Statistical Inference in Distributed or Constrained Settings: **Techniques and Recipes**

An Easy Dinner Classic

Classic Split Pea Soup
Serves: 8 | *Serving Size:* 1 and 1/2 cups

Ingredients:

2 cups split peas
8 cups water
1 bay leaf
1/4 teaspoon salt
2 cups chopped carrots
1 cup chopped celery
1 cup chopped onion
1 large potato, diced
1 teaspoon thyme leaves
1/2 teaspoon pepper

Directions:

Combine split peas, water, bay leaf, and salt in a large kettle.

Bring to a boil, reduce heat and simmer for 2 hours, stirring occasionally.

Add the remaining ingredients and continue to simmer until the vegetables are tender, about 30 minutes longer.

Nutrition Information:

Serves 8. Each 1 and 1/2 cup serving has 89 calories, 0 g fat, 0 g saturated fat, 0 g trans fat, 0 mg cholesterol, 233 mg sodium, 18 g carbohydrate, 6 g fiber, 4 g sugar, and 5 g protein.

Each serving also contains 108% DV vitamin A, 10% DV vitamin C, 4% DV calcium, and 5% DV iron.

Chef's Tips:

Frozen peas make a great garnish. You can add them to the soup during the last 10 minutes of cooking.

As the split peas are cooking, check to make sure that there is enough water and that the split peas do not stick. Add more water if the soup becomes too thick.

Peas are low in fat, but naturally high in protein and fiber.

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Conference on Learning Theory 2021

What’s on the menu?

I.	Appetizers	Jayadev
II.	MC 1	Jayadev
III.	MC 2	Himanshu
IV.	DIY Desserts	Clément

Chefs: Jayadev Acharya, Clément Canonne, Himanshu Tyagi

COLT 2021

Appetizers

- Statistical Inference
- Distributed / constrained settings
- Problems and examples
- Related work and pointers

Main Course – I: Discrete distributions

- A puzzle to solve **all** problems under communication constraints
- Lower bounds for interactive estimation for arbitrary channels
  - Tight bounds under communication, privacy as application

Main Course – II: General distributions

Unified method to prove “interactive” lower bounds

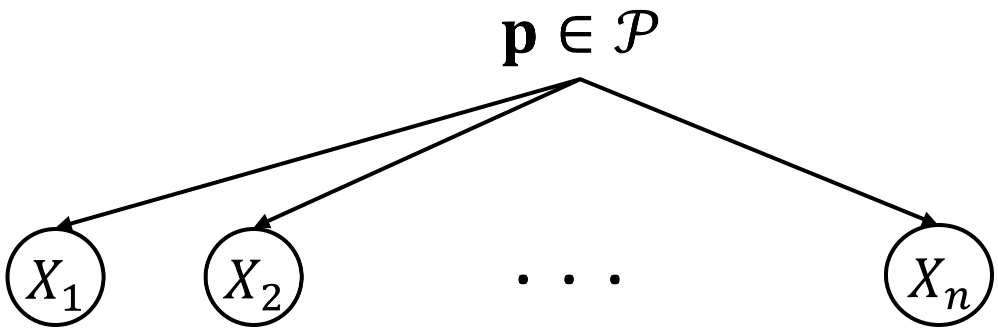
- Discrete, high-dimensional, nonparametric, etc
- Communication, privacy, etc
- General plug-n-play methods

DIY desserts: Recitation Clément

- Example of how to apply the lower bounds
- Several exercises

Statistical Inference

$\mathcal{P}$: family of distributions over $\mathcal{X}$



Given $X^n := (X_1, \dots, X_n)$: i.i.d. samples from an unknown $\mathbf{p}$

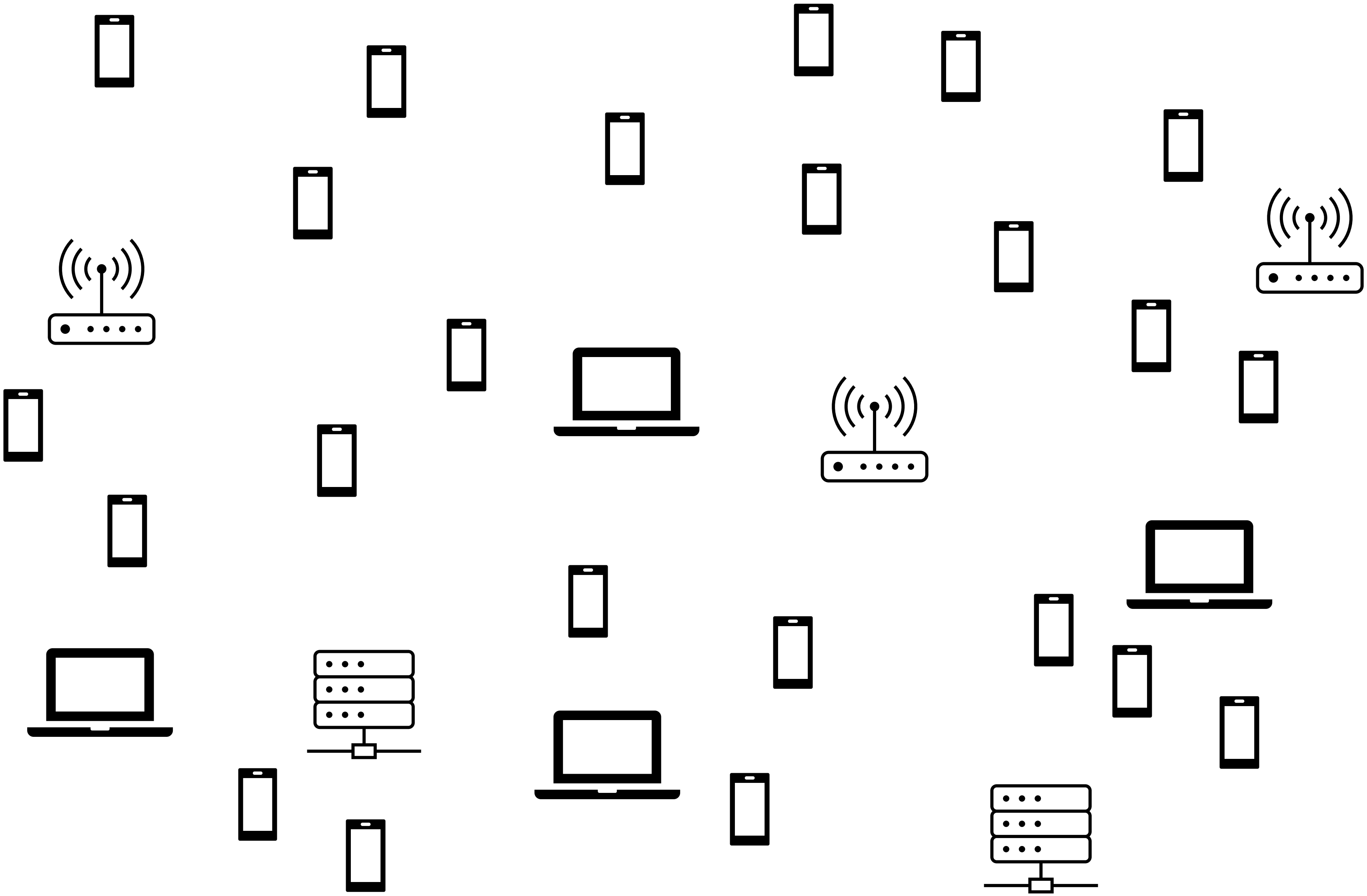
Solve some inference task about $\mathbf{p}$

This is inference in central setting

Information Constraints

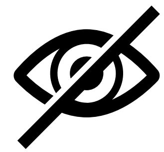
Distributed or Constrained Settings

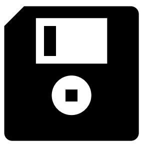
No direct access to X_is



("Motivation" slide)

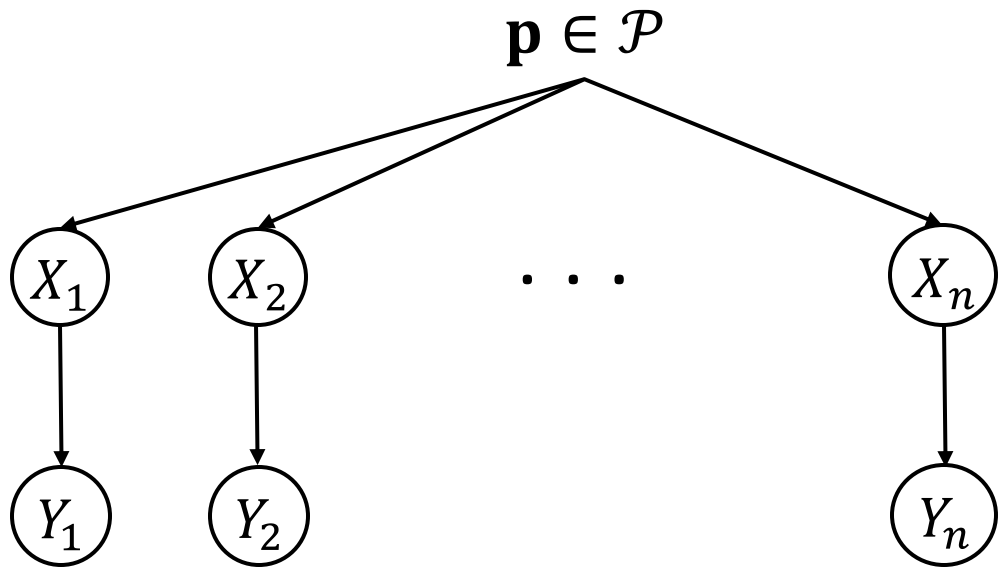
Techniques and **Statistical Inference** under **constraints**





Local constraints

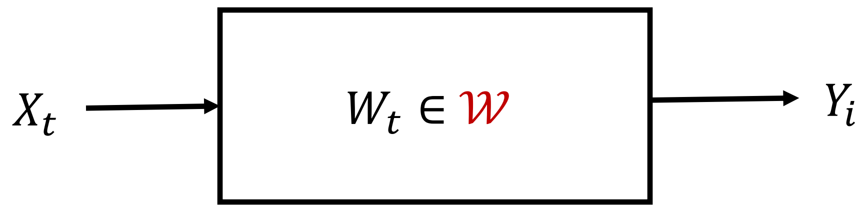
Statistical Inference



Modeling the constraints

[ACT20c]

n users, user t observes X_t and sends message Y_t

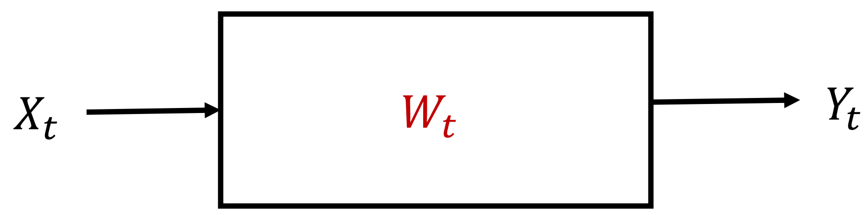


$$W_t(y|x) := \Pr(Y_t = y|X_t = x)$$

$\mathcal{W}$: a set of **allowed** (randomized) channels $\Leftrightarrow$ the constraints

The algorithm/protocol dictates how user t chooses W_t from $\mathcal{W}$

Modeling the local information constraints [ACT20c]



When $X_t \sim \mathbf{p}$

$$\mathbf{p}^{W_t}(y) := \sum_x \mathbf{p}(x) W_t(y|x) = \mathbb{E}[W_t(y|X)]$$

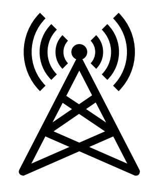
Example 1: Communication constraints

[Shamir14,HMÖW18,ACT20d…]

$$\mathcal{W}_\ell = \{W : \mathcal{X} \to \{0,1\}^\ell\}$$

Each X_t is mapped to ℓ bits.

