ITMO

Filter theory

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The Butterworth Approximation

parameters that fully describe the filter transfer function

$$\{H_o, H_C, H_S, \Omega_S\} \qquad (\Omega_C = 1)$$

In terms of logarithmic gain

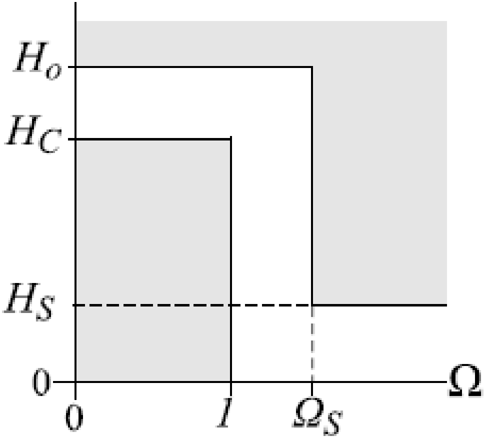
$$\{\alpha_o, \alpha_{\max}, \alpha_{\min}, \Omega_S\} \qquad (\Omega_C = 1)$$

If $H_0 = 1$, the filter requirements can be determined by three parameters

$$\Omega_S \quad \text{and} \quad \{H_C, H_S\} \quad \text{or} \quad \{\alpha_{\max}, \alpha_{\min}\}$$

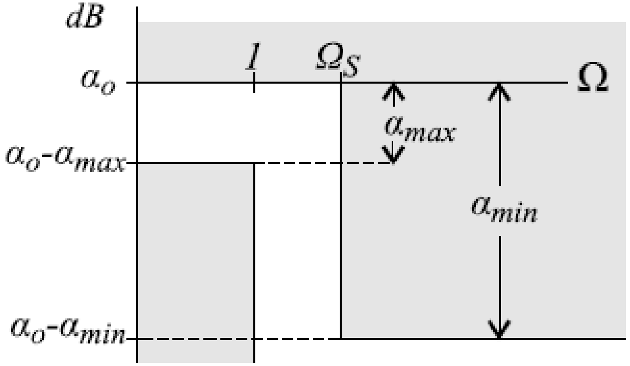
Plain gain

$G(\Omega)=|H(j\Omega)|$



Logarithmic gain

$G_{dB}(\Omega)=20\log|H(j\Omega)|$



Butterworth proposed the monotonic function

$$G(\Omega) = \frac{H_o}{\sqrt{1+\beta^2\Omega^{2N}}}$$

with N, the order of the approximation, a positive integer, and β a design parameter related to the passband tolerance.

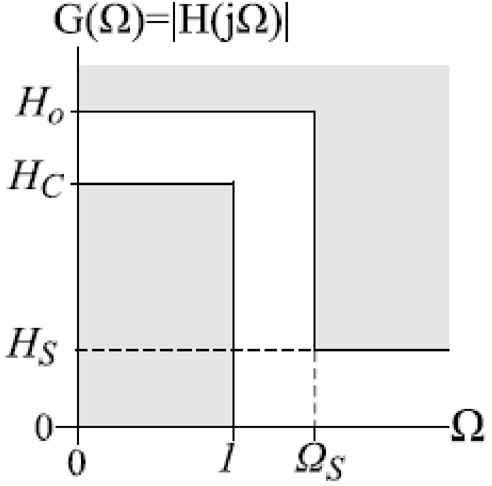
Gain and Attenuation Functions

$$G(\Omega) = \frac{H_o}{\sqrt{1+\beta^2\Omega^{2N}}}$$

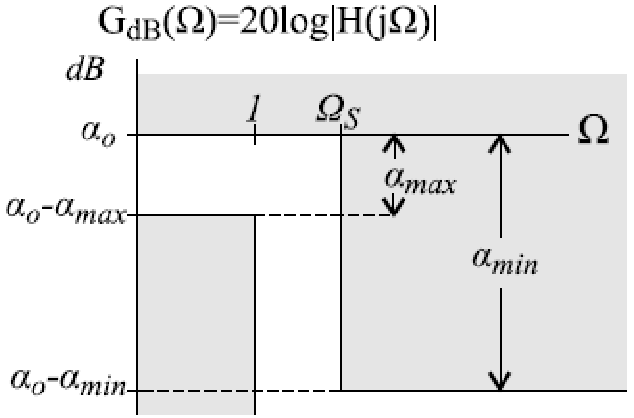
For $\Omega = 0$

$$G(0) = H_o$$

Plain gain



Logarithmic gain



$$G(1) = \frac{H_o}{\sqrt{1+\beta^2}} \geq H_C \quad \Leftrightarrow \quad \beta^2 \leq (H_o/H_C)^2 - 1$$

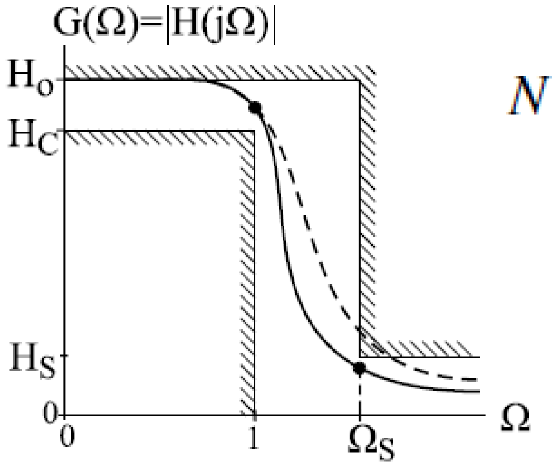
For $\beta = \beta_{max}$ the gain $G(1) = H_C$

$$\beta \leq \beta_{\max} = \sqrt{\left(\frac{H_o}{H_C}\right)^2 - 1} = \sqrt{10^{\frac{a_{\max}}{10}} - 1}$$

The Butterworth Approximation

$$G(\Omega) = \frac{H_o}{\sqrt{1+\beta^2\Omega^{2N}}}$$

$$G(\Omega_S) = \frac{H_o}{\sqrt{1+\beta^2\Omega_S^{2N}}} \le H_S$$

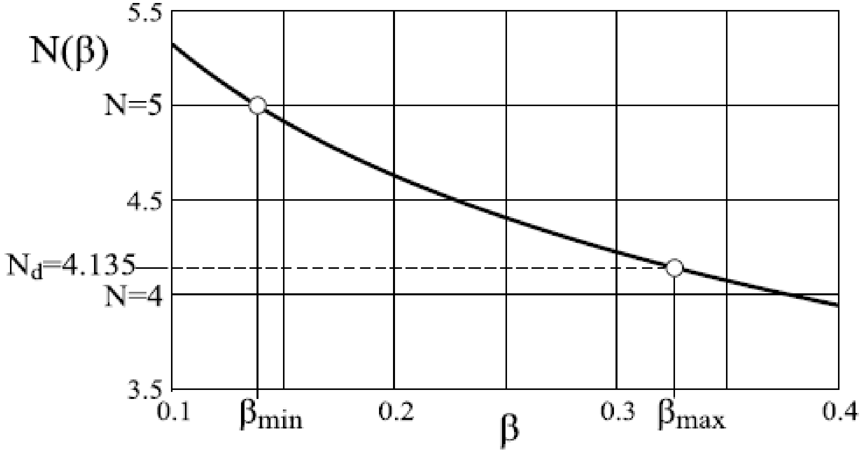


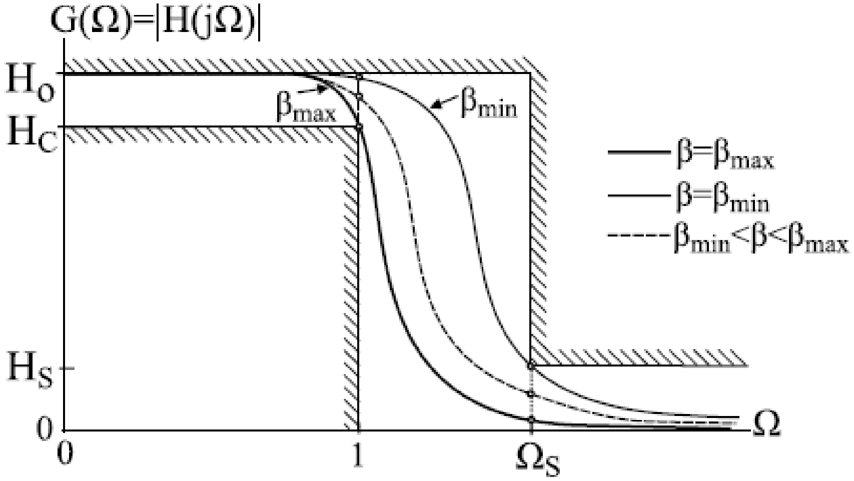
$$N \ge \frac{\log\left(\frac{(H_o/H_S)^2-1}{\beta^2}\right)}{2\log\Omega_S}$$

$$\boxed{N \ge N_d = \frac{\log\left(\frac{(H_o/H_S)^2-1}{\beta^2}\right)}{2\log\Omega_S}}$$

$$n_{f\,\min} = \frac{\log\left(\frac{\frac{H_o^2}{H_S^2}-1}{\frac{H_o^2}{H_C^2}-1}\right)}{2\log\Omega_S} = \frac{\log\left(\frac{10^{\frac{a_{\min}}{10}}-1}{10^{\frac{a_{\max}}{10}}-1}\right)}{2\log\Omega_S}$$

The Butterworth Approximation





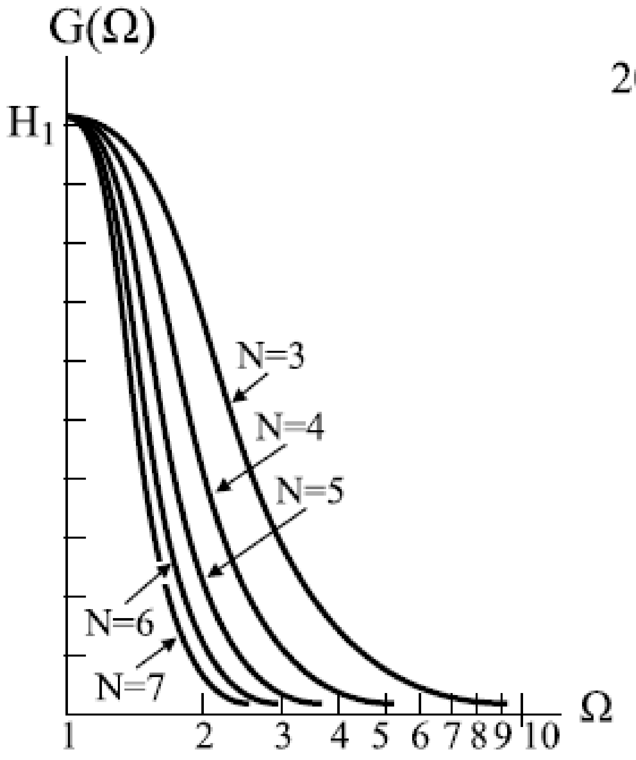
filter designed with normalized specifications

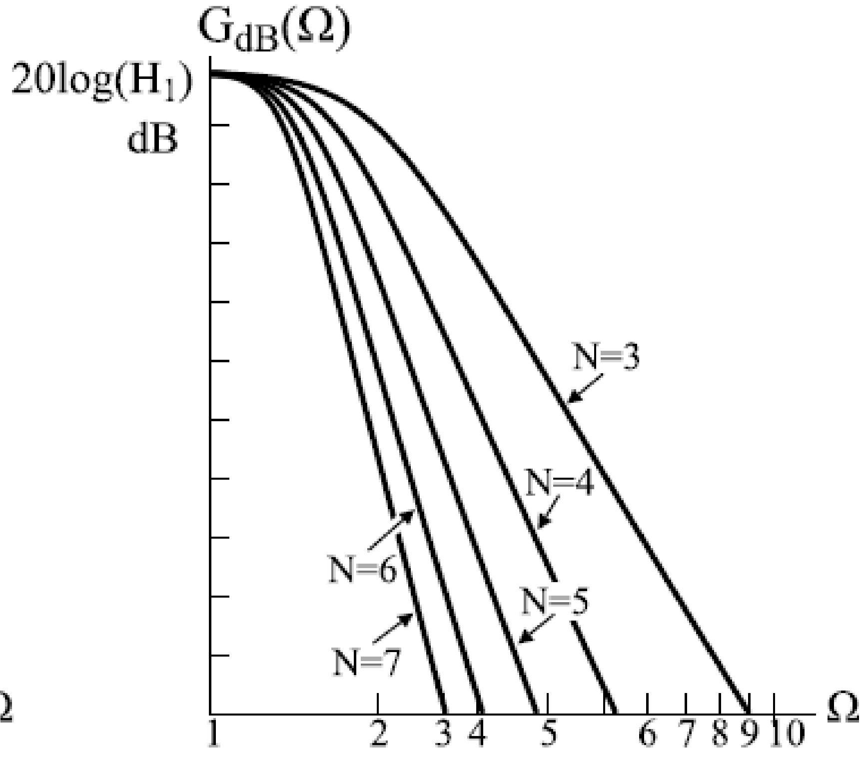
$H_o = 1$, $H_C = 0.95$, $H_S = 0.05$ *and* $\Omega_S = 2.7$

$$\beta_{\min} = \frac{\sqrt{\frac{H_o^2}{H_S^2} - 1}}{\Omega_S^N} \leq \beta \leq \sqrt{\frac{H_o^2}{H_C^2} - 1} = \beta_{\max}$$

$$\beta_{\min} = \frac{\sqrt{10^{\frac{\alpha_{\min}}{10}} - 1}}{\Omega_S^N} \leq \beta \leq \sqrt{10^{\frac{\alpha_{\max}}{10}} - 1} = \beta_{\max}$$

