

COMMONWEALTH OF AUSTRALIA

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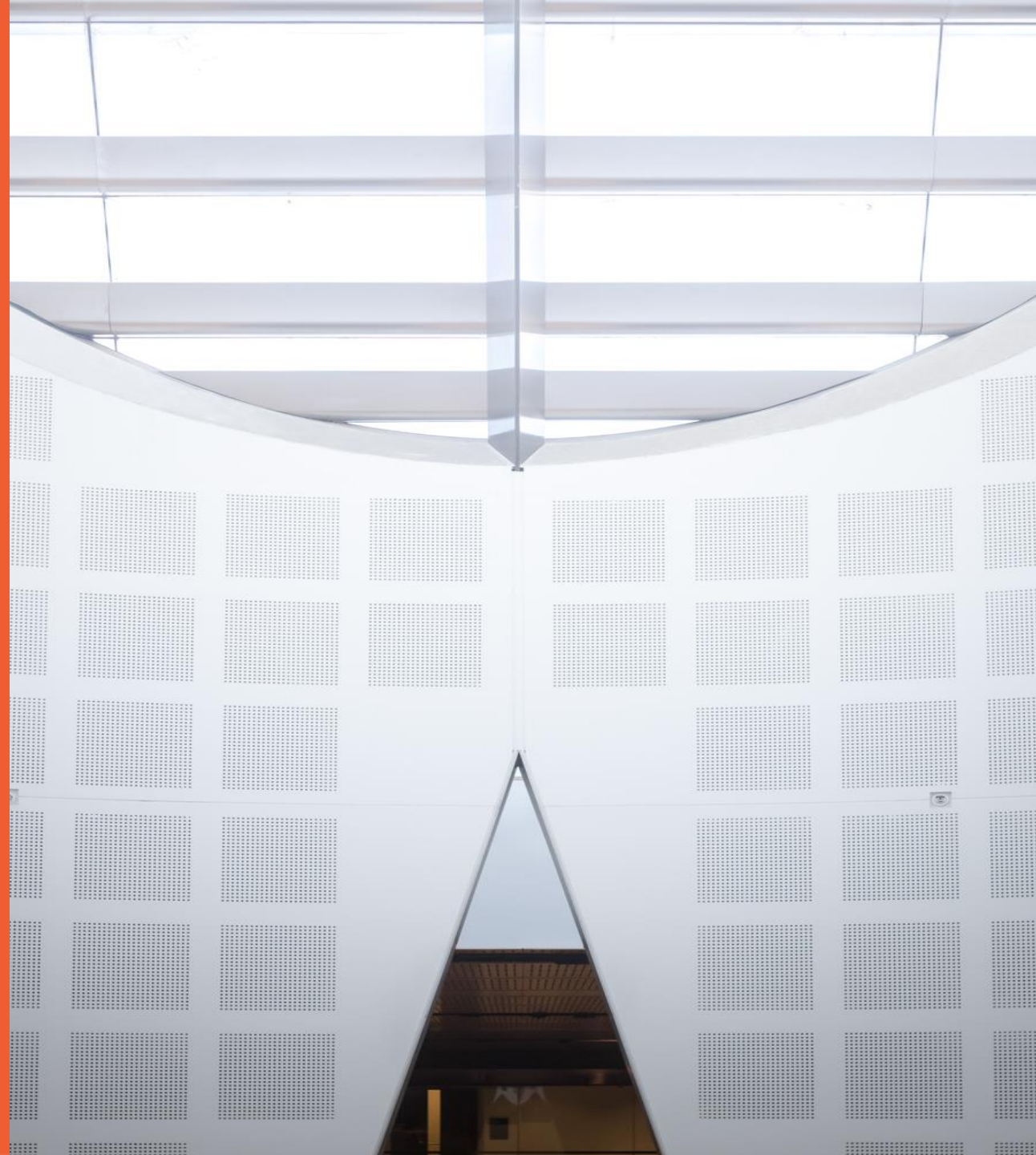
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COMPx270: Randomised and
Advanced Algorithms
Lecture 12: Learning from
experts

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School of Computer Science



THE UNIVERSITY OF
SYDNEY



A question

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However, you don't know anything about the stock market.

