COMPx270: Randomised and Advanced Algorithms
Lecture 9: Streaming and Sketching II

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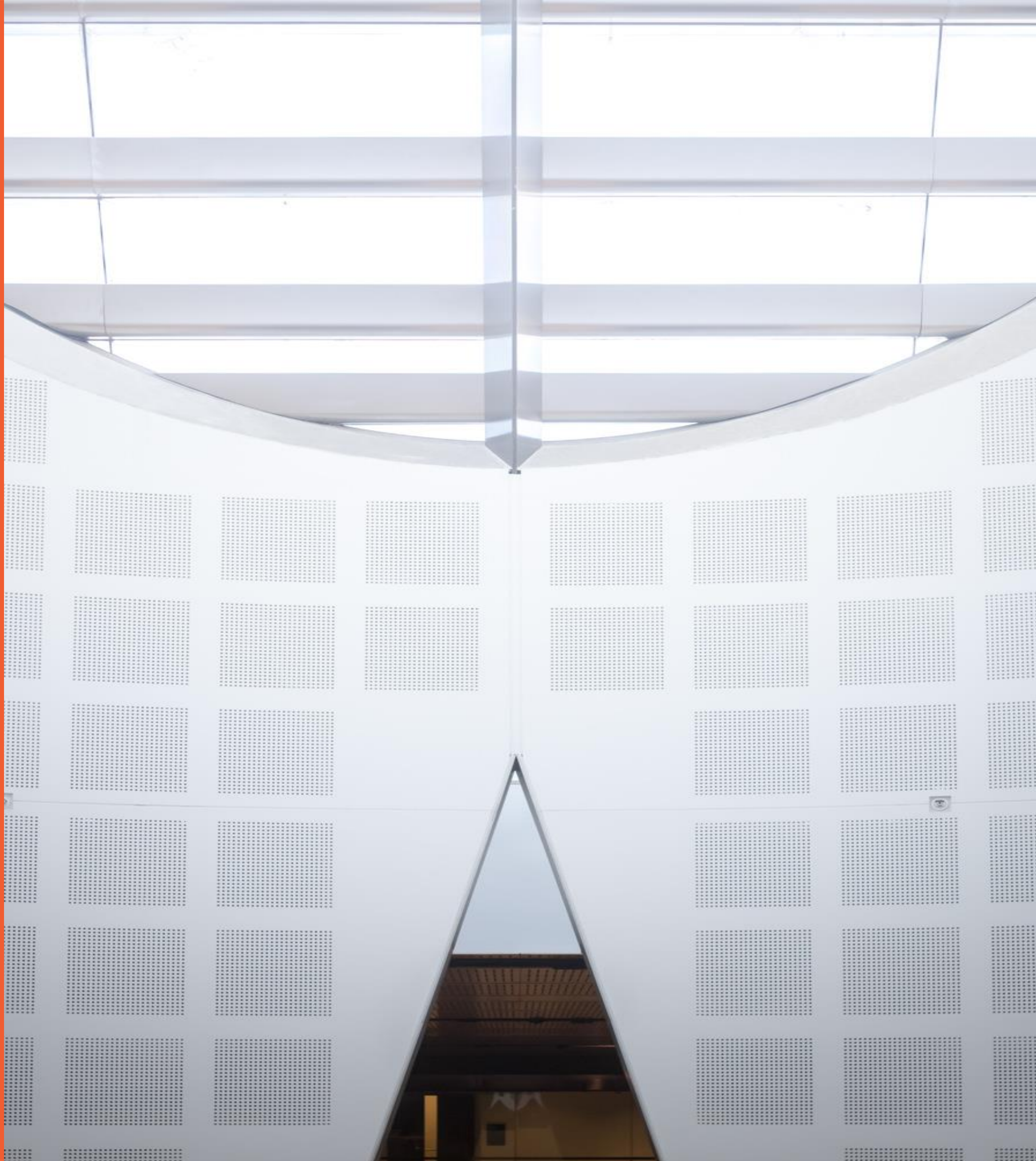

A question 🪡

You design a streaming algorithm A to solve some problem. The stream of data arrives.