The Butterworth Approximation





The All-Pole Chebyshev Approximation

parameters that fully describe the filter transfer function

$\{H_o, H_C, H_S, \Omega_S\} \qquad (\Omega_C = 1)$

In terms of logarithmic gain

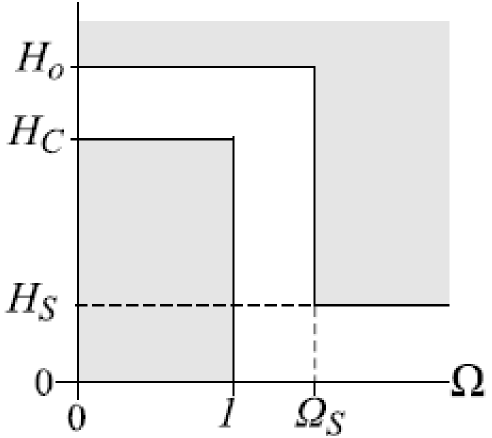
$\{\alpha_o, \alpha_{\max}, \alpha_{\min}, \Omega_S\} \qquad (\Omega_C = 1)$

If $H_0 = 1$, the filter requirements can be determined by three parameters

$\Omega_S \quad \text{and} \quad \{H_C, H_S\} \quad \text{or} \quad \{\alpha_{\max}, \alpha_{\min}\}$

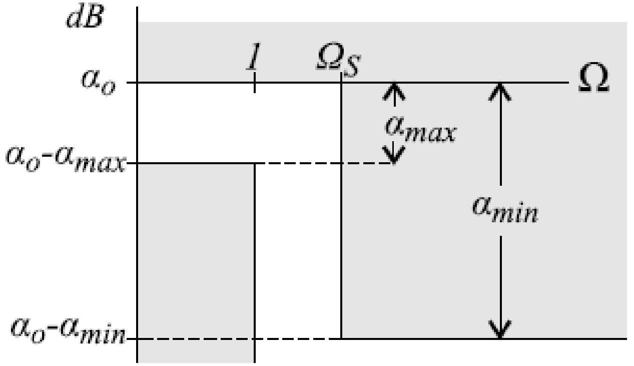
Plain gain

$G(\Omega)=|H(j\Omega)|$



Logarithmic gain

$G_{dB}(\Omega)=20\log|H(j\Omega)|$



Chebyshev approximation

$$G_{CH}(\Omega) = \frac{H_o}{\sqrt{1 + \varepsilon^2 C_N^2(\Omega)}}$$

The ripple factor ε and order N are so chosen to keep the response $G_{CH}(\Omega)$ within the specifications.

The All-Pole Chebyshev Approximation

$$G_{CH}(\Omega) = \frac{H_o}{\sqrt{1+\varepsilon^2 C_N^2(\Omega)}}$$

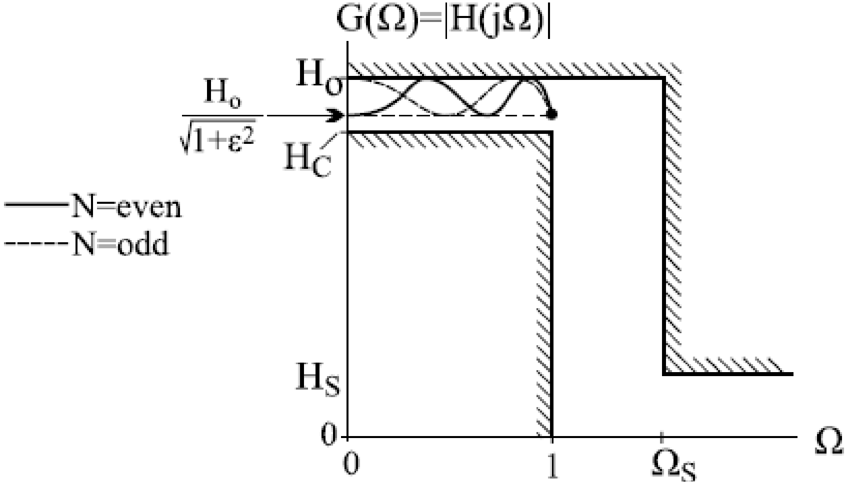
$$\varepsilon \leq \sqrt{\frac{H_o^2}{H_C^2} - 1} = \sqrt{10^{\frac{\alpha_{\max}}{10}} - 1} = \varepsilon_{\max}$$

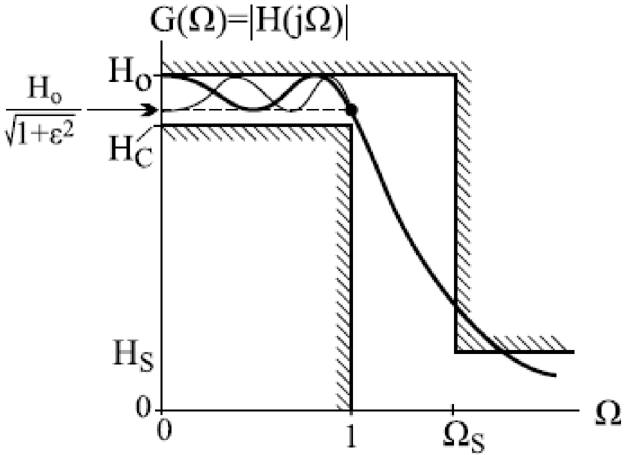
For $\varepsilon \leq \varepsilon_{max}$

$$H_0 \geq G_{CH}(\Omega) \geq \frac{H_0}{\sqrt{1+\varepsilon^2}} \geq H_C$$

For $\Omega = 1$

$$G_{CH}(1) = \frac{H_0}{\sqrt{1+\varepsilon^2}} \geq H_C$$





The All-Pole Chebyshev Approximation

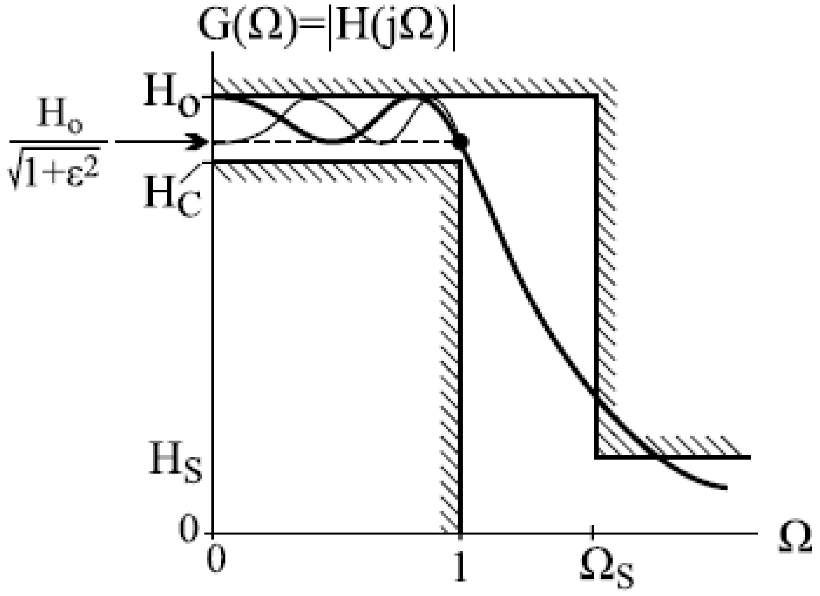
$$G_{CH}(\Omega_s) = \frac{H_o}{\sqrt{1+\varepsilon^2 C_N^2(\Omega_S)}} \leq H_S$$

$$\Leftrightarrow \quad C_N^2(\Omega_S) \geq \frac{(H_o/H_S)^2 - 1}{\varepsilon^2}$$

$$\Leftrightarrow \quad N \cosh^{-1}(\Omega_S) \geq \cosh^{-1}\sqrt{\frac{(H_o/H_S)^2 - 1}{\varepsilon^2}}$$

for N: $$N \geq N_d = \frac{\cosh^{-1}\left(\sqrt{\frac{(H_o/H_S)^2-1}{\varepsilon^2}}\right)}{\cosh^{-1}(\Omega_S)}$$

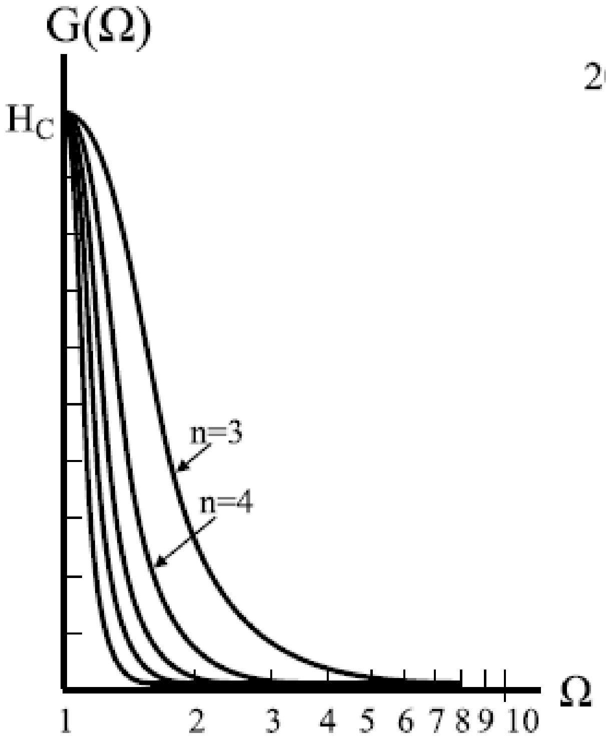
$$N \geq N_d = \frac{\cosh^{-1}\left(\sqrt{\frac{(H_o/H_S)^2-1}{\varepsilon^2_{\max}}}\right)}{\cosh^{-1}(\Omega_S)} = \frac{\cosh^{-1}\left(\sqrt{\frac{(H_o/H_S)^2-1}{(H_o/H_C)^2-1}}\right)}{\cosh^{-1}(\Omega_S)}$$

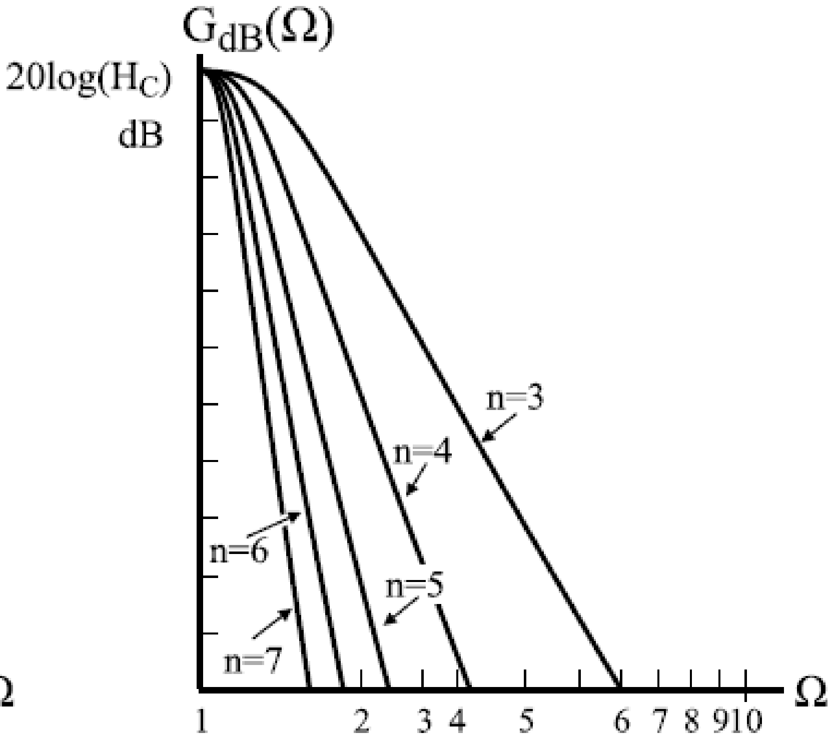


logarithmic gain specifications

$$N \geq N_d = \frac{\cosh^{-1}\left(\sqrt{\frac{10^{\frac{\alpha_{\min}}{10}}-1}{10^{\frac{\alpha_{\max}}{10}}-1}}\right)}{\cosh^{-1}(\Omega_S)}$$