Bandwidth constraints



Example 2: Local Differential Privacy (LDP)

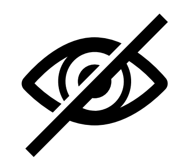
[Warner65, EPR03, KLNRS11]

$W: \mathcal{X} \to \{0,1\}^*$ is ϱ-**LDP** if $\forall x, x' \in \mathcal{X}, \forall y,$

$$\frac{W(y|x)}{W(y|x')} \leq e^{\varrho} \approx 1 + \varrho$$

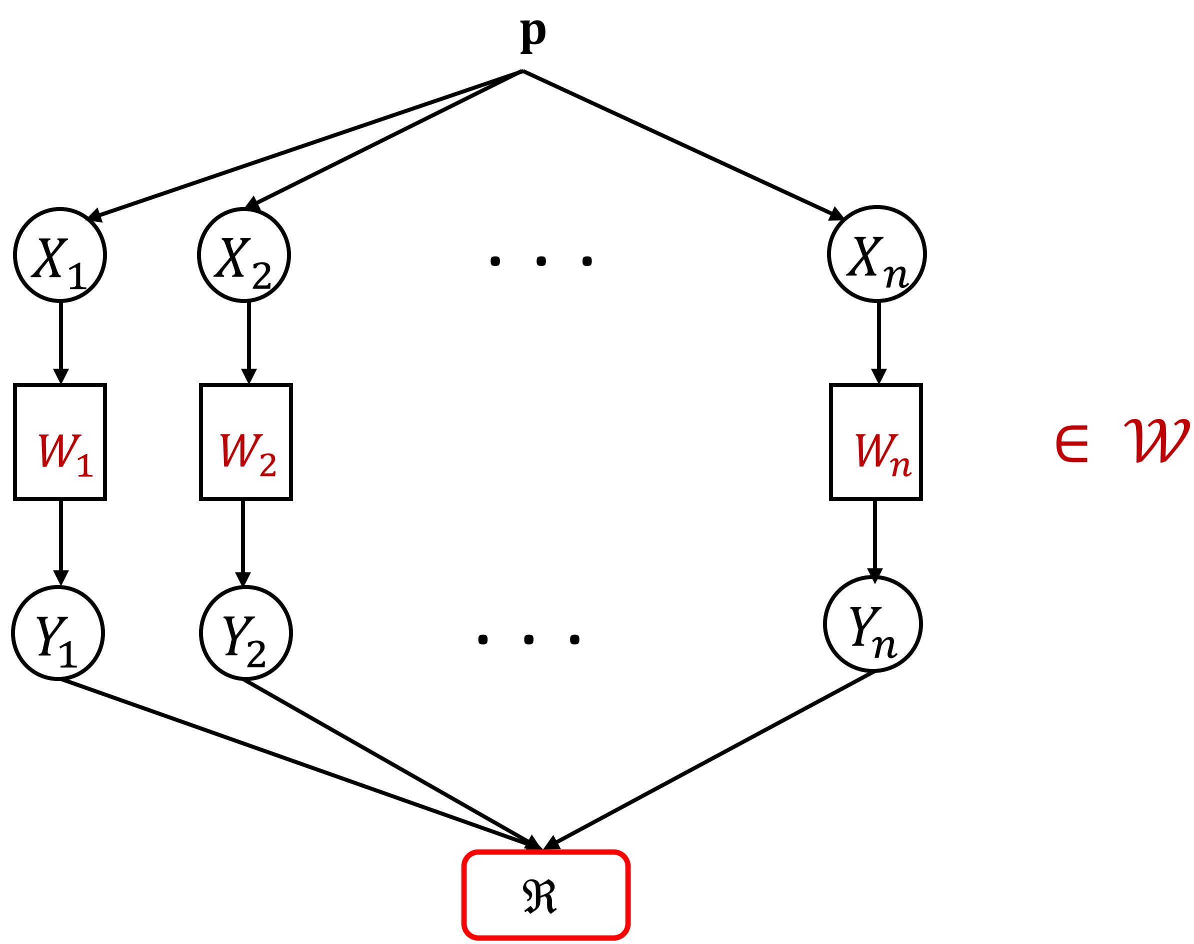
$$\mathcal{W}_{\varrho} = \{\text{all } \varrho - \text{LDP channels}\}$$

Privacy guarantees even "against" the server



The Protocols

Distributed Statistical Inference



Given $Y^n := Y_1, \ldots, Y_n$, solve the inference task

Distributed statistical inference

For $W^n := W_1, \ldots, W_n$,

$$\mathbf{p}^{W^n}(Y^n) = \prod_t \mathbf{p}^{W_t}(Y_t)$$

How to choose $W_1, W_2, \ldots, W_n \in \mathcal{W}$ to minimize n?

The protocols

Simultaneous Message Passing (SMP)/Non-interactive schemes

W_is are chosen simultaneously

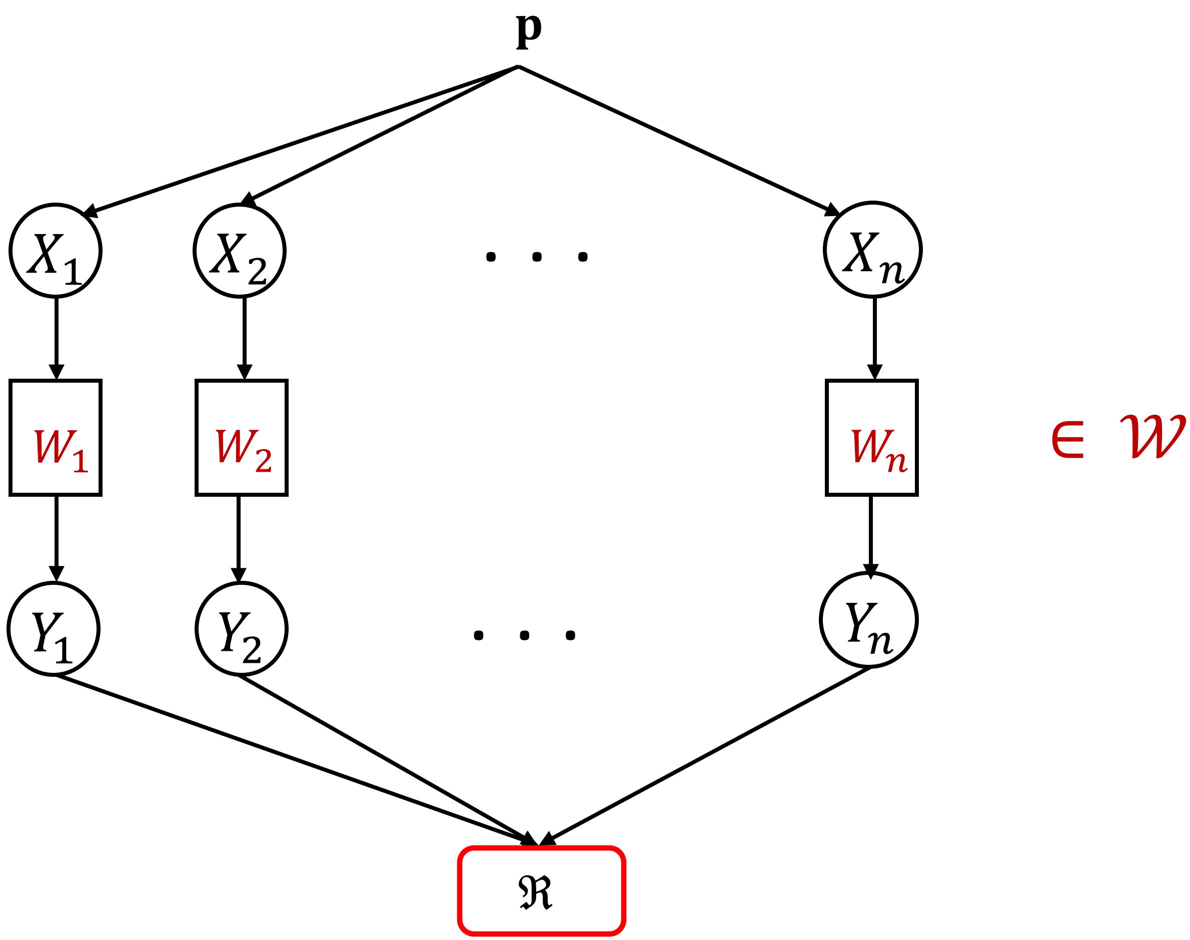
private-coin SMP (no shared randomness)

