Testing Distributions

Candidacy Talk

Clément Canonne

Columbia University – 2015

Introduction

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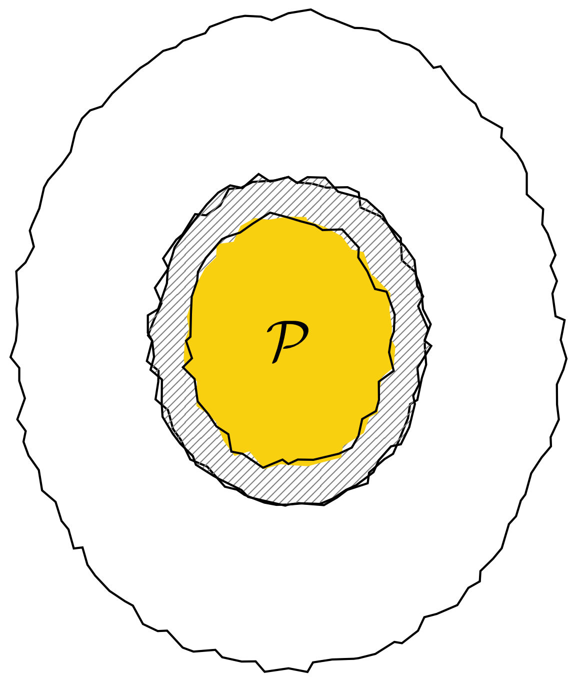
Property testing: what can we say about an object while barely looking at it?



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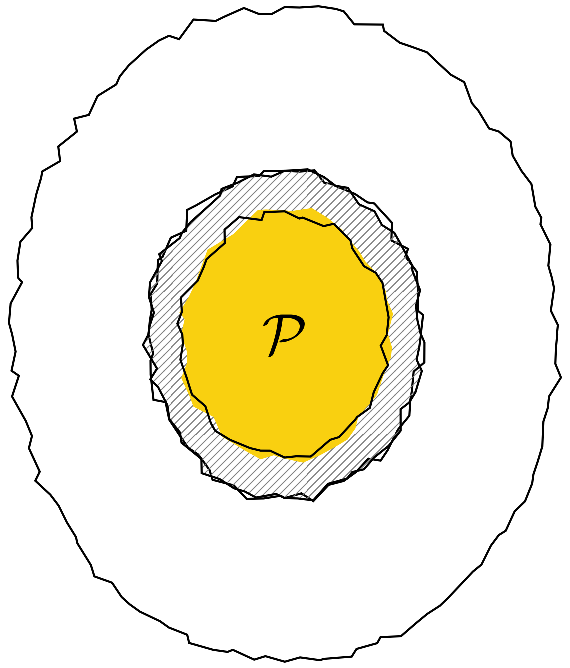

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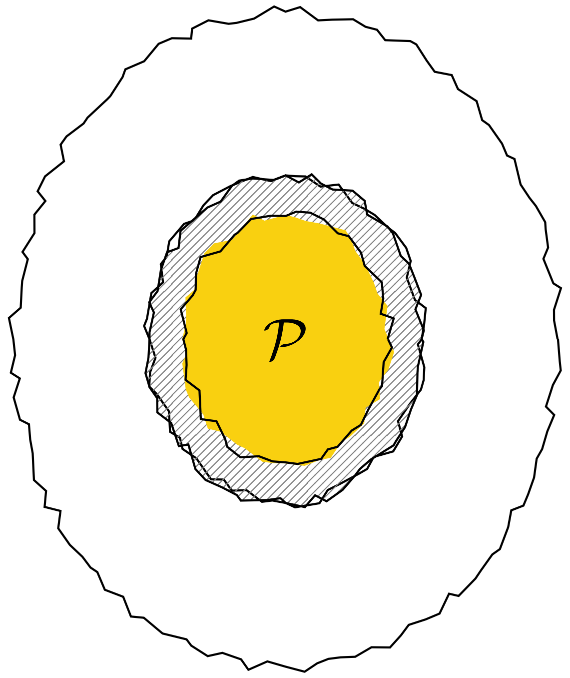

