}$$

$$Z(s) = \frac{s^2 + (\frac{R_S}{L} + \frac{1}{R_L C})s + \frac{1+(R_S/R_L)}{LC}}{s + \frac{1}{R_L C}} L$$

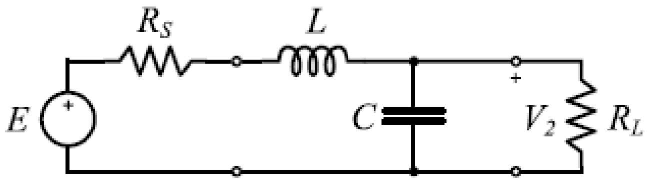
impedances divided (scaled) by *k*



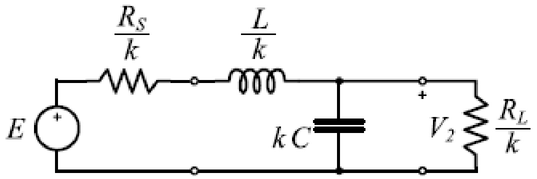
$$H(s) = \frac{1/LC}{s^2 + (\frac{R_S}{L} + \frac{1}{R_L C})s + \frac{1+R_S/R_L}{LC}}$$

$$Z_n(s) = \frac{Z(s)}{k}$$

Impedance Scaling



impedances divided (scaled) by *k*



If $k = R_0$ → the scaled resistors will be dimensionless quantities

$$R_{Sn} = \frac{R_S}{R_0}$$
$$R_{Ln} = \frac{L}{R_0}$$

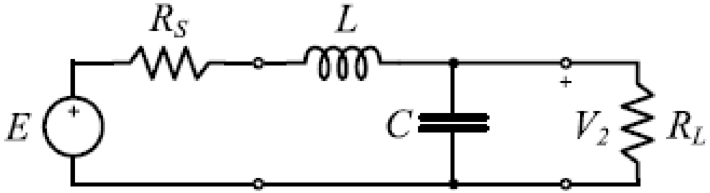
dimensionless

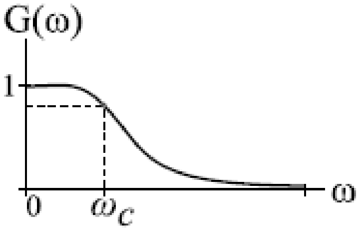
$$L_n = \frac{L}{R_0}$$

measured in seconds

$$\frac{1}{sC} = \frac{1}{sC}\frac{1}{R_0}$$

Frequency Scaling



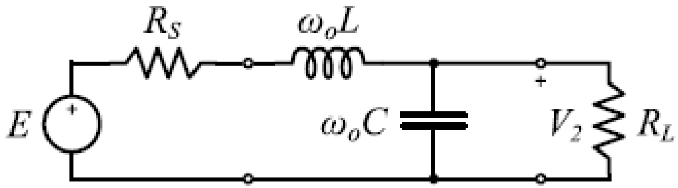


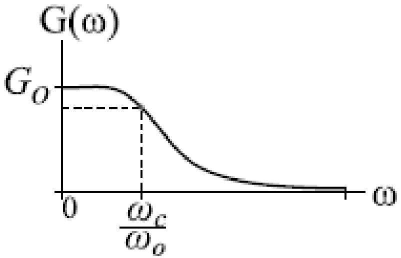
Defined

$$L = \omega_0 L,$$
$$C = \omega_0 C$$

inductances and capacitances are multiplied by ω_0

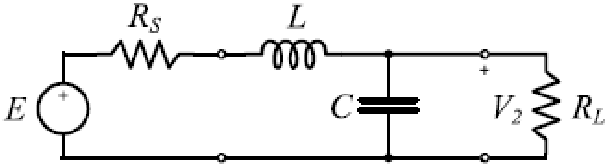
$$H_n(s) = \frac{1}{LC(\omega_o s)^2 + (R_S C + \frac{L}{R_L})\omega_o s + 1 + \frac{R_S}{R_L}} = H(\omega_o s)$$

