Sum of exponential and laplace distributions

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I proved the following result while working on my thesis. The result turned out to be not relevant to my thesis, but I wanted to record the result here since working through it was instructive.

1 Problem

Suppose we have two independent random variables $\mathcal{X} \sim \mathrm{Exp}(\lambda)$ and $\mathcal{Y} \sim \mathrm{Laplace}(\mu = 0, \gamma)$. Let $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$. What is the probability density function $f_\mathcal{Z}(z)$ of $\mathcal{Z}$?

$\mathrm{Exp}(\lambda)$ is the exponential distribution, defined such that λ is its mean:

$$\mathrm{Exp}(\lambda) = f_\mathcal{X}(x) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}} \qquad x \in [0, \infty) \tag{1}$$

where $f_\mathcal{X}$ is the probability density function (PDF), and x is the parameter of $\mathcal{X}$.

$\mathrm{Laplace}(\mu = 0, \gamma)$ is the laplace distribution, with its mean μ defined to be zero and scale parameter γ:

$$\mathrm{Laplace}(\mu, \gamma) = f_\mathcal{Y}(y) = \frac{1}{2\gamma} e^{-\frac{|y-\mu|}{\gamma}} \qquad y \in (\infty, \infty) \tag{2}$$

2 Solution

Since $\mathcal{X}, \mathcal{Y}$ are independent, we can define the joint probability distribution $f_{\mathcal{X}\mathcal{Y}}(x, y)$ as the product of the two PDFs:

$$\begin{align} f_{\mathcal{X}\mathcal{Y}}(x, y) &= f_\mathcal{X}(x) f_\mathcal{Y}(y) \tag{3} \\ &= \frac{1}{2\lambda\gamma} e^{-\frac{x}{\lambda}} e^{-\frac{|y|}{\gamma}} \qquad x \in [0, \infty) \quad y \in (\infty, \infty) \tag{4} \end{align}$$

We will start with the cumulative distribution function (CDF) $F_\mathcal{Z}(z)$. The CDF is defined as the probability that the random variable $\mathcal{Z}$ takes a value that is less than z: $F_\mathcal{Z}(z) = P(\mathcal{Z} < z)$. Since as $z \to \infty$ all values must be less than z, $F_{\mathcal{Z}}(z) \to 1$, and we can write $F_{\mathcal{Z}}(z) = 1 - P(\mathcal{Z} \geq z)$. Recall that the PDF is equal to the derivative of the CDF: $f_{\mathcal{Z}} = \dfrac{dF_{\mathcal{Z}}}{dz}$.