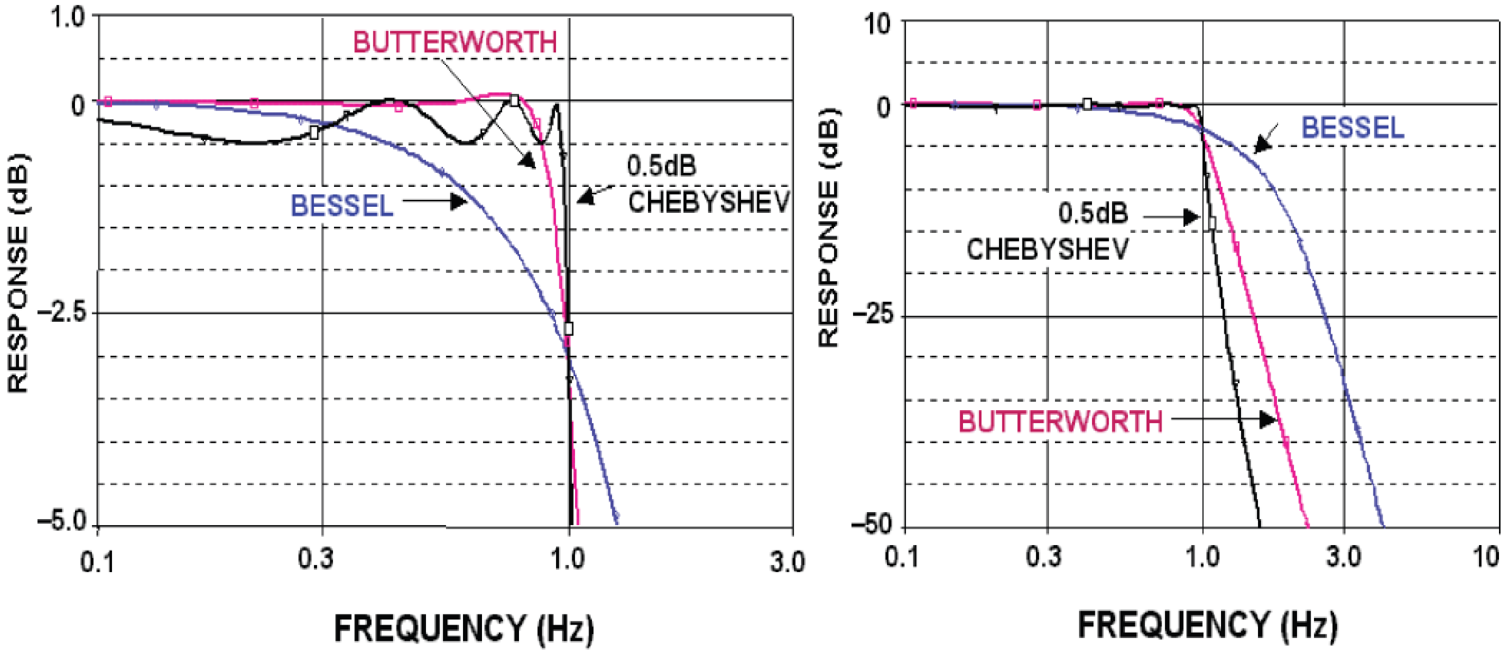
The All-Pole Chebyshev Approximation





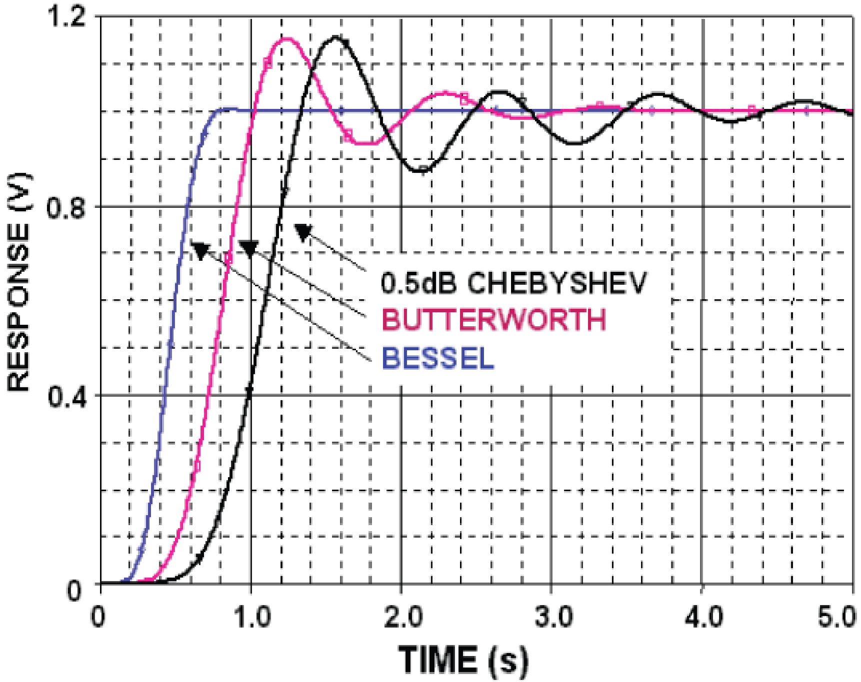
Comparison of All-Pole Responses

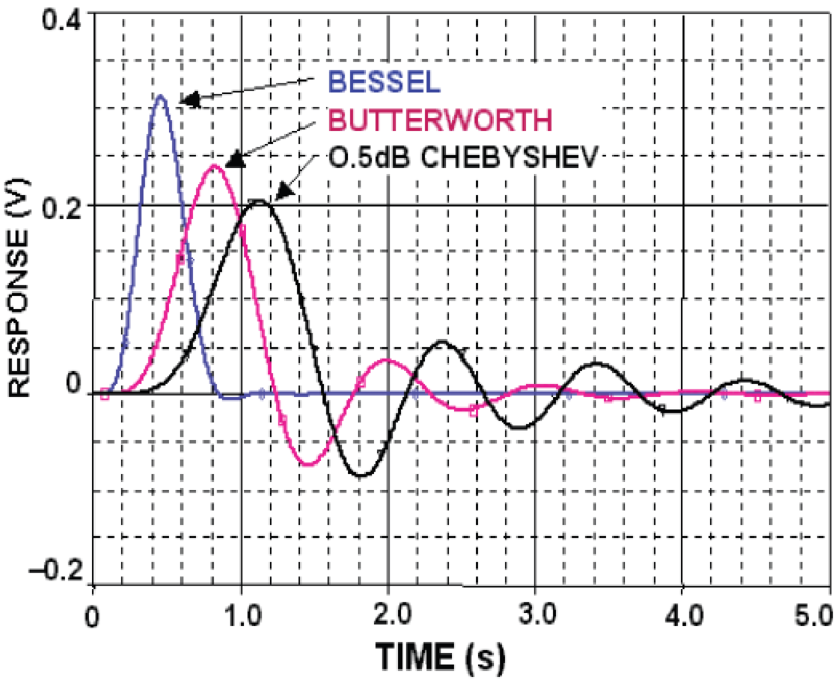
Comparison of Amplitude Response



Comparison of All-Pole Responses

Comparison of Step and Impulse Responses





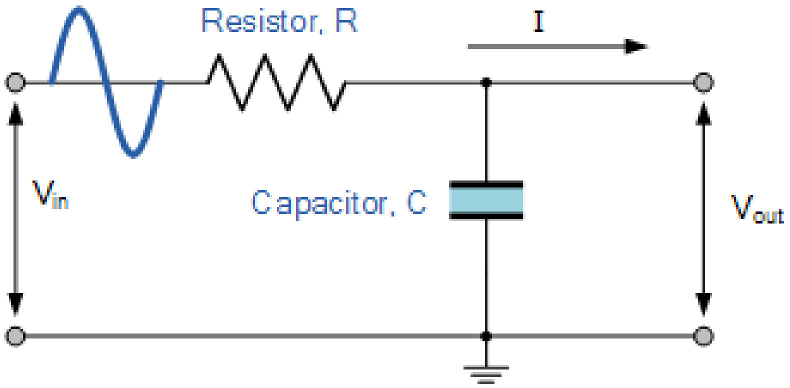
Butterworth VS Chebyshev

	Butterworth Filter	Chebyshev Filter
Order of Filter	The order of the Butterworth filter is higher than the Chebyshev filter for the same desired specifications.	The order of the Chebyshev filter is less compared to the Butterworth filter for the same desired specifications.
Hardware	It requires more hardware.	It requires less hardware.
Ripple	There is no ripple in passband and stopband of frequency response.	There is either ripple in passband or stopband.
Poles	All poles lie on a circle having a radius of the cutoff frequency.	All poles lie on ellipse having major axis R, ξ, minor axis r.
Transition band	The Butterworth filter has a wider transition band compared to the Chebyshev filter.	The Chebyshev filter has a narrow transition band compared to the Butterworth filter.
Types	It doesn't have any types.	It has two types; type-1 and type-2.
Cutoff Frequency	The cutoff frequency of this filter is not equal to the passband frequency.	The cutoff frequency of this filter is equal to the passband frequency.

Active filter circuits design

RC Low Pass Filter Circuit

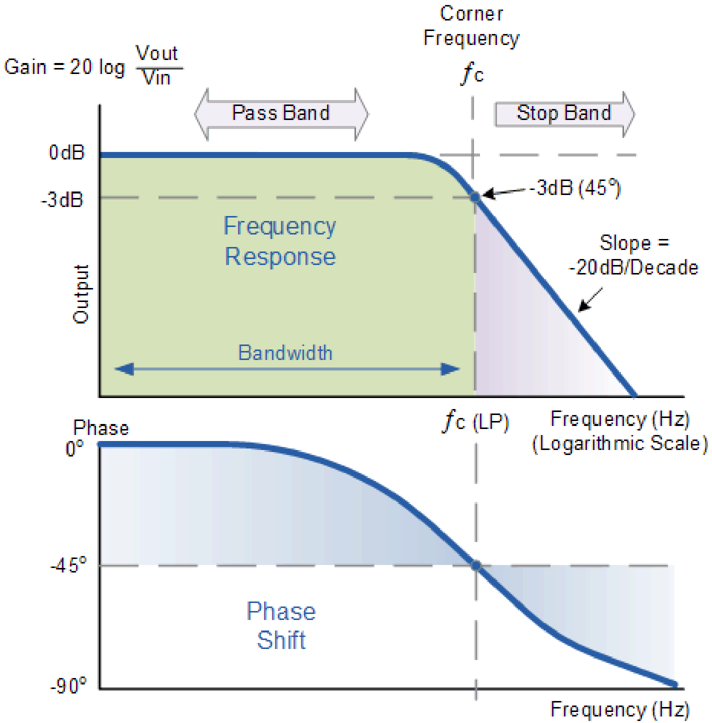
First-order Filter



The cut-off frequency point and phase shift angle can be found by using the following equation:

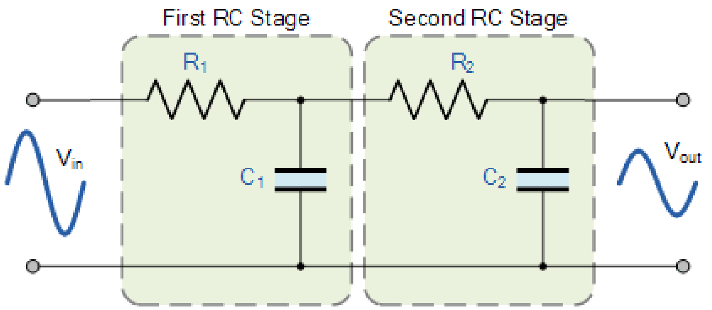
$$f_C = \frac{1}{2\pi RC}$$

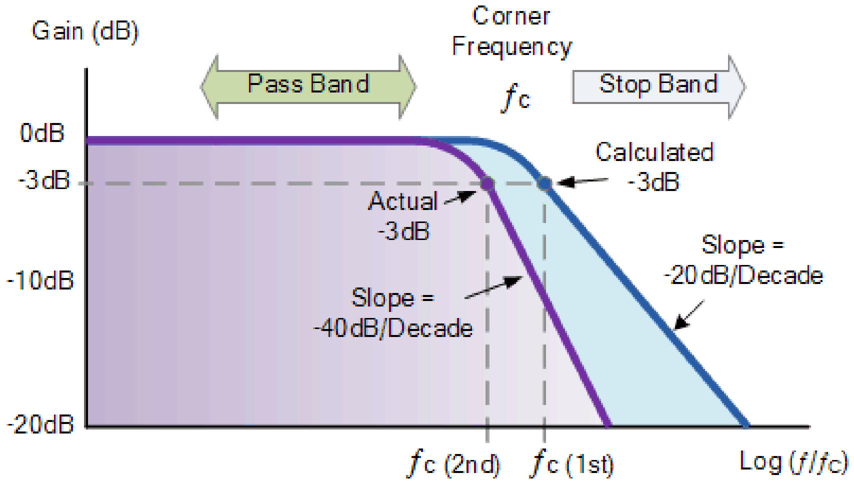
phase shift $\phi = -\,\mathrm{arctg}(2\pi f RC)$



RC Low Pass Filter Circuit

Second-order Low Pass Filter





Passive Low Pass Filter Gain at f_c *is proportional*

$$\left(\frac{1}{\sqrt{2}}\right)^n$$

where "*n*" is order of filter or the number of filter stages.