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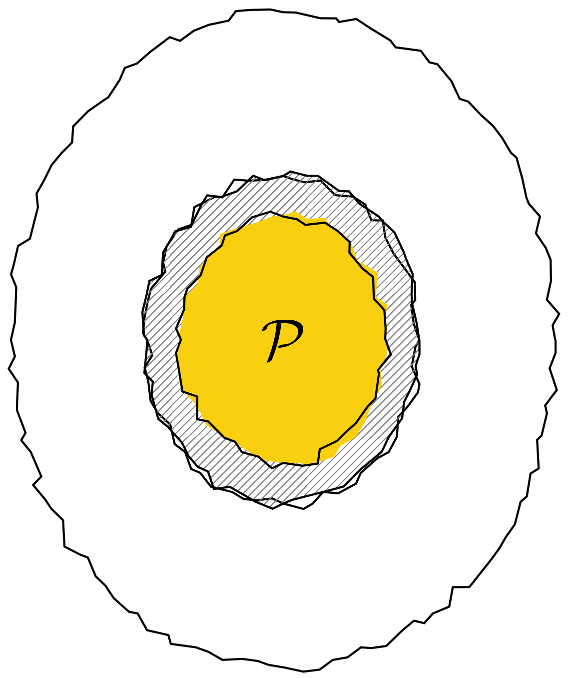

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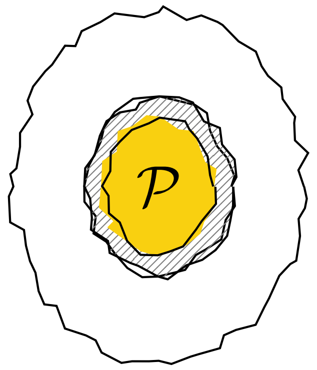
This talk: distribution testing, for various types of properties and settings.
(what is known, what is impossible, and under which assumptions can it still be done)

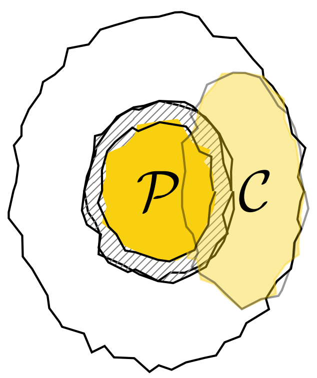
Outline of the talk

Plan in more detail

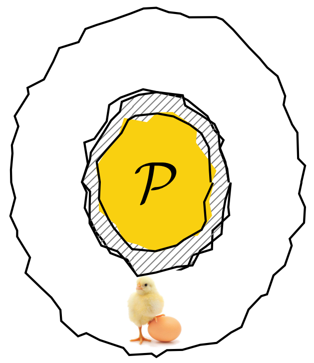
- Testing From Samples: the standard model, upper and lower bounds



- Testing Under Assumptions: "testing for $\mathcal{P}$ while knowing $\mathcal{C}$"



- Testing Differently: some other access (stronger or incomparable), or some other goal



Testing From Samples

The setting

$\Delta(\Omega)$: all distributions over (finite) domain Ω of size n. **Property:** subset $\mathcal{P} \subseteq \Delta(\Omega)$. **Tester:** randomized algorithm (knows n, $\mathcal{P}$).

Given independent samples from a distribution $D \in \Delta(\Omega)$, and parameter $\varepsilon \in (0,1)$, output accept or reject:

- If $D \in \mathcal{P}$, accept with probability at least $2/3$; *(in the yolk)*
- If $\ell_1(D, \mathcal{P}) > \varepsilon$, reject with probability at least $2/3$; *(definitely white)*
- otherwise, whatever (make an omelet).

Goal: take $o(n)$ samples, ideally $O_\varepsilon(1)$.

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[BFF+01, BKR04, BFR+10, GGR98]

Results: a representative sample

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Tolerant testing uniformity (and a range of other interesting properties) has sample complexity $\Theta(n/\log n)$ [Pan04, RRSS09, Val11, VV10a, VV10b, VV11].

