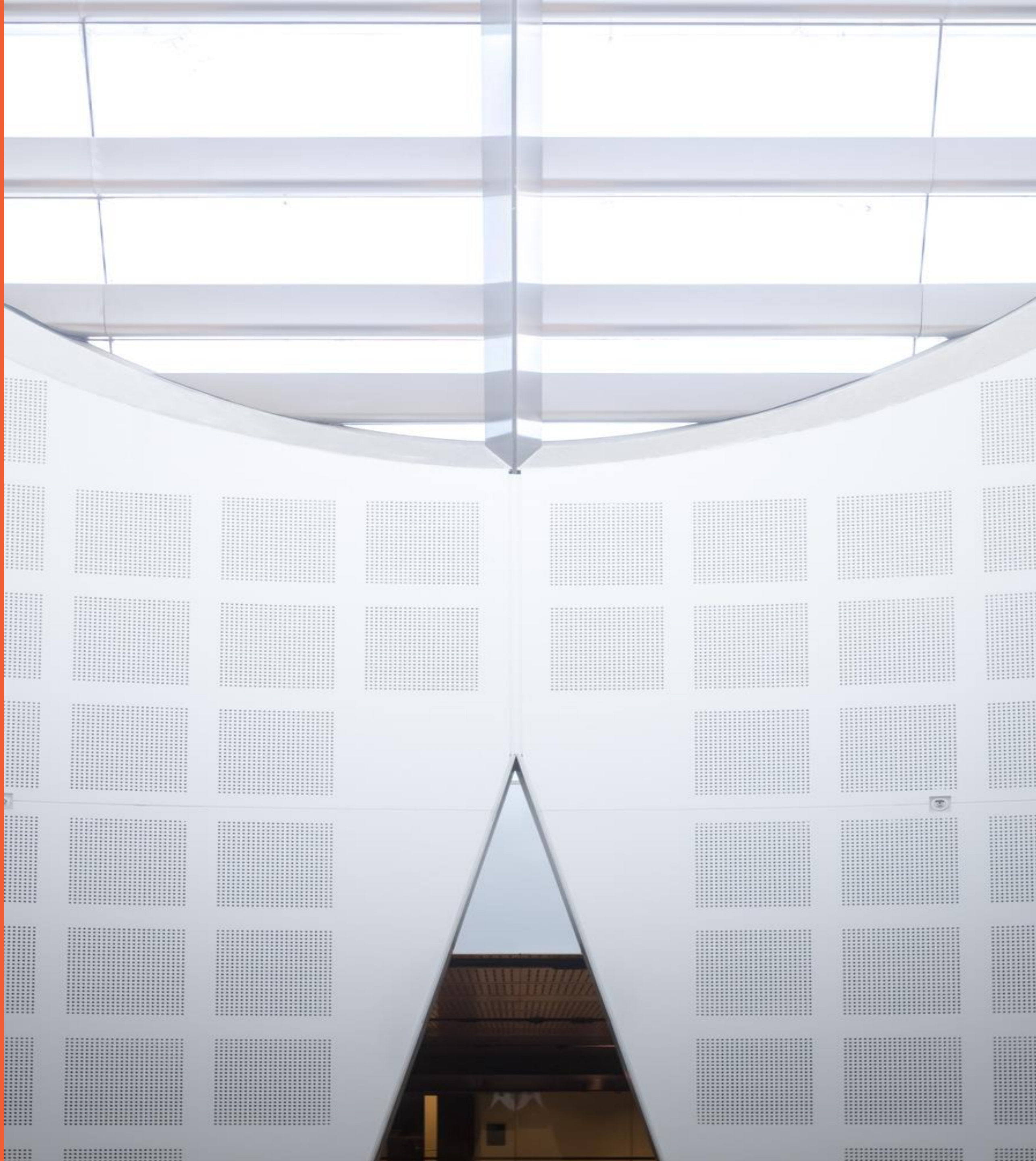
COMPx270: Randomised and Advanced Algorithms

Lecture 5: Graph algorithms

Clément Canonne

School of Computer Science

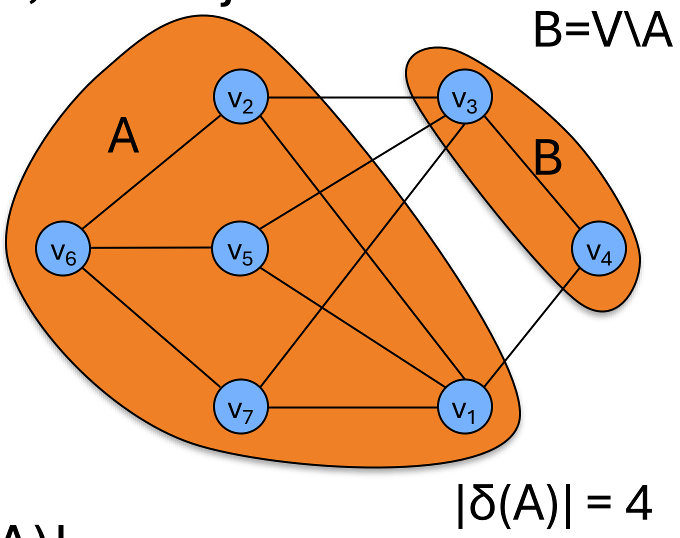
THE UNIVERSITY OF SYDNEY



Global Minimum Cut

Input: A connected, undirected graph $G = (V, E)$.

For a set $A \subseteq V$ let $\delta(A) = \{(u,v) \in E : u \in A, v \in V \setminus A\}$.

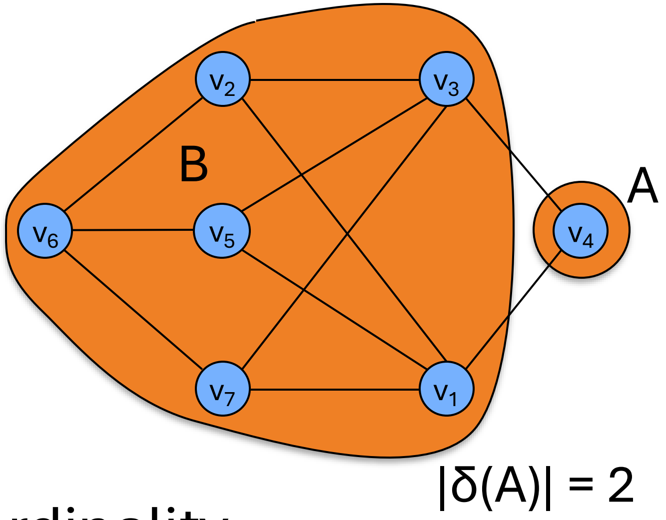


Aim: Find a cut (A, B) minimizing $|\delta(A)|$.