2nd-Order Filter Corner Frequency

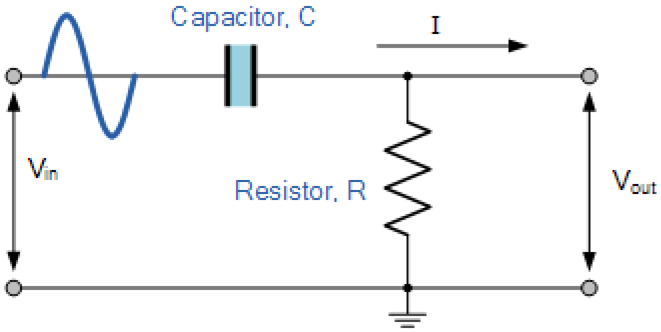
$$f_C = \frac{1}{2\pi\sqrt{R_1 C_1 R_2 C_2}}$$

2nd-Order Low Pass Filter -3dB Frequency

$$f_{-3dB} = f_c\sqrt{2^{1/n} - 1}$$

where f_c is the calculated cut-off frequency, *n* is the filter order and f_{-3dB} is the new -3dB pass band frequency

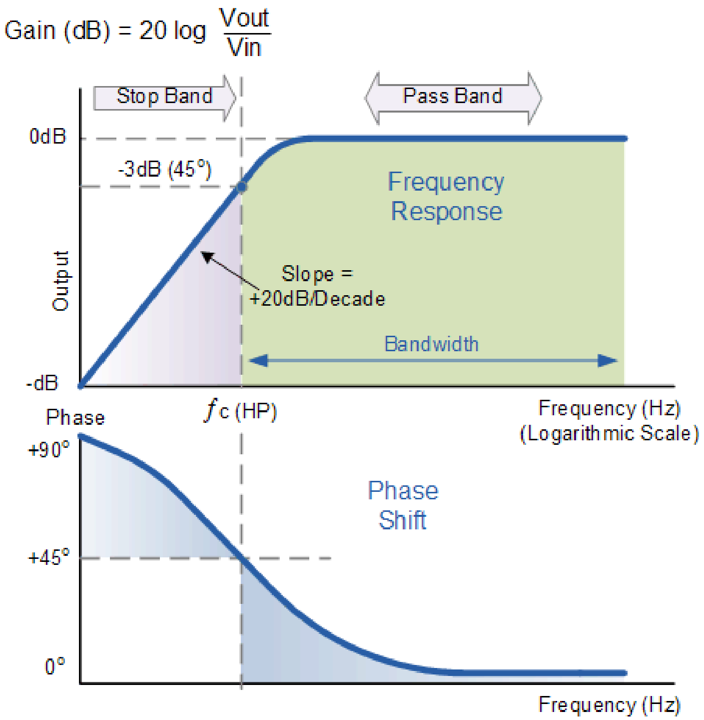
First-order Filter



Cut-off Frequency and Phase Shift

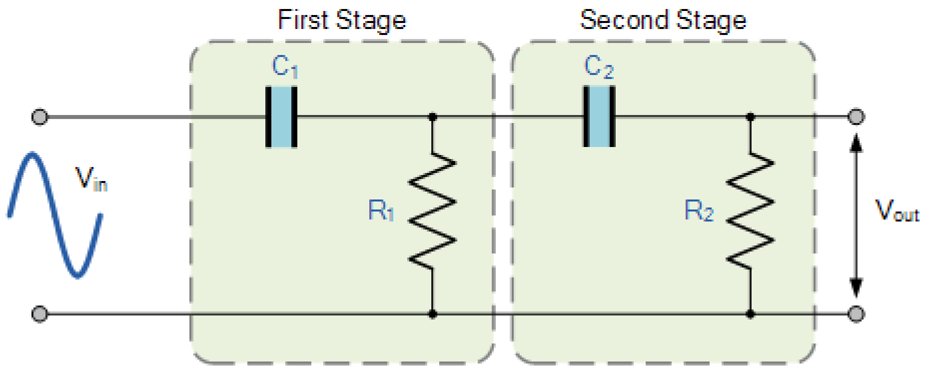
$$f_C = \frac{1}{2\pi RC}$$

phase shift ϕ = arctg$(2\pi f RC)$



The High Pass Filter Circuit

Second-order Low Pass Filter



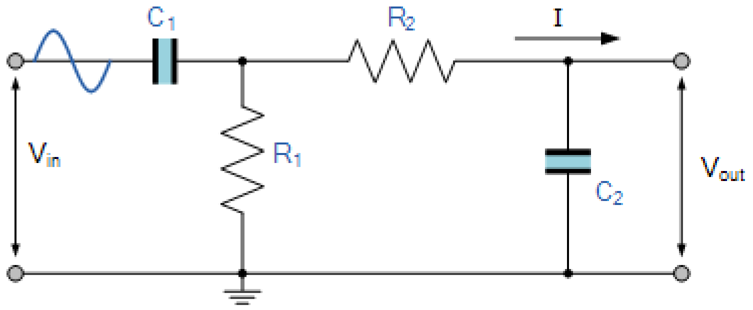
2nd-Order Filter Corner Frequency

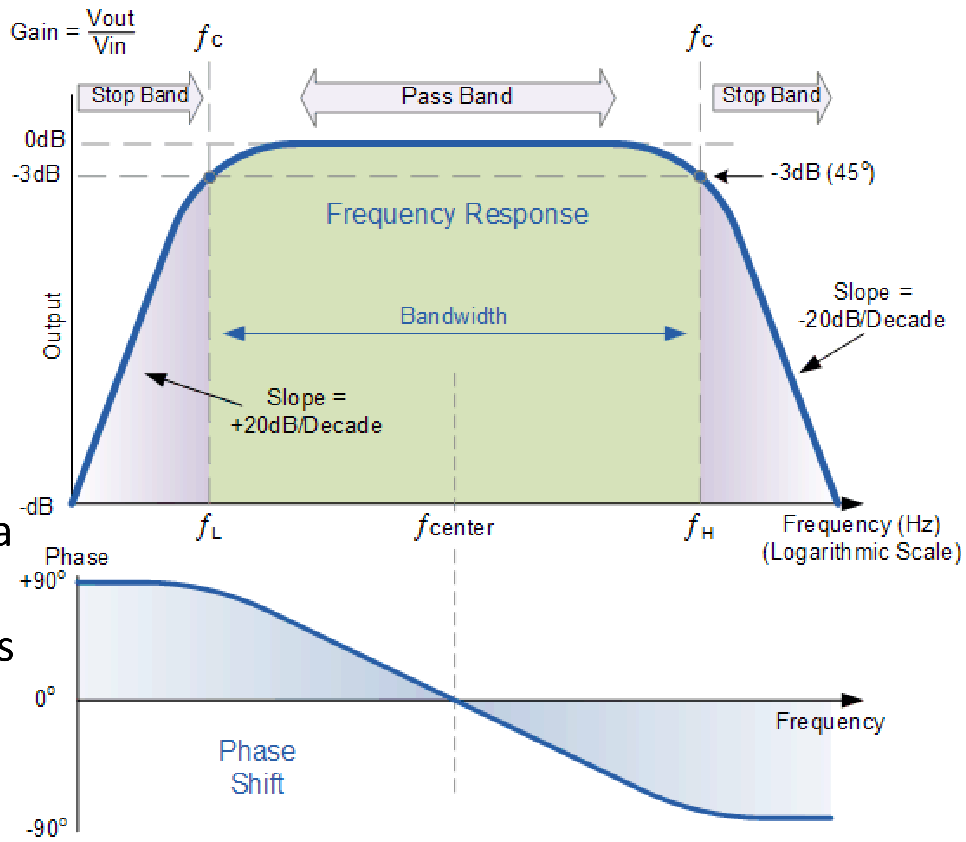
$$f_C = \frac{1}{2\pi\sqrt{R_1 C_1 R_2 C_2}}$$

to reduce the loading effect the impedance of each following stage 10x the previous stage

$$R_2 = 10R_1$$
$$C_2 = \frac{1}{10}C_1$$

Band Pass Filter Circuit





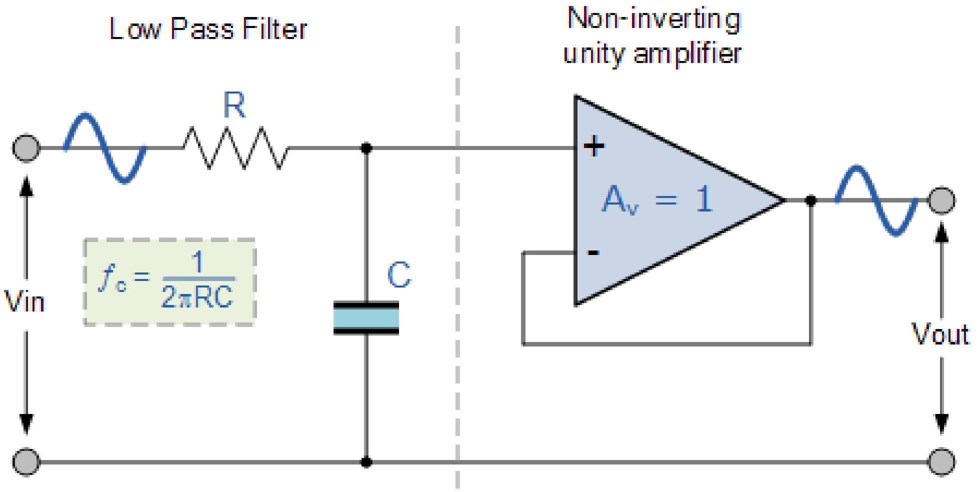
Bandwidth is frequency range that exists between two cut-off frequency

$$BW = f_H - f_L$$

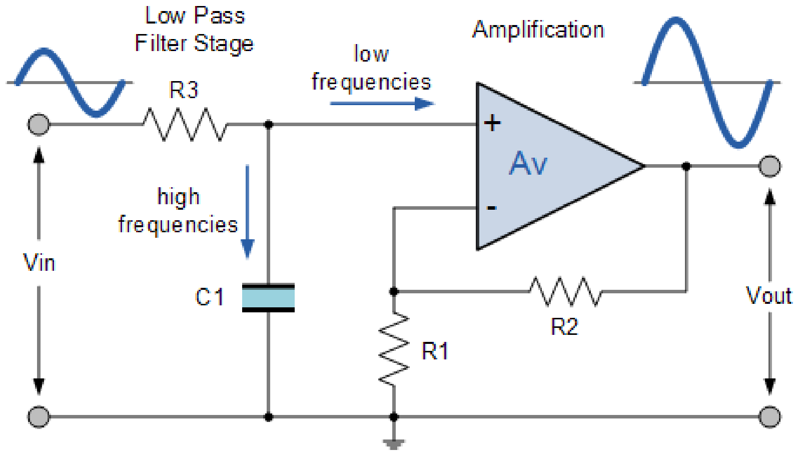
The upper and lower cut-off frequencies for a band pass filter can be found using the same formula as that for both the low and high pass filters