W_ts are chosen independently

$Y_1, Y_2, \dots, Y_n$ are independent

e.g., $W_1, \dots, W_n$ are fixed

Private-coin SMP protocols



Noninteractive ("simultaneous message-passing"), no common randomness

The protocols

Simultaneous Message Passing (SMP)/Non-interactive schemes

W_is are chosen simultaneously

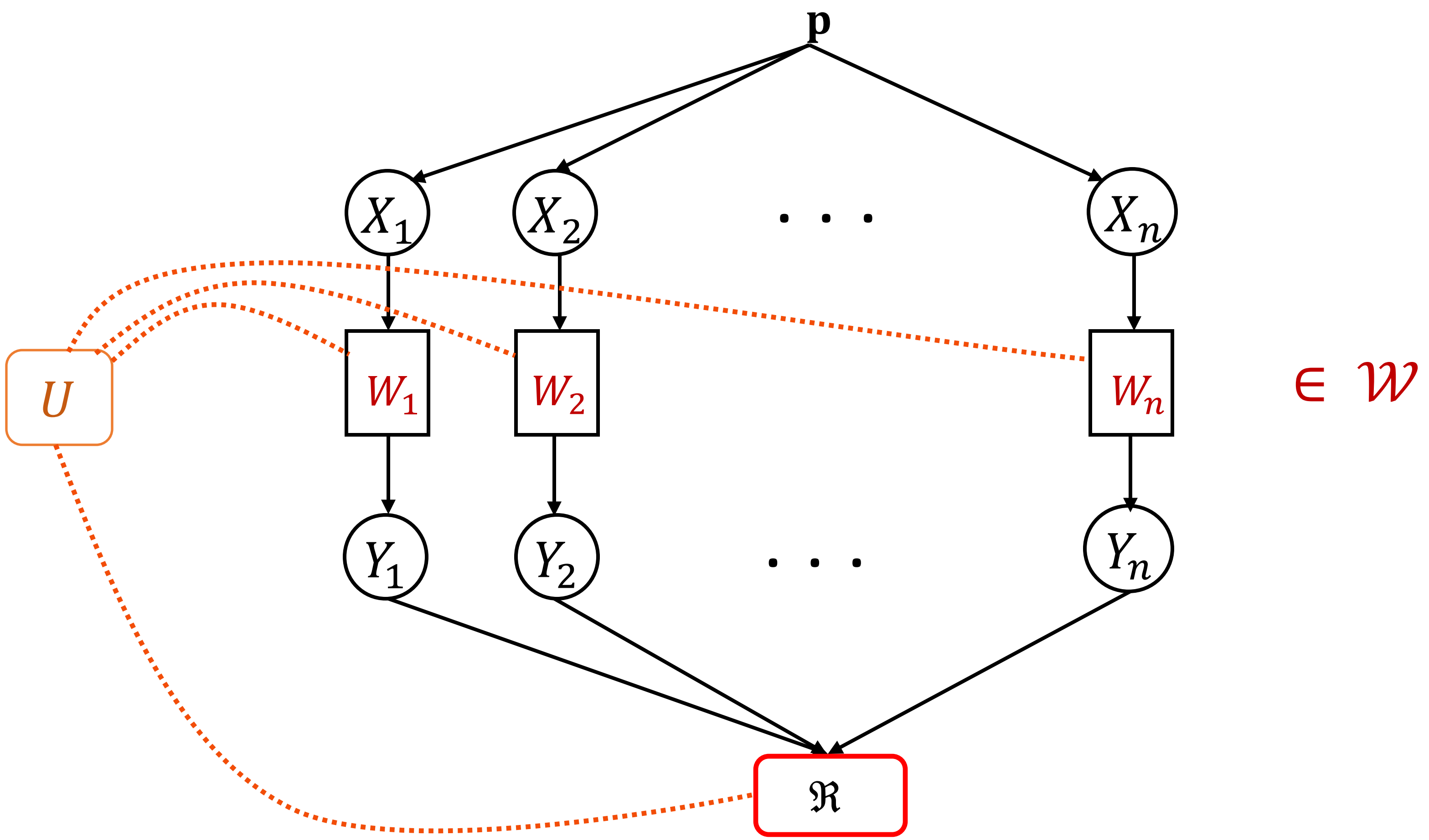
public-coin SMP (shared randomness)

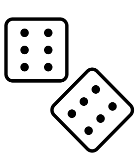
U: common random string available to all users and referee

$W_t s$ is a function of U

$Y_1, Y_2, \dots, Y_n$ are independent **given** U

Public-coin SMP protocols





Noninteractive ("simultaneous message-passing"), *but* common random seed

The protocols

Interactive schemes

W_is can depend on previous messages

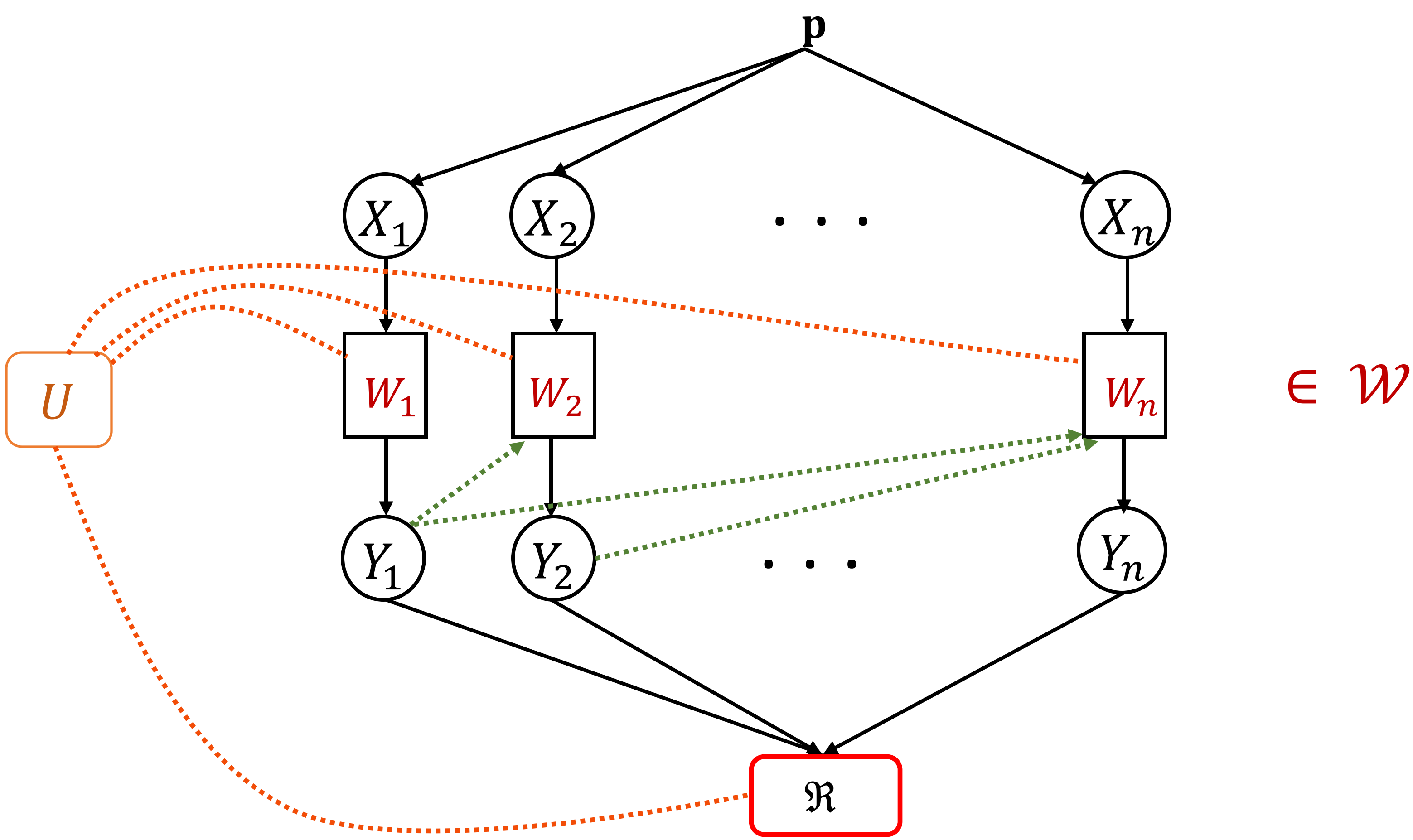
sequentially interactive protocols

U: common random string available to all users and referee

for $t = 1, \dots, n$

W_t is a function of (U, Y^{t-1})

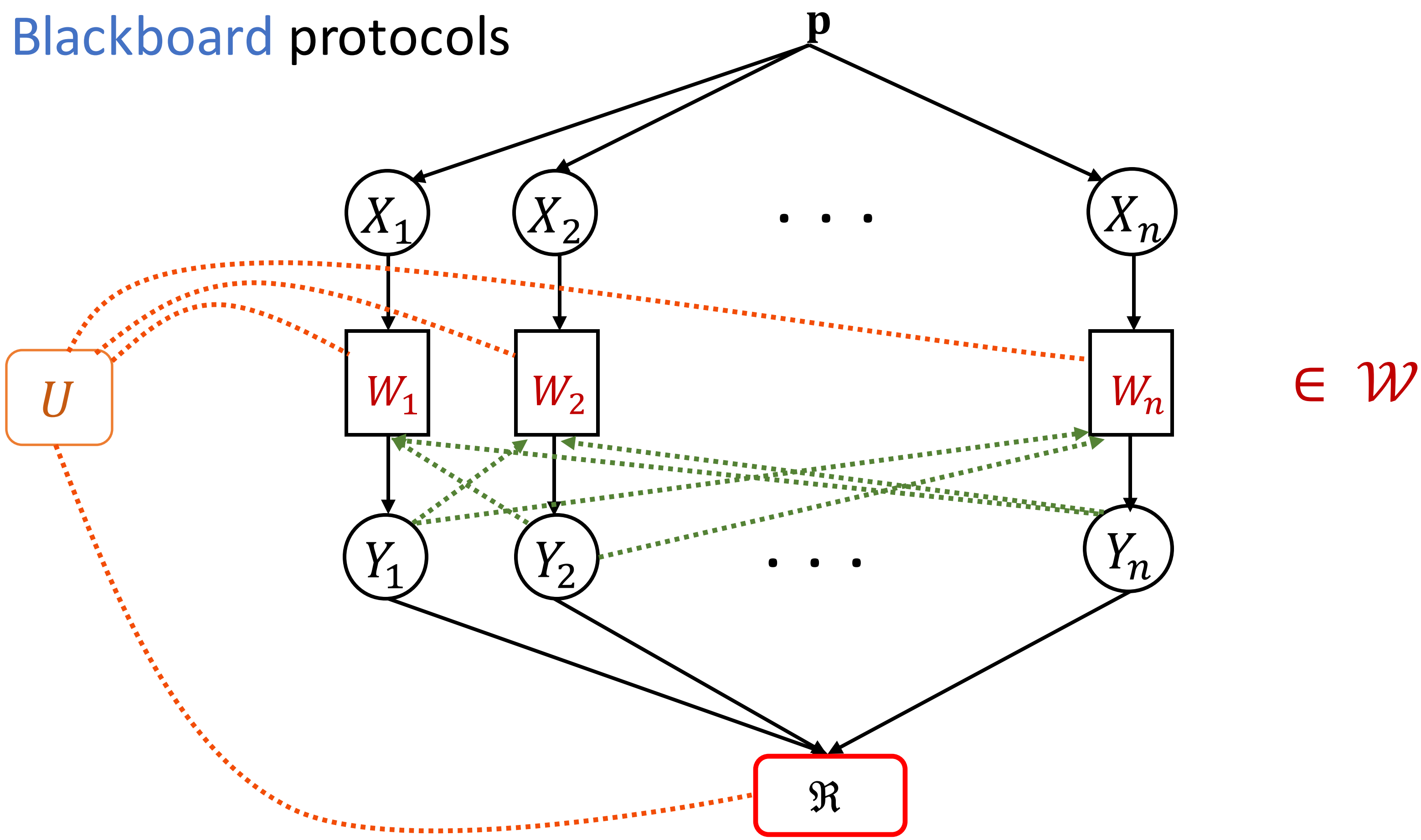
Sequentially Interactive protocols



Interactive ("one-pass, sequential"),
and common random seed

Types of protocols

Blackboard protocols



Fully interactive ("many passes"),
and common random seed

Types of protocols

Each of these models is **at least as powerful** as the previous

private-coin $\preccurlyeq$ public-coin $\preccurlyeq$ sequentially interactive $\preccurlyeq$ blackboard

Each has its pros and cons (both in theory *and* practice), and may require different techniques to analyze.