Global Minimum Cut

Input: A connected, undirected graph $G = (V, E)$.

For a set $A \subseteq V$ let $\delta(A) = \{(u,v) \in E : u \in A, v \in V \setminus A\}$.



Aim: Find a cut (A, B) of minimum cardinality.

Global Minimum Cut

Applications: Partitioning items in a database, identifying clusters of related documents, network reliability, network design, circuit design, TSP solvers.

Network flow solution.

- Replace every edge (u, v) with two directed edges (u, v) and (v, u).
- Pick some vertex s and compute min s-v cut separating s from each other vertex $v \in V$.

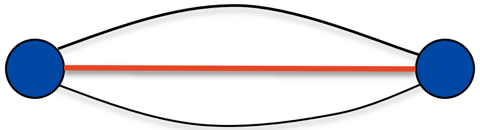
Running time: $O((n-1) \cdot \text{MaxFlows})$

Global Minimum Cut

Running time: $O((n-1) \cdot \text{MaxFlows})$

Karger's Contraction Algorithm

Definition. A multigraph is a graph that allows multiple edges between a pair of vertices.



Karger’s Contraction Algorithm

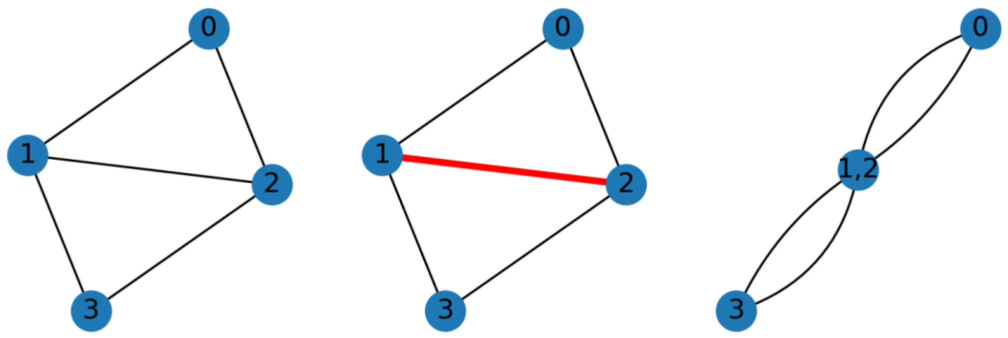
Definition. A multigraph is a graph that allows multiple edges between a pair of vertices.

Karger's Contraction Algorithm

Let G=(V,E) be a multigraph (without self-loops).

The contraction of an edge e=(u,v)∈E gives G\e

- Replace u and v by single new super-node w
- Replace all edges (u,x) or (v,x) with an edge (w,x)
- Remove self-loops to w



Karger's Contraction Algorithm

Require: multigraph $G = (V, E)$

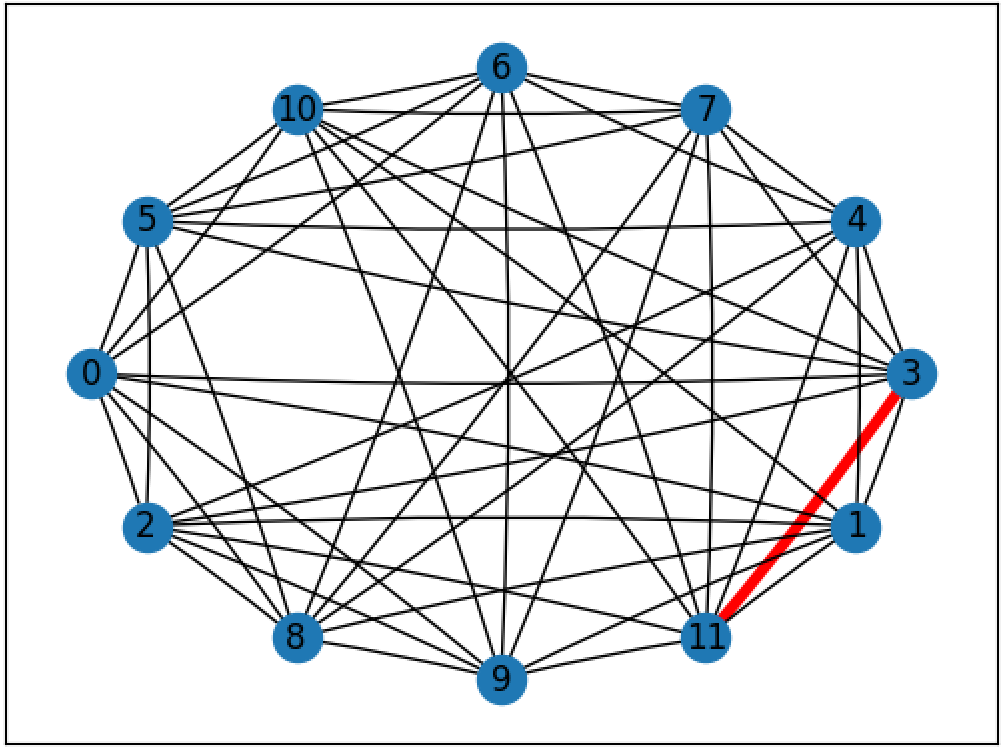
while $|V| > 2$ **do**

  Pick an edge $e \in E$ uniformly at random

  Contract it, and let $G \leftarrow G/e$

return the cut defined by the remaining two vertices.