First-order Filter

Active Low Pass Filter without Amplification

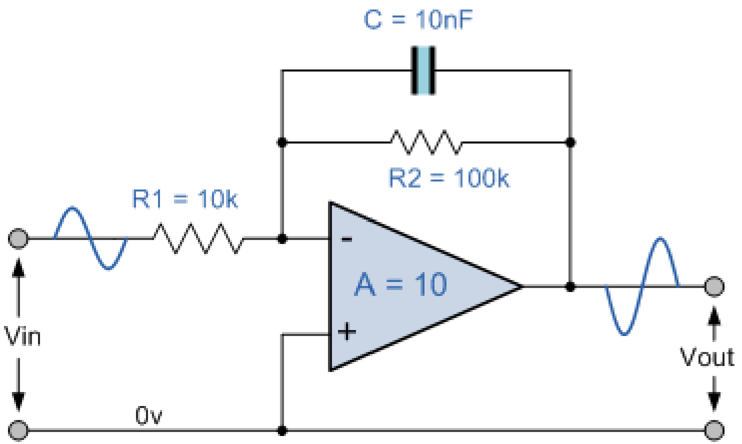


Active Low Pass Filter with Amplification

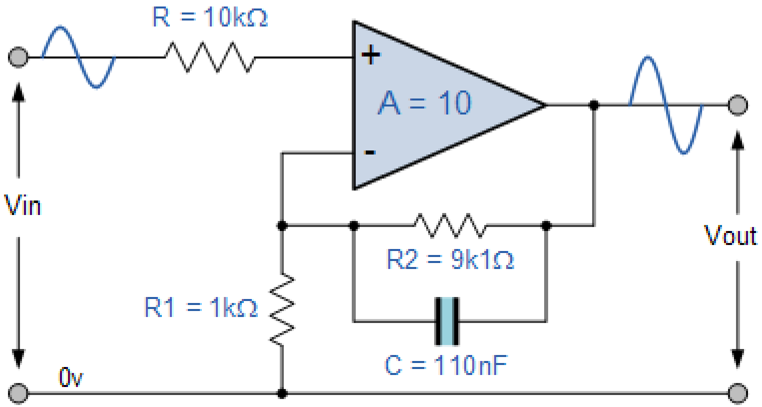


Active Low Pass Filter

Simplified Inverting Amplifier Filter Circuit



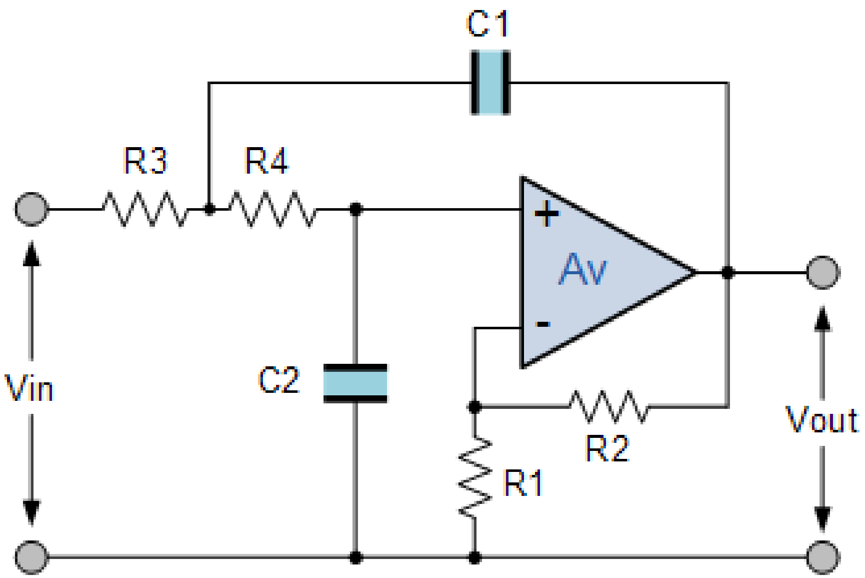
Unity Gain Non-inverting Amplifier Filter Circuit



Cut-off Frequency $$f_C = \frac{1}{2\pi R_2 C}$$

Active Low Pass Filter

Second-order Low Pass Active Filter

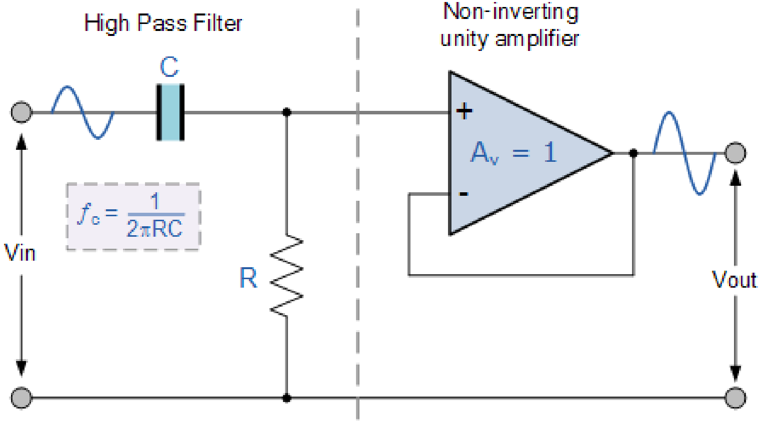


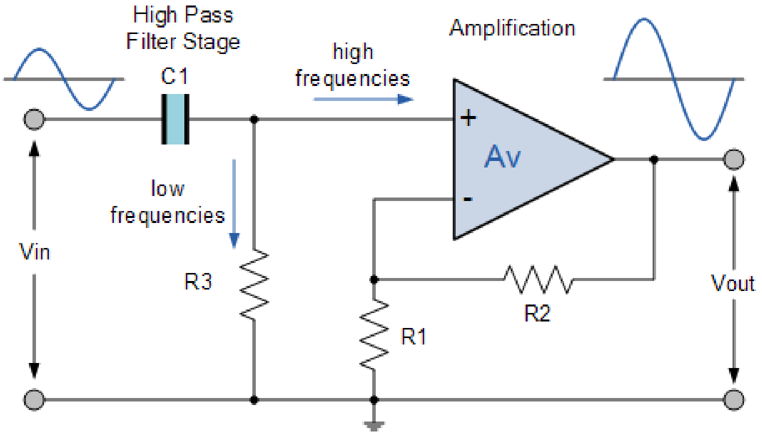
$$A_v = 1 + \frac{R_2}{R_1}$$

$$f_C = \frac{1}{2\pi\sqrt{R_3 R_4 C_1 C_2}}$$

Active High Pass Filter

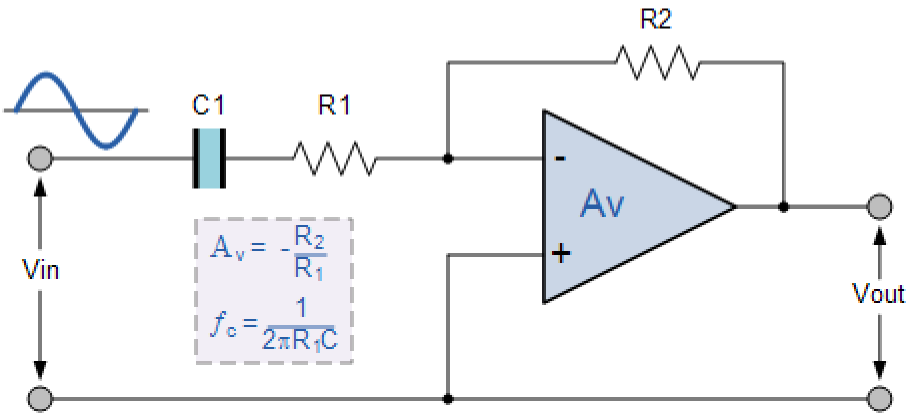
First-order Filter

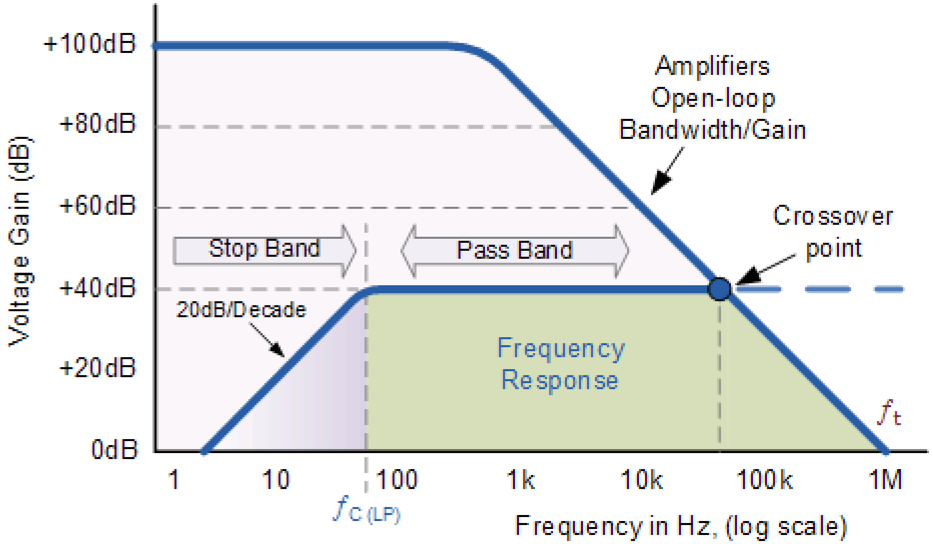




Active High Pass Filter

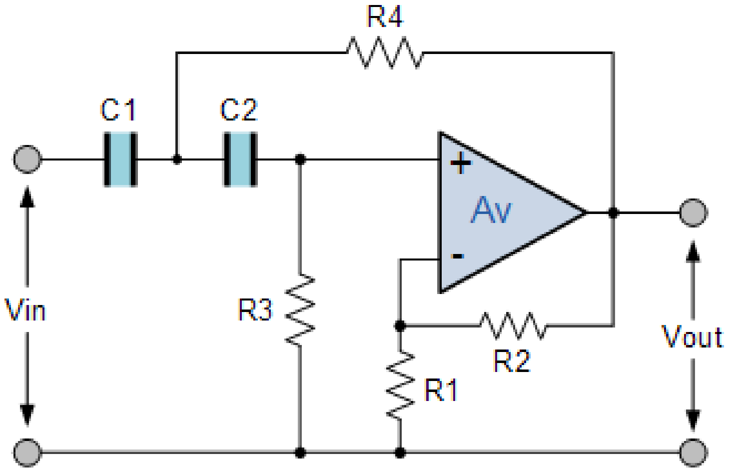
First-order Filter





Active High Pass Filter

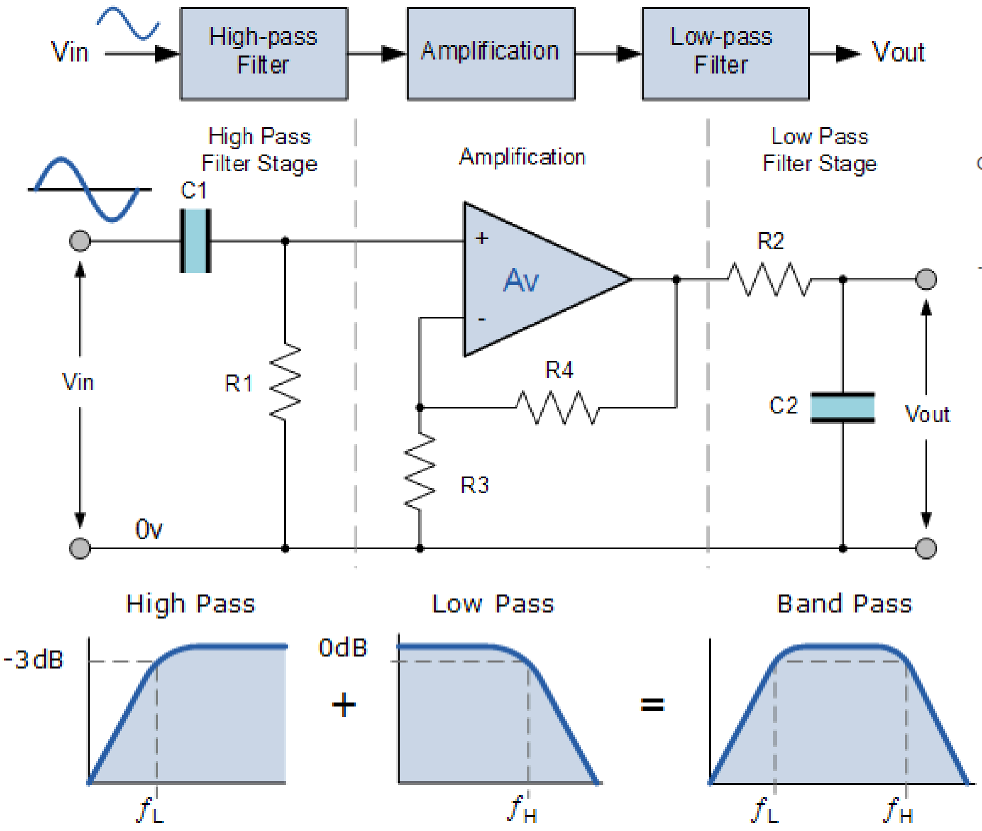
Second-order Active High Pass Filter Circuit