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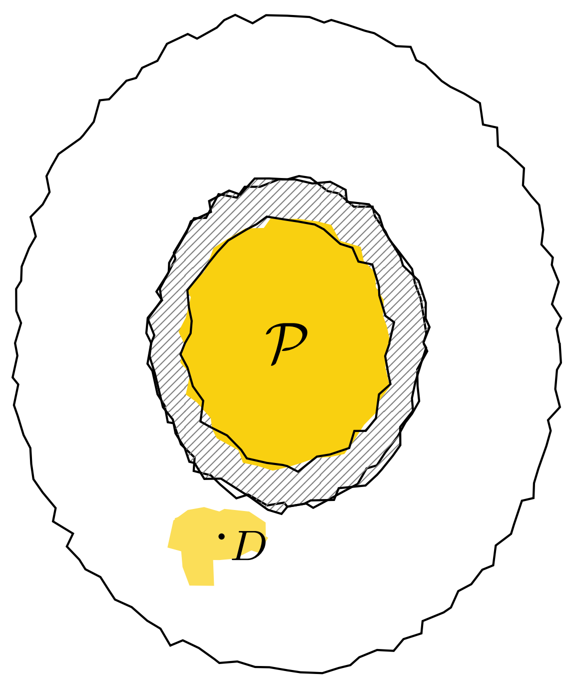
Testing Under Assumptions: Changing The Goal

The twist

Intuitively, one key difficulty here is that in the negative case, *the distribution could be absolutely anything.* No structure to exploit!



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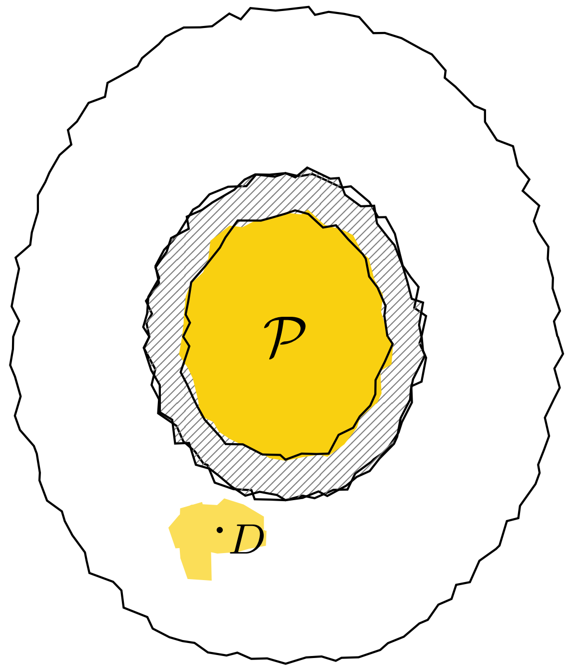

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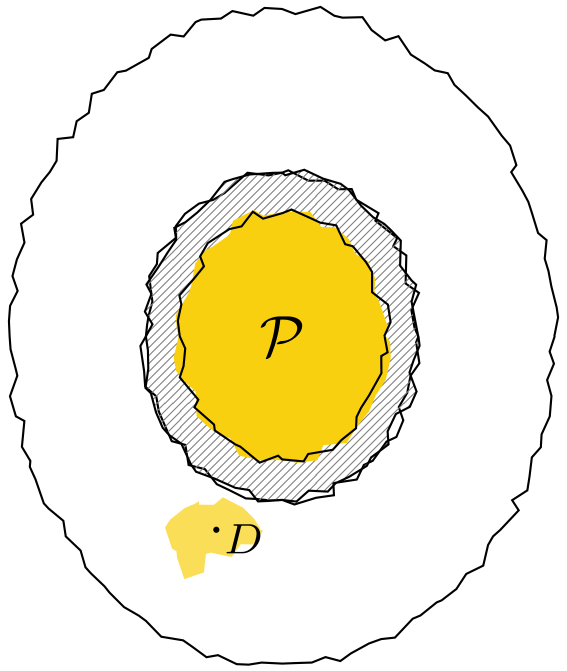

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But what if D was not arbitrary? E.g., the distribution is *known* to have some structure $\mathcal{C}$ – does it make it easier to test if it *also* has the property $\mathcal{P}$?