Karger's Contraction algorithm



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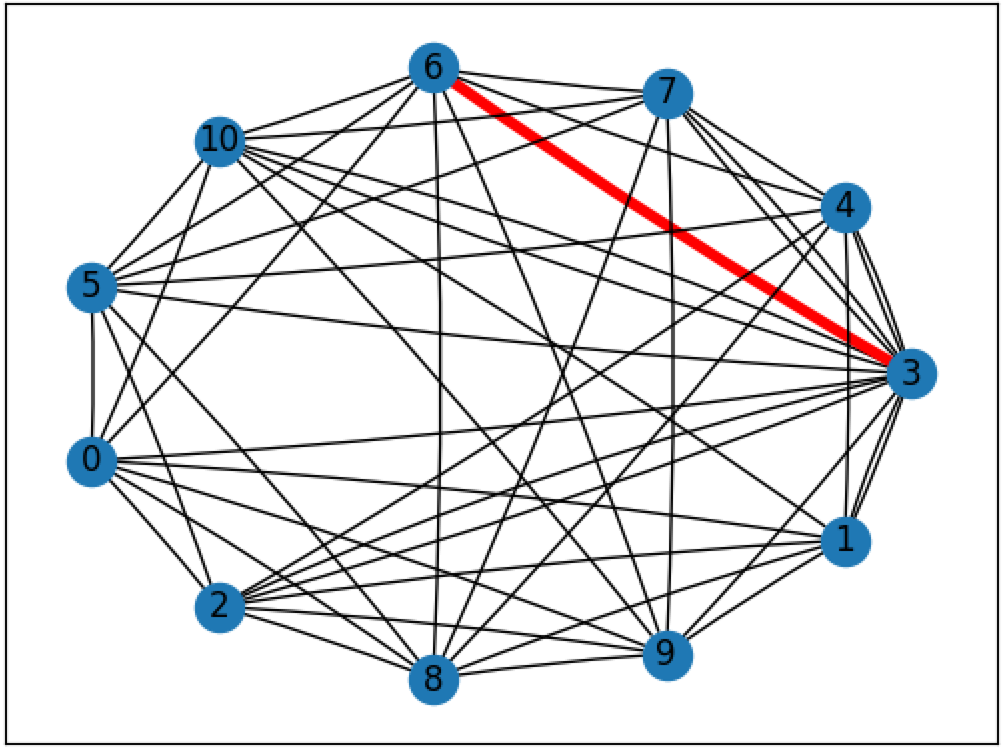

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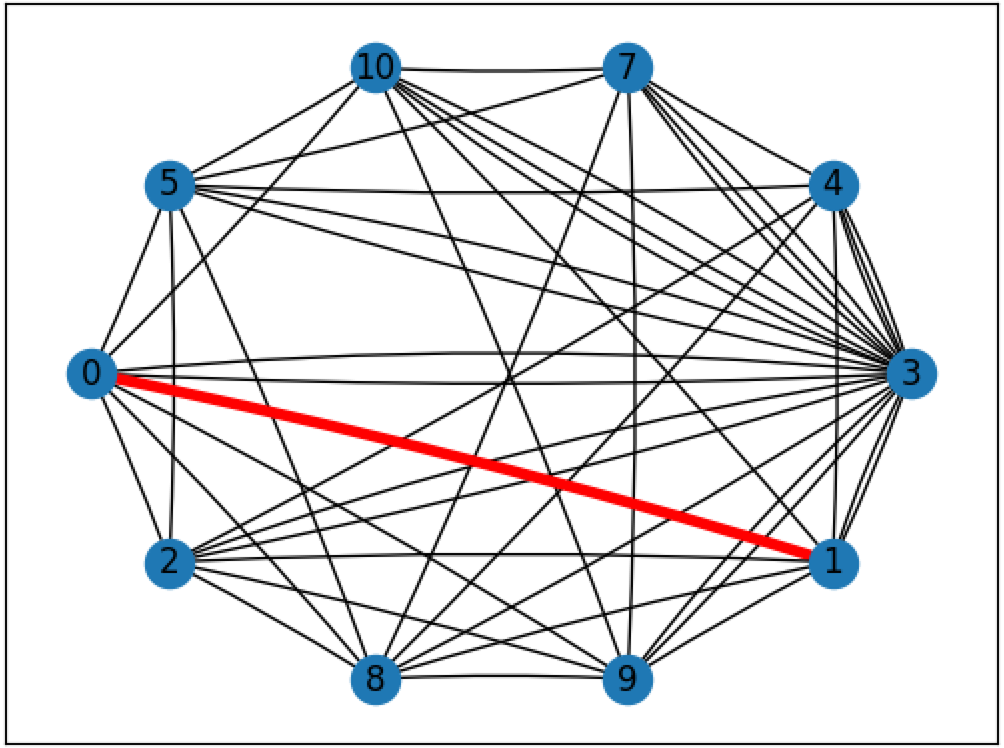

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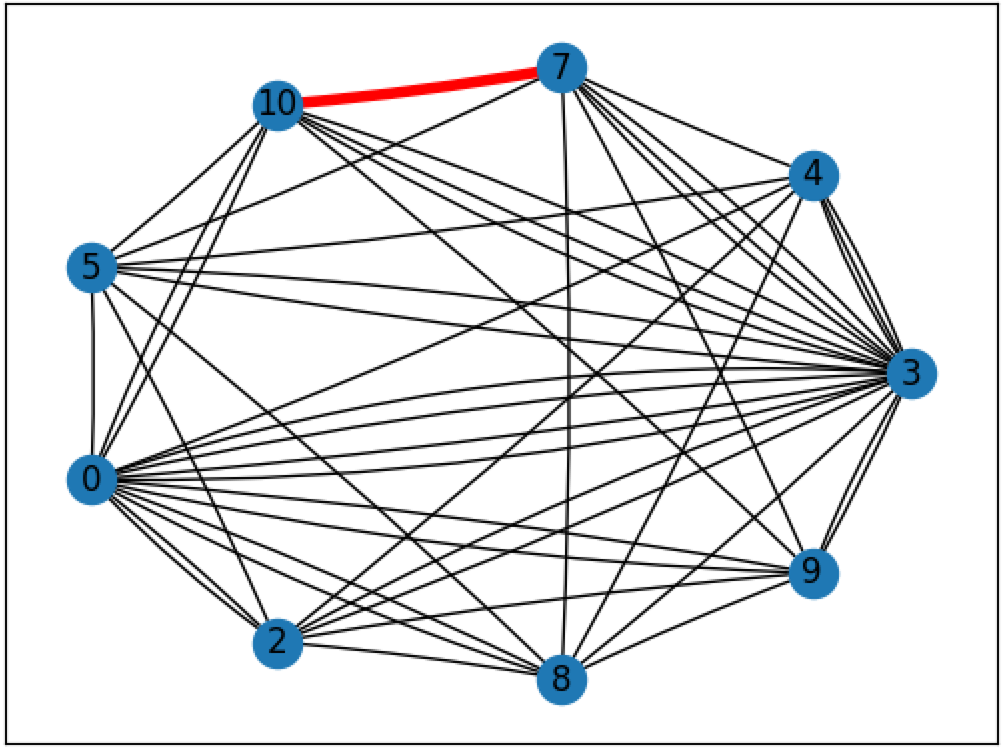

