Warm-up

Problem 1. Check your understanding: recall their definitions, and summarise the key differences between an LP and an ILP.

Problem 2. Formulate MAX-CUT as an ILP.

a) Give its LP relaxation, and suggest a randomised rounding strategy.
b) Show that $y^* = (1, 1, \dots, 1)$ and $x^* = (1/2, 1/2, \dots, 1/2)$ is always an optimal solution to the LP relaxation.
c) What does your rounding scheme become in this case?

Problem 3. Describe how to derandomise the 3/4-approximation algorithm for MAX-SAT given in class.

Problem solving

Problem 4. Consider the KNAPSACK problem, where the goal is to select a subset of n items that fit in the knapsack (which can only store total weight W) in order to maximise total value, where item i has value $v_i \geq 0$ and weight $w_i > 0$.

a) Give the corresponding ILP.
b) Provide the LP relaxation, which corresponds to the *Fractional* Knapsack.
c) Solve the LP relaxation (using, e.g., Matlab with the function linprog) on the following set of 10 items, with weight limit $W = 20$: $(v_i, w_i) = (i^2, i)$, $1 \leq i \leq 10$. See how this changes as you vary W from 20 to 55.
d) Compare to the solution obtained by the Greedy algorithm for Fractional Knapsack.
e) Compare to the optimal solution of the ILP (for $W = 20$, then varying W as before), also obtained by solving the ILP (on Matlab, with the function intlinprog).

Problem 5. Suppose the instance of MAX-SAT has no negated "unit clause" (that is, either a clause has length at least 2, or it is a non-negated variable x_i). Instead of setting each variable to 1 independently with probability $1/2$ in the "obvious" randomised algorithm, do the analysis when this is done with some (fixed) probability $p > 1/2$.

a) Show that this gives (in expectation) a $\min(p, 1 - p^2)$-approximation.
b) Optimise the choice of p to obtain the best approximation possible.

c) ($\star$) Show how to remove the "no negated unit clause" assumption: let $S \subseteq [n]$ be the set of variables such that both the unit clause $\neg x_i$ *and* the unit clause x_i exist in the instance ϕ, and $T \subseteq [n]$ be the set of variables for which only the unit clause $\neg x_i$ is in ϕ. Then consider the randomised rounding scheme with sets each variable i independently to 1 with probability p if $i \notin T$, and with probability p (as before) otherwise, where p is the value found in the previous subquestion. Show that $\mathrm{opt}(\phi) \leq m - |S|$. Use this to conclude that $\mathbb{E}[\mathrm{value}(\phi)] \geq p \cdot \mathrm{opt}(\phi)$.

d) Compare this with the $1 - 1/e$ approximation guarantee obtained by LP rounding in the lecture.

Problem 6. Show that one can also obtain (directly) an expected $\frac{3}{4}$-approximation to Max-SAT by using only randomised rounding: in Algorithm 20, instead of having $x_i \sim \mathrm{Bern}(y_i^*)$ (independently), we will set set them independently to 1 with probability

$$p_i \coloneqq f(y_i^*),$$

where $f\colon [0,1] \to [0,1]$ is any function such that $1 - \frac{1}{4^x} \leq f(x) \leq \frac{1}{4^{1-x}}$.

a) Draw the plot of both upper and lower bounds on f, to see what the conditions look like (and that such functions f do exist).

b) In what follows, we fix any such function f. With the notation of Theorem 48, show that, for any $1 \leq j \leq m$,

$$\Pr[\, C_j \text{ not satisfied} \,] \leq \frac{1}{4^{z_j^*}}$$

c) Deduce that, for any $1 \leq j \leq m$,

$$\Pr[\, C_j \text{ satisfied} \,] \geq \frac{3}{4} z_j^*$$

Hint: use concavity.

d) Conclude.

Advanced

Problem 7. Show that one can also obtain (directly) an expected $\frac{3}{4}$-approximation to Max-SAT by using only randomised rounding with a *linear* function of y_i^*: in Algorithm 20, instead of having $x_i \sim \mathrm{Bern}\, y_i^*$ (independently), set them independently to 1 with probability

$$p_i \coloneqq \frac{y_i^*}{2} + \frac{1}{4}\,.$$