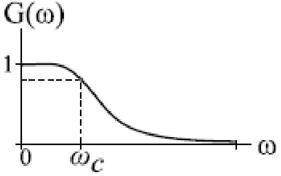




$$L, C \Longrightarrow L_n = \omega_0 L, C_n = \omega_0 C,$$

Full Normalization

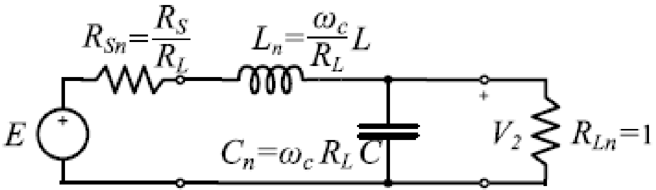


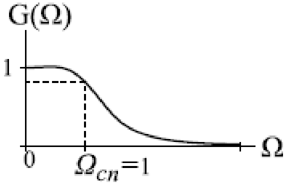


Impedance Frequency Scaling

$R_0 = R_L$

$\omega_0 = \omega_C$





dimensionless normalized values

$$R_{Sn} = \frac{R_S}{R_L}, R_{Ln} = \frac{R_L}{R_L} = 1,$$

$$L_n = \frac{\omega_0}{R_L} L, C_n = \omega_C R_L \mathrm{C}$$

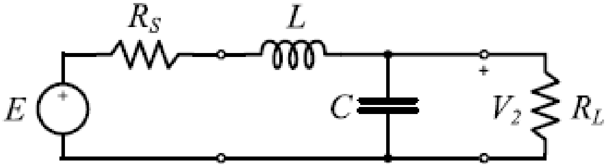
normalized dimensionless angular frequency, time and frequency

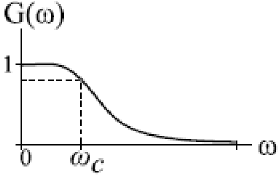
$$\Omega = \frac{\omega}{\omega_C}, t_n = \frac{\omega_C}{2\pi} t, F_n = \frac{2\pi}{\omega_C} f$$

Denormalized value

$$R_n = R_L R_n, L = \frac{R_L}{\omega_0} L_n, C = \frac{1}{\omega_C R_L} C_n$$

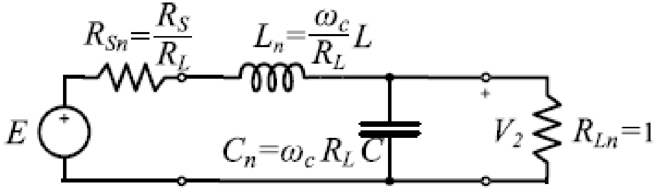
Prototype Filters

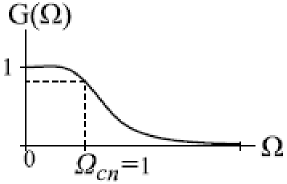




Impedance
Frequency
Scaling

$$R_0 = R_L$$
$$\omega_0 = \omega_C$$





dimensionless normalized values

$$R_{Sn} = \frac{R_S}{R_L}, R_{Ln} = \frac{R_L}{R_L} = 1,$$
$$L_n = \frac{\omega_0}{R_L}L, C_n = \omega_C R_L \mathrm{C}$$

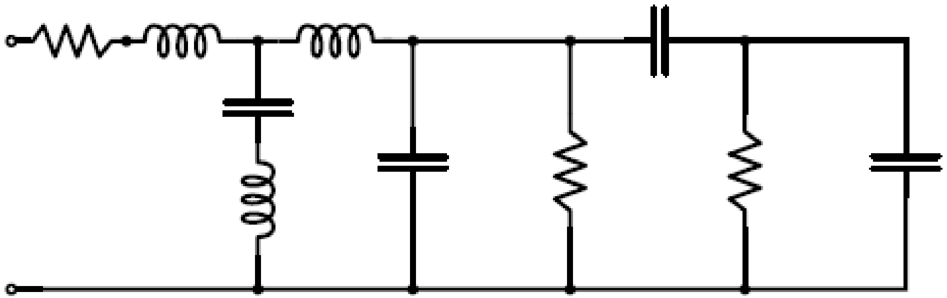
normalized dimensionless angular frequency, time and frequency

$$\Omega = \frac{\omega}{\omega_C}, t_n = \frac{\omega_C}{2\pi}t, F_n = \frac{2\pi}{\omega_C}f$$

Denormalized value

$$R_n = R_L R_n, L = \frac{R_L}{\omega_0}L_n, C = \frac{1}{\omega_C R_L}C_n$$

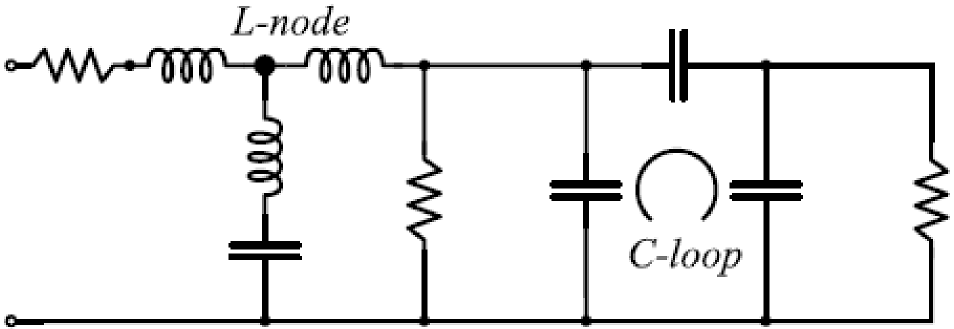
Circuit Order



number of reactive elements: 7
inductors: 3
capacitors: 4
maximum order (as expected): 7

