tl;dr: Answers for a quiz on anticoncentration held on Twitter. Overall, the upshot? We cannot say much in most cases; but Paley–Zygmund is a good friend.

QUESTIONS AND ANSWERS

Q 1. Let X be a real-valued random variable with finite expectation $\mathbb{E}[X]$. What can we say about $\Pr[\, X \geq \mathbb{E}[X]\,]$?

Answer. As 55.8% of you replied, the answer is a resounding "not much." The probability has to be strictly positive, basically by an averaging argument: indeed, suppose that $\Pr[\, X \geq \mathbb{E}[X]\,] = 0$. We can write $\{X \geq \mathbb{E}[X]\} = \bigcap_{n=1}^{\infty} \{X > \mathbb{E}[X] - 1/n\}$ which implies that, for any fixed $\varepsilon \in (0,1)$, $\Pr[\, X > \mathbb{E}[X] - 1/n\,] \leq \varepsilon$ for large enough n. But then, fixing any $\varepsilon > 0$ and any corresponding such large enough n, and letting $p \coloneqq \Pr[\, X > \mathbb{E}[X] - 1/n\,] > 0$,

$$\mathbb{E}[X] = \mathbb{E}\big[X \mathbb{1}_{\{X \leq \mathbb{E}[X] - 1/n\}}\big] + \mathbb{E}\big[X \mathbb{1}_{\{X > \mathbb{E}[X] - 1/n\}}\big] \leq (\mathbb{E}[X] - 1/n) \cdot p + \mathbb{E}[X] \cdot (1-p) < \mathbb{E}[X]$$

contradiction. So $\Pr[\, X \geq \mathbb{E}[X]\,] > 0$.
However, that's *all* we can say: that probability could be arbitrarily small! Consider, for $n \geq 2$,

$$X_n = \begin{cases} n & \text{with probability } \frac{1}{n} \\ -\frac{n}{n-1} & \text{with probability } 1 - \frac{1}{n} \end{cases}$$

and check that $\mathbb{E}[X_n] = 0$, but $\Pr[\, X_n \geq \mathbb{E}[X_n]\,] = \frac{1}{n} \xrightarrow{n \to \infty} 0$. □

Q 2. Let X be a real-valued r.v. with finite variance $\operatorname{Var}[X]$. What can we say about $\Pr[\, X \geq \mathbb{E}[X]\,]$?

Answer. This one is really a bummer. It really *feels* like we should be able to say $\Pr[\, X \geq \mathbb{E}[X]\,] \geq c \cdot \operatorname{Var}[X]$, or even $\Pr[\, X \geq \mathbb{E}[X]\,] \geq c \cdot \sqrt{\operatorname{Var}[X]}$ for some absolute constant $c > 0$. However, as 47% of you answered, we cannot say anything more that for **Q1**: it's strictly positive.

There is a nice counterexample by Iosif Pinellis on this MathOverflow answer, but considering $\Pr[\, X > \mathbb{E}[X]\,]$; let's modify it a little bit for the case $\Pr[\, X \geq \mathbb{E}[X]\,]$. First, by replacing X by $X/\sqrt{c \operatorname{Var}[X]}$, we can assume the variance is equal to any constant of our choosing, so we'll give something assuming the variance is upper bounded by say 2.1.
Fix any $n \geq 2$, set $\varepsilon \coloneqq \frac{n}{n^2-2}$, and consider X_n given by

$$X_n = \begin{cases} n & \text{with probability } \frac{3}{2n^2} \\ -\varepsilon_n & \text{with probability } 1 - \frac{2}{n^2} \\ -n & \text{with probability } \frac{1}{2n^2} \end{cases}$$

If I didn't mess up, we have $\mathbb{E}[X_n] = 0$, $\operatorname{Var}[X_n] = 2 + \frac{1}{n^2-2} = 2 + o(1)$, but $\Pr[\, X_n \geq \mathbb{E}[X_n]\,] = \frac{3}{2n^2} \xrightarrow{n \to \infty} 0$. □

Q 3. Let X be a real-valued r.v. with finite moments of all orders, and such that $\mathbb{E}\left[|X|^k\right] \leq 1$ for all $k \geq 0$. What can we say about $\Pr[\, X \geq \mathbb{E}[X]\,]$?

Answer. Sorry, did I say the *previous* question was a bummer? That one must be the bummest then. I really, really wanted to believe we could say something like $\Pr[\, X \geq \mathbb{E}[X]\,] \geq c]$ for some absoltue constant $c > 0$, but as 34.2% of you answered, it's still only as good as **Q1**: it's strictly positive, we cannot say more.