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628, 516, 163, 509, 15, 499, 772, 588, 737, 439, 79, 866, 186, 18, 854, 459, 146, 518, 748, 737, 685, 188, 939, 724, 27, 719, 263, 795, 120, 573, 853, 132, 522, 3, 298, 123, 932, 993, 180, 674, 1, 619, 989, 142, 496, 178, 191, 524, 716, 501, 677, 712, 452, 768, 591, 551, 439, 397, 229, 214, 43, 639, 353, 610, 737, 203, 933, 279, 877, 30, 513, 518, 616, 714, 633, 804, 422, 731, 867, 184, 124, 881, 595, 193, 254, 240, 4, 260, 303, 319, 757, 723, 309, 365, 278, 512, 658, 233, 393, 875

A question 🪡

You design a streaming algorithm A to solve some problem. The stream of data arrives. A outputs its answer:

21.5

A question 🪡

Oh, no! That wasn't the end. More data arrives.

A question

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834, 992, 528, 12, 181, 274, 159, 150, 716, 71, 755, 4, 324, 398, 802, 176, 302, 941, 678, 934, 546, 753, 812, 47, 755, 721, 893, 53, 410

A question 🪡

Oh, no! That wasn't the end. More data arrives. A outputs its answer on this:

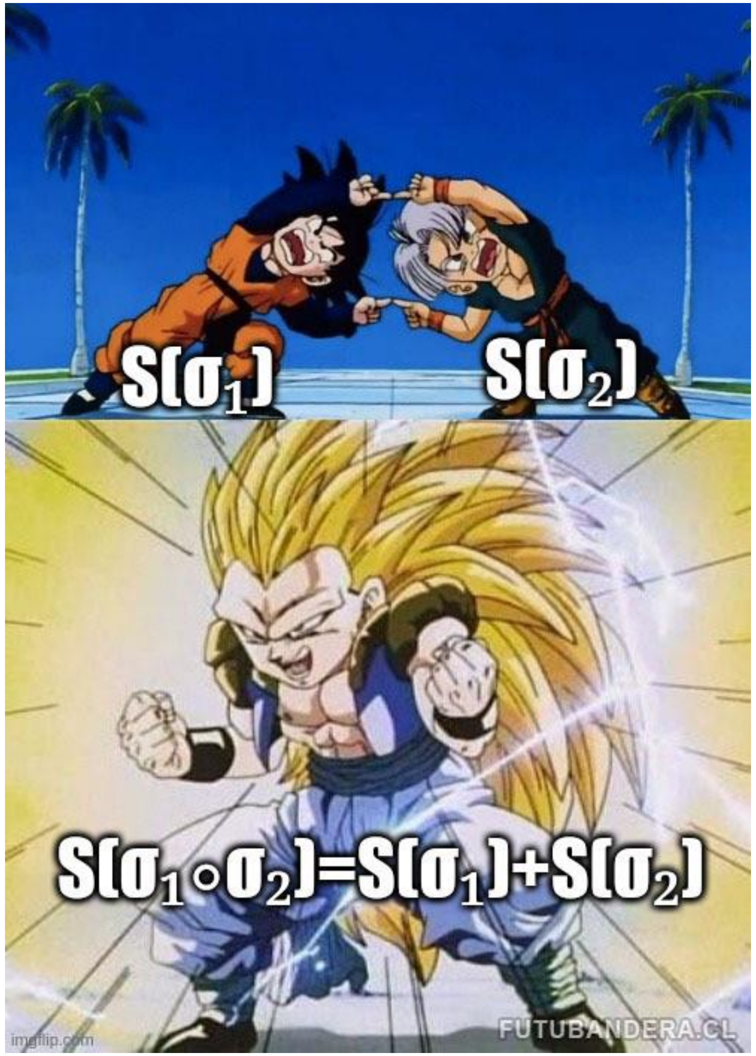
18.1

A question 🪡

How do you combine 21.5 and 18.1 to get the answer on the whole data stream?