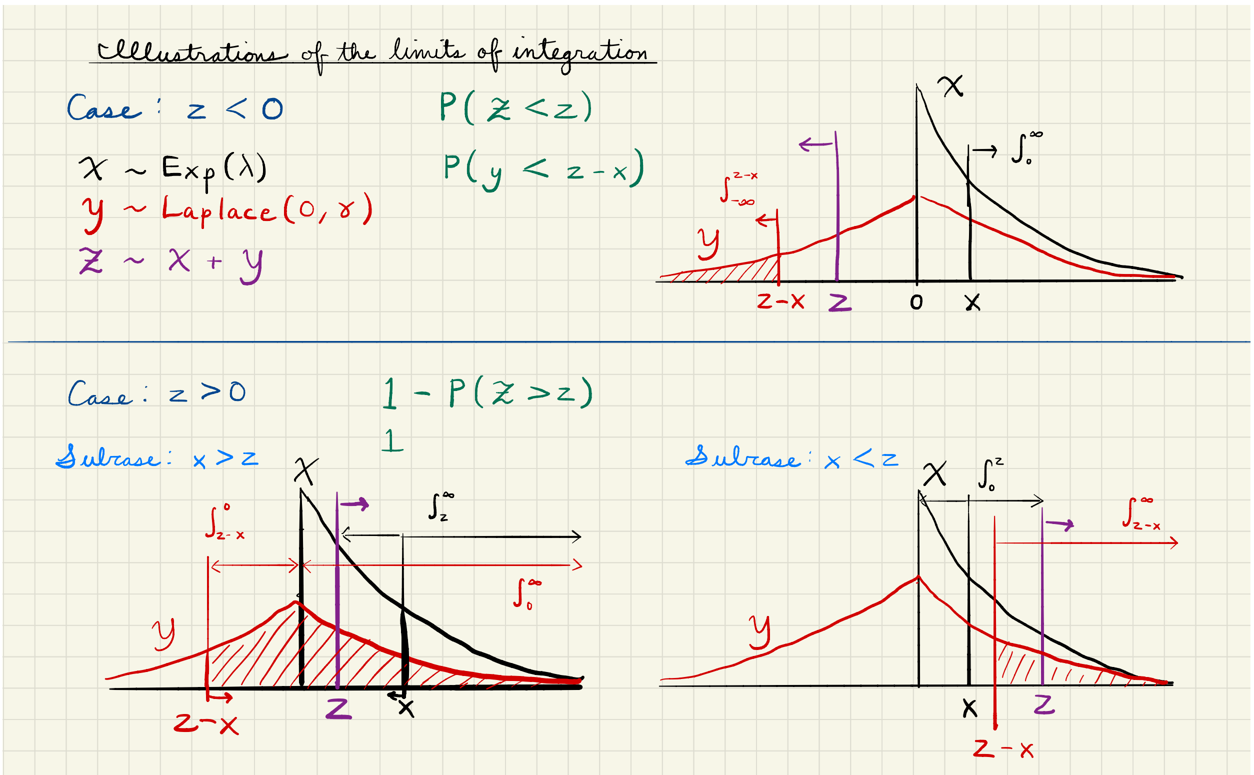

Figure 1: Illustrations of the limits of integration for Eqs. 5, 7, and 8.

We split the solution into parts. The first part is finding $F_{\mathcal{Z}}(z)$ for $z < 0$. The only way for $z = x + y < 0$ to be less than zero is if $y < -x$, because $x \in [0, \infty)$. Also, since we know $y < 0$, we can write $|y| = -y$ for the joint probability distribution. Thus we have

$$
\begin{aligned}
F_{\mathcal{Z}}(z < 0) &= P(x + y < z) \\
&= P(y < z - x) \\
&= \int_0^{\infty} dx \int_{-\infty}^{z-x} dy\, f_{\mathcal{X}\mathcal{Y}}(x, y) \\
&= \frac{1}{2\lambda\gamma} \int_0^{\infty} dx \int_{-\infty}^{z-x} dy\, e^{-\frac{x}{\lambda}} e^{\frac{y}{\gamma}} \\
&= \frac{e^{\frac{z}{\gamma}}}{2\lambda} \int_0^{\infty} dx e^{-\frac{x(\lambda+\gamma)}{\gamma\lambda}} \\
&= \frac{\gamma e^{\frac{z}{\gamma}}}{2(\lambda + \gamma)} \qquad z \in [0, \infty)
\end{aligned} \tag{5}
$$

This concludes the first part, $z < 0$.