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For monotone distributions

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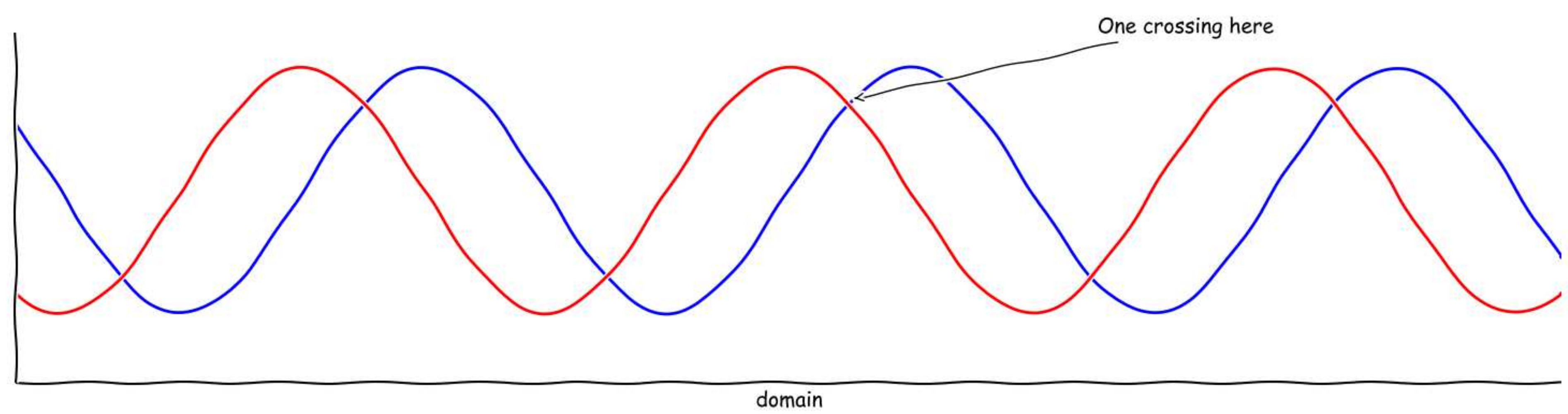
Main idea: reductions to (and from) the general case *via* structural results.

Under structural assumptions: m is the new n.

A different flavor of results is obtained in [DKN15]: no assumption on the "shape," but rather on the "structure" of the unknown distribution.

Theorem: Let $\mathcal{C} \subseteq \Delta([n])$ be a distribution class such that the probability mass functions (pmf) of any two $D, D' \in \mathcal{C}$ cross "essentially" at most m times. Then, given sampling access to an unknown $D \in \mathcal{C}$, one can test identity to an explicit D^* with $O(\sqrt{m}/\varepsilon^2)$ samples.