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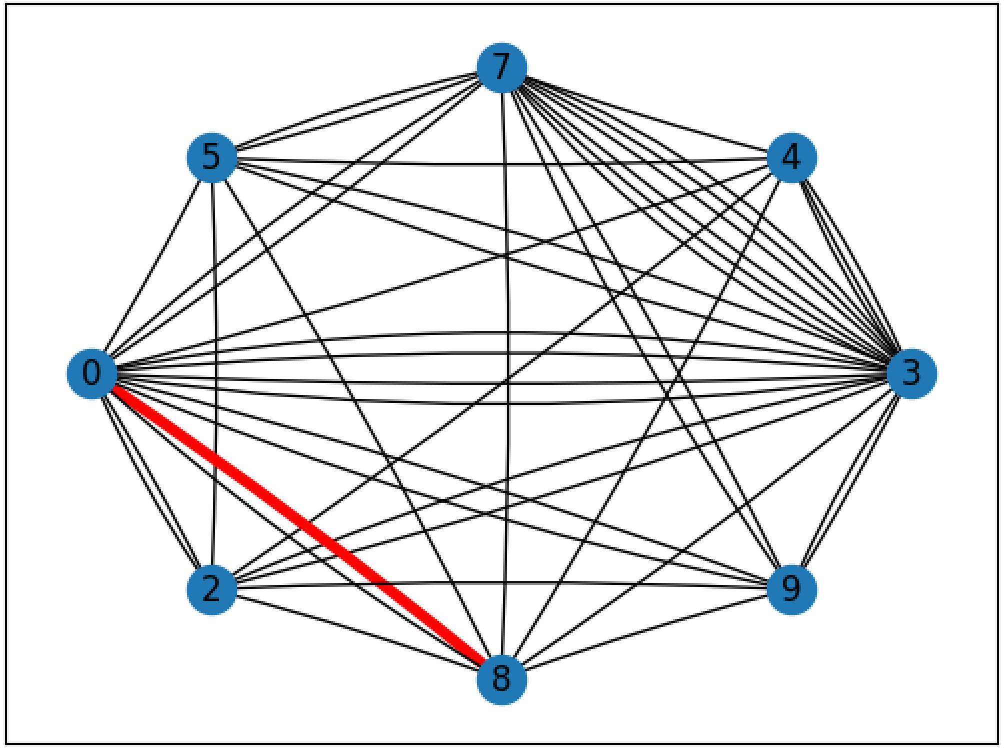

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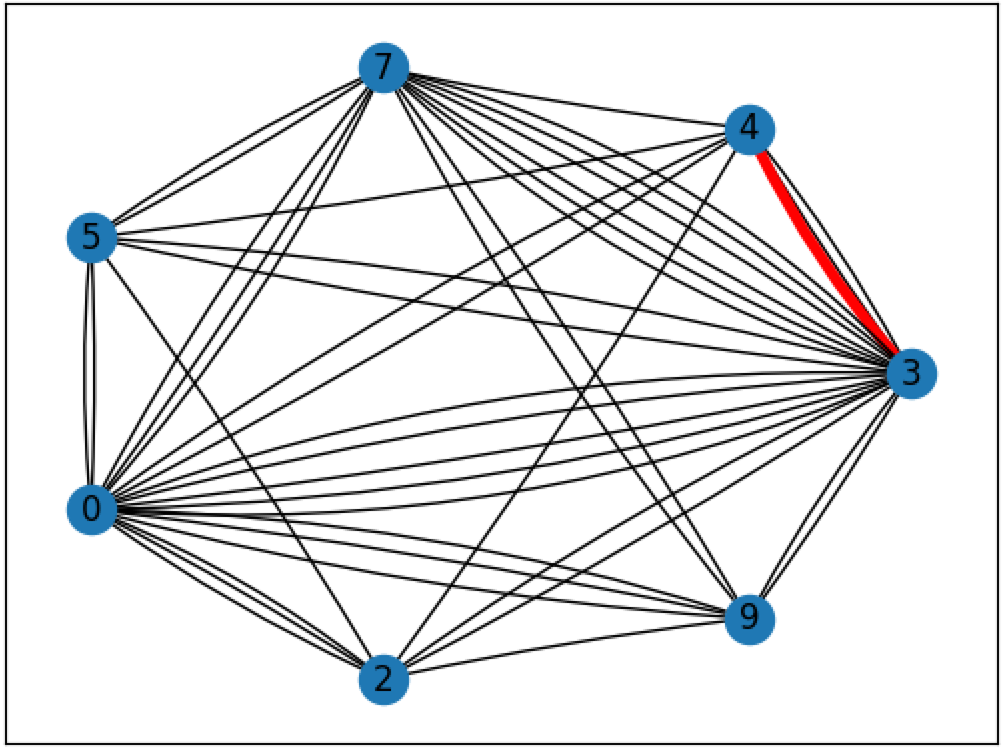

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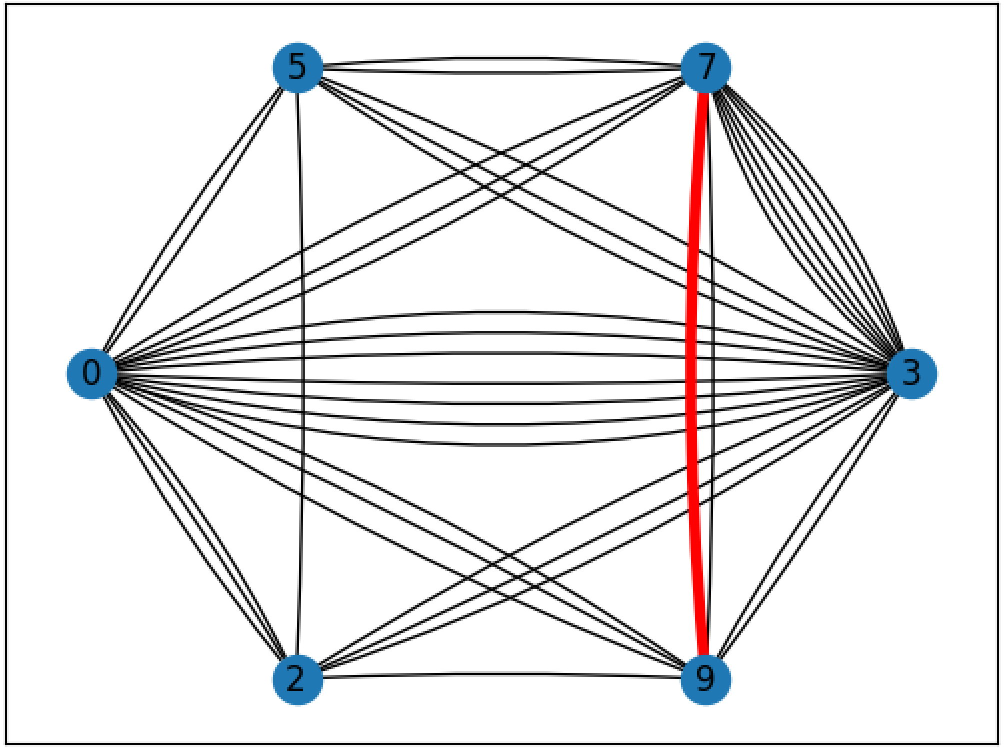

