Testing probability distributions underlying aggregated data

"Please, sir, I want some more."

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When? July 11th, 2014

Plan of the talk

Introduction: distribution testing

Two new models: Dual and Cumulative Dual access

Spoiler: the results

Main techniques

Background and motivation

Linear is the new exponential.

"Recently there has been a lot of glorious hullabaloo about Big Data and how it is going to revolutionize the way we work, play, eat and sleep." (R. Servedio)

Background and motivation

What is distribution testing?

Property testing

Big, hidden "object" X only accessible by local, expensive inspections (queries), and property $\mathcal{P}$: check in sublinear number of queries if (a) X has the property or (b) X is "far" from all objects having it.

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Testing distributions (standard model)

X is an unknown probability distribution D over some n-element set; the testing algorithm has blackbox sample access to D.

Distribution testing (1)

In more details.

Distance: total variation distance ($\propto \ell_1$). ORACLE_D: type of access to D (e.g. sampling).

Definition (Tester)

Tester for property $\mathcal{P}$: algorithm T which is given ε, n, makes $q(\varepsilon, n)$ calls to ORACLE_D, and:

- if $D \in \mathcal{P}$ then w.h.p. T outputs Yes;
- if $d_{\mathrm{TV}}(D, \mathcal{P}) \geq \varepsilon$ then w.h.p. T outputs No.

Distribution testing (1)

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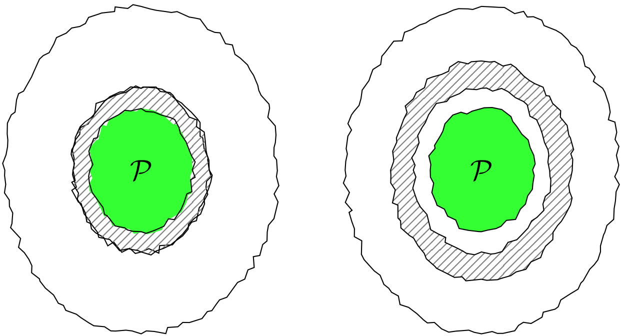
Definition (*Tolerant* tester)

Tolerant tester for property $\mathcal{P}$: algorithm T which is given $\varepsilon_1, \varepsilon_2, n$, makes $q(\varepsilon_1, \varepsilon_2, , n)$ calls to ORACLE_D, and:

- if $d_{\mathrm{TV}}(D, \mathcal{P}) \leq \varepsilon_1$ then w.h.p. T outputs Yes;
- if $d_{\mathrm{TV}}(D, \mathcal{P}) \geq \varepsilon_2$ then w.h.p. T outputs No.

Distribution testing (2)

Testing vs. Tolerant testing, in an egg-shell.



Distribution testing (3)

Comments

A few remarks

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A few remarks

- "gray" area for $d_{\mathrm{TV}}(D, \mathcal{P}) \in (0, \varepsilon)$
- tolerant testing usually much harder than testing.
- focuses on the sample complexity (*not* the runtime).

Distribution testing (4)

Concrete example: testing uniformity

General outline

1. Draw a bunch of samples from D;
2. "Process" them (e.g. count the number of points seen more than once *(collisions)*);
3. Compare to what the uniform distribution would give;
4. Reject if it differs too much; accept otherwise.

Background and motivation

So what is the problem with that?

Fact *In the standard sampling model, most (natural) properties are "hard" to test; that is, require a strong dependence on n (at least $\Omega(\sqrt{n})$).*

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Example Testing *uniformity* has $\Theta(\sqrt{n}/\varepsilon^2)$ sample complexity [GR00, BFR+10, Pan08], *equivalence to a known distribution* $\Theta(\sqrt{n}/\varepsilon^2)$ [BFF+01, Pan08]; *equivalence of two unknown distributions* $\Omega(n^{2/3})$ [BFR+10, Val11, CDVV14] (essentially tight)...

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and more depressing for tolerant testing: $\Omega\left(n^{1-o(1)}\right)$ for entropy, support size... even for uniformity! [VV11, VV10a]

Background and motivation

Bypassing the lower bounds: changing the adversary

First idea — Focusing on subclasses of distributions: structure may help!

Shape: monotone distributions, k-modal, log-concave. . .

Mixtures: Gaussian mixtures, Poisson Binomial Distributions, Sums of Independent Integer R.V.s. . .

([BKR04, DDS$^+$13], [DDS12, DDO$^+$13] (learning). . .)

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COND — can ask for samples *conditioned on a subset* $S \subseteq [n]$
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This work — can sample from D and *query* it: have either PMF (probability mass function) or CDF (cumulative distribution function) access.

