The Price of Tolerance in Distribution Testing

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WALD(O) 2021

(Joint work with Ayush Jain (UCSD), Gautam Kamath (U Waterloo), and Jerry Li (MSR))

arXiv:2106.13414

Distribution Testing?

Given m i.i.d. samples from some unknown p over $[n]$, a reference q, and distance parameter ε, distinguish between

- Completeness: $p = q$
- Soundness: $TV(p,q) \geq \varepsilon$

with probability at least 1/10, where

$$TV(p,q) := \sup(p(S) - q(S)) = \frac{1}{2} \ell_1(p,q)$$

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Identity testing/one-sample goodness-of-fit

[GR00, BFFKRW01, Pan08, VV14, DKN15, ADK15, DGPP18...]

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$$m = \Theta(\sqrt{n}/\varepsilon^2)$$

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Given m i.i.d. samples from some unknown p over $[n]$, a reference q, and distance parameter ε, distinguish between

- Completeness: $TV(p,q) = 0$
- Soundness: $TV(p,q) \geq \varepsilon$

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Distribution Testing?

Given m i.i.d. samples from some unknown p over $[n]$, a reference q, and distance parameters ε_1, ε_2, distinguish between

- Completeness: $TV(p,q) \leq \varepsilon_1$
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Tolerant identity testing

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$O(n/((\varepsilon_2-\varepsilon_1)^2 \log n))$

$\Theta(n/\log n)$ for constant $\varepsilon_1 < \varepsilon_2$

$\Omega(n/((\varepsilon_2-\varepsilon_1)^2 \log n))$ for distance estimation

$\Omega(\sqrt{n}/\varepsilon_2^2)$ always

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+ Analogous statement for tolerant **closeness** testing

Some remarks

New result even in the "low tolerance" regime: Cauchy–Schwarz wasn't tight!

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$\Theta(\sqrt{n}/\varepsilon_2^2)$ for $\varepsilon_1 \leq \varepsilon_2/\sqrt{n}$ $\leadsto$ $\Theta(\sqrt{n}/\varepsilon_2^2)$ for $\varepsilon_1 \leq \mathbf{1/\sqrt{n}}$

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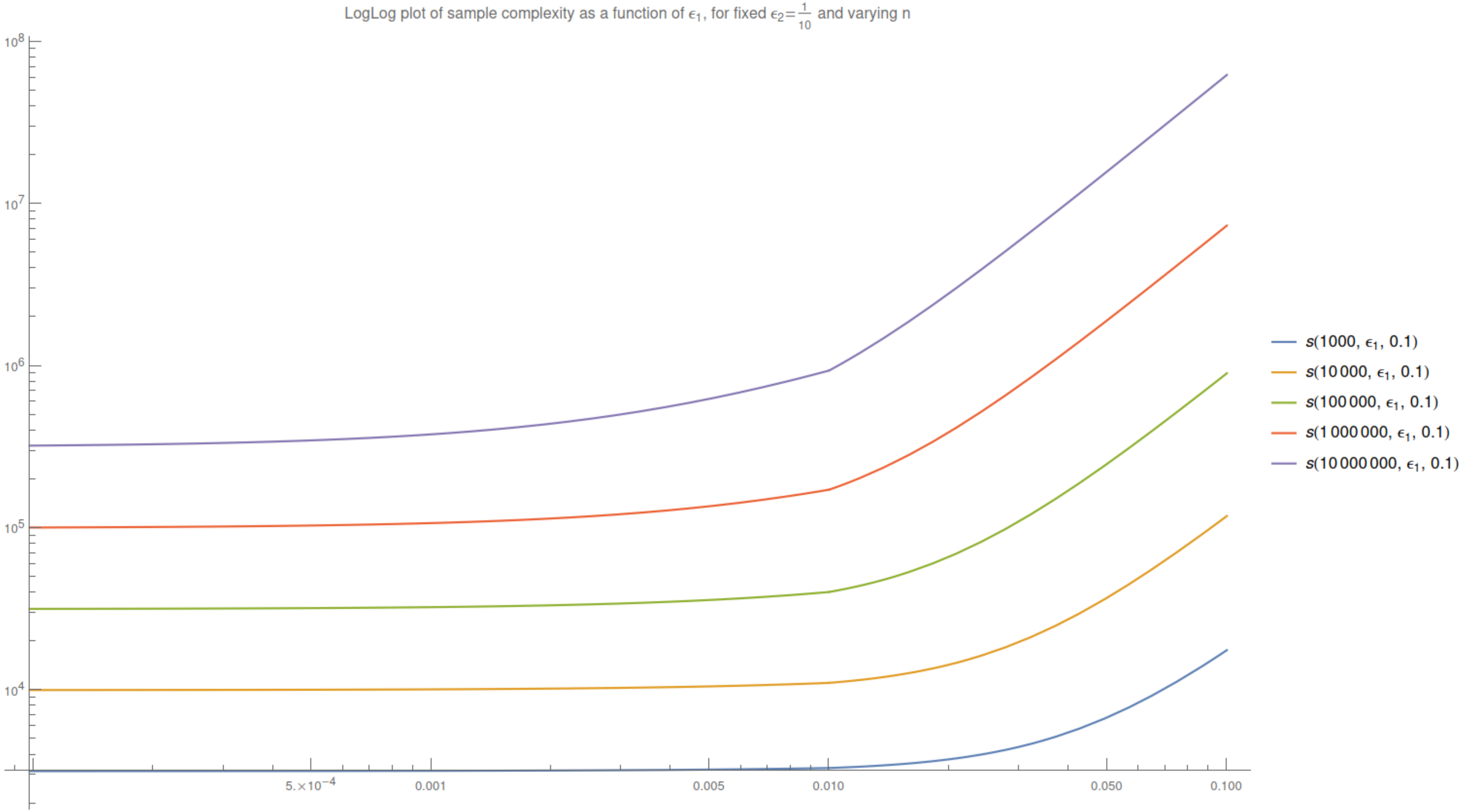