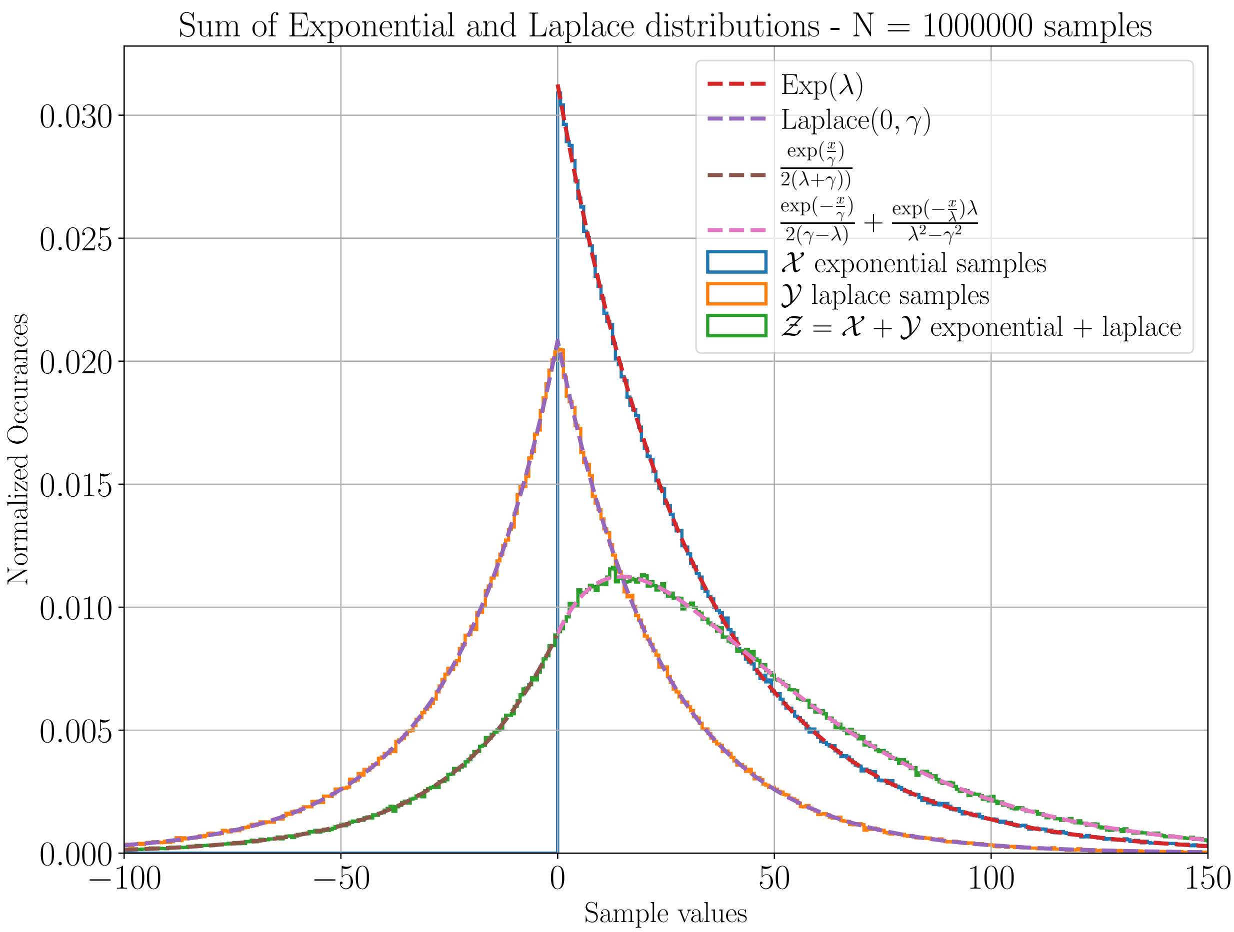

Figure 2: Plot of 1000000 samples from a exponential distribution $f_{\mathcal{X}}$, laplace distribution $f_{\mathcal{Y}}$, and their sum $f_{\mathcal{Z}}$. In this example, $\lambda = 32$ and $\gamma = 24$.

The second part is finding $F_{\mathcal{Z}}(z)$ for $z > 0$.

$$\begin{aligned} F_{\mathcal{Z}}(z > 0) &= P(x + y < z) \\ &= 1 - P(x + y > z) \\ &= 1 - P(y > z - x) \\ &= 1 - \frac{1}{2\lambda\gamma} \int_0^\infty dx \int_{z-x}^\infty dy\, e^{-\frac{x}{\lambda}} e^{\frac{|y|}{\gamma}} \end{aligned} \tag{6}$$

There are two ways for $z = x + y > 0$: $y < 0$ but $x > -y$, and simply $y > 0$. The difficulty lies in the change in PDF $f_{\mathcal{Y}}$ at $y = 0$. We will split this integral into two more parts: $x > z$ and $x < z$, at which point we will switch between PDFs.