But you have many friends who do: they're all "experts."

A question

So every morning, before you make your decision, all those friends will give you their advice.



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Some might collude, or be completely wrong, or even try to make you lose money. But each of them will tell you to either **sell** or **buy**.

A question

Then, based on those many pieces of advice, **you** decide.

(And you do that again, every day.)



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What is a good strategy to make money?



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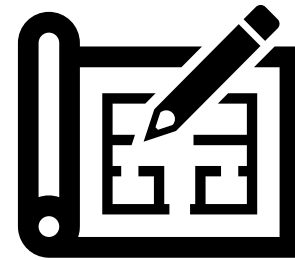
Then, based on those many pieces of advice, **you** decide.

(And you do that again, every day.)

What is a **provably** good strategy to make money?



Let's make this formal



- There are n experts.
- Each day, $t = 1, \dots, T$, each of them makes a prediction $v_{i,t} \in \{0,1\}$
- Based on those, you make your own prediction $\hat{u}_t \in \{0,1\}$
- Then the “true” value $u_t \in \{0,1\}$ is revealed
- If $\hat{u}_t \neq u_t$, this counts as a **mistake** (mistakes are bad)



Goal: minimise **total number of mistakes** $M = \sum_{t=1}^T 1_{\hat{u}_t \neq u_t}$

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But what do we mean by this? We don't assume **anything** on the experts or on the true values. **They could even all be adversarial!**

- There are n experts.
- Each day, $t = 1, \dots, T$, each of them makes a prediction $v_{i,t} \in \{0,1\}$
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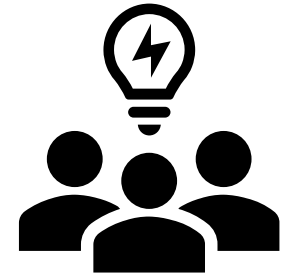
Goal: minimise **total number of mistakes** $M = \sum_{t=1}^T 1_{\hat{u}_t \neq u_t}$ compared to the **best expert** (whoever that is).

Not make much more mistakes than the **best advice in hindsight**.

Warmup: a Perfect Expert

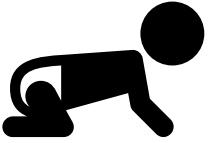


- There are n experts. **Suppose one of them (unknown) is always right.**
- Each day, $t = 1, \dots, T$, each of them makes a prediction $v_{i,t} \in \{0,1\}$
- Based on those, you make your own prediction $\hat{u}_t \in \{0,1\}$
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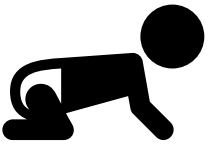
Goal: minimise **total number of mistakes** $M = \sum_{t=1}^T 1_{\hat{u}_t \neq u_t}$

Theorem. There is a strategy guaranteeing $M \leq n - 1$, regardless of T (even for $T = \infty$).



```
Set  $S \leftarrow [n]$ 
for all  $1 \leq t \leq T$  do
  Receive  $v_{1,t}, \dots, v_{n,t}$ 
  if  $|S| \geq 1$  then
    Pick any  $i \in S$                                 ▷ Lexicographically, for instance
    Choose  $\hat{u}_t \leftarrow v_{i,t}$ 
  else
    Choose  $\hat{u}_t \leftarrow 0$                                 ▷ Arbitrary
  Receive  $u_t$                                 ▷ Observe the truth
   $S \leftarrow S \setminus \{i \in S : v_{i,t} \neq u_t\}$     ▷ Remove all mistaken experts
```

Theorem. There is a strategy guaranteeing $M \leq n - 1$, regardless of T (even for $T = \infty$).



Proof.

- ① Every time I make a mistake, $|S|$ decreases by 1.
- ② At first, $|S| = n$.
- ③ Also, $|S| \geq 1$ always (there is a perfect expert!).