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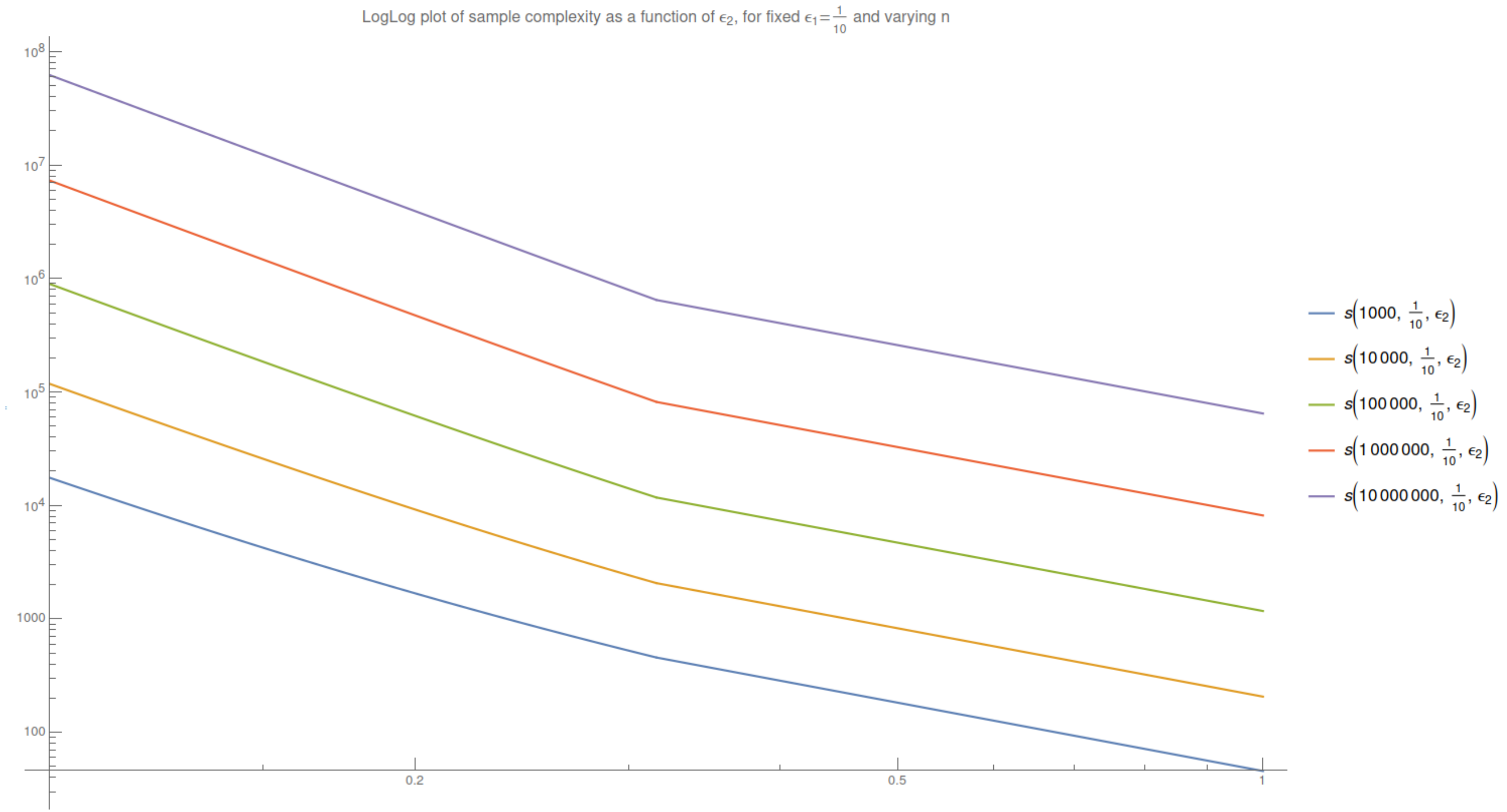