The Problems

Parameter/density
estimation

Goodness-of-fit /
Hypothesis testing

Sample complexity: smallest n to solve the task

Example 1: Discrete distributions

$$\mathcal{P} = \Delta_d\text{: distbs on } [d] \coloneqq \{1 \dots d\}$$

Goal: output $\widehat{\mathbf{p}}$ such that

$$\mathbb{E}[\mathrm{TV}(\widehat{\mathbf{p}}, \mathbf{p})] \le \varepsilon$$

Sample complexity = $\Theta\left(\frac{d}{\varepsilon^2}\right)$

(without constraints)

$\mathbf{q}$**:** a reference distribution

Goal: Test

$$\mathbf{p} = \mathbf{q} \text{ vs } \mathrm{TV}(\mathbf{p}, \mathbf{q}) > \varepsilon$$

Sample complexity = $\Theta\left(\frac{\sqrt{d}}{\varepsilon^2}\right)$

(without constraints)

$$TV(\mathbf{p}, \mathbf{q}) \coloneqq \sup_{S \subseteq [k]} \big(\mathbf{p}(S) - \mathbf{q}(S)\big) = \frac{1}{2} \ell_1(\mathbf{p}, \mathbf{q})$$

Example 2: High dimensional distributions

$$\mathcal{P} = \{\mathcal{N}(\boldsymbol{\mu}, \mathbf{I}_d) : \boldsymbol{\mu} \in \mathrm{R}^d\}$$

Goal: output $\widehat{\boldsymbol{\mu}}$ such that

$$\mathbb{E}[|\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu}|_2^2] \leq \varepsilon^2$$

Sample complexity = $\Theta\left(\frac{d}{\varepsilon^2}\right)$

(without constraints)

Goal: Test

$$\boldsymbol{\mu} = \mathbf{0} \text{ vs } |\boldsymbol{\mu}|_2 > \varepsilon$$

Sample complexity = $\Theta\left(\frac{\sqrt{d}}{\varepsilon^2}\right)$

(without constraints)

*detecting signal vs noise

Other families: Bernoulli product

Research goals

Establish sample complexity bounds for …

- Different $\mathcal{W}$s
- Estimation/Testing/other properties
- Private-coin SMP/public-coin SMP/interactive
- Discrete/high-dimensional/non-parametric

Already a bit too much … each interesting in its own right … !

For example … discrete distribution testing

$\mathcal{W}_{\varrho}$, [AminJosephMao’20, BerrettButucea’20, AcharyaCanonneLiuSunTyagi’20]:

Private-coin SMP $\ll$ public-coin SMP $\approx$ SMP/interactive

$\mathcal{W}_{\ell}$, [AcharyaCanonneLiuSunTyagi’20]:

Private-coin SMP $\ll$ public-coin SMP $\approx$ SMP/interactive

General $\mathcal{W}$, [AcharyaCanonneLiuSunTyagi’20]:

Private-coin SMP $\ll$ public-coin SMP $\ll$ SMP/interactive

Similarly for Gaussian mean testing … [AcharyaCanonneTyagi’20, SzaboVuursteenVanZanten’20]

Parameter/density estimation

~~Goodness-of-fit / Hypothesis testing~~

Part 3 of tutorial (link)

Learn about Ingster's method from HT!

Establishing tight results for SMP protocols generally easier … why?

$Y_1, \ldots, Y_n$ independent (given U)

See general discussion in

[ACLST20] J. Acharya, C. Canonne, Y. Liu, Z. Sun, H. Tyagi, "Interactive inference under information constraints" *arXiv: 2007.10976 (in submission)*

Methods to establish interactive lower bounds

1. Cramer-Rao/van Trees inequality [BarnesHanOzgur19, BarnesChenOzgur20, SarbuZaidi21]
   - Unified results for $\Delta_d, \mathcal{B}_d, \mathcal{G}_d$
   - Results hold for ℓ_2 loss

2. Strong Data Processing + Assouad's method [BravermanGardMaNguyenWoodruff16, DuchiRogers19]
   - Lower bounds for $\mathcal{B}_d, \mathcal{G}_d$ under ℓ_2 loss
   - Naturally extends to other ℓ_p loss functions

3. Chi-squared contractions + Assouad's method [AcharysCanonneLiuSunTyagi20, AcharyaCanonneSunTyagi20]
   - Unified bounds for $\Delta_d, \mathcal{B}_d, \mathcal{G}_d$
   - Works under ℓ_p for $p \geq 1$
   - For arbitrary channels

Pointers

Part 2 of tutorial at FOCS'20 ([link]())

Cramer-Rao/van Trees inequality

Strong Data Processing + Assouad's method

Next two parts …

- Discrete distributions
  - Simulate and infer for upper bounds
  - Lower bounds

- Unified method for general distributions and channel families

Part 2: Discrete Distributions

Discrete distribution estimation

$\mathcal{P} = \Delta_d$: distbs on $[d] \coloneqq \{1 \ldots d\}$

Goal: output $\widehat{\mathbf{p}}$ such that

$$\mathbb{E}[\mathrm{TV}(\widehat{\mathbf{p}}, \mathbf{p})] \leq \varepsilon$$

Sample complexity = $\Theta\left(\frac{d}{\varepsilon^2}\right)$ (without constraints)

Empirical distribution works - DIY

$X_1, \ldots, X_n \sim \mathbf{p}$, $N_x :=$ # times x appears

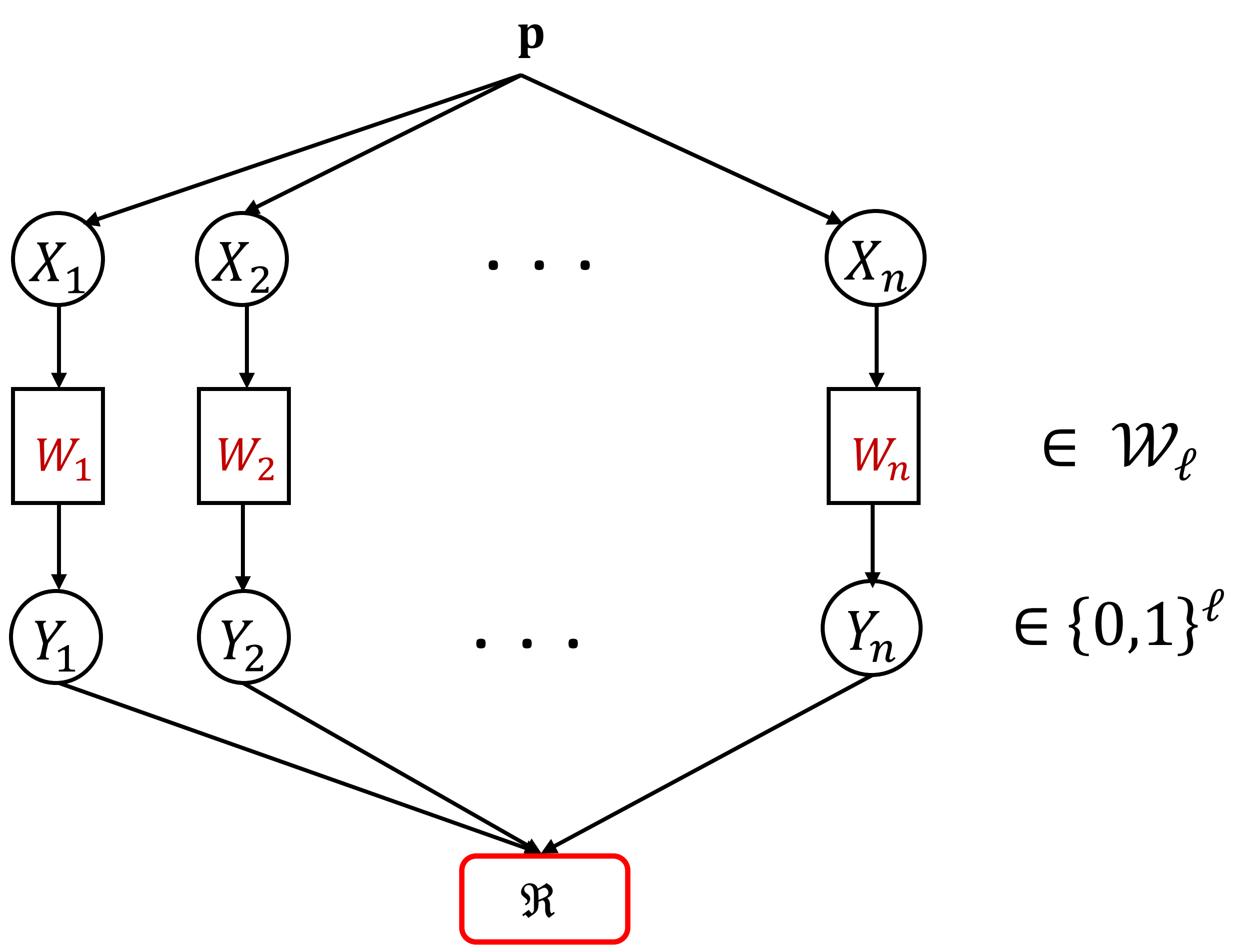
Empirical distribution: $\widehat{\mathbf{p}}(x) = N_x / n$

$N_x \sim \mathrm{Bin}\big(n, \mathbf{p}(x)\big)$

$$\mathbb{E}\left[\big(\widehat{\mathbf{p}}(x) - \mathbf{p}(x)\big)^2\right] = \frac{\mathbf{p}(x)\big(1 - \mathbf{p}(x)\big)}{n} \Rightarrow \mathbb{E}[\ell_2^2(\widehat{\mathbf{p}}, \mathbf{p})] \le \frac{1}{n}$$

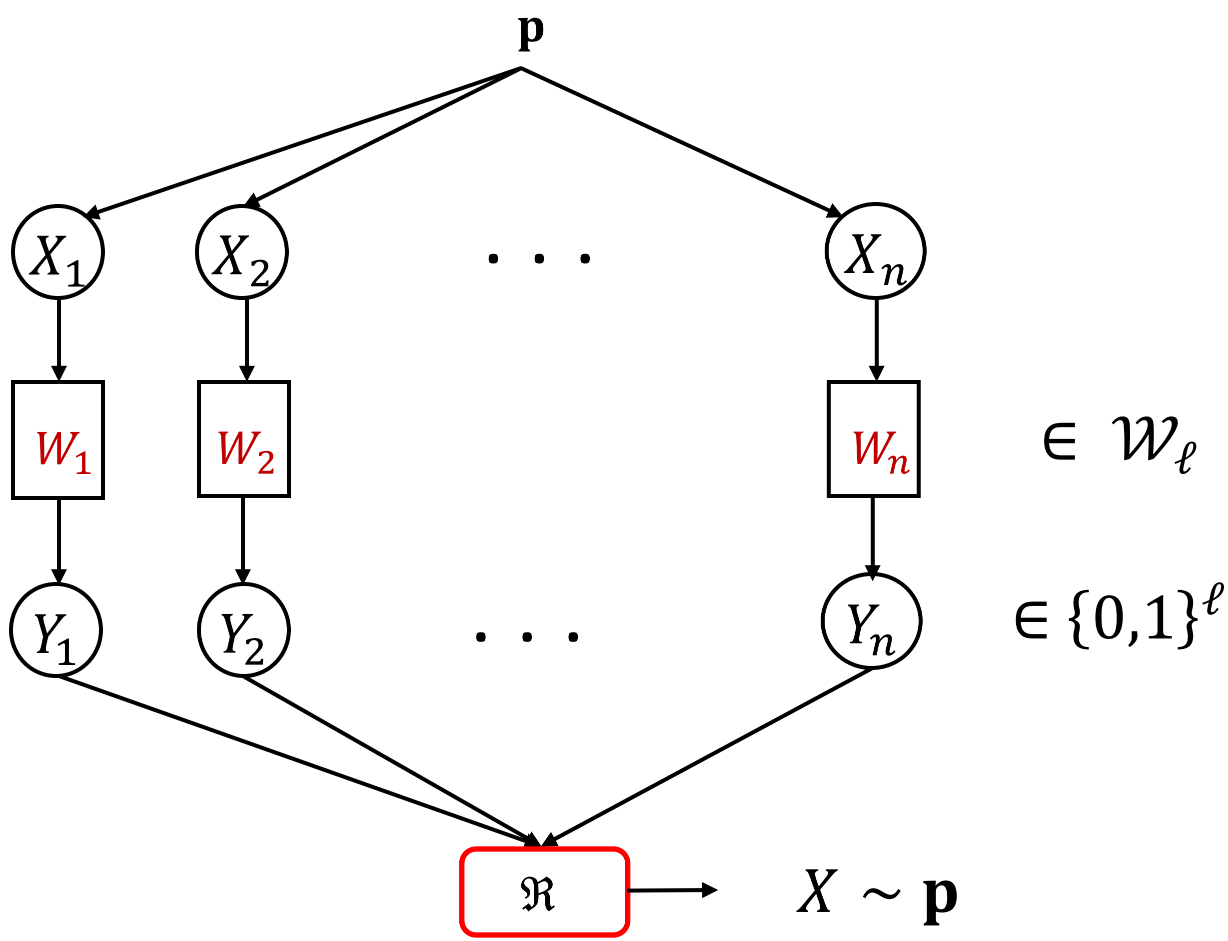
$$\begin{aligned} \mathbb{E}[\ell_1(\widehat{\mathbf{p}}, \mathbf{p})]^2 &\le \mathbb{E}[\ell_1(\widehat{\mathbf{p}}, \mathbf{p})^2] && \textit{(Jensen)} \\ &\le d \cdot \mathbb{E}[\ell_2^2(\widehat{\mathbf{p}}, \mathbf{p})] && \textit{(Cauchy Schwarz)} \\ &\le \frac{d}{n} \end{aligned}$$