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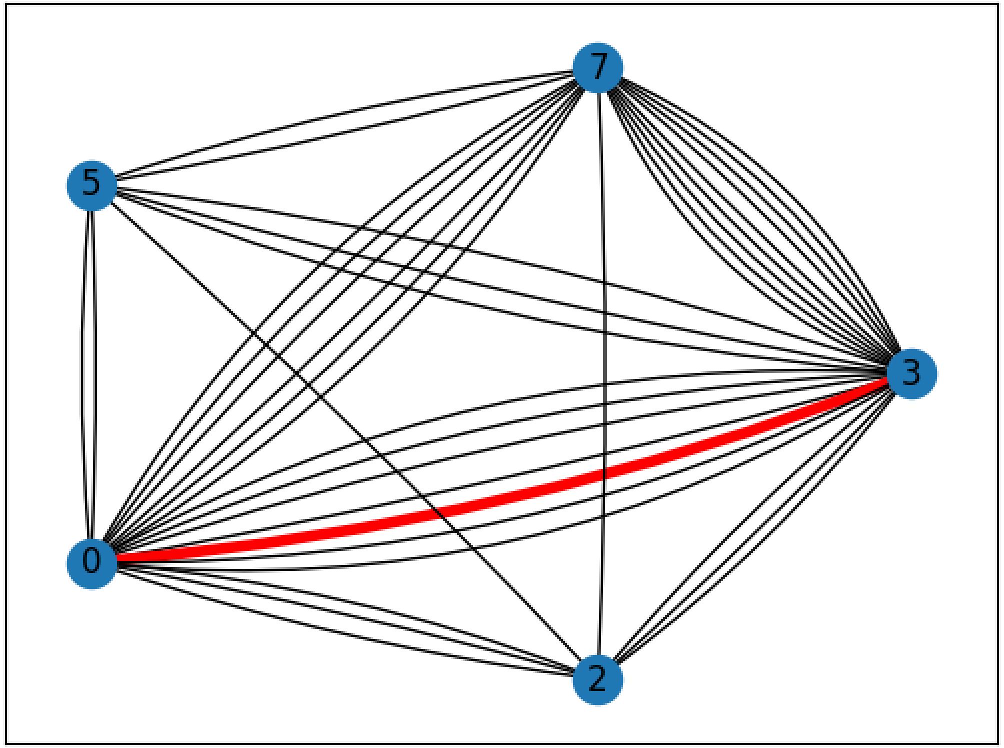

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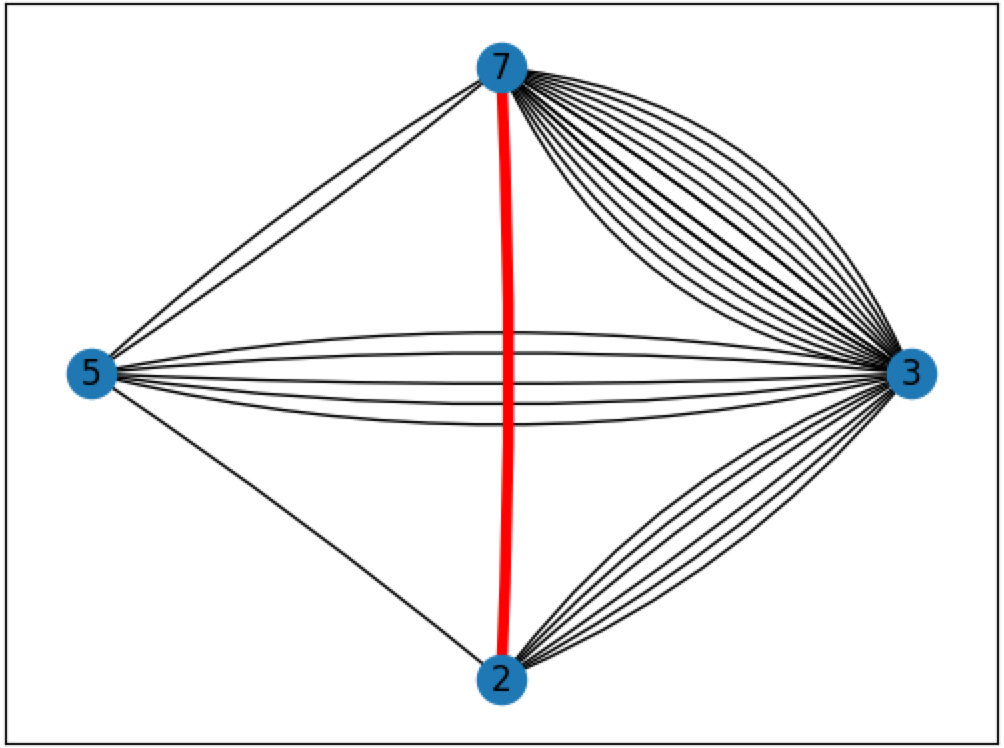