Potential argument.

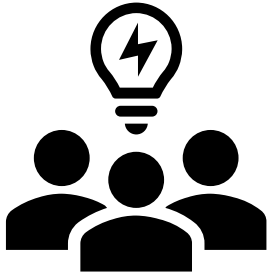
Potential function: $\phi = |S|$

$$\phi_0 = n$$

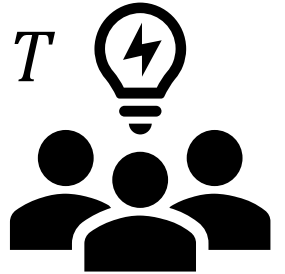
$$\phi_t \geq 1 \quad \forall t$$

$$\Delta \phi$$

Theorem. There is a strategy guaranteeing $M \leq \log_2 n$, regardless of T (even for $T = \infty$).

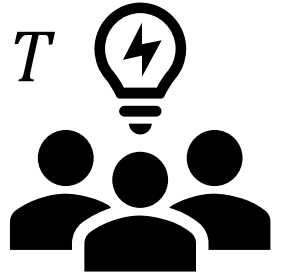


Claim. There is a strategy guaranteeing $M \leq \log_2 n$, regardless of T (even for $T = \infty$).



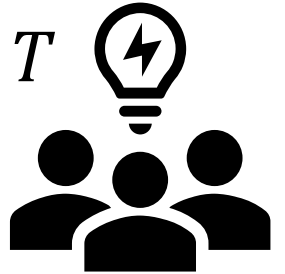
Algorithm: Start with $S = \{1, 2, \dots, n\}$. Each day, choose \hat{u}_t to be the **majority** of advices from experts still in S . At the end of the day, remove from S all experts who predicted wrong.

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for all  $1 \leq t \leq T$  do
    Receive  $v_{1,t}, \dots, v_{n,t}$ 
    if  $|S| \geq 1$  then
        Choose  $\hat{u}_t \leftarrow \text{maj}_{i \in S} v_{i,t}$            ▷ Take the majority advice
    else
        Choose  $\hat{u}_t \leftarrow 0$                                ▷ Arbitrary
    Receive  $u_t$                                              ▷ Observe the truth
     $S \leftarrow S \setminus \{i \in S : v_{i,t} \neq u_t\}$     ▷ Remove all mistaken experts
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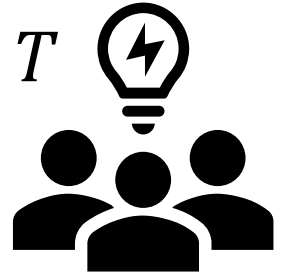
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Proof of correctness. Every time we make a mistake, at least half the experts in S must have been wrong (we took the majority vote). So after each mistake the size of S is at least halved.

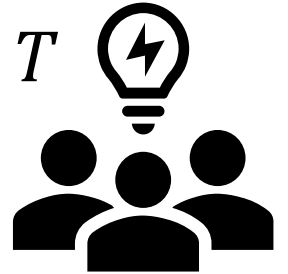
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Proof of correctness. Every time we make a mistake, **at least half** the experts in S must have been wrong (we took the majority vote). So after each mistake the size of S is at least **halved**. But we always have $|S| \geq 1$, since (by assumption) there exists an expert who is always right (and therefore never gets removed).

Claim. There is a strategy guaranteeing $M \leq \log_2 n$, regardless of T (even for $T = \infty$).



Algorithm: Start with $S = \{1, 2, \dots, n\}$. Each day, choose \hat{u}_t to be the **majority** of advices from experts still in S . At the end of the day, remove from S all experts who predicted wrong.

Proof of correctness. Since we started with $|S| = n$, our total number M of mistakes must then satisfy

$$\frac{n}{2^M} \geq 1$$

HALVING ALGORITHM

that is, $M \leq \log_2 n$.

Nobody's Perfect



This is great! But... things completely fail if there is no “perfect expert.”

What if even the **best** expert made some mistakes? Can we make things **robust**?