Rearranged

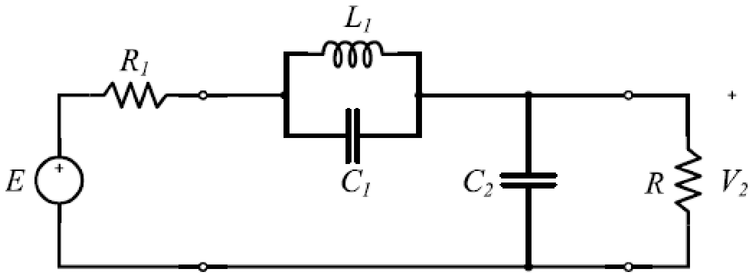
each L-node, C-node, L-loop or C-loop reduces the Circuit order by 1



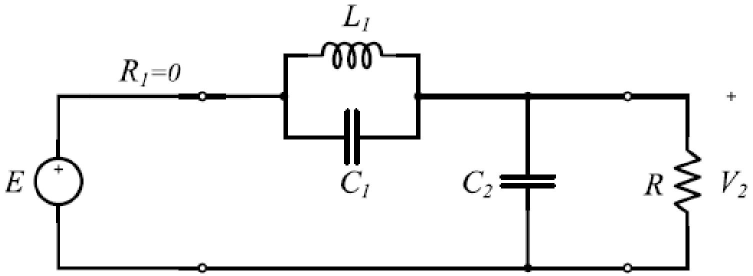
number of reactive elements: 7
inductors: 3
capacitors: 4
L and C-node: 1
L and C-loop: 1
maximum order: 5

All functions of the circuit will have a polynomial of maximum degree 5.

Circuit Order. Example



Removing resistor R1



number of reactive elements: 3

Inductors and capacitors : 1+2

maximum order (as expected): 3

$$H(s) = \frac{L_1C_1s^2+1}{L_1C_1C_2R_1s^3 + [L_1C_1 + L_1C_1(1+\frac{R_1}{R})]s^2 + (R_1C_2 + \frac{L_1}{R})s + 1 + \frac{R_1}{R}}$$