Under communication constraints



A simulation puzzle …

Goal: To simulate a sample from messages



One simulation to solve them all …

Theorem. Suppose simulation is possible with $f(d, \ell)$ samples.

Let T be some task with sample complexity $T(d, \varepsilon)$.

Then T can be solved with $f(d, \ell) \cdot T(d, \varepsilon)$ samples under $\mathcal{W}_\ell$.

What is $f(d, \log_2 d) = ?$

One simulation to solve them all …

Theorem. There is a private-coin SMP protocol with

$$f(d,\ell) \approx \max\left\{\frac{k}{2^\ell}, 1\right\}.$$

No protocol (even interactive) can do better!

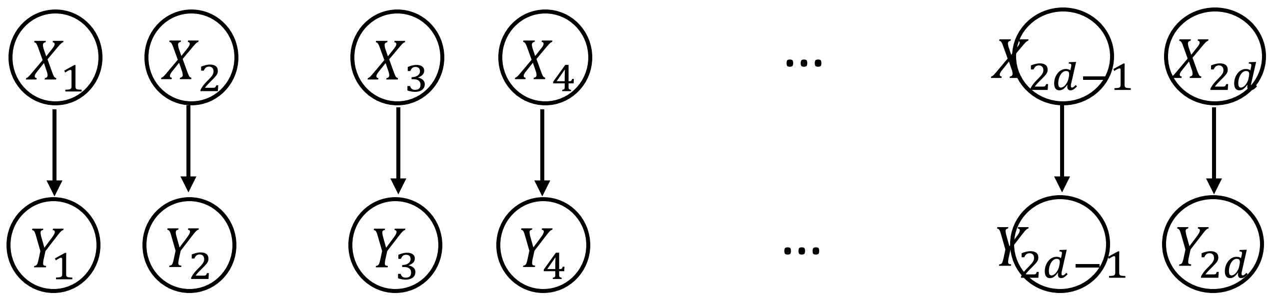
Estimation with $\Theta\left(\frac{d}{\varepsilon^2}\cdot\frac{d}{2^\ell}\right)$ and testing with $\Theta\left(\frac{\sqrt{d}}{\varepsilon^2}\cdot\frac{d}{2^\ell}\right)$

Algorithm for one-bit

Take $2d$ players:

- First two tell if their input is symbol 1
- Next two tell if their input is symbol 2
- And so on …

Algorithm for one-bit



$$Y_{2t-1} = I\{X_{2t-1} = t\}$$
$$Y_{2t} = I\{X_{2t} = t\}$$

Algorithm for one-bit

- Output t if:
  - Player $2t - 1$ is the **only** odd player sending 1
  - Player $2t$ sends 0
- If no such i, output $\perp$

Conditioned on not outputting $\perp$, a sample from p

Algorithm for one-bit

Player $2t - 1$ is the **only** odd player sending 1

$$\Pr(Y_{2t-1} = 1, Y_{2t'-1} = 0 \; for \; t' \neq t) = \mathbf{p}(t) \prod_{t' \neq t} (1 - \mathbf{p}(t'))$$

Player $2t$ sends 0

$$\Pr(Y_{2t} = 0) = (1 - \mathbf{p}(t))$$

$$\Pr(\text{output } t \mid \text{not } \perp) = \mathbf{p}(t) \cdot \prod_{t' \neq t} \big(1 - \mathbf{p}(t')\big) \propto \mathbf{p}(t)$$

Corollary

Inference Task	Centralized	One-bit private-SMP
Estimation	$\Theta\left(\frac{d}{\varepsilon^2}\right)$	$O\left(\frac{d^2}{\varepsilon^2}\right)$
Testing	$\Theta\left(\frac{\sqrt{d}}{\varepsilon^2}\right)$	$O\left(\frac{d^{3/2}}{\varepsilon^2}\right)$

Corollary

Inference Task	Centralized	One-bit private-SMP	One-bit public-SMP
Estimation	$\Theta\left(\frac{d}{\varepsilon^2}\right)$	$\Theta\left(\frac{d^2}{\varepsilon^2}\right)$	$\Theta\left(\frac{d^2}{\varepsilon^2}\right)$
Testing: $I_u(k, \varepsilon)$	$\Theta\left(\frac{\sqrt{d}}{\varepsilon^2}\right)$	$\Theta\left(\frac{d^{3/2}}{\varepsilon^2}\right)$	$\Theta\left(\frac{d}{\varepsilon^2}\right)$

Bounds are tight … simulate and infer is optimal for private-coin SMP

Related work

Under SMP protocols these bounds are tight for communications [HanMukherjeeOzgur19, AcharyaCanonneTyagi'19] and LDP [DuchiJordanWainwright14]

Sample complexity with interactivity and general channels?

[ACLST20] J. Acharya, C. Canonne, Y. Liu, Z. Sun, H. Tyagi, "Interactive inference under information constraints" *arXiv: 2007.10976 (in submission)*

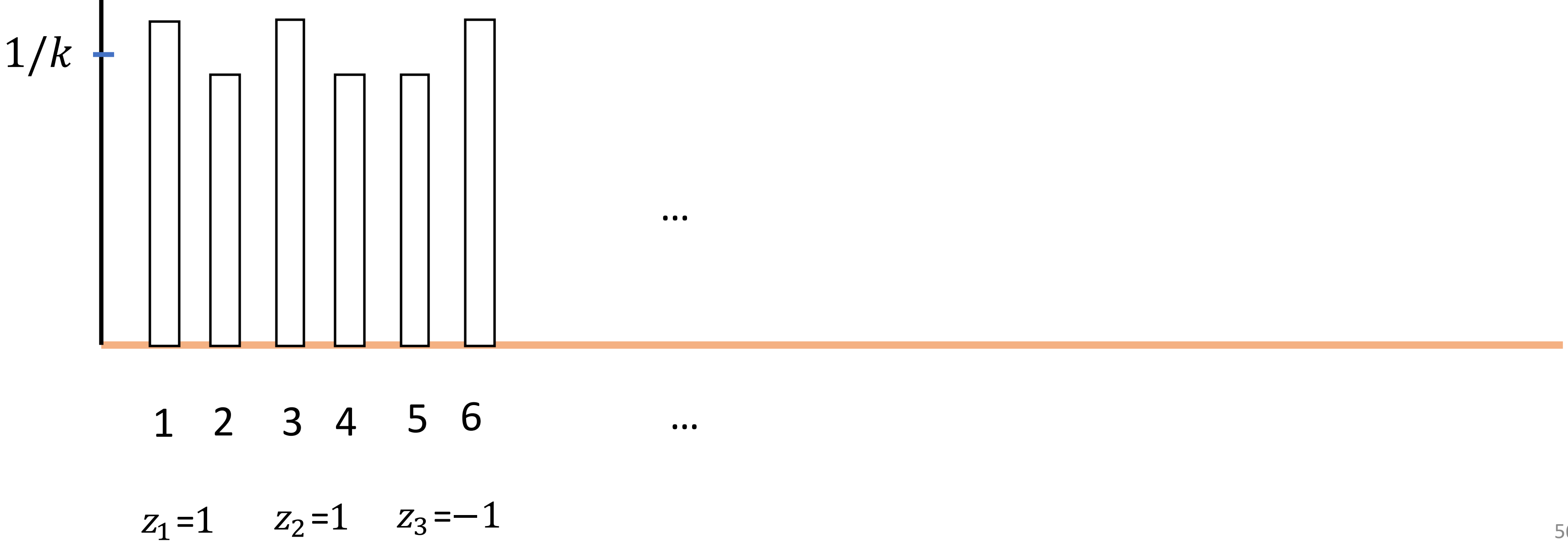
A hard instance

A hard instance

[Paninski'08] Let $\mathcal{Z} = \{-1,1\}^{d/2}$, and $\mathcal{P}_{\mathcal{Z}} = \{\mathbf{p}_z : z \in \mathcal{Z}\}$, where