How come? Well, the link to the MathOverflow post by Iosif Pinellis above shows that, given our assumptions (which imply $X \in [-1, 1]$ a.s.) we have

$$\Pr[\, X \geq \mathbb{E}[X]\,] \geq \Pr[\, X > \mathbb{E}[X]\,] \geq \frac{\mathrm{Var}[X]}{2}$$

which, frankly, looked promising (*also, it's a neat proof, check it out!*). But our assumption is on the raw moments, not the centered ones, so... $\mathrm{Var}[X]$ can still be arbitrarily small (think of $\mathbb{E}\left[X^2\right] \approx \mathbb{E}[X]^2$, "when Jensen is tight-ish."). For instance: fix any $n \geq 1$, and consider X_n given by

$$X_n = \begin{cases} 0 & \text{with probability } \frac{1}{n} \\ -1 & \text{with probability } 1 - \frac{1}{n} \end{cases}$$

We have $\mathbb{E}[X_n] = -(1 - \frac{1}{n})$, $\mathbb{E}\left[|X_n|^k\right] \leq 1$ for all k, but $\Pr[\, X_n \geq \mathbb{E}[X_n]\,] = \frac{1}{n} \xrightarrow{n \to \infty} 0$. □

Q 4. Let X be a non-negative real-valued r.v. with finite variance. What can we say about $\Pr\left[\, X \geq \frac{1}{2}\mathbb{E}[X]\,\right]$?

Answer. I have good and bad news. The good news is that there *is* something we can say here. The bad news is that the best option, among those suggested, is still the very disappointing strictly positive, as 34.8% of you answered.

It cannot be $\Pr\left[\, X \geq \frac{1}{2}\mathbb{E}[X]\,\right] \geq 1/2$, as taking X to be Bernoulli with parameter $p < 1/2$ shows. It cannot be $\Pr\left[\, X \geq \frac{1}{2}\mathbb{E}[X]\,\right] \geq c \cdot \mathrm{Var}[X]$ some some absolute constant $c > 0$, as the variance could be arbitrarily big, but probabilities tend to be at most one. *(They're stubborn like that.)*

But we *still* can say something! Just something not in the list. Namely, the wonderful–yet–basic–yet–so–useful *Paley–Zygmund inequality*, essentially the single most useful anticoncentration inequality I know, guarantees that for non-negative X, letting $\rho(X) \coloneqq \frac{\mathrm{Var}[X]}{\mathbb{E}[X]^2}$,

$$\Pr[\, X > \theta\mathbb{E}[X]\,] \geq \frac{(1-\theta)^2}{\rho(X) + (1-\theta)^2}, \qquad \theta \in [0, 1]$$

which in our case boilds down to

$$\Pr\left[\, X > \frac{1}{2}\mathbb{E}[X]\,\right] \geq \frac{1}{4\rho(X) + 1}\,.$$

Thinking of it differently: "if the standard deviation and the expectation are comparable, then the random variable cannot be *too* small all the time." □

Finally, last question: let's no longer assume $X \geq 0$, and ask for an *anti-Chebyshev*:

Q 5. Let X be a real-valued r.v. with finite variance. What can we say about $\Pr\left[\, |X - \mathbb{E}[X]\,| \geq \frac{\sqrt{\mathrm{Var}[X]}}{100}\,\right]$?

Answer. First, recall that Chebyshev's inequality ensures that $\Pr\left[\,|X-\mathbb{E}[X]\,|\geq 100\sqrt{\mathrm{Var}[X]}\,\right]\leq\frac{1}{100^2}$, so we're really asking if some non-trivial converse-type statement holds in general.

I am so, so sorry. The answer is no, as 40.6%. It's strictly positive, we cannot say more. One quick and sad way to see it is to consider the non-negative random variable $Y\coloneqq(X-\mathbb{E}[X])^2$, which has $\mathbb{E}[Y]=\mathrm{Var}[X]$ by definition. Then we are asking about

$$\Pr\left[\sqrt{Y}\geq\frac{\sqrt{\mathbb{E}[Y]}}{100}\right]=\Pr\left[Y\geq\frac{\mathbb{E}[Y]}{10000}\right]$$

and then it's clear we cannot say more without extra assumptions on Y (such as an almost sure upper bound, or if we want to use our new friend Paley–Zygmund, some bound on $\mathrm{Var}[Y]=\mathbb{E}\left[(X-\mathbb{E}[X]^4\right]$). For instance, one could take Y to be 0 with probability $1-1/n$ and $n\mathbb{E}[Y]$ with probability $1/n$, for arbitrarily large n… □