This is great! But... things completely fail if there is no “perfect expert.”

What if even the **best** expert made some mistakes? Can we make things **robust**?

Let's revisit the algorithm.

$$S = \{i \in [n] : w_i = 1\}$$

We had n **weights** w_1, \dots, w_n initialised to 1.

At day t , our prediction was $\hat{u}_t \leftarrow \text{Maj}(w_1 v_{1,t} + \dots + w_n v_{n,t})$

Whenever expert i made a mistake, we set $w_i \leftarrow 0 \cdot w_i$.

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Whenever expert i made a mistake, we set $w_i \leftarrow \frac{1}{2} \cdot w_i$.

Algorithm (Multiplicative Weights Update).

Start with n **weights** w_1, \dots, w_n initialised to 1.

Each day, choose the **weighted majority** $\hat{u}_t \leftarrow \text{Maj}(w_1 v_{1,t} + \dots + w_n v_{n,t})$

At the end of the day, set $w_i \leftarrow \frac{1}{2} \cdot w_i$ for expert i made a mistake.

Set $w_1, \dots, w_n \leftarrow 1$

for all $1 \leq t \leq T$ **do**

Receive $v_{1,t}, \dots, v_{n,t}$

Choose $\hat{u}_t \leftarrow \text{sign}\left(\sum_{i=1}^n w_i v_{i,t} \geq \frac{1}{2} \sum_{i=1}^n w_i\right)$ \triangleright Weighted majority

Receive u_t

\triangleright Observe the truth

for all $1 \leq i \leq n$ **do**

\triangleright Penalise all mistaken experts

$$w_i \leftarrow \begin{cases} \frac{1}{2} w_i & \text{if } v_{i,t} \neq u_t \\ w_i & \text{otherwise.} \end{cases}$$

