COMPx270: Randomised and Advanced Algorithms
Lecture 3: Balls in Bins

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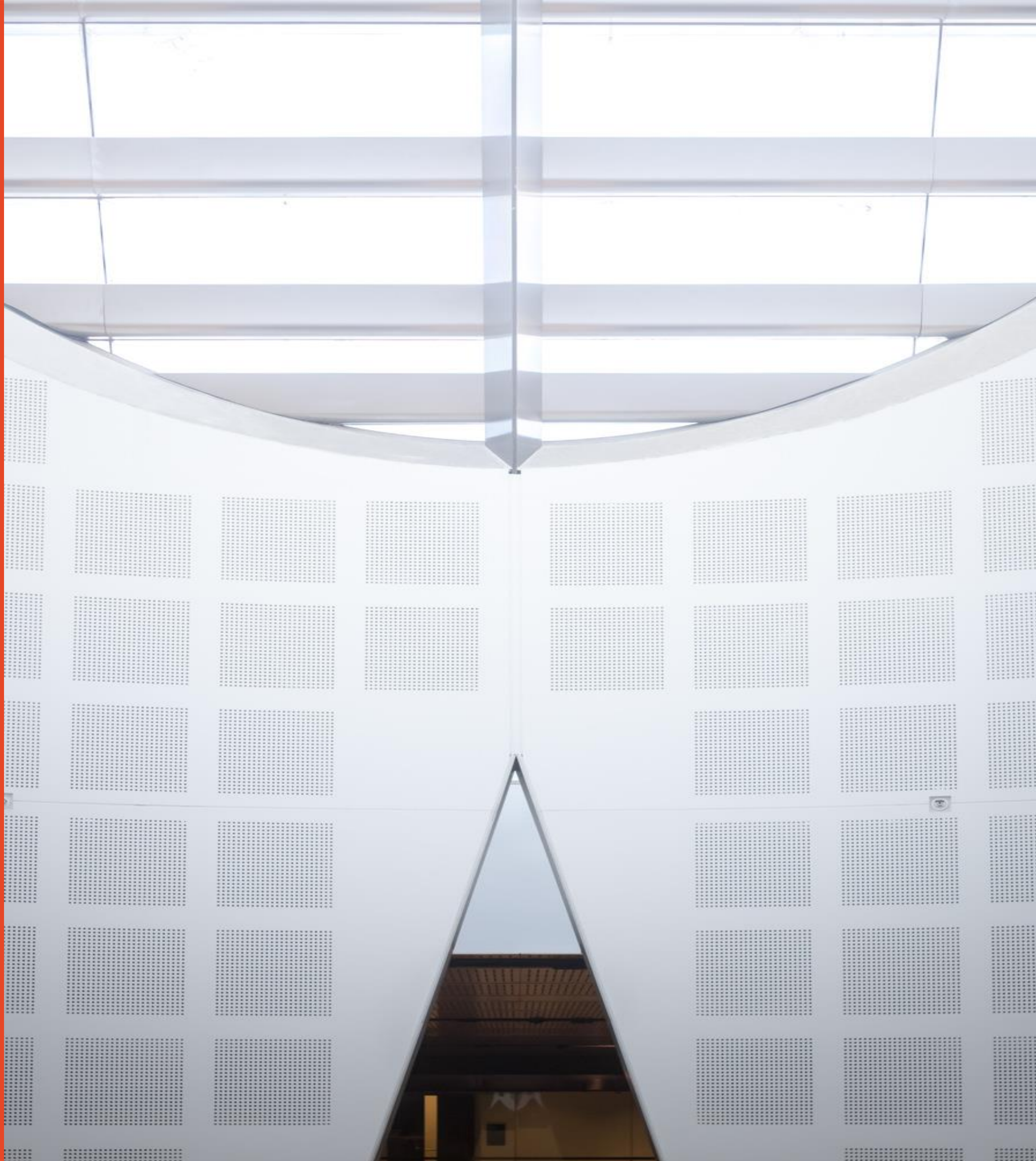

A question 🎂

There are quite a few people in the classroom right now. What are the odds two of you (at least) have the same birthday?