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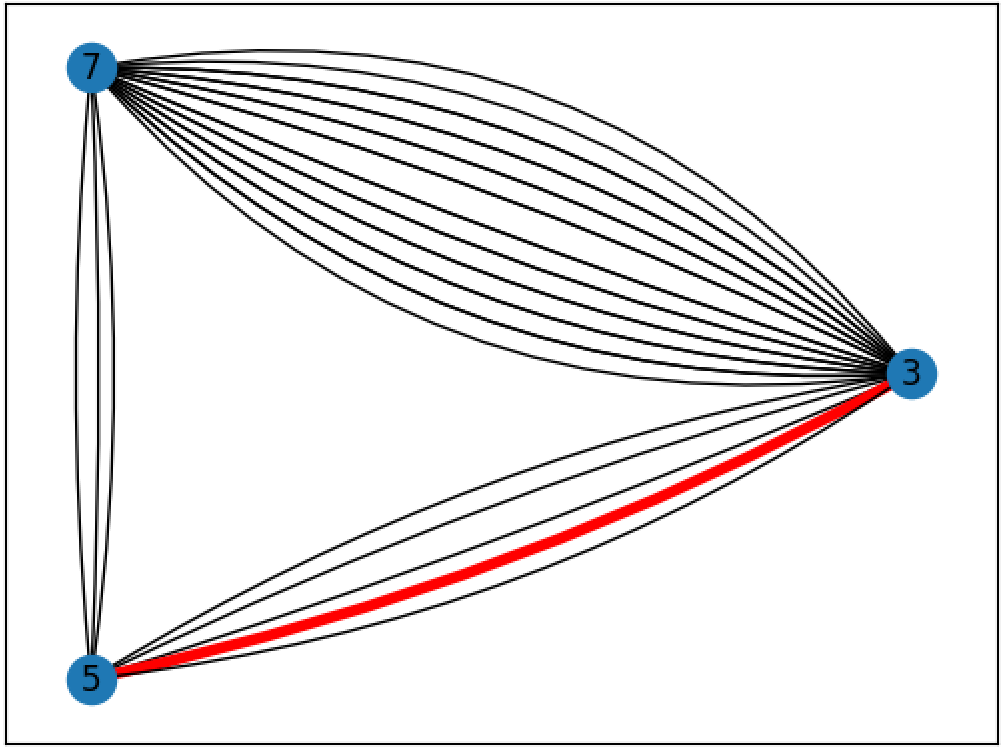

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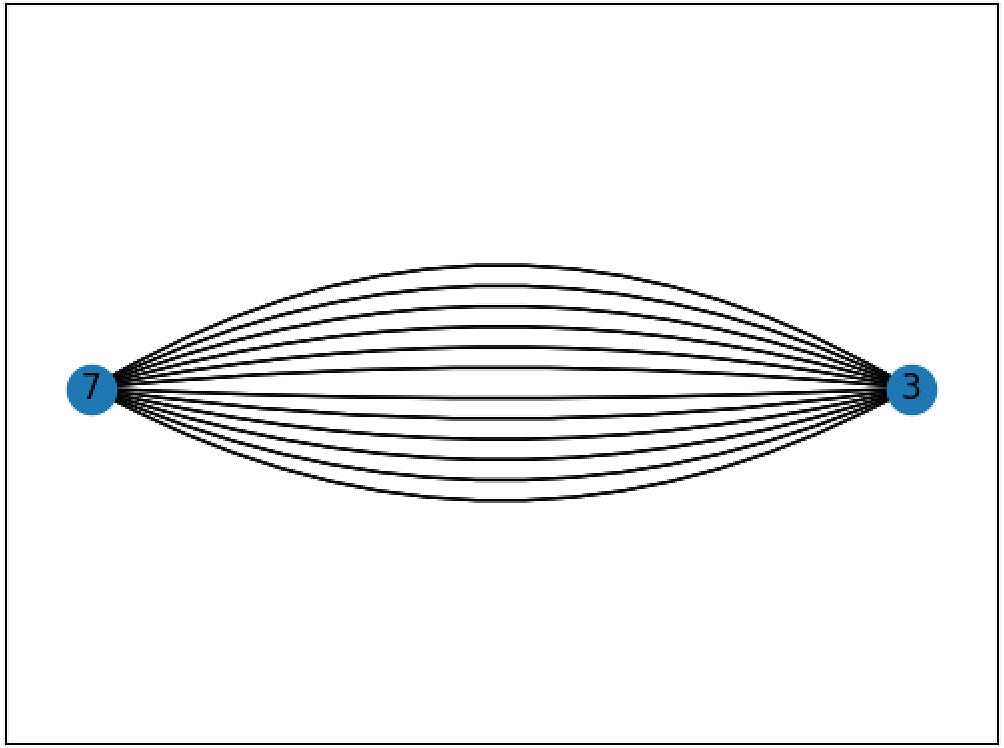
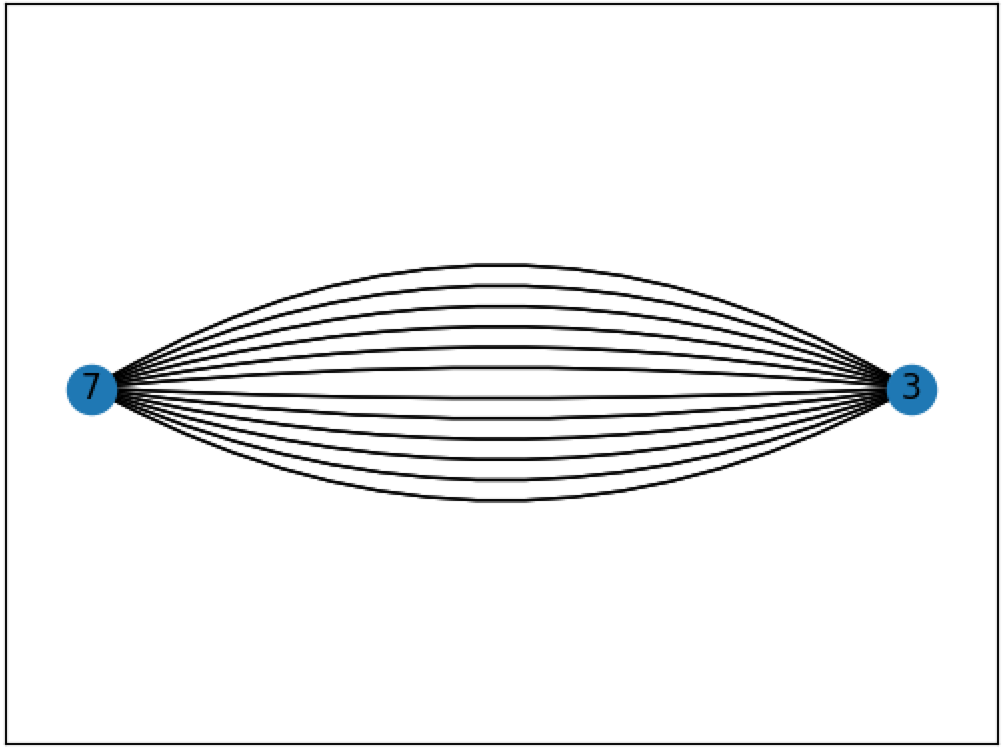