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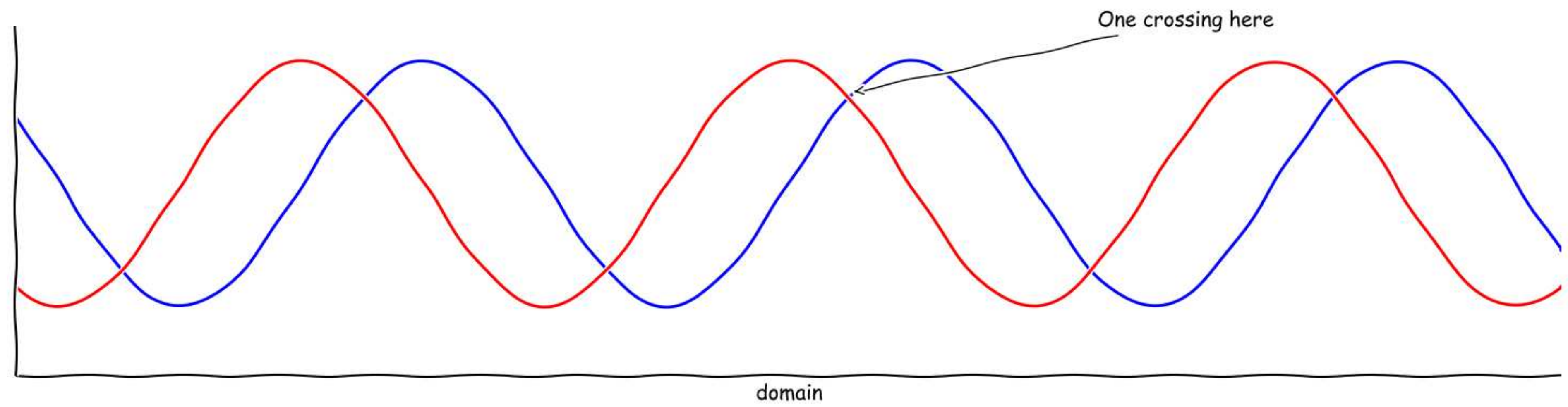

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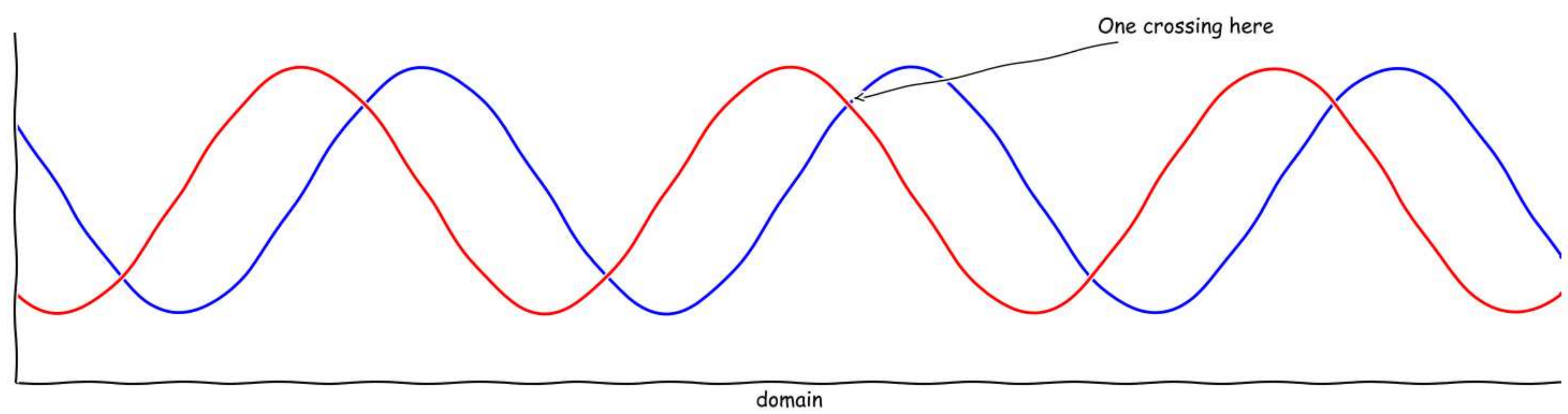

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Main idea: "testing in $\mathcal{A}_m$-norm," and reduction to testing uniformity in this distance (over a much bigger domain).

Testing Differently: Changing the Rules

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- with conditional sampling: [CFGM13, CRS15]

$$S \subseteq \Omega \rightsquigarrow x \sim D_S$$

Results: the Sunny Side (Up)

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Challenges: Understanding the intrinsic power and limitations of the models, how they relate, and whether there exist generic tools to analyze them.