A question 🎂

Theorem. (The 🎂 paradox) If you gather 23 people in a room, then with probability at least 50% a pair will sharing their birthday.

An answer? 🎂

Theorem. If you gather m people and give each a number uniform between 1 and n, then the probability p(m,n) that at least two have the same number is...

$$p_{m,n} = 1 - \frac{n!}{n^m (n-m)!} = 1 - \frac{m!}{n^m}\binom{n}{m}$$

Proof.

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DiscretePlot[{1/2, p[m, 250], p[m, 1000], p[m, 4000], p[m, 8000]}, {m, 1, 1000}, PlotRange → {0, 1}, PlotLegends → "Expressions"]
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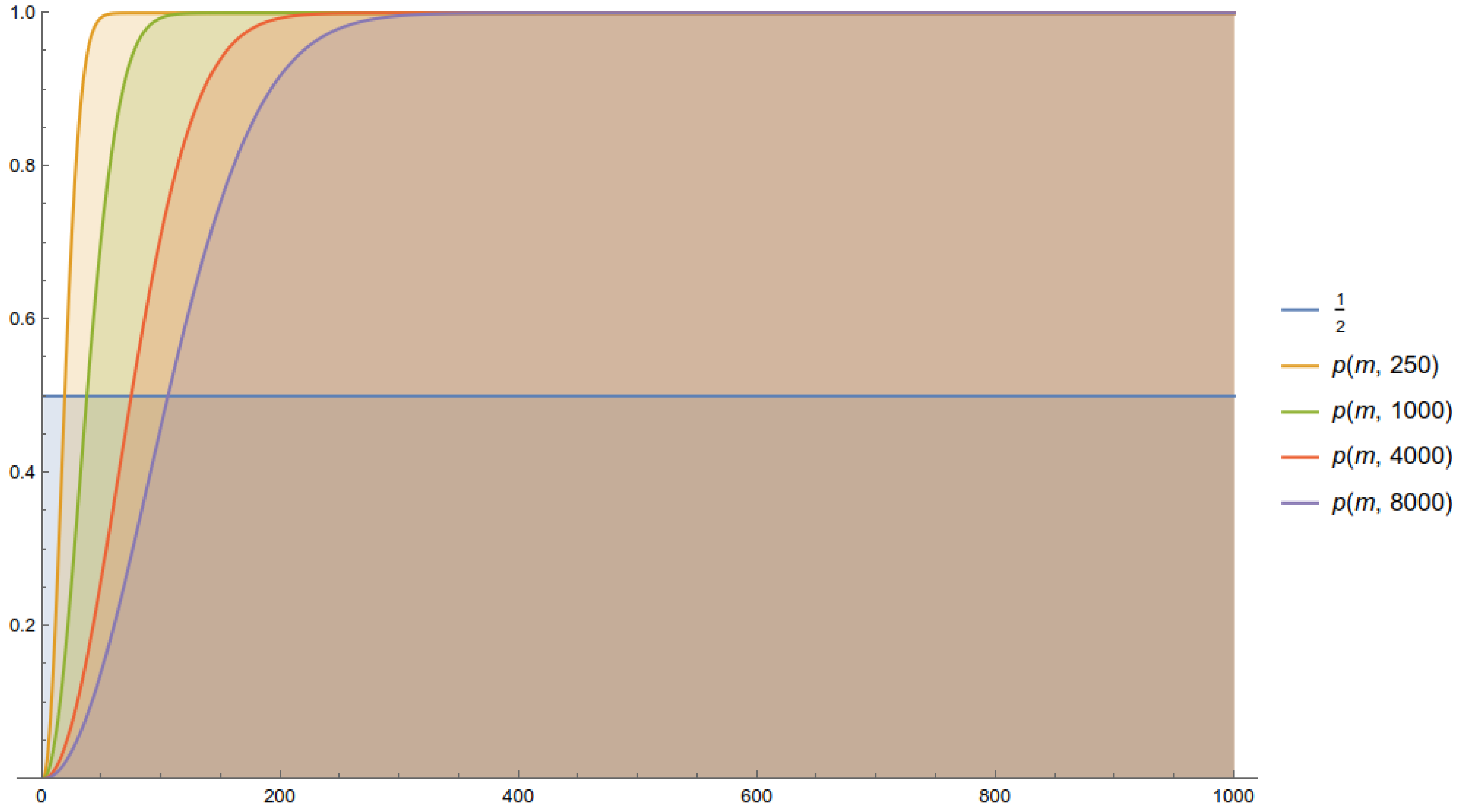

Let's start simple: $m=2$ ⚾ ⚾

Now, for large values of "2"...