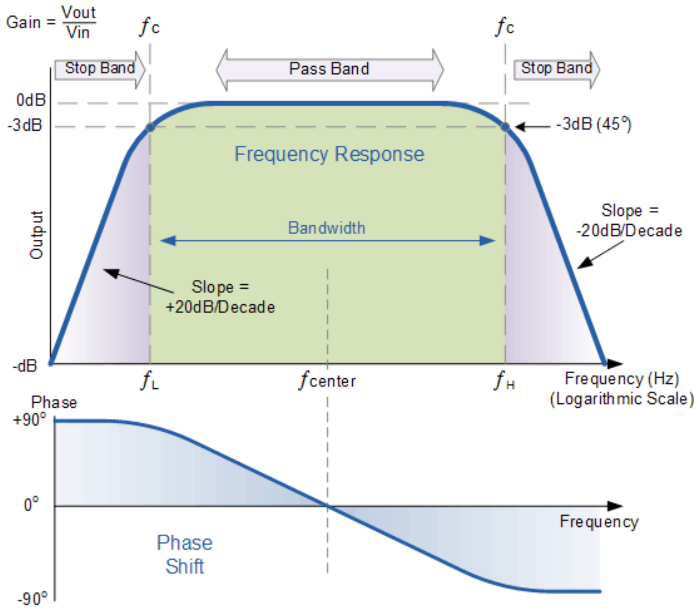


$$A_V = 1 + \frac{R_2}{R_1}$$

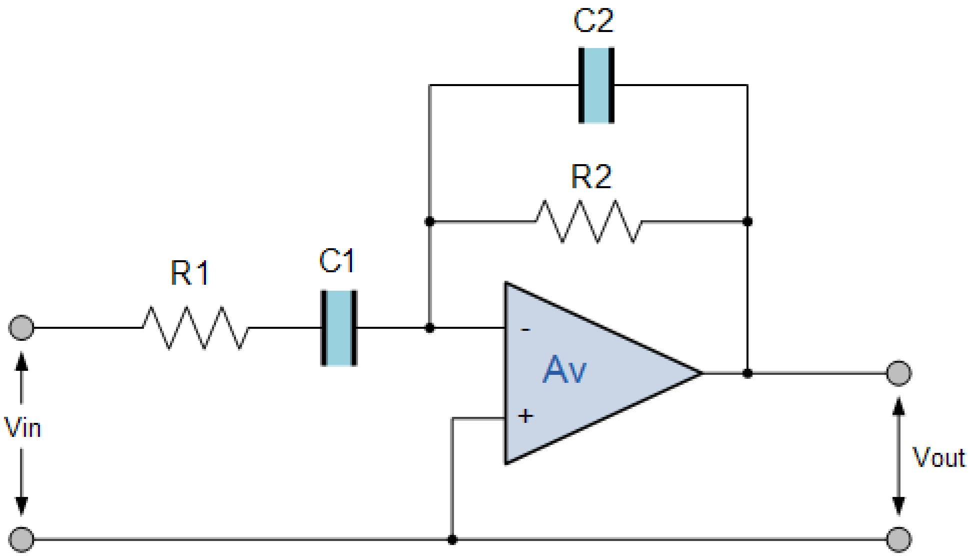
$$f_C = \frac{1}{2\pi\sqrt{R_3 R_4 C_1 C_2}}$$

Active Band Pass Filter





Active Band Pass Filter

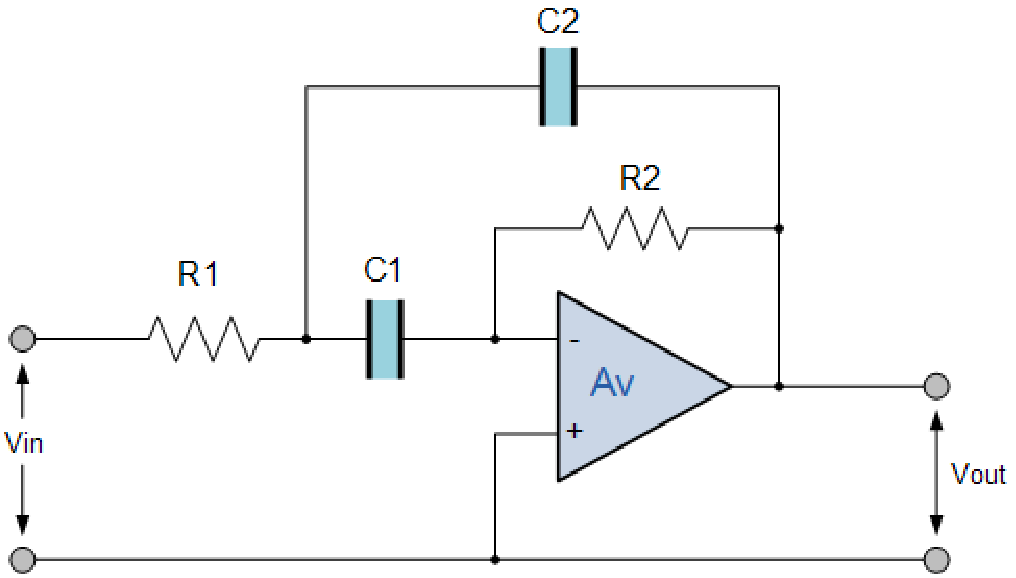


$$f_{C1} = \frac{1}{2\pi R_1 C_1}$$

$$f_{C2} = \frac{1}{2\pi R_2 C_2}$$

Active Band Pass Filter

Multiple Feedback Band Pass Active Filter



infinite-gain multiple-feedback (IGMF) band pass filter

the characteristics of the IGMF filter

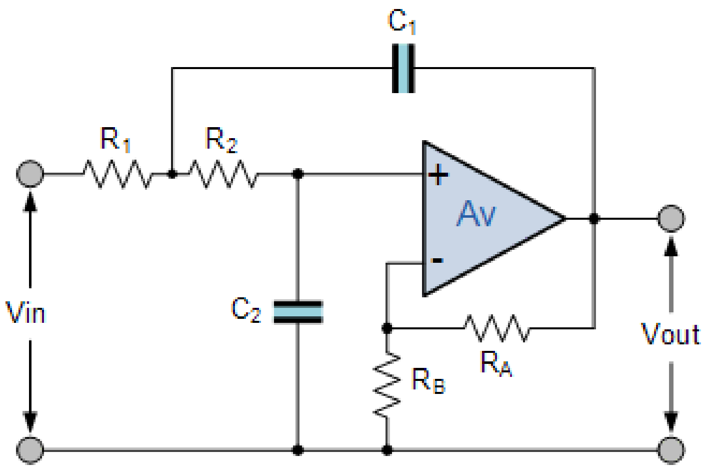
$$f_r = \frac{1}{2\pi\sqrt{R_1 C_1 R_2 C_2}}$$

$$Q = \frac{f_r}{BW_{-3dB}} = \frac{1}{2}\sqrt{\frac{R_2}{R_1}}$$

$$Max\ gain \sim -\frac{R_2}{2R_1} = 2Q^2$$

Second Order Filters

Second Order Low Pass Filter



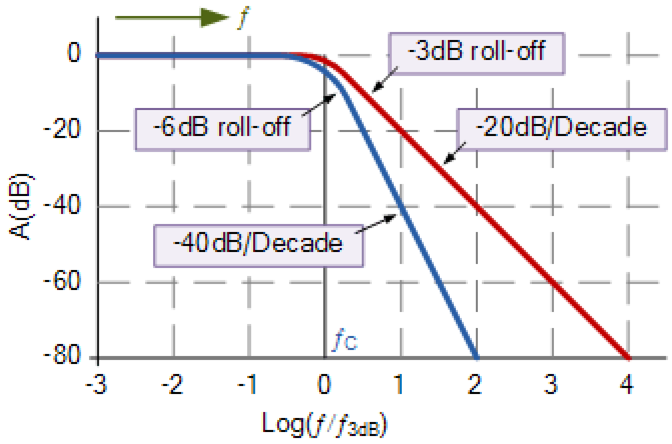
Gain (Av) = $1 + \frac{R_A}{R_B}$

If Resistor and Capacitor values are different:

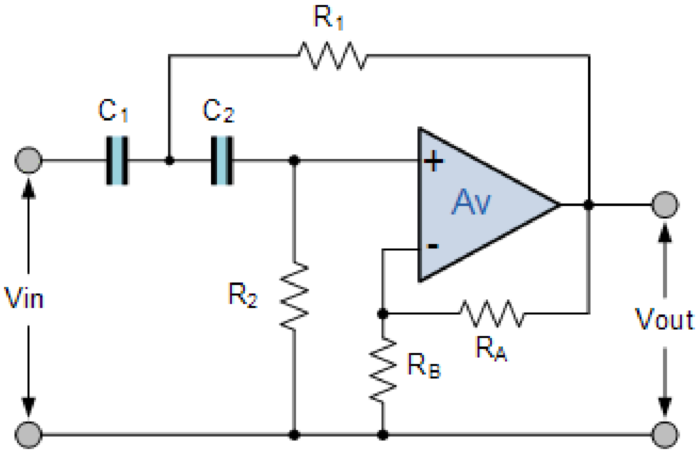
$$f_C = \frac{1}{2\pi\sqrt{R_1 R_2 C_1 C_2}}$$

If Resistor and Capacitor values are the same:

$$f_C = \frac{1}{2\pi RC}$$



Second Order High Pass Filter



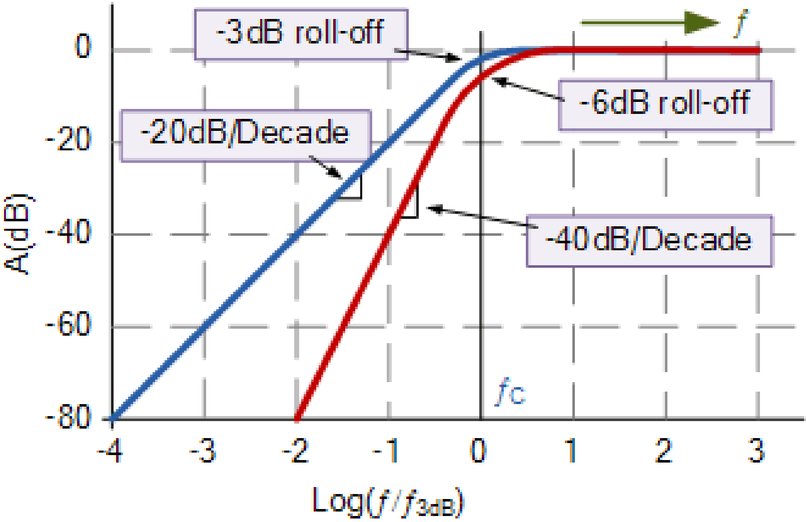
$$\text{Gain } (A_v) = 1 + \frac{R_A}{R_B}$$

If Resistor and Capacitor values are different:

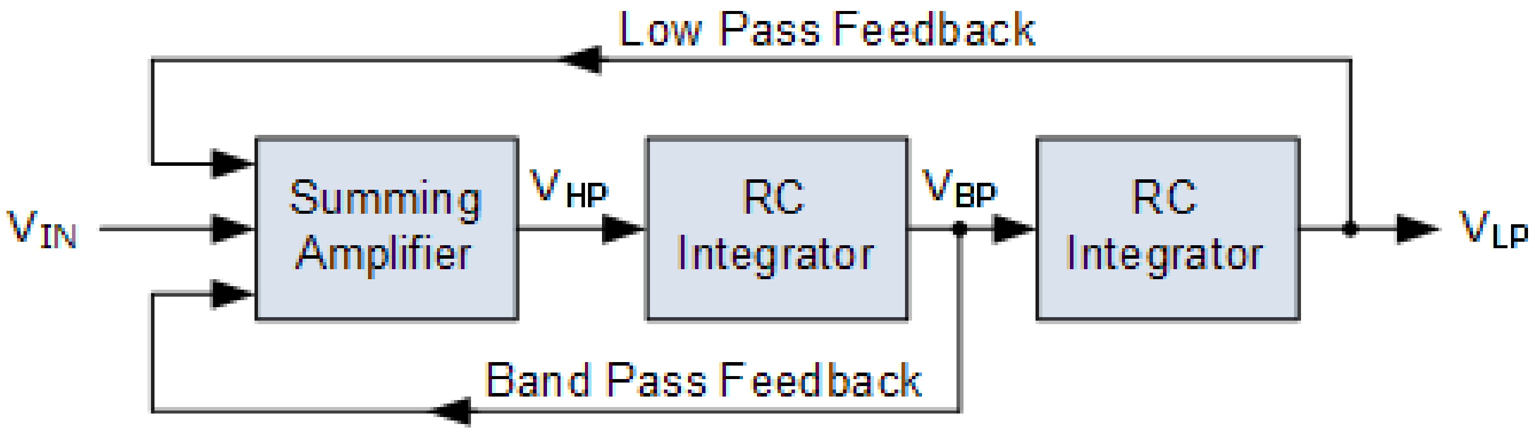
$$f_C = \frac{1}{2\pi\sqrt{R_1 R_2 C_1 C_2}}$$

If Resistor and Capacitor values are the same:

