ALICE AND BOB SHOW DISTRIBUTION TESTING LOWER BOUNDS

They don’t talk to each other anymore.

Clément Canonne (Columbia University)

December 8, 2016

Joint work with **Eric Blais** (UWaterloo) and **Tom Gur** (Weizmann Institute)

“DISTRIBUTION TESTING?”

Property testing of probability distributions:

WHY?

Property testing of probability distributions: sublinear, Property testing of probability distributions: sublinear, approximate,

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Property testing of probability distributions: sublinear, approximate, randomized algorithms that take random samples

- Big Dataset: too big
- Expensive access: pricey data
- “Model selection”: many options

Need to infer information – one bit – from the data: fast, or with very few samples.


12
22
5
18
26
49
30

(Property) Distribution Testing:

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in an (egg)shell.

HOW?

Known domain (here $[n] = \{1, \ldots, n\}$)

Property $\mathcal{P} \subseteq \Delta([n])$

Independent samples from unknown $p \in \Delta([n])$

Distance parameter $\varepsilon \in (0, 1]$

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Must decide:

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(and be correct on any p with probability at least 2/3)

AND?

Many results on many properties:

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- Uniformity [GR00, BFR$^+$00, Pan08]

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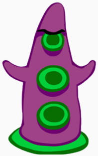

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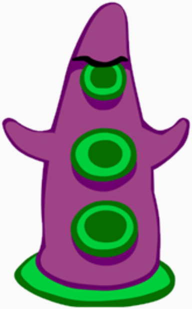

BUT?

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“COMMUNICATION COMPLEXITY?”

WHAT NOW?

$$f(x, y)$$

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But communicating is hard.

- f known by all parties
- Alice gets x, Bob gets y
- Private randomness

Goal: minimize communication (worst case over x, y, randomness) to compute $f(x, y)$.

ALSO...

...in our setting, Alice and Bob do not get to communicate.

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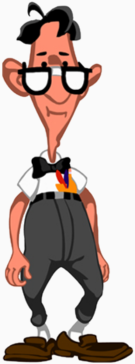
SMP

Simultaneous Message Passing model.

REFEREE MODEL (SMP).



Upshot

$$\mathrm{SMP}(\mathrm{EQ}_n) = \Omega(\sqrt{n})$$

(Only $O(\log n)$ with one-way communication!)

WELL, SURE, BUT WHY?

- Introduced by Blais, Brody, and Matulef [BBM12] for Boolean functions
- Elegant reductions, generic framework
- Carry over very strong communication complexity lower bounds

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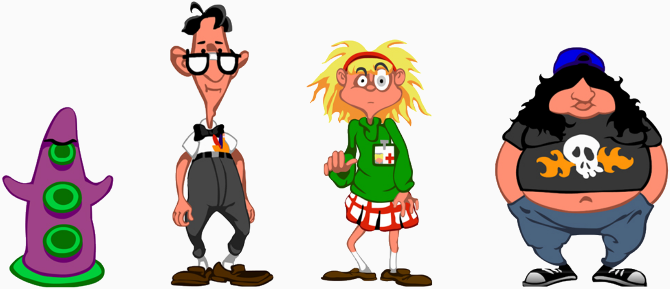
Can we...

... have the same for distribution testing?

DISTRIBUTION TESTING VIA COMM. COMPL.

THE TITLE SHOULD MAKE SENSE NOW.



1. The general methodology.
2. Application: testing uniformity, and the struggle for EQUALITY
3. Testing identity, an unexpected connection
   - The [VV14] result and the 2/3-pseudonorm
   - Our reduction, p-weighted codes, and the K-functional
   - Wait, what is this thing?
4. Conclusion

Theorem

Let $\varepsilon > 0$, and let Ω be a domain of size n. Fix a property $\Pi \subseteq \Delta(\Omega)$ and $f\colon \{0,1\}^k \times \{0,1\}^k \to \{0,1\}$. Suppose there exists a mapping $p\colon \{0,1\}^k \times \{0,1\}^k \to \Delta(\Omega)$ that satisfies the following conditions.

1. Decomposability: $\forall x, y \in \{0,1\}^k$, there exist $\alpha = \alpha(x), \beta = \beta(y) \in [0,1]$ and $p_A(x), p_B(y) \in \Delta(\Omega)$ such that
$$p(x,y) = \frac{\alpha}{\alpha+\beta} \cdot p_A(x) + \frac{\beta}{\alpha+\beta} \cdot p_B(y)$$
and α, β can each be encoded with $O(\log n)$ bits.
2. Completeness: For every $(x,y) = f^{-1}(1)$, it holds that $p(x,y) \in \Pi$.
3. Soundness: For every $(x,y) = f^{-1}(0)$, it holds that $p(x,y)$ is ε-far from Π in ℓ_1 distance.