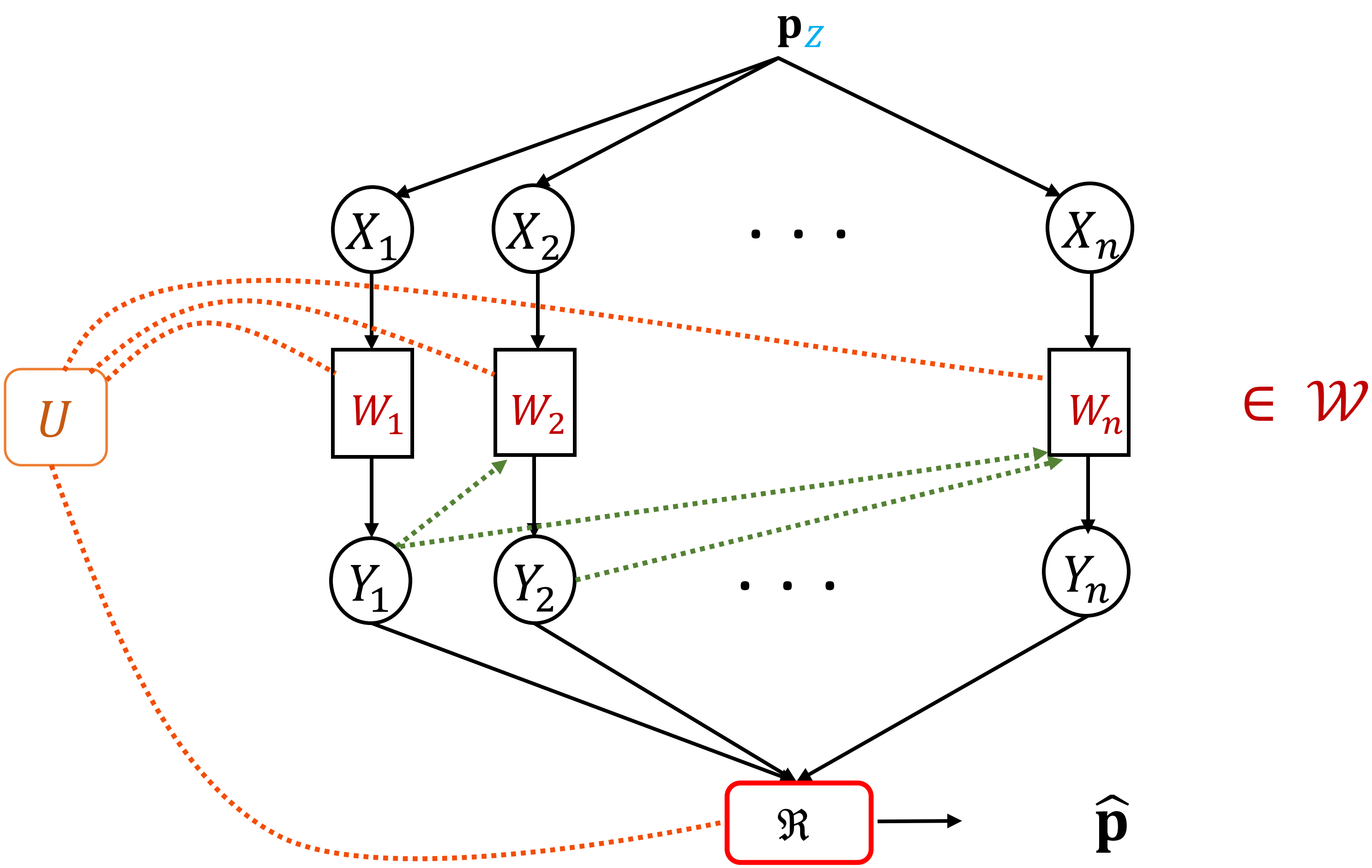
$$\mathbf{p}_z(2i-1) = \frac{1 + z_i \cdot 2\varepsilon}{d}, \qquad \mathbf{p}_z(2i) = \frac{1 - z_i \cdot 2\varepsilon}{d}, \qquad i = 1, \dots, d/2.$$

$$\mathrm{d_{TV}}(\mathbf{p}_z, \mathbf{p}_{z'}) = \frac{2\varepsilon}{d} \cdot \mathrm{Ham}(z, z')$$



Learning lower bounds

$Z = (Z_1, \dots, Z_{d/2}) \sim_{uar} \mathcal{Z}$, ie, each $Z_i \sim^{iid} \mathrm{Bern}(0.5)$



Learning lower bounds –

Exercise: Let $z \in \mathcal{Z}$ and $\widehat{\mathbf{p}}$ satisfies $\mathrm{d_{TV}}(\widehat{\mathbf{p}}, \mathbf{p}_z) < \frac{\varepsilon}{10}$.

Then,

$$z^* = \arg\min_{z'} \mathrm{d_{TV}}(\widehat{\mathbf{p}}, \mathbf{p}_{z'})$$

satisfies

$$\mathrm{Ham}(z, z^*) < \frac{d}{10}.$$

Assouad's method

If we can estimate $\mathbf{p}_Z \in_{\text{uar}} \mathcal{P}_{\mathcal{Z}}$, then we can estimate Z!

Theorem. Pick $Z \sim_{uar} \mathcal{Z}$.

If

$$\mathbb{E}_Z\left[\mathbb{E}_{\mathbf{p}_Z}[\mathrm{d}_{\mathrm{TV}}(\widehat{\mathbf{p}}(Y^n, U), \mathbf{p}_Z)]\right] < \frac{\varepsilon}{10}$$

then there exists an estimator $\hat{Z}(Y^n, U)$ such that

$$\sum_{1 \leq i \leq d/2} \Pr(\hat{Z}_i = Z_i) > 0.8 \times \frac{d}{2}.$$

- Note: We could write this as $\sum_i I(Z_i \wedge Y^n | U) = \Omega(d)$

Assouad's method

Exercise. If

$$\sum_{1 \leq i \leq d/2} \Pr(\hat{Z}_i = Z_i) > 0.8 \times \frac{d}{2},$$

then there exists a subset $S \subseteq \{1, \ldots, d/2\}$ with $|S| > d/6$ s.t. if i ∈ S,

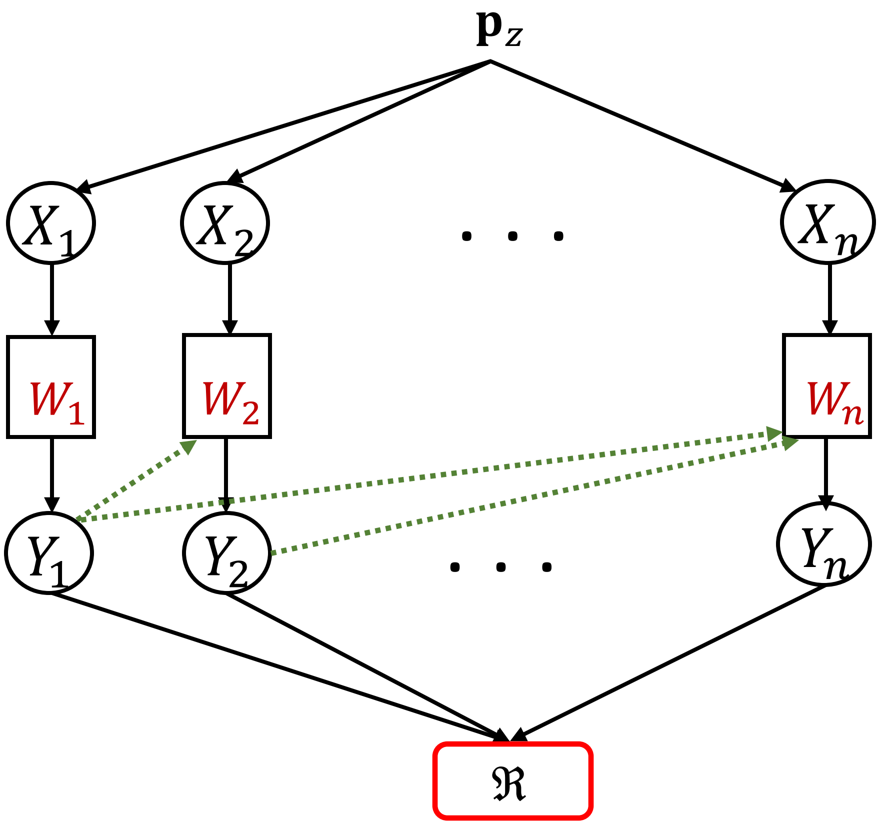
$$\Pr(\hat{Z}_i = Z_i) > 0.7.$$

Now we need a lower bound on n for this to happen

Notation

Fix $i \in [d/2]$, when can we figure Z_i?

$\mathbf{p}_z^{Y^n}$: distribution of Y^n when input distribution $\mathbf{p}_z$



Information bound on one coordinate

average output distribution fixing $Z_i = \pm 1$:

When $Z_i = 1$: $\mathbf{p}_{+i}^{Y^n} := \frac{1}{2^{d/2-1}} \sum_{z: z_i = +1} \mathbf{p}_z^{Y^n}$

When $Z_i = -1$: $\mathbf{p}_{-i}^{Y^n} := \frac{1}{2^{d/2-1}} \sum_{z: z_i = -1} \mathbf{p}_z^{Y^n}$

If we can guess Z_i from Y^n

$\Leftrightarrow$ $\mathrm{d_{TV}}\left(\mathbf{p}_{+i}^{Y^n}, \mathbf{p}_{-i}^{Y^n}\right)$ must be large

$\Rightarrow$ bound distance between $\mathbf{p}_{+i}^{Y^n}$ and $\mathbf{p}_{-i}^{Y^n}$

Total variation and hypothesis testing

$\mathbf{p}_1, \mathbf{p}_2$ be any two distributions over $\mathcal{Y}$

$j \in \{0,1\}$ be picked at random

Given $Y \sim \mathbf{p}_j$, design a $\hat{\jmath}(Y)$ that is a guess for j

For any $\hat{\jmath}(Y)$:

$$\Pr(\hat{\jmath}(Y) = j) \leq \frac{1}{2}\left(1 + d_{\mathrm{TV}}(\mathbf{p}_1, \mathbf{p}_2)\right)$$

Information bound on one coordinate

In our case, $\mathbf{p}_1 = \mathbf{p}_{+i}^{Y^n}, \mathbf{p}_2 = \mathbf{p}_{-i}^{Y^n}$, and

$$\Pr(\hat{Z}_i = Z_i) > 0.7 \Rightarrow d_{\mathrm{TV}}\left(\mathbf{p}_{+i}^{Y^n}, \mathbf{p}_{-i}^{Y^n}\right) \geq 0.4$$

Since this holds for at least $d/6$ coordinates,

$$\sum_i d_{\mathrm{TV}}\left(\mathbf{p}_{+i}^{Y^n}, \mathbf{p}_{-i}^{Y^n}\right)^2 \geq \frac{d}{6} \times 0.16.$$

Some ingredients

$$D(\mathbf{p}_1||\mathbf{p}_2) \coloneqq \sum_y \mathbf{p}_1(y) \log \frac{\mathbf{p}_1(y)}{\mathbf{p}_2(y)}, \chi^2(\mathbf{p}_1, \mathbf{p}_2) \coloneqq \sum_y \frac{\left(\mathbf{p}_1(y) - \mathbf{p}_2(y)\right)^{\mathbf{2}}}{\mathbf{p}_2(y)}$$

Pinsker's inequality, convexity of logarithms:

$$2 \cdot \mathrm{d}_{\mathrm{TV}}(\mathbf{p}_1, \mathbf{p}_2)^2 \leq D(\mathbf{p}_1||\mathbf{p}_2) \leq \chi^2(\mathbf{p}_1, \mathbf{p}_2)$$

Chain rule of KL divergence: If $\mathbf{p}_1$ and $\mathbf{p}_2$ are over $\mathcal{Y}_1 \times \mathcal{Y}_2$:

$$\begin{aligned} &D(\mathbf{p}_1(Y_1, Y_2)||\mathbf{p}_2(Y_1, Y_2)) \\ &= D(\mathbf{p}_1(Y_1)||\mathbf{p}_2(Y_1)) + \mathbb{E}_{Y_1}[D(\mathbf{p}_1(Y_2|Y_1)||\mathbf{p}_2(Y_2|Y_1))] \end{aligned}$$

KL $\leq$ chi-squared (DIY)

Since $\log(1+x) \leq x$ (why?)

$$D(\mathbf{p}||\mathbf{q}) \coloneqq \sum_x \mathbf{p}(x) \log\left(1 + \frac{\mathbf{p}(x) - \mathbf{q}(x)}{\mathbf{q}(x)}\right)$$

$$\leq \sum_x \mathbf{p}(x) \frac{\left(\mathbf{p}(\boldsymbol{x}) - \mathbf{q}(\boldsymbol{x})\right)}{\mathbf{q}(\boldsymbol{x})} = \boldsymbol{\chi}^2(\mathbf{p}, \mathbf{q})$$

Exercise: Prove the chain rule of KL.

Why go to KL?

