SIZE DISTRIBUTION FOR PLANETESIMALS

Suppose the number of plamls with radii between r & dr

is $$dN = k\, r^{-q}\, dr \qquad (1)$$

Then from radius r_1 to r_2 is $$r_1 \le N \le r_2 = \int_{r_1}^{r_2} k\, r^{-q}\, dr$$
$$= \frac{k}{1-q}\left[r_2^{\,1-q} - r_1^{\,1-q} \right] \qquad (2) \qquad (q \neq 1)$$

However, a planl of radius r has mass $m(r) = \frac{4\pi}{3}\rho r^3$ so the total amount of mass dM in objects between radii r & $r+dr$ is

$$dM = m(r)\, dN \qquad (3)$$

So $\left.\begin{matrix}\text{total}\\ \text{mass in}\\ \text{object}\\ \text{between}\\ r_1 \,\&\, r_2\end{matrix}\right\}$ $$r_1 \le M \le r_2 = \int_{r_1}^{r_2} \frac{4\pi}{3} r^3\, k\, r^{-q}\, dr$$

$$r_1 \le M \le r_2 = \frac{4\pi k \rho}{3(4-q)}\left[r_2^{\,4-q} - r_1^{\,4-q} \right] \qquad (4)$$

[Check: if instead we used bins in terms of masses $m_1 = \frac{4}{3}\pi\rho r_1^3$ & $m_2 = \frac{4}{3}\pi\rho r_2^3$ then $r(m) = \left(\frac{3}{4\pi\rho}\right)^{1/3} m^{1/3}$

so $$dr = \left(\frac{3}{4\pi\rho}\right)^{1/3} \frac{1}{3}\, m^{-2/3}\, dm$$

$$\therefore\ dN = \frac{dN}{dr}\frac{dr}{dm}\, dm$$
$$= k\left(\frac{3}{4\pi\rho}\right)^{-q/3} m^{-q/3} \left(\frac{3}{4\pi\rho}\right)^{1/3} \frac{1}{3}\, m^{-2/3}\, dm$$
$$\propto m^{-\frac{(q+2)}{3}}\, dm$$

Thus $$dM = m\,dN \propto m^{\frac{1-q}{3}}\,dm$$

$$\therefore \quad m_1 : M < m_2 = \int_{m_1}^{m_2} m\,dN \propto \left(m_2^{\frac{4-q}{3}} - m_1^{\frac{4-q}{3}} \right)$$

$$\propto \left(r_2^{4-q} - r_1^{4-q} \right)$$

Agrees with the statement above

Note also that if we create bins for the planetesimals in which $\frac{r_{i+1}}{r_i} = f$, independent of i

$$r_0 \quad r_1 \quad r_2 \quad r_3 \quad \cdots \quad r_k$$

then $$m_{i+1} < M < m_i = \text{const.} \times \left[r_{i+1}^{4-q} - r_i^{4-q} \right]$$

$$= r_{i+1}^{4-q} \times \text{const} \times \left[1 - \left(\frac{r_i}{r_{i+1}} \right)^{4-q} \right]$$

$$= r_{i+1}^{4-q} \times \text{const} \times \left[1 - \left(\frac{1}{f} \right)^{4-q} \right]$$

Thus we can assign a mass to each bin in proportion to the (radius of its upper edge)$^{4-q}$, add up all the assigned masses, then mult. the mass in each bin by the desired total mass divided by the sum of of the previously assigned values.

There remains the issue of what radius to assign to all particles in a given bin. We could for example work out the mean mass in a bin

$$\bar{m} = \frac{\int_{m_1}^{m_2} m \frac{dN}{dm} dm}{\int_{m_1}^{m_2} \frac{dN}{dm} dm}$$

$$\text{or} \quad \bar{r} = \frac{\int_{r_1}^{r_2} r \frac{dN}{dr} dr}{\int_{r_1}^{r_2} \frac{dN}{dr} dr}.$$

For now it's perhaps sufficient to take

$$r = \frac{r_{max} + r_{min}}{2}$$ for the bin, which might have an error on order of $\left(\frac{r_{max} - r_{min}}{r_{max} + r_{min}}\right)$.

$\Rightarrow$ Prob. means want $\frac{r_{i+1}}{r_i} \lesssim \sqrt{2}$