C: number of collisions when throwing m ⚾ into n 🗑️ . What is

$$c(m,n) = E[C]?$$

Number of collisions

... and what is Var[C]?

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Number of collisions

$$\mathrm{Var}[C] = \binom{m}{2}\frac{1}{n}\left(1 - \frac{1}{n}\right)$$

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Number of collisions

Now, we can use Chebyshev:

$$\Pr[\,X = 0\,] \leq 1/2$$

for $m = \Omega(\sqrt{n})$.

Number of collisions

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Now, we can use Chebyshev:

$$\Pr[\, X = 0 \,] \leq 1/2$$

for $m = \Omega(\sqrt{n})$. **Is it tight?**

Number of collisions

$$\mathrm{Var}[C] = \binom{m}{2}\frac{1}{n}\left(1-\frac{1}{n}\right)$$

Now, we can use Chebyshev:

$$\Pr[\, X = 0 \,] \le 1/2$$

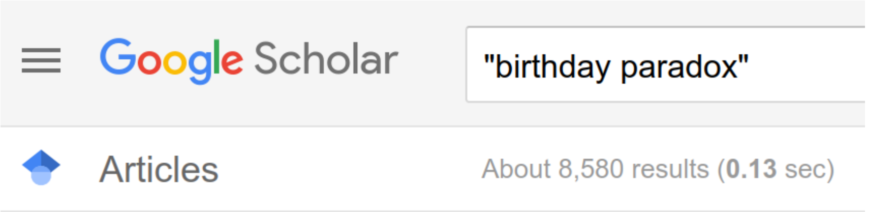
for $m = \Omega(\sqrt{n})$.

By Markov, we also have

$$\Pr[\, X \ne 0 \,] = \Pr[\, X \ge 1 \,] \le E[X] \le 1/2$$

for $m = O(\sqrt{n})$.

Applications?



Bounding the variance: is it always that bad?

Two tricks (and even 3).

Coverage (Coupon collector)

"What is the expected number of balls $M(n)$ we need to throw before each of the n bins contains at least one ball?"