By Pinsker's inequality,

$$4 \cdot d_{\mathrm{TV}}\left(\mathbf{p}_{+i}^{Y^n}, \mathbf{p}_{-i}^{Y^n}\right)^2 \leq \left(D\left(\mathbf{p}_{+i}^{Y^n} || \mathbf{p}_{-i}^{Y^n}\right) + D\left(\mathbf{p}_{+i}^{Y^n} || \mathbf{p}_{-i}^{Y^n}\right)\right)$$

Summing over i,

$$\sum_i \left(D\left(\mathbf{p}_{+i}^{Y^n} || \mathbf{p}_{-i}^{Y^n}\right) + D\left(\mathbf{p}_{+i}^{Y^n} || \mathbf{p}_{-i}^{Y^n}\right)\right)$$

$$\geq \sum_i 4 \cdot d_{\mathrm{TV}}\left(\mathbf{p}_{+i}^{Y^n}, \mathbf{p}_{-i}^{Y^n}\right)^2 \geq 4 \cdot \frac{d}{6} \times 0.16 \geq \frac{d}{10}$$

$\mathbf{p}_{+i}^{Y^n}$ are mixture distributions!

Handling mixtures is painful, leads to issues to extend SMP lower bounds to interactive setting

Convexity to the rescue

Exercise: KL divergence is convex.

For any distributions $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{q}_1, \mathbf{q}_2$ and $\lambda \in [0,1]$,

$$D(\lambda \mathbf{p}_1 + (1-\lambda)\mathbf{q}_1 || \lambda \mathbf{p}_2 + (1-\lambda)\mathbf{q}_2)$$
$$\leq \lambda \cdot D(\mathbf{p}_1 || \mathbf{p}_2) + (1-\lambda) \cdot D(\mathbf{p}_1 || \mathbf{p}_2)$$

Prove using concavity of logarithms

Convexity to handle mixtures

$z \in \{-1,1\}^{k/2}$, $z^{\oplus i}$ obtained by flipping the ith coordinate of z

Theorem.

$$\frac{1}{2}\left(D\left(\mathbf{p}_{+i}^{Y^n} || \mathbf{p}_{-i}^{Y^n}\right) + D\left(\mathbf{p}_{+i}^{Y^n} || \mathbf{p}_{-i}^{Y^n}\right)\right) \leq \mathbb{E}_Z\left[D\left(\mathbf{p}_Z^{Y^n} || \mathbf{p}_{Z^{\oplus i}}^{Y^n}\right)\right]$$

Proof. Convexity of divergence to the definitions of $\mathbf{p}_{+i}^{Y^n}$ and $\mathbf{p}_{-i}^{Y^n}$ ∎

Information about Z_i bounded by average divergence in message distribution upon changing only Z_i when all others are fixed!

Convexity to handle mixtures

Summing over i

$$\frac{d}{20} \leq \mathbb{E}_Z\left[\sum_i D\left(\mathbf{p}_Z^{Y^n} || \mathbf{p}_{Z\oplus i}^{Y^n}\right)\right]$$

What do we have here … Fix a Z and then change one coordinate at a time …

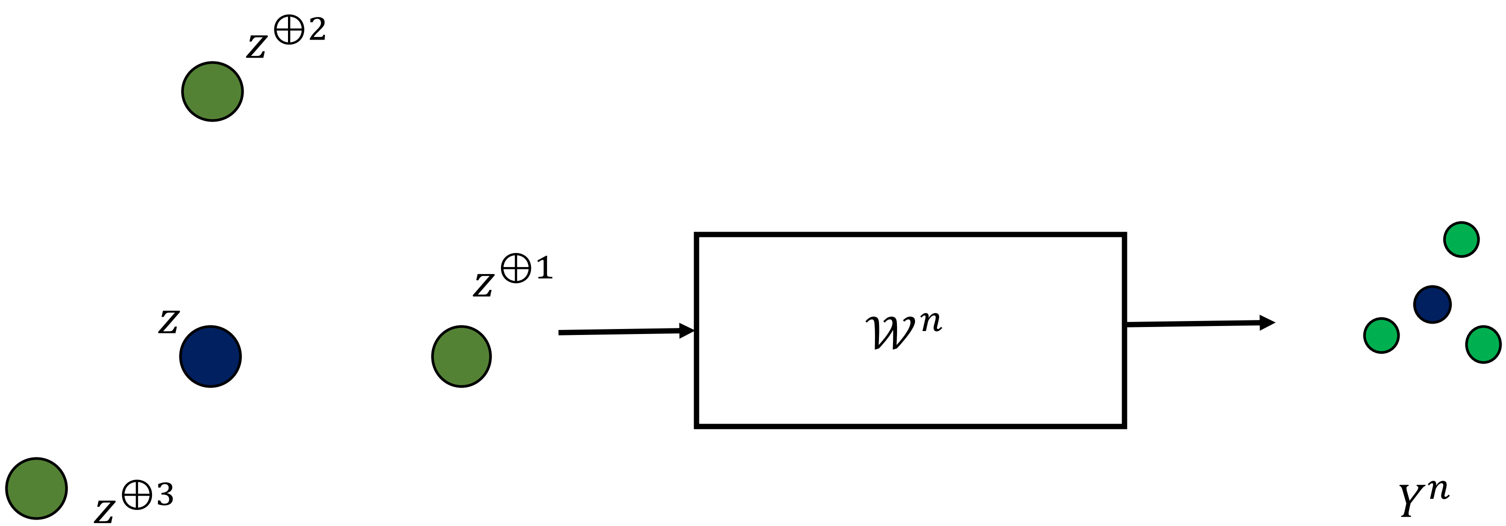
Focus on one z

By expectation<max, and linearity of expectations,

$$\frac{d}{20} \leq \max_z \left[\sum_i D\left(\mathrm{p}_z^{Y^n} || \mathrm{p}_{z\oplus i}^{Y^n}\right) \right]$$

** the following is the original bound in terms of MI:

$$\sum_i I(Z_i \wedge Y^n) \leq \frac{1}{2} \cdot \max_z \left[\sum_i D\left(\mathrm{p}_z^{Y^n} || \mathrm{p}_{z\oplus i}^{Y^n}\right) \right]$$

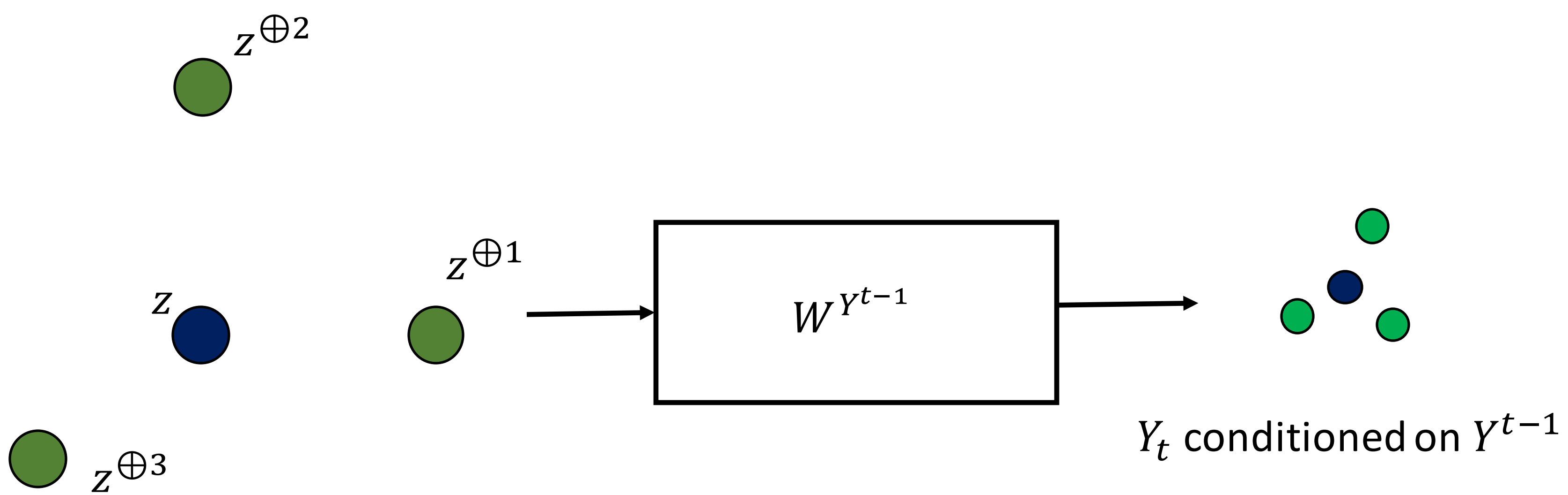


Bounding $\sum_i D\left(\mathrm{p}_z^{Y^n} \| \mathrm{p}_{z\oplus i}^{Y^n}\right)$

By the chain rule of divergence

$$\sum_i D\left(\mathbf{p}_z^{Y^n} \| \mathbf{p}_{z\oplus i}^{Y^n}\right) = \sum_t \mathbb{E}_{\mathbf{p}_z^{Y^{t-1}}}\left[\sum_i D\left(\mathbf{p}_z^{Y_t|Y^{t-1}} \| \mathbf{p}_{z\oplus i}^{Y_t|Y^{t-1}}\right)\right].$$

- $\mathbf{p}_z^{Y_t|Y^{t-1}}$: Distribution of Y_t with input $\mathbf{p}_z$ conditioned on Y^{t-1}
- Channel at player t a function only of Y^{t-1}, denoted $W^{Y^{t-1}}$