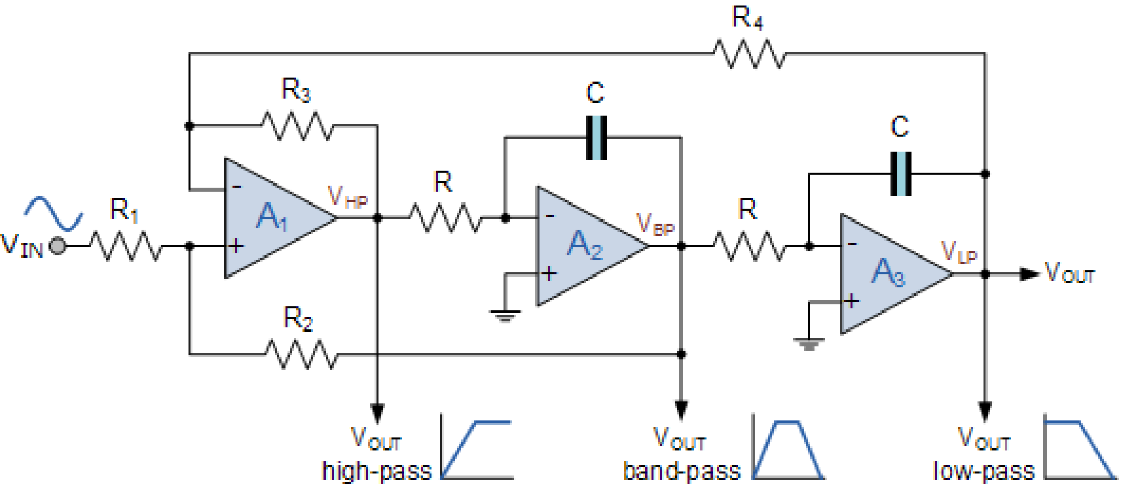
$$f_C = \frac{1}{2\pi RC}$$

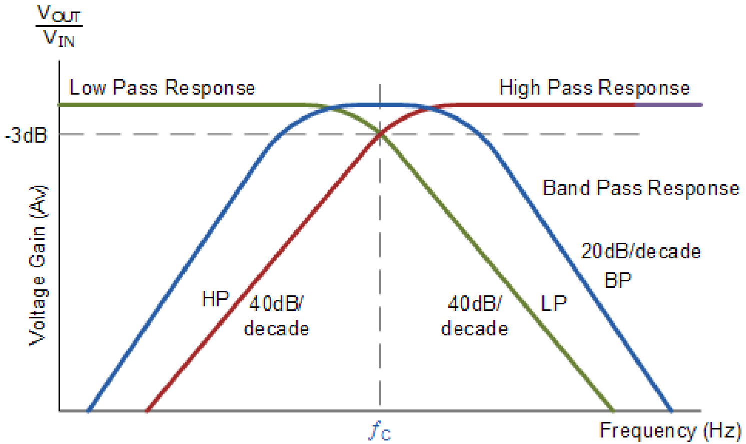


State Variable Filter



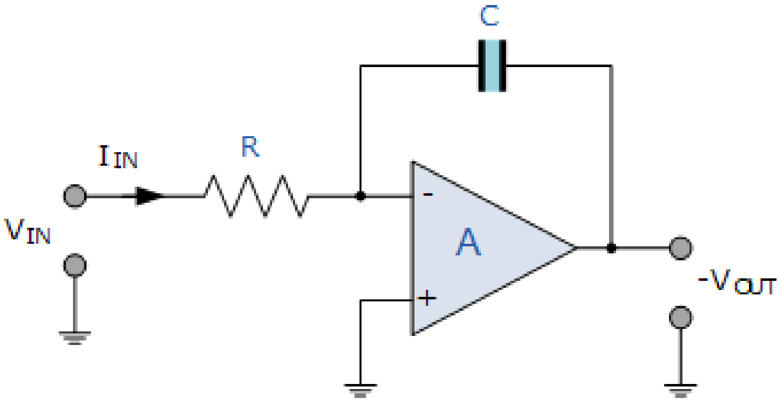
State Variable Filter Circuit





State Variable Filter Circuit

Op-amp Integrator Circuit



in the time domain $$V_{out} = \frac{1}{RC}\int_0^t V_{in}dt$$

in the frequency domain $$V_{out} = -\frac{1}{2\pi f_C RC}V_{in}$$

Op-amp A2 Transfer Function

$$\frac{V_{BP}}{V_{HP}} = -\frac{1}{2\pi f_C RC}$$

Op-amp A3 Transfer Function

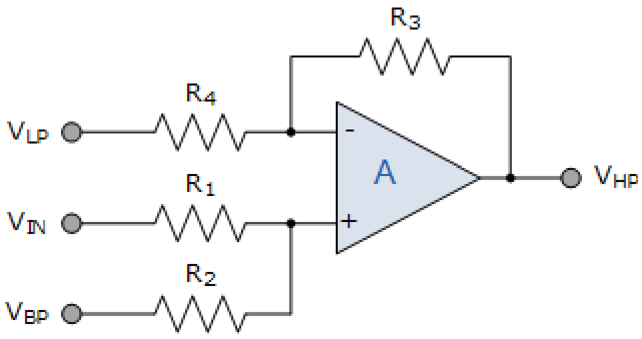
$$\frac{V_{LP}}{V_{BP}} = -\frac{1}{2\pi f_C RC}$$

Transfer function between V_{HP} and V_{LP}

$$\frac{V_{LP}}{V_{HP}} = -\frac{1}{2\pi f_C RC} \times -\frac{1}{2\pi f_C RC}$$

$$\frac{V_{LP}}{V_{HP}} = \frac{1}{(2\pi f_C RC)^2}$$

Amplifier Summing Circuit



$$V_+ = \frac{V_{in}R_2 + V_{BP}R_1}{R_1 + R_2}$$

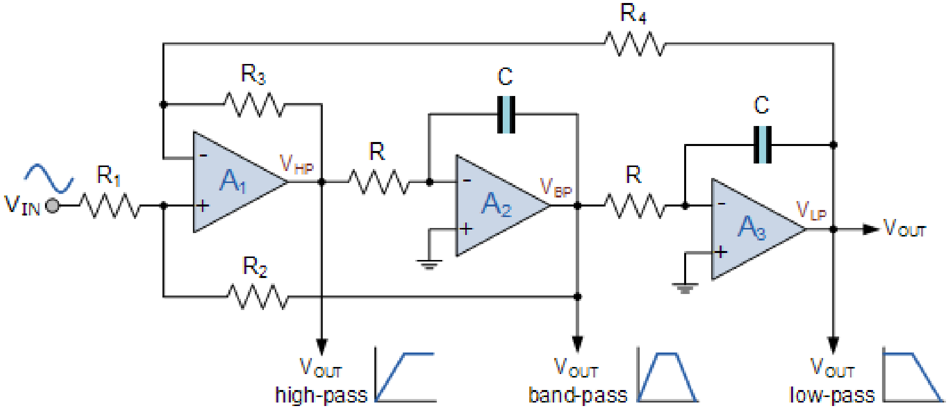
$$V_- = \frac{V_{LP}R_3 + V_{HP}R_4}{R_3 + R_4}$$

transfer function for the output of A1

$$V_{HP} = V_{In}\frac{R_2(R_3 + R_4)}{R_3(R_1 + R_2)} + V_{BP}\frac{R_1(R_3 + R_4)}{R_4(R_1 + R_2)} - V_{LP}\frac{R_3}{R_4}$$

$$\frac{V_{OUT}}{V_{IN}} = \frac{V_{LP}}{V_{IN}} = \frac{\frac{R_2(R_3 + R_4)}{R_3(R_1 + R_2)} \times \frac{1}{RC}}{\frac{R_3}{R_4RC} + \frac{R_1(R_3 + R_4)}{R_4(R_1 + R_2)} \times \frac{1}{2\pi RC} + \left(\frac{1}{2\pi RC}\right)^2} = \frac{A_0\frac{f}{f_0}}{1 + 2\zeta\frac{f}{f_0} + \left(\frac{f}{f_0}\right)^2}$$

State Variable Filter Circuit



State Variable Filter Corner Frequency

$$f_C = \sqrt{\frac{R_3}{R_4(2\pi RC)^2}}$$

If $R_3 = R_4$, then

$$f_{C(HP)} = f_{C(BP)} = f_{C(LP)} = \sqrt{\frac{1}{(2\pi RC)^2}}$$

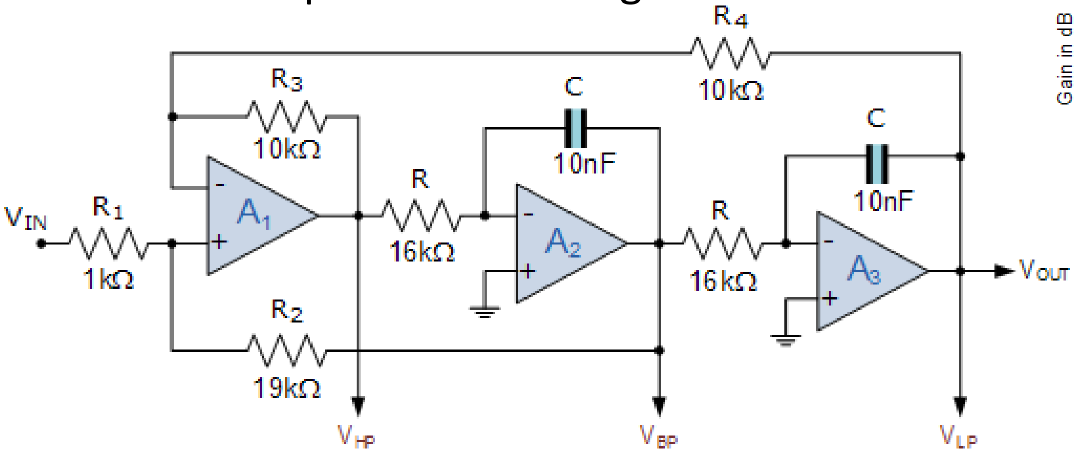
$$Q = \frac{f_C}{BW} = \frac{1}{2\zeta} = \frac{R_1(R_3 + R_4)}{R_4(R_1 + R_2)}\sqrt{\frac{R_3}{R_4} \times \frac{RC}{RC}}$$

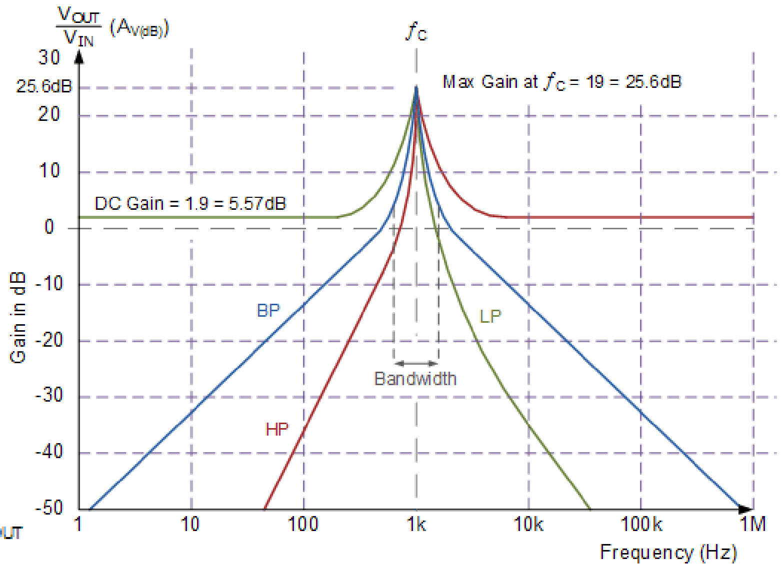
State Variable Filter Circuit. Example

Design a State Variable Filter which has

- f_C of 1kHz;
- quality factor, Q of 10;
- Assume both the frequency determining resistors and capacitors are equal.