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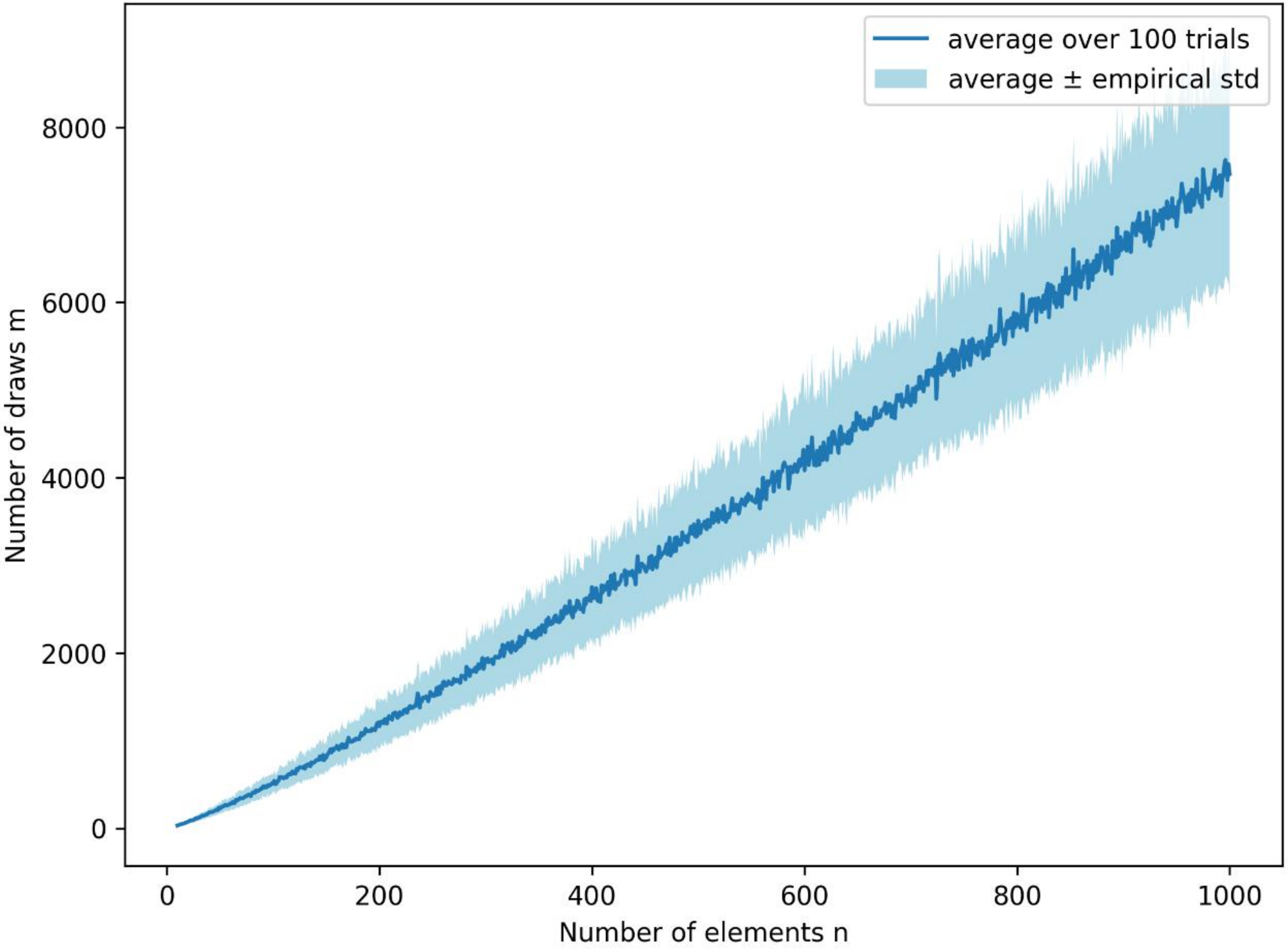
"What is the expected number of balls $M(n)$ we need to throw before each of the n bins contains at least one ball?"

- $\Theta(n)$?
- $\Theta(n \log n)$?
- $\Theta(n^2)$?
- Something else?



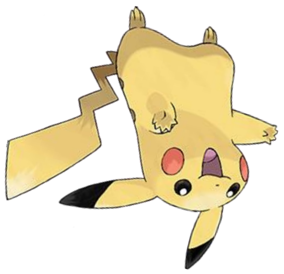
average over 100 trials
average ± empirical std
Number of draws m
Number of elements n
0
2000
4000
6000
8000
0
200
400
600
800
1000

Coverage (Coupon collector)

"What is the expected number of balls $M(n)$ we need to throw before each of the n bins contains at least one ball?"

Theorem. In expectation, $M(n) = \Theta(n \log n)$ balls. (Even more precisely: $n \ln n + O(n)$.)

Proof.



What about the variance?

Load balancing

"What is the expected number of balls $L(n)$ the fullest of the n bins contains after throwing n balls?"