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

```

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```

Theorem 59. *There is a (deterministic) algorithm (Algorithm 24) such that*

$$C(T) \leq \frac{C^*(T) + \log_2 n}{\log_2 \frac{4}{3}} \leq 2.41 (\underbrace{C^*(T)}_{\substack{\uparrow \\ \text{\# errors of the} \\ \text{best expert in} \\ \text{hindsight}}} + \log_2 n).$$

Moreover, this holds even when $T = \infty$.

$$C^*(T) = \min_{1 \leq i \leq n} \sum_{t=1}^T \mathbb{1}_{v_{i,t} \neq \hat{u}_t}$$



Theorem. The MWU algorithm guarantees $M \leq 2.41(M^* + \log_2 n)$, where M^* is the # of mistakes made by the best expert. This holds regardless of T (even for $T = \infty$).

Potential

Proof. Let W_t be the total weights of experts on day t . Initially, $W_0 = n$. Every time we make a mistake, this means **at least half the weight** was on experts who did a mistake (since we took the weighted majority). So if we made a mistake at day t ,

$$W_{t+1} = W_t^{\text{good}} + \frac{1}{2} W_t^{\text{bad}} \stackrel{(*)}{\leq} \frac{1}{2} W_t + \frac{1}{2} \cdot \frac{1}{2} W_t = \frac{3}{4} W_t$$

Now, look at the **best expert** (in hindsight). They made M^* mistakes, so their final weight is $(1/2)^{M^*}$.

$$\begin{aligned} W_t &= p \cdot W_t^{\text{good}} + (1-p) W_t^{\text{bad}} \quad (p \leq \frac{1}{2}) \\ W_t^{\text{good}} + \frac{1}{2} W_t^{\text{bad}} &= W_t^{\text{good}} + \frac{1}{2} (W_t - W_t^{\text{good}}) \\ &= \frac{1}{2} W_t^{\text{good}} + \frac{1}{2} W_t \leq \frac{1}{4} W_t + \frac{1}{2} W_t \end{aligned}$$

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Proof. Putting it all together:

$$\left(\frac{1}{2}\right)^{M^*} \leq W_T \leq \left(\frac{3}{4}\right)^M W_0 = \left(\frac{3}{4}\right)^M n$$

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Now, we take the logarithm:

$$-M^* \leq \underbrace{M \log_2 \left(\frac{3}{4}\right)}_{< 0} + \log_2 n$$

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and get

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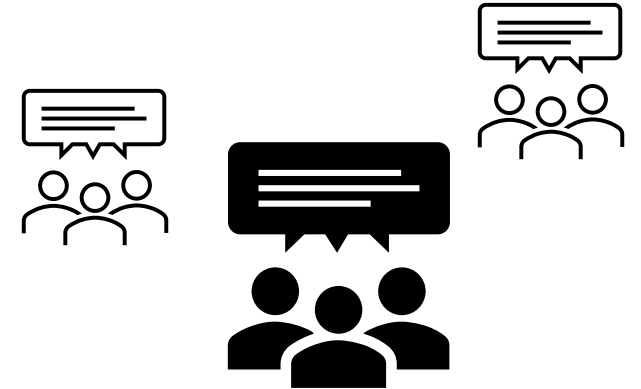
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and get

$$M \leq \frac{M^* + \log_2 n}{-\log_2 \left(\frac{3}{4}\right)} \leq 2.41(M^* + \log_2 n)$$



Let's go further!



This is what we proved:

Algorithm (Multiplicative Weights Update).

Start with n **weights** w_1, \dots, w_n initialised to 1.

Each day, choose the **weighted majority** $\hat{u}_t \leftarrow \text{Maj}(w_1 v_{1,t} + \dots + w_n v_{n,t})$

At the end of the day, set $w_i \leftarrow \frac{1}{2} \cdot w_i$ for expert i made a mistake.

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Using exactly the same argument (*try it!*), we get, for any $\beta \in (0,1)$:

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At the end of the day, set $w_i \leftarrow \beta \cdot w_i$ for expert i made a mistake.

Theorem. The MWU algorithm guarantees $M \leq \frac{M^* \log_2(1/\beta) + \log_2 n}{\log_2(\frac{2}{1+\beta})}$,

where M^* is the # of mistakes made by the best expert. This holds regardless of T (even for $T = \infty$).

Using exactly the same argument we get, for any $\beta \in (0,1)$:

Theorem. The MWU algorithm guarantees

$$M \leq \frac{M^* \log_2 \left(\frac{1}{\beta} \right) + \log_2 n}{\log_2 \left(\frac{2}{1 + \beta} \right)}$$

where M^* is the # of mistakes made by the best expert. This holds regardless of T (even for $T = \infty$).