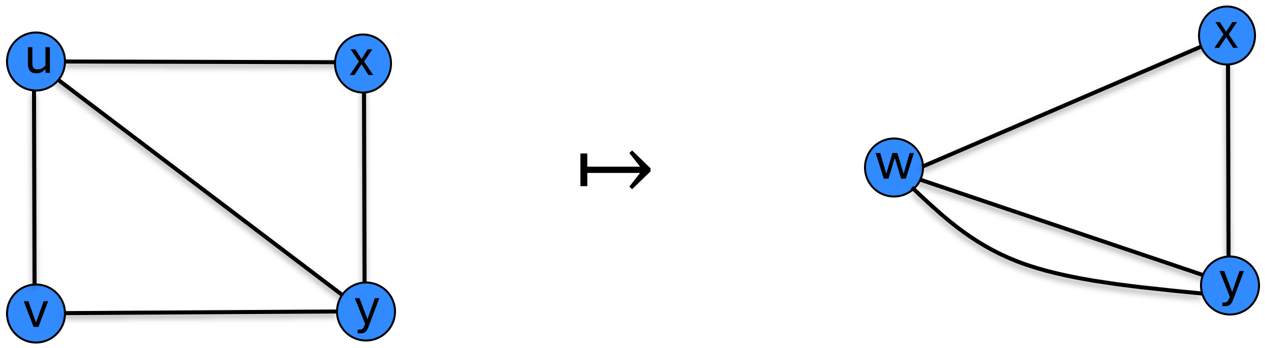
Karger's Contraction algorithm



The End.

Karger's Contraction Algorithm

Observation: An edge (u,v) contraction preserves the cuts (A,B) where u and v are both in A or both in B.



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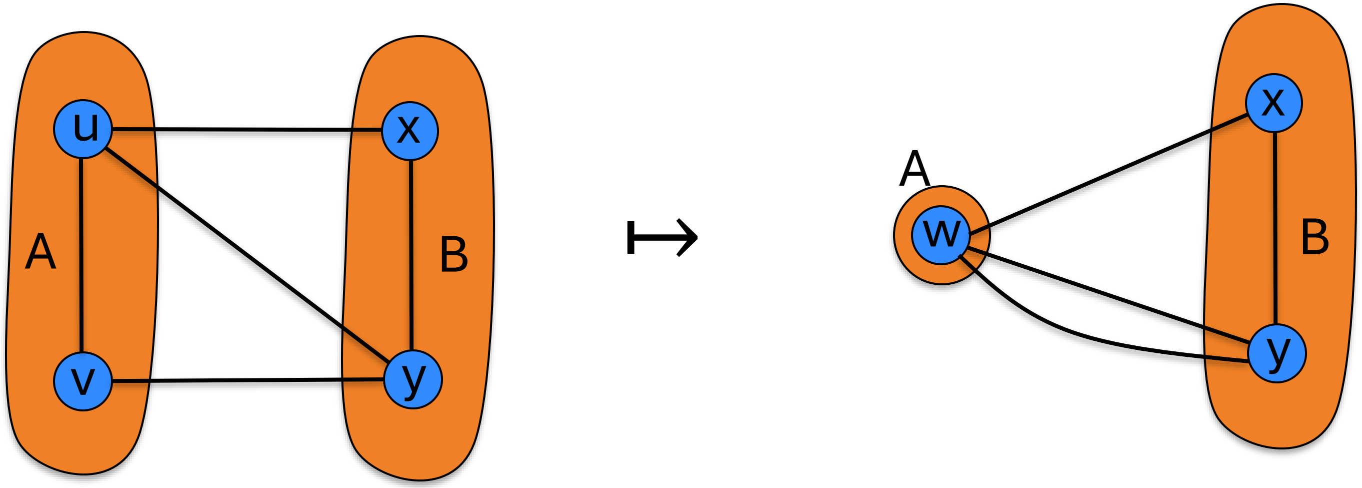

Karger's Contraction Algorithm

Observation: An edge (u,v) contraction preserves the cuts (A,B) where u and v are both in A or both in B.

$$\boxed{\begin{array}{c}\text{If } u,v \in A \text{ then } \delta_G(A) = \delta_{G\backslash e}(A).\\ \text{(with u and v replaced with ``uv'')}\end{array}}$$

Karger's Contraction Algorithm

Observation: If (A,B) is a minimum cut, then we are less likely to choose an edge (u,v) crossing it!

Karger’s Contraction Algorithm

Require: multigraph $G = (V, E)$

while $|V| > 2$ **do**

  Pick an edge $e \in E$ uniformly at random

  Contract it, and let $G \leftarrow G/e$

return the cut defined by the remaining two vertices.

Claim. This algorithm has a reasonable chance of finding a min cut.

Key claim

Claim. If C is a min-cut, then the algorithm returns it with probability at least $2/n^2$.

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Claim. If C is a min-cut, then the algorithm returns it with probability at least $2/n^2$.

Proof.

Amplification

To amplify the probability of success, run the contraction algorithm many times.

Require: multigraph $G = (V, E)$, integer T

1: **for** $1 \leq t \leq T$ **do** ▷ Use fresh (independent) random bits for each

2: Run Algorithm on G, let C_t be the output

3: **return** the smallest cut among all cuts $C_1, \ldots, C_T$ obtained

---

Amplification

To amplify the probability of success, run the contraction algorithm many times.

Claim. If we repeat the contraction algorithm $r\binom{n}{2}$ times with independent random choices, the probability that all runs fail is at most $(1/e)^r$.

Karger’s Contraction Algorithm

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while $|V| > 2$ **do**

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return the cut defined by the remaining two vertices.

Running time?

Karger's Contraction Algorithm

Require: multigraph $G = (V, E)$

1: **while** $|V| > 2$ **do**

2: Pick an edge $e \in E$ uniformly at random

3: Contract it, and let $G \leftarrow G/e$

4: **return** the cut defined by the remaining two vertices.

Running time?

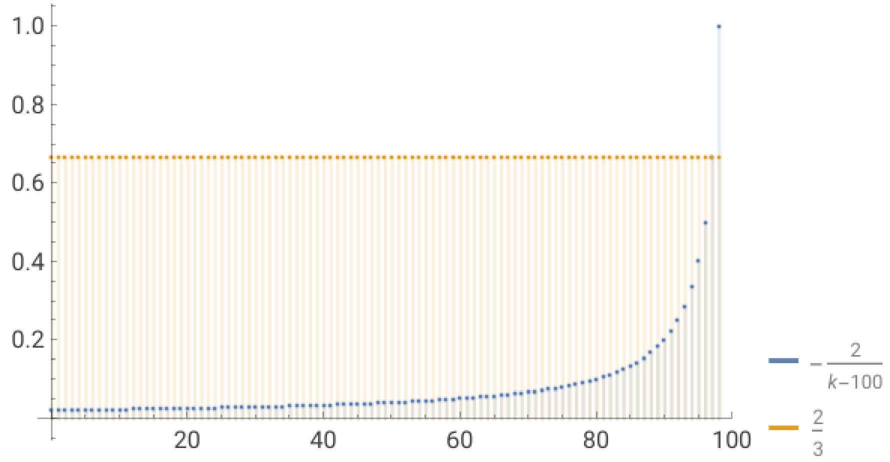
The algorithm is iterated $O(n^2 \log(1/\delta))$ times… total running time $O(n^4 \log(1/\delta))$.