Then, every ε-tester for Π needs $\Omega\left(\frac{\mathrm{SMP}(f)}{\log(n)}\right)$ samples.

Take the "equality" predicate $\textsc{Eq}_k$ as f:

Theorem (Newman and Szegedy [NS96])

For every $k \in \mathbb{N}$, $\mathrm{SMP}(\textsc{Eq}_k) = \Omega\left(\sqrt{k}\right)$.

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Theorem (Newman and Szegedy [NS96])

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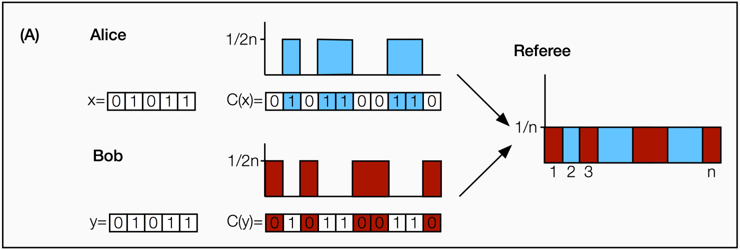
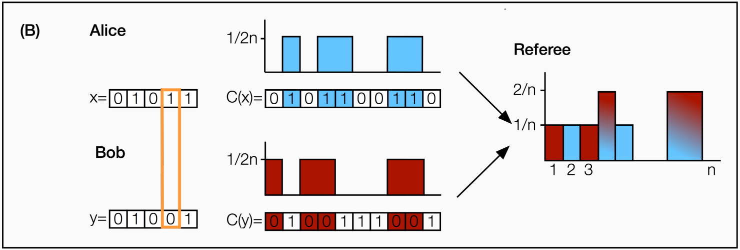
Goal:

Will (re)prove an $\tilde{\Omega}(\sqrt{n})$ lower bound on testing uniformity.


(A)
Alice
1/2n
x= 0 1 0 1 1
C(x)= 0 1 0 1 1 0 0 1 1 0
Bob
1/2n
y= 0 1 0 1 1
C(y)= 0 1 0 1 1 0 0 1 1 0
Referee
1/n
1 2 3 n


(B)
Alice
1/2n
x= 0 1 0 1 1
C(x)= 0 1 0 1 1 0 0 1 1 0
Bob
1/2n
y= 0 1 0 0 1
C(y)= 0 1 0 0 1 1 1 0 0 1
Referee
2/n
1/n
1 2 3 n

Statement

Explicit description of $p \in \Delta([n])$, parameter $\varepsilon \in (0, 1]$. Given samples from (unknown) $q \in \Delta([n])$, decide

$$p = q \qquad \text{vs.} \qquad \ell_1(p, q) > \varepsilon$$

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Actually...

Theorem ([VV14])

Identity testing requires $\Omega\left(\frac{\|p_{-\varepsilon}^{-\max}\|_{2/3}}{\varepsilon^2}\right)$ samples (and this is "tight").

An issue: how to interpret this $\|p^{-\max}_{-\varepsilon}\|_{2/3}$?

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Will prove an(other) $\tilde{\Omega}(\Phi(p, \varepsilon))$ lower bound on testing identity, via communication complexity.

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(and it will be "tight" as well.)

- p-weighted codes

$$\mathrm{dist}_p(x,y) := \sum_{i=1}^{n} p(i)\cdot |x_i - y_i| \qquad\qquad (x,y\in\{0,1\}^n)$$

  A p-weighted code has distance guarantee w.r.t. this p-distance: $\mathrm{dist}_p(C(x),C(y)) > \gamma$ for all distinct $x,y\in\{0,1\}^k$.

- Volume of the p-ball:

$$\mathrm{Vol}_{\mathbb{F}_2^n,\mathrm{dist}_p}(\varepsilon) := \left|\{\, w\in\mathbb{F}_2^n : \ \mathrm{dist}_p(w,0^n)\le\varepsilon \,\}\right|.$$

Lemma (Balanced p-weighted exist)

Fix $p \in \Delta([n])$ and ε. There exists a p-weighted (nearly) balanced code $C\colon \{0,1\}^k \to \{0,1\}^n$ with relative distance ε such that $k = \Omega(n - \log \mathrm{Vol}_{\mathbb{F}_2^n, \mathrm{dist}_p}(\varepsilon))$.

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Recall

Our reduction will give a lower bound of $\Omega\left(\frac{\sqrt{k}}{\log n}\right)$: so we need to analyze $\mathrm{Vol}_{\mathbb{F}_2^n,\mathrm{dist}_p}(\varepsilon)$.