A Collections of Eggs

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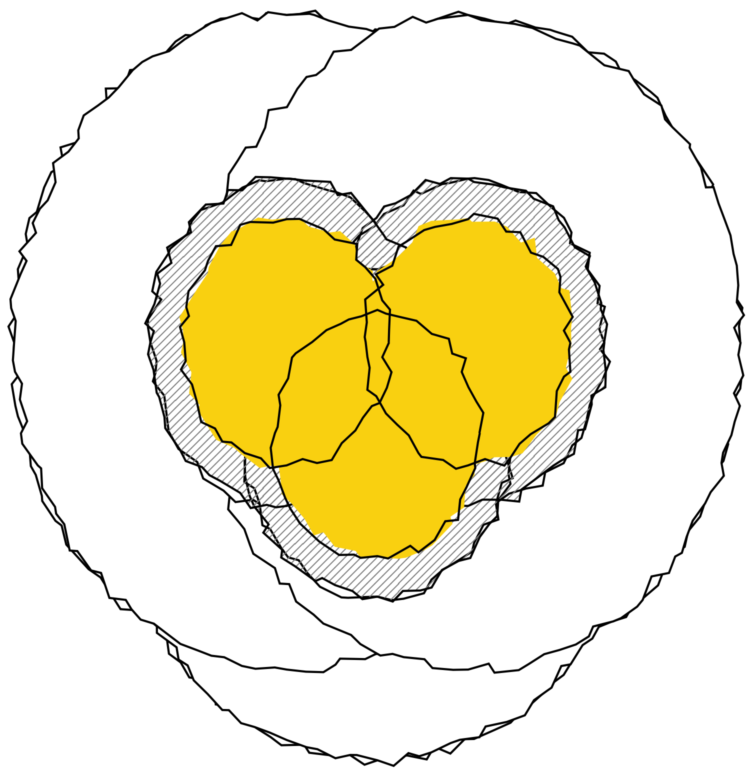

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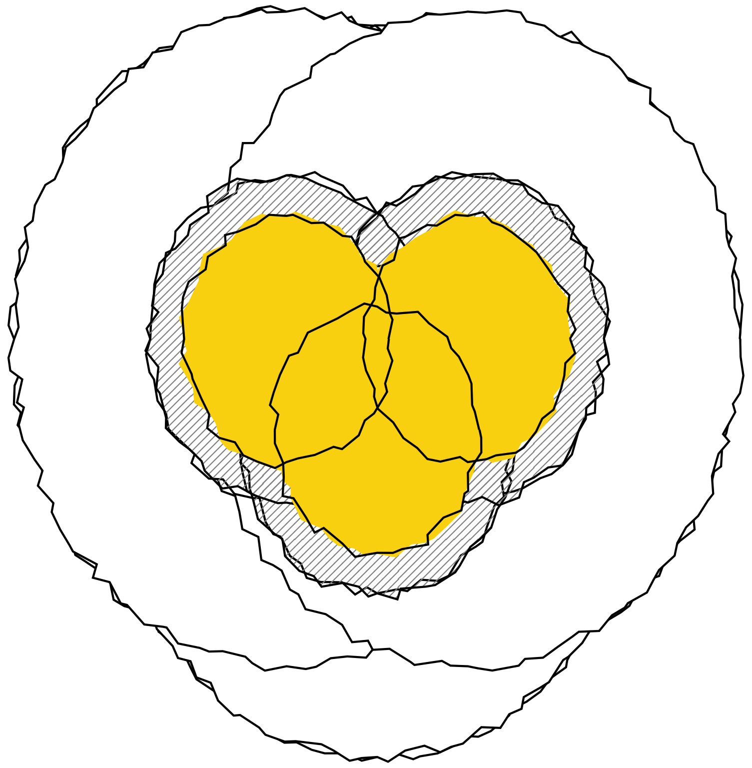

Testing collections: given "sampling access" to a family $\mathcal{D} = (D_1, \dots, D_m)$ of m distributions over Ω, test whether they satisfy a joint property or are far from it (in average ℓ_1 distance). E.g., equivalence: $D_1 = \cdots = D_m$.