First, let $x > z$. Then we have the sum of two integrals, one with $f_{\mathcal{Y}}(y < 0)$ and the other with $f_{\mathcal{Y}}(y > 0)$:

$$
\begin{aligned}
P(y > z - x | x > z) &= \frac{1}{2\lambda\gamma} \int_z^\infty dx \int_{z-x}^\infty dy\, e^{-\frac{x}{\lambda}} e^{\frac{|y|}{\gamma}} \\
&= \frac{1}{2\lambda\gamma} \int_z^\infty dx \left[\int_{z-x}^0 dy\, e^{-\frac{x}{\lambda}} e^{\frac{y}{\gamma}} + \int_0^\infty dy\, e^{-\frac{x}{\lambda}} e^{-\frac{y}{\gamma}} \right] \\
&= \frac{1}{2\lambda} \int_z^\infty dx \left[e^{-x\left(\frac{1}{\gamma}+\frac{1}{\lambda}\right)} \left(e^{x/\gamma} - e^{z/\gamma} \right) + e^{-\frac{x}{\lambda}} \right] \\
&= \frac{1}{2\lambda} \int_z^\infty dx \left[e^{-x\left(\frac{1}{\gamma}+\frac{1}{\lambda}\right)} \left(2e^{x/\gamma} - e^{z/\gamma} \right) \right] \\
&= \frac{(\gamma + 2\lambda) e^{-\frac{z}{\lambda}}}{2(\gamma + \lambda)}
\end{aligned} \tag{7}
$$

Second, let $x < z$. Now for $z > x + y$, $y > 0$, so we can just consider $f_{\mathcal{Y}}(y > 0)$:

$$
\begin{aligned}
P(y > z - x | x < z) &= \frac{1}{2\lambda\gamma} \int_0^z dx \int_{z-x}^\infty dy\, e^{-\frac{x}{\lambda}} e^{\frac{|y|}{\gamma}} \\
&= \frac{1}{2\lambda\gamma} \int_0^z dx \int_{z-x}^\infty dy\, e^{-\frac{x}{\lambda}} e^{-\frac{y}{\gamma}} \\
&= \frac{1}{2\lambda} \int_0^z dx e^{\frac{x}{\gamma} - \frac{x}{\lambda} - \frac{z}{\gamma}} \\
&= \frac{\gamma \left(e^{-\frac{z}{\gamma}} - e^{-\frac{z}{\lambda}} \right)}{2(\gamma - \lambda)}
\end{aligned} \tag{8}
$$

Summing Eqs. 7 and 8:

$$
\begin{aligned}
F_{\mathcal{Z}}(z > 0) &= 1 - P(y > z - x) \\
&= 1 - P(y > z - x | x > z) + P(y > z - x | x < z) \\
&= 1 - \frac{(\gamma + 2\lambda) e^{-\frac{z}{\lambda}}}{2(\gamma + \lambda)} - \frac{\gamma \left(e^{-\frac{z}{\gamma}} - e^{-\frac{z}{\lambda}} \right)}{2(\gamma - \lambda)} \\
&= 1 + \frac{\lambda^2 e^{-\frac{z}{\lambda}}}{\gamma^2 - \lambda^2} - \frac{\gamma e^{-\frac{z}{\gamma}}}{2(\gamma - \lambda)} \qquad z \in [0, \infty]
\end{aligned} \tag{9}
$$

This concludes the second part, $z > 0$.

Thus the total CDF is given by

$$
F_{\mathcal{Z}}(z) = \begin{cases} \dfrac{\gamma e^{\frac{z}{\gamma}}}{2(\lambda + \gamma)} & z < 0 \\[2ex] 1 + \dfrac{\lambda^2 e^{-\frac{z}{\lambda}}}{\gamma^2 - \lambda^2} - \dfrac{\gamma e^{-\frac{z}{\gamma}}}{2(\gamma - \lambda)} & z \geq 0 \end{cases} \tag{10}
$$

Taking the derivative of Eq 10 gives the PDF $f_{\mathcal{Z}}(z)$:

$$f_{\mathcal{Z}}(z) = \begin{cases} \dfrac{e^{\frac{z}{\gamma}}}{2(\lambda+\gamma)} & z < 0 \\ \dfrac{\lambda e^{-\frac{z}{\lambda}}}{\lambda^2-\gamma^2} + \dfrac{e^{-\frac{z}{\gamma}}}{2\gamma-2\lambda} & z \geq 0 \end{cases} \tag{11}$$

These formula can be seen as the brown and pink dashed curves in Figure 2.