Sketching

Even better: linear sketching



Even better: linear sketching

Frequent Elements (Heavy Hitters)

Theorem 39. *The* MISRA-GRIES *algorithm is a* deterministic *one-pass algorithm which, for any given parameter* $\varepsilon \in (0, 1]$, *provides* $\hat{f}_1, \ldots, \hat{f}_n$ *of all element frequencies such that*

$$f_j - \varepsilon m \leq \hat{f}_j \leq f_j, \qquad j \in [n]$$

with space complexity $s = O(\log(mn)/\varepsilon)$. *(In particular, it can be used to solve the* MAJORITY *problem in two passes.)*

Frequent Elements (Heavy Hitters): ℓ_1, ℓ_2, etc.

Frequent Elements (Heavy Hitters): CountSketch (1/5)

Input: Parameters $\varepsilon, \delta \in (0, 1]$

Set $k \leftarrow O(1/\varepsilon^2)$, and initialize an array C of size k to zero

Pick $h : [n] \to [k]$ from a strongly universal hashing family

Pick $g : [n] \to \{-1, 1\}$ from a strongly universal hashing family

for all $1 \le i \le m$ **do**

  Get item $a_i = (j, c) \in [n] \times \{-B, \ldots, B\}$ ▷ Assume $B = O(1)$

  $C[h(j)] \leftarrow C[h(j)] + c \cdot g(j)$

Output: On query $j \in [n]$, **return** $\widehat{f}_j \leftarrow g(j) \cdot C[h(j)]$

Frequent Elements (Heavy Hitters): CountSketch (2/5)

Input: Parameters $\varepsilon, \delta \in (0,1]$

Set $k \leftarrow O(1/\varepsilon^2)$, and initialize an array C of size k to zero

Pick $h\colon [n] \to [k]$ from a strongly universal hashing family

Pick $g\colon [n] \to \{-1,1\}$ from a strongly universal hashing family

for all $1 \le i \le m$ **do**

  Get item $a_i = (j,c) \in [n] \times \{-B, \ldots, B\}$ $\triangleright$ Assume $B = O(1)$

  $C[h(j)] \leftarrow C[h(j)] + c \cdot g(j)$

Output: On query $j \in [n]$, **return** $\widehat{f}_j \leftarrow g(j) \cdot C[h(j)]$

Frequent Elements (Heavy Hitters): CountSketch (3/5)

Input: Parameters $\varepsilon, \delta \in (0, 1]$

Set $k \leftarrow O(1/\varepsilon^2)$, and initialize an array C of size k to zero
Pick $h\colon [n] \to [k]$ from a strongly universal hashing family
Pick $g\colon [n] \to \{-1, 1\}$ from a strongly universal hashing family
for all $1 \leq i \leq m$ **do**
  Get item $a_i = (j, c) \in [n] \times \{-B, \dots, B\}$ ▷ Assume $B = O(1)$
  $C[h(j)] \leftarrow C[h(j)] + c \cdot g(j)$

Output: On query $j \in [n]$, **return** $\widehat{f}_j \leftarrow g(j) \cdot C[h(j)]$

Frequent Elements (Heavy Hitters): CountSketch (3/5)

Input: Parameters $\varepsilon, \delta \in (0,1]$

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Output: On query $j \in [n]$, **return** $\widehat{f}_j \leftarrow g(j) \cdot C[h(j)]$

Frequent Elements (Heavy Hitters): CountSketch (5/5)