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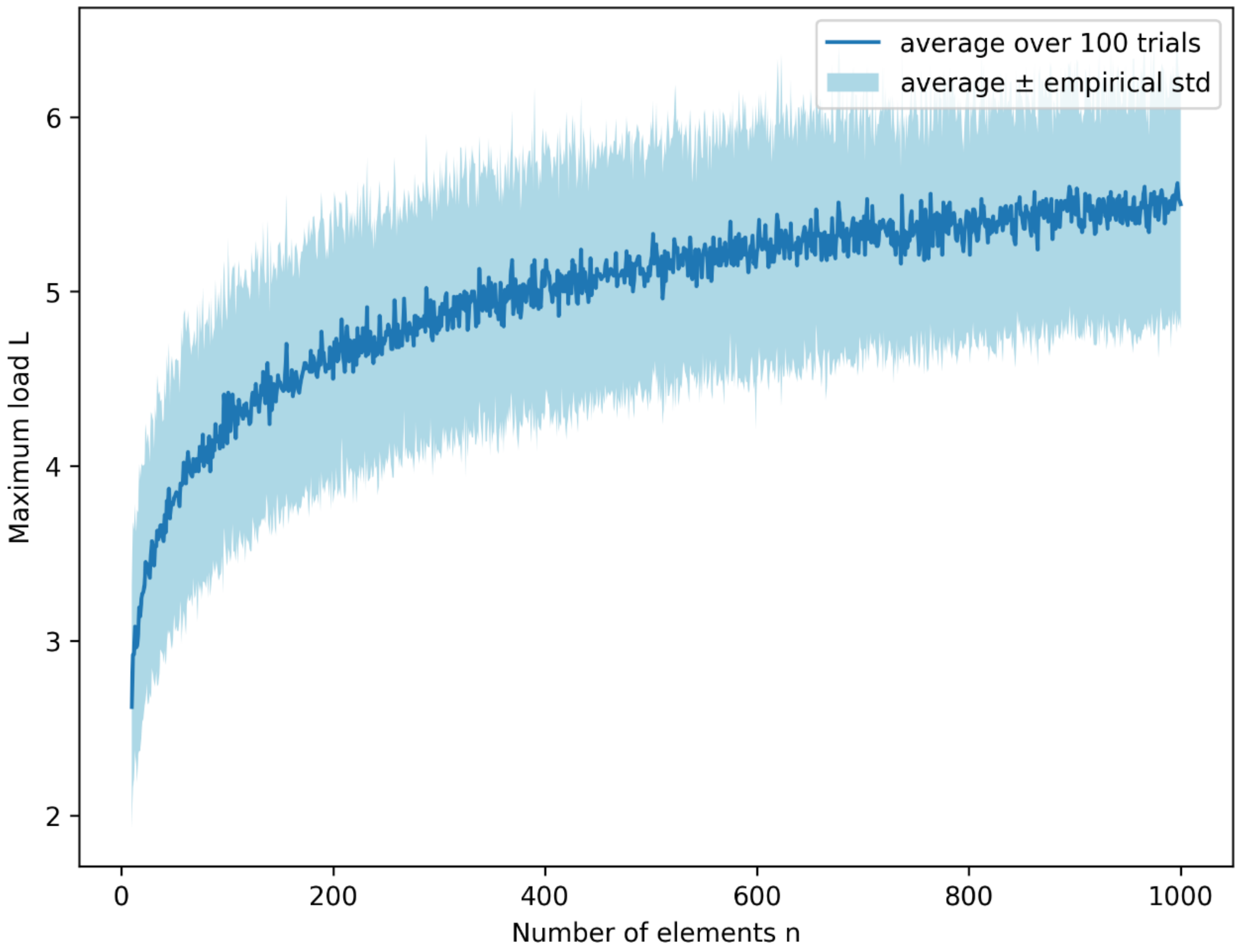
"What is the expected number of balls $L(n)$ the fullest of the n bins contains after throwing n balls?"

- $\Theta(1)$?
- $\Theta(\log n)$?
- $\Theta(\sqrt{n})$?
- Something else?


average over 100 trials
average ± empirical std
Maximum load L
Number of elements n
2
3
4
5
6
0
200
400
600
800
1000

Load balancing

"What is the expected number of balls $L(n)$ the fullest of the n bins contains after throwing n balls?"

Theorem. The expected maximum load is $L(n) = \Theta(\log n / \log \log n)$.

Proof.

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{e \cdot n}{k}\right)^k$$

Proof.

Load balancing (a twist)

"Now, every time you throw a ball, it selects two bins at random, and goes to the least full of the two. What is the maximum expected load?"