$$R_1 = 0$$

$$H(s) = \frac{L_1C_1s^2+1}{s^2 + L_1(C_1+C_1)s + \frac{L_1}{R} + 1}$$

number of reactive elements: 3

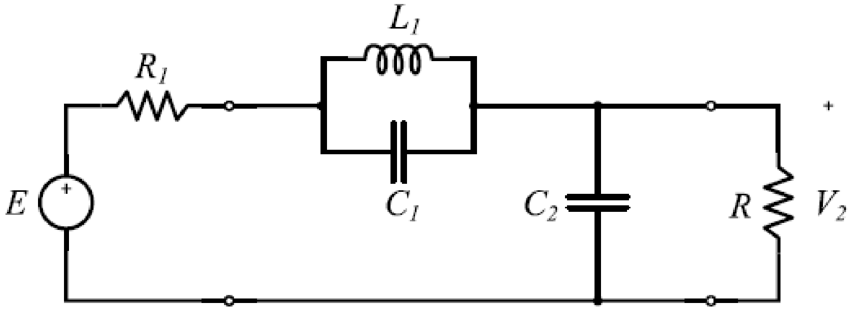
Inductors and capacitors : 1+2

node and loop: 1

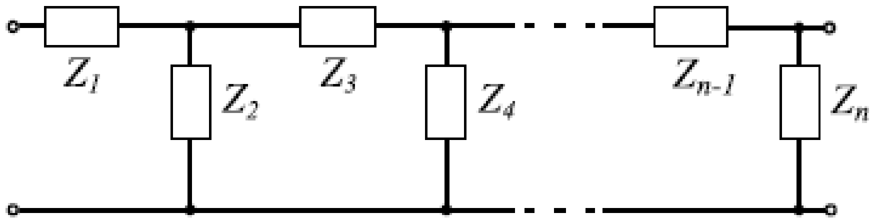
maximum order: 2

Circuit Order

not canonic circuits



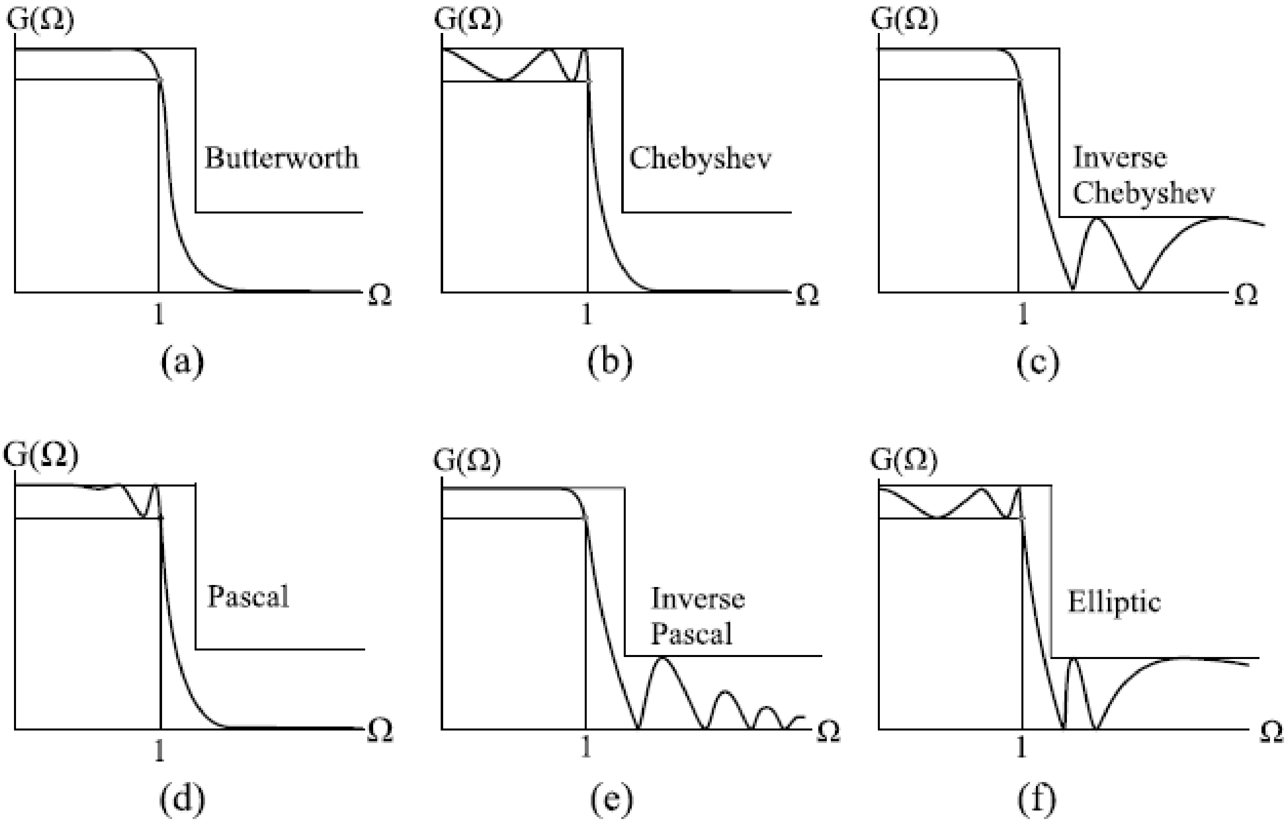
Canonic circuits



do not have L-nodes, C-nodes, L-loops or C-loops

All-Pole Approximations

All-Pole Approximations



All-Pole Transfer Functions and Approximations

transfer function

$$H(s) = \frac{k}{D(s)} = \frac{k}{s^N + B_{N-1}s^{N-1} + \cdots + B_1 s + B_0}$$

where *D(s)* is in general a polynomial in s of degree *N*:

acceptable of poles

$$s_p = -c \ (c > 0)$$

$s_p = -a \pm jb$, with negative real part

general form of the magnitude response of
all-pole lowpass filters

$$|H(j\Omega)| = \frac{|k|}{\sqrt{1 + [|D(j\Omega)|^2 - 1]}} = \frac{|k|}{\sqrt{1 + Q(\Omega)}}$$

$$G(\Omega) = |H(j\Omega)| = \frac{H_o}{\sqrt{1 + P_a(\Omega)}}$$

All-Pole Transfer Functions and Approximations

$$G(\Omega) = |H(j\Omega)| = \frac{H_o}{\sqrt{1 + P_a(\Omega)}}$$

$$\Downarrow$$

$$G(\Omega) = \frac{H_o}{\sqrt{1 + \gamma^2 P_N^2(\Omega)}}$$

with $P_N(\Omega)$ being the *approximating polynomial* (complete even or odd) and γ a design parameter.

References

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6. https://www.electronics-tutorials.ws/
7. https://en.wikipedia.org/

Thank you for your attention!