Karger's Contraction Algorithm

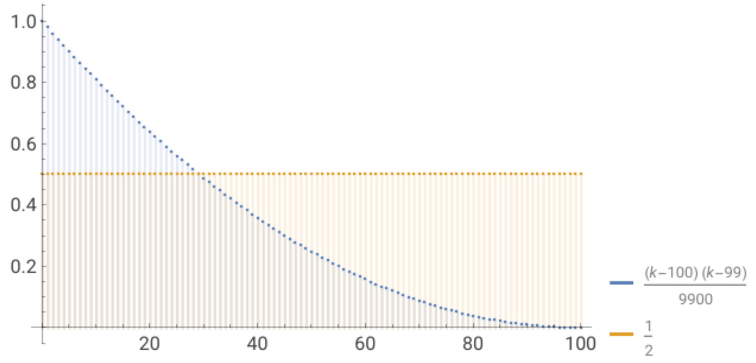
Can we do better?

Karger’s Contraction Algorithm

Can we do better?



Improved algorithm



Improvement. [Karger-Stein 1996]

- Early iterations are less risky than later ones: probability of contracting an edge in min cut hits 50% when $n/\sqrt{2}$ nodes remain.
- Run contraction algorithm until $n/\sqrt{2}$ nodes remain.
- Run contraction algorithm twice on resulting graph, and return best of two cuts.

Improved algorithm

```
procedure ModifiedKarger(G = (V, E), s)
    while |V| > s do
        Pick an edge e ∈ E uniformly at random
        Contract it, and let G ← G/e
    return G
procedure KargerStein(G = (V, E))
    if |V| ≤ 6 then
        return a minimum cut                ▷ Brute-force computation
    Set s ← ⌈n/√2 + 1⌉
    ▷ Contraction
        G_1 ← ModifiedKarger(G, s)
        G_2 ← ModifiedKarger(G, s)
    ▷ Recursion
        C_1 ← KargerStein(G_1)
        C_2 ← KargerStein(G_2)
    return the smallest cut among C_1, C_2
```

Improved algorithm: Karger-Stein

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Running time?

Improved algorithm: Karger-Stein

Running time?

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    return the smallest cut among C1, C2
```

Improved algorithm: Karger-Stein

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    while $|V| > s$ **do**
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    if $|V| \leq 6$ **then**
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    Set $s \leftarrow \lceil n/\sqrt{2} + 1 \rceil$
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        $G_1 \leftarrow$ ModifiedKarger(G, s)
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    ▷ **Recursion**
        $C_1 \leftarrow$ KargerStein(G_1)
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    return the smallest cut among C_1, C_2

Success probability?

Improved algorithm: Karger-Stein

Success probability?

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procedure KargerStein($G = (V, E)$)
  if $|V| \leq 6$ **then**
    return a minimum cut ▷ Brute-force computation
  Set $s \leftarrow \lceil n/\sqrt{2} + 1 \rceil$
  ▷ **Contraction**
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  ▷ **Recursion**
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Improved algorithm: Karger-Stein

Theorem. [Karger-Stein 1996] The Karger-Stein algorithm runs in time $O(n^2 \log n)$ and returns a min cut with probability at least $\Omega(1/\log n)$.

Corollary. The "best-of-T" Karger-Stein algorithm runs in time $O(n^2 \log^2 n)$ and returns a min cut with probability at least 99%.

Improved algorithm: Karger-Stein

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Best known. [Karger 2000] $O(m \log^3 n)$.

And now, for something completely different

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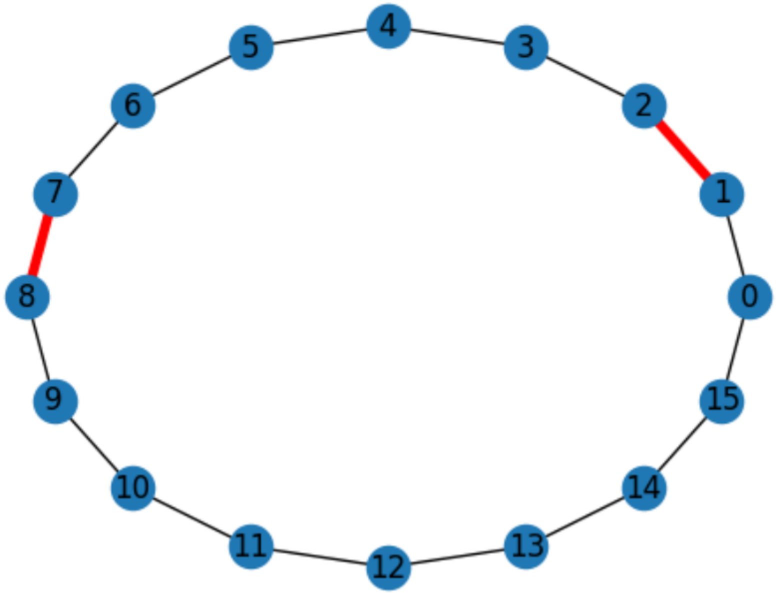
Theorem. An undirected graph G=(V,E) has at most _______ distinct min cuts.

And now, for something completely different?

Theorem. An undirected graph G=(V,E) has at most $n(n-1)/2$ distinct min cuts.

And now, for something completely different?

Theorem. An undirected graph G=(V,E) has at most n(n-1)/2 distinct min cuts. And this is tight.



And now, for something completely different?

Theorem. An undirected graph G=(V,E) has at most n(n-1)/2 distinct min cuts.

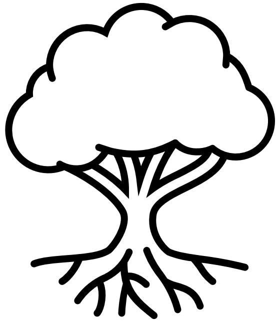
Proof.

And now, for something completely different?

How mind-blowing is this? Analyzing an algorithm proved a seemingly unrelated mathematical statement. 🤯

Compare this with last time, and the probabilistic method ("proof without algorithm!").

And now, for something else completely different!



Minimum Spanning Tree in (Expected) Linear Time: Karger-Klein-Tarjan.