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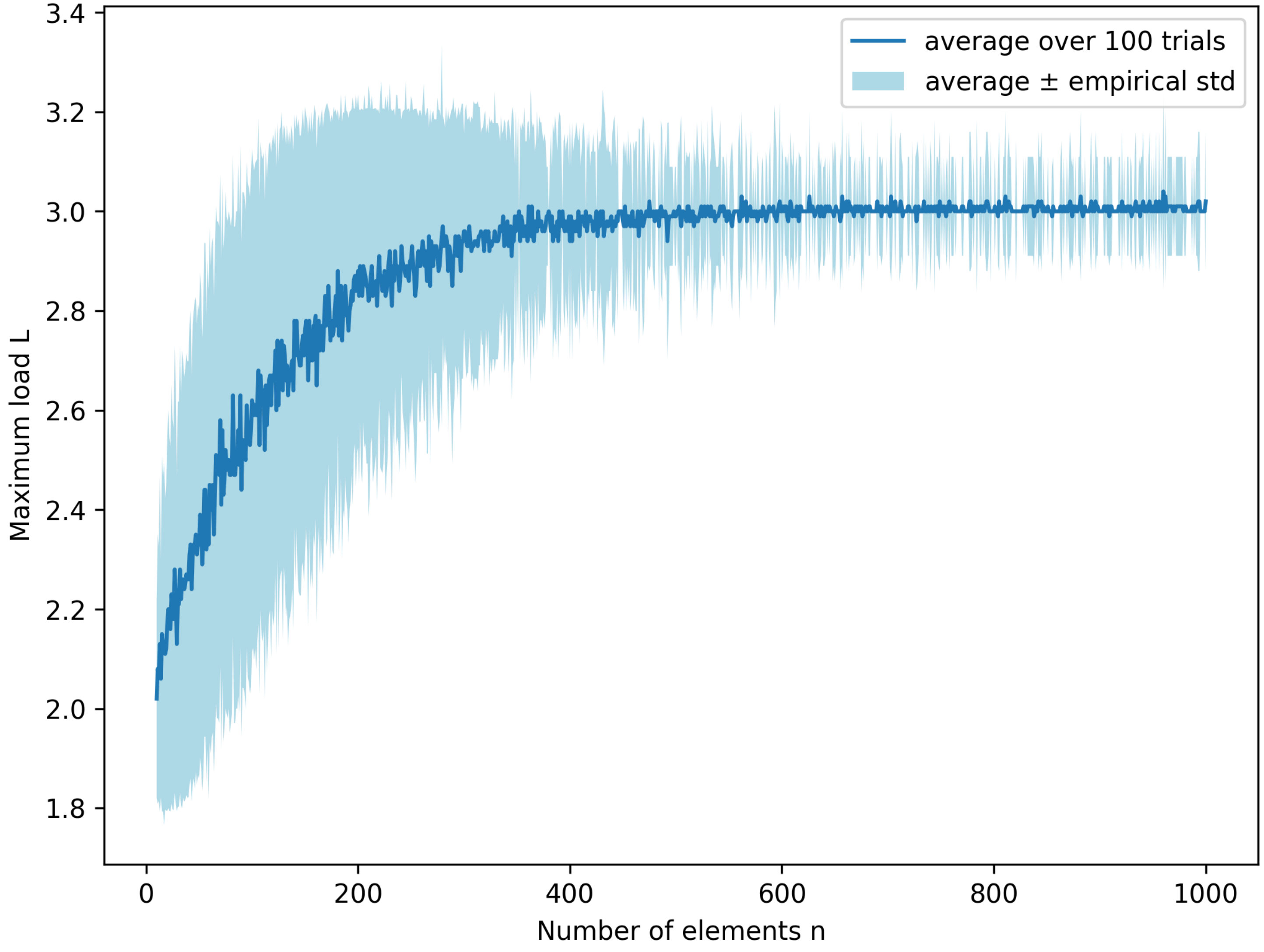
"Now, every time you throw a ball, it selects two bins at random, and goes to the least full of the two. What is the maximum expected load?"

- $\Theta(1)$?
- $\Theta(\log \log n)$?
- $\Theta(\sqrt{\log n})$?
- Something else?


average over 100 trials
average ± empirical std
Maximum load L
Number of elements n
3.4
3.2
3.0
2.8
2.6
2.4
2.2
2.0
1.8
0
200
400
600
800
1000

Load balancing (a twist)

"Now, every time you throw a ball, it selects two bins at random, and goes to the least full of the two. What is the maximum expected load?"

Theorem. The expected maximum load now $\Theta(\log \log n)$.