Theorem 44. *The (median trick version of the)* CountSketch *algorithm is a* randomised *one-pass sketching algorithm which, for any given parameters* $\varepsilon, \delta \in (0, 1]$, *provides a (succinctly represented) estimate* $\widehat{f}$ *of frequency vector* f *of the stream such that, for every* $j \in [n]$

$$\Pr\left[\left| \widehat{f}_j - f_j \right| \leq \varepsilon \|f_{-j}\|_2 \right] \geq 1 - \delta$$

with space complexity

$$s = O\left(\frac{\log(nm)}{\varepsilon^2} \log \frac{1}{\delta} \right).$$

Frequent Elements (Heavy Hitters): CountMinSketch (1/4)

Input: Parameters $\varepsilon, \delta \in (0,1]$

Set $k \leftarrow O(1/\varepsilon)$ and $T \leftarrow O(\log(1/\delta))$, and initialize a two-dimensional array C of size $T \times k$ to zero

Pick $h_1, \ldots, h_T : [n] \rightarrow [k]$ independently from a strongly universal hashing family

for all $1 \leq i \leq m$ **do**

    Get item $a_i = (j, c) \in [n] \times \{0, \ldots, B\}$ $\triangleright$ Assume $B = O(1)$

    for all $1 \leq t \leq T$ **do**

        $C[t][h_t(j)] \leftarrow C[t][h_t(j)] + c$

Output: On query $j \in [n]$, **return** $\widehat{f}_j \leftarrow \min_{1 \leq t \leq T} C[t][h_t(j)]$

Frequent Elements (Heavy Hitters): CountMinSketch (2/4)

Input: Parameters $\varepsilon, \delta \in (0,1]$

Set $k \leftarrow O(1/\varepsilon)$ and $T \leftarrow O(\log(1/\delta))$, and initialize a two-dimensional array C of size $T \times k$ to zero

Pick $h_1, \dots, h_T : [n] \to [k]$ independently from a strongly universal hashing family

for all $1 \le i \le m$ **do**

  Get item $a_i = (j, c) \in [n] \times \{0, \dots, B\}$ ▷ Assume $B = O(1)$

  for all $1 \le t \le T$ **do**

    $C[t][h_t(j)] \leftarrow C[t][h_t(j)] + c$

Output: On query $j \in [n]$, **return** $\widehat{f}_j \leftarrow \min_{1 \le t \le T} C[t][h_t(j)]$

Frequent Elements (Heavy Hitters): CountMinSketch (3/4)

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  for all $1 \leq t \leq T$ **do**

    $C[t][h_t(j)] \leftarrow C[t][h_t(j)] + c$

Output: On query $j \in [n]$, **return** $\widehat{f}_j \leftarrow \min_{1 \leq t \leq T} C[t][h_t(j)]$

Frequent Elements (Heavy Hitters): CountMinSketch (4/4)

Theorem 45. *The* COUNTMINSKETCH *algorithm is a* randomised *one-pass sketching algorithm which, for any given parameters $\varepsilon, \delta \in (0, 1]$, provides a (succinctly represented) estimate $\widehat{f}$ of frequency vector f of the stream such that, for every $j \in [n]$*

$$\Pr\left[\left| \widehat{f}_j - f_j \right| \leq \varepsilon \|f_{-j}\|_1 \right] \geq 1 - \delta$$

with space complexity

$$s = O\left(\frac{\log(nm)}{\varepsilon} \log \frac{1}{\delta} \right).$$

(Moreover, $\widehat{f}_j$ is always an overestimate: $\widehat{f}_j \geq f_j$ for all $j \in [n]$.)

Wait a minute…

This seems strictly worse than Misra-Gries!

- Randomised instead of deterministic!
- Uses more space!
- Also in the cash register model!
- Also an ℓ_1 guarantee!

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– Also an ℓ_1 guarantee!

Yes, but:

– Linear sketch!
– Much faster per time step!
– Can be extended to the strict turnstile model!

https://sites.google.com/site/countminsketch/

Recap