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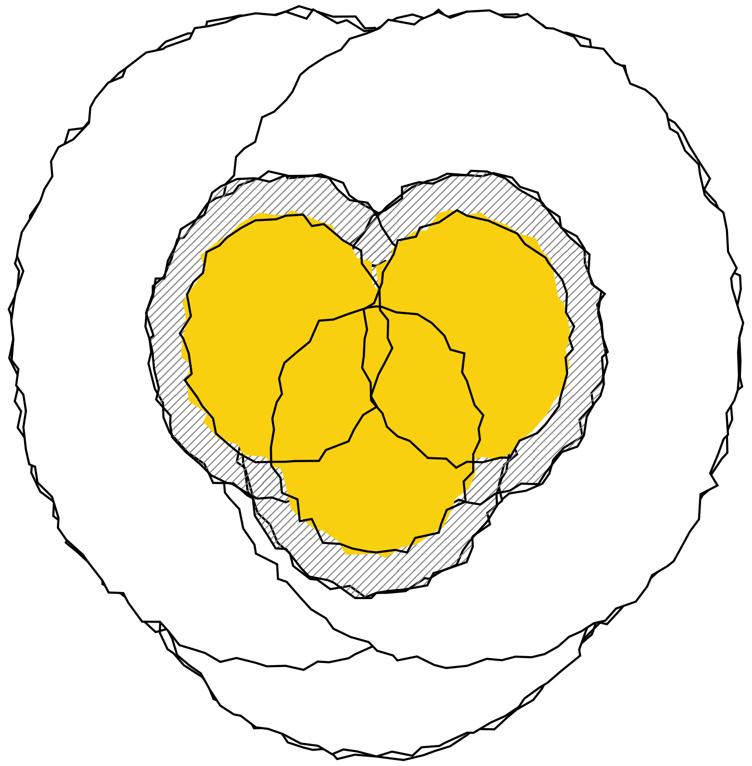

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[LRR13] (equivalence and clustering), [LRR14] (similarity of means)

Some collected results

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Bonus: Strong connections between equivalence for collections and independence for distributions.

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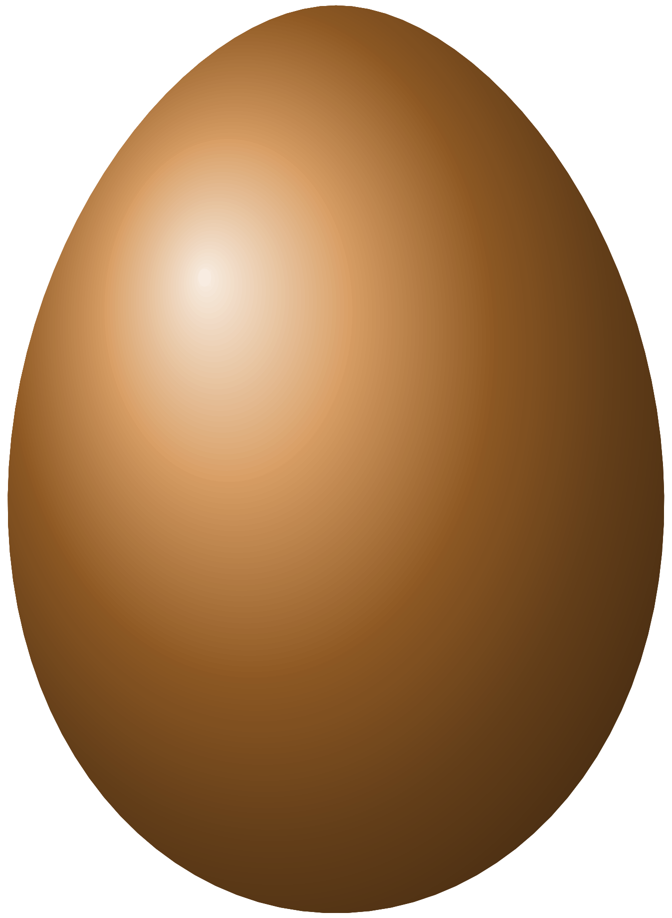
Bonus: Strong connections between equivalence for collections and independence for distributions. Testing collections related to conditional sampling.

That's All, (Y)olks!



Thank you.

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