$\beta \rightarrow 0$ $\nearrow \sim M^* \log \frac{1}{\beta} + \log_2 n$
 $\searrow \infty$

$\beta = \frac{1}{2}$ $\rightarrow \frac{M^* \cdot 1 + \log_2 n}{\log_2 \frac{4}{3}}$

$\beta = 1 - \varepsilon$
 $\varepsilon \rightarrow 0 \searrow \frac{M^* \ln \frac{1}{1 - \varepsilon} + \ln n}{\ln \left(\frac{2}{2 - \varepsilon} \right)}$
 $\sim \frac{M^* \varepsilon + \ln n}{\varepsilon^{1/2}}$
 $= 2M^* + O\left(\frac{\ln n}{\varepsilon}\right)$

Using exactly the same argument we get, for any $\beta = 1 - \varepsilon \in (0,1)$:

Theorem. The MWU algorithm guarantees

$$M \leq \frac{M^* \log_2 \left(\frac{1}{\beta} \right) + \log_2 n}{\log_2 \left(\frac{2}{1 + \beta} \right)} \approx 2 \left(M^* + \frac{\ln n}{\varepsilon} \right)$$

where M^* is the # of mistakes made by the best expert. This holds regardless of T (even for $T = \infty$).

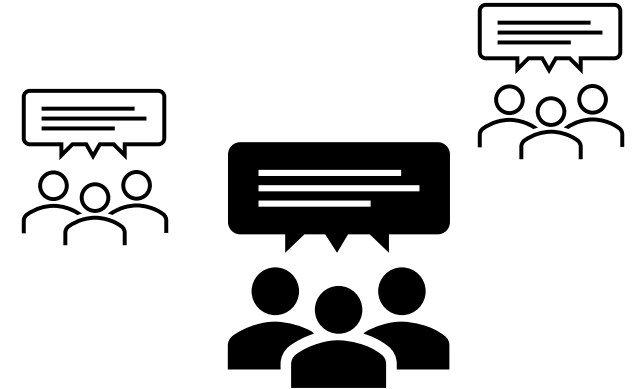
① $W_0 = n$

③ $W_T \geq \beta^{M^*}$

② $W_{t+1} = W_t^{\text{good}} + \beta W_t^{\text{bad}} = (1 - \beta) W_t^{\text{good}} + \beta W_t$
 $\leq \frac{1 + \beta}{2} W_t$

$\rightarrow \beta^{M^*} \leq W_T \leq \left(\frac{1 + \beta}{2} \right)^T n$

Is that tight?



Theorem. The MWU algorithm guarantees

$$M \approx 2 \left(M^* + \frac{\ln n}{\varepsilon} \right)$$

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Can we improve that factor **2**?

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Can we improve that factor **2**? **No.**

Theorem. The MWU algorithm guarantees

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where M^* is the # of mistakes made by the best expert. This holds regardless of T (even for $T = \infty$).

Can we improve that factor **2**? **No**. Consider two sets of $n/2$ experts, where experts in the first set are wrong on odd-numbered days, and those in the second set are wrong on even days. That will force T mistakes (while the best experts make $T/2$).

Theorem. The MWU algorithm guarantees

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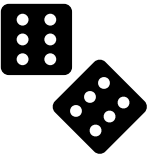
Can we improve that factor **2**? **Yes.**

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where M^* is the # of mistakes made by the best expert. This holds regardless of T (even for $T = \infty$).

Can we improve that factor **2**? **Yes.** With randomisation! Instead of deterministically choosing the weighted majority, pick the answer **at random according to the weights.**

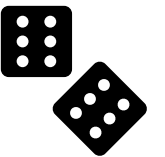


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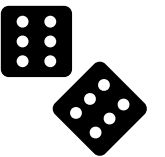


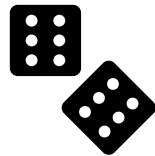
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Input: Penalty parameter $\beta \in (0, 1)$

Set $w_1, \dots, w_n \leftarrow 1$

for all $1 \leq t \leq T$ **do**

Receive $v_{1,t}, \dots, v_{n,t}$

Draw $I \in [n]$ according to the weights:

$$\Pr[I = i] = \frac{w_i}{\sum_{i=1}^n w_i}, \quad i \in [n]$$

Choose $\hat{u}_t \leftarrow v_{I,t}$

▷ One expert gets the vote

Receive u_t

▷ Observe the truth

for all $1 \leq i \leq n$ **do**

▷ Penalise all mistaken experts

$$w_i \leftarrow \begin{cases} \beta w_i & \text{if } v_{i,t} \neq u_t \\ w_i & \text{otherwise.} \end{cases}$$

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Theorem 61. *There is a (randomised) algorithm (Algorithm 26) such that*