ENTER CONCENTRATION INEQUALITIES

$$
\begin{aligned}
\mathrm{Vol}_{\mathbb{F}_2^n,\mathrm{dist}_p}(\gamma) &= \left| \left\{ w \in \mathbb{F}_2^n : \sum_{i=1}^{n} p_i w_i \leq \gamma \right\} \right| \\
&= 2^n \Pr_{Y \sim \{0,1\}^n} \left[\sum_{i=1}^{n} p_i Y_i \leq \gamma \right] \\
&= 2^n \Pr_{X \sim \{-1,1\}^n} \left[\sum_{i=1}^{n} p_i X_i \geq 1 - 2\gamma \right]
\end{aligned}
$$

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\end{aligned}$$

Concentration inequalities for weighted sums of Rademacher r.v.'s?

Definition (K-functional)

Fix any two Banach spaces $(X_0, \|\cdot\|_0), (X_1, \|\cdot\|_1)$. The K-functional between X_0 and X_1 is the function $K_{X_0,X_1} \colon (X_0 + X_1) \times (0, \infty) \to [0, \infty)$ defined by

$$K_{X_0,X_1}(x, t) := \inf_{\substack{(x_0,x_1)\in X_0\times X_1 \\ x_0+x_1=x}} \|x_0\|_0 + t\|x_1\|_1.$$

For $a \in \ell_1 + \ell_2$, we write κ_a for the function $t \mapsto K_{\ell_1,\ell_2}(a, t)$.

Theorem ([MS90])

Let $(X_i)_{i\geq 0}$ be a sequence of independent Rademacher random variables, i.e. uniform on $\{-1,1\}$. Then, for any $a \in \ell_2$ and $t > 0$,

$$\Pr\left[\sum_{i=1}^{\infty} a_i X_i \geq \kappa_a(t)\right] \leq e^{-\frac{t^2}{2}}. \tag{1}$$

and

$$\Pr\left[\sum_{i=1}^{\infty} a_i X_i \geq \frac{1}{2}\kappa_a(t)\right] \geq e^{-(2\ln 24)t^2}. \tag{2}$$

Theorem ([BCG16])

Identity testing to $p \in \Delta([n])$ requires $\Omega(t_\varepsilon / \log(n))$ samples, where $t_\varepsilon := \kappa_p^{-1}(1 - 2\varepsilon)$.

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But...

...is it tight?

Theorem ([BCG16])

Identity testing to $p \in \Delta([n])$ can be done with $O\left(\frac{t_{\varepsilon/18}}{\varepsilon^2}\right)$ samples and requires $\Omega\left(\frac{t_\varepsilon}{\varepsilon}\right)$ of them, where $t_\varepsilon := \kappa_p^{-1}(1-2\varepsilon)$.

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Upper bound established by a new connection between K-functional and "effective support size."

Theorem ([Ast10, MS90])

For arbitrary $a \in \ell_2$ and $t \in \mathbb{N}$, define the norm

$$\|a\|_{Q(t)} := \sup \left\{ \sum_{j=1}^{t} \left(\sum_{i \in A_j} a_i^2 \right)^{1/2} \ : \ (A_j)_{1 \leq j \leq t} \text{ partition of } \mathbb{N} \right\}.$$

Then, for any $a \in \ell_2$, and $t > 0$ such that $t^2 \in \mathbb{N}$, we have

$$\|a\|_{Q(t^2)} \leq \kappa_a(t) \leq \sqrt{2}\|a\|_{Q(t^2)}. \tag{3}$$

Lemma ([BCG16])

For any $a \in \ell_2$ and t such that $t^2 \in \mathbb{N}$, we have

$$\|a\|_{Q(t^2)} \leq \kappa_a(t) \leq \|a\|_{Q(2t^2)}. \tag{4}$$

Lemma (Sparsity Lemma)

If $\|p\|_{Q(T)} \geq 1 - 2\varepsilon$, then there is a subset S of T elements such that $p(S) \geq 1 - 6\varepsilon$.

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Proof idea.

By monotonicity, $\sum_{j=1}^{T} \left(\sum_{i \in A_j} p_i^2\right)^{1/2} \leq \sum_{j=1}^{T} \sum_{i \in A_j} p_i = \|p\|_1 = 1$. So we have

$$1 - 2\varepsilon \leq \sum_{j=1}^{T} \left(\sum_{i \in A_j} p_i^2\right)^{1/2} \leq 1$$

which (morally) implies that p is "close to a singleton" on each A_j. $\square$

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- **Codes are great!**

Thank you

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