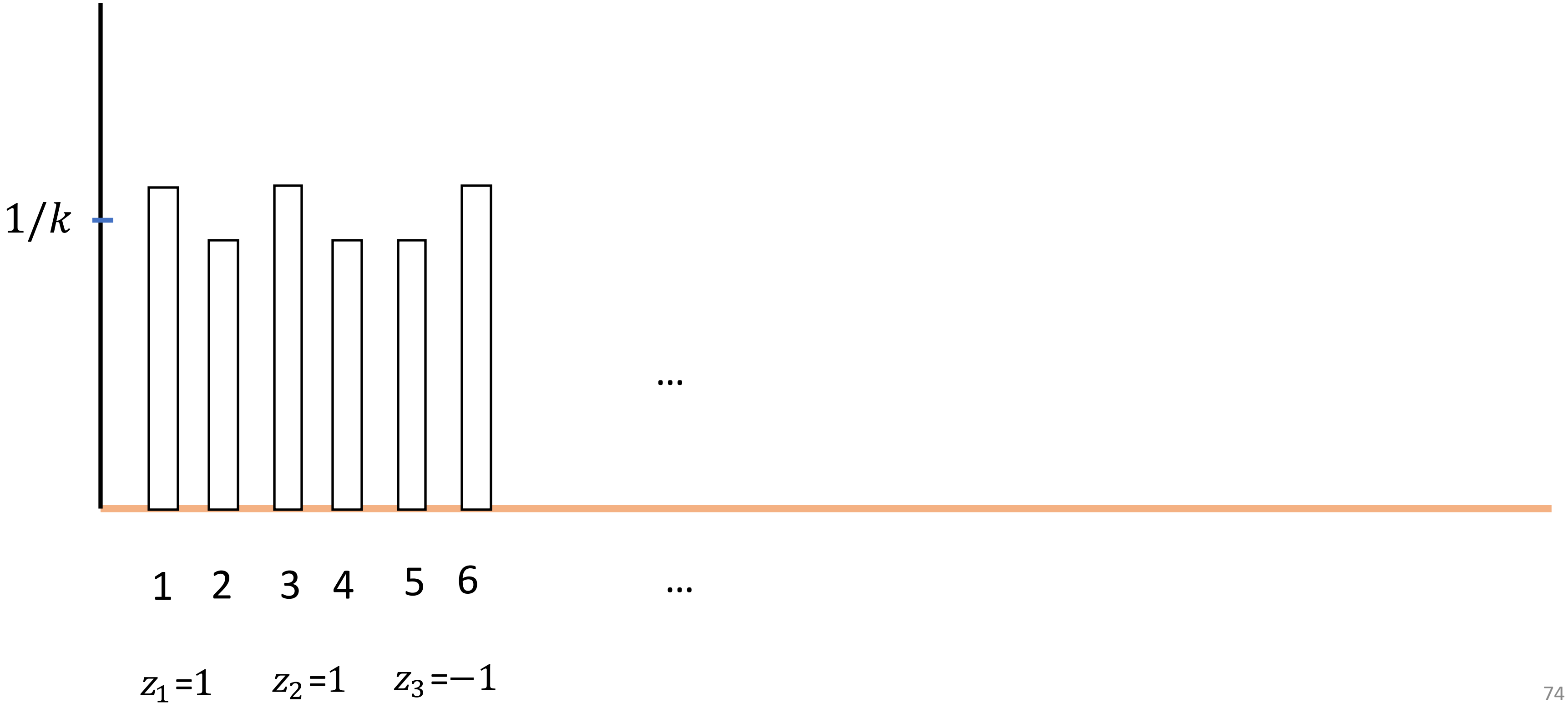


Recall

For $z \in \{-1,1\}^{k/2}$,

$$\mathbf{p}_z(2i-1) = \frac{1+z_i\varepsilon}{k}, \qquad \mathbf{p}_z(2i) = \frac{1-z_i\varepsilon}{k}, \qquad i = 1, \dots, k/2.$$

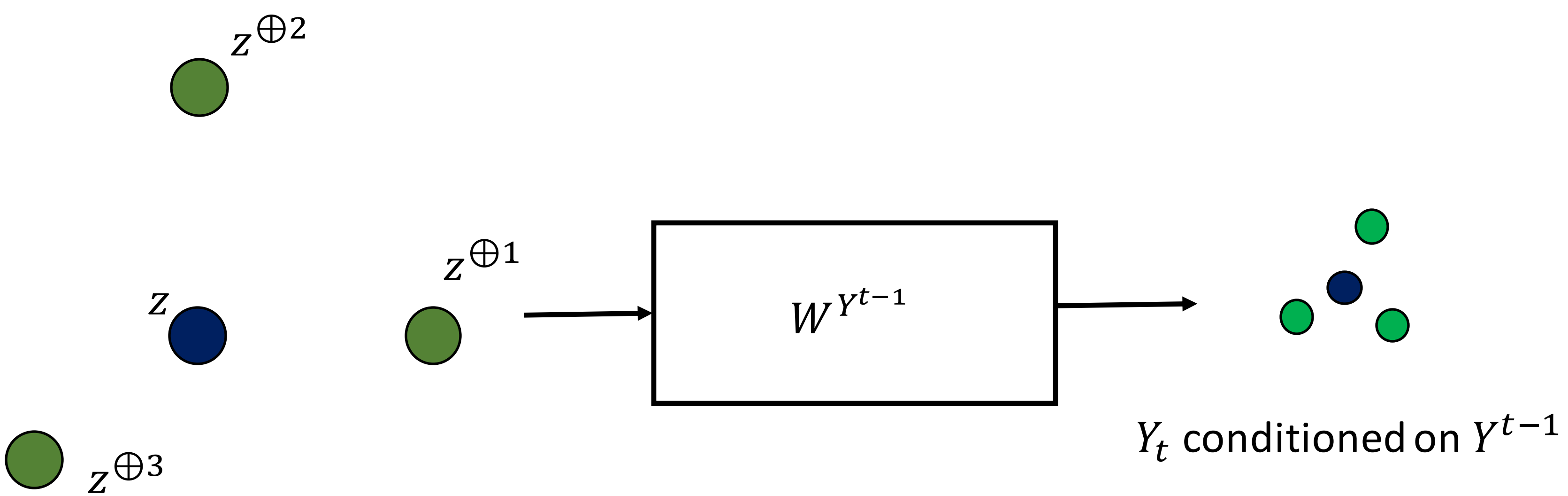
$\mathbf{p}_z$ and $\mathbf{p}_{z \oplus i}$ differ only on $2i-1$ and $2i$



Bounding $\sum_i D\left(\mathbf{p}_z^{Y_t|Y^{t-1}} \| \mathbf{p}_{z\oplus i}^{Y_t|Y^{t-1}}\right)$

- Fix Y^{t-1}

$$\mathbf{p}_z^{Y_t|Y^{t-1}}(y) = \mathbf{p}_{z\oplus i}^{Y_t|Y^{t-1}}(y) + \frac{2\varepsilon z_i}{k}\left(W^{Y^{t-1}}(y|2i-1) - W^{Y^{t-1}}(y|2i)\right)$$



Bounding $\sum_i D\left(\mathbf{p}_z^{Y_t|Y^{t-1}} \,||\, \mathbf{p}_{z\oplus i}^{Y_t|Y^{t-1}}\right)$

Since $\mathrm{KL} \leq \chi^2$, plugging the expression above

$$\sum_i D\left(\mathbf{p}_z^{Y_t|Y_{t-1}} \,||\, \mathbf{p}_{z\oplus i}^{Y_t|Y_{t-1}}\right) \leq \sum_i \sum_y \frac{\left(\mathbf{p}_z^{Y_t}(y) - \mathbf{p}_{z\oplus i}^{Y_t}(y)\right)^2}{\mathbf{p}_{z\oplus i}^{Y_t}(y)}$$

$$\leq \frac{8\varepsilon^2}{k} \cdot \sum_i \sum_y \frac{\left(W(y|2i-1) - W(y|2i)\right)^2}{\sum_x W(y|x)}$$

An average information bound

Theorem.

$$\sum_i I(Z_i \wedge Y^n) \le n \cdot \frac{8\varepsilon^2}{d} \cdot \sup_{W \in \mathcal{W}} \sum_i \sum_y \frac{\left(W(y|2i-1) - W(y|2i)\right)^2}{\sum_x W(y|x)}$$

Recall

$$\mathbf{p}_Z(2i-1) = \frac{1 + Z_i \varepsilon}{d}, \qquad \mathbf{p}_Z(2i) = \frac{1 - Z_i \varepsilon}{d}$$

$|W(y|2i-1) - W(y|2i)|$ large $\Leftrightarrow$ seeing y tells about input
$\Leftrightarrow$ tells about Z_i

An average information bound

Theorem. [ACLST20] Under any **interactive protocol**,

$$\sum_i I(Z_i \wedge Y^n) \leq n \cdot \frac{8\varepsilon^2}{k} \cdot \sup_{W \in \mathcal{W}} \sum_i \sum_y \frac{\big(W(y|2i-1) - W(y|2i)\big)^2}{\sum_x W(y|x)}$$

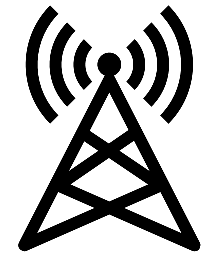
Theorem. If there exists an estimator then

$$\frac{d}{20} \leq n \cdot \frac{8\varepsilon^2}{k} \cdot \sup_{W \in \mathcal{W}} \sum_i \sum_y \frac{\big(W(y|2i-1) - W(y|2i)\big)^2}{\sum_x W(y|x)}$$

Applications

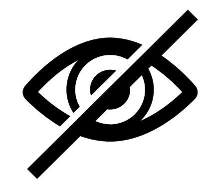
For any $W \in \mathcal{W}_\ell$

$$\sum_i \sum_y \frac{\left(W(y|2i-1) - W(y|2i)\right)^2}{\sum_x W(y|x)} \leq 2^\ell$$



For any $W \in \mathcal{W}_\varrho, \varrho \leq 1$

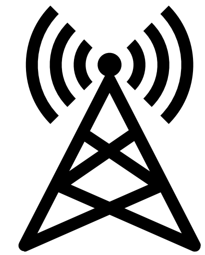
$$\sum_i \sum_y \frac{\left(W(y|2i-1) - W(y|2i)\right)^2}{\sum_x W(y|x)} = O(\varrho^2)$$



Interactive lower bound for estimation

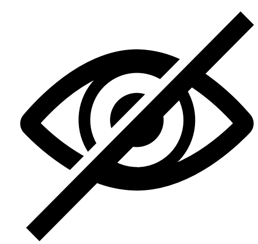
$$\frac{d}{20} \leq n \cdot \frac{8\varepsilon^2}{d} \cdot 2^{\ell}$$

$$n = \Omega\left(\frac{d^2}{2^{\ell}\varepsilon^2}\right)$$



$$\frac{d}{20} \leq n \cdot \frac{8\varepsilon^2}{d} \cdot \varrho^2$$

$$n = \Omega\left(\frac{d^2}{\varepsilon^2\varrho^2}\right)$$



Plug-n-play bounds

$H(W)$ is a $\frac{d}{2} \times \frac{d}{2}$ PSD matrix:

$$\big(H(W)\big)_{ij} :=$$
$$\sum_{y \in Y} \frac{\big(W(y|2i-1) - W(y|2i)\big)\big(W(y|2j-1) - W(y|2j)\big)}{\sum_j W(y|j)}$$

$$\sum_i \sum_y \frac{\big(W(y|2i-1) - W(y|2i)\big)^2}{\sum_x W(y|x)} = \| H(W) \|_*$$

Plug-n-play bounds

$$\| \mathcal{W} \| \stackrel{\text{def}}{=} \max_{W \in \mathcal{W}} \| H(W) \|$$

Testing:

Classic	Private-coin SMP	Public-coin SMP	Sequentially Interactive
$\Omega\left(\frac{\sqrt{d}}{\varepsilon^2}\right)$	$\Omega\left(\frac{d^{3/2}}{\varepsilon^2 \| \mathcal{W} \|_*}\right)$	$\Omega\left(\frac{d}{\varepsilon^2 \| \mathcal{W} \|_{\mathrm{F}}}\right)$	$\Omega\left(\frac{d}{\varepsilon^2 \sqrt{\| \mathcal{W} \|_{OP} \| \mathcal{W} \|_*}}\right)$

Estimation

Classic	Sequentially Interactive
$\Omega\left(\frac{d}{\varepsilon^2}\right)$	$\Omega\left(\frac{d^2}{\varepsilon^2 \| \mathcal{W} \|_*}\right)$

Next 45 minutes:

Reinforcement Learning by Himanshu Tyagi …

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Some references and previous work

Some references and previous work

(not in that order)

Some references and previous work

Too many for a single slide, or two. Starts, more or less, with Tsitsiklis'89, picks up again in the mid-2000's with a slightly different focus: local privacy, various types of communication constraints, ML-related motivations…

For a detailed bibliography:

www.cs.columbia.edu/~ccanonne/tutorial-focs2020/bibliography.html

THE END >>>>