$$\mathbb{E}[C(T)] \leq \frac{C^*(T) \ln(1/\beta) + \ln n}{1 - \beta}.$$

Moreover, this holds even when $T = \infty$.

Before :

$$C(T) \leq \frac{C^*(T) \ln\left(\frac{1}{\beta}\right) + \ln n}{\ln\left(\frac{2}{1+\beta}\right)}$$

Claim.

$$1 - \beta > \ln \frac{2}{1+\beta}$$

so the new bound is better!

Theorem 61. There is a (randomised) algorithm (Algorithm 26) such that

$$\mathbb{E}[C(T)] \leq \frac{C^*(T) \ln(1/\beta) + \ln n}{1 - \beta}.$$

Moreover, this holds even when $T = \infty$.

Pf: $\mathbb{E}[C(t)] = \sum_{s=1}^t \Pr[\hat{u}_s \neq u_s] = \sum_{s=1}^t \frac{W_s^{\text{bad}}}{W_s} \triangleq F_s$

Fraction of the weight on wrong advice

At every time step

$$W_t = (1 - F_t)W_t + F_t W_t$$

$$\textcircled{2} \quad W_{t+1} = (1 - F_t)W_t + \beta F_t W_t$$

$$= W_t \cdot (1 - (1 - \beta)F_t)$$

$$\beta^{C^*} \leq W_T \leq n \cdot \prod_{t=1}^T (1 - (1 - \beta)F_t)$$

"Take logarithms"

Input: Penalty parameter $\beta \in (0, 1)$

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$$w_i \leftarrow \begin{cases} \beta w_i & \text{if } v_{i,t} \neq u_t \\ w_i & \text{otherwise.} \end{cases}$$

$$\textcircled{1} \quad W_0 = n$$

$$\textcircled{3} \quad V_T \geq \beta^{C^*}$$

Recall

$$* \quad \mathbb{E}[C(T)] = \sum_{t=1}^T F_t$$

$$* \quad \ln(1+x) \leq x$$

$$- \ln(1-x) \geq x$$

"Convexity"

Theorem 61. *There is a (randomised) algorithm (Algorithm 26) such that*

$$\mathbb{E}[C(T)] \leq \frac{C^*(T) \ln(1/\beta) + \ln n}{1 - \beta}.$$

Moreover, this holds even when $T = \infty$.

$$\begin{aligned} \underbrace{\ln(\beta^{C^*})}_{C^* \ln \beta} &\leq \ln n + \sum_{t=1}^T \ln(1 - (1-\beta)F_t) \\ C^* \ln \frac{1}{\beta} &\geq -\ln n + \sum_{t=1}^T (-\ln(1 - (1-\beta)F_t)) \\ &\geq -\ln n + (1-\beta) \underbrace{\sum_{t=1}^T F_t}_{\mathbb{E}[C(T)]} \end{aligned}$$

$$(1-\beta) \mathbb{E}[C(T)] \leq \ln n + C^* \ln \frac{1}{\beta}$$

Input: Penalty parameter $\beta \in (0, 1)$
 Set $w_1, \dots, w_n \leftarrow 1$
for all $1 \leq t \leq T$ **do**
 Receive $v_{1,t}, \dots, v_{n,t}$
 Draw $I \in [n]$ according to the weights:
 $\Pr[I = i] = \frac{w_i}{\sum_{i=1}^n w_i}, \quad i \in [n]$
 Choose $\hat{u}_t \leftarrow v_{I,t}$ ▷ One expert gets the vote
 Receive u_t ▷ Observe the truth
 for all $1 \leq i \leq n$ **do** ▷ Penalise all mistaken experts
 $w_i \leftarrow \begin{cases} \beta w_i & \text{if } v_{i,t} \neq u_t \\ w_i & \text{otherwise.} \end{cases}$

$$\boxed{-\ln(1-x) \geq x}$$

□

Concluding remarks



- This was a **short** intro to the Multiplicative Weights Update Algorithms. Much more to say!
 - Different **predictions** (not only binary)
 - Different **payoffs** (not just 0-1 loss: correct/incorrect)
 - **Randomised** version!
- Discovered/rediscovered in many areas: **learning theory**, **game theory**/economics, **computational geometry**, **convex optimisation**...
- Many (sometimes unexpected) **applications**: online learning/bandits, semidefinite programming, flow algorithms, zero-sum games, algorithmic takes on evolution (!)

Some pointers if you have questions or want to know more about any of those (or connections to some of those topics):

- *The Multiplicative Weights Update Method: a Meta-Algorithm and Applications*. Arora, Hazan, Kale (2012):
<https://theoryofcomputing.org/articles/v008a006/>
- Lecture notes by Daniel Hsu (2017), Chapter 1:
<https://www.cs.columbia.edu/~djhsu/coms6998-f17/notes.pdf>