Definition (Dual oracle)

Fix a distribution D over $[n]$. A dual oracle for D is a pair of oracles $(\mathrm{SAMP}_D, \mathrm{EVAL}_D)$:

- *sampling* oracle SAMP_D returns $i \in [n]$ drawn from D;
- *evaluation* oracle EVAL_D takes $j \in [n]$, and returns $D(j)$.

Definition (Cumulative Dual oracle)

A cumulative dual oracle for D is a pair of oracles $(\mathrm{SAMP}_D, \mathrm{CEVAL}_D)$:

- *sampling* oracle SAMP_D as above;
- *cumulative evaluation* oracle CEVAL_D takes $j \in [n]$, and returns $D([j]) = \sum_{i=1}^{j} D(i)$.

Dual and Cumulative Dual access models

A couple remarks

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- Key point

  SAMP: can get $D(i) \pm \varepsilon$
  COND: can get $(1 \pm \varepsilon)D(i)$
  here: can get $D(i)$

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  here: can get $D(i)$
- *How to motivate such a model?*

Dual and Cumulative Dual access models

Is that even a thing?

Broke Arthur & Greedy Merlin

- Free but huge dataset out there
- Long and expensive analysis of it held by Merlin
- Computationally limited Arthur working on the data

Dual and Cumulative Dual access models

Is that even a thing?

Someone did the work!

- Google n-gram data: pdf for sequences of n words + samples of sequences
- Sorted files: samples in $O(1)$ time, cdf and pdf queries in $O(\log n)$

Dual and Cumulative Dual access models

Is that even a thing?

. . . and more.

- Connection between dual model and datastream algorithms [GMV05]
- Further understanding of distribution testing (*what* is hard in it, and *why*?)

Our results

(and comparison with the original sampling model)

<table>
<tr><th>Problem</th><th>SAMP</th><th>Dual</th><th>Cumulative Dual</th></tr>
<tr><td>Testing uniformity</td><td>$\Theta\left(\frac{\sqrt{n}}{\varepsilon^2}\right)$</td><td rowspan="3">$\Theta\left(\frac{1}{\varepsilon}\right)$</td><td rowspan="3">$\Theta\left(\frac{1}{\varepsilon}\right)$</td></tr>
<tr><td>Testing $\equiv D^*$</td><td>$\Theta \frac{\sqrt{n}}{\varepsilon^2}$</td></tr>
<tr><td>Testing $D_1 \equiv D_2$</td><td>$\Theta\left(\max\left(\frac{n^{2/3}}{\varepsilon^{4/3}}, \frac{\sqrt{n}}{\varepsilon^2}\right)\right)$</td></tr>
<tr><td>Tolerant uniformity</td><td>$O\left(\frac{1}{(\varepsilon_2-\varepsilon_1)^2}\frac{n}{\log n}\right)$
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<tr><td>Tolerant D_1, D_2</td></tr>
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<tr><td>Estimating support size to $\pm\varepsilon n$</td><td>$\Theta\left(\frac{n}{\log n}\right)$</td><td>$\Theta\left(\frac{1}{\varepsilon^2}\right)$</td><td>$O\left(\frac{1}{\varepsilon^2}\right)$</td></tr>
</table>

Techniques (1)

Upper bounds: Hey, we've got a hammer!

Main technique

With Dual access: rewrite the quantity to estimate as

$$\mathbb{E}_{i\sim D}\left[\Phi(i, D(i))\right]$$

for *bounded* Φ.

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Examples Entropy, support size, distance to D^* or D_2...

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Examples

Entropy, support size, distance to D^* or D_2...

$$H(D) = -\sum_{i\in[n]} D(i)\log D(i) = -\mathbb{E}_{i\sim D}\left[\log D(i)\right]$$

Techniques (2)

Lower bounds: if I had a hammer…

Fact *To distinguish between D^+ and D^- with constant probability, any* SAMP *algorithm needs*

$$\Omega\left(\frac{1}{\mathrm{d}_{\mathrm{TV}}(D^+, D^-)}\right)$$

samples.

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Separation

Is Cumulative Dual any better?

Question Do we have $(\mathsf{SAMP}, \mathsf{EVAL}) \not\preceq (\mathsf{SAMP}, \mathsf{CEVAL})$?

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Answer Yes: for entropy of monotone distributions.

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Answer: Yes: for entropy of close to monotone distributions.

Conclusion

- Two new models for studying distributions
- Significant savings for property testing
- A general technique to get upper bounds with dual access

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- Significant savings for property testing
- A general technique to get upper bounds with dual access

- Stronger separation between dual and cumulative dual oracles?
- More lower bounds for cumulative dual?
- What about other properties? (monotonicity (†), log-concavity. . .)
- What about learning? What about a "Lower Bound Hammer"?

Thank you.

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