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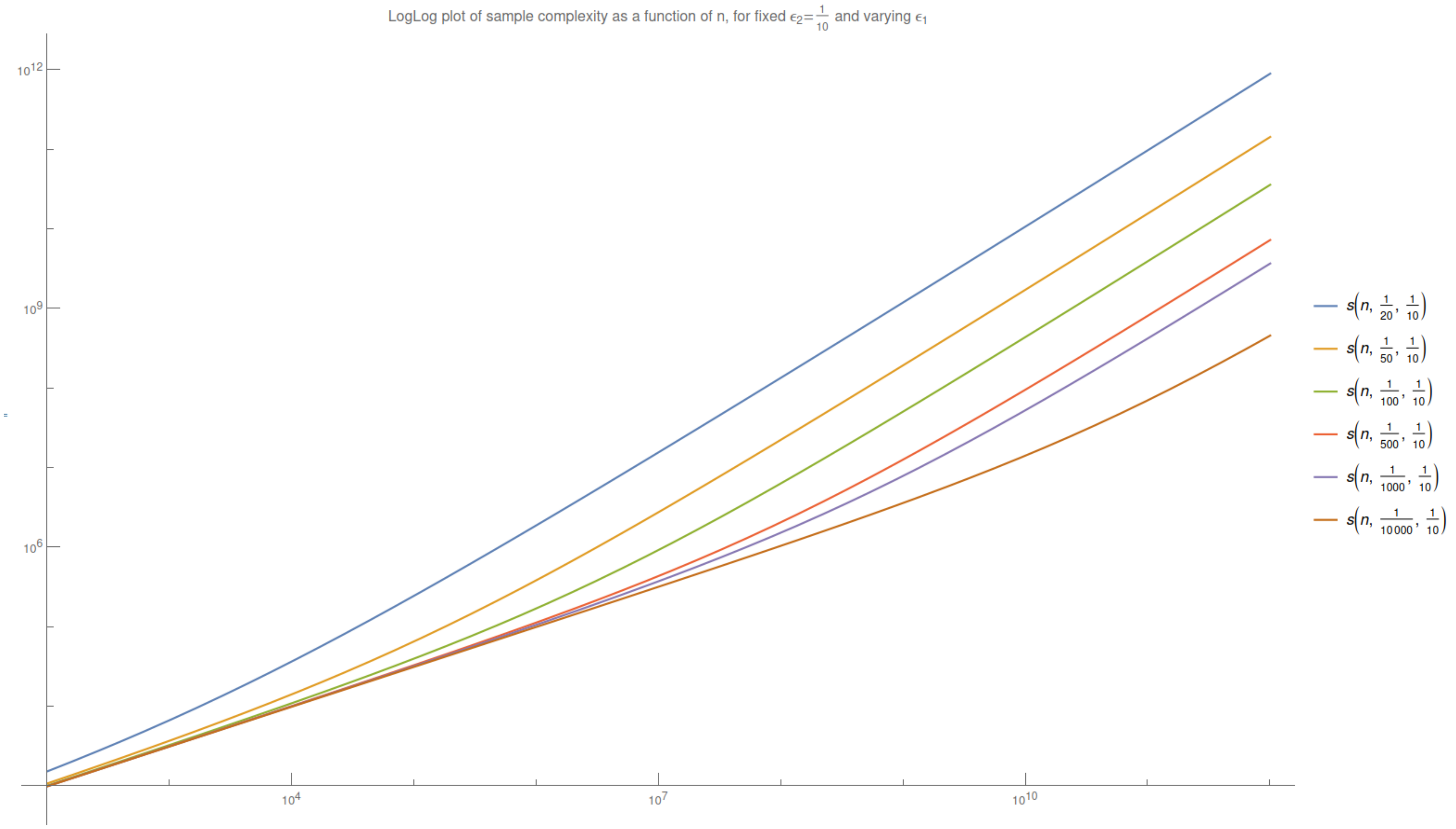

More remarks

Can use our results on tolerant **identity** and **closeness** to obtain tolerant testing for/under **structure**:

- Monotonicity, unimodality, log-concavity, independence
- Under structural assumptions ("A_k distances")
- Instance-optimal tolerant testing
- (insert favourite property here)

A glimpse of our techniques 👀

Upper bounds via carefully **rescaled** χ^2/ℓ_2-type tester

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Algorithm 1 Tolerant testing algorithm.

Require: $0 \leq \varepsilon_1 < \varepsilon_2 \leq 1$, m, n, two sets of $\mathrm{Poi}(m)$ samples from both p and q

Set the threshold

$$\tau \leftarrow c \cdot \min\left(\frac{m^{3/2}\varepsilon_2}{n^{1/2}}, \frac{m^2\varepsilon_2^2}{n}\right)$$

Compute Z from the sets of samples, as per (1).

if $Z \geq \tau$ **then return** $\|p-q\|_1 \geq \varepsilon_2$

else return $\|p-q\|_1 \leq \varepsilon_1$

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Does not require knowledge of ε_1!

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- Write this as an LP ✍: need to lower bound its optimal value
- Massage this LP, take the dual 🪄
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Recap

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