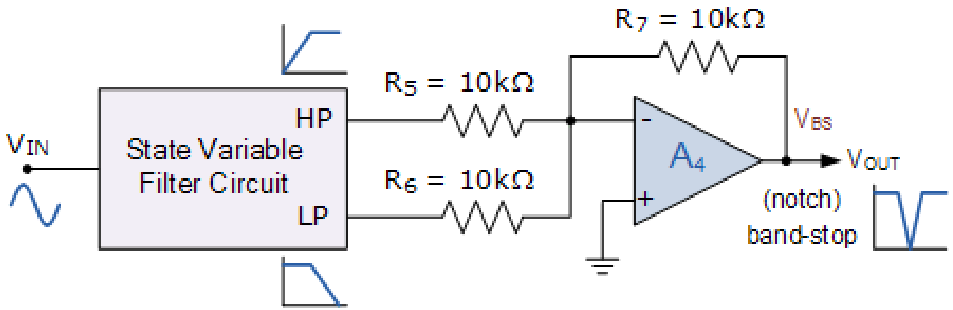
Assuming in calculations that C= 10nF, R3 and R4 are the same and equal to 10kΩ we get:



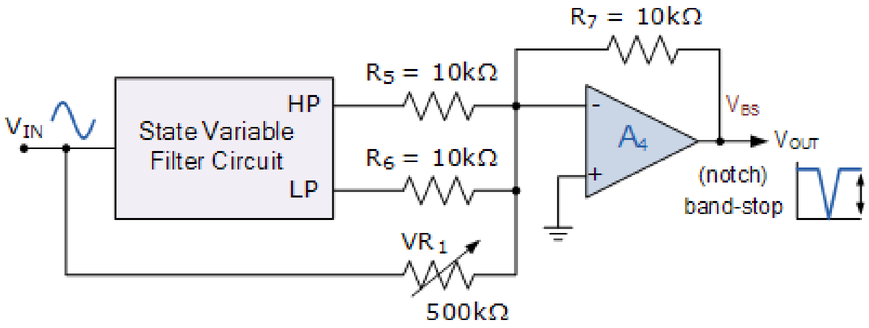


State Variable Filter Circuit

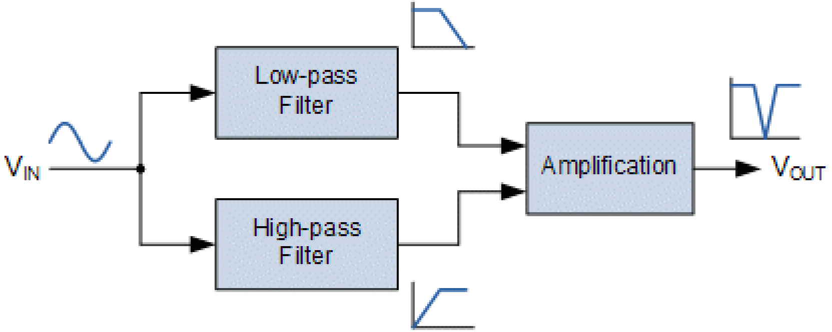
Notch Filter Design

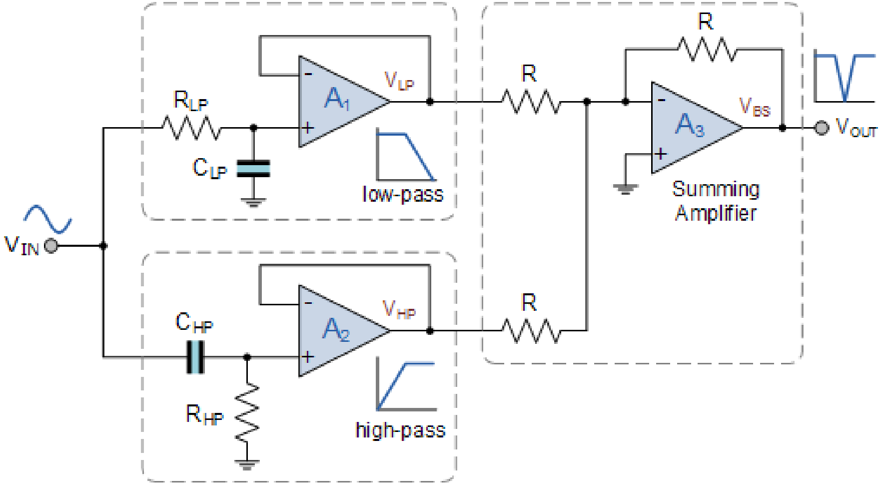


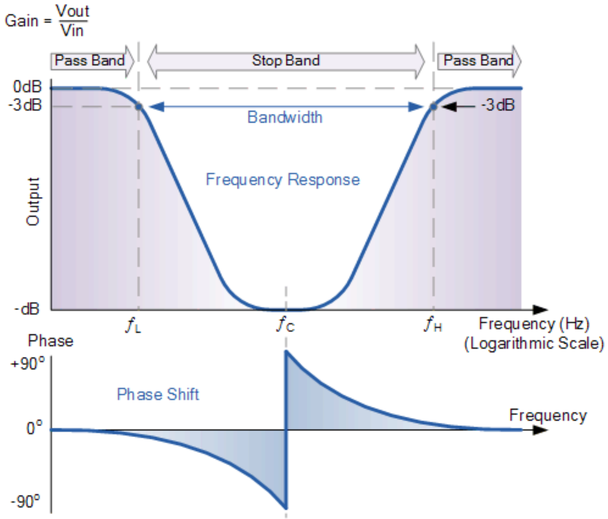
Variable Notch Filter Depth



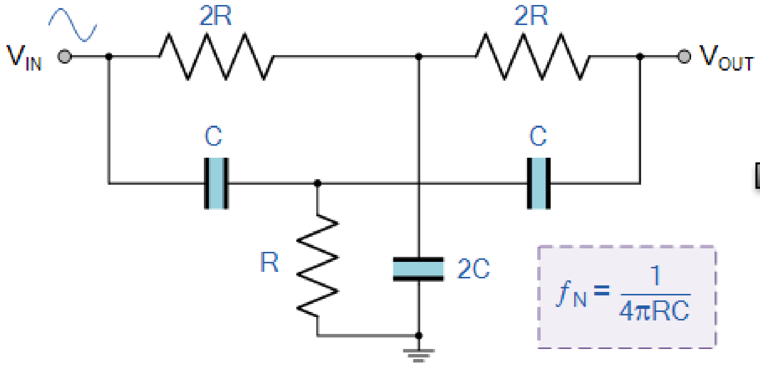
Band Stop Filter

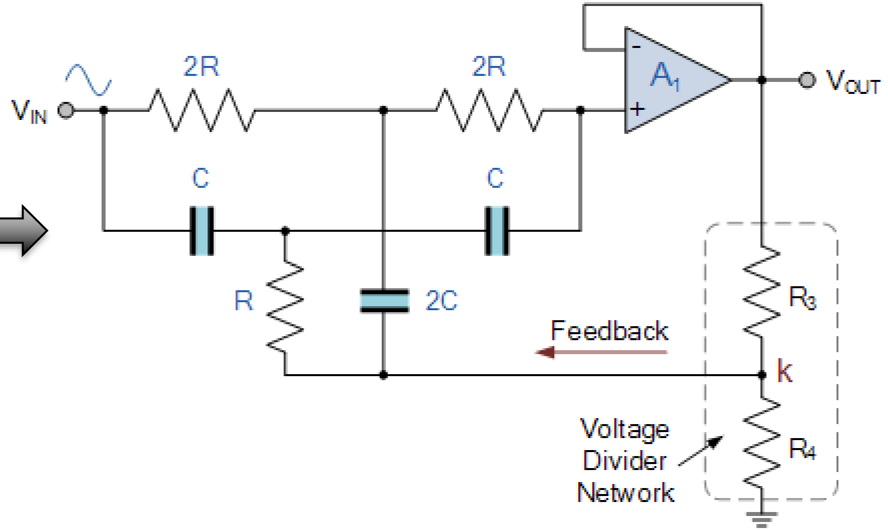




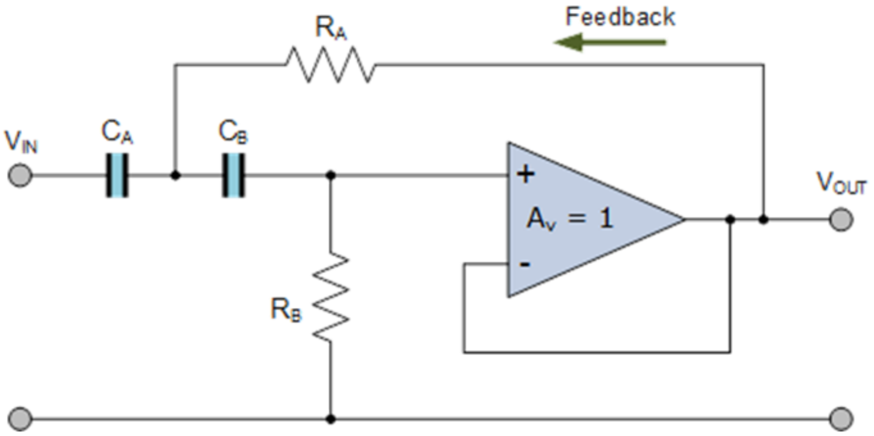


Notch Filters





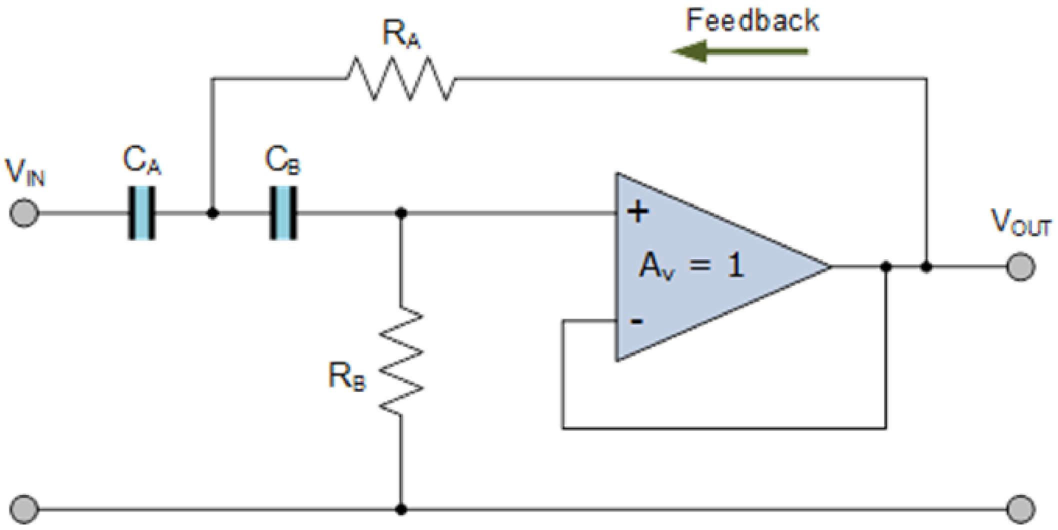
Sallen-Key Filter



The main advantages of the Sallen-key filter design are:

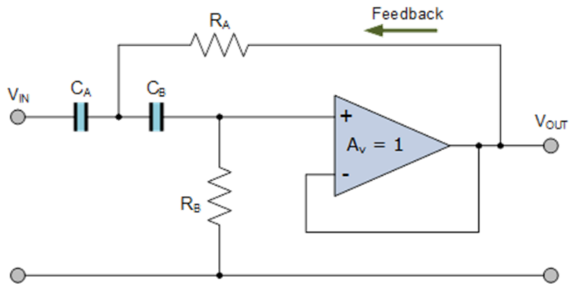
- Simplicity and Understanding of their Basic Design
- The use of a Non-inverting Amplifier to Increase Voltage Gain
- First and Second-order Filter Designs can be Easily Cascaded Together
- Low-pass and High-pass stages can be Cascaded Together
- Each RC stage can have a different Voltage Gain
- Replication of RC Components and Amplifiers
- Second-order Sallen-key Stages have Steep 40dB/decade roll-off than cascaded RC

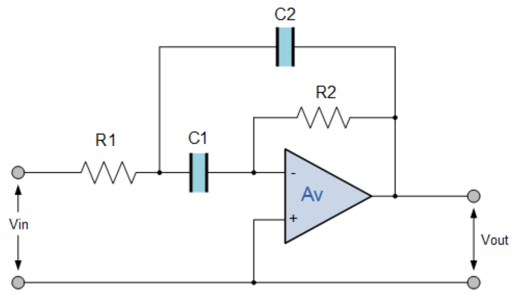
Sallen-key High Pass Filter Circuit



$$f_C = \frac{1}{2\pi\sqrt{R_A C_A R_B C_B}}$$

Sallen-Key Filter VS Multiple Feedback Filter





Sallen-Key	Multiple Feedback
Non-inverting	Inverting
Very precise DC-gain of 1	Any gain is dependent on the resistor precision
Less components for gain = 1	Less components for gain > 1 or < 1
Op-amp input capacitance must possibly be taken into account	Op-amp input capacitance has almost no effect
Resistive load for sources even in high-pass filters	Capacitive loads can become very high for sources in high-pass filters

Frequency Transformation

Frequency Transformation

Type of Transformation	Frequency transform
The Lowpass to Highpass (LP-HP) Frequency Transformation	$s \Leftrightarrow \frac{1}{s}$ $H_{HP}(s) = H_{LP}\left(\frac{1}{s}\right)$
The Lowpass to Bandpass (LP-BP) Frequency Transformation	$s \Leftrightarrow \frac{s^2 + \omega_0^2}{sBW}$ $H_{BP}(s) = H_{LP}\left(\frac{s^2 + \omega_0^2}{sBW}\right)$
The Lowpass to Band-Reject (LP-BR) Frequency Transformation	$s \Leftrightarrow \frac{sBW}{s^2 + \omega_0^2}$ $H_{BR}(s) = H_{LP}\left(\frac{sBW}{s^2 + \omega_0^2}\right)$

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7. https://en.wikipedia.org/

Thank you for your attention!